Arithmetic $\mathcal{D}$-Modules

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Contents

Introduction

0.1 Motivations

Let $q = p^s$ be a power of a prime number, $k = \mathbb{F}_q$ be a finite field with q elements and X be an algebraic variety over k of dimension d, i.e. a separated k-scheme of finite type. We want to compute

$$N_r(X) := |X(\mathbb{F}_{q^r})|$$

for all $r \in \mathbb{N}$. André Weil made in 1949 ([Wei49]) few conjectures concerning theses numbers. Weil conjectures are easier to deal with via the Zeta function of X which is the generating function

$$\zeta(X,t) = \exp\left(\sum_{r=1}^{\infty} N_r(X)\frac{t^r}{r}\right) = \prod_{x\in X^0} \frac{1}{1-t^{d(x)}},$$

where x going through the set X^0 of closed points of X, and $d(x)$ denotes the degree of the residual field $k(x)$ of x over k. Three main parts of Weil conjectures can be formulated in the following way:

(a) Rationality: the function $\zeta(X,t)$ can be written of the form

$$\zeta(X,t) = \frac{P_1(t)\dots P_{2d-1}(t)}{P_0(t)\dots P_{2d}(t)},$$

where $P_i(t) \in \mathbb{Q}[t]$ is a polynomial of degree b_i, where b_i coincide with the ith Betti-number of a lifting of X to characteristic zero, is one exists.

(b) Functional equation: if X is proper and smooth, then

$$\zeta(X, \frac{1}{q^d t}) = \pm q^{d\chi(X)/2} t^{\chi(X)} \zeta(X,t)$$

with $\chi(X) = \sum_{i=1}^{2d} (-1)^i b_i$ (Euler characteristic).

(c) Purity: if X is proper and smooth, then the polynomials $P_i(t)$ have integral coefficients. Moreover, if $P_i(t) = \prod_{j_i=1}^{b_i}(1-\omega_{j_i}t)$, the complex numbers ω_{j_i} have archimedean absolute values $q^{i/2}$ (such numbers are called Weil number of weight i).

More concretely, the rationality means that there exists finitely many algebraic integers α_j and β_j such that for all r, we have $N_r(X) = \sum \beta_j^r - \alpha_j^r$ (the β_j are equal to the ω_{j_i} with i odd and α_j are equal to the ω_{j_i} with i even). When X is proper and smooth, the functional equation is translated by the property that the application $\gamma \mapsto q^d/\gamma$ induces a permutation of the α_j's and a permutation of the β_j's. The purity implies that the algebraic integers α_j and β_j are Weil numbers with weight in $[0, 2d]$.

Using p-adic methods, Bernard Dwork in 1960 gave the first proof of the rationality ([Dwo60]). Later, using l-adic methods, where l is a prime number different from p, the characteristic of the algebraic variety, Alexandre Grothendieck proved the rationality. More precisely, let $\overline{k}$ be an algebraic closure of k, $\overline{X} := X \otimes_k \overline{k}$ and F be the Frobenius endomorphism on $\overline{X}$. He obtained the cohomological interpretation of the zeta function (e.g. [SGA4.1, SGA4.2, SGA4.3] or [Mil80, VI.13.1]):

Theorem (Grothendieck). *$Z(X,t)$ is rational and we have the equality:*

$$Z(X,t) = \prod_{i=0}^{2d_X} \det(1 - tF | H^i_{\text{ét,c}}(\overline{X}, \mathbb{Q}_l))^{(-1)^{i+1}}$$

where $H^i_{\text{ét,c}}(\overline{X}, \mathbb{Q}_l)$ designates the spaces of étale cohomology with compact support of the constant sheaf $\mathbb{Q}_l$.

He also proved the functional equation holds in 1965 (see [Gro95]). Using Grothendieck's theory of ℓ-adic étale cohomology of varieties over a field of characteristic $p \neq \ell$, Pierre Deligne in 1974 established the purity which fully completed the Weil conjectures ([Del80]). Moreover, he built a theory of mixedness and weights for constructible ℓ-adic sheaves which is compatible with the "six functors formalism" of the ℓ-adic cohomology, namely the mixedness and weight are stable under the functors $f_!$, $f_*, f^*, f^!, \otimes$ and $\mathcal{H}om$ (see [Del80]). Later, using the theory of perverse sheaves, in [BBD82] Gabber proved the stability of purity and mixedness under intermediate extensions, with which the theory of weights for ℓ-adic cohomology may be regarded as complete.

However, the problem of obtaining a p-adic Weil cohomology, i.e. getting similar results within a p-adic cohomological framework attached to a separated scheme of finite type over a perfect field k of characteristic p remained opened. Inspired by the Dwork's proof of the rationality of the zeta function, Monsky and Washnitzer introduced the notion of p-adic weak completion and constructed the so-called Monsky-Washintzer cohomology (see [MW68, Mon68, Mon71]). The is the first attempt towards a p-adic Weil cohomology (as nice as the l-adic étale cohomology for $l \neq p$) and is well suited for affine and smooth k-varieties. In the continuation of the notion of crystals and infinitesimal site formulated by Grothendieck in the zero characteristic (see [Grot68]), Berthelot constructed the crystalline cohomology over schemes of characteristic p, which is a p-adic Weil cohomology well adapted to proper and smooth k-varieties ([Ber74, Ill76, BO78]). In order to unify both crystalline and Monsky-Washnitzer cohomologies and to get a p-adic Weil cohomology at least well suited for any k-variety X, Berthelot introduced the rigid cohomology (see [LS07] and [Ber96b]). For any k-variety X, the coefficients on X studied in rigid cohomology, are the isocrystals on X, specially those endowed with a Frobenius action which are called F-isocrystals. They constitute a p-adic avatar of the notion of integrable connections (which are of finite type over the structural sheaf of the variety) on a variety of characteristic zero. This cohomology behaves well for the following multiple reasons. For any k-variety X and any F-isocrystal E on X, Kedlaya proved the finiteness of the rigid cohomological spaces with or without compact support denoted respectively by $H^*_{\text{rig}}(Y,E)$ and $H^*_{\text{rig,c}}(Y,E)$ (see [Ked06a]). In addition, Etesse and Le Stum defined (see [ÉLS93]) the L-functions associated with overconvergent F-isocrystals and obtained the following rigid cohomological interpretation of the zeta function:

Theorem (Etesse-Le Stum). *Let E be an overconvergent F-isocrystal on a smooth k-scheme Y of pure dimension d. We then have the equality*

$$L(Y,E,t) = \prod_{r=1}^{2d} \det_K(1 - tF^{-1} | H^r_{\text{rig,c}}(Y,E)).$$

When E is the constant F-isocrystal, we retrieve the Weil zeta function.

Concerning the purity, the first attempt to calculate the weights of some p-adic cohomology was made by Katz and Messing in their famous paper [KM74]. Using Deligne's deep results on weights, especially *"Le théorème du pgcd"*, they showed that, for projective smooth varieties, the weight of crystalline cohomology is the same as that of ℓ-adic one. It was reasonable to hope that the coefficient theory of weights parallel to ℓ-adic cohomology should exist in the spirit of the *petit camarade conjecture* [Del80, 1.2.10], even though there were many obstacles that prevented the construction of such a theory. After the work of Katz-Messing, efforts were made until Kedlaya finally obtained in [Ked06b] the expected estimation of weights of rigid cohomology. We do not go into more details of the history, and recommend the reader to consult the excellent explanation in the introduction of [Ked06b].

The theory of modules over suitable rings of differential operators are generically called theory of "$\mathcal{D}$-modules" and are an essential tool in the study of de Rham cohomology and other theories deriving from it. Overconvergent F-isocrystals can be seen as a p-adic avatar of the notion of smooth constructible

l-adic sheaves. For instance, tensor products, pullbacks, internal homomorphisms of overconvergent F-isocrystals give overconvergent F-isocrystals. However, the property of being stable by direct images by closed immersions are missing for such p-adic coefficients. In order to get a p-adic Weil cohomology stable under six functors on k-varieties, being inspired by the notion of p-adic weak completion appearing in Monsky-Washnitzer cohomology or rigid cohomology, Berthelot constructed a p-adic avatar of a theory of modules over the ring of differential operators. The objects appearing in his theory are called arithmetic $\mathcal{D}$-modules or complexes of arithmetic $\mathcal{D}$-modules. In the framework of Berthelot's theory of arithmetic $\mathcal{D}$-modules, several constructions of p-adic coefficients stable by Grothendieck's six functors satisfying few extra properties, e.g. with finite de Rham cohomology, that we might call six functors formalisms, have been verified. With N. Tsuzuki (see [CT12]), we established such a formalism for overholonomic F-complexes of arithmetic $\mathcal{D}$-modules (i.e. complexes together with a Frobenius structure) over realizable k-varieties (i.e. k-varieties which can be embedded into a proper smooth $W(k)$-formal scheme). Another example was given by holonomic F-complexes of arithmetic $\mathcal{D}$-modules over quasi-projective varieties ([Car11d]). In a wider geometrical context, T. Abe established a six functors formalism for admissible stacks, namely algebraic stacks of finite type with finite diagonal morphism (see [Abe18, 2.3]). The starting point of his work was the case of quasi-projective k-varieties. Again, some Frobenius structures are involved in his construction. Finally, without Frobenius structure, in [Car18], we showed how to build such a p-adic formalism of Grothendieck's six functors, e.g. with quasi-unipotent complexes of arithmetic $\mathcal{D}$-modules (see [Car18]). When the field k is not perfect, such a formalism has been extended in [Car21].

In [AC18], with T. Abe we use systematically Berthelot's arithmetic $\mathcal{D}$-modules to complete the program of establishing a p-adic theory of weights stable under six functors. In many applications, such a theory should play as important roles as suggested by the classical situations; for example, the theory of intersection cohomology and its purity, the theory of Springer representations, Lafforgue's proof of Langlands correspondence, etc.

0.2 Goals

In his introduction to arithmetic $\mathcal{D}$-modules of [Ber02], Berthelot gave an excellent overview of the state of the art of his theory in 2002. However, some proofs were missing there and were still unpublished, specially the most technical and fundamental ones concerning his notion of quasi-coherent or coherent complexes over a projective or inductive system of differential rings. One first main goal of this book is to fill this gap by giving to the reader the details of Berthelot's proofs and therefore to understand the solid foundations of the theory of arithmetic $\mathcal{D}$-modules of Berthelot.

Another main objective is to gather everything needed in the theory of arithmetic $\mathcal{D}$-modules to get in the case where the field k is perfect a six functors formalism for 1) overholonomic complexes with Frobenius structure (or more generally of Frobenius type) on realizable varieties (see here 19.1.2.19), 2) holonomic complexes with Frobenius structures which solves Berthelot's conjecture in this context (see 19.2.4) or 3) more recently some slightly more technical ingredients to construct a stable coefficient without Frobenius structure (see 19.1.3.10). We think this represents a sufficiently interesting mathematical goal to prove the maturity of Berthelot's theory. In this book, we are content to present a valid theory for realisable (e.g. quasi-projective) k-varieties (with or without Frobenius structure). We mention at the end (see 19.2.4.6) the extension (from the case of quasi-projective schemes) to certain algebraic stacks by T. Abe which allowed him to establish a striking proof of the conjecture of Deligne's little crystalline companion.

We have very good foundational papers by P. Berthelot on arithmetic $\mathcal{D}$-modules (e.g. see [Ber96c, Ber00]), but they are mostly written without logarithmic structures. Let us explain why log-structures are mandatory in the theory of arithmetic $\mathcal{D}$-modules. The proof concerning the stability of the overholonomicity with Frobenius structure reduces by devissage to check the overholonomicity of overconvergent F-isocrystals on smooth k-varieties (see 18.3.2.2 for a proof of this later result). The main ingredient is the semistable reduction of Kedlaya which is one of the most profound structure results concerning overconvergent F-isocrystals. This allows us to reduce to the case of log extendable isocrystals, which requires a log version of arithmetic $\mathcal{D}$-modules. Hence, another objective of the book is to properly rewrite all of Berthelot's constructions within this framework as general as possible and check the log version of theorems when they are valid. We tried to have complete proofs and to be as self-contained as possible. However, we have sometimes omitted some proofs by simply giving a few references or the main ingredients of the checks. Let us cite in particular Berthelot's Frobenius descent of the level (see [Ber00]), the finite homological dimension of the sheaf of differential operators of finite level when the log structure comes from a strict normal crossing divisor proved by C. Montagnon (see 6.1.4.2), with overconvergent singularities and with trivial log structure established by C. Huyghe (see 8.7.7.9) or the commutation of the extraordinary pullbacks of a smooth morphism to duality proved by T. Abe (see 11.3.5.1.1).

0.3 Overview

Let $X \to S$ be a smooth morphism of schemes. We recall in the first chapter Grothendieck's construction of the usual sheaf of differential operators $\mathcal{D}_{X/S}$ of X/S, which (in contract with Berthelot construction) can be clarified by adding "of infinite level and finite order" and denoted by $\mathcal{D}^{(\infty)}_{X/S}$. Then we introduce the sheaf of differential operators $\mathcal{D}^{(m)}_{X/S}$ of finite order and of finite level m for some integer $m \in \mathbb{N}$ as defined in ([Ber96c] §2, [Ber02] §1.2.2, [Ber90] §1.2) by Berthelot. In order to obtain coherent sheaf of rings, the idea is to replace in Grothendieck's construction the diagonal immersion by its m-PD envelope whose notion is explained here in details. For instance, in the level 0 case, the sheaf $\mathcal{D}^{(0)}_{X/S}$ is locally generated by $\mathcal{O}_X$ and the derivations. Next, we study the notion of left and right $\mathcal{D}^{(m)}_{X/S}$-modules (for m finite or not) via respectively m-PD-stratifications and m-PD-costratifications. Beware that contrary to the infinite level case, $\mathcal{D}^{(m)}_{X/S}$ is not a subring $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$ but we have ring homomorphisms $\mathcal{D}^{(m)}_{X/S} \to \mathcal{D}^{(m+1)}_{X/S} \to \mathcal{D}_{X/S}$ which gives for instance a canonical structure of left $\mathcal{D}^{(m)}_{X/S}$-modules on $\mathcal{O}_X$. We pay particular attention to the case of level zero, which had already been studied in [Ber74]. For instance, we give a detailed proof of the equivalence for an $\mathcal{O}_X$-module $\mathcal{E}$ between the data of an integrable connection relative to X/S on $\mathcal{E}$ and a structure of left $\mathcal{D}^{(0)}_{X/S}$-module on $\mathcal{E}$ extending its structure of $\mathcal{O}_X$-module (see 2.3.2.6).

We then move on to the logarithmic context. To make things easier, we introduce the notion of nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ for some integer i and we exclusively works in this context: a fine log scheme $S^\sharp$ over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ is nice if there exists a scheme Z such that $S^\sharp$ is a log flat Z-log-scheme (see the definition [Ogu18, IV.4.1.1] of log-flatness, which we can also simply be called flatness). The useful point which simplify the comparison between logarithmic and non logarithmic derivations is that when $S^\sharp$ is a nice fine log scheme then, denoting by $U := (S^\sharp)^*$ the open in $S^\sharp$ subscheme with trivial log-structure and $j\colon U \to S^\sharp$ the canonical inclusion, the canonical morphism $\mathcal{O}_{S^\sharp} \to j_*\mathcal{O}_U$ is injective (see 3.1.1.3). Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0, p)$, with perfect residue field k and field of fraction K. We also benefit from a notion of nice fine log $\mathcal{V}$-formal scheme. Let $X^\sharp \to S^\sharp$ be a log smooth morphism of nice fine log schemes (or log $\mathcal{V}$-formal schemes). We construct similarly the sheaf of differential operators $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ of finite order and of finite level m for some integer $m \in \mathbb{N}$. Again, the notion of left and right $\mathcal{D}^{(m)}_{X/S}$-modules can be translated in terms of m-PD-stratifications and m-PD-costratifications. In order to describe the right structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module of the sheaf $\omega_{X^\sharp/S^\sharp}$, we introduce the notion of logarithmic transposition (see 3.4.1). We have treated separately the non logarithmic case because for instance in the case where the log structure is trivial, the correspondance between logarithmic and non-logarithmic properties or formulas are not always obvious (e.g. the logarithmic inverse formula of the Taylor formula of 3.4.5.3.2 or the logarithmic transposition version describing the tensor product formula of 3.4.2.7.2), or because logarithmic formulas are much more complicated than the non logarithmic ones (e.g. see 1.4.2.7 vs 3.2.3.13.1).

Next, we explain in the forth chapter how to obtain "coefficients" of differential operators more general than the constant coefficient $\mathcal{O}_X$. More precisely, let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra which is endowed with a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{B}_X$ which is "compatible" with its $\mathcal{O}_X$-algebra structure. This compatibility makes it possible to obtain a canonical ring structure on the sheaf $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. In this context, $\mathcal{B}_X$ play in $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ the role of the constant coefficient $\mathcal{O}_X$ in $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. We can view $\mathcal{B}_X$ as the coefficients of the differential operators of $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. We will mainly need later such a generalization in order to construct the sheaf of differential operators with overconvergent singularities along a divisor. Beware that this notion of compatibility requires the full understanding of the constant case (i.e. $\mathcal{B}_X = \mathcal{O}_X$), so the reading of the first chapters are mandatory before going to the general case of log $\mathcal{D}$-modules with coefficients and we had to treat the constant case separately.

Once the differential operators *of finite order* are well defined in their full generality, we then define the cohomological functors in the case of differential operators *of finite orders*: the pushforwards f_+ and extraordinary pullbacks $f^!$ by a morphism f, dual functor $\mathbb{D}$, external tensor product $\boxtimes$, internal tensor product $\otimes$. In the case where log structures are trivial, we then give the Frobenius descent of Berthelot whose reader can find complete proofs in [Ber00]. The main application of such a Frobenius descent is the finiteness of the homological dimension of the sheaf of differential operators of level m: Berthelot reduces by descent to the level 0 case which is checked using the same computations as in the characteristic zero case.

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme. Moreover, let $\mathfrak{P}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes (the underlying formal schemes are denoted by $\mathfrak{S}$ and $\mathfrak{P}$). Let π be a uniformizer of $\mathcal{V}$ and $P^\sharp \to S^\sharp$ be the induced modulo π morphism of log schemes over $\operatorname{Spec} k$. For any integer $i \geq 0$, set $P_i^\sharp := \mathfrak{P}^\sharp \times_{\operatorname{Spf} \mathcal{V}} \operatorname{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$, $P^\sharp := P_0^\sharp$. For simplicity, suppose $\mathfrak{P}$ is p-torsion free, noetherian of finite Krull dimension and P is regular.

Taking the p-adic completion of $\mathcal{D}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$, we get the sheaf $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$ of differential operators of level m and *infinite* order (roughly speaking locally we get p-adically converging power series on the derivations), which remains a coherent sheaf of rings with Noetherian sections. When one wants to extend cohomological operations to complexes of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$-modules, technical difficulties related to completions appear. To obtain the desired transitivity formulas when composing several functors that do not necessarily preserve coherence (in particular the transitivity of inverse and direct images), we have to integrate completion into the definition of these functors, and thus also to impose a completion condition on the complexes with which we are working. We call quasi-coherent in the sense of Berthelot the complexes $\mathcal{E}$ of $D^-(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$ such that $\mathcal{O}_P \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}} \mathcal{E}$ has quasi-coherent cohomology, and that the adjunction morphism $\mathcal{E} \to \mathbb{R}\varprojlim_i \mathcal{O}_{P_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}} \mathcal{E}$ is an isomorphism. These conditions are verified by bounded complexes with coherent cohomology. We can then extend the cohomological operations studied (the finite order case) above to these complexes, and deduce from them by completion the expected properties or theorems. This is written in detail in chapter 7 and is mostly based on the unpublished notes of Berthelot.

Taking the inductive limit on the level of the inductive system $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp} = (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})_{m\in\mathbb{N}}$, we get the sheaf of differential operators of finite level and infinite order $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$, which is a subsheaf of the p-adic completion of the sheaf of usual differential operators $\mathcal{D}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$. Let $\mathfrak{U}^\sharp$ be an affine open of $\mathfrak{P}^\sharp$ endowed with logarithmic coordinates. Let $P \in \Gamma(\mathfrak{U}, \widehat{\mathcal{D}}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$. We can uniquely write

$$P = \sum_{\underline{k}\in\mathbb{N}^d} a_{\underline{k}} \underline{\partial}_\sharp^{[\underline{k}]},$$

with $a_{\underline{k}} \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P}})$ a sequence converging to 0 for the p-adic topology when $|\underline{k}|$ goes to infinity. For any $i \in \mathbb{N}$, let $P_i \in \Gamma(U, \mathcal{D}_{P_i^\sharp/S_i^\sharp})$ be the image of P, where $P_i^\sharp/S_i^\sharp$ is the reduction modulo π^{i+1} of $\mathfrak{P}^\sharp/\mathfrak{S}^\sharp$. The following conditions are equivalent (see 8.7.1.8):

(a) $P \in \Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$;

(b) $\exists \alpha, \beta \in \mathbb{R}$ such that $\operatorname{ord}(P_i) \leq \alpha i + \beta$ for any $i \in \mathbb{N}$;

(c) $\exists c, \eta \in \mathbb{R}_+$ such that $\eta < 1$ and $\|b_{\underline{k}}\| \leq c\eta^{|\underline{k}|}$, for any $\underline{k} \in \mathbb{N}^d$.

This can be interpreted by saying that $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$ is the weak completion of $\mathcal{D}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$ as $\mathcal{O}_{\mathfrak{P}}$-ring, the dagger symbole meaning "p-adic completion". The sheaf $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$, its tensorisation with $\mathbb{Q}$, is not any more noetherian but is at least coherent, thanks to a theorem of flatness of the transition homomorphisms $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$. Montagnon proved that it is also of finite homological dimension when $\mathfrak{P}$ is a smooth $\mathcal{V}$-formal scheme and log structure comes from a relative to $\mathfrak{P}/\mathfrak{S}$ normal crossing divisor (see [Mon02]).

A key fundamental idea of Berthelot's theory is the need to work with more evolved categories which we might be called "LD-type" category. Let us first recall what these categories introduced by Berthelot consist of. Consider first the derived category of $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$-modules, i.e. of inductive systems of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$-modules. By tensorizing by $\mathbb{Q}$ and taking the inductive limit on the level, we obtain the functor denoted by $\underrightarrow{l}^*_{\mathbb{Q}}\colon D^{\mathrm{b}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}) \to D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp,\mathbb{Q}})$. In order to obtain a fully faithful functor which factorises this functor $\underrightarrow{l}^*_{\mathbb{Q}}$, Berthelot has introduced the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$ which is an appropriate localization (by ind-isogenies and lim-isomorphisms) of $D^{\mathrm{b}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$. He has also defined a full subcategory of "coherent" (in the sense of Berthelot) complexes of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$ that he denotes by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$. Then he has established that the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ induces the equivalence of categories

$$(*)\qquad \underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp,\mathbb{Q}}).$$

This Berthelot theorem is at the heart of the foundations of the theory of arithmetic $\mathcal{D}$-modules. Such theorem and related to the categories of the form $\underrightarrow{LD}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$ results were announced in [Ber02]. In this book, the reader will found complete and detailed proofs of theses unpublished Berthelot's foundational theorems.

Berthelot also defines the full subcategory $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$ formed by "quasi-coherent" complexes. The pushforwards and extraordinary pullbacks can be defined for both categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$ and $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ and coincide for coherent complexes via the above equivalence of categories $(*)$ induced by $\underrightarrow{l}^*_{\mathbb{Q}}$. However, even if LD-type categories are technically more involved, the LD-type categories were constructed so that the verification of the expected properties can be formally reduced to the case of schemes (i.e. to the context of finite order operators, without completion or passage to the inductive limit on the level). For instance, we get without further effort that pushforwards and extraordinary pullbacks preserve the quasi-coherence in the sense of Berthelot and are transitive with respect to the composition. Another huge advantage to work with LD-type categories is the possibility of properly and canonically defining the internal tensor product, which is translated for categories of the form $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ by a sort of derived p-adic weak completion of the usual tensor product (such derived p-adic weak completions need a priori the choice of a model in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$, i.e. we have to choose an inverse of the equivalence $\underrightarrow{l}^*_{\mathbb{Q}}$ which do not seem canonical). More comprehensively, in the chapter 9 we give the following cohomological operations for quasi-coherent inductive systems of arithmetic $\mathcal{D}$-modules: pushforwards, extraordinary pullbacks, base changes, dual functors, exterior or internal tensor products.

Using internal tensor products, we define next localisation outside a divisor as follows. Let T be a divisor of P. Let $\mathfrak{Y}^\sharp$ be the open of $\mathfrak{P}^\sharp$ complementary to T and $j\colon \mathfrak{Y}^\sharp \subset \mathfrak{P}^\sharp$ be the inclusion. Let $m \in \mathbb{N}$ be an integer such that $p^m \geq e/(p-1)$. We get a p-adically complete $\mathcal{O}_{\mathbb{P}}$-algebra $\mathcal{B}^{(m)}_{\mathfrak{P}}(T)$ which is endowed with compatible canonical structure of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$-module together with a canonical $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}$-linear inclusion $\mathcal{B}^{(m)}_{\mathfrak{P}}(T) \to j_*\mathcal{O}_{\mathfrak{Y}}$. We define respectively the sheaf of "functions on $\mathfrak{P}$ with overconvergent singularities along the divisor T" and the sheaf of "differential operators of finite level on $\mathfrak{P}^\sharp/\mathfrak{S}^\sharp$ with overconvergent singularities along of T" by setting

$$\mathcal{O}_{\mathfrak{P}}({}^\dagger T) := \varinjlim_m \mathcal{B}^{(m)}_{\mathfrak{P}}(T),$$
$$\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T) = \varinjlim_m \mathcal{B}^{(m)}_{\mathfrak{P}}(T)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{P}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}.$$

Locally, if $\mathfrak{U} = \operatorname{Spf} A$ is some open of $\mathfrak{P}$ such that there exist a lifting $f \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P}})$ of an equation of $T \subset P$, then $\mathcal{B}^{(m)}_{\mathfrak{U}}(T\cap U)$ is the p-adic completion of $\mathcal{O}_{\mathfrak{U}}[T]/(f^{p^{m+1}}T - \pi)$. A version with π replaced by p is available in the litterature (see [Ber96c]), but by default we prefer to use Huyghe's version. This is harmless since the ring inductive systems $\mathcal{B}^{(\bullet)}_{\mathfrak{P}}(T)$ are isomorphic (up to a canonical lim-isomorphism). After tensorising by $\mathbb{Q}$ the sheaf $\mathcal{O}_{\mathfrak{P}}({}^\dagger T)$ (indicated by the index $\mathbb{Q}$), we have the following nice description:

$$\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \xrightarrow{\sim} \left\{ \sum_{n=0}^{\infty} \frac{a_n}{f^n} : a_n \in A_K,\ \exists\, c, \eta \in \mathbb{R}, \eta < 1 \text{ such that } \|a_n\| \leq c\eta^n \right\},$$

where A_K is endowed with the topology induced by the p-adic topology of A which induces a p-adic norm on A_K (see 8.7.1.6).

The localisation outside T is by definition the functor $({}^\dagger T) := \mathcal{B}^{(\bullet)}_{\mathfrak{P}}(T)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{P}}} -$ which preserves the quasi-coherence in the sense of Berthelot. This functor can simply be described for coherent complexes: if $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$ and $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ is the associated complex via the equivalence $\underrightarrow{l}^*_{\mathbb{Q}}$, then $\underrightarrow{l}^*_{\mathbb{Q}}\left(({}^\dagger T)(\mathcal{E}^{(\bullet)})\right) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}$.

The local cohomological functor with strict support over T is the functor $\mathbb{R}\underline{\Gamma}^\dagger_T$ so that the triangle

$$\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)})[1].$$

is exact for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$ (see 13.1.1.5.1).

More generally, both functors can uniquely up to canonical isomorphism be extended by a local functor $\mathbb{R}\underline{\Gamma}^\dagger_U$ over any subscheme U of P as follows: if U is the complementary of a divisor T of P then $\mathbb{R}\underline{\Gamma}^\dagger_Y = ({}^\dagger T)$, for any subschemes U and U' of P we have the natural isomorphism $\mathbb{R}\underline{\Gamma}^\dagger_U \circ \mathbb{R}\underline{\Gamma}^\dagger_{U'} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{U\cap U'}$ and for any closed subscheme U' of a subscheme U of P, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$ we have the exact triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp})$:

$$\mathbb{R}\underline{\Gamma}^\dagger_{U'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_U(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{U\setminus U'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{U}(\mathcal{E}^{(\bullet)})[1].$$

Let us now explain the link between overconvergent isocrystals and arithmetic $\mathcal{D}$-modules in the following lifted case, which is the starting point of building a bridge between the rigid cohomology world with arithmetic $\mathcal{D}$-modules one. Suppose $\mathfrak{S}$ is a smooth $\mathcal{V}$-formal scheme, $\mathfrak{P}/\mathfrak{S}$ is smooth and there exists a relative strict normal crossing divisor $\mathfrak{Z}$ of $\mathfrak{P}/\mathfrak{S}$ such that $\mathfrak{P}^\sharp = (\mathfrak{P}, M_{\mathfrak{Z}})$ is the logarithmic $\mathcal{V}$-formal scheme whose underlying logarithmic structure $M_{\mathfrak{Z}}$ is the one associated with $\mathfrak{Z}$. We have the morphism of ringed spaces $\mathrm{sp}\colon (\mathfrak{P}_K, \mathcal{O}_{\mathfrak{P}_K}) \to (\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$ induced by the specialization morphism. We get the inverse image functor sp^* by setting $\mathrm{sp}^*(\mathcal{E}) := \mathcal{O}_{\mathfrak{P}_K} \otimes_{\mathrm{sp}^{-1}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}} \mathrm{sp}^{-1}(\mathcal{E})$, for any $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-modules $\mathcal{E}$.

The functors sp_* and sp^* induce quasi-inverse equivalences between the category of (resp. locally free of finite type) coherent $\mathcal{O}_{\mathfrak{P}_K}$-modules together with a convergent (i.e. satisfying a convergent condition: see 10.3.2.3.1) logarithmic connection relative to $\mathfrak{P}^\sharp_K/\mathfrak{S}_K$ and that of left $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S},\mathbb{Q}}$-modules which are (resp. locally projective of finite type) coherent as $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module (see 11.1.1.2). When $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$, the category consisting of left $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S},\mathbb{Q}}$-modules which are locally projective of finite type as $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module is equivalent to the full subcategory of $I_{\mathrm{conv}}(P^\sharp/\mathrm{Spf}\,\mathcal{V})$ consisting of locally free isocrystals on the log convergent site $((P, M)/\mathrm{Spf}\,\mathcal{V})_{\mathrm{conv}}$ defined by Shiho at [Shi02, Definition 2.1.5] (see here 10.3.1.1).

Let T be a divisor of P and $\mathrm{MIC}^\dagger(\mathfrak{P}^\sharp_K, T/\mathfrak{S}^\sharp_K)$ be the category of overconvergent along T log-isocrystal on $\mathfrak{P}^\sharp/\mathfrak{S}^\sharp$ (see the chapter 10 and 11). We denote by $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$ the category of $\mathcal{D}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules which are coherent as $\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module and such that the underlying connection is overconvergent along T. We get almost by definition that the functors sp_* and sp^* induce quasi-inverse equivalences of categories between $\mathrm{MIC}^\dagger(\mathfrak{P}^\sharp_K, T/\mathfrak{S}^\sharp_K)$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$.

From now let $\mathfrak{S} := \mathrm{Spf}\,\mathcal{V}$ and let $\mathfrak{P}$ is a separated and smooth $\mathfrak{S}$-formal scheme. In that case, the category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$ is equal to the full category of that of coherent $\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-modules consisting of objects which are coherent as $\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module (see 11.2.1.14). In fact, we can check that the objects of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$ are overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-modules, i.e., for any smooth morphism of the form $f\colon \mathfrak{P}' \to \mathfrak{P}$, for any divisor T' containing $f^{-1}(T)$ we have $({}^\dagger T') \circ f^!(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger f^{-1}(T))_{\mathbb{Q}})$. For instance, when T is empty, this means that the localisation functor outside a divisor preserves the $\mathcal{D}^\dagger$-coherence and this latter property is stable under pullbacks by smooth morphisms.

We would like to extend the category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$ with T is replaced by a closed subscheme Z of P. When T is not empty, an object of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$ is not (in general) a coherent $\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}$-module. Moreover, the sheaf $\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}$ has a priori no meaning because Z is not a divisor. So, in order to well define $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, Z/\mathcal{V})$, it seems unavoidable to work with quasi-coherence complexes in the sense of Berthelot and to build it as a full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$. For this purpose, we construct a strictly full subcategory $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$ consisting of objects $\mathcal{E}^{(\bullet)}$ satisfying certain overconvergent condition which is required to be stable by base change (see 15.3.7.1). The objects of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)))$ are called $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)$-overcoherent $(F\text{-})$complexes after any base change (this is an abuse of notation since $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)$ do not a priori exists). When Z is the support of a divisor T, we retrieve the above notion of $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T)$-overcoherence. It has a canonical t-structure, its heart is written $(F\text{-})$ $\underrightarrow{LM}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. If $U = P \setminus Z$, we get an equivalence between $\mathrm{MIC}^{\dagger}(Y, P/K)$ and $\mathrm{MIC}^{(\bullet)}(\mathfrak{P}, Z/\mathcal{V})$, where $\mathrm{MIC}^{(\bullet)}(\mathfrak{P}, Z/\mathcal{V})$ is the subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ of objects which are roughly speaking $\mathcal{O}_{\mathfrak{P}}$-coherent outside Z.

More generally, let Z and X be two closed subschemes of P such that $Y = X \setminus Z$ is smooth. We construct $(F\text{-})\mathrm{MIC}^{(\bullet)}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$ (see 16.2.1.1), a strictly full abelian subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ consisting of objects with support in X and which can be seen as $\mathcal{O}$-coherent on Y, together with the canonical equivalence of categories (see 16.2.1.10.1):

$$\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +} \colon \mathrm{MIC}^{\dagger}(Y, X/K) \cong \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}).$$

The category $\mathrm{MIC}^{(\bullet)}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$ can be written $\mathrm{MIC}^{(\bullet)}(Y, \mathfrak{P}/\mathfrak{S})$ since this only depends on $Y \subset \mathfrak{P}$.

Let Y be a subvariety of P. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. Via the above equivalence of categories of the form $\mathrm{sp}^{(\bullet)}_{+}$, we get a notion of devissability into overconvergent isocrystals as follows: The complex $\mathcal{E}^{(\bullet)}$ "splits into overconvergent isocrystals" if there exists a smooth stratification of P of the form $P = \sqcup_{i=1,\ldots,r} Y_i$ such that, for any $i = 1, \ldots, r$, the cohomological spaces of $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y_i}(\mathcal{E}^{(\bullet)})$ belong to $\mathrm{MIC}^{(\bullet)}(Y_i, \mathfrak{P}/\mathcal{V})$.

We then focus on the construction of stable properties by the 6 Grothendieck functors with a Frobenius structure. Let us now explain further notions stable by Grothendieck's six functors with a Frobenius structure. Suppose from now that k is perfect and $\mathfrak{P}$ is a smooth $\mathcal{V}$-formal scheme. We give Berthelot's construction of the characteristic variety associated with a coherent F-$\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. This characteristic variety satisfies Bernstein's inequality which yields as usual to the notion of holonomicity. In fact, without Frobenius structure, even if the notion of characteristic variety associated with a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module is problematic because we canot reduce canonically to the level 0 case, we can still define a notion of dimension of a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module so that Bernstein's inequality holds, which extends the notion of holonomicity (see chapter 15). Beware that without Frobenius structure, we do not know if the holonomicity is stable under pullbacks by a closed immersion for instance. In order to get a notion stable by duality and stronger (a priori) to the overcoherence (in the strong sense: as complexes of $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules, i.e. without overconvergent singularities added in $\mathcal{D}^{\dagger}$), we introduce the notion of overholonomicity (see [Car09b]) and that of overholonomicity after any base change to get a notion stable by base change (see 18.1). This notion of overholonmicity is stable by duality, (extraordinary) pushforwards and (extraordinary) pullbacks. Without Frobenius structure, the stability of the overholonomicity under tensor products is an open question. However, with Tsuzuki, we proved that for any smooth subvariety Y of $\mathfrak{P}$ the following theorem (see [CT12] or 18.3.2.2 for a slight simplication of the proof):

Theorem (C.-Tsuzuki)**.** *Let Y be a smooth subvariety of $\mathfrak{P}$. The objects of F-$\mathrm{MIC}^{(\bullet)}(Y, \mathfrak{P}/\mathcal{V})$ are overholonomic after any base change.*

This yields that for F-complexes the notion of overcoherent, overholonomicity after any base change, devissability into overconvergent isocrystals are equal (see 18.3.2.3). Since the tensor product of isocrystals gives isocrystals, this implies that the notion of overholonomicity (after any base change) for F-complexes is stable by tensor products. These notions are then stable under Grothendieck's six functors.

When $\mathfrak{P}$ is projective (which is possible when Y is quasi-projective), we prove furthermore that in the case of F-complexes these notions are equal to that of holonomic F-complexes. The proof relies on a construction by hand (using affine $\mathcal{V}$-weak formal schemes) for certain affine and smooth k-schemes Y of a functor sp_{Y+} from the category of overconvergent isocrystal on Y to that of arithmetic $\mathcal{D}$-modules, studying specially the case where Y has a finite étale map over an affine space (see chapter 17).

Finally, in the last chapter we explain how to build a formalism of Grothendieck six functors for (without Frobenius structure) arithmetic $\mathcal{D}$-modules associated to realisable k-varieties. The rough idea is the same but more technical than for the overholonmicity since we have to define stable by six functors (including tensor products) categories, the stability being formal "by construction".

Notation

Without further mentioning, all occuring modules will be left modules. We let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of non-negative integers.

We fix $\mathcal{V}$ a complete discrete valuation ring of mixed characteristics $(0, p)$, e its absolute ramification, k its residual field, K its field of fractions and π a uniformizer.

Logarithmic formal schemes (resp. $\mathcal{V}$-weak formal schemes, resp. schemes) will be denoted by gothic letters (resp. roman letters with the exponent † or without if no confusion is possible, resp. roman letters). The special fibers of $\mathcal{V}$-formal schemes will be denoted by the corresponding straight letters with or without 0 as index. The $\mathcal{V}$-formal scheme induced by p-adic completion of a $\mathcal{V}$-weak formal scheme will be denoted by the gothic corresponding letter, e.g., if $P^\dagger$ a $\mathcal{V}$-weak formal schemes, we set $\mathfrak{P} := \widehat{P^\dagger}$ and $P := \mathfrak{P} \otimes_{\mathcal{V}} k$ or $P_0 := \mathfrak{P} \otimes_{\mathcal{V}} k$. Moreover, if u is a morphism of logarithmic $\mathcal{V}$-formal schemes, the induced modulo π morphism between special fibers will be denoted by u_0 or simply u.

If $\mathcal{E}$ is a sheaf of $\mathcal{V}$-modules, we set $\mathcal{E}_{\mathbb{Q}} := \mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{V}} K$ and $\widehat{\mathcal{E}}$ is the p-adic completion of $\mathcal{E}$.

For any $\underline{k} \in (k_1, \dots, k_d) \in \mathbb{N}^d$, we set $|\underline{k}| = k_1 + \cdots + k_d$.

If D, D' are two rings, we say that E is a (D, D')-bimodule (resp. left bimodule, resp. right bimodule) if E is endowed with two compatible structures of left (resp. left, resp. right) D -module and right (resp. left, resp. right) D'-module. If $D = D'$, we simply say (resp. left, resp. right) D-bimodule.

If $\phi \colon \mathcal{A} \to \mathcal{B}$ is a homomorphism of sheaves of rings, for any $\mathcal{A}$-module $\mathcal{M}$ (resp. $\mathcal{B}$-module $\mathcal{N}$), we will denote by $\phi^\flat(\mathcal{M}) := \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M})$ (resp. $\phi_*(\mathcal{N})$ is $\mathcal{N}$ viewed as $\mathcal{A}$-module). The convention is to work in the derived categories but the functors of the form $\phi^\flat$ that we will use will be exact.

Let $\mathfrak{A}$ be an abelian category. We denote by $C(\mathfrak{A})$ the category of complexes of $\mathfrak{A}$. We say that a complex $X^\bullet \in C(\mathfrak{A})$ is bounded from below (resp. bounded from above) if there exists $n_0 \in \mathbb{Z}$ such that $X^n = 0$ for $n < n_0$ (resp. $X^n = 0$ for $n > n_0$). The complex $X^\bullet$ is bounded if it is bounded from above and below. We denote by $C^+(\mathfrak{A})$ (resp. $C^-(\mathfrak{A})$, resp. $C^{\mathrm{b}}(\mathfrak{A})$) the full subcategory of $C(\mathfrak{A})$ consisting of bounded below complexes (resp. bounded above complexes, resp. bounded complexes). Let $*$ be one of the symbol $\emptyset$, $+$, $-$, or b. We denote by $K^*(\mathfrak{A})$ the homotopic category of complexes of $C^*(\mathfrak{A})$, i.e., $K^*(\mathfrak{A})$ is a triangulated category with the same objects as $C^*(A)$ and classes of homotopic morphisms as morphisms. The class S of all quasi-isomorphisms in $K^*(\mathfrak{A})$ is a localizing class compatible with the triangulation. Localize by this class S, we get triangulated category $D^*(\mathfrak{A}) := K^*(\mathfrak{A})[S^{-1}]$ together with a triangulated functor $Q \colon K^*(\mathfrak{A}) \to D^*(\mathfrak{A})$.

Let $\mathcal{A}$ be a sheaf of rings on a topological space X (or a topos). If $*$ is one of the symbol $\emptyset$, $+$, $-$, or b, we denote by $D^*({}^{\mathrm{l}}\mathcal{A})$ (resp. $D^*({}^{\mathrm{r}}\mathcal{A})$) the derived category of complexes of left (resp. right) $\mathcal{A}$-modules satisfying the corresponding vanishing conditions. If $*$ is one of the symbol $-$, or b, we write $D^*_{\mathrm{coh}}(\mathcal{A})$ (resp. $D^*_{\mathrm{tdf}}(\mathcal{A})$, resp. $D^*_{\mathrm{perf}}(\mathcal{A})$) for the full subcategory of $D^*(\mathcal{A})$ consisting of pseudo-coherent (resp. finite tor dimension, resp. perfect) complexes

If X is a scheme and $\mathcal{A}$ is an $\mathcal{O}_X$-algebra, then for $\star \in \{\emptyset, +, -, b\}$, we denote by $D^\star_{\mathrm{qc}}(\mathcal{A})$ the full subcategory of $D^\star(\mathcal{A})$ consisting of complexes whose cohomology sheaves are $\mathcal{O}_X$-quasi-coherent.

Chapter 1

Sheaves of differential operators of finite order

1.1 Sheaves of differential operators of infinite level and finite order

1.1.1 nth infinitesimal neighborhood

Let S be a scheme.

Definition 1.1.1.1. (a) We denote by $\mathscr{C}$ the category whose objects are S-immersions of schemes and whose morphisms $u' \to u$ are commutative diagrams of the form

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ Z' & \longrightarrow & Z. \end{array} \qquad (1.1.1.1.1)$$

We say that $u' \to u$ is flat (resp. cartesian) if f is flat (resp. the square 1.1.1.1.1 is cartesian).

(b) Let $n \in \mathbb{N}$. We denote by $\mathscr{C}_n$ the full subcategory of $\mathscr{C}$ whose objects are nilpotent closed immersions of order n.

(c) Let u be an object of $\mathscr{C}$. A "nilpotent closed immersion of order n induced by u" is an object u' of $\mathscr{C}_n$ endowed with a morphism $u' \to u$ of $\mathscr{C}$ satisfying the following universal property: for any object u'' of $\mathscr{C}_n$ endowed with a morphism $f : u'' \to u$ of $\mathscr{C}$ there exists a unique morphism $u'' \to u'$ in $\mathscr{C}_n$ whose composition with $u' \to u$ is f. The existence and the uniqueness up to canonical isomorphism of the nilpotent closed immersion of order n induced by u are obvious. We will denote by $P^n(u)$ the nilpotent closed immersion of order n induced by u. We also say that $P^n(u)$ is the "nth infinitesimal neighbourhood of u"

Proposition 1.1.1.2. *Let $n \geq 0$ be an integer.*

(a) The inclusion functor $For_n : \mathscr{C}_n \to \mathscr{C}$ has a right adjoint functor which we will denote by $P^n : \mathscr{C} \to \mathscr{C}_n$.

(b) Let $u : Z \hookrightarrow X$ be an object of $\mathscr{C}$. By abuse of notation we can also write $P^n(u)$ for the target of the arrow $P^n(u)$.

(i) Then Z is also the source of $P^n(u)$.

(ii) If $U \subset X$ is an open subset containing Z such that $v : Z \hookrightarrow U$ is a closed immersion then $P^n(u) = P^n(v)$ and the morphism $P^n(u) \to U$ is affine. In that case, we denote by $\mathcal{P}^n(u)$ the quasi-coherent $\mathcal{O}_U$-algebra so that $P^n(u) = \mathrm{Spec}(\mathcal{P}^n(u))$. The sheaf $\mathcal{P}^n(u)$ has his support in Z and we can simply denote $v^{-1}\mathcal{P}^n(u)$ by $\mathcal{P}^n(u)$. If X is noetherian, then so is $P^n(u)$.

Proof. Let U be an open subset of X containing $u(Z)$ such that the induced morphism $v\colon Z \to U$ is a closed immersion. Let $\mathcal{I}$ be the ideal of $\mathcal{O}_X$ given by v and $U_n := \mathrm{Spec}(\mathcal{O}_X/\mathcal{I}^{n+1})$. Then the closed immersion $Z \hookrightarrow U_n$ satisfies the universal property of $P^n(u) \to u$ (and hence do not depend on the choice of U). □

Let us recall the following standard definitions.

Definition 1.1.1.3. Let $f\colon X \to Y$ be an S-morphism.

(a) We say that f is "formally étale" (resp. "formally unramified") if it satisfies the following property: for any commutative diagram of schemes of the form

$$\begin{array}{ccc} U & \xrightarrow{u_0} & X \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{v} & Y \end{array} \tag{1.1.1.3.1}$$

such that ι is an object of $\mathscr{C}_1$, there exists a unique morphism (resp. there exists at most one morphism) $u\colon T \to X$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

(b) We say that f is "étale" if f is formally étale and Zariski locally of finite presentation.

Definition 1.1.1.4. Let $f\colon X \to Y$ be a morphism of S-schemes.

(a) We say that a finite set $t_1, \ldots, t_d$ of elements of $\Gamma(X, \mathcal{O}_X)$ are "formal coordinates of f" if the corresponding Y-morphism $X \to \mathbb{A}^d_Y$ is formally étale.

(b) We say that a finite set $t_1, \ldots, t_d$ of elements of $\Gamma(X, \mathcal{O}_X)$ are "coordinates of f" if the corresponding Y-morphism $X \to \mathbb{A}^d_Y$ is étale.

(c) We say that f is "weakly smooth" if, étale locally on X, f has formal coordinates. Notice that this notion of weak smoothness is étale local on Y.

Recall that f is "smooth" if, étale locally on X, f has coordinates. Notice that this notion of smoothness is étale local on Y.

Lemma 1.1.1.5. *Let $n \in \mathbb{N}$, $f\colon X \to Y$ be a formally étale morphism of S-schemes, $u\colon Z \hookrightarrow X$ and $v\colon Z \hookrightarrow Y$ be two S-immersions such that $v = f \circ u$. Then we have $P^n(u) = P^n(v)$.*

Proof. By using the universal property of the formal etaleness, we get a unique Y-morphism $P^n(v) \to u$ (see 1.1.1.3). It is sufficient to prove that $P^n(v) \to v$ satisfies the universal property corresponding to $P^n(u) \to u$, which is easy. □

Lemma 1.1.1.6. *Let $u \to v$ be a morphism of $\mathscr{C}$. Let $w := P^n(v) \times_v u$ (this is the product in $\mathscr{C}$). Then $P^n(w) = P^n(u)$.*

Proof. We can easily check that the composition $P^n(w) \to w \to u$ satisfies the universal property of $P^n(u) \to u$. Hence, we are done.

□

Proposition 1.1.1.7. *Let $f\colon X \to Y$ be an S-morphism of schemes, $(t_\lambda)_{\lambda=1,\ldots,r}$ be formal coordinates of f. Let $u\colon Z \hookrightarrow X$ and $v\colon Z \hookrightarrow Y$ be two S-immersions of schemes such that $v = f \circ u$. Suppose that there exist $y_\lambda \in \Gamma(Y, \mathcal{O}_Y)$ whose images in $\Gamma(Z, \mathcal{O}_Z)$ coincide with the images of t_λ. Then we have the following isomorphism of $\mathcal{O}_{P^n(v)}$-algebras*

$$\mathcal{O}_{P^n(v)}[T_1, \ldots, T_r]_n \xrightarrow{\sim} \mathcal{O}_{P^n(u)}$$
$$T_\lambda \mapsto t_\lambda - f^*(y_\lambda), \tag{1.1.1.7.1}$$

with $\mathcal{O}_{P^n(v)}[T_1, \ldots, T_r]_n := \mathcal{O}_{P^n(v)}[T_1, \ldots, T_r]/\left(I_n + (T_1, \ldots, T_r)\right)^{n+1}$, where I_n is the ideal defined by the closed immersion $P^n(v)$.

Proof. 1) Using Lemma 1.1.1.5 (and [Gro60, 5.3.13]), we may assume that $X = Y \times_S \mathbb{A}^r_S$, $f : X \to Y$ is the first projection, and that the family $(t_\lambda)_{\lambda=1,\ldots,r}$ are the elements of $\Gamma(X, \mathcal{O}_X)$ corresponding to the coordinates of $\mathbb{A}^r$. Using Lemma 1.1.1.6, we may furthermore assume that $P^n(v) = v$.

Let $\phi : Y \times_S \mathbb{A}^r_S \to Y \times_S \mathbb{A}^r_S$ be the Y-morphism given by $t_1 - f^*(y_1), \ldots, t_r - f^*(y_r)$. Let $i : Y \hookrightarrow Y \times_S \mathbb{A}^r_S$ be the closed immersion defined by $t_\lambda \mapsto 0$. Since ϕ is etale, as $\phi \circ u = i \circ v$, using Lemma 1.1.1.5, we reduce to the case where u is equal to $i \circ v$, and $(y_\lambda)_{\lambda=1,\ldots,r}$ are equal to 0. Then 1.1.1.7.1 is obvious. □

1.1.2 Sheaf of principal parts relative to weakly smooth morphisms

Let S be a scheme.

Notation 1.1.2.1. Let X be an S-scheme. Let $\Delta_{X/S}(r) : X \hookrightarrow X^{r+1}_{/S}$ denote the diagonal immersion. With notation 1.1.1.2, we set

$$\Delta^n_{X/S}(r) := P^n(\Delta_{X/S}(r)).$$

Let us write $\mathcal{P}^n_{X/S}(r)$ for the nth infinitesimal neighborhood $\mathcal{P}^n(\Delta_{X/S}(r))$ (see 1.1.1.2). Let U be an open of $X^{r+1}_{/S}$ containing the image of $\Delta_{X/S}(r)$ such that the induced immersion $v : X \to U$ is closed. Let $\mathcal{I}(r)$ be the ideal given by v. Recall, by definition, we have $v_* \mathcal{P}^n_{X/S}(r) = \mathcal{P}^n(\mathcal{I}(r))$. For $i = 0, \ldots, r$, let $p_i : X \hookrightarrow X^{r+1}_{/S} \to X$ be the morphisms induced by the projections. Let $p^U_i : U \to X$ be the morphism induced by the projection p_i. We get the $\mathcal{O}_X$-algebra $p^U_{i*}\mathcal{P}^n(\mathcal{I}(r))$. Since $p^U_i \circ v = \mathrm{id}$ then as a sheaf of sets $\mathcal{P}^n_{X/S}(r) = p^U_{i*}\mathcal{P}^n(\mathcal{I}(r))$. This yields $r+1$-structures of $\mathcal{O}_X$-algebras on $\mathcal{P}^n_{X/S}(r)$. To clarify which $\mathcal{O}_X$-algebra structure we consider, we set $p_{i*}\mathcal{P}^n_{X/S}(r) := p^U_{i*}\mathcal{P}^n(\mathcal{I}(r))$. By composing p^U_i with the canonical morphism $\Delta^n_{X/S} \to U$, we get the projection $p^n_i : \Delta^n_{X/S}(r) \to X$, for $i = 0, \ldots, r$. We get the ring homomorphisms $p^n_i : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$.

When $r = 1$, we simply write $\mathcal{I}$, $\Delta^n_{X/S}$, $\Delta_{X/S}$, $\mathcal{P}^n_{X/S}$, $\mathcal{P}_{X/S}$. The left (resp. right) structure of $\mathcal{O}_X$-module on $\mathcal{P}^n_{X/S}$ is by definition the one given by $p_{0*}\mathcal{P}^n_{X/S}$ (resp. $p_{1*}\mathcal{P}^n_{X/S}$).

Definition 1.1.2.2. We say that $\mathcal{P}^n_{X/S}$ is the *sheaf of principal parts of order n of X/S.*

Proposition 1.1.2.3 (Local description of $\mathcal{P}^n_{X/S}$)**.** Suppose f has formal coordinates $(t_\lambda)_{\lambda=1,\ldots,d}$. Let $\tau_{\lambda,n}$ be the image of $1 \otimes t_\lambda - t_\lambda \otimes 1$ in $\mathcal{P}^n_{X/S}$. For any $i = 0, 1$, we have the following isomorphism of $\mathcal{O}_X$-algebras:

$$\begin{aligned} \mathcal{O}_X[T_1, \ldots, T_d]_n &\xrightarrow{\sim} p_{i*}\mathcal{P}^n_{X/S} \\ T_\lambda &\mapsto \tau_{\lambda,n}, \end{aligned} \tag{1.1.2.3.1}$$

with $\mathcal{O}_X[T_1, \ldots, T_d]_n := \mathcal{O}_X[T_1, \ldots, T_d] / (T_1, \ldots, T_d)^{n+1}$.

Proof. Since the case of where $i = 1$ is checked symmetrically, let us compute the case where $i = 0$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{A}^d_S & \xleftarrow{p_1} & X \times_S \mathbb{A}^d_S & & \\ \uparrow t & \square & \uparrow \mathrm{id} \times t & \searrow q & \\ X & \xleftarrow{p_1} & X \times_S X & \xrightarrow{p_0} & X, \end{array} \tag{1.1.2.3.2}$$

where p_0, p_1 means respectively the left and right projection, where q is the canonical projection, where t is the S-morphism induced by $t_1, \ldots, t_d$, where $id \times t$ is the X-morphism induced by $p_1^*(t_1), \ldots, p_1^*(t_d)$. Since $(p_1^*(t_\lambda))_{\lambda=1,\ldots,d}$ are formal coordinates of p_0 (because the square of the diagram 1.1.2.3.2 is cartesian), we can apply Proposition 1.1.1.7 in the case where f is p_0, u is $\Delta_{X/S}$, v is the identity, t_λ is $t_\lambda \otimes 1$ and y_λ is t_λ. □

Remark 1.1.2.4. From the local description of 1.1.2.3, we get that the morphisms $p^n_1 : \Delta^n_{X/S}(r) \to X$ and $p^n_0 : \Delta^n_{X/S}(r) \to X$ are finite when X/S is weakly smooth.

1.1.2.5. The closed immersions $\Delta^n_{X/S}$ and $\Delta^{n'}_{X/S}$ induce $\Delta^{n,n'}_{X/S} := (\Delta^n_{X/S}, \Delta^{n'}_{X/S})\colon X \hookrightarrow \Delta^n_{X/S} \times_X \Delta^{n'}_{X/S}$. We get $\Delta^{n,n'}_{X/S} \in \mathscr{C}_{n+n'}$. Using the universal property of the $n+n'$ infinitesimal neighborhood of $\Delta_{X/S}$, we get a unique morphism $\Delta^n_{X/S} \times_X \Delta^{n'}_{X/S} \to \Delta^{n+n'}_{X/S}$ of $\mathscr{C}_{n+n'}$ inducing the commutative diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \Delta^n_{X/S} \times_X \Delta^{n'}_{X/S} & \longrightarrow & X \times_S X \times_S X \\ \| & & \downarrow & & \downarrow{\scriptstyle p_{02}} \\ X & \hookrightarrow & \Delta^{n+n'}_{X/S} & \longrightarrow & X \times_S X. \end{array} \tag{1.1.2.5.1}$$

We denote by $\delta^{n,n'}\colon \mathcal{P}^{n+n'}_{X/S} \to \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S}$ the corresponding morphism, where $\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S} := p_{1*}\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} p_{0*}\mathcal{P}^{n'}_{X/S}$. We compute $\delta^{n,n'}(a \otimes b) = (a \otimes 1) \otimes (1 \otimes b)$ and we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{p^n_1} & \mathcal{P}^n_{X/S} \\ {\scriptstyle p^{n+n'}_1}\downarrow & & \downarrow{\scriptstyle 1\otimes p^{n'}_1} \\ \mathcal{P}^{n+n'}_{X/S} & \xrightarrow{\delta^{n,n'}} & \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S}. \end{array} \tag{1.1.2.5.2}$$

By replacing p_{02} by p_{01} (resp. p_{12}), we get a unique morphism $\Delta^n_{X/S} \times_X \Delta^{n'}_{X/S} \to \Delta^{n+n'}_{X/S}$ making commutative the diagram 1.1.2.5.1. We denote by $q^{n,n'}_0\colon \mathcal{P}^{n+n'}_{X/S} \to \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S}$ (resp. $q^{n,n'}_1\colon \mathcal{P}^{n+n'}_{X/S} \to \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S}$) the corresponding morphism (or simply $q^{n,n'}_0$ or q_0). We notice that $q^{n,n'}_0 = \pi^{n+n',n}_{X/S} \otimes 1$ and $q^{n,n'}_1 = 1 \otimes \psi^{n+n',n'}_{X/S}$, where $\psi^{n_1,n_2}_{X/S}$ is the projection $\mathcal{P}^{n_1}_{X/S} \to \mathcal{P}^{n_2}_{X/S}$ for any integers $n_1 \geq n_2$.

1.1.2.6. We have the following proprieties making some links between the sheaf of relative differentials and the sheaf of principal parts of order ≤ 1 of X/S.

(a) We denote by $\mathcal{I}^1_{X/S}$ the ideal of the closed immersion $\Delta^1_{X/S}$ and by $\Omega^1_{X/S} := (\Delta^1_{X/S})^{-1}(\mathcal{I}^1_{X/S})$ the corresponding $\mathcal{O}_X$-module (recall $\Delta^1_{X/S*}$ is an homeomorphism). In other words, $\Omega^1_{X/S}$ is the kernel of the canonical morphism of $\mathcal{O}_X$-algebras (for both structures) $\psi^{1,0}_{X/S}\colon \mathcal{P}^1_{X/S} \to \mathcal{P}^0_{X/S} = \mathcal{O}_X$. The sheaf $\Omega^1_{X/S}$ is the "sheaf of relative differentials of X/S". Since $\Omega^1_{X/S}$ is an ideal of $\mathcal{P}^1_{X/S}$ of order 2 then the left and right structure of $\mathcal{O}_X$-module of $\Omega^1_{X/S}$ (recall $\psi^{1,0}_{X/S}$ is $\mathcal{O}_X$-linear for both structures) are in fact identical.

(b) We have the exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_{X/S} \xrightarrow{j} p_{0*}\mathcal{P}^1_{X/S} \xrightarrow{\psi^{1,0}_{X/S}} \mathcal{O}_X \to 0, \tag{1.1.2.6.1}$$

where j is the canonical inclusion (recall p_{0*} means that $\mathcal{P}^1_{X/S}$ is considered as an $\mathcal{O}_X$-algebra for its left structure). The exact sequence 1.1.2.6.1 splits via the section $p^1_0\colon \mathcal{O}_X \to p_{0*}\mathcal{P}^1_{X/S}$, which yields the isomorphism of $\mathcal{O}_X$-modules

$$(p^1_0, j)\colon \mathcal{O}_X \oplus \Omega^1_{X/S} \xrightarrow{\sim} p_{0*}\mathcal{P}^1_{X/S}. \tag{1.1.2.6.2}$$

(c) Via the isomorphism 1.1.2.6.2, we get the $\mathcal{O}_X$-linear homomorphism

$$\varpi_{X/S}\colon p_{0*}\mathcal{P}^1_{X/S} \twoheadrightarrow \Omega^1_{X/S} \tag{1.1.2.6.3}$$

which is a left inverse of the inclusion $\Omega^1_{X/S} \subset p_{0*}\mathcal{P}^1_{X/S}$. We compute $\varpi_{X/S} = \mathrm{id} - p^1_0 \circ \psi^{1,0}_{X/S}$.

(d) We denote by $d_{X/S}\colon \mathcal{O}_X \to \Omega^1_{X/S}$ (or simply d) the constant $\mathcal{O}_S$-derivation which is given by $a \mapsto p^1_1(a) - p^1_0(a)$ for any local section a of $\mathcal{O}_X$ (see notation 1.1.2). The composition of $\varpi_{X/S}$ with $p^1_1\colon \mathcal{O}_X \to \mathcal{P}^1_{X/S}$ is the constant $\mathcal{O}_S$-derivation $d_{X/S}$. (Indeed, since $\psi^{1,0}_{X/S} \circ p^1_1 = \mathrm{id}$, then $\varpi_{X/S} \circ p^1_1 = p^1_1 - p^1_0 = d_{X/S}$.) Since $\varpi_{X/S}$ is $\mathcal{O}_X$-linear, this means that $d_{X/S}$ is a differential operator of order ≤ 1 relatively to X/S (see definition 1.1.3.2).

(e) We have the exact sequence of $\mathcal{O}_X$-modules

$$0 \xrightarrow{j} \Omega^1_{X/S} \to p_{1*}\mathcal{P}^1_{X/S} \xrightarrow{\psi^{1,0}_{X/S}} \mathcal{O}_X \to 0. \tag{1.1.2.6.4}$$

The exact sequence 1.1.2.6.4 splits via the section $p^1_1 \colon \mathcal{O}_X \to p_{1*}\mathcal{P}^1_{X/S}$, which yields the isomorphism of $\mathcal{O}_X$-modules

$$(p^1_1, j) \colon \mathcal{O}_X \oplus \Omega^1_{X/S} \xrightarrow{\sim} p_{1*}\mathcal{P}^1_{X/S}. \tag{1.1.2.6.5}$$

(f) Suppose X/S has coordinates $(t_\lambda)_{\lambda=1,\dots,d}$. According to notation 1.1.2.3, let $\tau_{\lambda,1}$ be the image of $1 \otimes t_\lambda - t_\lambda \otimes 1$ in $\mathcal{P}^1_{X/S}$ for $\lambda = 1,\dots,d$. Then $\varpi_{X/S}(\tau_{\lambda,1}) = \tau_{\lambda,1} = dt_\lambda$. Moreover, $\varpi_{X/S}(1) = 0$. Hence, $\ker \varpi_{X/S} = \mathcal{O}_X$.

1.1.3 Differential operators

We remind few elements on the concept of differential operators (see the section [Gro67, IV.16.8]). We only collect information to understand the construction of the de Rham complex (see the paragraph below 2.3.2.1). The reader may find further properties that we will not need in [Gro67, IV.16.8]. Let $X \to S$ be a morphism of schemes. Let $\mathcal{E}, \mathcal{F}$ be two $\mathcal{O}_X$-modules, $n \geq 0$ be an integer.

Notation 1.1.3.1. By convention, $\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ means $p_{1*}(\mathcal{P}^n_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S}$ means $\mathcal{E} \otimes_{\mathcal{O}_X} p_{0*}(\mathcal{P}^n_{X/S})$. For instance, $\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S}$ is $p_{1*}(\mathcal{P}^n_{X/S}) \otimes_{\mathcal{O}_X} p_{0*}(\mathcal{P}^{n'}_{X/S})$.

We have two structures of $\mathcal{O}_X$-module on the sheaf $\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$: the "left structure" given by functoriality from the left structure of $\mathcal{P}^n_{X/S}$ and the "right structure" given by the internal tensor product. We denote by $p_{0*}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$ (resp. $p_{1*}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$) to clarify we are considering the left structure (resp. right structure).

Similarly, we denote by $p_{0*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S})$ (resp. $p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S})$) the $\mathcal{O}_X$-module given by the internal tensor product (resp. by functoriality from the right $\mathcal{O}_X$-module structure of $\mathcal{P}^n_{X/S}$) which is called the left (resp. right) structure.

We denote by $p^n_{0,\mathcal{E}} \colon \mathcal{E} \to p_{0*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S})$ the canonical $\mathcal{O}_X$-linear map given by $x \mapsto x \otimes \mathbb{1}$, i.e. is the composition of $\mathrm{id}_{\mathcal{E}} \otimes p^n_0$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^0_{X/S}$. We denote by $p^n_{1,\mathcal{E}} \colon \mathcal{E} \to p_{1*}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$ the canonical $\mathcal{O}_X$-linear map given by $x \mapsto \mathbb{1} \otimes x$, i.e. is the composition of $p^n_1 \otimes \mathrm{id}_{\mathcal{E}}$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{P}^0_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$.

Definition 1.1.3.2. We say that a $f^{-1}\mathcal{O}_S$-linear homomorphism $D \colon \mathcal{E} \to \mathcal{F}$ is a "differential operator of order $\leq n$ (relatively to X/S)" if there exists a homomorphism of $\mathcal{O}_X$-modules $u \colon p^n_{0*}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}) \to \mathcal{F}$ such that $D = u \circ p^n_{1,\mathcal{E}}$ (see notation 1.1.3.1).

1.1.3.3 (Local description)**.** Let $D \colon \mathcal{E} \to \mathcal{F}$ be a $f^{-1}\mathcal{O}_S$-linear homomorphism. The property that D is a differential operator of order $\leq n$ is better understood locally. Let us describe it below. Suppose $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$. Then $\Gamma(X, \mathcal{P}^n_{X/S}) = (A \otimes_R A)/I^{n+1}$ where I is the kernel of the multiplication $A \otimes_R A \to A$. Set $E := \Gamma(X, \mathcal{E})$, $F := \Gamma(X, \mathcal{F})$. Then $\Gamma(X, p^n_{0,\mathcal{E}})$ is the canonical map $E \to (A \otimes_R A)/I^{n+1} \otimes_A E$ given by $x \mapsto \mathbb{1} \otimes x$. Let us denote by $p_{0,E} \colon E \to (A \otimes_R A) \otimes_A E$ the canonical map given by $x \mapsto \mathbb{1} \otimes x$. We have the A-linear isomorphism $(A \otimes_R A) \otimes_A E \xrightarrow{\sim} A \otimes_R E$ (for the left structure of $A \otimes_R A$ by default). Since D is R-linear, this yields by extension a unique A-linear map $u \colon (A \otimes_R A) \otimes_A E \to F$ such that $D = \widetilde{D} \circ p_{0,E}$. Then D is a differential operator of order $\leq n$ is equivalent to saying that $u(I^{n+1}) = 0$. Remark also that the A-linear map u such that $D = u \circ p^n_{0,\mathcal{E}}$ is therefore unique if it exists.

Proposition 1.1.3.4. *Let $D \colon \mathcal{E} \to \mathcal{F}$ be a $f^{-1}\mathcal{O}_S$-linear homomorphism. The following conditions are equivalent:*

(a) D is a differential operator of order $\leq n$.

(b) For any section a of $\mathcal{O}_X$ above an open subset U, the homomorphism $D_a \colon \mathcal{E}|U \to \mathcal{F}|U$ such that, for any section t of $\mathcal{E}$ above a open subset $V \subset U$, we have

$$D_a(t) = D(at) - aD(t) \tag{1.1.3.4.1}$$

is is a differential operator of order $\leq n - 1$.

(c) *For any open subset U of X, for any family $(a_i)_{1\le i\le n+1}$ of $n+1$ sections of $\mathcal{O}_X$ above U and any section t of $\mathcal{E}$ above U, we have the identity*

$$\sum_{H\subset\{1,\dots,n+1\}}(-1)^{\mathrm{Card}\,H}(\prod_{i\in H}a_i)D\left((\prod_{i\notin H}a_i)t\right)=0. \tag{1.1.3.4.2}$$

Proof. Let us sketch the proof (see [Gro67, IV16.8.8)] for a complete one). These properties are local. Hence, we can use the local description of 1.1.3.3 of a differential operator of order $\le n$. With notation 1.1.3.3, the ideal I^{n+1} is generated by the elements of the form

$$\prod_{i=1}^{n+1}(a_i\otimes 1-1\otimes a_i)=\sum_{H\subset\{1,\dots,n+1\}}(-1)^{\mathrm{Card}\,H}(\prod_{i\in H}a_i)\otimes(\prod_{i\notin H}a_i).$$

Hence, we easily check via some computation the equivalence between the three statements. □

1.1.4 Sheaf of differential operators relative to weakly smooth morphisms

Let $f\colon X\to S$ be a weakly smooth morphism (see 1.1.1.4).

Definition 1.1.4.1. In the case where $\mathcal{E}$ and $\mathcal{F}$ are equal to $\mathcal{O}_X$ in the definition (1.1.3.2), we get the sheaf of differential operators of order $\le n$ of f (of infinite level) is defined by putting $\mathcal{D}_{X/S,n}:=\mathcal{H}om_{\mathcal{O}_X}(p^n_{0*}\mathcal{P}^n_{X/S},\mathcal{O}_X)$. The sheaf of differential operators of f is defined by putting $\mathcal{D}_{X/S}:=\cup_{n\in\mathbb{N}}\mathcal{D}_{X/S,n}$. The tautological structure of $\mathcal{O}_X$-module on $\mathcal{D}_{X/S,n}$ is said to be the left one. For any $a\in\mathcal{O}_X$, we have the map $\mathcal{D}_{X/S,n}\to\mathcal{D}_{X/S,n}$ induced by the $\mathcal{O}_X$-linear map $p^n_1(a)\colon p^n_{0*}\mathcal{P}^n_{X/S}\to p^n_{0*}\mathcal{P}^n_{X/S}$. This yields another structure of $\mathcal{O}_X$-module on $\mathcal{D}_{X/S,n}$ which is called the right structure of $\mathcal{O}_X$-module. For any $n'\ge n$, the homomorphisms $\mathcal{D}_{X/S,n}\to\mathcal{D}_{X/S,n'}$ are $\mathcal{O}_X$-linear for both structures. This yields two structures of $\mathcal{O}_X$-modules on $\mathcal{D}_{X/S}$, the left one and the right one.

Since the image of $p^n_1\colon\mathcal{O}_X\to\mathcal{P}^n_{X/S}$ generates $p^n_{0*}\mathcal{P}^n_{X/S}$ as $\mathcal{O}_X$-module, then the morphism

$$\mathcal{D}_{X/S,n}=\mathcal{H}om_{\mathcal{O}_X}(p^n_{0*}\mathcal{P}^n_{X/S},\mathcal{O}_X)\xrightarrow{p^n_1}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{O}_X)=\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$$

is injective. We get the injection $\mathcal{D}_{X/S}\hookrightarrow\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$. The sheaf $\mathcal{D}_{X/S}$ is in fact a subring of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$. Indeed, let $P\in\mathcal{D}_{X/S,n}$, $P'\in\mathcal{D}_{X/S,n'}$. Then it follows from the commutativity of 1.1.2.5.2 that the product PP' in $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$ corresponds to the element of $\mathcal{D}_{X/S,n+n'}$ defines by the composition:

$$PP'\colon\mathcal{P}^{n+n'}_{X/S}\xrightarrow{\delta^{n,n'}}\mathcal{P}^n_{X/S}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X/S}\xrightarrow{\mathrm{id}\otimes P'}\mathcal{P}^n_{X/S}\xrightarrow{P}\mathcal{O}_X. \tag{1.1.4.1.1}$$

Since $\mathcal{D}_{X/S}$ is a subring of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$, this yields a canonical structure of left $\mathcal{D}_{X/S}$-module on $\mathcal{O}_X$. By definition, the action of $P\in\mathcal{D}_{X/S,n}$ on $f\in\mathcal{O}_X$ is denoted by $P(f)$ or $P\cdot f$ and is given by the formula

$$P(f)=P\circ p^n_1(f). \tag{1.1.4.1.2}$$

Remark 1.1.4.2. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. We can set $\mathcal{D}iff_{X/S}(\mathcal{E},\mathcal{E}):=\cup_{n\in\mathbb{N}}\mathcal{H}om_{\mathcal{O}_X}(p^n_{0*}(\mathcal{P}^n_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}),\mathcal{E})$. Then we get the injection $\mathcal{D}iff_{X/S}(\mathcal{E},\mathcal{E})\hookrightarrow\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$ via the map

$$\mathcal{H}om_{\mathcal{O}_X}(p^n_{0*}(\mathcal{P}^n_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}),\mathcal{E})\xrightarrow{p^n_1}\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E},\mathcal{E})=\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}).$$

We check similarly that the sheaf $\mathcal{D}iff_{X/S}(\mathcal{E},\mathcal{E})$ is in fact a subring of $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$ (see [Gro67, IV.16.8.9]).

Notation 1.1.4.3. Let $d\ge 1$ be an integer. By convention, an element $\underline{k}\in\mathbb{N}^d$ is of the form $\underline{k}=(k_1,\dots,k_d)$. We consider $(\mathbb{N}^d,+)$ as a monoid and we put $|\underline{k}|:=k_1+\cdots+k_d$. We endow $\mathbb{N}^d$ with the partial order as follows. For any $\underline{k},\underline{k}'\in\mathbb{N}^d$, by definition the inequality $\underline{k}'\le\underline{k}$ means that we have $k'_i\le k_i$, for any $i=1,\dots,d$. For any $\underline{k},\underline{l}\in\mathbb{N}^d$, we remark that $\sup\{\underline{k},\underline{l}\}$ exists and $\underline{n}:=\sup\{\underline{k},\underline{l}\}$ is given by $n_i=\max\{k_i,l_i\}$ for any $i=1,\dots d$ (see 3.2.3.13 to get an example of formula involving such computation). Moreover, for any $\underline{k}'\le\underline{k}'$ in $\mathbb{N}^d$, we write

$$\underline{k}!:=\prod_{i=1}^d k_i!,\qquad\binom{\underline{k}}{\underline{k}'}:=\prod_{i=1}^d\binom{k_i}{k'_i}. \tag{1.1.4.3.1}$$

1.1.4.4 (Local formulas)**.** Suppose f has the formal coordinates $(t_\lambda)_{\lambda=1,\ldots,r}$. Put $\tau_\lambda := 1 \otimes t_\lambda - t_\lambda \otimes 1$. Following 1.1.2.3, the elements $\{\underline{\tau}^{\underline{k}}\}_{|\underline{k}|\leq n}$ form a basis of $\mathcal{P}^n_{X/S}$. The corresponding dual basis of $\mathcal{D}_{X/S,n}$ will be denoted by $\{\underline{\partial}^{[\underline{k}]},\ |\underline{k}| \leq n\}$. Hence, $\mathcal{D}_{X/S}$ is a free $\mathcal{O}_X$-module (for both structures) with the basis $\{\underline{\partial}^{[\underline{k}]},\ \underline{k} \in \mathbb{N}^r\}$. We have the formulas:

$$\underline{\partial}^{[\underline{k}']}\underline{\partial}^{[\underline{k}'']} = \binom{\underline{k}'+\underline{k}''}{\underline{k}'}\underline{\partial}^{[\underline{k}'+\underline{k}'']}, \qquad \text{product rule,} \tag{1.1.4.4.1}$$

$$\underline{\partial}^{[\underline{k}]}f = \sum_{\underline{k}'+\underline{k}''=\underline{k}} \underline{\partial}^{[\underline{k}']}(f)\underline{\partial}^{[\underline{k}'']}, \qquad \text{Leibniz rule} \tag{1.1.4.4.2}$$

for $\underline{k}, \underline{k}', \underline{k}'' \in \mathbb{N}^d$ and any $f \in \Gamma(U, \mathcal{O}_U)$. The product formula says the operators $\underline{\partial}^{[\underline{k}]}$ commute.

For any $i = 1, \ldots, d$, let $\underline{\epsilon}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is at the ith place. We set $\partial_i := \underline{\partial}^{[\underline{\epsilon}_i]}$. Write $\underline{\partial}^{\underline{k}} = \underbrace{\partial_1 \cdots \partial_1}_{k_1} \cdots \underbrace{\partial_d \cdots \partial_d}_{k_d}$. Then we get $\underline{k}!\underline{\partial}^{[\underline{k}]} = \underline{\partial}^{\underline{k}}$.

Proposition 1.1.4.5. *Let X be a smooth scheme over $S := \operatorname{Spec} k$, with k a perfect field. Any $\mathcal{D}_{X/S}$-module which is coherent over $\mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module.*

Proof. Let $\mathcal{M}$ be a $\mathcal{D}_{X/S}$-module coherent over $\mathcal{O}_X$. Let x be a closed point of X. We have to check that the stalk $\mathcal{M}_x$ is free (or projective) over $\mathcal{O}_{X,x}$. There exists an open neighborhood U of x, such that $U/\operatorname{Spec} k$ has coordinates $t_1, \ldots, t_d$. Since $k(x)/k$ is separable, then $t_{1x}, \ldots, t_{dx}$ generate $\mathcal{O}_{X,x}$ which number is the minimal (see [Gro67, IV.17.12.2]) i.e. $t_{1x}, \ldots, t_{dx}$ is a regular system of parameter of $\mathfrak{m}_{X,x}$. Then it follows from Nakayama's lemma that there exist $s_1, s_2, \ldots, s_m \in \mathcal{M}_x$ such that $\mathcal{M}_x m \sum_{i=1}^m \mathcal{O}_{X,x}s_i$ and $\overline{s}_1, \ldots, \overline{s}_m \in \mathcal{M}_x/\mathfrak{m}_{X,x}\mathcal{M}_x$ are free generators of the vector space $\mathcal{M}_x/\mathfrak{m}_{X,x}\mathcal{M}_x$ over $k = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. We will show that $\{s_1, s_2, \ldots, s_m\}$ is a free generator of the $\mathcal{O}_{X,x}$ -module $\mathcal{M}_x$. Now we define the order of each $f \in \mathcal{O}_{X,x}$ at $x \in X$ by $\operatorname{ord}(f) := \max\{l \in \mathbb{N} \parallel f \in \mathfrak{m}_{X,x}\}$ if $f \neq 0$ and $\operatorname{ord}(f) = \infty$ otherwise. Let ν be the minimum of the order of $f_1, \ldots, f_m \in \mathcal{O}_{X,x}$ such that

$$\sum_{i=1}^m f_i s_i = 0.$$

To arrive to a contradiction, suppose $\nu \in \mathbb{N}$. Reordoring if necessarily, let $f_1, \ldots, f_m\mathcal{O}_{X,x}$ such that $\sum_{i=1}^m f_i s_i = 0$ and $\operatorname{ord}(f_1) = \nu$. There exists $\underline{k} \in \mathbb{N}^d$ such that $|\underline{k}| = \nu$ and $\underline{\partial}^{[\underline{k}]}(f_1) \in \mathcal{O}^*_{X,x}$. Using 1.1.4.4.2, we get $\underline{\partial}^{[\underline{k}]}f_i - \underline{\partial}^{[\underline{k}]}(f_i) \in \mathfrak{m}_{X,x}$. This yields $0 = \underline{\partial}^{[\underline{k}]} \cdot (\sum_{i=1}^m f_i s_i) = \sum_{i=1}^m (\underline{\partial}^{[\underline{k}]} f_i) \cdot s_i \equiv \sum_{i=1}^m \underline{\partial}^{[\underline{k}]}(f_i) \cdot s_i \mod \mathfrak{m}_{X,x}\mathcal{M}_x$. Since $\underline{\partial}^{[\underline{k}]}(f_1) \in \mathcal{O}^*_{X,x}$, then this is a contradiction with the fact that $\overline{s}_1, \ldots, \overline{s}_m \in \mathcal{M}_x/\mathfrak{m}_{X,x}\mathcal{M}_x$ are free generators of the k-vector space $\mathcal{M}_x/\mathfrak{m}_{X,x}\mathcal{M}_x$. □

1.2 Partial divided power

1.2.1 Modified binomial coefficients

Let $m \in \mathbb{N}$.

1.2.1.1. For any integer $k \geq 0$, we recall the formula

$$v_p(k!) = (k - \sigma(k))/(p-1), \tag{1.2.1.1.1}$$

where we set $k = \sum_{i=0}^{n_k} a_i p^i$, with $1 \leq a_i \leq p-1$, and $\sigma(k) = \sum_{i=0}^{n_k} a_i$. Since $n_k < \log_p(k+1) \leq n_k + 1$, we have the estimate

$$(p-1)\log_p(k+1) \leq \sigma(k) < (p-1)(\log_p(k+1)+1). \tag{1.2.1.1.2}$$

For $k = k' + k''$ with $k', k'' \in \mathbb{N}$, from 1.2.1.1.1 we get

$$v_p\left(\binom{k}{k'}\right) = (\sigma(k') + \sigma(k'') - \sigma(k))/(p-1). \tag{1.2.1.1.3}$$

Notation 1.2.1.2. Let $k \in \mathbb{N}$.

(a) We denote by $q_k^{(m)}$ and $r_k^{(m)}$ the nonnegative integers such that $k = p^m q_k^{(m)} + r_k^{(m)}$ and $r_k^{(m)} < p^m$.

(b) Let $k' \in \mathbb{N}$ such that $k' \leq k$. We set

$$\left\{ \begin{matrix} k \\ k' \end{matrix} \right\}_{(m)} = \frac{q_k^{(m)}!}{q_{k'}^{(m)}! q_{k-k'}^{(m)}!}, \quad \left\langle \begin{matrix} k \\ k' \end{matrix} \right\rangle_{(m)} = \binom{k}{k'} \left\{ \begin{matrix} k \\ k' \end{matrix} \right\}_{(m)}^{-1}. \tag{1.2.1.2.1}$$

Notation 1.2.1.3. Let $d \geq 1$ be an integer. We complete the notation 1.1.4.3 as follows. For any $\underline{k'} \leq \underline{k}'$ in $\mathbb{N}^d$, we write

$$\underline{q}_{\underline{k}}^{(m)} := (q_{k_1}^{(m)}, \ldots, q_{k_d}^{(m)}), \qquad \left\{ \begin{matrix} \underline{k} \\ \underline{k'} \end{matrix} \right\}_{(m)} := \prod_{i=1}^{d} \left\{ \begin{matrix} k_i \\ k'_i \end{matrix} \right\}_{(m)}, \qquad \left\langle \begin{matrix} \underline{k} \\ \underline{k'} \end{matrix} \right\rangle_{(m)} := \prod_{i=1}^{d} \left\langle \begin{matrix} k_i \\ k'_i \end{matrix} \right\rangle_{(m)}. \tag{1.2.1.3.1}$$

1.2.1.4. Let $0 \leq k' \leq k$ be two integers. From 1.2.1.1.3, we compute

$$v_p \left(\left\langle \begin{matrix} k \\ k' \end{matrix} \right\rangle_{(m)} \right) = (q_{k'}^{(m)} + q_{k-k'}^{(m)} - q_k^{(m)} - \sigma(r_k^{(m)}) + \sigma(r_{k'}^{(m)}) + \sigma(r_{k-k'}^{(m)}))/(p-1). \tag{1.2.1.4.1}$$

Lemma 1.2.1.5. *We have the following properties.*

(a) For any integers $m, k, k' \geq 0$, we have

$$\left\{ \begin{matrix} k \\ k' \end{matrix} \right\}_{(m)} \in \mathbb{N}, \qquad \left\langle \begin{matrix} k \\ k' \end{matrix} \right\rangle_{(m)} \in \mathbb{Z}_{(p)}.$$

(b) Let $0 \leq j$, and $0 \leq q' \leq q$ be three integers. If $j \geq m$, or if $q < p$, we have

$$\left\langle \begin{matrix} p^j q \\ p^j q' \end{matrix} \right\rangle_{(m)} \in \mathbb{Z}_{(p)}^*.$$

(c) For any nonnegative integers j, r and q such that $r < p^j$ we have

$$\left\langle \begin{matrix} p^j q + r \\ p^j q \end{matrix} \right\rangle_{(m)} \in \mathbb{Z}_{(p)}^*.$$

(d) For any $k \geq p^m$, we have

$$\left\langle \begin{matrix} k \\ p^m \end{matrix} \right\rangle_{(m)} \in \mathbb{Z}_{(p)}^*. \tag{1.2.1.5.1}$$

Proof. The Lemma is an easy consequence of the formula 1.2.1.4.1. □

Notation 1.2.1.6. Let $k', k'' \in \mathbb{N}$. We set

$$C_{k'',k'}^{(m)} := q_{k'k''}^{(m)}!/((q_{k'}^{(m)}!)^{k''} q_{k''}^{(m)}!). \tag{1.2.1.6.1}$$

It follows from [Ber96c, 1.1.5] that $C_{k'',k'}^{(m)} \in \mathbb{N}$.

1.2.2 Divided power

We begin with a summary on divided power envelope which we need to construct differential operators of finite level. Details of proofs can be found in [Ber96c] §1, [Ber74], [BO78].

Definition 1.2.2.1. Let A be a commutative ring with identity and $I \subset A$ an ideal. By a *divided power structure* on I we mean a family $\gamma = (\gamma_i)$ of maps from I to A indexed by integers $i \geq 0$ satisfying the following conditions for $x, y \in I$ and $a \in A$

(1) $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_i(x) \in I$ for all $i \geq 2$.

(2) $\gamma_k(x + y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$ for all $k \geq 0$.

(3) $\gamma_k(ax) = a^k \gamma_k(x)$ for all $k \geq 0$.

(4) $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$ for all $i, j \geq 0$.

(5) $\gamma_i(\gamma_j(x)) = \frac{(ij)!}{i!(j!)^i}\gamma_{ij}(x)$ for all $i, j \geq 0$.

We say that (I, γ) is a PD ideal, and (A, I, γ) is a PD ring. If there is no ambiguity on γ, we write n-th *divided power* $\gamma_n(x)$ of x as $x^{[n]}$.

Examples 1.2.2.2. (a) 0 is a PD-ideal with $\gamma_i(0) = 0$ for any integer $i \geq 1$.

(b) If A is a $\mathbb{Q}$-algebra, every ideal has a unique PD structure, given by $\gamma_i(x) = x^i/i!$.

(c) If $\mathcal{V}$ is a discrete valuation ring of unequal characteristic $(0, p)$ and uniformizing parameter π, write $p = u\pi p^e$, with u a unit. Then $\pi^k\mathcal{V}$ has a PD-structure if and only if $e/(p-1) \leq k$. In particular, $\pi\mathcal{V}$ has a PD-structure if and only if $e \leq p - 1$.

Definition 1.2.2.3. Given PD rings (A, I, γ) and (A', I', γ') we call a ring homomorphism $f : A \to A'$ a *PD homomorphism*, if $f(I) \subset I'$ and $\gamma'_n \circ f = f \circ \gamma_n$ $(\forall n)$.

Definition 1.2.2.4. Let (A, I, γ) be a PD ring. We say an ideal $J \subseteq I$ is a *sub PD ideal* if and only if $\gamma_k(x) \in J$ for any $x \in J$, $k \geq 1$.

Definition 1.2.2.5. Let (A, I, γ) be a PD ring. For any positive integer n, let us set $I^{[n]} \subset A$ to be the ideal generated by the set of elements $\gamma_{i_1}(x_1)\gamma_{i_2}(x_2)\cdots\gamma_{i_k}(x_k)$ with $x_i \in I$ and $i_1 + \cdots i_k \geq n$. Clearly $I^{[n]}$ is a PD ideal, and $I^{[n]}I^{[m]} \subset I^{[n+m]}$. If for some positive integer n we have $I^{[n]} = 0$, then say I is *PD nilpotent*.

Definition 1.2.2.6. Let (A, I, γ) be a PD ring and B be an A-algebra. We say that "γ extends to B" iff there is a (a necessarily unique) PD structure $\overline{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \overline{\gamma})$ is a PD-morphism.

Example 1.2.2.7. Let (A, I, γ) be a PD ring and B be an A-algebra. When I is principal, then γ extends to B.

Proposition 1.2.2.8. *Let (A, I, γ) be a PD ring. Let B be an A-algebra. (J, δ) be a PD ideal in B. Then the following are equivalent:*

(1) γ extends to B and $\overline{\gamma} = \delta$ on $IB \cap J$.

(2) The ideal $IB + J$ has (a necessarily unique) structure $\overline{\delta}$ such that $(A, I, \gamma) \to (B, IB + J, \overline{\delta})$ and $(B, J, \delta) \to (B, IB + J, \overline{\delta})$ are PD-morphisms.

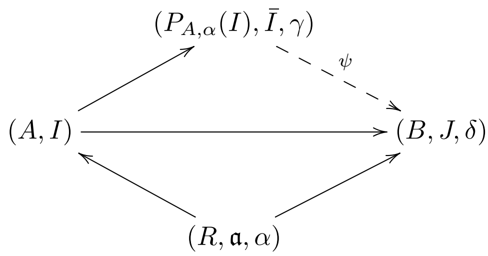
Definition 1.2.2.9. If the equivalent conditions of the proposition 1.2.2.8 are fulfilled, we say that γ and δ are compatible.

Proposition 1.2.2.10. *If $(R, \mathfrak{a}, \alpha)$ is a PD-ring, A is a R-algebra and I is an ideal of A, then there exists an A algebra $P_{A,\alpha}(I)$ with a PD ideal $(\bar{I}, \gamma)$ such that*

(a) $IP_{A,\alpha}(I) \subseteq \bar{I}$,

(b) γ is compatible with α

(c) and the following universal property holds: if (B, J, δ) is any PD A-algebra with $IB \subseteq J$ and δ is compatible with α, then there is a unique PD homomorphism $\tilde{\psi} : (P_{A,\alpha}(I), \bar{I}, \gamma) \to (B, J, \delta)$ such that the following commutes



Definition 1.2.2.11. Under the hypotheses of the proposition, we say (A, I) has *PD envelope* $(P_{A,\alpha}(I), \bar{I}, \gamma)$ compatible with α. We also write $P_\alpha(I)$ or $P_A(I)$ for $P_{A,\alpha}(I)$ ([BO78] Theorem 3.19;[Ber74] 2.4.2).

Examples 1.2.2.12. Take any ring R, put $A = R[T_1, \ldots, T_d]$, let $I = (T_1, \ldots, T_d)$. Then PD envelope $P_A(I)$ is the free R-module with basis consists of products of divided powers $T_1^{[k_1]} \cdots T_d^{[k_d]}$ (in the notation for divided powers above).

If R is a $\mathbb{Q}$-algebra, it follows from the uniqueness of PD-structure in an ideal in a $\mathbb{Q}$-algebra that $A \cong P_A(I)$. If R is not a $\mathbb{Q}$-algebra then $P_A(I)$ is not finitely generated as an R-algebra and is not noetherian.

1.2.3 Divided power compatible with (p)

Let $\mathbb{Z}_{(p)}$ denote the ring of integers localized at the ideal generated by p.

Definition 1.2.3.1. If A is a $\mathbb{Z}_{(p)}$-algebra, the canonical PD-structure of the ideal $(p) \subset \mathbb{Z}_{(p)}$ extends to A ; we will denote by $p^{[n]} = p^n/n!$. If $(I, \gamma) \subset A$ is a PD-ideal, we say that γ is **compatible with (p)** if γ is compatible with the canonical PD-structure of (p) (see 1.2.2.9).

The following Lemma will be applied in the case where the ideal $\mathfrak{a}$ is (p).

Lemma 1.2.3.2. *Let A be a $\mathbb{Z}_{(p)}$-algebra, $(I, \gamma) \subset A$, $(I', \gamma') \subset A$ be two ideals endowed with a PD-structure compatible with (p). We denote by $I_1 := I + pA$, $I_1' := I' + pA$. Let α be the canonical PD-structure on pA. Let δ (resp. δ') be the PD-structure on I_1 (resp. I_1') extending γ (resp. γ') and α. The following conditions are equivalent:*

(a) The PD-structure δ and δ' coincide on $I_1 \cap I_1'$.

(b) The PD-structure δ and γ' coincide on $I_1 \cap I'$.

(c) The PD-structure δ and δ' coincide on $I_1 \cap I_1'$.

(d) There exists on $I + I' + pA$ a PD structure inducing γ, γ' and α.

(e) There exists on $I + I'$ a PD structure which is compatible with (p) and inducing γ and γ'.

In that case we say that γ and γ' are "strictly compatible" (see below 1.2.3.3 for an extension of this notion).

Definition 1.2.3.3. Let $A \to A'$ be a morphism of commutative $\mathbb{Z}_{(p)}$-algebras.

1. Let $(I, \gamma) \subset A$ be an ideal endowed with a PD-structure compatible with (p). We say that γ "strictly extends" to A' if γ extends to A' and if $\overline{\gamma}$ is compatible with (p), where $\overline{\gamma}$ is the PD-structure on IA' which is induced by γ. Beware that γ can extend to A' but not strictly.
2. Let $(I, \gamma) \subset A$ be an ideal endowed with a PD-structure compatible with (p). Let $(I', \gamma') \subset A'$ be an ideal endowed with a PD-structure compatible with (p). We say that γ and γ' are "strictly compatible" if γ strictly extends to A' and if $\overline{\gamma}$ and γ' are strictly compatible (see 1.2.3.2), where $\overline{\gamma}$ is the PD-structure on IA' which is induced by γ. Beware that γ and γ' can be compatible but not strictly compatible.

Lemma 1.2.3.4. *Let A be a $\mathbb{Z}_{(p)}$-algebra and I be an ideal of A.*

(a) The ideal $pA \cap I$ is a sub-PD ideal of pA.

(b) If $(J, \gamma) \subset I$ is a PD-ideal compatible with (p) then $(J + pA) \cap I$ is a sub-PD-ideal of $J + pA$.

Proof. a) If $a \in A$ is such that $pa \in I$, then for any integer $n \geq 1$ we have $p^{[n]}a \in pA \cap I$ and then $(pa)^{[n]} = p^{[n]}a^n \in pA \cap I$.

b) The second statement is a consequence of the first one and of $(J + pA) \cap I = J + pA \cap I$. □

1.2.4 Divided powers of level m

The definition of divided powers is modified by Berthelot to give partial divided powers. More details can be found in [Ber96c] §1.3, 1.4. In the rest of this book, a PD-structure of an ideal of a commutative $\mathbb{Z}_{(p)}$-algebra will always be supposed compatible with (p). Fix an integer m.

Definition 1.2.4.1. We have the following notions.

(a) Let A be a $\mathbb{Z}_{(p)}$-algebra and I be an ideal of A. A "partial PD structure of level m on I" or a "m-PD structure on I" is the data of a PD-ideal (J, γ) such that

$$I^{(p^m)} + pI \subset J \subset I \tag{1.2.4.1.1}$$

where $I^{(p^m)}$ denotes the ideal generated by x^{p^m} for $x \in I$. We say (I, J, γ) is an "m-PD ideal of A" and that (A, I, J, γ) is an m-PD-$\mathbb{Z}_{(p)}$-algebra.

(b) Let (A, I, J, γ) and (A', I', J', γ') be two m-PD-$\mathbb{Z}_{(p)}$-algebras. An "m-PD morphism" or "morphism of m-PD-$\mathbb{Z}_{(p)}$-algebras" of the form $(A, I, J, \gamma) \to (A', I', J', \gamma')$ is a homomorphism of $\mathbb{Z}_{(p)}$-algebras $\phi\colon A \to A'$ such that $\phi(I) \subset I'$, and $(A, J, \gamma) \to (A', J', \gamma')$ is a PD-morphism. When ϕ is the identity, we say that (I, J, γ) is a "sub-m-PD-ideal" of (I', J', γ') ; if moreover $I' = I$, we say that (J, γ) is a "sub-m-PD-structure" of (J', γ').

Remark that a 0-PD-structure (resp. 0-PD-ideal, 0-PD morphism) is the same than a PD-structure (resp. PD-ideal, PD morphism) since when $m = 0$ the equality 1.2.4.1.1 implies $J = I$. Moreover, for any $m' \geq m$, if (I, J, γ) is an m-PD ideal of A then it is also an m'-PD ideal of A. In other words, an m-PD-$\mathbb{Z}_{(p)}$-algebra can be viewed as an m'-PD-$\mathbb{Z}_{(p)}$-algebra.

Examples 1.2.4.2. We have the following m-PD structures.

(a) Let $\mathcal{V}$ be a discrete valuation ring of unequal characteristic $(0, p)$ and uniformizing parameter π, write $p = u\pi p^e$, with u a unit. Let $\mathfrak{a} := \pi^h \mathcal{V}$ and $\mathfrak{b} := \pi^k \mathcal{V}$. Following 1.2.2.2.c, $\mathfrak{b}$ has a canonical PD-structure if and only if $e/(p-1) \leq k$. It follows from the inclusions 1.2.4.1.1, that $(\mathfrak{a}, \mathfrak{b}, \square)$ is an m-ideal if and only if $e/(p-1) \leq k$, $e + h \geq k$, $hp^m \geq k$ and $k \geq h$. In particular, $\mathfrak{a}$ has an m-PD-structure if and only if $hp^m \geq e/(p-1)$.

(b) Let A be a $\mathbb{Z}_{(p)}$-algebra and I be an ideal of A. Let (J_0, γ) be a PD-ideal such that $J_0 \cap I$ is a sub PD-ideal of J_0 and such that $I^{(p^m)} \subset J_0$. We denote by $(J_0 + pA, \overline{\gamma})$ the PD-ideal induced by the PD-ideal (J_0, γ) (compatible with (p)). Then $J := (J_0 + pA) \cap I$ is a sub PD-ideal of $(J_0 + pA, \overline{\gamma})$ (see 1.2.3.4). We get the m-PD ideal $(I, J, \overline{\gamma})$.

For instance, if $I^{(p^m)} \subset pA$ then by taking $J_0 = J = pA \cap I$ endowed with the PD-structure $\square$ induced by the canonical one of pA (see 1.2.3.4), we get the m-PD ideal $(I, pA \cap I, \square)$. We say that $(pA \cap I, \square)$ is the "trivial" m-PD structure of I.

(c) If (J, γ) is an m-PD structure of an ideal I then it is also an m'-PD structure for any $m' \geq m$.

Definition 1.2.4.3. We introduce the notion of *compatible* m-PD structures as follows.

(i) Let $(R, \mathfrak{a}, \mathfrak{b}, \alpha)$ be an m-PD-$\mathbb{Z}_{(p)}$-algebra. Let A be a commutative R-algebra. We say that the "m-PD structure $(\mathfrak{b}, \gamma)$ extends to A" if α is extends strictly to A. If we denote by $\overline{\alpha}$ the PD structure on $\mathfrak{b}A$ extending α, then $(A, \mathfrak{a}A, \mathfrak{b}A, \overline{\alpha})$ is an m-PD-$\mathbb{Z}_{(p)}$-algebra.

(ii) Let $(R, \mathfrak{a}, \mathfrak{b}, \alpha)$ and (A, I, J, γ) be two m-PD-$\mathbb{Z}_{(p)}$-algebras such that A is an R-algebra. We say that "the m-PD structures $(\mathfrak{b}, \alpha)$ and (J, γ) are compatible" if the following conditions are satisfied

  (a) the PD-structures α and γ are strictly compatible,

  (b) $(\mathfrak{b}A + pA) \cap I$ is a sub PD ideal of $\mathfrak{b}A + pA$.

Remark 1.2.4.4. Let $(R, \mathfrak{a}, \mathfrak{b}, \alpha)$ and (A, I, J, γ) be two m-PD-$\mathbb{Z}_{(p)}$-algebras such that A is an R-algebra and such that the m-PD structures $(\mathfrak{b}, \alpha)$ and (J, γ) are compatible. Then $J + \mathfrak{b}A$ endowed with the PD structure extending γ and α is an m-PD-structure on $I + \mathfrak{a}A$ which is also compatible with $(\mathfrak{b}, \alpha)$. For more details, see [Ber96c, 1.3.2.(ii)].

1.2.4.5. Let (A, I, J, γ) be an m-PD-$\mathbb{Z}_{(p)}$-algebra. For $x \in I$ and $k \in \mathbb{N}$, the k-th *partial divided power* of x is

$$x^{\{k\}} = \gamma_{q_k^{(m)}}(x^{p^m})x^{r_k^{(m)}}.$$

We can simply write $x^{\{k\}}$. The map $x \mapsto x^{\{k\}_{(m)}}$ satisfies properties similar to that of $x \mapsto x^{[k]}$. We list some of these properties:

(a) For any $m' \geq m$ and $k \in \mathbb{N}$, we have

$$q_k^{(m)}!x^{\{k\}} = x^k, \qquad x^{\{k\}_{(m')}} = \frac{q_k^{(m)}!}{q_k^{(m')}!}x^{\{k\}_{(m)}}. \tag{1.2.4.5.1}$$

(b) $\forall x \in I$, $x^{\{0\}_{(m)}} = 1$, $x^{\{1\}_{(m)}} = x$; $\forall k \geq 1$, $x^{\{k\}_{(m)}} \in I$; $\forall k \geq p^m$, $x^{\{k\}_{(m)}} \in J$.

(c) $\forall x \in I$, $\forall k \in \mathbb{N}$, $\forall a \in A$, $(ax)^{\{k\}_{(m)}} = a^k x^{\{k\}_{(m)}}$.

(d) $\forall\, x, y \in I$ we have binomial expansion

$$(x+y)^{\{k\}_{(m)}} = \sum_{k'+k''=k} \left\langle {k \atop k'} \right\rangle_{(m)} x^{\{k'\}_{(m)}} y^{\{k''\}_{(m)}}. \tag{1.2.4.5.2}$$

(e) For any $k, k' \in \mathbb{N}$,

$$x^{\{k'\}_{(m)}} x^{\{k''\}_{(m)}} = \left\{ {k'+k'' \atop k'} \right\}_{(m)} x^{\{k'+k''\}_{(m)}}. \tag{1.2.4.5.3}$$

(f) For any $k', k'' \in \mathbb{N}$, with notation 1.2.1.6.1, we have

$$(x^{\{k'\}_{(m)}})^{\{k''\}_{(m)}} = C^{(m)}_{k'',k'} x^{\{k'k''\}_{(m)}}.$$

1.3 m-PD envelopes

1.3.1 m-PD adic filtration

We give the definition of an m-PD adic filtration following [Ber00] Appendice.

Proposition 1.3.1.1. *Let $(R, \mathfrak{a}, \mathfrak{b}, \alpha)$ and (A, I, J, γ) be two m-PD-$\mathbb{Z}_{(p)}$-algebras such that A is an R-algebra and the m-PD structures $(\mathfrak{b}, \alpha)$ and (J, γ) are compatible. Consider the set $\mathcal{F}$ of decreasing filtrations $(F^n A)_{n \geq 0}$ of A by ideals satisfying the following conditions:*

(a) $F^0 A = A$, $F^1 A = I$ and for $n, n' \geq 0$, we have $F^n A \cdot F^{n'} A \subseteq F^{n+n'} A$;

(b) for any $n \geq 1$, $x \in F^n A$ and $k \geq 0$, $x^{\{k\}} \in F^{kn} A$;

(c) for all $n \geq 0$, $(J + pA) \cap F^n A$ is a sub PD ideal in $(J + pA)$.

Then there exists a finer filtration $(I^{\{n\}_{(m)}})_{n \in \mathbb{N}}$ of $\mathcal{F}$, i.e. for any $(F^n A)_{n \in \mathbb{N}} \in \mathcal{F}$ then $I^{\{n\}_{(m)}} \subseteq F^n A$ for all n.

Definition 1.3.1.2. Under the hypotheses of 1.3.1.1, the m-PD filtration of (A, I, J, γ) is $(I^{\{n\}_{(m)}})_{n \in \mathbb{N}}$. Say I is *m-PD nilpotent* if there exists n such that $I^{\{n\}_{(m)}} = 0$. Say I is *topologically (for the p-adic topology) m-PD nilpotent* if the filtration given by the m-PD-filtration is finer than the p-adic topology on A.

We give some example.

Proposition 1.3.1.3. *Let $\mathcal{V}$ be a discrete valuation ring of characteristic $(0, p)$ with maximal ideal $\mathfrak{m}$ and ramification index e. Let $k, h \in \mathbb{N}$, $\mathfrak{a} := \mathfrak{m}^h$, $\mathfrak{b} := \mathfrak{m}^k$ such that $(\mathfrak{a}, \mathfrak{b}, {}^{[]})$ is canonically an m-PD ideal (see 1.2.4.2.a).*

(a) *The m-PD filtration of* $\mathfrak{a}$ *is given by*

$$\mathfrak{a}^{\{n\}} = ((\pi^h)^{\{n'\}})_{n'\geq n}.$$

This is independent of $\mathfrak{b}$*, and is topologically m-PD nilpotent if and only if* $hp^m > \frac{e}{p-1}$.

(b) *Let* A *be a* $\mathcal{V}$*-algebra,* $I = \mathfrak{a}A$*,* $J = \mathfrak{b}A$*. Then the m-PD adic filtration of* $(I, J, ^{[]})$ *is given by* $I^n = \mathfrak{a}^{\{n\}}I$.

Proposition 1.3.1.4. *With the notation and hypotheses of 1.3.1.2, the m-PD adic filtration satisfies the following properties:*

(a) *Let* $n \geq 1$*,* $\overline{A} := A/I^{\{n\}}$*,* $\varpi\colon A \to \overline{A}$ *be the canonical homomorphism,* $\overline{I} := I\overline{A}$*,* $\overline{J} := J\overline{A}$*,* $\overline{\gamma}$ *be the quotient PD-structure on* $\overline{J}$*. Then* $(\overline{J}, \overline{\gamma})$ *defines on* $\overline{\gamma}$ *an m-PD structure compatible with* $(\mathfrak{b}, \alpha)$*,* ϖ *is a strict morphism for the m-PD-adic filtrations, and* ϖ *is universal for m-PD-morphism* $(A, I, J, \gamma) \to (A', I', J', \gamma')$ *such that* (J', γ') *is compatible with* $(\mathfrak{b}, \alpha)$ *and* $I'^{\{n\}_{(m)}} = 0$.

(b) *For any flat morphism* $A \to A'$*, the ideal* $I' := IA$ *can be endowed with an m-PD structure* $(J' = JA', \gamma')$ *extending* (J, γ) *to* A' *and compatible with* $(\mathfrak{b}, \alpha)$*. Moreover, the m-PD adic filtration of* (I', J', γ') *is given by* $I'^{\{n\}_{(m)}} = I^{\{n\}_{(m)}}A'$.

Proof. See [Ber00, A.5]. □

1.3.2 m-PD envelopes of an ideal

Proposition 1.3.2.1. *Let* R *be a* $\mathbb{Z}_{(p)}$*-algebra,* $(\mathfrak{a}, \mathfrak{b}, \alpha)$ *an m-PD ideal of* R*,* A *a* R*-algebra,* I *an ideal of* A*. Then there exists a ring homomorphism*

$$\varphi : A \to A_1$$

an ideal I_1 *of* A_1 *such that* $\varphi(I) \subset I_1$*, and a m-PD structure* $(J_1, ^{[\,]})$ *of* I_1 *which is compatible with* $(\mathfrak{b}, \alpha)$ *such that for any R-homomorphism*

$$f\colon A \to A'$$

satisfying $f(I) \subset I'$ *and for any m-PD structure* (J', γ') *in* A' *which is compatible with* $(\mathfrak{b}, \alpha)$*, there exists a unique m-PD morphism*

$$g : (A_1, I_1, J_1, ^{[]}) \to (A', I', J', \gamma')$$

such that $g \circ \varphi = f$.

Proof. A proof of the existence is given in [Ber96c, 1.4.1]. □

Definition 1.3.2.2. With notation 1.3.2.1, A_1 is denoted by $P_{A,(m),\alpha}(I)$ or simply by $P_{(m),\alpha}(I)$ and is called the *level m partial divided power envelope* of (A, I) (compatible with $(\mathfrak{b}, \alpha)$). The m-PD ideal of $P_{(m),\alpha}(I)$ will be denoted by $(I_{(m)\alpha}, \widetilde{I}_{(m)\alpha}, ^{[\,]})$.

Corollary 1.3.2.3. *Under the hypotheses of 1.3.2.1, for any integer* $n \geq 0$*, the R-algebra* $P^n_{A,(m),\alpha}(I) := P_{A,(m),\alpha}(I)/I^{\{n+1\}}_{(m)\alpha}$ *is endowed with the m-PD-ideal* $(I^n_{(m)\alpha}, \widetilde{I}^n_{(m)\alpha}, ^{[\,]}) := (I_{(m)\alpha}P^n_{A,(m),\alpha}(I), \widetilde{I}_{(m)\alpha}P^n_{A,(m),\alpha}(I), ^{[\,]})$ *which is compatible with* $(\mathfrak{b}, \alpha)$ *and satisfies* $(I^n_{(m)\alpha})^{\{n+1\}_{(m)}} = 0$*, and is endowed with an R-homomorphism* $\phi_n\colon A \to P^n_{A,(m),\alpha}(I)$ *such that* $\phi_n(I) \subset I^n_{(m)\alpha}$ *and which is universal for R-homomorphisms* $R \to (A', I', J', \gamma')$ *sending* I *into an m-PD ideal* I' *compatible with* $(\mathfrak{b}, \alpha)$ *and such that* $I'^{\{n+1\}_{(m)}} = 0$.

Proof. This is a consequence of 1.3.2.1 and 1.3.1.4. □

1.3.2.4. With notation 1.3.2.2, followgin the proof of 1.3.2.1 of Berthelot, we have the equality

$$P_{(m),\alpha}(I) = P_{(0),\alpha}(I^{(m)}). \tag{1.3.2.4.1}$$

The construction of the m-PD ideal $(I_{(m)\alpha}, \widetilde{I}_{(m)\alpha}, ^{[\,]})$ reduce to the level 0 case (but this is too technical to be described here in the general case in few words). For the detailed descriptions, see the proof of [Ber96c, 1.4.1].

Examples 1.3.2.5. With notation 1.3.2.1, the obvious extreme examples are $P_{A,(m),\alpha}(A) = 0$ and $P_{A,(m),\alpha}(0) = A$, $(I_{(m)\alpha}, \widetilde{I}_{(m)\alpha}, {}^{[\,]}) = (0, 0, {}^{[\,]})$.

We have the following fundamental example.

Proposition 1.3.2.6. *Let $m \geq 0$ be an integer, let R be a $\mathbb{Z}_{(p}$-algebra, $(\mathfrak{a}, \mathfrak{b}, \alpha)$ an m-PD ideal of R, $A = R[t_1, \ldots, t_d]$ be a polynomial algebra with coefficients in R, $I = (t_1, \ldots, t_d)$.*

(a) The m-PD-envelope $P_{A,(m),\alpha}(I)$ is independent of α. The canonical morphism $A \to P_{A,(m),\alpha}(I)$ is injective and $P_{A,(m),\alpha}(I)$ is a free R-module having the basis $t^{\{\underline{k}\}} := t_1^{\{k_1\}} \cdots t_d^{\{k_d\}}$, $(k_1, \ldots, k_d) \in \mathbb{N}^d$.

(b) The m-PD-ideal $I_{(m)\alpha}$ is the free R-submodule having the basis $t_1^{\{k_1\}} \cdots t_d^{\{k_d\}}$, with $\sum_{i=1}^d k_i \geq 1$.

(c) The PD-ideal $\widetilde{I}_{(m)\alpha}$ is the R-submodule generated by the elements $pt_1^{\{k_1\}} \cdots t_d^{\{k_d\}}$ with $\sum_{i=1}^d k_i \geq 1$ and by the elements $t_1^{\{k_1\}} \cdots t_d^{\{k_d\}}$, such that there exists i satisfying $k_i \geq p^m$.

(d) The nth term $I_{(m)\alpha}^{\{n\}}$ of the m-PD-filtration of $P_{A,(m),\alpha}(I)$ is the free R-submodule having the basis $t_1^{\{k_1\}} \cdots t_d^{\{k_d\}}$, with $\sum_{i=1}^d k_i \geq n$.

(e) The m-PD-envelope $P^n_{A,(m),\alpha}(I)$ is independent of α. The canonical morphism $A \to P^n_{A,(m),\alpha}(I)$ is injective and $P^n_{A,(m),\alpha}(I)$ is a free R-module having the basis $t^{\{\underline{k}\}} := t_1^{\{k_1\}} \cdots t_d^{\{k_d\}}$, $(k_1, \ldots, k_d) \in \mathbb{N}^d$ with $\sum_{i=1}^d k_i \geq n$.

Proof. See [Ber96c, 1.5.1]. □

Definition 1.3.2.7. Let $m \geq 0$ be an integer, let R be a $\mathbb{Z}_{(p}$-algebra, $(\mathfrak{a}, \mathfrak{b}, \alpha)$ an m-PD ideal of R. Let $A = R[t_1, \ldots, t_d]$ and $I = (t_1, \ldots, t_d)$. We say that $P_{A,(m),\alpha}(I)$ $(P^n_{A,(m),\alpha}(I))$ is the m-PD polynomial algebra with coefficients in R (resp. the m-PD polynomial algebra with coefficients in R of order n) in d variables given by $t_1, \ldots, t_d$, and is denoted by $R\langle t_1, \ldots, t_d\rangle_{(m)}$ (resp. $R\langle t_1, \ldots, t_d\rangle_{(m),n}$).

1.3.2.8. We keep Notation 1.3.2.1. Let K be an ideal of A. By using the universal property of the m-PD envelope, we have the canonical homomorphism

$$P_{A,(m),\alpha}(I) \to P_{A/K,(m),\alpha}(I/I \cap K), (\text{resp. } P^n_{A,(m),\alpha}(I) \to P^n_{A/K,(m),\alpha}(I/I \cap K)). \tag{1.3.2.8.1}$$

Suppose now the image of K in $P_{A,(m),\alpha}(I)$ (resp. $P^n_{(m),\alpha}(I)$) is null. Then, by using the universal property of the m-PD envelope, we construct an m-PD morphism which is an inverse of 1.3.2.8.1. Hence, 1.3.2.8.1 is an m-PD-isomorphism. In particular, taking $K = I^{n+1}$, we get the canonical m-PD-isomorphism:

$$P^n_{A,(m),\alpha}(I) \xrightarrow{\sim} P^n_{A/I^{n+1},(m),\alpha}(I/I^{n+1}). \tag{1.3.2.8.2}$$

1.3.2.9. We keep Notation 1.3.2.1. By using the universal property of the m-PD envelope, we get the m-PD homomorphism $P_{A,(m),\alpha}(I) \to P_{A,(m),\alpha}(I + \mathfrak{a}A)$. It follows from 1.2.4.4, $I_{(m)\alpha} + \mathfrak{a}P_{A,(m),\alpha}(I)$ is endowed with the m-PD-structure $\widetilde{I}_{(m)\alpha} + \mathfrak{b}P_{A,(m),\alpha}(I)$ compatible with $(\mathfrak{b}, \alpha)$. Hence, this yields the m-PD morphism

$$(P_{A,(m),\alpha}(I), I_{(m)\alpha} + \mathfrak{a}P_{A,(m),\alpha}(I), \widetilde{I}_{(m)\alpha} + \mathfrak{b}P_{A,(m),\alpha}(I)) \to P_{A,(m),\alpha}(I + \mathfrak{a}A). \tag{1.3.2.9.1}$$

On the other hand, by using again the universal property of the m-PD envelope, we get the m-PD-morphism

$$P_{A,(m),\alpha}(I + \mathfrak{a}A) \to (P_{A,(m),\alpha}(I), I_{(m)\alpha} + \mathfrak{a}P_{A,(m),\alpha}(I), \widetilde{I}_{(m)\alpha} + \mathfrak{b}P_{A,(m),\alpha}(I)). \tag{1.3.2.9.2}$$

By composing 1.3.2.9.1 with 1.3.2.9.2 or 1.3.2.9.2 with 1.3.2.9.1 we get the identity. Hence 1.3.2.9.1 and 1.3.2.9.2 are m-PD isomorphism.

1.3.2.10. For any $m' \geq m$, since an m-PD-$\mathbb{Z}_{(p)}$-algebra can be viewed as an m'-PD-$\mathbb{Z}_{(p)}$-algebra, then from the universal property we get the homomorphisms of m'-PD-$\mathbb{Z}_{(p)}$-algebra and of A-algebras of the form

$$\psi_{m,m'} \colon P_{(m'),\alpha}(I) \to P_{(m),\alpha}(I), \qquad \psi^n_{m,m'} \colon P^n_{(m'),\alpha}(I) \to P^n_{(m),\alpha}(I). \tag{1.3.2.10.1}$$

From 1.2.4.5.1, we get for any $x \in I_{(m)\alpha}$ and $k \in \mathbb{N}$

$$\psi_{m,m'}(x^{\{k\}_{(m')}}) = \frac{q_k^{(m)}!}{q_k^{(m')}!} x^{\{k\}_{(m)}}. \tag{1.3.2.10.2}$$

For any $m'' \geq m'$, from the universal property, we get the transitive formula $\psi_{m,m'} \circ \psi_{m',m''} = \psi_{m,m''}$ and $\psi^n_{m,m'} \circ \psi^n_{m',m''} = \psi^n_{m,m''}$.

Proposition 1.3.2.11. *Under the hypotheses of 1.3.2.1, suppose the m-PD structure $(\mathfrak{b}, \alpha)$ extends to A/I. Then the canonical homomorphisms*

$$A/I \to P_{A,(m),\alpha}(I)/I_{(m)\alpha} \to P^n_{A,(m),\alpha}(I)/I_{(m)\alpha} \tag{1.3.2.11.1}$$

are isomorphisms for any $n \in \mathbb{N}$.

Proof. See [Ber96c, 1.4.5]. □

Proposition 1.3.2.12. *Under the hypotheses of 1.3.2.1, let A' be flat commutative A-algebra. Then the homomorphisms*

$$P_{A,(m),\alpha}(I) \otimes_A A' \to P_{A',(m),\alpha}(IA'), \tag{1.3.2.12.1}$$

$$P^n_{A,(m),\alpha}(I) \otimes_A A' \to P^n_{A',(m),\alpha}(IA') \tag{1.3.2.12.2}$$

are m-PD-isomorphisms, compatibles with the m-PD filtrations.

Proof. See [Ber96c, 1.4.6]. □

1.3.2.13. Let S be a $\operatorname{Spec} \mathbb{Z}_{(p)}$-scheme. Let $(\mathfrak{a}, \mathfrak{b}, \alpha)$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_S$. Let X be an S-scheme and $\mathcal{I}$ be a quasi-coherent ideal and Z be the closed subscheme of X induced by $\mathcal{I}$. It follows from 1.3.2.12 that the presheaves on X given by $U \mapsto P_{\Gamma(U,\mathcal{O}),(m),\alpha}(\Gamma(U,\mathcal{I}))$, $U \mapsto I_{\Gamma(U,\mathcal{O}),(m),\alpha}(\Gamma(U,\mathcal{I}))$, $U \mapsto J_{\Gamma(U,\mathcal{O}),(m),\alpha}(\Gamma(U,\mathcal{I}))$, are $\mathcal{O}_X$-quasi-coherent. We denote theses sheaves by $\mathcal{P}_{(m),\alpha}(\mathcal{I})$, $\mathscr{I}_{(m),\alpha}(\mathcal{I})$, $\mathscr{J}_{(m),\alpha}(\mathcal{I})$. Similarly, we get the $\mathcal{O}_X$-quasi-coherent sheaves $\mathcal{P}^n_{(m),\alpha}(\mathcal{I})$, $\mathscr{I}^n_{(m),\alpha}(\mathcal{I})$, $\mathscr{J}^n_{(m),\alpha}(\mathcal{I})$.

It follows from the first example of 1.3.2.5 and of Proposition 1.3.2.12 that $\mathcal{P}_{(m),\alpha}(\mathcal{I})$ has its support in Z.

1.3.3 m-PD envelopes of an immersion

Let S be a $\operatorname{Spec} \mathbb{Z}_{(p)}$-scheme. Let $(\mathfrak{a}, \mathfrak{b}, \alpha)$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_S$.

Definition 1.3.3.1. Let $n \geq 1$ be an integer. Let $\mathscr{C}^{(m)}_\alpha$ (resp. $\mathscr{C}^{(m)}_{\alpha,n}$) be the category whose objects are pairs (u, δ) where u is a closed S-immersion $Z \hookrightarrow X$ of schemes and δ is an m-PD-structure on the ideal $\mathcal{I}$ of $\mathcal{O}_X$ given by u which is compatible (see definition 1.2.4.3) with α (resp. and such that $\mathcal{I}^{\{n+1\}_{(m)}} = 0$), where $\mathcal{I}^{\{n+1\}_{(m)}}$ is defined in proposition 1.3.1.1; whose morphisms $(u', \delta') \to (u, \delta)$ are commutative diagrams of the form

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \uparrow{\scriptstyle u'} & & \uparrow{\scriptstyle u} \\ Z' & \longrightarrow & Z \end{array} \tag{1.3.3.1.1}$$

such that f is an m-PD-morphism with respect to the m-PD-structures δ and δ' (i.e., denoting by $\mathcal{I}'$ the sheaf of ideals of $\mathcal{O}_{X'}$ defined by u', for any affine opens U' of X' and U of X such that $f(U') \subset U$, the morphism f induces the m-PD-morphism $(\mathcal{O}_X(U), \mathcal{I}(U), \delta) \to (\mathcal{O}_{X'}(U'), \mathcal{I}'(U'), \delta')$). Beware that theses categories depend on S and also on the quasi-coherent m-PD-ideal (I_S, J_S, α). The objects of $\mathscr{C}^{(m)}_\alpha$ (resp. $\mathscr{C}^{(m)}_{\alpha,n}$) are called m-PD-S-immersions compatible with α (resp. m-PD-S-immersions of order n

compatible with α). We remark that we have the inclusions $\mathscr{C}_{\alpha}^{(m)} \subset \mathscr{C}_{\alpha}^{(m')}$ for any integer $m' \geq m$ (recall an m-PD-structure is also an m'-PD-structure).

We say that a morphism $(u', \delta') \to (u, \delta)$ of $\mathscr{C}_{\alpha}^{(m)}$ (resp. $\mathscr{C}_{\alpha,n}^{(m)}$) is flat (resp. cartesian) if f is flat (resp. the square 1.3.3.1.1 is cartesian).

Notation 1.3.3.2. In this paragraph, suppose $\mathfrak{b} = p\mathcal{O}_S$. Then, there is a unique PD-structure on J_S which we will denote by $\alpha_{\emptyset}$. Let $u\colon Z \hookrightarrow X$ be a closed immersion of S-schemes and δ be an m-PD-structure on the ideal $\mathcal{I}$ of $\mathcal{O}_X$ defined by u. It follows from Lemma 1.2.3.4 that the m-PD-structure δ of $\mathcal{I}$ is always compatible with $\alpha_{\emptyset}$. Hence, in the description of $\mathscr{C}_{\alpha_{\emptyset}}^{(m)}$ (resp. $\mathscr{C}_{\alpha_{\emptyset},n}^{(m)}$) we can remove "compatible with $\alpha_{\emptyset}$" without changing the respective categories. For this reason, we put $\mathscr{C}^{(m)} := \mathscr{C}_{\alpha_{\emptyset}}^{(m)}$ (resp. $\mathscr{C}_{n}^{(m)} := \mathscr{C}_{\alpha_{\emptyset},n}^{(m)}$). But, recall these categories depend on S even if this is not written in the notation. Finally, for any quasi-coherent m-PD-ideal (I_S, J_S, α) of $\mathcal{O}_S$, we have the inclusions

$$\mathscr{C}_{\alpha}^{(m)} \subset \mathscr{C}^{(m)} \text{ and } \mathscr{C}_{\alpha,n}^{(m)} \subset \mathscr{C}_{n}^{(m)}. \tag{1.3.3.2.1}$$

Lemma 1.3.3.3. *The inclusion functor $For_n\colon \mathscr{C}_{\alpha,n}^{(m)} \to \mathscr{C}_{\alpha}^{(m)}$ has a right adjoint. We denote by $Q^n_{(m),\alpha}\colon \mathscr{C}_{\alpha}^{(m)} \to \mathscr{C}_{\alpha,n}^{(m)}$ this right adjoint functor. The functor $Q^n_{(m),\alpha}$ preserves the sources.*

Proof. Let (u, δ) be an object of $\mathscr{C}_{\alpha}^{(m)}$ and $\mathcal{I}$ be the ideal defined by the closed immersion $u\colon Z \hookrightarrow X$. Let $Q^n \hookrightarrow X$ be the closed immersion which is defined by $\mathcal{I}^{\{n+1\}_{(m)}}$. It follows from 1.3.1.4 that $Q^n_{(m),\alpha}(u)$ exists and is equal to the closed immersion $Z \hookrightarrow Q^n$. □

Proposition 1.3.3.4. *Let $u\colon Z \hookrightarrow X$ be an object of $\mathscr{C}$.*

(a) The canonical functor $For^{(m)}\colon \mathscr{C}_{\alpha}^{(m)} \to \mathscr{C}$ has a right adjoint. We denote this right adjoint functor by $P_{(m),\alpha}\colon \mathscr{C} \to \mathscr{C}_{\alpha}^{(m)}$. We say that $P_{(m),\alpha}(u)$ is the m-PD-envelope compatible with α of u.

Similarly, we have the right adjoint functor $P^n_{(m),\alpha}\colon \mathscr{C} \to \mathscr{C}_{\alpha,n}^{(m)}$ of the canonical functor $For_n^{(m)}\colon \mathscr{C}_{\alpha,n}^{(m)} \to \mathscr{C}$. We say that $P^n_{(m),\alpha}(u)$ is the m-PD-envelope of order n compatible with α of u. We have the relation $P^n_{(m),\alpha} = Q^n_{(m),\alpha} \circ P_{(m),\alpha}$.

(b) By an abuse of notation we let $P_{(m),\alpha}(u)$ (resp. $P^n_{(m),\alpha}(u)$) denote the target of the arrow $P_{(m),\alpha}(u)$ (resp. $P^n_{(m),\alpha}(u)$). Let U be an open of X containing Z such that the induced immersion $v\colon Z \hookrightarrow U$ is closed. Then the morphism of schemes of $P_{(m),\alpha}(u) \to U$ (resp. $P^n_{(m),\alpha}(u) \to U$) is affine. Let $\mathcal{I}$ be the ideal of $\mathcal{O}_U$ given by v. Then $P_{(m),\alpha}(u) = \mathrm{Spec}(\mathcal{P}_{(m),\alpha}(\mathcal{I}), (\mathscr{I}_{(m),\alpha}(\mathcal{I}), \mathscr{J}_{(m),\alpha}(\mathcal{I}), ^{[\,]}))$ (see notation 1.3.2.13).

(c) If α extends to Z then the source of $P_{(m),\alpha}(u)$ is Z.

(d) Suppose that $J_S + p\mathcal{O}_S$ is locally principal and that X is noetherian. Then $P^n_{(m),\alpha}(u)$ is a noetherian scheme.

Proof. Let U be an open of X containing Z such that $v\colon Z \hookrightarrow U$ is a closed immersion. Let $\mathcal{I}$ be the ideal of $\mathcal{O}_U$ induced by v. Then we check that $\mathrm{Spec}(\mathcal{P}_{(m),\alpha}(\mathcal{I}), (\mathscr{I}_{(m),\alpha}(\mathcal{I}), \mathscr{J}_{(m),\alpha}(\mathcal{I}), ^{[\,]}))$ satisfies the corresponding universal property. Moreover, $P^n_{(m),\alpha} = Q^n_{(m),\alpha} \circ P_{(m),\alpha}$ satisfies the desired universal property. The other properties are easy. □

Notation 1.3.3.5. Let $u\colon Z \hookrightarrow X$ be an S-immersions of schemes. Let U be an open set of X containing Z such that the induced immersion $v\colon Z \hookrightarrow U$ is closed. Let $\mathcal{I}$ be the ideal given by v. Let $\mathcal{P}_{(m),\alpha}(u) := v^{-1}\mathcal{P}_{(m),\alpha}(\mathcal{I})$ and $\mathcal{P}^n_{(m),\alpha}(u) := v^{-1}\mathcal{P}^n_{(m),\alpha}(\mathcal{I})$. Since $\mathcal{P}_{(m),\alpha}(\mathcal{I})$ and $\mathcal{P}^n_{(m),\alpha}(\mathcal{I})$ have their support in Z (see 1.3.2.13), then $\mathcal{P}_{(m),\alpha}(u)$ and $\mathcal{P}^n_{(m),\alpha}(u)$ do not depend on the choice of U. Beware that even if $\mathcal{P}_{(m),\alpha}(\mathcal{I})$ is quasi-coherent with support in Z, we do not expect that $\mathcal{P}_{(m),\alpha}(u)$ is a quasi-coherent $\mathcal{O}_Z$-module. The sheaf $\mathcal{P}_{(m),\alpha}(u)$ is endowed with the m-PD ideal of $(\mathscr{I}_{(m),\alpha}(u), \mathscr{J}_{(m),\alpha}(u), ^{[\,]})$, where $\mathscr{I}_{(m),\alpha}(u) := v^{-1}\mathscr{I}_{(m),\alpha}(\mathcal{I})$ and $\mathscr{J}_{(m),\alpha}(u) := v^{-1}\mathscr{J}_{(m),\alpha}(\mathcal{I})$.

For any $m' \geq m$, from the universal property

Example 1.3.3.6. From 1.3.2.6, we get the following straightforward example. Let D be an S-scheme such that α extends to Z. Let $u\colon D \hookrightarrow \mathbb{A}^r_D$ be the closed immersion induced by $T_1 = 0, \ldots, T_r = 0$, where $T_1, \ldots, T_r$ are the coordinates of $\mathbb{A}^r_D$. Let $\varpi\colon \mathbb{A}^r_D \to D$ be the projection, which yields the quasi-coherent $\mathcal{O}_D$-algebra $\varpi_* \mathcal{P}_{(m),\alpha}(u)$. Then we get the m-PD polynomial $\mathcal{O}_D$-algebra in the variables $T_1, \ldots, T_r$ by setting:

$$\varpi_* \mathcal{P}_{(m),\alpha}(u) = \mathcal{O}_D\langle T_1, \ldots, T_r\rangle_{(m)},$$

where $\mathcal{O}_D\langle T_1, \ldots, T_r\rangle_{(m)}$ is the sheafification of 1.3.2.7. In particular, this does not depend on the compatibility with α. This is a free $\mathcal{O}_D$-module with basis $T^{\{\underline{k}\}} = T_1^{\{k_1\}} \cdots T_d^{\{k_d\}}$ such that

$$T^{\{\underline{k}'\}} T^{\{\underline{k}''\}} = \left\{ \begin{matrix} \underline{k}' + \underline{k}'' \\ \underline{k}' \end{matrix} \right\} T^{\{\underline{k}' + \underline{k}''\}}.$$

We denote by $(\mathcal{I}_{D,(m),r}, \mathcal{J}_{D,(m),r}, {}^{[\,]})$ the canonical m-PD-structure of $\mathcal{P}_{(m),\alpha}(u)$. We have

$$\varpi_* \mathcal{I}_{D,(m),r} = \oplus_{\underline{k} \neq 0} \mathcal{O}_D T^{\{\underline{k}\}}, \quad \varpi_* \mathcal{I}^{\{n\}}_{D,(m),r} = \oplus_{\underline{k} \geq n} R T^{\{\underline{k}\}},$$

$$\varpi_* \mathcal{J}_{D,(m),r} = \left\{ \sum_{\underline{k}} a_{\underline{k}} T^{\{\underline{k}\}} : a_{\underline{k}} \in p\mathcal{O}_D \text{ if } k_i < p^m \ \forall\, i \right\}$$

Similarly $\varpi_* \mathcal{P}^n_{(m),\alpha}(u) = \mathcal{O}_D\langle T_1, \ldots, T_r\rangle_{(m),n}$.

The purpose of the following Lemma is to introduce the notation 1.3.3.8.

Lemma 1.3.3.7. *Let $r \geq 0$ be an integer, $(v, \delta) \in \mathscr{C}^{(m)}_\alpha$ where $v\colon T \hookrightarrow D$ is a closed S-immersion and $(\widetilde{\mathcal{K}}, \delta)$ is an m-PD-structure compatible with α on the ideal $\mathcal{K}$ of $\mathcal{O}_D$ given by v. Let $(t_\lambda)_{\lambda=1\ldots r}$ be the canonical coordinates of $\mathbb{A}^r_S/S$. Let $i\colon D \to D \times_S \mathbb{A}^r_S$ be the closed D-immersion defined by $t_\lambda \mapsto 0$. Let $\varpi\colon \mathbb{A}^r_D \to D$ be the projection, which yields the quasi-coherent $\mathcal{O}_D$-algebra $\varpi_* \mathcal{P}_{(m),\delta}(i \circ v)$.*

With notation 1.3.3.4 and 1.3.3.6, we have the following properties.

(a) We have the isomorphism of $\mathcal{O}_D$-m-PD-algebras

$$\mathcal{O}_D\langle T_1, \ldots, T_r\rangle_{(m)} \to \varpi_* \mathcal{P}_{(m),\delta}(i \circ v)$$

given by $T_\lambda \mapsto t_\lambda$.

(b) The structural m-PD ideal $(\mathcal{I}_{(m),\alpha}(i \circ v), \mathcal{J}_{(m),\alpha}(i \circ v), {}^{[\,]})$ of $\mathcal{P}_{(m),\delta}(i \circ v)$ is given by

$$\mathcal{I}_{(m),\alpha}(i \circ v) = \mathcal{I}_{D,(m),r} + \mathcal{K}\mathcal{P}_{(m),\delta}(i \circ v),$$
$$\mathcal{J}_{(m),\alpha}(i \circ v) = \mathcal{J}_{D,(m),r} + \widetilde{\mathcal{K}}\mathcal{P}_{(m),\delta}(i \circ v).$$

Proof. By using the remark 1.3.2.9, we can suppose that $v = \mathrm{id}$. Then, this is a consequence of 1.3.2.6. □

Notation 1.3.3.8. With notation 1.3.3.7, we set $\mathcal{O}_{(v,\delta)}\langle T_1, \ldots, T_r\rangle_{(m)} := \varpi_* \mathcal{P}_{(m),\delta}(i \circ v)$ and $\mathcal{O}_{(v,\delta)}\langle T_1, \ldots, T_r\rangle_{(m),n} := \varpi_* \mathcal{P}^n_{(m),\delta}(i \circ v)$.

Lemma 1.3.3.9. *Let $f\colon X \to Y$ be an étale S-morphism, $u\colon Z \hookrightarrow X$ and $v\colon Z \hookrightarrow Y$ be two S-immersions such that $v = f \circ u$.*

(a) $P^n_{(m),\alpha}(u) = P^n_{(m),\alpha}(v)$.

(b) Suppose p is locally nilpotent. Then we have the equality $P_{(m),\alpha}(u) = P_{(m),\alpha}(v)$.

Proof. Let us prove the first (resp. second) assertion. Let $(P(u), \delta)$ be the m-PD envelope (resp. of order n) of u. Let us check that the composition of the canonical morphism $P(u) \to u$ with the morphism $u \to v$ (induced by f) satisfies the universal property of the m-PD envelope (resp. of order n). Let (v', δ') be an object of $\mathscr{C}^{(m)}_\alpha$ (resp. $\mathscr{C}^{(m)}_{\alpha,n}$) and $g\colon v' \to v$ be a morphism of $\mathscr{C}$. Using the universal property of étaleness, since v' is a nil-immersion, then we get a unique morphism $h\colon v' \to u$ of $\mathscr{C}$ whose composition with $u \to v$ gives g. Using the universal property of the m-PD-envelope of u compatible with α that there exists a unique morphism $(v', \delta') \to (P(u), \delta)$ of $\mathscr{C}^{(m)}_\alpha$ (resp. $\mathscr{C}^{(m)}_{\alpha,n}$) such that the composition of $v' \to P(u)$ with $P(u) \to u$ is h. □

Lemma 1.3.3.10. *Let $u \to v$ be a morphism of $\mathscr{C}$. Let δ be the m-PD-structure of $P_{(m),\alpha}(v)$ and $w := P_{(m),\alpha}(v) \times_v u$ (this is the product in $\mathscr{C}$). We denote by $P_{(m),\delta}(w)$ the m-PD-envelope of w compatible with δ. Then $P_{(m),\delta}(w)$ and $P_{(m),\alpha}(u)$ are isomorphic in $\mathscr{C}_\alpha^{(m)}$. Moreover, $P^n_{(m),\delta}(w)$ and $P^n_{(m),\alpha}(u)$ are isomorphic in $\mathscr{C}^{(m)}_{\alpha,n}$.*

Proof. Since the second assertion is checked in the same way, let us prove the first one. Let us check that the composition $P_{(m),\delta}(w) \to w \to u$ satisfies the universal property of $P_{(m),\alpha}(u) \to u$. Let $(u', \delta') \in \mathscr{C}_\alpha^{(m)}$ and $f: u' \to u$ be a morphism of $\mathscr{C}$. First let us check the existence. Composing f with $u \to v$ we get a morphism $g: u' \to v$. Using the universal property of the m-PD envelope, there exists a morphism $\phi: (u', \delta') \to (P_{(m),\alpha}(v), \delta)$ of $\mathscr{C}_\alpha^{(m)}$ such that the composition $u' \to P_{(m),\alpha}(v) \to v$ is g. Hence, we get the morphism $(\phi, f): u' \to w$. Using the universal property of $P_{(m),\delta}(w)$, we get a morphism $u' \to P_{(m),\delta}(w)$ of $\mathscr{C}_\delta^{(m)}$ (and then of $\mathscr{C}_\alpha^{(m)}$) whose composition with $P_{(m),\delta}(w) \to w \to u$ is f. Let us check the unicity. Let $\alpha: u' \to P_{(m),\delta}(w)$ be a morphism of $\mathscr{C}_\alpha^{(m)}$ whose composition with $P_{(m),\delta}(w) \to w \to u$ is f. This implies that the composition of α with $P_{(m),\delta}(w) \to w \to P_{(m),\alpha}(v) \to v$ is g. Since the composition $P_{(m),\delta}(w) \to w \to P_{(m),\alpha}(v)$ is a morphism of $\mathscr{C}_\delta^{(m)}$, then so is the composition of α with $P_{(m),\delta}(w) \to w \to P_{(m),\alpha}(v)$ (in particular, this implies that $\alpha \in \mathscr{C}_\delta^{(m)}$). Using the universal property of $P_{(m),\alpha}(v)$, this latter composition morphism is uniquely determined by g. Hence, the composition of α with $P_{(m),\delta}(w) \to w$ is a morphism of $\mathscr{C}$ uniquely determined by f. Since α is a morphism of $\mathscr{C}_\delta^{(m)}$, we conclude using the universal property of $P_{(m),\delta}(w)$. □

Proposition 1.3.3.11. *Let $f: X \to Y$ be an S-morphism of schemes. Suppose f is endowed with the coordinates $(t_\lambda)_{\lambda=1,\ldots,r}$. Let $u: Z \hookrightarrow X$ and $v: Z \hookrightarrow Y$ be two S-immersions of schemes such that $v = f \circ u$. Suppose that there exist $y_\lambda \in \Gamma(Y, \mathcal{O}_Y)$ whose images in $\Gamma(Z, \mathcal{O}_Z)$ coincide with the images of t_λ. Let $\varpi: P_{(m),\alpha}(u) \to P_{(m),\alpha}(v)$ and $\varpi_n: P^n_{(m),\alpha}(u) \to P^n_{(m),\alpha}(v)$ be the canonical morphisms.*

(a) Suppose p is locally nilpotent. We have the following isomorphism of m-PD-$\mathcal{O}_{P_{(m),\alpha}(v)}$-algebras (see notation 1.3.3.8):

$$\mathcal{O}_{P_{(m),\alpha}(v)}\langle T_1, \ldots, T_r\rangle_{(m)} \xrightarrow{\sim} \varpi_* \mathcal{O}_{P_{(m),\alpha}(u)}$$
$$T_\lambda \mapsto t_\lambda - f^*(y_\lambda), \tag{1.3.3.11.1}$$

where by abuse of notation we denote by t_λ and $f^*(y_\lambda)$ the canonical image in $\mathcal{O}_{P_{(m),\alpha}(u)}$.

(b) We have the following isomorphism of m-PD-$\mathcal{O}_{P^n_{(m),\alpha}(v)}$-algebras (see notation 1.3.3.8):

$$\mathcal{O}_{P^n_{(m),\alpha}(v)}\langle T_1, \ldots, T_r\rangle_{(m),n} \xrightarrow{\sim} \varpi_{n*} \mathcal{O}_{P^n_{(m),\alpha}(u)}$$
$$T_\lambda \mapsto t_\lambda - f^*(y_\lambda), \tag{1.3.3.11.2}$$

where on the left side $P^n_{(m),\alpha}(v)$ means the object of $\mathscr{C}_\alpha^{(m)}$ (and not the target of the closed immersion), where by abuse of notation we denote by t_λ and $f^*(y_\lambda)$ the canonical image in $\mathcal{O}_{P^n_{(m),\alpha}(u)}$.

Proof. Since 1.3.3.11.2 is checked similarly, let us prove 1.3.3.11.1. Using Lemma 1.3.3.9 (and [Gro60, 5.3.13]), we may assume that $X = Y \times_S \mathbb{A}^r_S$, $f: X \to Y$ is the first projection, and that the family $(t_\lambda)_{\lambda=1,\ldots,r}$ are the elements of $\Gamma(X, \mathcal{O}_X)$ corresponding to the coordinates of $\mathbb{A}^r$. Using Lemma 1.3.3.10, we may furthermore assume that $P_{(m),\alpha}(v) = v$ and that the target of $P_{(m),\alpha}(v)$ is $(S, \mathfrak{a}, \mathfrak{b}, \alpha)$. By using Lemma 1.3.3.7, we can suppose that $v = \mathrm{id}$.

Let $\phi: Y \times_S \mathbb{A}^r_S \to Y \times_S \mathbb{A}^r_S$ be the Y-morphism given by $t_1 - f^*(y_1), \ldots, t_r - f^*(y_r)$. Let $i: Y \hookrightarrow Y \times_S \mathbb{A}^r_S$ be the closed immersion defined by $t_\lambda \mapsto 0$. Since ϕ is etale, since $\phi \circ u = i$, using Lemma 1.3.3.9, we reduce to the case where u is equal to i, and $y_\lambda = 0$ for any $\lambda = 1, \ldots, r$. Using 1.3.2.6 this yields that the morphism $\mathcal{O}_Y\langle T_1, \ldots, T_r\rangle_{(m)} \to \mathcal{O}_{P_{(m),\alpha}(\iota)} = \mathcal{O}_{P_{(m),\alpha}(i)}$ given by $T_\lambda \mapsto t_\lambda$ is an isomorphism.

□

Lemma 1.3.3.12. *We have the equality $P^n_{(m),\gamma} \circ For_n \circ P^n = P^n_{(m),\gamma}$, where $For_n: \mathscr{C}_n \to \mathscr{C}$ is the canonical functor and $P^n: \mathscr{C} \to \mathscr{C}_n$ is its right adjoint (see 1.1.1.2).*

Proof. Let $u: Z \hookrightarrow X$ be an object of $\mathscr{C}$. Looking at the construction of P^n and $P^n_{(m),\gamma}$ (see the proof of 1.1.1.2 and 1.3.3.4), the Lemma is a reformulation of 1.3.2.8.2. □

1.4 Sheaves of differential operators of finite level m and finite order

Let S be a $\operatorname{Spec}\mathbb{Z}_{(p)}$-scheme. Let $m \in \mathbb{N}$. Let $(\mathfrak{a}, \mathfrak{b}, \alpha)$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_S$.

1.4.1 Sheaf of principal parts of level m

Let $f\colon X \to S$ be a smooth morphism.

Remark 1.4.1.1. Since α extends to X, then the m-PD envelope compatible with α (of order n) of the identity of X is the identity of X. Indeed, using the arguments given in the proof of 3.2.1.3, we can check that the ideal 0 of $\mathcal{O}_X$ is endowed with a (unique) m-PD structure compatible with α.

Notation 1.4.1.2. Suppose X is a S-scheme. Let $\Delta_{X/S}(r)\colon X \hookrightarrow X^{r+1}_{/S}$ denote the diagonal immersion. With notation 1.3.3.5, we set

$$\Delta^n_{X/S,(m),\alpha}(r) := P^n_{(m),\alpha}(\Delta_{X/S}(r)), \qquad \Delta_{X/S,(m),\alpha}(r) := P_{(m),\alpha}(\Delta_{X/S}(r)).$$

Let us write $\mathcal{P}^n_{X/S,(m),\alpha}(r)$ for the m-PD envelope $\mathcal{P}^n_{(m),\alpha}(\Delta_{X/S}(r))$ of order n of $\Delta_{X/S}(r)$ (see 1.3.3.5). Let U be an open of $X^{r+1}_{/S}$ containing the image of $\Delta_{X/S}(r)$ such that the induced immersion $v\colon X \to U$ is closed. Let $\mathcal{I}(r)$ be the ideal given by v. Recall, by definition, we have $v_*\mathcal{P}^n_{X/S,(m),\alpha}(r) = \mathcal{P}^n_{(m),\alpha}(\mathcal{I}(r))$. For $i = 0, \ldots, r$, let $p_i\colon X^{r+1}_{/S} \to X$ be the projections. Let $p^U_i : U \hookrightarrow X^{r+1}_{/S} \to X$ be the morphism induced by the projection p_i. We get the $\mathcal{O}_X$-algebra $p_{i*}\mathcal{P}^n_{(m),\alpha}(\mathcal{I}(r))$. Since $p^U_i \circ v = \mathrm{id}$ then as a sheaf of sets $\mathcal{P}^n_{X/S,(m),\alpha}(r) = p^U_{i*}\mathcal{P}^n_{(m),\alpha}(\mathcal{I}(r))$. This yields $r+1$-structures of $\mathcal{O}_X$-algebras on $\mathcal{P}^n_{X/S,(m),\alpha}(r)$. To clarify which $\mathcal{O}_X$-algebra structure we consider, we set $p_{i*}\mathcal{P}^n_{X/S,(m),\alpha}(r) := p^U_{i*}\mathcal{P}^n_{(m),\alpha}(\mathcal{I}(r))$. By composing p^U_i with the canonical morphism $\Delta^n_{X/S,(m)}(r) \to U$, we get the projection $p^n_{i(m)}\colon \Delta^n_{X/S,(m),\alpha}(r) \to X$ and $p_{i(m)}\colon \Delta_{X/S,(m),\alpha}(r) \to X$, for $i = 0, \ldots, r$. We get the ring homomorphisms $p^n_{i(m)}\colon \mathcal{O}_X \to \mathcal{P}^n_{X/S,(m)}(r)$ and $p_{i(m)}\colon \mathcal{O}_X \to \mathcal{P}_{X/S,(m)}(r)$. We can simply denote $p^n_{i(m)}$ by p^n_i and $p_{i(m)}$ by p_i. The m-PD-ideal of $\mathcal{P}^n_{X/S,(m)}(r)$ will be denoted by $(\mathcal{I}^n_{X/S,(m)}(r), \mathcal{J}^n_{X/S,(m)}(r), ^{[\,]})$.

When $r = 1$, we simply write $\mathcal{I}$, $\Delta^n_{X/S,(m),\alpha}$, $\Delta_{X/S,(m),\alpha}$, $\mathcal{P}^n_{X/S,(m),\alpha}$, $\mathcal{P}_{X/S,(m),\alpha}$. The left (resp. right) structure of $\mathcal{O}_X$-algebra on $\mathcal{P}^n_{X/S,(m),\alpha}$ is by definition the one given by $p_{0*}\mathcal{P}^n_{X/S,(m),\alpha}$ (resp. $p_{1*}\mathcal{P}^n_{X/S,(m),\alpha}$).

Definition 1.4.1.3. We say that $\mathcal{P}^n_{X/S,(m)}$ is the *sheaf of principal parts of level m of order n of X/S.*

Proposition 1.4.1.4 (Local description)**.** *Let $g\colon X \to \mathbb{A}^d_S$ be an étale morphism. Let $t_1, \ldots, t_d$ be the elements of $\Gamma(X, \mathcal{O}_X)$ defining g. Set $\tau_\lambda := 1\otimes t_\lambda - t_\lambda \otimes 1 \in \mathcal{I}(1)$. For any $\mu = 0, \ldots, r$, let $p_\mu\colon X^{r+1}_{/S} \to X$ be the index μ projection. For any $1 \le \lambda \le d$, $1 \le \mu \le r$, set $\tau_{\lambda\mu} = p^*_\mu(t_\lambda) - p^*_{\mu-1}(t_\lambda) = 1\otimes\cdots\otimes\tau_\lambda\otimes\cdots\otimes 1$. Let $\tau_{\lambda\mu(m),\alpha}$ (resp. $\tau_{\lambda\mu(m),n,\alpha}$) be the image of $\tau_{\lambda\mu}$ in $\mathcal{P}_{X/S,(m),\alpha}(r)$ (resp. $\mathcal{P}^n_{X/S,(m),\alpha}(r)$).*

(a) For any $i = 0, \ldots r$, we have the following isomorphism of $\mathcal{O}_X$-m-PD-algebras

$$\begin{aligned} \mathcal{O}_X\langle T_{\lambda\mu},\, 1 \le \lambda \le d,\, 1 \le \mu \le r\rangle_{(m),n} &\xrightarrow{\sim} p_{i*}\mathcal{P}^n_{X/S,(m),\alpha}(r),\\ T_{\lambda\mu} &\mapsto \tau_{\lambda\mu,(m),n,\alpha}, \end{aligned} \tag{1.4.1.4.1}$$

(b) Suppose p is nilpotent in S. For any $i = 0, \ldots r$, we have the following isomorphism of $\mathcal{O}_X$-m-PD-algebras

$$\begin{aligned} \mathcal{O}_X\langle T_{\lambda\mu},\, 1 \le \lambda \le d,\, 1 \le \mu \le r\rangle_{(m)} &\xrightarrow{\sim} p_{i*}\mathcal{P}_{X/S,(m),\alpha}(r),\\ T_{\lambda\mu} &\mapsto \tau_{\lambda\mu,(m),\alpha}, \end{aligned} \tag{1.4.1.4.2}$$

Proof. By symmetry, we can focus on the case where the structure of $\mathcal{O}_X$-algebra of $\mathcal{P}^n_{X/S,(m),\alpha}(r)$ (resp. $\mathcal{P}_{X/S,(m),\alpha}(r)$) is given by p^n_0 (resp. p_0). Consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{A}^{dr}_S & \xleftarrow{\;q_1\;} & X \times_S \mathbb{A}^{dr}_S & \searrow^{q} & \\ \uparrow{\scriptstyle g^r_{/S}} & \square & \uparrow{\scriptstyle \mathrm{id}\times g^r_{/S}} & & \\ X^r_{/S} & \xleftarrow{\;q_1\;} & X \times_S X^r_{/S} & \xrightarrow{\;p_0\;} & X, \end{array} \tag{1.4.1.4.3}$$

where p_0, q_1 means respectively the left and right projection, where q is the canonical projection, where $g^r_{/S}$ is the S-morphism induced by g, where $id \times g^r_{/S}$ is the X-morphism induced by $(p^*_\mu(t_\lambda))_{\lambda=1,\dots,d,\mu=1,\dots,r}$. Since étaleness is stable under base change, then $(p^*_\mu(t_\lambda))_{\lambda=1,\dots,d,\mu=1,\dots,r}$ are coordinates of p_0. Since the m-PD envelope compatible with α of order n (resp. m-PD envelope compatible with α) of the identity of X is X (see remark 1.4.1.1), we can apply in the first assertion (resp. the second one), we are in the situation to use formula 1.3.3.11.2 (resp. 1.3.3.11.1) in the case where $u = \Delta$ and f is the left projection $p_0 \colon X \times_S X^r_{/S} \to X$. Hence, we get an isomorphism of the form 1.4.1.4.2 where $\tau_{\lambda\mu,(m),\alpha}$ is replaced by the image of $p^*_\beta(t_\lambda) - p^*_0(t_\lambda)$ in $\mathcal{P}_{X/S,(m),\alpha}(r)$ (resp. $\mathcal{P}^n_{X/S,(m),\alpha}(r)$) for $\lambda = 1, \dots, d, \mu = 1, \dots, r$. Since $p^*_\mu(t_\lambda) - p^*_0(t_\lambda) = \tau_{\lambda 0} + \tau_{\lambda 1} + \cdots + \tau_{\lambda\mu}$, we are done. □

In particular, we get the following local description.

Proposition 1.4.1.5 (Local description of $\mathcal{P}_{X/S,(m),\alpha}$). *Let $(t_\lambda)_{\lambda=1,\dots,r}$ be coordinates of f. Let $\tau_{\lambda(m),\alpha}$ (resp. $\tau_{\lambda(m),n,\alpha}$) be the image of $1 \otimes t_\lambda - t_\lambda \otimes 1$ in $\mathcal{P}_{X/S,(m),\alpha}$ (resp. $\mathcal{P}^n_{X/S,(m),\alpha}$).*

(a) For any $i = 0, 1$, we have the following $\mathcal{O}_X$-m-PD isomorphism

$$\begin{aligned} \mathcal{O}_X\langle T_1, \dots, T_r\rangle_{(m),n} &\xrightarrow{\sim} p_{i*}\mathcal{P}^n_{X/S,(m),\alpha} \\ T_\lambda &\mapsto \tau_{\lambda,(m),n,\alpha}. \end{aligned} \tag{1.4.1.5.1}$$

(b) Suppose p is nilpotent in S. For any $i = 0, 1$, we have the following $\mathcal{O}_X$-m-PD isomorphism

$$\begin{aligned} \mathcal{O}_X\langle T_1, \dots, T_r\rangle_{(m)} &\xrightarrow{\sim} p_{i*}\mathcal{P}_{X/S,(m),\alpha} \\ T_\lambda &\mapsto \tau_{\lambda,(m),\alpha}. \end{aligned} \tag{1.4.1.5.2}$$

Corollary 1.4.1.6. *Suppose p is nilpotent in S. Let $f \colon X \to Y$ be an étale morphism of smooth S-schemes. Then the canonical homomorphism $f^*\mathcal{P}_{Y/S,(m)}(r) \to \mathcal{P}_{X/S,(m)}(r)$ is an isomorphism.*

Proof. Since this is local then we can suppose there exists an étale morphism of S-schemes the form $Y \to \mathbb{A}^d_S$. Hence, this follows from 1.4.1.4. □

Remark 1.4.1.7. (a) From the local description 1.4.1.4.1, we get that $\mathcal{P}^n_{X/S,(m),\alpha}(r)$ does not depend on the m-PD-structure (satisfying the conditions of the subsection). Hence, from now, we reduce to the case where $\alpha = \alpha_\emptyset$ (see Notation 1.3.3.2) and we remove α in the notation: we simply write $\mathcal{P}^n_{X/S,(m)}(r)$, $\Delta^n_{X/S,(m)}(r)$, $\tau_{\lambda\beta(m),n}$, $\tau_{\lambda(m),n}$.

(b) Suppose p is nilpotent in S. From 1.4.1.4.2, $\mathcal{P}_{X/S,(m),\alpha}(r)$ does not depend on the m-PD-structure (satisfying the conditions of the subsection). Hence, we can remove α in the corresponding notation.

Remark 1.4.1.8. From the local description of 1.4.1.5, we get that the morphisms p^n_i are finite.

Notation 1.4.1.9. For any integer $m' \geq m$ and $n' \geq n$, we remark that the canonical map $\mathcal{P}^n_{X/S,(m')}(r) \to \mathcal{P}^n_{X/S,(m)}(r)$ (resp. $\mathcal{P}^{n'}_{X/S,(m)}(r) \to \mathcal{P}^n_{X/S,(m)}(r)$) sends $\tau_{\lambda\mu(m'),n}$ to $\tau_{\lambda\mu(m),n}$ (resp. $\tau_{\lambda\mu(m),n'}$ to $\tau_{\lambda\mu(m),n}$). Hence, this will be harmless to denote abusively $\tau_{\lambda\mu(m),n}$ by $\tau_{\lambda\mu}$. Similarly, we can simply denote abusively $\tau_{\lambda(m),n}$ by τ_λ.

1.4.1.10. Suppose p is nilpotent in S. Let $g \colon S' \to S$ be a morphism of schemes, let $(\mathfrak{a}', \mathfrak{b}', \alpha')$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_{S'}$ such that g becomes an m-PD-morphism. Put $X' := X \times_S S'$. Then, the m-PD-morphism $\Delta_{X'/S',(m)} \to \Delta_{X/S,(m)}$ induces the isomorphism $\Delta_{X'/S',(m)} \xrightarrow{\sim} \Delta_{X/S,(m)} \times_S S'$. Indeed, this is equivalent to check that the morphism $g^*\mathcal{P}_{X/S,(m)} \to \mathcal{P}_{X'/S',(m)}$ is an isomorphism. This can be checked by using the local description of 1.4.1.4.1.

1.4.1.11. Let $m' \geq m$ be two integers. Since $\mathscr{C}'^{(m)}_n \subset \mathscr{C}'^{(m')}_n$, then by using the universal property defining $\Delta^n_{X/S,(m')}$ we get a morphism $\psi^n_{m,m'} \colon \Delta^n_{X/S,(m)} \to \Delta^n_{X/S,(m')}$ and then the homomorphism $\psi^n_{m,m'} \colon \mathcal{P}^n_{X/S,(m')} \to \mathcal{P}^n_{X/S,(m)}$. We also get the equalities

$$p^n_{i(m)} \circ \psi^n_{m,m'} = p^n_{i(m')} \text{ and } p^n_i \circ \psi^n_m = p^n_{i(m)}. \tag{1.4.1.11.1}$$

From 1.3.3.12, we get $P^n_{(m)}(P^n(\Delta_{X/S})) = P^n_{(m)}(\Delta_{X/S})$. By using the universal property of the m-PD envelopes we can check that $\psi^n_{m,m'} \circ \psi^n_{m',m''} = \psi^n_{m,m''}$ for $m \le m' \le m''$. Hence, we get a canonical map $\psi^n_m \colon \Delta^n_{X/S,(m)} \to \Delta^n_{X/S}$ and then the homomorphism $\psi^n_m \colon \mathcal{P}^n_{X/S} \to \mathcal{P}^n_{X/S,(m)}$.

Now, suppose that $X \to S$ is endowed with coordinates $(t_\lambda)_{\lambda=1,\dots,r}$. With the notation of 1.4.1.5 and 1.4.1.9, following 1.3.2.10 we have

$$\psi^n_{m,m'}(\underline{\tau}^{\{\underline{k}\}_{(m')}}) = \frac{q^{(m)}_{\underline{k}}!}{q^{(m')}_{\underline{k}}!}\underline{\tau}^{\{\underline{k}\}_{(m)}}, \tag{1.4.1.11.2}$$

where $\underline{\tau}^{\{\underline{k}\}_{(m)}} := \prod_{\lambda=1}^r \tau_\lambda^{\{k_\lambda\}_{(m)}}$, $q^{(m)}_{\underline{k}}! := \prod_{\lambda=1}^r q^{(m)}_{k_\lambda}!$ and similarly with some primes. Moreover, we compute $\psi^n_m(\underline{\tau}^{\underline{k}}) = q^{(m)}_{\underline{k}}!\underline{\tau}^{\{\underline{k}\}_{(m)}}$.

1.4.1.12. By composing the canonical morphism $\Delta_{X/S,(m)}(r) \to X^{\sharp r+1}_{/S}$ and $\Delta^n_{X/S,(m)}(r) \to X^{\sharp r+1}_{/S}$ with the ith projection $p_i \colon X^{\sharp r+1}_{/S} \to X$ for $i = 0, \dots, r$, we get

$$p_{i,(m)}(r) \colon \Delta^n_{X/S,(m)}(r) \to X, \qquad p^n_{i,(m)}(r) \colon \Delta^n_{X/S,(m)}(r) \to X. \tag{1.4.1.12.1}$$

If there are no risk of confusion, we can simply write p_i. We denote by $\Delta_{X/S,(m)}(r) \times_{p_i,X,p_{i'}} \Delta_{X/S,(m)}(r')$ the base change of $p_{i,(m)} \colon \Delta_{X/S,(m)}(r) \to X$ by $p_{i',(m)} \colon \Delta_{X/S,(m)}(r') \to X$. The immersion $X \hookrightarrow \Delta_{X/S,(m)}(r) \times_{p_i,X,p_{i'}} \Delta_{X/S,(m)}(r')$ induced by $X \hookrightarrow \Delta_{X/S,(m)}(r)$ and $X \hookrightarrow \Delta_{X/S,(m)}(r')$ is an closed immersion endowed with a canonical m-PD structure. Indeed, this is an easy consequence of 1.4.1.5 (for more details, see [Ber96c, 2.1.3.(i)]). Moreover, by using the universal property of m-PD-envelopes, we get the m-PD-morphism $q_{(m)}(r, r')$ making commutative the diagram:

$$\begin{array}{ccccccc}
X & \hookrightarrow & \Delta_{X/S,(m)}(r) \times_{p_i,X,p_{i'}} \Delta_{X/S,(m)}(r') & \longrightarrow & X^{r+1}_{/S} \times_{p_i,X,p_{i'}} X^{r'+1}_{/S} & \xrightarrow{p_i \times p_{i'}} & X \times_X X \\
\| & & \downarrow{\scriptstyle q_{(m)}(r,r')} & & \downarrow\wr & & \downarrow\wr \\
X & \hookrightarrow & \Delta_{X/S,(m)}(r+r') & \longrightarrow & X^{r+r'+1}_{/S} & \xrightarrow{p_i} & X.
\end{array} \tag{1.4.1.12.2}$$

By using again 1.4.1.5, we check that this arrow $q_{(m)}(r, r')$ is in fact an m-PD-isomorphism. Similarly, the immersion $X \hookrightarrow \Delta^n_{X/S,(m)}(r) \times_X \Delta^{n'}_{X/S,(m)}(r')$ induced by $X \hookrightarrow \Delta^n_{X/S,(m)}(r)$ and $X \hookrightarrow \Delta^{n'}_{X/S,(m)}(r')$ is a closed immersion endowed with a canonical m-PD structure of order $n + n'$ and we have the m-PD-morphism $q^{n,n'}_{(m)}(r, r')$ making commutative the diagram

$$\begin{array}{ccccccc}
X & \hookrightarrow & \Delta^n_{X/S,(m)}(r) \times_{p_i,X,p_{i'}} \Delta^{n'}_{X/S,(m)}(r') & \longrightarrow & X^{r+1}_{/S} \times_{p_i,X,p_{i'}} X^{r'+1}_{/S} & \xrightarrow{p_i \times p_{i'}} & X \times_X X \\
\| & & \downarrow{\scriptstyle q^{n,n'}_{(m)}(r,r')} & & \downarrow\wr & & \downarrow\wr \\
X & \hookrightarrow & \Delta^{n+n'}_{X/S,(m)}(r+r') & \longrightarrow & X^{r+r'+1}_{/S} & \xrightarrow{p_i} & X.
\end{array} \tag{1.4.1.12.3}$$

When $r = r' = 1$, we simply write $q_{(m)}$ and $q^{n,n'}_{(m)}$.

Notation 1.4.1.13. For any integer n and any integers $0 \le i < j \le 2$, it follows from the universal property of m-PD-envelopes of order n that we get a unique m-PD-morphism $q^n_{ij,(m)} \colon \Delta^n_{X/S,(m)}(2) \to \Delta^n_{X/S,(m)}$ making commutative the diagram

$$\begin{array}{ccccccc}
X & \hookrightarrow & \Delta^n_{X/S,(m)}(2) & \longrightarrow & X \times_S X \times_S X & \overset{p_i}{\underset{p_j}{\rightrightarrows}} & X \\
\| & & \downarrow{\scriptstyle q^n_{ij,(m)}} & & \downarrow{\scriptstyle p_{ij}} & & \| \\
X & \hookrightarrow & \Delta^n_{X/S,(m)} & \longrightarrow & X \times_S X & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & X
\end{array} \tag{1.4.1.13.1}$$

We still denote by $q^n_{ij,(m)} \colon p_{0*}\mathcal{P}^n_{X/S,(m)} \to p_{i*}\mathcal{P}^n_{X/S,(m)}(2)$ or $q^n_{ij,(m)} \colon p_{1*}\mathcal{P}^n_{X/S,(m)} \to p_{j*}\mathcal{P}^n_{X/S,(m)}(2)$ the corresponding homomorphism of m-PD-$\mathcal{O}_X$-algebras. We can simply write the m-PD-morphism $q^n_{ij,(m)} \colon \mathcal{P}^n_{X/S,(m)} \to \mathcal{P}^n_{X/S,(m)}(2)$ and recall that this homomorphism is also a homomorphism of m-PD-$\mathcal{O}_X$-algebras for two structures. Similarly we denote by $q_{ij,(m)} \colon \Delta_{X/S,(m)}(2) \to \Delta_{X/S,(m)}$ making commutative the diagram 1.4.1.13.1 without the order n condition.

Notation 1.4.1.14. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. By convention, $\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ means $p^n_{1*}(\mathcal{P}^n_{X/S,(m)}) \otimes_{\mathcal{O}_X} \mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)}$ means $\mathcal{E} \otimes_{\mathcal{O}_X} p^n_{0*}(\mathcal{P}^n_{X/S,(m)})$. For instance, $\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$ is $p^n_{1*}(\mathcal{P}^n_{X/S,(m)}) \otimes_{\mathcal{O}_X} p^{n'}_{0*}(\mathcal{P}^{n'}_{X/S,(m)})$.

We have two structures of $\mathcal{O}_X$-module on the sheaf $\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$: the "left structure" given by functoriality from the left structure of $\mathcal{P}^n_{X/S,(m)}$ and the "right structure" given by the internal tensor product. We denote by $p_{0*}(\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E})$ (resp. $p_{1*}(\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E})$) to clarify we are considering the left structure (resp. right structure).

Similarly, we denote by $p_{0*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)})$ (resp. $p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)})$) the $\mathcal{O}_X$-module given by the internal tensor product (resp. by functoriality from the right $\mathcal{O}_X$-module structure of $\mathcal{P}^n_{X/S,(m)}$) which is called the left (resp. right) structure.

We denote by $p^n_{0,\mathcal{E}} \colon \mathcal{E} \to p_{0*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)})$ the canonical $\mathcal{O}_X$-linear map given by $x \mapsto x \otimes \mathbb{1}$, i.e. is the composition of $\mathrm{id}_{\mathcal{E}} \otimes p^n_0$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^0_{X/S,(m)}$. We denote by $p^n_{1,\mathcal{E}} \colon \mathcal{E} \to p_{1*}(\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E})$ the canonical map given by $x \mapsto \mathbb{1} \otimes x$, i.e. is the composition of $p^n_1 \otimes \mathrm{id}_{\mathcal{E}}$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{P}^0_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$.

Notation 1.4.1.15. We simply denote by $\Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)}$ the base change of $p^{n'}_{0,(m)} \colon \Delta^{n'}_{X/S,(m)} \to X$ by $p^n_{1,(m)} \colon \Delta^n_{X/S,(m)} \to X$. Similarly, $X^2_{/S} \times_{p_1,X,p_0} X^2_{/S}$ is simply denoted by $X^2_{/S} \times_X X^2_{/S}$. By composition of 1.4.1.12.3 and 1.4.1.13.1, we get

$$\begin{array}{ccccccccc}
X & \hookrightarrow & \Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} & \longrightarrow & X^2_{/S} \times_X X^2_{/S} & \xrightarrow{\sim} & X \times_S X \times_S X & & \\
\| & & \downarrow{\scriptstyle q^{n,n'}_{(m)}} & & & & \| & & \\
X & \hookrightarrow & \Delta^{n+n'}_{X/S,(m)}(2) & \longrightarrow & & & X \times_S X \times_S X & \overset{p_i}{\underset{p_j}{\rightrightarrows}} & X \\
\| & & \downarrow{\scriptstyle q^{n+n'}_{ij,(m)}} & & & & \downarrow{\scriptstyle p_{ij}} & & \| \\
X & \hookrightarrow & \Delta^{n+n'}_{X/S,(m)} & \longrightarrow & & & X \times_S X & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & X.
\end{array} \tag{1.4.1.15.1}$$

We get the morphism $p^{n,n'}_{ij,(m)} := q^{n+n'}_{ij,(m)} \circ q^{n,n'}_{(m)} \colon \Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to \Delta^{n+n'}_{X/S,(m)}$ (which satisfies also a universal property). The morphism $p^{n,n'}_{01,(m)}$ is the composition

$$p^{n,n'}_{01,(m)} \colon \Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to \Delta^n_{X/S,(m)} \overset{\psi^{n+n',n}_{X/S,(m)}}{\hookrightarrow} \Delta^{n+n'}_{X/S,(m)},$$

where the first morphism is given by the left projection. Similarly, $p^{n,n'}_{12,(m)}$ is the composition $\Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to \Delta^{n'}_{X/S,(m)} \to \Delta^{n+n'}_{X/S,(m)}$, where the first morphism is given by the right projection.

By composing the canonical morphism $\Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to X \times_S X \times_S X$ with the ith projection $p_i \colon X \times_S X \times_S X \to X$ for $i = 0, 1, 2$, we get

$$p^{n,n'}_{i,(m)} \colon \Delta^n_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to X. \tag{1.4.1.15.2}$$

This yields three ring homomorphisms $p^{n,n'}_{i,(m)} \colon \mathcal{O}_X \to \mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X,(m)}$. When $i = 0$ (resp. $i = 1$, resp. $i = 2$), this is said to be the left (resp. middle, resp right) $\mathcal{O}_X$-algebra structure of $\mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X,(m)}$ and this is equal to the $\mathcal{O}_X$-algebra structure given by the left structure of $\mathcal{P}^n_{X/S,(m)}$ (resp. the tensor product, resp. the right structure of $\mathcal{P}^{n'}_{X/S,(m)}$). Using the associated universal properties, we have the equalities $p^{n,n'}_{i,(m)} = p^{n+n'}_{i,(m)}(2) \circ p^{n,n'}_{ij,(m)}$ for any $i, j = 0, 1, 2$.

We denote by $\delta^{n,n'}_{(m)} \colon \mathcal{P}^{n+n'}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$ (resp. $q^{n,n'}_{0(m)} \colon \mathcal{P}^{n+n'}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$, resp. $q^{n,n'}_{1(m)} \colon \mathcal{P}^{n+n'}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$) the morphism of m-PD-algebras associated to the morphism $p^{n,n'}_{02,(m)}$ (resp. $p^{n,n'}_{01,(m)}$, resp. $p^{n,n'}_{12,(m)}$). If there is no doubt on the level, we can also simply write $\delta^{n,n'}$, $q^{n,n'}_0$, $q^{n,n'}_1$.

The morphism $q^{n,n'}_{0(m)}$ is equal to the composition $q^{n,n'}_{0(m)} \colon \mathcal{P}^{n+n'}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$ (the last morphism is $\tau \mapsto \tau \otimes 1$). Moreover, $q^{n,n'}_{1(m)}$ is equal to the composition $q^{n,n'}_{1(m)} \colon \mathcal{P}^{n+n'}_{X/S,(m)} \to \mathcal{P}^{n'}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$ (the last morphism is $\tau \mapsto 1 \otimes \tau$). In order words, we have the relation $q^{n,n'}_{0(m)} = \varpi^{n,n'}_{i(m)} \circ \psi^{n+n',n}_{X/S,(m)}$ and $q^{n,n'}_{1(m)} = \varpi^{n,n'}_{i(m)} \circ \psi^{n+n',n'}_{X/S,(m)}$, where $\varpi^{n,n'}_{0(m)} \colon \mathcal{P}^{n}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$ and $\varpi^{n,n'}_{1(m)} \colon \mathcal{P}^{n'}_{X/S,(m)} \to \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}$ are the homomorphisms associated with the projections. The morphism $q^{n,n'}_{0(m)}$ is $\mathcal{O}_X$-linear for the left (resp. right) structure of $\mathcal{P}^{n+n'}_{X/S(m)}$ and the left structure (resp. of the center) of $\mathcal{P}^{n}_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S(m)}$. Finally, the morphism $q^{n,n'}_{0(m)}$ is $\mathcal{O}_X$-linear for the left (resp. right) structure of $\mathcal{P}^{n+n'}_{X/S(m)}$ and the structure of the center (resp. right) of $\mathcal{P}^{n}_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S(m)}$.

Using the commutativity of the diagram 1.4.1.15.1, we see that $\delta^{n,n'}_{(m)}$ is also an $\mathcal{O}_X$-algebra homomorphism for the respective left structures and for the respective right structures, i.e., $\delta^{n,n'}_{(m)} \circ p^{n+n'}_{0,(m)} = p^{n,n'}_{0,(m)}$ and $\delta^{n,n'}_{(m)} \circ p^{n+n'}_{1,(m)} = p^{n,n'}_{2,(m)}$. Using the commutativity of the diagram 1.4.1.15.1, we see that the morphism $q^{n,n'}_{0(m)}$ is $\mathcal{O}_X$-linear for the left (resp. right) structure of $\mathcal{P}^{n+n'}_{X/S(m)}$ and the left (resp. middle) structure of $\mathcal{P}^{n}_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S(m)}$.

By using 1.4.1.11 and the universal property of the m-PD envelopes, these morphisms are compatible with the change of level and we have the commutative diagram:

$$\begin{array}{ccccc} \mathcal{P}^{n+n'}_{X/S} & \longrightarrow & \mathcal{P}^{n+n'}_{X/S,(m')} & \xrightarrow{\psi^{n+n'}_{m,m'}} & \mathcal{P}^{n+n'}_{X/S,(m)} \\ \downarrow{\scriptstyle \delta^{n,n'}} & & \downarrow{\scriptstyle \delta^{n,n'}_{(m')}} & & \downarrow{\scriptstyle \delta^{n,n'}_{(m)}} \\ \mathcal{P}^{n}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S} & \longrightarrow & \mathcal{P}^{n}_{X/S,(m')} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m')} & \xrightarrow{\psi^{n}_{m,m'} \otimes \psi^{n'}_{m,m'}} & \mathcal{P}^{n}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}. \end{array} \tag{1.4.1.15.3}$$

Remark 1.4.1.16. The canonical m-PD structure on $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)}$ is characterized by the following property: the projections $q^{n,n'}_{0(m)} \colon \Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to \Delta^{n}_{X/S,(m)}$ and $q^{n,n'}_{1(m)} \colon \Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \to \Delta^{n'}_{X/S,(m)}$ are morphisms of $\mathscr{C}^{(m)}_{n+n'}$.

1.4.1.17. Suppose that f has coordinates $(t_\lambda)_{\lambda=1,\dots,r}$. With the notation of 1.4.1.5 and 1.4.1.9, since $\delta^{n,n'}(a \otimes b) = (a \otimes 1) \otimes (1 \otimes b)$ for any $a, b, \in \mathcal{O}_X$, then

$$\delta^{n,n'}_{(m)}(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i. \tag{1.4.1.17.1}$$

The following Lemma will be useful to check the associativity of the product law of the sheaf of differential operator:

Lemma 1.4.1.18. *We denote by $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)}$ the base change of $p^{n'}_0 \circ q^{n',n''}_0 \colon \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)} \to X$ by $p^n_1 \colon \Delta^{n}_{X/S,(m)} \to X$. The closed immersion $X \hookrightarrow \Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)}$ induced by $X \hookrightarrow \Delta^{n}_{X/S,(m)}$, $X \hookrightarrow \Delta^{n'}_{X/S,(m)}$ and $X \hookrightarrow \Delta^{n''}_{X/S,(m)}$ is endowed with a canonical m-PD structure. By abuse of notation, we denote by $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)}$ this object of $\mathscr{C}^{(m)}_{n+n'+n''}$. This m-PD structure on $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)}$ is characterized by the following property: the projections $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)} \to \Delta^{n}_{X/S,(m)}$, $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)} \to \Delta^{n'}_{X/S,(m)}$, and $\Delta^{n}_{X/S,(m)} \times_X \Delta^{n'}_{X/S,(m)} \times_X \Delta^{n''}_{X/S,(m)} \to \Delta^{n''}_{X/S,(m)}$ are morphisms of $\mathscr{C}^{(m)}_{n+n'+n''}$.*

Proof. This is checked similarly to 1.4.1.12. □

1.4.2 Differential operators of level m

Let $f\colon X \to S$ be a smooth morphism.

Definition 1.4.2.1. (a) The *sheaf of differential operators on X/S of level m and order n* is

$$\mathcal{D}^{(m)}_{X/S,n} := \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X/S,(m)}, \mathcal{O}_X). \tag{1.4.2.1.1}$$

(b) For any $n' \geq n$, the surjections $\psi^{n',n*}_{X/S,(m)}\colon \mathcal{P}^{n'}_{X/S,(m)} \hookrightarrow \mathcal{P}^n_{X/S,(m)}$ induces the injections

$$\rho^{(m)}_{n',n}\colon \mathcal{D}^{(m)}_{X/S,n} \hookrightarrow \mathcal{D}^{(m)}_{X/S,n'}. \tag{1.4.2.1.2}$$

The *sheaf of differential operators on X/S of level m* is

$$\mathcal{D}^{(m)}_{X/S} := \bigcup_{n\geq 0} \mathcal{D}^{(m)}_{X/S,n}.$$

(c) The tautological structure of $\mathcal{O}_X$-module on $\mathcal{D}^{(m)}_{X/S,n}$ is said to be the left one. For any $a \in \mathcal{O}_X$, we have the map $\mathcal{D}^{(m)}_{X/S,n} \to \mathcal{D}^{(m)}_{X/S,n}$ induced by the $\mathcal{O}_X$-linear map $p^n_{1(m)}(a)\colon p^n_{0(m)*}\mathcal{P}^n_{X/S,(m)} \to p^n_{0(m)*}\mathcal{P}^n_{X/S,(m)}$. This yields another structure of $\mathcal{O}_X$-module on $\mathcal{D}^{(m)}_{X/S,n}$ which is called the right structure of $\mathcal{O}_X$-module. For any $n' \geq n$, the homomorphisms $\mathcal{D}^{(m)}_{X/S,n} \to \mathcal{D}^{(m)}_{X/S,n}$ are $\mathcal{O}_X$-linear for both structures. This yields two structures of $\mathcal{O}_X$-modules on $\mathcal{D}^{(m)}_{X/S}$, the left one and the right one. We write $p_{0*}\mathcal{D}^{(m)}_{X/S,n}$ (resp. $p_{1*}\mathcal{D}^{(m)}_{X/S,n}$) when we consider the left structure (resp. the right one).

(d) Let $P \in \mathcal{D}^{(m)}_{X/S,n}$, $P' \in \mathcal{D}^{(m)}_{X/S,n'}$. We define the product $PP' \in \mathcal{D}^{(m)}_{X/S,n+n'}$ to be the composition

$$PP'\colon \mathcal{P}^{n+n'}_{X/S,(m)} \xrightarrow{\delta^{n,n'}_{(m)}} \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)} \xrightarrow{\mathrm{id}\otimes P'} \mathcal{P}^n_{X/S,(m)} \xrightarrow{P} \mathcal{O}_X. \tag{1.4.2.1.3}$$

Proposition 1.4.2.2. *The sheaf $\mathcal{D}^{(m)}_{X/S}$ is a sheaf of rings with the product as defined in 1.4.2.1.3.*

Proof. We have to check the product as defined in 1.4.2.1.3 is associative. To simplify notation, let us denote $\mathcal{P}_{X/S,(m)}$ (resp. $\mathcal{O}_X$) by $\mathcal{P}$ (resp. $\mathcal{O}$), i.e. we remove the index in the notation. One checks the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathcal{P}^{n+n'+n''} & = & \mathcal{P}^{n+n'+n''} & \xrightarrow{\delta^{n,n'+n''}_{(m)}} & \mathcal{P}^n\otimes_{\mathcal{O}}\mathcal{P}^{n'+n''} & = & \mathcal{P}^n\otimes_{\mathcal{O}}\mathcal{P}^{n'+n''} \\
 & & \downarrow{\scriptstyle \delta^{n+n',n''}_{(m)}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\delta^{n',n''}_{(m)}} & & \\
 & & \mathcal{P}^{n+n'}\otimes_{\mathcal{O}}\mathcal{P}^{n''} & \xrightarrow{\delta^{n,n'}_{(m)}\otimes\mathrm{id}} & \mathcal{P}^n\otimes_{\mathcal{O}}\mathcal{P}^{n'}\otimes_{\mathcal{O}}\mathcal{P}^{n''} & & \\
\downarrow{\scriptstyle (PP')P''} & & \downarrow{\scriptstyle \mathrm{id}\otimes P''} & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}\otimes P''} & & \downarrow{\scriptstyle \mathrm{id}\otimes P'P''} \\
 & & \mathcal{P}^{n+n'} & \xrightarrow{\delta^{n,n'}_{(m)}} & \mathcal{P}^n\otimes_{\mathcal{O}}\mathcal{P}^{n'} & & \\
 & & \downarrow{\scriptstyle PP'} & & \downarrow{\scriptstyle \mathrm{id}\otimes P'} & & \\
\mathcal{O} & = & \mathcal{O} & \xleftarrow[P]{} & \mathcal{P}^n & = & \mathcal{P}^n.
\end{array} \tag{1.4.2.2.1}$$

Indeed, let us check the commutativity of the top square of the middle. Since this is local, we can suppose that f has coordinates $(t_\lambda)_{\lambda=1,\dots,r}$. With the notation of 1.4.1.5 and 1.4.1.9, by using 1.4.1.17.1, we compute that the images of $\tau_1, \dots, \tau_r$ by both maps $\mathcal{P}^{n+n'+n''}_{X/S,(m)} \to \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X/S,(m)}$ are the same. Since both maps are m-PD-morphisms (see 1.4.1.18 for the m-PD-structure), then by using 1.4.1.5 we get the desired commutativity. Since the commutativity of the other squares are obvious, we conclude the proof.

□

Notation 1.4.2.3 (Local description)**.** Let X/S is equipped with coordinates $t_1, \dots, t_d$. The $\mathcal{P}^n_{X/S,(m)}$ is isomorphic to an m-PD polynomial algebra with coefficients in $\mathcal{O}_X$ of order n in d variables given by $\tau_1, \dots, \tau_d$ (see 1.4.1.5 and 1.4.1.9). In particular, $\mathcal{P}^n_{X/S,(m)}$ is a free $\mathcal{O}_X$-module with basis $\{\underline{\tau}^{\{\underline{k}\}_{(m)}}$:

$|\underline{k}| \leq n\}$. Let $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} : |\underline{k}| \leq n\}$ be the dual basis for $\mathcal{D}^{(m)}_{X/S,n} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S,(m)}, \mathcal{O}_X)$. Remark that the notation $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ does not depends on n, since the monomorphisms $\mathcal{D}^{(m)}_{X/S,n} \to \mathcal{D}^{(m)}_{X/S,n+1}$ sends $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ to $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$. We get the basis $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} : \underline{k} \in \mathbb{N}^d\}$ of $\mathcal{D}^{(m)}_{X/S}$.

1.4.2.4. We have $\mathcal{O}_X = \mathcal{P}^0_{X/S,(m)}$ and by duality $\mathcal{O}_X = \mathcal{D}^{(m)}_{X/S,0}$. This yields the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{D}^{(m)}_{X/S}$. From 1.4.2.1.3, we remark that $\mathcal{O}_X$ is in fact a subring of $\mathcal{D}^{(m)}_{X/S}$. Moreover, since $\mathcal{D}^{(m)}_{X/S}$ is a $\mathcal{D}^{(m)}_{X/S}$-bimodule, this yields that $\mathcal{D}^{(m)}_{X/S}$ is a $\mathcal{O}_X$-bimodule. We compute that $\mathcal{D}^{(m)}_{X/S,n}$ is a sub $\mathcal{O}_X$-bimodule of $\mathcal{D}^{(m)}_{X/S}$. Moreover, the structure of $\mathcal{O}_X$-module on $\mathcal{D}^{(m)}_{X/S,n}$ coming from the multiplication on the left (resp. the right) is equal to the left structure (resp. right structure) which was defined in 1.4.2.1.1.c.

1.4.2.5. For any $m' \geq m$, from the homomorphisms $\psi^n_{m,m'} \colon \mathcal{P}^n_{X/S,(m')} \to \mathcal{P}^n_{X/S,(m)}$ and $\psi^n_m \colon \mathcal{P}^n_{X/S} \to \mathcal{P}^n_{X/S,(m)}$ of 1.4.1.11, we get by duality, the maps

$$\mathcal{D}^{(m)}_{X/S} \xrightarrow{\rho_{m',m}} \mathcal{D}^{(m')}_{X/S} \xrightarrow{\rho_{m'}} \mathcal{D}_{X/S}. \tag{1.4.2.5.1}$$

It follows from 1.1.4.1.1, 1.4.2.1.3 and 1.4.1.15.3 that the maps of 1.4.2.5.1 are homomorphisms of filtered rings.

When X/S is equipped with coordinates $t_1, \ldots, t_d$, we obtain by duality from 1.4.1.11.2 the formula

$$\rho_{m',m}(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}) = \frac{q^{(m)}_{\underline{k}}!}{q^{(m')}_{\underline{k}}!} \underline{\partial}^{\langle \underline{k} \rangle_{(m')}} \text{ and } \rho_m(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}) = \underline{q}^{(m)}_{\underline{k}}! \underline{\partial}^{[\underline{k}]}. \tag{1.4.2.5.2}$$

By taking the inductive limit of ρ_m, we get the filtered ring morphism

$$\rho \colon \varinjlim_m \mathcal{D}^{(m)}_{X/S} \xrightarrow{\sim} \mathcal{D}_{X/S}, \tag{1.4.2.5.3}$$

which is an isomorphism (since this is local, this is an easy consequence of the formula 1.4.2.5.2). However, beware that the homomorphisms 1.4.2.5.1 are not necessarily injective.

Since $\mathcal{O}_X$ is endowed with a canonical structure of left $\mathcal{D}_{X/S}$-module, this induces a structure of left $\mathcal{D}^{(m)}_{X/S}$-module on $\mathcal{O}_X$. It follows from 1.1.4.1.2 and 1.4.2.5.1 that the structure of $\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{O}_X$ is given via the formula

$$P(f) := P \circ p^n_{1(m)}(f). \tag{1.4.2.5.4}$$

Notation 1.4.2.6 (Infinite level)**.** When $f \colon X \to S$ be a weakly smooth morphism, we write $\mathcal{D}^{(\infty)}_{X/S} := \mathcal{D}_{X/S}$, $\mathcal{P}^n_{X/S,(\infty)} = \mathcal{P}^n_{X/S,(m)}$. The notation is justified by the isomorphism 1.4.2.5.3, i.e. we can viewed the usual sheaf of differential operators as the sheaf of differential operators of infinite level.

Proposition 1.4.2.7. *With notation 1.4.2.3, the following rules hold in* $\mathcal{D}^{(m)}_{X/S}$.

(a) For any $x \in \Gamma(X, \mathcal{O}_X)$, $p^n_{1,(m)}(x) = \sum_{|\underline{k}| \leq n} p^n_{0,(m)}(\underline{\partial}^{\langle \underline{k} \rangle}(x)) \underline{\tau}^{\{\underline{k}\}} \in \mathcal{P}^n_{X,(m)}$ *(Taylor formula).*

(b) For any $x \in \Gamma(X, \mathcal{O}_X)$ *and* $\underline{k} \in \mathbb{N}^r$, *we have in* $\Gamma(X, \mathcal{D}^{(m)}_{X/S})$

$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} x = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}(x) \underline{\partial}^{\langle \underline{i} \rangle_{(m)}}. \tag{1.4.2.7.1}$$

(c) $\forall\ \underline{k}', \underline{k}'' \in \mathbb{N}^d$, $\underline{\partial}^{\langle \underline{k}' \rangle} \underline{\partial}^{\langle \underline{k}'' \rangle} = \left\langle \begin{smallmatrix} \underline{k}' + \underline{k}'' \\ \underline{k}' \end{smallmatrix} \right\rangle \underline{\partial}^{\langle \underline{k}' + \underline{k}'' \rangle}$,

(d) $\forall\ \underline{k}, \underline{i} \in \mathbb{N}^d$, *if* $\underline{i} \geq \underline{k}$, $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}(\underline{t}^{\underline{i}}) = q^{(m)}_{\underline{k}}! \left(\begin{smallmatrix} \underline{i} \\ \underline{k} \end{smallmatrix} \right) t^{\underline{i} - \underline{k}}$.

Proof. a) Taylor formula 3.2.3.7.1 comes from the fact that $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}, |\underline{k}| \leq n\}$ is the dual basis of $\{\underline{\tau}^{\{\underline{k}\}_{(m)}}, |\underline{k}| \leq n\}$.

b) Let $\underline{k} \in \mathbb{N}^r$ and $n = |\underline{k}|$. Since $\delta_{(m)}^{n,0} = \mathrm{id}$, then by definition of the multiplication (see 1.4.2.1.3), we get

$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} x(\underline{\tau}^{\{\underline{i}\}_{(m)}}) = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} (p_{1,(m)}^{n*}(x)\underline{\tau}^{\{\underline{i}\}_{(m)}}) = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \left(\sum_{|\underline{j}| \leq n} \underline{\partial}^{\langle \underline{j} \rangle}(x) \underline{\tau}^{\{\underline{j}\}} \underline{\tau}^{\{\underline{i}\}_{(m)}} \right) \quad (1.4.2.7.2)$$

$$\underset{1.2.4.5.3}{=} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \left(\sum_{|\underline{j}| \leq n} \left\{ \begin{matrix} \underline{i} + \underline{j} \\ \underline{j} \end{matrix} \right\}_{(m)} \underline{\partial}^{\langle \underline{j} \rangle}(x) \underline{\tau}^{\{\underline{j}+\underline{i}\}} \right) = \left\{ \begin{matrix} \underline{k} \\ \underline{i} \end{matrix} \right\}_{(m)} \underline{\partial}^{\langle \underline{k}-\underline{i} \rangle_{(m)}}(x) \underline{\tau}^{\{\underline{j}+\underline{i}\}}. \quad (1.4.2.7.3)$$

c) Since $\delta_{(m)}^{n,n'}(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i$ (see 1.4.1.17.1), then $\forall\ \underline{k}', \underline{k}'' \in \mathbb{N}^d$, setting $n = |\underline{k}|$ and $n' = |\underline{k}'|$, we get

$$\underline{\partial}^{\langle \underline{k}' \rangle} \underline{\partial}^{\langle \underline{k}'' \rangle}(\underline{\tau}^{\{\underline{i}\}}) = \underline{\partial}^{\langle \underline{k}' \rangle} \circ (\mathrm{id} \otimes \underline{\partial}^{\langle \underline{k}'' \rangle}) \circ \delta_{(m)}^{n,n'}(\underline{\tau}^{\{\underline{i}\}}) = \underline{\partial}^{\langle \underline{k}' \rangle} \circ (\mathrm{id} \otimes \underline{\partial}^{\langle \underline{k}'' \rangle})(\underline{\tau} \otimes 1 + 1 \otimes \underline{\tau})^{\{\underline{i}\}})$$

$$\underset{1.2.4.5.2}{=} \underline{\partial}^{\langle \underline{k}' \rangle} \circ (\mathrm{id} \otimes \underline{\partial}^{\langle \underline{k}'' \rangle}) \left(\sum_{\underline{i}' + \underline{i}'' = \underline{i}} \left\langle \begin{matrix} \underline{i} \\ \underline{i}' \end{matrix} \right\rangle_{(m)} (\underline{\tau}^{\{\underline{i}'\}_{(m)}} \otimes \underline{\tau}^{\{\underline{i}''\}_{(m)}}) \right). \quad (1.4.2.7.4)$$

Hence, if $\underline{i} = \underline{k}' + \underline{k}''$, then $\underline{\partial}^{\langle \underline{k}' \rangle} \underline{\partial}^{\langle \underline{k}'' \rangle}(\underline{\tau}^{\{\underline{i}\}}) = \left\langle \begin{smallmatrix} \underline{k}' + \underline{k}'' \\ \underline{k}' \end{smallmatrix} \right\rangle$, otherwise this is null.

d) The last computation follows from the Tayor formula and the computation

$$p_{1,(m)}^{n*}(\underline{t}^{\underline{i}}) = \left(p_{1,(m)}^{n*}(\underline{t})\right)^{\underline{i}} = \left(p_{0,(m)}^{n*}(\underline{t}) + \underline{\tau}\right)^{\underline{i}} \underset{1.2.4.5.1}{=} \sum_{\underline{k} \leq \underline{i}} q_{\underline{k}}^{(m)}! \begin{pmatrix} \underline{i} \\ \underline{k} \end{pmatrix} \underline{t}^{\underline{i}-\underline{k}} \underline{\tau}^{\{\underline{k}\}}.$$

□

1.4.2.8. We have the following properties.

(a) For any integer k, we denote by $(k)_i \in \mathbb{N}^d$ the element $(k)_i = (0, \ldots, k, \ldots, 0)$, where k is at the ith place. We set $\partial_i^{\langle k \rangle} := \underline{\partial}^{\langle (k)_i \rangle}$.

(b) It follows from 1.4.2.7.c that $\forall\ \underline{k}', \underline{k}'' \in \mathbb{N}^d$, we have $\underline{\partial}^{\langle \underline{k}' \rangle} \underline{\partial}^{\langle \underline{k}'' \rangle} = \underline{\partial}^{\langle \underline{k}'' \rangle} \underline{\partial}^{\langle \underline{k}' \rangle}$. Hence, for any $\underline{k}$ we have

$$\underline{\partial}^{\langle \underline{k} \rangle} = \prod_{i=1}^{d} \partial_i^{\langle k_i \rangle}. \quad (1.4.2.8.1)$$

For $k \in \mathbb{N}$, $k = \sum_{j=0}^{m-1} a_j p^j + a p^m$, with $0 \leq a_j < p$ for all j. It follows from 1.2.1.5 and 1.4.2.7.c that

$$\partial_i^{\langle k \rangle} = u \left(\prod_{j=0}^{m-1} (\partial_i^{\langle p^j \rangle})^{a_j} \right) (\partial_i^{\langle p^m \rangle})^a \quad (1.4.2.8.2)$$

for some invertible elements u in $\mathbb{Z}_{(p)}$ ($u = 1$ if $m = 0$).

(c) If $\underline{k} = p^m \underline{q} + \underline{r}$ with $0 \leq r_i < p^m$ If $r_i = \sum_{j=0}^{m-1} a_{i,j} p^j$ with $0 \leq a_{i,j} < p$, then

$$\underline{\partial}^{\langle \underline{k} \rangle} = u_{\underline{k}} \prod_{i=1}^{p} \left(\prod_{j=0}^{m-1} (\partial_i^{\langle p^j \rangle})^{a_{i,j}} \right) (\partial_i^{[p^m]})^{q_i} \quad (1.4.2.8.3)$$

for some invertible elements $u_{\underline{k}}$ in $\mathbb{Z}_{(p)}^*$.

We have the well known and easy result (e.g. see [LvO96, II.1.2 Proposition 3] or [BGK⁺87, II.5.2]) that we recall for the reader.

Lemma 1.4.2.9. *Let $D = \cup_{n \in \mathbb{N}} D_n$ be a filtered ring (i.e. D_n are abelian groups such that $D_n \cdot D_{n'} \subset D_{n+n'}$) such that $\mathrm{gr}\, D = D_0 \oplus \oplus_{n \geq 1} D_n / D_{n-1}$ is noetherian. Then D is noetherian.*

Proposition 1.4.2.10. *Under the hypotheses 1.4.2.3, the sheaf $\mathcal{D}^{(m)}_{X/S}$ is generated as $\mathcal{O}_S$-algebra by $\mathcal{O}_X$ and the two by two commuting operators $\partial_i, \partial_i^{\langle p \rangle}, \cdots, \partial_i^{\langle p^m \rangle}$, $1 \leq i \leq r$.*

More precisely, fix $1 \leq i \leq r$. Then, for any $0 \leq j \leq m$, for any $0 \leq k < p^j$, the operator $\partial_i^{\langle k \rangle}$ belongs to the sub $\mathbb{Z}_{(p)}$-algebra of $\mathcal{D}^{(m)}_{X/S}$ generated by $\partial_i, \partial_i^{\langle p \rangle}, \cdots, \partial_i^{\langle p^{j-1} \rangle}$. Moreover, for any $k \in \mathbb{N}$, $\partial_i^{\langle k \rangle}$ belongs to the sub $\mathbb{Z}_{(p)}$-algebra of $\mathcal{D}^{(m)}_{X/S}$ generated by $\partial_i, \partial_i^{\langle p \rangle} \cdots, \partial_i^{\langle p^m \rangle}$.

Proof. This is a straightforward consequence of the formula 1.4.2.8.3. □

Proposition 1.4.2.11. *We have the following properties.*

(a) The graded ring gr $\mathcal{D}^{(m)}_{X/S}$ associated to the order filtration $(\mathcal{D}^{(m)}_{X/S,n})_{n \in \mathbb{N}}$ is a commutative ring.

(b) Suppose S be a locally noetherian scheme. Let U be an affine open of X (resp. $x \in X$). Then $\Gamma(U, \mathrm{gr}\ \mathcal{D}^{(m)}_{X/S})$ (resp. $D^{(m)}_U := \Gamma(U, \mathcal{D}^{(m)}_{X/S})$, resp. $\mathrm{gr}\ \mathcal{D}^{(m)}_{X/S,x}$, resp. $\mathcal{D}^{(m)}_{X/S,x}$) is left and right noetherian.

Proof. Let us check (a). Since this is local, we can suppose Let X/S is equipped with coordinates $t_1, \ldots, t_d$. Then this follows from the formulas 1.4.2.7.c and 1.4.2.7.b.

Let us prove (b). As U is affine, it is a coherent topological space, so that the functor $\Gamma(U, -)$ commutes with filtered inductive limits (see [SGA4.2] VI, 5.2). Thus the ring $D^{(m)}_U$ has filtration by order and its associated graded ring is commutative and is noetherian because it is generated as $\Gamma(U, \mathcal{O}_X)$-algebra by the finitely generated $\Gamma(U, \mathcal{O}_X)$-modules $\mathrm{gr}_i\, D^{(m)}_U = \Gamma(U, \mathrm{gr}_i\, \mathcal{D}^{(m)}_{X/S})$ for $i < p^m$ (use 1.4.2.10). This implies that the filtered ring $D^{(m)}_U$ is left and right noetherian (see 1.4.2.9). We proceed similarly for the two remaining cases. □

Remark 1.4.2.12. Beware that the proposition 1.1.4.5 is false in the case of finite level. For instance, $\mathcal{E} = \mathcal{O}_X / t^p \mathcal{O}_X$ is a $\mathcal{D}^{(0)}_X$-module which is $\mathcal{O}_X$-coherent but it is not a locally free $\mathcal{O}_X$-module.

1.4.3 Tor dimension, Cartan's theorems A and B, perfection

Let us recall some facts on tor dimension and perfection which will be useful later.

Definition 1.4.3.1. Let $\mathcal{D}$ be a sheaf of rings on a topological space. Let $\mathcal{E}^\bullet \in C({}^{\mathrm{l}}\mathcal{D})$.

1. Let $a, b \in \mathbb{Z}$ with $a \leq b$. We say $\mathcal{E}^\bullet$ has tor-amplitude in $[a, b]$ if $H^i(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) = 0$ for all right $\mathcal{D}$-modules $\mathcal{M}$ and all $i \notin [a, b]$.
2. We say $\mathcal{E}^\bullet$ has finite tor dimension if it has tor-amplitude in $[a, b]$ for some integer $a \leq b$. We denote by $D_{\mathrm{tdf}}(\mathcal{D})$ the strictly full subcategory of $D(\mathcal{D})$ consisting of complexes having finite tor dimension on $\mathcal{D}$.
3. We say $\mathcal{E}^\bullet$ *locally has finite tor dimension* if there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that $\mathcal{E}^\bullet|_{U_\lambda}$ has finite tor dimension and for each $\lambda \in \Lambda$.

1.4.3.2. Let $\mathcal{D}$ be a sheaf of rings on a topological space X. Let $\mathcal{E}^\bullet \in C({}^{\mathrm{l}}\mathcal{D})$. Let $a, b \in \mathbb{Z}$ with $a \leq b$. The following properties are equivalent.

(a) The complex $\mathcal{E}^\bullet$ has tor-amplitude in $[a, b]$.

(b) There exists a complex $\mathcal{P}^\bullet$ of flat left $\mathcal{D}$-modules with $\mathcal{P}^i = 0$ for $i \notin [a, b]$ together with a quasi-isomorphism of $K({}^{\mathrm{l}}\mathcal{D})$ of the form $\mathcal{P}^\bullet \to \mathcal{E}^\bullet$.

(c) For any $x \in X$, the complex $\mathcal{E}^\bullet_x$ has tor-amplitude in $[a, b]$.

Indeed, for the equivalence between (a) and (b) (resp. (a) and (c)), we can follow the proof of [Sta22, 08CI] (resp. [Sta22, 09U9]).

We will need later (see 5.3.2.12) the following lemma.

Lemma 1.4.3.3. *Let X be a scheme and $\mathcal{D}$ be a sheaf of rings on X endowed with a morphism of rings $\mathcal{O}_X \to \mathcal{D}$. Let $\mathcal{E}^\bullet \in K({}^1\mathcal{D})$. Let $a, b \in \mathbb{Z}$ with $a \leq b$. The following properties are equivalent.*

(a) The complex $\mathcal{E}^\bullet$ has tor-amplitude in $[a, b]$.

(b) We have $H^n(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) = 0$ for all right $\mathcal{D}$-modules $\mathcal{M}$ which are quasi-coherent as $\mathcal{O}_X$-modules, and all $n \notin [a, b]$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious by definition. Let us prove that the property (b) implies the property 1.4.3.2.(c). Suppose $\mathcal{E}^\bullet$ satisfies the condition (b). Let M be a right $\mathcal{D}_x$-modules and let $n \notin [a, b]$ be an integer. Let $i_x \colon \operatorname{Spec} k(x) \to X$ be the morphism induced by x. Since $i_{x*}M$ is a quasi-coherent right $\mathcal{D}$-module, then $H^n(i_{x*}M \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) = 0$. Since i_x^{-1} is exact and since $i_x^{-1}(i_{x*}M \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) \xrightarrow{\sim} M \otimes^{\mathbb{L}}_{\mathcal{D}_x} \mathcal{E}^\bullet_x$, then $H^n(M \otimes^{\mathbb{L}}_{\mathcal{D}_x} \mathcal{E}^\bullet_x) = 0$. □

Lemma 1.4.3.4. *Let $\rho \colon \mathcal{D} \to \mathcal{D}'$ be an homomorphism of sheaf of rings on a topological space. We get the functors $\rho_* \colon D({}^1\mathcal{D}') \to D({}^1\mathcal{D})$ and $\mathbb{L}\rho^* = \mathcal{D}' \otimes^{\mathbb{L}}_{\mathcal{D}} - \colon D({}^1\mathcal{D}) \to D({}^1\mathcal{D}')$. Let $a, b \in \mathbb{Z}$ with $a \leq b$.*

(a) If $\mathcal{E}^\bullet \in D({}^1\mathcal{D})$ has tor amplitude in $[a, b]$ then $\mathbb{L}\rho^*(\mathcal{E}^\bullet)$ has tor-amplitude in $[a, b]$.

(b) Suppose that $\mathcal{D}'$ has tor-amplitude in $[-d, 0]$ for some $d \in \mathbb{N}$ as a complex of $K({}^{\mathrm{r}}\mathcal{D})$. If $\mathcal{E}'^\bullet \in D({}^1\mathcal{D}')$ has tor amplitude in $[a, b]$ then $\rho_*(\mathcal{E}'^\bullet)$ has tor-amplitude in $[a - d, b]$.

Proof. Let $\mathcal{M}'$ be a right $\mathcal{D}'$-module. The part (a) is a consequence of the isomorphism $\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{D}'} (\mathcal{D}' \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) \xrightarrow{\sim} \mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet$. Let $\mathcal{M}$ be a right $\mathcal{D}$-module. The part (b) of lemma follows from the isomorphism $\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}'^\bullet \xrightarrow{\sim} (\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{D}') \otimes^{\mathbb{L}}_{\mathcal{D}'} \mathcal{E}'^\bullet$. □

Definition 1.4.3.5 (tor-dimension of a ring)**.** Let $\mathcal{D}$ be a sheaf of rings on a topological space X. We define the "tor-dimension of the sheaf of ring $\mathcal{D}$" to be the following number (possibly ∞):

$$\operatorname{tor.dim}(\mathcal{D}) := \sup\{n \in \mathbb{N} \ : \ \operatorname{Tor}^{\mathcal{D}}_n(\mathcal{M}, \mathcal{E}) \neq 0 \text{ for some left (resp. right) } \mathcal{D}\text{-module } \mathcal{E} \text{ (resp. } \mathcal{M})\}.$$

We say that $\mathcal{D}$ has *finite tor dimension* if $\operatorname{tor.dim}(\mathcal{D}) \in \mathbb{N}$. We say that $\mathcal{D}$ has *locally finite tor dimension* if there exists an open covering $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ such that $\mathcal{D}|U_\lambda$ has finite tor dimension for any $\lambda \in \Lambda$.

1.4.3.6. Let $\mathcal{D}$ be a sheaf of rings on a topological space X. Let $n \in \mathbb{N}$. By definition, the following conditions are equivalent:

(a) $\operatorname{tor.dim}(\mathcal{D}) \leq n$;

(b) We have $\operatorname{Tor}^{\mathcal{D}}_i(\mathcal{M}, \mathcal{E}) = 0$ for any integer $i > n$ and any left (resp. right) $\mathcal{D}$-module $\mathcal{E}$ (resp. $\mathcal{M}$).

(c) Any left (resp. right) $\mathcal{D}$-module $\mathcal{E}$ (resp. $\mathcal{M}$) has tor-amplitude in $[-n, 0]$.

Remark 1.4.3.7. With notation 1.4.3.4, suppose ρ is flat. Then the functors ρ_* and ρ^* preserve the tor-amplitude. However, since they are not essentially surjective, we cannot compare a priori the tor dimensions of $\mathcal{D}$ and $\mathcal{D}'$ (however, sometimes this might happen: see 1.4.3.24).

Remark 1.4.3.8. Let $\mathcal{D}$ be a sheaf of rings on a topological space X. If $\mathcal{D}$ has locally finite tor dimension, then any complex of $C^{\mathrm{b}}(\mathcal{D})$ has locally finite tor-dimension.

1.4.3.9 (Point case)**.** Let D be a ring. Let $d \in \mathbb{N}$. The following assertions are equivalent.

(a) $\operatorname{tor.dim}(D) \leq d$;

(b) Any left D-module E has tor-amplitude in $[-d, 0]$;

(c) Any left D-module E of finite presentation has tor-amplitude in $[-d, 0]$;

(d) For any left D-module E of finite presentation, for any right D-module M of finite presentation, for any $n > d$, we have $\operatorname{Tor}^D_n(M, E) = 0$.

Indeed, the equivalence between (a) and (b) is tautological. The implications $(b) \Rightarrow (c) \Rightarrow (d)$ are obvious. The converse implications follow from the fact that any left (right) D-module is a small filtered inductive limit of modules of finite presentation (e.g. see [Bou61b, I.§2.Exercice 10]), from the fact that tensor products commute with inductive limits and that small filtered inductive limits are exact in the category of abelian groups.

Consider the assertions.

(i) There exists a left D-module E of finite presentation, a right D-module M of finite presentation, such that $\mathrm{Tor}_d^D(M,E) \neq 0$;

(ii) There exists a left D-module E of finite presentation, a right D-module M of finite presentation, there exists $n \geq d$ such that $\mathrm{Tor}_n^D(M,E) \neq 0$.

(iii) $\mathrm{tor}\,.\dim(D) \geq d$.

It follows from the equivalence between (a) and (d) we have the last relation $(i) \Rightarrow (ii) \Leftrightarrow (iii)$. We have moreover:

$$\mathrm{tor}\,.\dim(D) = \sup\{n \in \mathbb{N}, \exists \text{ a left (right) } D\text{-module } E\ (M) \text{ of f.p. such that } \mathrm{Tor}_n^D(M,E) \neq 0\}.$$

Lemma 1.4.3.10. *Let $\mathcal{D}$ be a sheaf of rings on a topological space X. We have the equality* $\mathrm{tor}\,.\dim(\mathcal{D}) = \sup_{x\in X} \mathrm{tor}\,.\dim(\mathcal{D}_x)$.

Proof. To check $\mathrm{tor}\,.\dim(\mathcal{D}) \leq \sup_{x\in X} \mathrm{tor}\,.\dim(\mathcal{D}_x)$, we can suppose $\sup_{x\in X} \mathrm{tor}\,.\dim(\mathcal{D}_x)$ is an integer denoted by n. Let $\mathcal{E}$ be a left $\mathcal{D}$-module. Then for any $x \in X$, the left $\mathcal{D}_x$-module $\mathcal{E}_x$ has tor-amplitude in $[-n,0]$. Hence, following 1.4.3.2, this implies that the left $\mathcal{D}$-module $\mathcal{E}$ has tor-amplitude in $[-n,0]$. With 1.4.3.6, this yields $\mathrm{tor}\,.\dim(\mathcal{D}) \leq n$.

Let $x \in X$. It remains to check $\mathrm{tor}\,.\dim(\mathcal{D}) \geq \mathrm{tor}\,.\dim(\mathcal{D}_x)$. We reduce to the case where $\mathrm{tor}\,.\dim(\mathcal{D})$ is an integer denoted by n. Let E be a left $\mathcal{D}_x$-module. Then the left $\mathcal{D}$-module $i_{x*}E$ has tor-amplitude in $[-n,0]$. Hence, so is $E = i_x^{-1} i_{x*}E$. □

Lemma 1.4.3.11. *Let $(D_i)_{i\in I}$ be a filtered inductive system of rings. Set $D := \varinjlim_{i\in I} D_i$. Suppose that $D_i \to D$ is right flat for any $i \in I$. Then* $\mathrm{tor}\,.\dim(D) \leq \sup\{\mathrm{tor}\,.\dim(D_i)\ ;\ i \in I\}$.

Proof. Set $n := \sup\{\mathrm{tor}\,.\dim(D_i)\colon\ i \in I\}$ that we can suppose finite. Let E be a left D-module of finite presentation. Then there exist $i_0 \in I$ and a left D_{i_0}-module E_{i_0} of finite presentation together with an isomorphism of left D-modules $D \otimes_{D_{i_0}} E_{i_0} \xrightarrow{\sim} E$. Since $\mathrm{tor}\,.\dim(D_{i_0}) \leq n$ then E_{i_0} has tor amplitude in $[-n,0]$. Since $D_{i_0} \to D$ is right flat this yields that $D \otimes_{D_{i_0}} E_{i_0}$ has tor amplitude in $[-n,0]$ (use 1.4.3.4.(a)). We conclude by using 1.4.3.9. □

Lemma 1.4.3.12. *Let $\mathcal{D}$ be a sheaf of rings on a topological space X. Let $x \in X$ and $\mathfrak{B}$ be a basis of open neighborhoods of x in X such that for any $U \in \mathfrak{B}$ the ring homomorphism $\mathcal{D}(U) \to \mathcal{D}_x$ is right flat. We have* $\mathrm{tor}\,.\dim(\mathcal{D}_x) \leq \sup\{\mathrm{tor}\,.\dim(\mathcal{D}(U))\ ;\ U \in \mathfrak{B}\}$.

Proof. This is a consequence of 1.4.3.11. □

Notation 1.4.3.13. By abuse of notation, for any set A we still denote by A the sheaf on a topological space Y associated with the constant preasheaf with value A.

Definition 1.4.3.14. Let $\mathcal{D}$ be a sheaf of rings on a scheme or a formal scheme X. We say that $\mathcal{D}$ satisfies "theorem A for coherent left modules" if the following conditions hold.

(i) The sheaf of rings $\mathcal{D}$ is left coherent ;

(ii) For any affine opens U, V of X such that $V \subset U$, the extension $\Gamma(U,\mathcal{D}) \to \Gamma(V,\mathcal{D})$ is right flat ;

(iii) For any affine open U of X, the functors $\Gamma(U,-)$ and $\mathcal{D}|_U \otimes_{\Gamma(U,\mathcal{D})} -$ induces canonically quasi-inverse equivalences between the category of coherent left $\mathcal{D}|_U$-modules and that of coherent left $\Gamma(U,\mathcal{D})$-modules ;

We say that $\mathcal{D}$ satisfies "theorem B for coherent left modules" if the following conditions hold.

(i) The sheaf of rings $\mathcal{D}$ is left coherent ;

(iv) For any affine open U of X, for any coherent left $\mathcal{D}|_U$-module $\mathcal{E}$, for any $i \geq 1$, we have $H^i(U, \mathcal{E}) = 0$.

We define similarly the notion of "theorems A and B for coherent right modules" and of "theorems A and B for coherent modules" when both left and right versions are satisfied.

1.4.3.15. By convention in this book, when we add "for coherent modules", it is understood that the ring $\mathcal{D}$ is coherent and for convenience we have added the flatness condition 1.4.3.14.(ii) to get the exactness of the functor $\mathcal{D}_U \otimes_{\Gamma(U,\mathcal{D})} -$. But we do not require affine sections to be noetherian (remark this latter condition together with 1.4.3.14.ii implies the coherence of $\mathcal{D}$: see 1.4.5.2). Such generality is needed in Theorem 8.7.5.5.

The word "canonically" in the condition (iii) of 1.4.3.14 means that the canonical morphisms $M \to \Gamma(U, \mathcal{D}|_U \otimes_{\Gamma(U,\mathcal{D})} M)$ and $\mathcal{D}|_U \otimes_{\Gamma(U,\mathcal{D})} \Gamma(U, \mathcal{M}) \to \mathcal{M}$ are isomorphisms for any affine open set U of X, any coherent $\Gamma(U, \mathcal{D})$-module M and any coherent $\mathcal{D}|_U$-module $\mathcal{M}$.

Remark 1.4.3.16. Let $\mathcal{D}$ be a sheaf of rings on a affine scheme or an affine formal scheme X. Put $D := \Gamma(X, \mathcal{D})$. We have the following properties (and its right version).

i) If $\mathcal{D}$ satisfies theorem A for coherent left modules then D is also left coherent from the property 1.4.3.14.(iii).

ii) Suppose $\mathcal{D}$ satisfies theorem B for coherent left modules. Since X is quasi-compact and quasi-separated, then it follows from [SGA4.2, VI.5.2] that the functors $H^i(X, -)$ for any $i \geq 0$ commute with filtered inductive limits of abelian groups. Since any left D-module is a filtered inductive limits of coherent left D-modules (see [Bou61b, I.§2.Exercice 10]), since the functor $\mathcal{D} \otimes_D -$ commutes also with filtered inductive limits of abelian groups, then for any left D-module E, $H^i(X, \mathcal{D} \otimes_D E) = 0$ for any $i \geq 1$.

iii) Suppose $\mathcal{D}$ satisfies theorems A and B for coherent left modules. Then, using the same arguments as above in ii), we get that the canonical map $E \to \Gamma(X, \mathcal{D} \otimes_D E)$ is an isomorphism for any left D-module E.

Proposition 1.4.3.17. *Let X be a noetherian affine (formal) scheme of finite Krull dimension. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorems A and B for left coherent modules (see 1.4.3.14). Set $D = \mathcal{D}(X)$. Let E be a coherent left D-module E, and $n \in \mathbb{N}$. The following conditions are equivalent.*

(a) The left D-module E has tor-amplitude in $[-n, 0]$ *;*

(b) The left $\mathcal{D}$-module $\mathcal{D} \otimes_D E$ has tor-amplitude in $[-n, 0]$.

In particular, we get the inequality $\operatorname{tor.dim} D \leq \operatorname{tor.dim} \mathcal{D}$.

Proof. I) Let us prove the equivalence $(a) \Leftrightarrow (b)$.

0) By right flatness of $D \to \mathcal{D}$, we get $(a) \to (b)$ (see 1.4.3.4.(a)). Conversely, suppose the left $\mathcal{D}$-module $\mathcal{D} \otimes_D E$ has tor-amplitude in $[-n, 0]$. Let $L^\bullet$ be a complex of free left D-modules of finite type such that $L^i = 0$ for $i > 0$ together with a quasi-isomorphism of $K({}^l D)$ of the form $L^\bullet \to E$.

1) Let us prove the case where $n = 0$, i.e. suppose $\mathcal{D} \otimes_D E$ is $\mathcal{D}$-flat. Let M be a left D-module. Set $\mathcal{E} := \mathcal{D} \otimes_D E$, $\mathcal{M} := M \otimes_D \mathcal{D}$, $\mathcal{L}^\bullet := \mathcal{D} \otimes_D L^\bullet$; $G^\bullet := M \otimes_D L^\bullet$ and $\mathcal{G}^\bullet := \mathcal{M} \otimes_{\mathcal{D}} \mathcal{L}^\bullet$. Following [Gro57, 3.6.5], since X is noetherian of finite Krull dimension then the functor $\Gamma(X, -)$ has bounded cohomology (see definition 4.6.1.4). Hence, it follows from 4.6.1.6.3 that we get the spectral sequence

$$E_1^{r,s} = R^s\Gamma(X, \mathcal{G}^r) \Rightarrow H^{r+s}\mathbb{R}\Gamma(X, \mathcal{G}^\bullet). \qquad (1.4.3.17.1)$$

Since $\mathcal{D}$ satisfies theorem B, since $\mathcal{G}^r$ is of the form $\mathcal{M}^{n_r}$ for some integer n_r, then if follows from 1.4.3.16.ii) that we get $E_1^{r,s} = 0$ if $s \neq 0$. It follows from 1.4.3.16.iii) that the canonical map $G^r \to \Gamma(X, \mathcal{G}^r)$ is an isomorphism. Hence, $E_1^{r,0}$ is canonically isomorphic to G^r and we get $H^n(G^\bullet) \xrightarrow{\sim} H^n\mathbb{R}\Gamma(X, \mathcal{G}^\bullet)$. Since $\mathcal{E}$ is flat, then $\mathcal{G}^\bullet \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{D}} \mathcal{E}$ and therefore $H^n\mathbb{R}\Gamma(X, \mathcal{G}^\bullet) = 0$ for any $n \leq -1$. Since $H^n(G^\bullet) = Tor^D_{-n}(M, E)$, then we are done (use [Bou61b, I.§4. Proposition 1]).

2) Let us return to the general case. Let Q be the image of $L^{-n} \to L^{-n+1}$. By flatness of $D \to \mathcal{D}$, we get the flat left resolution $\mathcal{L}^\bullet := \mathcal{D} \otimes_D L^\bullet$ of $\mathcal{E} := \mathcal{D} \otimes_D E$ and the flat left resolution $\mathcal{Q}^\bullet := \mathcal{L}^{\bullet - n}$ of $\mathcal{Q} := \mathcal{D} \otimes_D Q$. For any right $\mathcal{D}$-module $\mathcal{M}$, this yields $\operatorname{Tor}_1^{\mathcal{D}}(\mathcal{M}, \mathcal{Q}) = H^{-1}(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{Q}^\bullet) = H^{-n-1}(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{L}^\bullet) = H^{-n-1}(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}) = 0$. Hence $\mathcal{Q}$ is flat. This yields that Q is flat from the step 1). Hence we get

the exact sequence $0 \to Q \to L^{-n+1} \to L^{-n+2} \cdots \to L^0 \to E \to 0$ with flat left D-modules except E and we are done.

II) By choosing $n = \mathrm{tor.dim}\,\mathcal{D}$, the inequality $\mathrm{tor.dim}\, D \leq \mathrm{tor.dim}\,\mathcal{D}$ follows from the part I) and from 1.4.3.9.(c). □

By removing the hypothesis "satisfies theorem B", we can get closed to 1.4.3.17 results below (see 1.4.3.20). However, beware that 1.4.3.19 and 1.4.3.20 below are not enough to get a theorem A version of the inequality $\mathrm{tor.dim}\, D \leq \mathrm{tor.dim}\,\mathcal{D}$. Indeed, a priori we cannot descent the projectivity of finite presentation because D is not commutative.

Lemma 1.4.3.18. *Let $D \to D'$ be a right flat ring homomorphism such that D is a coherent ring. Then the following assertions are equivalent.*

(a) The map $D \to D'$ be a right faithfully flat ;

(b) For any coherent left D-module E such that $D' \otimes_D E = 0$, we have necessarily $E = 0$;

(c) Any sequence of coherent left D-modules $E \to F \to G$ such $D' \otimes_D E \to D' \otimes_D F \to D' \otimes_D G$ is exact is necessarily exact.

Proof. The equivalence between (b) and (c) is easy. The implication $(a) \Rightarrow (b)$ is tautological. Conversely, suppose (c). Let E be a left D-module such that $D' \otimes_D E = 0$. We have to prove that $E = 0$. Then $E \xrightarrow{\sim} \varinjlim_{i \in I} E_i$ with I a small filtered set and where E_i are coherent left D-modules (e.g. see [Bou61b, I.§2.Exercice 10] and recall that D is coherent). Hence, $\varinjlim_{i \in I} D' \otimes_D E_i = 0$. Let $i_0 \in I$. Since E_i is a left D-module of finite type, then there exists $i \geq i_0$ such that $D' \otimes_D E_{i_0} \to D' \otimes_D E_i$ is the null morphism, i.e. the sequence $D' \otimes_D E_{i_0} \to D' \otimes_D E_{i_0} \to D' \otimes_D E_i$ is exact. By using the property (c), we get that $E_{i_0} \to E_{i_0} \to E_i$ is exact, i.e. $E_{i_0} \to E_i$ is the null morphism. Hence, $\varinjlim_{i \in I} E_i = 0$. □

Lemma 1.4.3.19. *Let X be a scheme or a formal scheme. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorem A for left coherent modules (see 1.4.3.14). For any affine open U of X, for any finite covering $(U_i)_{i \in I}$ of U by affine opens, the extension $\mathcal{D}(U) \to \prod_{i \in I} \mathcal{D}(U_i)$ is left faithfully flat.*

Proof. Let E be a coherent left $\mathcal{D}(U)$-module such that $(\prod_{i \in I} \mathcal{D}(U_i)) \otimes_{\mathcal{D}(U)} E = 0$. Following 1.4.3.18, we reduce to prove that $E = 0$. Since I is finite, $\prod_{i \in I} \mathcal{D}(U_i) = \oplus_{i \in I} \mathcal{D}(U_i)$ and then for any $i \in I$ we get $\mathcal{D}(U_i) \otimes_{\mathcal{D}(U)} E = 0$. Set $\mathcal{E} := \mathcal{D}|U \otimes_{\mathcal{D}(U)} E$. For any $i \in I$, we get $\mathcal{E}|U_i = \mathcal{D}|U_i \otimes_{\mathcal{D}(U)} E \xrightarrow{\sim} \mathcal{D}|U_i \otimes_{\mathcal{D}(U_i)} \mathcal{D}(U_i) \otimes_{\mathcal{D}(U)} E = 0$. Hence $\mathcal{E} = 0$. By using the property 1.4.3.14.(iii), this yields $E = 0$. □

Proposition 1.4.3.20. *Let X be an affine scheme or an affine formal scheme. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorem A for left coherent modules (see 1.4.3.14). Set $D = \mathcal{D}(X)$. Let E be a coherent left D-module E, and $n \in \mathbb{N}$. The following conditions are equivalent.*

(a) There exists a finite covering $(U_i)_{i \in I}$ of X by affine opens such that the left $\mathcal{D}(U_i)$-module $\mathcal{D}(U_i) \otimes_D E$ has tor-amplitude in $[-n, 0]$ for any $i \in I$;

(b) The left $\mathcal{D}$-module $\mathcal{D} \otimes_D E$ has tor-amplitude in $[-n, 0]$.

Proof. 0) By right flatness of $\mathcal{D}(U_i) \to \mathcal{D}$, we get $(a) \to (b)$ (see 1.4.3.4.(a)). Conversely, suppose the left $\mathcal{D}$-module $\mathcal{D} \otimes_D E$ has tor-amplitude in $[-n, 0]$.

1) Let us prove the case where $n = 0$, i.e. suppose $\mathcal{D} \otimes_D E$ is $\mathcal{D}$-flat. Let $x \in X$. By hypothesis, $E_x := \mathcal{D}_x \otimes_D E$ is $\mathcal{D}_x$-flat. Since E_x is a flat left $\mathcal{D}_x$-module of finite presentation, then it is also a projective left $\mathcal{D}_x$-modules (e.g. see [Laz69, 1.4]). We have an exact sequence of coherent left D-module of the form $0 \to K \xrightarrow{a} L \xrightarrow{b} E \to 0$, where L is a free left D-module of finite type. By applying the exact functor $\mathcal{D}_x \otimes_D -$, we get the exact sequence $0 \to K_x \xrightarrow{a_x} L_x \xrightarrow{b_x} E_x \to 0$. Since E_x is projective then we get the section $\gamma \colon E_x \to L_x$. Since this is a morphism of left $\mathcal{D}_x$-module of finite presentation then for an open of X containing x small enough, the map γ is (up to canonical isomorphism) the image by $\mathcal{D} \otimes_{\mathcal{D}(U)} -$ of a morphism of left $\mathcal{D}(U)$-module of the form $c \colon \mathcal{D}(U) \otimes_D E \to \mathcal{D}(U) \otimes_D L$. Shrinking U is necessary, we can suppose c is a section of $\mathcal{D}(U) \otimes_D L \xrightarrow{\mathrm{id} \otimes b} \mathcal{D}(U) \otimes_D E$. Hence, $\mathcal{D}(U) \otimes_D E$ is projective. Since X is quasi-compact, we get a finite covering $(U_i)_{i \in I}$ of X by affine opens such that $\mathcal{D}(U_i) \otimes_D E$ is projective of finite presentation and we are done.

2) We deduce from 1) the general case by copying the part I.2) of the proof of 1.4.3.17. □

1.4.3.21. Let X be an affine scheme or an affine formal scheme. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorem A for left coherent modules (see 1.4.3.14). Set $D = \mathcal{D}(X)$. Let E be a coherent left D-module E.

1. In fact, we have proved in 1.4.3.20 that the following conditions are equivalent.
   (a) There exists a finite covering $(U_i)_{i\in I}$ of X by affine opens such that the left $\mathcal{D}(U_i)$-module $\mathcal{D}(U_i) \otimes_D E$ is projective of finite finite type for any $i \in I$;
   (b) The left $\mathcal{D}$-module $\mathcal{D} \otimes_D E$ is flat.
2. Suppose now that for any $x \in X$, the sheaf $\mathcal{D}_x$ is a local (non-commutative) ring. We have therefore the equivalent conditions:
   (a) There exists a finite covering $(U_i)_{i\in I}$ of X by affine opens such that the left $\mathcal{D}(U_i)$-module $\mathcal{D}(U_i) \otimes_D E$ is free of finite finite type for any $i \in I$;
   (b) The left $\mathcal{D}$-module $\mathcal{D} \otimes_D E$ is flat.

   Indeed, suppose $\mathcal{D}$-module $\mathcal{E} := \mathcal{D} \otimes_D E$ is flat. Then $\mathcal{E}_x = \mathcal{D}_x \otimes_D E$ is a projective, finitely presented $\mathcal{D}_x$-module (e.g. see [Laz69, 1.4]). Hence, following [Rot09, Theorem 4.44], $\mathcal{E}_x$ is a free $\mathcal{D}_x$-module of finite type. We get an isomorphism of the $\mathcal{D}_x^n \xrightarrow{\sim} \mathcal{E}_x$. Since $\mathcal{D}^n$ and $\mathcal{E}$ are coherent, then (e.g. use 8.4.1.11) there exists an open U containing x such that $\mathcal{D}_x^n \xrightarrow{\sim} \mathcal{E}_x$ comes from an isomorphism of the form $\mathcal{D}^n(U) \xrightarrow{\sim} \mathcal{E}(U)$. Hence, via theorem A for coherent $\mathcal{D}$-modules, we are done.

Proposition 1.4.3.22. *Let X be a noetherian affine (formal) scheme of finite Krull dimension. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorems A and B for left coherent modules (see 1.4.3.14). Let $\mathfrak{B}$ be an open basis of X consisting of affine opens. We have the formula*

$$\operatorname{tor.dim} \mathcal{D} = \sup\{\operatorname{tor.dim}(\mathcal{D}(U)) \ ; \ U \in \mathfrak{B}\}.$$

Proof. It follows from 1.4.3.17 (resp. 1.4.3.10) that we have the first (resp. second) inequality $\operatorname{tor.dim}(\mathcal{D}(U)) \leq \operatorname{tor.dim} \mathcal{D}|U \leq \operatorname{tor.dim} \mathcal{D}$ for any $U \in \mathfrak{B}$. Hence, $\sup\{\operatorname{tor.dim}(\mathcal{D}(U)) \ ; \ U \in \mathfrak{B}\} \leq \operatorname{tor.dim} \mathcal{D}$. Conversely, let $x \in X$ such that $\operatorname{tor dim}(\mathcal{D}) = \operatorname{tor dim}(\mathcal{D}_x)$ (recall 1.4.3.10). By using 1.4.3.12, since $\{U \in \mathfrak{B}; x \in U\}$ is a basis of open neighborhoods of x in X, since the ring homomorphisms $\mathcal{D}(U) \to \mathcal{D}_x$ are right flat, then we get $\operatorname{tor dim}(\mathcal{D}_x) \leq \sup\{\operatorname{tor.dim}(\mathcal{D}(U)) \ ; \ U \in \mathfrak{B}\}$. Hence, we are done. □

1.4.3.23. Let X be a scheme and $\mathcal{D}$ be a sheaf of rings on X endowed with a morphism of rings $\mathcal{O}_X \to \mathcal{D}$ such that $\mathcal{D}$ is quasi-coherent for the *left* multiplication. For any $U \subset V$ affine opens, since $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat then $\mathcal{D}(V) \to \mathcal{D}(U)$ is *right* flat. Hence, for any affine open U, the map $\mathcal{D}(U) \to \mathcal{D}|U$ is right flat. This yields the exact functor $E \mapsto \mathcal{D}|U \otimes_{\mathcal{D}(U)} E$ from the category of left $\mathcal{D}(U)$-modules to the category of left $\mathcal{D}|U$-modules.

Proposition 1.4.3.24. *With the notation 1.4.3.23, let Y be a principal open subscheme of X. We get* $\operatorname{tor dim}(\mathcal{D}(Y)) \leq \operatorname{tor dim}(D)$.

Proof. We have $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A_f$ for some $f \in A$. Set $D_f := \Gamma(Y, \mathcal{D})$. Since $D \to D_f$ is right flat, then it follows from 1.4.3.4.(a) that we reduce to check the functor $D_f \otimes_D -$ from the category of left D-modules to that of left D_f-modules is essentially surjective.

By quasi-coherence hypothesis, the canonical morphism $A_f \otimes_A D \to D_f$ is an isomorphism of (A_f, D)-bimodule. Hence, for any left D-module E, the canonical homomorphism $A_f \otimes_A E \to D_f \otimes_D E$ is an isomorphism (in particular $D \to D_f$ is right flat). Since the canonical homomorphism $A_f \otimes_A D_f \to D_f$ is an isomorphism, this yield that so is the canonical morphism $D_f \otimes_D D_f \to D_f$. Hence, for any left D_f-module E, we get the isomorphisms $D_f \otimes_D E \xrightarrow{\sim} (D_f \otimes_D D_f) \otimes_{D_f} E) \xrightarrow{\sim} D_f \otimes_{D_f} E \xrightarrow{\sim} E$, and we are done. □

Corollary 1.4.3.25. *Let X be a noetherian scheme of finite Krull dimension. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorems A and B for left coherent modules (see 1.4.3.14). Suppose moreover $\mathcal{D}$ is endowed with a ring morphism $\mathcal{O}_X \to \mathcal{D}$ such that $\mathcal{D}$ is quasi-coherent for the left multiplication. Then for any affine open U of X we get* $\operatorname{tor dim}(\mathcal{D}(U)) = \operatorname{tor dim}(\mathcal{D}|U)$.

Proof. Let U be an affine open of X. By using 1.4.3.17, we get $\operatorname{tor}\dim(\mathcal{D}(U)) \leq \operatorname{tor}\dim(\mathcal{D}|U)$. Let $x \in U$ such that $\operatorname{tor}\dim(\mathcal{D}|U) = \operatorname{tor}\dim(\mathcal{D}_x)$ (recall 1.4.3.10). Let $\mathfrak{B}$ be the basis of open neighborhoods of x in X consisting in affine opens V including in U and containing x. By using 1.4.3.12 and 1.4.3.24, we get $\operatorname{tor}\dim(\mathcal{D}_x) \leq \operatorname{tor}\dim(\mathcal{D}(U))$. Hence, we are done. □

The following corollary improves 1.4.3.22 (which only gives $(b) \to (a)$) when X is a scheme and $\mathcal{D}$ is left quasi-coherent:

Corollary 1.4.3.26. *We keep notation and hypotheses of 1.4.3.25. Let U be an affine open of X and $n \in \mathbb{N}$. The following conditions are equivalent:*

(a) We have $\operatorname{tor}.\dim(\mathcal{D}(U)) = n$ *;*

(b) There exists an open basis $\mathfrak{B}$ of U consisting of affine opens such that $\operatorname{tor}.\dim(\mathcal{D}(V)) = n$*, for any $V \in \mathfrak{B}$.*

Proof. This is a consequence of 1.4.3.25. □

Definition 1.4.3.27. Let $\mathcal{D}$ be a sheaf of rings on a topological space X. Let $\mathcal{E}^\bullet \in C(\mathcal{D})$.

1. We say $\mathcal{E}^\bullet$ is "strictly perfect" if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{D}$-module for all i.
2. Let $n \in \mathbb{Z}$. According to [Sta22, 08FT], we say $\mathcal{E}^\bullet$ is n-pseudo-coherent if there exists an open covering $\{(U_\lambda \to X)\}_{\lambda\in\Lambda}$, and for each $\lambda \in \Lambda$ there exist a strictly perfect complex of $\mathcal{D}|_{U_\lambda}$-modules $\mathcal{E}^\bullet_\lambda$ and a morphism $\alpha_\lambda\colon \mathcal{E}^\bullet_\lambda \to \mathcal{E}^\bullet|_{U_\lambda}$ of $C(\mathcal{D}|_{U_\lambda})$ such that $H^j(\alpha_\lambda)$ is an isomorphism for $j > n$ and $H^n(\alpha_\lambda)$ is surjective.
3. We say $\mathcal{E}^\bullet$ is "pseudo-coherent" if it is n-pseudo-coherent for all $n \in \mathbb{Z}$.
4. Let $a, b \in \mathbb{Z}$ with $a \leq b$. We say $\mathcal{E}^\bullet$ is "has perfect amplitude in $[a, b]$" if there exists an open covering $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ and for each $\lambda \in \Lambda$ a morphism of complexes $\alpha_\lambda\colon \mathcal{E}^\bullet_\lambda \to \mathcal{E}^\bullet|_{U_\lambda}$ which is a quasi-isomorphism with $\mathcal{E}^\bullet_\lambda$ strictly perfect and such that $\mathcal{E}^i_\lambda$ for any $i \notin [a, b]$ (see [SGA6, I.4.8]).
5. We say $\mathcal{E}^\bullet$ is "perfect" if there exists an open covering $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ and for each $\lambda \in \Lambda$ a morphism of complexes $\alpha_\lambda\colon \mathcal{E}^\bullet_\lambda \to \mathcal{E}^\bullet|_{U_\lambda}$ which is a quasi-isomorphism with $\mathcal{E}^\bullet_\lambda$ strictly perfect (see [Sta22, 08CM]).
6. Let $n \in \mathbb{Z}$. We say an object of $D(\mathcal{D})$ is n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) if and only if it can be represented by a n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) complex of $\mathcal{D}$-modules. We denote by $D_{n-\mathrm{coh}}(\mathcal{D})$ (resp. $D_{\mathrm{coh}}(\mathcal{D})$, resp. $D_{\mathrm{perf}}(\mathcal{D})$) the full subcategory of $D(\mathcal{D})$ consisting of n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) complexes.

Remark 1.4.3.28. Let $\mathcal{D}$ be a sheaf of rings on a *quasi-compact* topological space X. With the quasi-compactness hypothesis, a complex $\mathcal{E}^\bullet$ of $C(\mathcal{D})$ is perfect if and only if there exists integers $a \leq b$ such that $\mathcal{E}^\bullet$ has perfect amplitude in $[a, b]$.

1.4.3.29. Let $\mathcal{D}$ a sheaf of rings on a topological space. Suppose $\mathcal{D}$ is left coherent. Let $n \in \mathbb{N}$. Let $\mathcal{E}$ be a coherent left $\mathcal{D}$-module. By using [SGA6, I.4.14 and I.5.8.1], the following assertions are therefore equivalent:

a) The left $\mathcal{D}$-module $\mathcal{E}$ has tor-amplitude in $[-n, 0]$;

b) The left $\mathcal{D}$-module $\mathcal{E}$ has perfect amplitude in $[-n, 0]$;

c) There exists an open covering $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ and for each $\lambda \in \Lambda$ there exists a complex $\mathcal{P}^\bullet_\alpha$ of projective left $\mathcal{D}|_{U_\lambda}$-modules of finite type with $\mathcal{P}^i = 0$ for $i \notin [a, b]$ together with a quasi-isomorphism of $K({}^l\mathcal{D}|_{U_\lambda})$ of the form $\mathcal{P}^\bullet \to \mathcal{E}|_{U_\lambda}$.

Let $\mathcal{E}^\bullet$ be an object of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D})$ (see 4.6.1.3.(b)). This yields (by devissage) that $\mathcal{E}^\bullet$ is a perfect if and only if $\mathcal{E}$ has locally finite tor dimension.

Suppose $\mathcal{D}$ has locally finite tor dimension (and $\mathcal{D}$ is still coherent). Then, it follows from 1.4.3.8 that we get the equality $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}) = D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D})$.

Lemma 1.4.3.30. *Let D be a left coherent ring. Let $d \in \mathbb{N}$. We get that the following assertions are equivalent.*

(a) $\mathrm{tor}\,.\dim(D) \leq d$ *;*

(b) Any left D-module E has tor-amplitude in $[-d, 0]$;

(c) Any left coherent D-module E has tor-amplitude in $[-d, 0]$;

(d) Any left coherent D-module E has perfect amplitude in $[-d, 0]$;

(e) For any left coherent D-module E, there exists a complex $P^\bullet$ of projective left D-modules of finite type with $P^i = 0$ for $i \notin [-d, 0]$ together with a quasi-isomorphism of $K({}^1D)$ of the form $P^\bullet \to E$.

(f) We have $\mathrm{Ext}^i_D(E, F) = 0$ *for any integer $i > d$, any left coherent D-module E and any left D-module F.*

(g) D has finite tor dimension and we have have $\mathrm{Ext}^i_D(E, D) = 0$ *for any integer $i > d$, any left coherent D-module E.*

Proof. By using 1.4.3.9 and 1.4.3.29, we get the equivalence between (a), (b), (c), (d), (e). The equivalence between (e) and (f) is easy. It remains to check the implication $(g) \Rightarrow (f)$. Let E be a left coherent D-module and F be a left D-module. Since D is left coherent and has finite tor dimension, then $E \in D^{\mathrm{b}}_{\mathrm{perf}}(D)$. Hence, it follows from 4.6.3.6.1 that the canonical morphism

$$\mathbb{R}\mathcal{H}om_D(E, D) \otimes^{\mathbb{L}}_D F \to \mathbb{R}\mathcal{H}om_D(E, D \otimes^{\mathbb{L}}_D F) = \mathbb{R}\mathcal{H}om_D(E, F)$$

is an isomorphism. By hypothesis $\mathbb{R}\mathcal{H}om_D(E, D) \in D^{\leq q}(D)$. Hence, we are done. □

The check of the following proposition is contained in the proofs of [Ber00, 4.4.4–5].

Proposition 1.4.3.31 (Berthelot)**.** *Let D be a p-torsion free π-adically complete noetherian $\mathcal{V}$-algebra. We get the k-algebra $\overline{D} := D/\mathfrak{m}D$ and we set $d :=$* $\mathrm{tor}\,.\dim(\overline{D})$*. We have the following properties.*

(a) We have $\mathrm{tor}\,.\dim D \leq d + 1$

(b) Let E be a p-torsion free left D-module of finite type. Then E admits a resolution by projective of finite type left D-modules of length $\leq d$.

(c) Let E be a left $D_{\mathbb{Q}}$-module of finite type. Then E admits a resolution by projective of finite type left $D_{\mathbb{Q}}$-modules of length $\leq d$.

(d) We have the inequality $\mathrm{tor}\,.\dim(D_{\mathbb{Q}}) \leq d$.

Proof. a) First remark that since the rings are noetherian, then left global dimension is equal to the right global dimension and to the tor dimension (see 2.3.4.2). Since the ring D is a π-adically complete noetherian ring, then by using the $\mathfrak{m}$-adic filtration it follows from [LvO96, I.7.2 Corollary 2] that we get the inequality

$$\mathrm{gl}\,.\dim D \leq \mathrm{gl}\,.\dim \mathrm{gr}\, D.$$

Since π is a regular sequence of D, then we get the canonical isomorphism of graded rings

$$\overline{D} \otimes_k k[t] \xrightarrow{\sim} \mathrm{gr}\, D$$

given by $t \to \pi \mod \mathfrak{m}^2 D$.

b) Remark that if N is a p-torsion free left D-module N of finite type such that $N/\pi N$ is a projective $\overline{D}$-module then N is a projective D-module (exercice or see the end of the proof of [Ber00, 4.4.5]). Hence, by truncating a resolution of E by p-torsion free left D-module of finite type, we get the part (b).

c) Since a left $D_{\mathbb{Q}}$-module of finite type comes from by extension a left D-module (see 7.4.5.3 and recall the noetherianity of D), since $D \to D_{\mathbb{Q}}$ is flat, then get (c) from (b).

d) The inequality $\mathrm{tor}\,.\dim D_{\mathbb{Q}} \leq d$ follows from 1.4.3.30 and the fact that D is noetherian. □

Proposition 1.4.3.32. *Let $(D^{(i)})_{i\in I}$ be an inductive system indexed by a filtered set I of left (resp. right) coherent rings such that the ring homomorphisms $D^{(i)} \to D^{(j)}$ are left (resp. right) flat. Set $D^\dagger := \varinjlim_i D_i$. Suppose there exists $d \in \mathbb{N}$ such that* $\operatorname{tor.dim} D^{(m)} \leq d$ *for m large enough. Then have the inequality* $\operatorname{tor.dim} D^\dagger \leq d$.

Proof. Since the proof is the same, let us consider the non-respective case. It follows from 1.4.3.30 that $D^\dagger$ is a left coherent ring. Let E be a coherent left $\mathcal{D}^\dagger$-module. Using 8.4.1.13, there exist $i \in I$ and a coherent left $D^{(i)}$-module $E^{(i)}$ such that $E \xrightarrow{\sim} D^\dagger \otimes_{D^{(i)}} E^{(i)}$. Since $D^{(i)}$ is a left coherent ring, then following 1.4.3.30 there exists a complex $P^\bullet$ of projective left $D^{(n)}$-modules of finite type with $P^n = 0$ for $n \notin [-d, 0]$ together with a quasi-isomorphism of $K({}^l D^{(i)})$ of the form $P^\bullet \to E^{(i)}$. Since E is quasi-isomorphic to $D^\dagger \otimes_{D^{(i)}} P^\bullet$, then we are done using loc. cit. □

1.4.4 Increasing the level: finiteness of the tor-dimension

Let $f: X \to S$ be a smooth morphism.

Proposition 1.4.4.1. Suppose S is of characteristic $p > 0$. Suppose that X/S have coordinates $t_1, \dots, t_d$.

(a) We have $\partial_i^{\langle p^{m+1}\rangle_{(m)}} = u_m \left(\partial_i^{[p^m]}\right)^p$, for $i = 1, \dots, d$, with $u_m = p!(p^m!)^p/p^{m+1}! \in \mathbb{Z}_{(p)}^*$. For any integer q, we have $\left(\partial_i^{\langle p^{m+1}\rangle_{(m)}}\right)^q = v_{m,q}\partial_i^{\langle p^{m+1}q\rangle_{(m)}}$, with $v_{m,q} \in \mathbb{Z}_{(p)}^*$.

(b) For any integer $q \geq 0$, $0 \leq r \leq p^{m+1} - 1$, by setting $k := p^{m+1}q + r$, we have $\partial_i^{\langle k\rangle_{(m)}} = u_{m,k}\left(\partial_i^{\langle p^{m+1}\rangle_{(m)}}\right)^q \partial_i^{\langle r\rangle_{(m)}}$, for $i = 1, \dots, d$, with $u_{m,k} \in \mathbb{Z}_{(p)}^*$.

(c) Let $\mathcal{K}^{(m)}$ be the set whose elements are the finite sums of the form $\sum_{\underline{k} \not< p^{m+1}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k}\rangle_{(m)}}$, with $a_{\underline{k}} \in \mathcal{O}_X$ and $\underline{k} \not< p^{m+1}$ meaning that $k_i \geq p^{m+1}$ for at least one $1 \leq i \leq d$. Then $\mathcal{K}^{(m)}$ is the two-sided ideal of $\mathcal{D}_{X/S}^{(m)}$ generated by the operators $\partial_i^{\langle p^{m+1}\rangle_{(m)}}$, for $i = 1, \dots, d$.

(d) The center of $\mathcal{D}_{X/S}^{(m)}$ is equal to the polynomial algebra with coefficients in $\mathcal{O}_{X^{(m+1)}}$ in the operators $\partial_i^{\langle p^{m+1}\rangle_{(m)}}$.

Proof. The formula $\partial_i^{\langle p^{m+1}\rangle_{(m)}} = u_m \left(\partial_i^{[p^m]}\right)^p$ is a consequence of 1.4.2.7.c. The fact that u_m is invertible follows from 1.2.1.4.1. Let $n \geq p^{m+1}$. Then $n = p^m q + r$ with $q \geq p$ and $0 \leq r < p^m$, $n - p^{m+1} = p^m(q-p) + r$ and $p^{m+1} = p^m p + 0$. By using 1.2.1.4.1, this yields

$$v_p\left(\left\langle \begin{matrix} n \\ p^{m+1} \end{matrix} \right\rangle_{(m)}\right) = ((q-p) + p - q - \sigma(r) + \sigma(r) + \sigma(0))/(p-1) = 0. \tag{1.4.4.1.1}$$

By using 1.4.2.7.c , this yields

$$\partial_i^{\langle n\rangle_{(m)}} = \alpha_{m,n}\partial_i^{\langle p^{m+1}\rangle_{(m)}}\partial_i^{\langle n-p^{m+1}\rangle_{(m)}},$$

with $\alpha_{m,n} \in \mathbb{Z}_{(p)}^*$. Hence, by iterating this decomposition, we get the second part of a and b. By using 1.4.2.7.c, we get c. Finally, the property d is checked in [Ber96c, 2.2.6]. □

Proposition 1.4.4.2. *If S has characteristic p. Suppose that X/S have local coordinates $t_1, \dots, t_d$. Let $m' \geq m+1$ be an integer.*

(a) The homomorphisms $\mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}^{(m')}$, and $\mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}$ have the same kernel equal to $\mathcal{K}^{(m)}$.

(b) Moreover, $\mathcal{D}_{X/S}^{(m')}$ is free on the image of $\rho_{m',m}$, with the operators of the form $\underline{\partial}^{\langle p^{m+1}\underline{n}\rangle_{(m')}}$, $\underline{n} \in \mathbb{N}^d$ as a basis.

Proof. By using 1.2.1.1.1, since $q_{\underline{k}}^{(m)} = pq_{\underline{k}}^{(m+1)} + \underline{s}$ with $0 \leq s_i \leq p-1$, then we compute

$$v_p(q_{\underline{k}}^{(m)}!/q_{\underline{k}}^{(m+1)}!) = \sum_{i=1}^{d}(q_{k_i}^{(m)} - \sigma(q_{k_i}^{(m)}) - q_{k_i}^{(m+1)} + \sigma(q_{k_i}^{(m+1)}))/(p-1) = \sum_{i=1}^{d} q_{k_i}^{(m+1)}.$$

This yields that $v_p(q_{\underline{k}}^{(m)}!/q_{\underline{k}}^{(m+1)}!) = 0$ if and only if for any integer i we have $k_i \geq p^{m+1}$, which is denoted by $\underline{k} < p^{m+1}$. This implies that $v_p(q_{\underline{k}}^{(m)}!/q_{\underline{k}}^{(m')}!) = 0$ if and only if for for any integer i we have $k_i \geq p^{m+1}$, which is simply denoted by $\underline{k} < p^{m+1}$. Hence $\rho_{m,m'}(\mathcal{K}^{(m)}) = 0$. Conversely, let $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \in \mathcal{D}_X^{(m)}$ be an operator such that $\rho_{m',m}(P) = 0$. From 1.4.2.5.2 this yields $\rho_{m',m}(P) = \sum_{\underline{k}<p^{m+1}} \frac{q_{\underline{k}}^{(m)}!}{q_{\underline{k}}^{(m+1)}!} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m')}}$, with $\frac{q_{\underline{k}}^{(m)}!}{q_{\underline{k}}^{(m')}!} \in \mathbb{Z}_{(p)}^*$ for any $\underline{k} < p^{m+1}$.This implies, for any $\underline{k} < p^{m+1}$, we have $a_{\underline{k}} = 0$. Hence, $P \in \mathcal{K}^{(m)}$. Then we have checked $\mathcal{K}^{(m)}$ is the kernel of $\rho_{m,m'}$ for any m'. By passing m' to the limit, this yields that $\mathcal{K}^{(m)}$ is also the kernel of ρ_m.

The image of $\rho_{m,m'}$ consists of elements of the form $\sum_{\underline{k}<p^{m+1}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m')}}$. Hence, it follows from 1.4.4.1.a and 1.4.4.1.c the last statement. □

Remark 1.4.4.3. Suppose that X/S have local coordinates $t_1, \ldots, t_d$ with $d \geq 1$. Then similarly to 1.4.4.2, we check that the homomorphisms $\mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}^{(m+1)}$, $\mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}$ are injective if and only if S is flat over $\mathbb{Z}$.

Proposition 1.4.4.4. *Suppose p is locally nilpotent on S. For any m, $\mathcal{D}_{X/S}^{(m+1)}$ has left and right tor-dimension d over $\mathcal{D}_{X/S}^{(m)}$.*

Proof. Since this is local, we can suppose that X/S has local coordinates $t_1, \ldots, t_d$.

1) Suppose S has characteristic $p > 0$. Denote by $\overline{\mathcal{D}}_{X/S}^{(m)} := \mathcal{D}_{X/S}^{(m)}/\mathcal{K}^{(m)}$. Since $\overline{\mathcal{D}}_{X/S}^{(m)}$ is isomorphic to the image of $\rho_{m',m}$, it follows from 1.4.4.2.b that we reduce to check that $\overline{\mathcal{D}}_{X/S}^{(m)}$ has left and right tor-dimension d over $\mathcal{D}_{X/S}^{(m)}$.

Let $\mathcal{K}_0^{(m)}$ be the zero ideal of $\mathcal{D}_{X/S}^{(m)}$. Set $E_0 := \mathbb{N}^d$. For any integer $1 \leq r \leq d$, denote by $\mathcal{K}_r^{(m)}$ the is the two-sided ideal of $\mathcal{D}_{X/S}^{(m)}$ generated by the operators $\partial_i^{\langle p^{m+1} \rangle_{(m)}}$, for $i = 1, \ldots, r$; and let E_r be the subset of $\mathbb{N}^d$ of elements $\underline{k}$ such that $k_1, \ldots, k_r < p^{m+1}$. For any $0 \leq r \leq d$, let $\mathcal{J}_r^{(m)}$ be the subset of $\mathcal{D}_{X/S}^{(m)}$ consisting of the elements which can (uniquely) be written of the form $\sum_{\underline{k} \in E_r} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$, with $a_{\underline{k}} \in \mathcal{O}_X$. If follows from 1.4.4.1.b that the composition morphism $\mathcal{J}_r^{(m)} \to \mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}^{(m)}/\mathcal{K}_r^{(m)}$ is an isomorphism and we can identify them. Moreover, for any $0 \leq r \leq d-1$, by using 1.4.4.1.1, we can easily check that the homomorphism $\partial_{r+1}^{\langle p^{m+1} \rangle_{(m)}} : \mathcal{D}_{X/S}^{(m)}/\mathcal{K}_r^{(m)} \to \mathcal{D}_{X/S}^{(m)}/\mathcal{K}_r^{(m)}$ is injective. Since $\mathcal{K}^{(m)} = \mathcal{K}_d^{(m)}$, we are done.

2) Let us go back to the general case. Suppose S is affine and p^{n+1} vanishes in S. We proceed by induction on $n \in \mathbb{N}$. For any integer $0 \leq i \leq n-1$, set $S_i := S \times_{\mathbb{Z}} \mathbb{Z}/p^{i+1}\mathbb{Z}$, $X_i = X \times_S S_i$. Let $\mathcal{E}$ be a left $\mathcal{D}_{X/S}^{(m)}$-module. We have to prove that $H^j(\mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \mathcal{E}) = 0$ for any $j \notin [-d, 0]$. For any $0 \leq i \leq n-1$, we have the isomorphisms

$$\mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \mathcal{D}_{X_i/S_i}^{(m)} \xrightarrow{\sim} \mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \left(\mathcal{D}_{X/S}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathcal{O}_{S_i}\right) \xrightarrow{\sim} \mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathcal{O}_{S_i} \xrightarrow{\sim} \mathcal{D}_{X_i/S_i}^{(m+1)}.$$

Hence, if $\mathcal{G}_i$ is a left $\mathcal{D}_{X_i/S_i}^{(m)}$-module, we get the isomorphisms

$$\mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \mathcal{G}_i \xrightarrow{\sim} \left(\mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \mathcal{D}_{X_i/S_i}^{(m)}\right) \otimes^{\mathbb{L}}_{\mathcal{D}_{X_i/S_i}^{(m)}} \mathcal{G}_i \xrightarrow{\sim} \mathcal{D}_{X_i/S_i}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X_i/S_i}^{(m)}} \mathcal{G}_i.$$

We have the exact sequence $0 \to p\mathcal{E} \to \mathcal{E} \to \mathcal{E}/p\mathcal{E} \to 0$, which yields the exact triangle

$$\mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} p\mathcal{E} \to \mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \mathcal{E} \to \mathcal{D}_{X/S}^{(m+1)} \otimes^{\mathbb{L}}_{\mathcal{D}_{X/S}^{(m)}} \mathcal{E}/p\mathcal{E} \to +1$$

Since $p\mathcal{E}$ is a left $\mathcal{D}^{(m)}_{X_{n-1}/S_{n-1}}$-module and $\mathcal{E}/p\mathcal{E}$ is a left $\mathcal{D}^{(m)}_{X_0/S_0}$-module, we are done.

□

1.4.4.5. The sheaf $\mathcal{D}^{(m)}_{X/S}$ admits a natural filtration $F_\bullet\mathcal{D}^{(m)}_{X/S}$ of sheaf of rings which we call the *filtration by order*. If P is a section of $\mathcal{D}^{(m)}_{X/S}$ on a neighborhood of a point $x \in X$,we denote by $\mathrm{ord}_x(P)$ the smallest integer n such that $P_x \in \mathcal{D}^{(m)}_{X/S,n,x}$. For any open set $U \subset X$ and any $P \in \Gamma(U, \mathcal{D}^{(m)}_{X/S})$, put

$$\mathrm{ord}_U(P) := \sup_{x\in U} \mathrm{ord}_x(P).$$

Set $F_k\mathcal{D}^{(m)}_{X/S}(U) = \{P \in \Gamma(U, \mathcal{D}^{(m)}_{X/S}) : \mathrm{ord}_U(P) \leq k\}$.

1.4.5 Coherence of the sheaf of differential operators

First, let us recall the notion of coherence.

Definition 1.4.5.1. Let T be a topological space, $\mathcal{D}$ a sheaf of rings on T. Let $\mathcal{E}$ be a left (resp. right) $\mathcal{D}$-module.

(i) We say that $\mathcal{E}$ is of finite type if for any $x \in T$, there exists an open neighborhood U of x such that is generated by finitely many sections, i.e. there exist a positive integer r and an exact sequence in the category of (sheaves of) left (resp. right) $\mathcal{D}$-modules of the form $\mathcal{D}^r_{|U} \to \mathcal{E}_{|U}$.

(ii) We say that $\mathcal{E}$ is of finite presentation if for any $x \in T$, there exists an open neighborhood U of x such that $\mathcal{E}_{|U}$ is the cokernel of a morphism of left (resp. right) $\mathcal{D}$-modules of the form $\mathcal{D}^a_{|U} \to \mathcal{D}^{\mathrm{b}}_{|U}$.

(iii) We say $\mathcal{E}$ is a coherent left (resp. right) $\mathcal{D}$-module if the following conditions are satisfied

  (a) $\mathcal{E}$ is of finite type ;

  (b) for any open sets $U \subset T$, for any morphism of left (resp. right) $\mathcal{D}$-modules $\alpha\colon \mathcal{D}^a_{|U} \to \mathcal{E}_{|U}$, the kernel $\ker\alpha$ is of finite type.

  In fact, we can check that the property (b) is equivalent to the following one

  (c) for any open sets $U \subset T$, any submodule $\mathcal{F}$ of $\mathcal{E}|U$ of finite type is of finite presentation.

(iv) We say that $\mathcal{D}$ is a left (resp. right) coherent sheaf of rings if $\mathcal{D}$ is coherent as a left (resp. right) $\mathcal{D}$-module.

Proposition 1.4.5.2. *Let T be a topological space, $\mathcal{D}$ a sheaf of rings on T and $\mathfrak{B}$ a basis of open subsets of T satisfying the following conditions:*

(a) For any $U \in \mathfrak{B}$, the ring $\Gamma(U, \mathcal{D})$ is left (resp. right) noetherian.

(b) For any $U, V \in \mathfrak{B}$ with $V \subset U$, the homomorphism $\Gamma(U, \mathcal{D}) \to \Gamma(V, \mathcal{D})$ is right (resp. left) flat.

Then $\mathcal{D}$ is a left (resp. right) coherent sheaf of rings.

Proof. It suffices to show that for all $U \in \mathfrak{B}$ and all $u\colon (\mathcal{D}|_U)^n \to \mathcal{D}|_U$, $\mathcal{N} = \mathrm{Ker}(u)$ is a left $\mathcal{D}|_U$-module of finite type. Write $D = \Gamma(U, \mathcal{D})$, $N = \Gamma(U, \mathcal{N})$. From the condition b), we can check the functor which associates from a left D-module M the left $\mathcal{D}|_U$-module $\mathcal{D}|_U \otimes_D M$ is exact. Since N is of finite type by a), this yields that $\mathcal{N} \xrightarrow{\sim} \mathcal{D}|_U \otimes_D N$ is of finite type. □

Corollary 1.4.5.3. *Let $f\colon X \to S$ be a smooth morphism.*

(a) Then for any inclusion of affine opens of X of the form $V \subset U$, the canonical morphism $\Gamma(U, \mathcal{D}^{(m)}_{X/S}) \to \Gamma(V, \mathcal{D}^{(m)}_{X/S})$ is flat.

(b) Suppose S is locally noetherian. The sheaf $\mathcal{D}^{(m)}_{X/S}$ is coherent.

Proof. Since $\mathcal{D}^{(m)}_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module, for any inclusion of affine opens of X of the form $V \subset U$, the canonical morphism

$$\Gamma(V, \mathcal{O}_X) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathcal{D}^{(m)}_{X/S}) \to \Gamma(V, \mathcal{D}^{(m)}_{X/S})$$

is an isomorphism. Hence, $\Gamma(U, \mathcal{D}^{(m)}_{X/S}) \to \Gamma(V, \mathcal{D}^{(m)}_{X/S})$ is flat. By using 1.4.2.11, we can apply proposition 1.4.5.2 to conclude. □

Chapter 2

$\mathcal{D}$-modules

Let $m \in \mathbb{N} \cup \{+\infty\}$.

(a) First case: $m \in \mathbb{N}$, S is a $\operatorname{Spec} \mathbb{Z}_{(p)}$-scheme and $m \in \mathbb{N}$, $(\mathfrak{a}, \mathfrak{b}, \alpha)$ is a quasi-coherent m-PD-ideal of $\mathcal{O}_S$ and $f\colon X \to S$ be a smooth morphism.

(b) Second case: $m = \infty$ and $f\colon X \to S$ is a weakly smooth morphism and we use notation 1.4.2.6.

2.1 Stratifications and costratifications

2.1.1 PD-stratifications of level m

The formula (1.4.2.8.3) shows that the data of a left $\mathcal{D}_X^{(m)}$-module structure extending an $\mathcal{O}_X$-module structure is determined by the action of the operators $\partial_i^{\langle p^j \rangle}$ for $0 \leq j \leq m$. When $m \geq 1$ it is not always easy to give explicitly this action. For this reason when we want to show an $\mathcal{O}_X$-module $\mathcal{M}$ has the structure of a left $\mathcal{D}_X$-module, we often construct on $\mathcal{M}$ stratifications which are just the data of infinitesimal descent. This coordinate-free method of using the crystalline interpretation of a $\mathcal{D}_X$-module in terms of stratifications was due to Grothendieck.

Definition 2.1.1.1. With notations 1.4.1.2 and 1.4.1.14, an m-PD *stratification relative to X/S* on an $\mathcal{O}_X$-module $\mathcal{E}$ is the data of a family (as n varies) of $\mathcal{P}^n_{X,(m)}$-linear homomorphisms

$$\varepsilon_n \colon \mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} = p^{n*}_{1(m)}(\mathcal{E}) \to p^{n*}_{0(m)}(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X,(m)}, \tag{2.1.1.1.1}$$

which satisfy the following conditions:

(a) $\varepsilon_0 = \mathrm{id}_{\mathcal{E}}$ and for any $n' \geq n$ in $\mathbb{N}$, ε_n and $\mathcal{P}^n_{X,(m)} \otimes_{\mathcal{P}^{n'}_{X,(m)}} \varepsilon_{n'}$ are canonically isomorphic, i.e. the following diagram

$$\begin{array}{ccc} \psi^{n',n*}_{X/S,(m)}(\mathcal{P}^{n'}_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) & \xrightarrow{\psi^{n',n*}_{X/S,(m)}(\varepsilon^{\mathcal{E}}_{n'})} & \psi^{n',n*}_{X/S,(m)}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S,(m)}) \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\varepsilon^{\mathcal{E}}_n} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)} \end{array} \tag{2.1.1.1.2}$$

whose vertical isomorphisms are the canonical ones, is commutative.

(b) (Cocycle condition) For all integers n, n' in $\mathbb{N}$, with notation 1.4.1.15, we have commutative diagrams of $\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}$-modules:

$$\begin{array}{ccc} \mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)} \otimes \mathcal{E} & \xrightarrow{\delta^{n,n',*}_{(m)}(\varepsilon_{n+n'})} & \mathcal{E} \otimes \mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}. \\ & {\scriptstyle q^{n,n',*}_{1,(m)}(\varepsilon_{n+n'})} \searrow \quad \nearrow {\scriptstyle q^{n,n',*}_{0,(m)}(\varepsilon_{n+n'})} & \\ & \mathcal{P}^n_{X,(m)} \otimes \mathcal{E} \otimes \mathcal{P}^{n'}_{X,(m)} & \end{array} \tag{2.1.1.1.3}$$

Say an $\mathcal{O}_X$-linear homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ between modules equipped with m-PD stratifications relative to X/S is *horizontal* if it commutes with all ε_n.

Remark 2.1.1.2. Following 1.4.1.15, we have three ring homomorphisms $p^{n,n'}_{i,(m)} \colon \mathcal{O}_X \to \mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X,(m)}$. The diagram 2.1.1.1.3 can be reinterpreted with the following commutative diagram

$$\begin{array}{ccc} p^{n,n'*}_{2,(m)}(\mathcal{E}) & \xrightarrow{\delta^{n,n',*}_{(m)}(\varepsilon_{n+n'})} & p^{n,n'*}_{0,(m)}(\mathcal{E}) \\ & {\scriptstyle q^{n,n',*}_{1,(m)}(\varepsilon_{n+n'})}\searrow \qquad \nearrow{\scriptstyle q^{n,n',*}_{0,(m)}(\varepsilon_{n+n'})} & \\ & p^{n,n'*}_{1,(m)}(\mathcal{E}). & \end{array} \qquad (2.1.1.2.1)$$

It follows from 1.4.1.12.3 and 1.4.1.13.1 that we have the commutative diagram

$$\begin{array}{ccccccc} X & \hookrightarrow & \Delta^n_{X/S,(m)} \times_{p_1,X,p_0} \Delta^{n'}_{X/S,(m)} & \longrightarrow & X^2_{/S} \times_{p_1,X,p_0} X^2_{/S} & \xrightarrow{p_1 \times p_0} & X \times_X X \\ \| & & \downarrow & & \downarrow \wr & & \downarrow \wr \\ X & \hookrightarrow & \Delta^{n+n'}_{X/S,(m)}(2) & \longrightarrow & X^3_{/S} & \xrightarrow{p_1} & X \\ \| & & \downarrow {\scriptstyle (q^{n+n'}_{01,(m)}, q^{n+n'}_{12,(m)})} & & {\scriptstyle (p_{01},p_{12})} \downarrow \wr & & \downarrow \wr \\ X & \hookrightarrow & \Delta^{n+n'}_{X/S,(m)} \times_{p_1,X,p_0} \Delta^{n+n'}_{X/S,(m)} & \longrightarrow & X^2_{/S} \times_{p_1,X,p_0} X^2_{/S} & \xrightarrow{p_1 \times p_0} & X \times_X X. \end{array} \qquad (2.1.1.2.2)$$

With notation 1.4.1.13, this yields that the above condition 2.1.1.1.b is equivalent to

$$\forall\, n \in \mathbb{N}, \qquad q^{n*}_{02,(m)}(\varepsilon_n) = q^{n*}_{01,(m)}(\varepsilon_n) \circ q^{n*}_{12,(m)}(\varepsilon_n). \qquad (2.1.1.2.3)$$

Proposition 2.1.1.3. *Let $\mathcal{E}$ be an $\mathcal{O}_X$-module together with an m-PD stratification $(\varepsilon^{\mathcal{E}}_n)$ relative to X/S. Then the homomorphisms $\varepsilon^{\mathcal{E}}_n$ are $\mathcal{P}^n_{X/S,(m)}$-linear isomorphisms.*

Proof. Let $\iota^n_{(m)} \colon \mathcal{P}^n_{X/S,(m)}(2) \to \mathcal{P}^n_{X/S,(m)}$ be the m-PD-homomorphism corresponding to the closed immersion $\iota \colon X^2_{/S} \to X^3_{/S}$ given by $(x,y) \mapsto (x,y,x)$. Let σ be the canonical involution of $X^2_{/S}$, i.e. $\sigma(x,y) = (y,x)$. This yields the m-PD-isomorphism $\sigma^n_{(m)} \colon \mathcal{P}^n_{X/S,(m)} \xrightarrow{\sim} \mathcal{P}^n_{X/S,(m)}$. Since $p_{01} \circ \iota = \mathrm{id}$, then by unicity of the m-PD-factorisation we get $q^n_{01,(m)} \circ \iota^n_{(m)} = \mathrm{id}$. Moreover, since $p_{12} \circ \iota = \sigma$ then $q^n_{12,(m)} \circ \iota^n_{(m)} = \sigma^n_{(m)}$. Since $p_{02} \circ \iota = \Delta_{X/S} \circ p_0$, then $q^n_{02,(m)} \circ \iota^n_{(m)}$ is equal to the composition $\mathcal{P}^n_{X/S,(m)} \twoheadrightarrow \mathcal{O}_X \xrightarrow{p^n_{1,(m)}} \mathcal{P}^n_{X/S,(m)}$. Hence, $\iota^{n*}_{(m)} \circ q^{n*}_{02,(m)}(\varepsilon_n) = p^{n*}_{1,(m)}(\mathrm{id}) = \mathrm{id}$. By applying $\iota^{n*}_{(m)}$ to the equality 2.1.1.2.3 we get $\mathrm{id} = \sigma^{n*}_{(m)}(\varepsilon_n) \circ \varepsilon_n$. Since $\sigma^{n*}_{(m)}$ is an involution, by applying $\sigma^{n*}_{(m)}$ to this equality we get $\mathrm{id} = \varepsilon_n \circ \sigma^{n*}_{(m)}(\varepsilon_n)$ and then $\sigma^{n*}_{(m)}(\varepsilon_n)$ is the inverse of ε_n. □

Notation 2.1.1.4. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. For any $n \in \mathbb{N}$, since $\mathcal{P}^n_{X,(m)}$ has two structures of $\mathcal{O}_X$-modules, then so is $\mathcal{D}^{(m)}_{X,n}$, which we call the left one and the right one. When we want to clarify which $\mathcal{O}_X$-module structure we choose, we write p_{0*} for the left one and p_{1*} for the right one. For instance, we write $\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n}, \mathcal{E})$ to clarify that we take the left structure of $\mathcal{O}_X$-module to compute the internal homomorphism sheaf. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n}, \mathcal{E})$ has also by functoriality two structures of $\mathcal{O}_X$-modules: the left (resp. right) one denoted by p_{0*} (resp. p_{1*}) equal to that coming from the left (resp. right) structure of $\mathcal{D}^{(m)}_{X,n}$.

We denote by $\mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{E}$ where to compute the tensor product we have chosen the right structure of $\mathcal{O}_X$-module of $\mathcal{D}^{(m)}_{X,n}$. By functoriality, $\mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{E}$ is also an $(\mathcal{O}_X, \mathcal{O}_X)$-module i.e. has a left structure of $\mathcal{O}_X$-module denoted by p_{0*} coming from the left structure of $\mathcal{D}^{(m)}_{X,n}$, and the internal one (i.e. given by the tensor product) denoted by p_{1*}. Similarly, $\mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'}$ where the $\mathcal{O}_X$-module structure of the tensor product comes from the right (resp. left) structure of $\mathcal{D}^{(m)}_{X,n}$ (resp. $\mathcal{D}^{(m)}_{X,n'}$) is an $(\mathcal{O}_X, \mathcal{O}_X, \mathcal{O}_X)$-module:

its left structure comes from the left one of $\mathcal{D}^{(m)}_{X,n}$, its middle structure is the internal one coming from the tensor product, its right structure comes from the right one of $\mathcal{D}^{(m)}_{X,n'}$. When we want to clarify which $\mathcal{O}_X$-module structure we choose, we write p_{i*} with $i = 0, 1, 2$ for the left, middle, right ones.

We have the Cartan isomorphism (see 4.6.3.9.1):

$$\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}\mathcal{D}^{(m)}_{X,n'}, p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n}, \mathcal{E})\right) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'}), \mathcal{E}\right). \qquad (2.1.1.4.1)$$

The right term of 2.1.1.4.1 can be endowed with three structures of $\mathcal{O}_X$-module: that coming from the internal structure denoted by p_0, that coming from the middle and right structures of $\mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'}$ denoted respectively by p_{i*} for $i = 1, 2$. Via the isomorphism 2.1.1.4.1, this yields three structures of $\mathcal{O}_X$-module for left term that we still denote by p_{i*} for $i = 0, 1, 2$.

It follows from 1.4.2.1.3 that we get the homomorphism

$$\mu_{n,n'} \colon \mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'} \to \mathcal{D}^{(m)}_{X,n+n'}, \qquad (2.1.1.4.2)$$

where the $\mathcal{O}_X$-module structure of the tensor product comes from the right (resp. left) structure of $\mathcal{D}^{(m)}_{X,n}$ (resp. $\mathcal{D}^{(m)}_{X,n'}$). Since $\mu_{n,n'}$ is $\mathcal{O}_X$-linear for the left (and the right) structure, we get the morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mu_{n,n'}, \mathcal{E}) \colon p_{1*}\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}^{(m)}_{X,n+n'}), \mathcal{E}\right) \to p_{2*}\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'}), \mathcal{E}\right). \qquad (2.1.1.4.3)$$

Proposition 2.1.1.5. *We have the following properties.*

(I) Given an $\mathcal{O}_X$-module $\mathcal{E}$. The following are equivalent.

(a) A left $\mathcal{D}^{(m)}_X$-module structure on $\mathcal{E}$ extending its $\mathcal{O}_X$-module structure.

(b) A family of $\mathcal{O}_X$-linear homomorphisms $\theta_n \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X,(m)})$ (the $\mathcal{O}_X$-module structure of this latter is induced by the right structure of $\mathcal{P}^n_{X,(m)}$) satisfying

(i) $\theta_0 = \mathrm{id}_{\mathcal{E}}$ (modulo the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes \mathcal{P}^0_{X,(m)}$) and for any $n, n' \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\theta_n} & \mathcal{E} \otimes \mathcal{P}^n_{X,(m)} \\ \| & & \uparrow{\scriptstyle \mathrm{id} \otimes \psi^{n+n',n}_{X/S,(m)}} \\ \mathcal{E} & \xrightarrow[\theta_{n+n'}]{} & \mathcal{E} \otimes \mathcal{P}^{n+n'}_{X,(m)} \end{array} \qquad (2.1.1.5.1)$$

is commutative.

(ii) for all n, n' we have commutative diagrams (cocycle condition)

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{P}^{n+n'}_{X,(m)} & \xrightarrow{\mathrm{id} \otimes \delta^{n,n'}_{(m)}} & \mathcal{E} \otimes \mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)} \\ {\scriptstyle \theta_{n+n'}}\uparrow & & \uparrow{\scriptstyle \theta_n \otimes \mathrm{id}} \\ \mathcal{E} & \xrightarrow{\theta'_n} & \mathcal{E} \otimes \mathcal{P}^{n'}_{X,(m)}. \end{array} \qquad (2.1.1.5.2)$$

(c) An m-PD stratification $\varepsilon = (\varepsilon^{\mathcal{E}}_n)$ on $\mathcal{E}$.

(II) Let $\mathcal{E}$ be left $\mathcal{D}^{(m)}_X$-module and let $\theta^{\mathcal{E}} = (\theta^{\mathcal{E}}_n)$, $\varepsilon^{\mathcal{E}} = (\varepsilon^{\mathcal{E}}_n)$ be the associated family or m-PD stratification.

(a) We retrieve from $\theta^{\mathcal{E}}$ (resp. $\varepsilon^{\mathcal{E}}$) the action by a section P of $\mathcal{D}^{(m)}_X$ on $\mathcal{E}$ via the following composition of the bottom (resp. top) horizontal morphisms of the commutative diagram:

$$\begin{array}{ccccccc} \mathcal{E} & \xrightarrow[1.4.1.14]{p^n_{1,(m),\mathcal{E}}} & \mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\varepsilon^{\mathcal{E}}_n} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X,(m)} & \xrightarrow{\mathrm{id} \otimes P} & \mathcal{E} \\ \| & & & & \| & & \| \\ \mathcal{E} & & \xrightarrow{\theta^{\mathcal{E}}_n} & & p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X,(m)}) & \xrightarrow{\mathrm{id} \otimes P} & \mathcal{E}. \end{array} \qquad (2.1.1.5.3)$$

For instance, when $\mathcal{E} = \mathcal{O}_X$ is endowed with the canonical m-PD-stratification (see 3.4.2.4), we retrieve its usual action of left $\mathcal{D}_X^{(m)}$-module (see 1.4.2.5.4).

(b) Suppose X/S is equipped with coordinates $t_1, \ldots, t_d$. Conversely, with notation 1.4.2.3, for any $x \in \mathcal{E}$, we have the Taylor expansion formula

$$\theta_n(x) = \sum_{|k| \leq n} \underline{\partial}^{\langle k \rangle} x \otimes \underline{\tau}^{\{k\}}, \qquad \varepsilon_n^{\mathcal{E}}(1 \otimes x) = \sum_{|\underline{k}| \leq n} \underline{\partial}^{\langle \underline{k} \rangle} x \otimes \underline{\tau}^{\{\underline{k}\}}. \tag{2.1.1.5.4}$$

The inverse is given by

$$(\varepsilon_n^{\mathcal{E}})^{-1}(x \otimes 1) = \sum_{|\underline{k}| \leq n} (-1)^{|\underline{k}|} \underline{\tau}^{\{\underline{k}\}} \otimes \underline{\partial}^{\langle \underline{k} \rangle} x. \tag{2.1.1.5.5}$$

(III) An $\mathcal{O}_X$-linear morphism $\phi \colon \mathcal{E} \to \mathcal{F}$ between two left $\mathcal{D}_{X/S}^{(m)}$-modules is $\mathcal{D}_{X/S}^{(m)}$-linear if and only if ϕ is horizontal.

Proof. I) 1) The idea of the proof is standard and comes from Grothendieck: Let us prove the equivalence between (a) and (b). Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. A left $\mathcal{D}_X^{(m)}$-module structure on $\mathcal{E}$ extending its $\mathcal{O}_X$-module structure is equivalent to the data a family $(\mu_n^{\mathcal{E}})_{n \in \mathbb{N}}$ of $\mathcal{O}_X$-linear homomorphisms $\mu_n^{\mathcal{E}} \colon p_{0*}(\mathcal{D}_{X,n}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) \to \mathcal{E}$ such that $\mu_0^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ (modulo some canonical identification) and for any $n, n' \in \mathbb{N}$ the following diagrams in the category of $\mathcal{O}_X$-modules

$$\begin{array}{ccc} \mathcal{D}_{X,n}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\mu_n^{\mathcal{E}}} & \mathcal{E} \\ \downarrow{\scriptstyle \rho_{n+n',n}^{(m)} \otimes \mathrm{id}} & & \| \\ \mathcal{D}_{X,n+n'}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\mu_{n+n'}^{\mathcal{E}}} & \mathcal{E} \end{array} \qquad \begin{array}{ccc} \mathcal{D}_{X,n}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{D}_{X,n'}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\mathrm{id} \otimes \mu_{n'}^{\mathcal{E}}} & \mathcal{D}_{X,n}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \\ \downarrow{\scriptstyle \mu_{n,n'} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu_n^{\mathcal{E}}} \\ \mathcal{D}_{X,n+n'}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\mu_{n+n'}^{\mathcal{E}}} & \mathcal{E} \end{array} \tag{2.1.1.5.6}$$

are commutative (see notation 1.4.2.1.2).

By using Cartan isomorphism (see 4.6.3.9.1 in the case where $\mathcal{A} = \mathcal{O}_X$ and correspond to the left structure on $\mathcal{D}_{X,n}^{(m)}$, $\mathcal{A}' = \mathcal{O}_X$ and correspond to the right structure on $\mathcal{D}_{X,n}^{(m)}$, $\mathcal{C} = \mathcal{O}_X$) the data of the family $(\mu_n)_{n \in \mathbb{N}}$ is equivalent to the data of a family $(\nu_n)_{n \in \mathbb{N}}$ where ν_n is of the form

$$\nu_n \colon \mathcal{E} \to p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}_{X,n}^{(m)}, \mathcal{E}). \tag{2.1.1.5.7}$$

The commutativity of the following diagram in the category of $\mathcal{O}_X$-modules

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nu_{n+n'}} & p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}_{X,n+n'}^{(m)}, \mathcal{E}) \\ \downarrow{\scriptstyle \nu_{n'}} & & \\ p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}_{X,n'}^{(m)}, \mathcal{E}) & & \downarrow{\scriptstyle 2.1.1.4.3} \\ \downarrow{\scriptstyle \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}_{X,n'}^{(m)}, \nu_n)} & & \\ p_{2*}\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}\mathcal{D}_{X,n'}^{(m)}, p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}_{X,n}^{(m)}, \mathcal{E})\right) & \underset{\sim}{\xrightarrow{2.1.1.4.1}} & p_{2*}\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}_{X,n}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{D}_{X,n'}^{(m)}), \mathcal{E}\right) \end{array} \tag{2.1.1.5.8}$$

is equivalent to that of the left one of 2.1.1.5.6. Indeed, the map $\mathcal{E} \to p_{2*}\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}_{X,n}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{D}_{X,n'}^{(m)}), \mathcal{E}\right)$ from the left and bottom path (resp. top and right path) sends $e \in \mathcal{E}$ to $P \otimes P' \mapsto \mu_n^{\mathcal{E}}(P \otimes \mu_{n'}^{\mathcal{E}}(P' \otimes e))$ (resp. to $P \otimes P' \mapsto \mu_{n+n'}^{\mathcal{E}}(PP' \otimes e)$) for any $P \in \mathcal{D}_{X,n}^{(m)}$, $P' \in \mathcal{D}_{X,n'}^{(m)}$ and $e \in \mathcal{E}$.

We have the canonical isomorphism

$$\iota_n \colon p_{1*}\left(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(m)}^n\right) \xrightarrow{\sim} p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}_{X,n}^{(m)}, \mathcal{E}) \tag{2.1.1.5.9}$$

given by $e \otimes \tau \mapsto (P \mapsto P(\tau)e)$, where the $\mathcal{O}_X$-module structure on $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(m)}^n$ is given by the right structure of $\mathcal{P}_{X,(m)}^n$. Set $\theta_n := (\iota_n)^{-1} \circ \nu_n$. Hence, by definition and functoriality, we get the commutative diagram:

$$\begin{array}{ccccc}
 & & & & \mathcal{E} \\
 & & \swarrow^{\theta_{n'}} & & \big\downarrow{\scriptstyle \sim\ \nu_{n'}} \\
 & \mathcal{E}\otimes\mathcal{P}^{n'}_{X,(m)} & \xrightarrow[\sim]{\iota_{n'}} & & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n'},\mathcal{E}) \\
 \swarrow^{\theta_n\otimes\mathrm{id}} & \big\downarrow{\scriptstyle \nu_n\otimes\mathrm{id}} & & & \big\downarrow{\scriptstyle \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n'},\nu_n)} \\
 \mathcal{E}\otimes\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)} \xrightarrow[\sim]{\iota_n\otimes\mathrm{id}} & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n},\mathcal{E})\otimes\mathcal{P}^{n'}_{X,(m)} & \xrightarrow[\sim]{\iota_{n'}} & & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n'},p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n},\mathcal{E}))
\end{array}$$

(2.1.1.5.10)

By composing the bottom arrows of the diagrams 2.1.1.5.10 and 2.1.1.5.8 we get the canonical isomorphism

$$\mathcal{E}\otimes\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}^{(m)}_{X,n}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n'}),\mathcal{E}\right), \tag{2.1.1.5.11}$$

which is $\mathcal{O}_X$-linear for the right structures and is given by $e\otimes x\mapsto (P\otimes P'\mapsto P\circ(\mathrm{id}\otimes P')(x)\cdot e)$ for any $P\in\mathcal{D}^{(m)}_{X,n}$ and $P'\in\mathcal{D}^{(m)}_{X,n'}$, $e\in\mathcal{E}$ and $x\in\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}$ (indeed, if $x=\tau\otimes\tau'$, then we compute $(PP'(\tau'))(\tau)=P(\tau p_1^*(P'(\tau'))=P\circ(\mathrm{id}\otimes P')(\tau\otimes\tau'))$. Hence, by construction of the product (see 1.4.2.1.3), we compute that the diagram

$$\begin{array}{ccc}
p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n+n'},\mathcal{E}) & \xleftarrow[\iota_{n+n'}]{\sim} & \mathcal{E}\otimes\mathcal{P}^{n+n'}_{X,(m)} \\
\big\downarrow{\scriptstyle 2.1.1.4.3} & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\delta^{n,n'}_{(m)}} \\
p_{2*}\mathcal{H}om_{\mathcal{O}_X}\left(p_{0*}(\mathcal{D}^{(m)}_{X,n}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n'}),\mathcal{E}\right) & \xleftarrow[\sim]{2.1.1.5.11} & \mathcal{E}\otimes\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}
\end{array} \tag{2.1.1.5.12}$$

is commutative. By composing (horizontally from left to right) the diagrams 2.1.1.5.10, 2.1.1.5.8 and 2.1.1.5.12, we get the diagram 2.1.1.5.2. Since the maps ι_n are isomorphisms, then the commutativity of 2.1.1.5.2 is equivalent to that of 2.1.1.5.8. Since the commutativity of 2.1.1.5.1 is equivalent to that of the left square of 2.1.1.5.6, then we get the equivalence between the assertions (a) and (b) of the proposition.

2) Now, let us prove the equivalence between (b) and (c). We have the morphism $p^{n*}_{1(m)}\otimes\mathrm{id}_{\mathcal{E}}\colon\mathcal{E}\to\mathcal{P}^{n}_{X,(m)}\otimes_{\mathcal{O}_X}\mathcal{E}$ given by $e\mapsto 1\otimes e$. By adjunction involving the ring morphism $p^{n*}_{1(m)}\colon\mathcal{O}_X\to\mathcal{P}^{n}_{X,(m)}$, a family of $\mathcal{O}_X$-linear homomorphisms $\theta_n\colon\mathcal{E}\to p_{1*}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{P}^{n}_{X,(m)})$ is equivalent to a family of $\mathcal{P}^{n}_{X,(m)}$-linear homomorphisms $\epsilon_n\colon\mathcal{P}^{n}_{X,(m)}\otimes_{\mathcal{O}_X}\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{P}^{n}_{X,(m)}$, the relation being $\epsilon_n\circ(p^{n*}_{1(m)}\otimes\mathrm{id}_{\mathcal{E}})=\theta_n$.

Moreover, $\varepsilon_0=\mathrm{id}_{\mathcal{E}}$ if and only if $\theta_0=\mathrm{id}_{\mathcal{E}}$ and for any $n'\geq n$, ε_n the diagram 2.1.1.1.2 is and only if so is 2.1.1.5.1. Suppose now $\theta_0=\mathrm{id}_{\mathcal{E}}$ and 2.1.1.5.1 is commutative. The morphism $\delta^{n,n'*}_{(m)}(\epsilon_{n+n'})$ is the unique $\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}$-linear morphism making commutative the contour of the diagram

$$\begin{array}{ccccc}
\mathcal{E} & \xrightarrow{p^{n+n'*}_{1(m)}\otimes\mathrm{id}_{\mathcal{E}}} & \mathcal{P}^{n+n'}_{X,(m)}\otimes\mathcal{E} & \xrightarrow{\delta^{n,n'}_{(m)}\otimes\mathrm{id}} & \mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}\otimes\mathcal{E} \\
\| & & \big\downarrow{\scriptstyle\epsilon_{n+n'}} & & \big\downarrow{\scriptstyle\delta^{n,n'*}_{(m)}(\epsilon_{n+n'})} \\
\mathcal{E} & \xrightarrow{\theta_{n+n'}} & \mathcal{P}^{n+n'}_{X,(m)}\otimes\mathcal{E} & \xrightarrow{\mathrm{id}\otimes\delta^{n,n'}_{(m)}} & \mathcal{E}\otimes\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}.
\end{array} \tag{2.1.1.5.13}$$

The morphism $(\epsilon_n\otimes\mathrm{id})\circ(\mathrm{id}\otimes\epsilon_{n'})$ is the unique $\mathcal{P}^{n}_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}$-linear morphism making commutative the contour of the diagram

$$\begin{array}{ccccc}
\mathcal{E} & \xrightarrow{p^{n'*}_{1(m)}\otimes \mathrm{id}_{\mathcal{E}}} & \mathcal{P}^{n'}_{X,(m)} \otimes \mathcal{E} & \xrightarrow{p^{n*}_{1(m)}\otimes \mathrm{id}_{\mathcal{P}^{n'}_{X,(m)}\otimes\mathcal{E}}} & \mathcal{P}^{n}_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)} \otimes \mathcal{E} \\
\| & & \downarrow{\scriptstyle \epsilon_{n+n'}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\epsilon_{n'}} \\
\mathcal{E} & \xrightarrow{\theta^{\mathcal{E}}_{n'}} & \mathcal{E}\otimes\mathcal{P}^{n'}_{X,(m)} & \xrightarrow{p^{n*}_{1(m)}\otimes \mathrm{id}_{\mathcal{E}}\otimes \mathrm{id}} & \mathcal{P}^{n}_{X,(m)} \otimes \mathcal{E} \otimes \mathcal{P}^{n'}_{X,(m)} \\
 & & & \searrow^{\theta^{\mathcal{E}}_{n}\otimes \mathrm{id}} & \downarrow{\scriptstyle \epsilon_{n}\otimes \mathrm{id}} \\
 & & & & \mathcal{E}\otimes\mathcal{P}^{n}_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}.
\end{array} \tag{2.1.1.5.14}$$

Since $\epsilon_n \otimes \mathrm{id} = q^{n,n',*}_{1,(m)}(\varepsilon_{n+n'})$ and $\mathrm{id}\otimes\epsilon_{n'} = q^{n,n',*}_{0,(m)}(\varepsilon_{n+n'})$ (remark we need the compatibility given by the commutativity of 2.1.1.1.2), since both composition of the top morphisms of 2.1.1.5.13 and 2.1.1.5.14 are equal (to $e \mapsto 1 \otimes 1 \otimes e$), then by unicity of the factorisation, we get that the diagram 2.1.1.5.2 is commutative if and only if the diagram 2.1.1.1.3 is commutative.

II) Let $\mathcal{E}$ be left $\mathcal{D}^{(m)}_X$-module and let $\varepsilon = (\varepsilon^{\mathcal{E}}_n)$ be the associated m-PD stratification. For any $P \in \mathcal{D}^{(m)}_{X,n}$, by construction the action of P on $\mathcal{E}$ is given by the composition of the bottom arrows of the diagram:

$$\begin{array}{ccccc}
\mathcal{P}^{n}_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\varepsilon^{\mathcal{E}}_n} & \mathcal{E}\otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X,(m)} & \xrightarrow{\mathrm{id}\otimes P} & \mathcal{E} \\
\uparrow{\scriptstyle p^{n*}_{1(m)}} & \nearrow^{\theta^{\mathcal{E}}_n} & {\scriptstyle 2.1.1.5.9}\downarrow{\scriptstyle \iota_n} & & \| \\
\mathcal{E} & \xrightarrow[\nu_n]{2.1.1.5.7} & p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{D}^{(m)}_{X,n}, \mathcal{E}) & \xrightarrow{\mathrm{ev}_P} & \mathcal{E}.
\end{array} \tag{2.1.1.5.15}$$

Since the diagram 2.1.1.5.15 is commutative, then we get the assertion II)a). When X/S is equipped with coordinates $t_1, \ldots, t_d$, then by using IIa) in the case where $P = \underline{\partial}^{\langle \underline{k}\rangle}$, we get that $(\mathrm{id}\otimes\underline{\partial}^{\langle \underline{k}\rangle})\circ\theta_n(x) = \underline{\partial}^{\langle \underline{k}\rangle}x$. This yields the formulas 2.1.1.5.4.

Let us check that the morphism $\zeta^{\mathcal{E}}_n$ defined by the formula 2.1.1.5.5 is the inverse of the morphism $\varepsilon^{\mathcal{E}}_n$. We compute

$$\begin{aligned}
\zeta^{\mathcal{E}}_n \circ \varepsilon^{\mathcal{E}}_n(1\otimes x) &= \zeta^{\mathcal{E}}_n\Big(\sum_{|\underline{i}|\le n} \underline{\partial}^{\langle \underline{i}\rangle}x \otimes \underline{\tau}^{\{\underline{i}\}}\Big) = \sum_{|\underline{i}|\le n}\sum_{|\underline{j}|\le n} (-1)^{|\underline{j}|}\underline{\tau}^{\{\underline{j}\}}\underline{\tau}^{\{\underline{i}\}} \otimes \underline{\partial}^{\langle \underline{j}\rangle}\underline{\partial}^{\langle \underline{i}\rangle}x \\
&\overset{1.2.4.5.3+1.4.2.7.c}{=} \sum_{|\underline{k}|\le n}\sum_{\underline{j}\le \underline{k}} (-1)^{|\underline{j}|}\left(\begin{matrix}\underline{k}\\ \underline{j}\end{matrix}\right)\underline{\tau}^{\{\underline{k}\}} \otimes \underline{\partial}^{\langle \underline{k}\rangle}x = 1\otimes x.
\end{aligned}$$

Similarly, we check $\varepsilon^{\mathcal{E}}_n \circ \zeta^{\mathcal{E}}_n(x\otimes 1)$. Hence, we are done.

III) To check the last assertion we can either use the functoriality in $\mathcal{E}$ of the diagrams appearing in the equivalences of (I) or make some local computations using (II). □

Remark 2.1.1.6. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module endowed with a family (as n varies) of $\mathcal{P}^n_{X,(m)}$-linear isomorphisms

$$\varepsilon_n \colon \mathcal{P}^{n}_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} = p^{n*}_{1(m)}(\mathcal{E}) \xrightarrow{\sim} p^{n*}_{0(m)}(\mathcal{E}) = \mathcal{E}\otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X,(m)},$$

satisfying the condition (a) of 2.1.1.1. We remark that the cocycle condition is local. Let us give a local description of this condition. Suppose X/S is equipped with coordinates $t_1, \ldots, t_d$. For any x of $\mathcal{E}$, for any $\underline{k} \in \mathbb{N}^d$, for any $n \ge |\underline{k}|$, let us denote by $\underline{\partial}^{\langle \underline{k}\rangle}(x)$ the section of $\mathcal{E}$ such that we get

$$\varepsilon^{\mathcal{E}}_n(1\otimes x) = \sum_{|\underline{k}|\le n} \underline{\partial}^{\langle \underline{k}\rangle}(x) \otimes \underline{\tau}^{\{\underline{k}\}}.$$

First remark that these elements do not depend on the choice of n which justifies the notation. Moreover, the cocycle condition is equivalent to the condition that the formula

$$\left\langle \begin{matrix}\underline{i}+\underline{j}\\ \underline{i}\end{matrix}\right\rangle \underline{\partial}^{\langle \underline{i}+\underline{j}\rangle}(x) = \underline{\partial}^{\langle \underline{i}\rangle}(\underline{\partial}^{\langle \underline{j}\rangle}(x)) \tag{2.1.1.6.1}$$

holds for any section x of $\mathcal{E}$, for any $\underline{i}, \underline{j} \in \mathbb{N}^d$ (to understand this formula, see 1.4.2.7.c). Indeed, this is an easy computation using $\delta^{n,n'}_{(m)}(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i$ (see 1.4.1.17.1) and the formula 1.2.4.5.2.

Example 2.1.1.7. Let Z be a closed subscheme of X/S defined by an ideal $\mathcal{K}$ of $\mathcal{O}_X$. With notation 1.3.2.13, 1.4.1.2, put

$$\mathcal{K}' = \mathcal{K} \cdot \mathcal{P}^n_{X/S,(m)} + \mathcal{I}^n_{X/S,(m)} = \mathrm{Ker}(\mathcal{P}^n_{X/S,(m)} \to \mathcal{O}_X/\mathcal{K}) = \mathcal{P}^n_{X/S,(m)} \cdot \mathcal{K} + \mathcal{I}^n_{X/S,(m)}. \tag{2.1.1.7.1}$$

Then the canonical map

$$\mathcal{P}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)} \to \mathcal{P}_{(m),\alpha}(\mathcal{K}') \tag{2.1.1.7.2}$$

is an isomorphism. Indeed, since this is local, we can suppose X/S is equipped with coordinates $t_1, \ldots, t_d$. First, we remark that this is sufficient to check that $\mathscr{I}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)} + \mathcal{P}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{I}^n_{X/S,(m)}$ has an m-PD structure extending that of $\mathscr{I}_{(m),\alpha}(\mathcal{K})$ and of $\mathcal{I}^n_{X/S,(m)}$ (indeed, this would yield an inverse of 2.1.1.7.2 by using the universal property of the m-PD envelope). By flatness of $p^n_{1,(m)} \colon \mathcal{O}_X \to \mathcal{P}^n_{X/S,(m)}$, the ideal $\mathscr{I}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)}$ is endowed with an m-PD structure extending $\mathscr{I}_{(m),\alpha}(\mathcal{K})$

From the description 1.4.2.3, $\mathcal{P}^n_{X/S,(m)}$ is isomorphic to an m-PD polynomial algebra with coefficients in $\mathcal{O}_X$ of order n in d variables given by $\tau_1, \ldots, \tau_d$. Hence, $\mathcal{P}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)}$ is isomorphic to an m-PD polynomial algebra with coefficients in $\mathcal{P}_{(m),\alpha}(\mathcal{K})$ of order n in d variables. Following 1.3.2.6, the m-PD structure of $\mathcal{P}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{I}^n_{X/S,(m)}$ is compatible with any m-PD-structure on $\mathscr{I}_{(m),\alpha}(\mathcal{K})$. Hence, we get the m-PD structure.

For the same reason we have the canonical isomorphism

$$\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}_{(m),\alpha}(\mathcal{K}) \to \mathcal{P}_{(m),\alpha}(\mathcal{K}'). \tag{2.1.1.7.3}$$

By composing 2.1.1.7.2 and 2.1.1.7.3, we get the isomorphisms

$$\varepsilon_n : \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}_{(m),\alpha}(\mathcal{K}) \xrightarrow{\sim} \mathcal{P}_{(m),\alpha}(\mathcal{K}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)}$$

which gives an m-PD stratification on $\mathcal{P}_{(m),\alpha}(\mathcal{K})$.

2.1.2 PD-costratifications of level m

Conventions on notations for derived categories are given at the beginning of Chapter 5.

2.1.2.1. We will use the following notation (for the second one, see 4.2.2).

(a) Let $\phi \colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of commutative rings. We denote by $\phi_* \colon : D(\mathcal{B}) \to D(\mathcal{A})$ the canonical exact forgetful functor. Recall that following the Cartan isomorphisms (see the general version 4.6.3.9) for any $\mathcal{B}$-module $\mathcal{N}$ and any $\mathcal{A}$-module $\mathcal{M}$ we have the bifunctorial in $\mathcal{M}$ and $\mathcal{N}$ canonical isomorphism

$$\mathcal{H}om_{\mathcal{A}}(\phi_*\mathcal{N}, \mathcal{M}) \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{N}, \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M})). \tag{2.1.2.1.1}$$

In other words, the forgetful functor ϕ_* has the right adjoint $? \mapsto \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, ?)$. We denote by $\phi^\flat \colon D(\mathcal{A}) \to D(\mathcal{B})$ the functor defined by setting $\phi^\flat(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M})$ for any $\mathcal{M} \in D(\mathcal{A})$. The convention here is to work in the derived categories but our functors of the form $\phi^\flat$ will often be exact.

(b) More generally, if $f \colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is a morphism of ringed spaces, then we define the functor $f^\flat \colon D(\mathcal{B}_Y) \to D(\mathcal{B}_X)$ by setting

$$f^\flat \mathcal{M} = \bar{f}^* \mathbb{R}\mathcal{H}om_{\mathcal{B}_Y}(f_*\mathcal{B}_X, \mathcal{M})$$

where $\bar{f}$ denotes the morphism $(X, \mathcal{B}_X) \to (Y, f_*\mathcal{B}_X)$.

2.1.2.2. Let $f \colon X \to Y$ be a finite morphism between two locally noetherian schemes. $\mathrm{Mod}(\mathcal{O}_X)$ denotes the category of $\mathcal{O}_X$-modules. Since $\bar{f}$ is flat, $\bar{f}^* : \mathrm{Mod}(f_*\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$ is exact and take quasi-coherent (resp. coherent) sheaves to quasi-coherent (resp. coherent), we get that $f^\flat$ takes $D^+_{\mathrm{qc}}(Y)$ (resp. $D^+_{\mathrm{coh}}(Y)$) to $D^+_{\mathrm{qc}}(X)$ (resp. $D^+_{\mathrm{coh}}(X)$). If f is a locally free finite homeomorphism then $f^\flat \mathcal{M} = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{M})$ and $f^\flat$ takes $D_{\mathrm{qc}}(Y)$ (resp. $D_{\mathrm{coh}}(Y)$) to $D_{\mathrm{qc}}(X)$ (resp. $D_{\mathrm{coh}}(X)$).

Definition 2.1.2.3. An m-PD-*costratification on* $\mathcal{M}$ *relatively to* S on an $\mathcal{O}_X$-module $\mathcal{M}$ is the data of a family of $\mathcal{P}^n_{X/S(m)}$-linear homomorphisms

$$\varepsilon_n \colon \mathcal{H}om_{\mathcal{O}_X}(p^n_{0,(m)*}\mathcal{P}^n_{X/S(m)},\, \mathcal{M}) = p^{n\flat}_{0,(m)}(\mathcal{M}) \to p^{n\flat}_{1,(m)}(\mathcal{M}) = \mathcal{H}om_{\mathcal{O}_X}(p^n_{1,(m)*}\mathcal{P}^n_{X/S(m)},\, \mathcal{M}),$$

this ones satisfying the following conditions :

(a) $\varepsilon_0 = \mathrm{id}_{\mathcal{M}}$ and for any $n' \geq n$ in $\mathbb{N}$, ε_n and $\psi^{n',n\flat}_{X/S,(m)}(\varepsilon_{n'})$ are canonically isomorphic, i.e. the following diagram

$$\begin{array}{ccc}
\psi^{n',n\flat}_{X/S,(m)}(p^{n'\flat}_{0,(m)}(\mathcal{M})) & \xrightarrow{\psi^{n',n\flat}_{X/S,(m)}(\varepsilon_{n'})} & \psi^{n',n\flat}_{X/S,(m)}(p^{n'\flat}_{1,(m)}(\mathcal{M})) \\
\downarrow \wr & & \downarrow \wr \\
p^{n\flat}_{0,(m)}(\mathcal{M}) & \xrightarrow{\varepsilon_n} & p^{n\flat}_{1,(m)}(\mathcal{M})
\end{array}, \tag{2.1.2.3.1}$$

whose vertical isomorphisms are the canonical ones, is commutative.

(b) For any n, n', the diagram

$$\begin{array}{ccc}
p^{n,n'\flat}_{0,(m)}(\mathcal{M}) & \xrightarrow{\delta^{n,n',\flat}_{(m)}(\varepsilon_{n+n'})} & p^{n,n'\flat}_{2,(m)}(\mathcal{M}) \\
{\scriptstyle q^{n,n',\flat}_{0,(m)}(\varepsilon_{n+n'})} \searrow & & \nearrow {\scriptstyle q^{n,n',\flat}_{0,(m)}(\varepsilon_{n+n'})} \\
& p^{n,n'\flat}_{1,(m)}(\mathcal{M}) &
\end{array} \tag{2.1.2.3.2}$$

is commutative.

Say an $\mathcal{O}_X$-linear homomorphism $\varphi\colon \mathcal{M} \to \mathcal{N}$ between modules equipped with m-PD costratifications relative to S is *horizontal* if it commutes with all ε_n.

2.1.2.4. With notation 1.4.1.13, similarly to 2.1.1.2, we check that the above condition 2.1.2.3.2 is equivalent to

$$\forall\, n, \qquad q^{n\flat}_{02,(m)}(\varepsilon_n) = q^{n\flat}_{01,(m)}(\varepsilon_n) \circ q^{n\flat}_{12,(m)}(\varepsilon_n). \tag{2.1.2.4.1}$$

Proposition 2.1.2.5. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module together with an m-PD costratification $(\varepsilon^{\mathcal{M}}_n)$ relative to X/S. Then the homomorphisms $\varepsilon^{\mathcal{M}}_n$ are $\mathcal{P}^n_{X/S,(m)}$-linear isomorphisms.*

Proof. Copy the proof of 2.1.1.3. □

Notation 2.1.2.6. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Since $p_{i*}\mathcal{P}^n_{X,(m)}$ is locally free as $\mathcal{O}_X$-modules, then the canonical homomorphism

$$\iota^{\mathcal{M}}_n \colon \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n} = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X,(m)}, \mathcal{M}) = p^{n\flat}_{0,(m)}(\mathcal{M}) \tag{2.1.2.6.1}$$

given by $x \otimes P \mapsto (\tau \mapsto xP(\tau))$ is an isomorphism. Similarly, we have the canonical isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^n_{X,(m)}, \mathcal{M}) = p^{n\flat}_{1,(m)}(\mathcal{M}). \tag{2.1.2.6.2}$$

2.1.2.7. Let $\alpha\colon \mathcal{A} \to \mathcal{B}$ and $\beta\colon \mathcal{B} \to \mathcal{C}$ be two homomorphisms of sheaves of commutative algebras. For any $\mathcal{A}$-module $\mathcal{M}$, we get the commutative triangle

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M})) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{A}}(\mathcal{C}, \mathcal{M}) \\
{\scriptstyle \mathrm{ev}_1} \searrow & & \swarrow {\scriptstyle -\circ\beta} \\
& \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{M}) &
\end{array} \tag{2.1.2.7.1}$$

where the horizontal map is the Cartan isomorphism, ev_1 is the evalutation at 1 morphism and $- \circ \beta$ is the morphism induced by composition with β.

Proposition 2.1.2.8. *Given an $\mathcal{O}_X$-module $\mathcal{M}$. The following are equivalent.*

(a) A right $\mathcal{D}^{(m)}_X$ module structure on $\mathcal{M}$ extending its $\mathcal{O}_X$-module structure.

(b) A family $\theta_n \colon p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X,(m)}, \mathcal{M}) \to \mathcal{M}$ which are $\mathcal{O}_X$-linear homomorphisms (for the $\mathcal{O}_X$ structure on $\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X,(m)}, \mathcal{M})$ induced by the right $\mathcal{O}_X$-algebra structure on $\mathcal{P}^n_{X,(m)}$) satisfying

(i) $\theta_0 = \mathrm{id}_{\mathcal{M}}$, and for any $n' \geq n$ in $\mathbb{N}$, θ_n and $\theta_{n'}$ are compatible i.e. the following diagram

$$\begin{array}{ccc} p^{n'\flat}_{0,(m)}(\mathcal{M}) & \xrightarrow{\theta_{n'}} & \mathcal{M} \\ \uparrow{\scriptstyle \psi^{n',n}} & & \| \\ p^{n\flat}_{0,(m)}(\mathcal{M}) & \xrightarrow{\theta_n} & \mathcal{M} \end{array} \quad (2.1.2.8.1)$$

whose the vertical morphism is given by adjunction of $(\psi^{n',n}_{X/S,(m)*} \dashv \psi^{n',n\flat}_{X/S,(m)})$, is commutative.

(ii) for all n, n' we have commutative diagrams

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^{n'}_{X,(m)}, \mathcal{M}) & \xrightarrow{\theta_{n'}} & \mathcal{M} \\ \uparrow{\scriptstyle \tilde{\theta}_n} & & \uparrow{\scriptstyle \theta_{n+n'}} \\ \mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) & \xrightarrow{\delta^{n,n'}_{(m)}} & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^{n+n'}_{X,(m)}, \mathcal{M}) \end{array} \quad (2.1.2.8.2)$$

where $\tilde{\theta}_n$ is the $\mathcal{P}^{n'}_{X,(m)}$-linear homomorphism corresponding by adjointness to the composed morphism

$$\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) \to \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X,(m)}, \mathcal{M}) \xrightarrow{\theta_n} \mathcal{M}, \quad (2.1.2.8.3)$$

where the fist map is induced by functoriality by $q^{n,n'}_{0(m)}$ (which is $\mathcal{O}_X$-linear for the structure defined by p_1 on $\mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X,(m)}$).

(c) An m-PD costratification (ε_n) relatively to X/S on $\mathcal{M}$.

An $\mathcal{O}_X$-linear morphism $\phi \colon \mathcal{M} \to \mathcal{N}$ between two right $\mathcal{D}^{(m)}_{X/S}$-modules is $\mathcal{D}^{(m)}_{X/S}$-linear if and only if ϕ is horizontal or if it commutes to the homomorphisms θ_n.

Proof. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. A right $\mathcal{D}^{(m)}_X$-module structure on $\mathcal{M}$ extending its $\mathcal{O}_X$-module structure is equivalent to the data a family $(\mu^{\mathcal{M}}_n)_{n\in\mathbb{N}}$ of $\mathcal{O}_X$-linear homomorphisms $\mu^{\mathcal{M}}_n \colon p_{1*}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n}) \to \mathcal{M}$ such that $\mu^{\mathcal{M}}_0 = \mathrm{id}_{\mathcal{M}}$ (modulo some canonical identification) and for any $n, n' \in \mathbb{N}$ the following diagrams in the category of $\mathcal{O}_X$-modules

$$\begin{array}{ccc} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n} & \xrightarrow{\mu^{\mathcal{M}}_n} & \mathcal{M} \\ \downarrow{\scriptstyle \rho^{(m)}_{n+n',n} \otimes \mathrm{id}} & & \| \\ \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n+n'} & \xrightarrow{\mu^{\mathcal{M}}_{n+n'}} & \mathcal{M}, \end{array} \qquad \begin{array}{ccc} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'} & \xrightarrow{\mu^{\mathcal{M}}_n \otimes \mathrm{id}} & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'} \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mu_{n,n'}} & & \downarrow{\scriptstyle \mu^{\mathcal{M}}_{n'}} \\ \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n+n'} & \xrightarrow{\mu^{\mathcal{M}}_{n+n'}} & \mathcal{M} \end{array} \quad (2.1.2.8.4)$$

are commutative (see notation 1.4.2.1.2). With notation 2.1.2.6.1, we denote by $\theta^{\mathcal{M}}_n$ the morphism

$p_{1*}(\mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^n_{X,(m)},\mathcal{M})) \to \mathcal{M}$ so that by setting $\mu_n = \iota_n \circ \theta^{\mathcal{M}}_n$. With notation 1.4.1.15

$$\begin{array}{ccccc}
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}),\mathcal{M}) & \xrightarrow{\varpi^{n,n'}_{0(m)}} & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)}),\mathcal{M}) & & \\
\star\downarrow\wr & \nearrow_{\mathrm{ev}_1} & & \searrow^{\theta^{\mathcal{M}}_n} & \\
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^{n'}_{X,(m)}),p_{1*}\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^n_{X,(m)},\mathcal{M})) & & & & \mathcal{M} \\
\wr\uparrow\iota_n & \searrow^{\theta^{\mathcal{M}}_n} & & \nearrow_{\mathrm{ev}_1} & \\
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^{n'}_{X,(m)}),p_{1*}(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n})) & \xrightarrow{\mu^{\mathcal{M}}_n} & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^{n'}_{X,(m)}),\mathcal{M}) & & \\
\wr\uparrow\iota_{n'} & & \wr\uparrow\iota_{n'} & \searrow^{\theta^{\mathcal{M}}_{n'}} & \\
\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n'} & \xrightarrow{\mu^{\mathcal{M}}_n\otimes\mathrm{id}} & \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n'} & \xrightarrow{\mu^{\mathcal{M}}_{n'}} & \mathcal{M}
\end{array} \tag{2.1.2.8.5}$$

where the $\star$ isomorphism is the Cartan isomorphism and the top left triangle is commutative following 2.1.2.7.1. By unicity of the factorisation, we get that $\theta^{\mathcal{M}}_n \circ \star = \tilde{\theta}_n$. By composing the left arrows of the diagrams 2.1.2.8.5, we get the canonical isomorphism

$$\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n'} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}),\mathcal{M}), \tag{2.1.2.8.6}$$

which is $\mathcal{O}_X$-linear for the right structures and is given by $m\otimes P\otimes P' \mapsto (\tau\otimes\tau' \mapsto mP(\tau p_1^*(P'(\tau')) = mP\circ(\mathrm{id}\otimes P')(\tau\otimes\tau')$ for any $P\in\mathcal{D}^{(m)}_{X,n}$, $P'\in\mathcal{D}^{(m)}_{X,n'}$, $m\in\mathcal{M}$. Hence, by construction of the product (see 1.4.2.1.3), we compute that the square of the diagram

$$\begin{array}{ccccc}
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}),\mathcal{M}) & \xrightarrow{\delta^{n,n'}_{(m)}} & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^{n+n'}_{X,(m)}),\mathcal{M}) & & \\
\wr\uparrow 2.1.2.8.6 & & \iota_{n+n'}\uparrow\wr & \searrow^{\theta^{\mathcal{M}}_{n+n'}} & \\
\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n'} & \xrightarrow{\mathrm{id}\otimes\mu_{n,n'}} & \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n+n'} & \xrightarrow{\mu^{\mathcal{M}}_{n+n'}} & \mathcal{M}
\end{array} \tag{2.1.2.8.7}$$

is commutative. Since its triangle is commutative by definition, then so is the diagram 2.1.2.8.7. Since the map 2.1.2.8.6 is an isomorphism, then using the commutative diagram 2.1.2.8.5 and 2.1.2.8.7 the right square of 2.1.2.8.4 is commutative if and only if the diagram 2.1.2.8.2 is commutative.

We go from μ_n to ϵ_n by adjointness (use 2.1.2.1.1 in the case of $p^n_{1,(m)}\colon \mathcal{O}_X \to p_{1*}\mathcal{P}^n_{X,(m)}$), i.e. ϵ_n is the unique $\mathcal{P}_{X/S(m)}$-linear morphism such that the composition map

$$\mu_n\colon \mathcal{H}om_{\mathcal{O}_X}(p^n_{0,(m)*}\mathcal{P}_{X/S(m)},\,\mathcal{M}) \xrightarrow{\varepsilon_n} \mathcal{H}om_{\mathcal{O}_X}(p^n_{1,(m)*}\mathcal{P}_{X/S(m)},\,\mathcal{M}) \xrightarrow{\mathrm{ev}_1} \mathcal{M} \tag{2.1.2.8.8}$$

is μ_n. Via this correspondance, by using some universal property, we can check that the commutativity of 2.1.2.8.1 is equivalent to that of 2.1.2.3.1. Now let us suppose the square 2.1.2.8.1 is commutative. Via this correspondance, it remains to check that the commutativity of 2.1.2.8.2 is equivalent to that of 2.1.2.3.2. Using our hypothesis, the commutativity of 2.1.2.8.2 is equivalent to saying that $\theta_{n+n'}\circ\delta^{n,n'}_{(m)} = \theta_{n+n'}\circ\psi^{n+n',n}\circ\tilde{\theta}_n$.

$$\begin{array}{ccccc}
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}),\mathcal{M}) & & \xrightarrow[\sim]{q^{n,n',\flat}_{0,(m)}(\varepsilon_{n+n'})} & & \mathcal{H}om_{\mathcal{O}_X}(p_{1*}(\mathcal{P}^n_{X,(m)}\otimes\mathcal{P}^{n'}_{X,(m)}),\mathcal{M}) \\
\downarrow q^{n,n'}_{0,(m)} & & & \swarrow_{q^{n,n'}_{0,(m)}} & \downarrow q^{n,n'}_{1,(m)} \\
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^{n+n'}_{X,(m)},\mathcal{M}) & \xrightarrow[\sim]{\varepsilon_{n+n'}} & \mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^{n+n'}_{X,(m)},\mathcal{M}) & & \mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^{n+n'}_{X,(m)},\mathcal{M}) \\
& \searrow_{\theta_{n+n'}} & \downarrow\mathrm{ev}_1 & \swarrow_{\mathrm{ev}_1} & \\
& & \mathcal{M} & &
\end{array} \tag{2.1.2.8.9}$$

Since $\theta_{n+n'} \circ q^{n,n'}_{0,(m)}$ is equal to the composition morphism 2.1.2.8.3 (use again the commutativity of 2.1.2.8.1), then by uniqueness of the factorization through ev_1, this yields $q^{n,n'}_{1,(m)} \circ q^{n,n',\flat}_{0,(m)}(\varepsilon_{n+n'}) = \psi^{n+n',n} \circ \tilde{\theta}_n$. Hence, by composition 2.1.2.8.9 with the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{O}_X}(p_{1*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) & \xrightarrow[\sim]{q^{n,n',\flat}_{1,(m)}(\varepsilon_{n+n'})} & \mathcal{H}om_{\mathcal{O}_X}(p_{2*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) \\
\downarrow q^{n,n'}_{1,(m)} & & \downarrow q^{n,n'}_{1,(m)} \\
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}\mathcal{P}^{n+n'}_{X,(m)}, \mathcal{M}) & \xrightarrow[\sim]{\varepsilon_{n+n'}} & \mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^{n+n'}_{X,(m)}, \mathcal{M}) \\
& \searrow^{\theta_{n+n'}} & \downarrow \mathrm{ev}_1 \\
& & \mathcal{M}
\end{array} \tag{2.1.2.8.10}$$

we get the commutative triangle:

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) & \xrightarrow[\sim]{q^{n,n',\flat}_{1,(m)} \circ q^{n,n',\flat}_{0,(m)}(\varepsilon_{n+n'})} & \mathcal{H}om_{\mathcal{O}_X}(p_{2*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) \\
& \searrow^{\theta_{n+n'} \circ \psi^{n+n',n} \circ \tilde{\theta}_n} & \downarrow \mathrm{ev}_1 \\
& & \mathcal{M}.
\end{array} \tag{2.1.2.8.11}$$

Similarly to 2.1.2.8.10 (replace $q^{n,n'}_{1,(m)}$ by $\delta^{n,n'}_{(m)}$), we get the commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{O}_X}(p_{0*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) & \xrightarrow[\sim]{\delta^{n,n',\flat}_{(m)}(\varepsilon_{n+n'})} & \mathcal{H}om_{\mathcal{O}_X}(p_{2*}(\mathcal{P}^n_{X,(m)} \otimes \mathcal{P}^{n'}_{X,(m)}), \mathcal{M}) \\
& \searrow^{\theta_{n+n'} \circ \delta^{n,n'}_{(m)}} & \downarrow \mathrm{ev}_1 \\
& & \mathcal{M}
\end{array} \tag{2.1.2.8.12}$$

Using 2.1.2.8.11 and 2.1.2.8.12, by unicity of the factorisation through ev_1, this yields the equality $\theta_{n+n'} \circ \delta^{n,n'}_{(m)} = \theta_{n+n'} \circ \psi^{n+n',n} \circ \tilde{\theta}_n$ is equivalent to $q^{n,n',\flat}_{1,(m)} \circ q^{n,n',\flat}_{0,(m)}(\varepsilon_{n+n'}) = \delta^{n,n',\flat}_{(m)}(\varepsilon_{n+n'})$.

□

Lemma 2.1.2.9. *Let $\mathcal{M}$ be a right $\mathcal{D}^{(m)}_X$-module. Let $(\varepsilon^{\mathcal{M}}_n)$ be the m-PD-costratification associated with $\mathcal{M}$.*

(a) With notation 2.1.2.6.1, the action of $P \in \mathcal{D}^{(m)}_{X,n}$ on $x \in \mathcal{M}$ is given from the costratification by the formula

$$xP = \mathrm{ev}_1 \circ \epsilon_n \circ \iota^{\mathcal{M}}_n(x \otimes P). \tag{2.1.2.9.1}$$

(b) Suppose $t_1, \ldots, t_d$ are coordinates of X/S. Let $\{\underline{\partial}^{\star\langle \underline{k}\rangle}, \|\underline{k}| \leq n\}$ be the dual basis to $\{\underline{\tau}^{\{\underline{k}\}}, \|\underline{k}| \leq n\}$ of $\mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X)$. Conversely, via the identification 2.1.2.6.1 and 2.1.2.6.2, the costratification ε_n of $\mathcal{M}$ satisfies the following formula for any $x \in \mathcal{M}$ and any $\underline{k} \in \mathbb{N}^d$:

$$\varepsilon_n(x \otimes \underline{\partial}^{\langle \underline{k}\rangle}) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}^{\star\langle \underline{h}\rangle} \otimes x \underline{\partial}^{\langle \underline{k}-\underline{h}\rangle}. \tag{2.1.2.9.2}$$

We have the following formula for the inverse

$$\varepsilon_n^{-1}(\underline{\partial}^{\star\langle \underline{k}\rangle} \otimes x) = \sum_{\underline{h} \leq \underline{k}} (-1)^{|\underline{k}-\underline{h}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} x \underline{\partial}^{\langle \underline{k}-\underline{h}\rangle} \otimes \underline{\partial}^{\langle \underline{h}\rangle}. \tag{2.1.2.9.3}$$

Proof. a) The formula 2.1.2.9.1 is straightforward by construction of the $\epsilon^{\mathcal{M}}_n$ from the right $\mathcal{D}^{(m)}_X$-module action (see 2.1.2.8.8).

b) Set $\delta_{\underline{k},x} := \iota_n^{\mathcal{M}}(x \otimes \underline{\partial}^{\langle \underline{k} \rangle})$. It follows from 1.2.4.5.3 that $\underline{\tau}^{\{\underline{h}\}}\delta_{\underline{k},x} = \left\{\begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix}\right\}_{(m)} \delta_{\underline{k}-\underline{h},x}$. We compute

$$\varepsilon_n(\delta_{\underline{k},x})(\underline{\tau}^{\{\underline{h}\}}) = \left(\underline{\tau}^{\{\underline{h}\}}\varepsilon_n(\delta_{\underline{k},x})\right)(1) = \varepsilon_n(\underline{\tau}^{\{\underline{h}\}}\delta_{\underline{k},x})(1) = \left\{\begin{matrix} \underline{k} \\ \underline{h} \end{matrix}\right\}_{(m)} \varepsilon_n(\delta_{\underline{k}-\underline{h},x})(1) = \left\{\begin{matrix} \underline{k} \\ \underline{h} \end{matrix}\right\}_{(m)} x\underline{\partial}^{\langle \underline{k}-\underline{h} \rangle}. \tag{2.1.2.9.4}$$

From 2.1.2.9.4, we get the formula 2.1.2.9.2. Finally, via an easy computation, we can check that the formula given by 2.1.2.9.3 do is the inverse. □

Remark 2.1.2.10. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module endowed with a family (as n varies) of $\mathcal{P}^n_{X,(m)}$-linear isomorphisms

$$\varepsilon_n \colon p_{0,(m)}^{n\flat}(\mathcal{M}) \to p_{1,(m)}^{n\flat}(\mathcal{M}),$$

satisfying the condition (a) of 2.1.2.3. We remark that the cocycle condition is local. Let us give a local description of this condition. Suppose X/S is equipped with coordinates $t_1, \ldots, t_d$. For any x of $\mathcal{M}$, for any $\underline{k} \in \mathbb{N}^d$, for any $n \geq |\underline{k}|$, we set

$$x \cdot \underline{\partial}^{\langle \underline{k} \rangle} := \mathrm{ev}_1 \circ \varepsilon_n(x \otimes \underline{\partial}^{\langle \underline{k} \rangle}). \tag{2.1.2.10.1}$$

First remark that these elements do not depend on the choice of n which justifies the notation. Moreover, the cocycle condition is equivalent to the condition that the formula

$$\left\langle \begin{matrix} \underline{i} + \underline{j} \\ \underline{i} \end{matrix} \right\rangle x \cdot \underline{\partial}^{\langle \underline{i}+\underline{j} \rangle} = (x \cdot \underline{\partial}^{\langle \underline{i} \rangle}) \cdot \underline{\partial}^{\langle \underline{j} \rangle} \tag{2.1.2.10.2}$$

holds for any section x of $\mathcal{E}$, for any $\underline{i}, \underline{j} \in \mathbb{N}^d$. Indeed, we have checked in the proof of 2.1.2.8 that the family (ε_n) is equivalent to the family of $\mathcal{O}_X$-linear homomorphisms of the form $\mu_n^{\mathcal{M}} \colon p_{1*}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n}) \to \mathcal{M}$ such that $\mu_0^{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$ and the left square of 2.1.2.8.4 is commutative. Moreover, we have checked the cocycle condition is equivalent to the commutativity of the right square of 2.1.2.8.4, i.e. of the associativity of the definition 2.1.2.10.1. By $\mathcal{O}_X$-linearity of $\mu_n^{\mathcal{M}}$, we reduce to the case where the operators are of the form $\underline{\partial}^{\langle \underline{i} \rangle}$ and we are done thanks to 1.4.2.7.c.

2.1.3 Internal tensor product of $\mathcal{D}$-modules of level m

We give a summary of the local formulas calculating the $\otimes$ of $\mathcal{D}^{(m)}_X$-modules. These will be used throughout the book implicitly. We give some details here as an illustration of calculations with left and right modules.

Proposition 2.1.3.1. *Let $\mathcal{E}$, $\mathcal{F}$ be two left $\mathcal{D}^{(m)}_X$-modules and let $\mathcal{M}$ be a right $\mathcal{D}^{(m)}_X$-module.*

(a) There exists on $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ a unique structure of left $\mathcal{D}^{(m)}_X$-module (extending its structure of $\mathcal{O}_X$-module) such that for any open set $U \subset X$ such that U/S has coordinates, for any sections $x \in \Gamma(U, \mathcal{E})$ and $y \in \Gamma(U, \mathcal{F})$ the formula holds

$$\underline{\partial}^{\langle \underline{k} \rangle} \cdot (x \otimes y) = \sum_{\underline{i} \leq \underline{k}} \left\{\begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix}\right\} \underline{\partial}^{\langle \underline{i} \rangle} x \otimes \underline{\partial}^{\langle \underline{k}-\underline{i} \rangle} y. \tag{2.1.3.1.1}$$

(b) There exists on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ a unique structure of right $\mathcal{D}^{(m)}_X$-module such that such that for any open set $U \subset X$ such that U/S has coordinates, $x \in \Gamma(U, \mathcal{E})$, $y \in \Gamma(U, \mathcal{M})$ the formula holds

$$(y \otimes x)\underline{\partial}^{\langle \underline{k} \rangle} = \sum_{\underline{h} \leq \underline{k}} (-1)^{|\underline{h}|} \left\{\begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix}\right\} y\underline{\partial}^{\langle \underline{k}-\underline{h} \rangle} \otimes \underline{\partial}^{\langle \underline{h} \rangle} x. \tag{2.1.3.1.2}$$

(c) Let $\mathcal{D}$ be a sheaf of rings on X. If the structure of left $\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{E}$ (resp. $\mathcal{E}$) extends to a structure of $(\mathcal{D}^{(m)}_{X/S}, \mathcal{D})$-bimodule then then the structure of left $\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ extends to a structure of $(\mathcal{D}^{(m)}_{X/S}, \mathcal{D})$-bimodule, where the structure of left $\mathcal{D}^{(m)}_{X/S}$-module is the tensor product structure. Moreover, if the structure of left (resp. right) $\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{E}$ (resp. $\mathcal{M}$) extends to a

structure of $(\mathcal{D}^{(m)}_{X/S}, \mathcal{D})$-bimodule (resp. $(\mathcal{D}, \mathcal{D}^{(m)}_{X/S})$-bimodule) then the structure of right $\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$ extends to a structure of right $(\mathcal{D}, \mathcal{D}^{(m)}_{X/S})$-bimodule (resp. $(\mathcal{D}, \mathcal{D}^{(m)}_{X/S})$-bimodule). In the same way, these structures of $\mathcal{D}^{(m)}_{X/S}$-module do not depend on the level m.

Proof. (a) Modulo the canonical isomorphisms of $\mathcal{P}^n_{X/S(m)}$-modules

$$p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)}\otimes_{\mathcal{O}_X}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}) \xrightarrow{\sim} (p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)}\otimes_{\mathcal{O}_X}\mathcal{E})\otimes_{\mathcal{P}^n_{X/S(m)}}(p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)}\otimes_{\mathcal{O}_X}\mathcal{F}), \quad (2.1.3.1.3)$$

we get the isomorphisms of $\mathcal{P}^n_{X/S(m)}$-modules

$$\epsilon^{\mathcal{E}\otimes\mathcal{F}}_n := \varepsilon^{\mathcal{E}}_n \otimes \varepsilon^{\mathcal{F}}_n \colon \mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\sim} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S(m)}.$$

By using the formulas 1.2.4.5.3 and 2.1.1.5.4, with the above identification 4.2.3.1.5 we get

$$\varepsilon^{\mathcal{E}\otimes\mathcal{F}}_n(1\otimes(x\otimes y)) = \left(\sum_{|\underline{i}|\leq n} \underline{\partial}^{\langle \underline{i}\rangle} x \otimes \underline{\tau}^{\{\underline{i}\}}\right)\otimes\left(\sum_{|\underline{j}|\leq n} \underline{\partial}^{\langle \underline{j}\rangle} y \otimes \underline{\tau}^{\{\underline{j}\}}\right) = \sum_{|\underline{k}|\leq n}\left(\sum_{\underline{i}\leq\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{i}\end{smallmatrix}\right\} \underline{\partial}^{\langle \underline{i}\rangle} x \otimes \underline{\partial}^{\langle \underline{k}-\underline{i}\rangle} y\right)\otimes\underline{\tau}^{\{\underline{k}\}},$$

which yields 2.1.3.1.1. Using the remark 2.1.1.6, we easily compute that the cocycle condition holds. Hence, $(\epsilon^{\mathcal{E}\otimes\mathcal{F}}_n)$ is an m-PD-stratification relative to X/S.

(b) Set $\mathcal{H} := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}$. Since $\mathcal{P}^n_{X,(m)}$ is locally free of finite rank, so for $i = 0, 1$, we have canonical isomorphisms

$$\begin{aligned} &(p^n_{i,(m)})^\flat(\mathcal{H}) = \mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{E} \\ &\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{M}) \otimes_{\mathcal{P}^n_{X,(m)}} (p_{i*}\mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) = (p^n_{i,(m)})^\flat(\mathcal{M}) \otimes_{\mathcal{P}^n_{X/S,(m)}} (p^n_{i,(m)})^*(\mathcal{E}) \end{aligned}$$

Via these identifications, we get the isomorphisms of $\mathcal{P}^n_{X/S(m)}$-modules

$$\epsilon^{\mathcal{H}}_n := \epsilon^{\mathcal{M}}_n \otimes (\epsilon^{\mathcal{E}}_n)^{-1}.$$

By using the formulas 2.1.2.9.2 and 2.1.1.5.5, for any open set $U \subset X$ such that U/S has coordinates, $x \in \Gamma(U, \mathcal{E})$, $y \in \Gamma(U, \mathcal{M})$, we compute

$$\begin{aligned} \varepsilon^{\mathcal{H}}_n(y \otimes x \otimes \underline{\partial}^{\langle \underline{k}\rangle}) &= \varepsilon^{\mathcal{M}}_n(y \otimes \underline{\partial}^{\langle \underline{k}\rangle}) \otimes_{\mathcal{P}^n_{X,(m)}} (\varepsilon^{\mathcal{E}}_n)^{-1}(x \otimes 1) \\ &= \left(\sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} \underline{\partial}^{\star\langle \underline{h}\rangle} \otimes y\underline{\partial}^{\langle \underline{k}-\underline{h}\rangle}\right) \otimes \left(\sum_{|\underline{i}|\leq n} (-1)^{|\underline{i}|}\underline{\tau}^{\{\underline{i}\}} \otimes \underline{\partial}^{\langle \underline{i}\rangle} x\right) \\ &= \sum_{\underline{h}\leq\underline{k}}\sum_{|\underline{i}|\leq n} (-1)^{|\underline{i}|} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} \underline{\tau}^{\{\underline{i}\}}\underline{\partial}^{\star\langle \underline{h}\rangle} \otimes y\underline{\partial}^{\langle \underline{k}-\underline{h}\rangle} \otimes \underline{\partial}^{\langle \underline{i}\rangle} x. \end{aligned} \quad (2.1.3.1.4)$$

Since $\underline{\tau}^{\{\underline{i}\}}\underline{\partial}^{\star\langle \underline{h}\rangle}(1) = 0$ for $\underline{h} \neq \underline{i}$ and $\underline{\tau}^{\{\underline{h}\}}\underline{\partial}^{\star\langle \underline{h}\rangle}(1) = 1$, then we compute

$$(y \otimes x)\underline{\partial}^{\langle \underline{k}\rangle} \overset{2.1.2.9.1}{=} \varepsilon^{\mathcal{M}\otimes\mathcal{E}}_n(y \otimes x \otimes \underline{\partial}^{\langle \underline{k}\rangle})(1) \overset{2.1.3.1.4}{=} \sum_{\underline{h}\leq\underline{k}} (-1)^{|\underline{h}|} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} y\underline{\partial}^{\langle \underline{k}-\underline{h}\rangle} \otimes \underline{\partial}^{\langle \underline{h}\rangle} x.$$

which gives the formula 2.1.3.1.2. Using the remark 2.1.1.6, we easily compute that the cocycle condition holds. Hence, $(\epsilon^{\mathcal{H}}_n)$ is an m-PD-stratification relative to X/S.

(c) The functoriality in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{M}$ of these structures of $\mathcal{D}^{(m)}_{X/S}$-modules is a consequence for instance of the formulas 2.1.3.1.2 and 2.1.3.1.1. It follows from this functoriality that the part (c) holds (this can also be proves directly via the formula 2.1.3.1.1).

□

2.1.4 Internal homorphism bifunctor of $\mathcal{D}$-modules of level m

We give the local formulas calculating Hom.

Proposition 2.1.4.1. *Let $\mathcal{E}$, $\mathcal{F}$ be two left $\mathcal{D}_X^{(m)}$-modules. Let $\mathcal{M}$, $\mathcal{N}$ be two right $\mathcal{D}_X^{(m)}$-modules.*

(a) There exists on the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ a unique structure of left $\mathcal{D}_X^{(m)}$-module such that for any open set $U \subset X$ such that U/S has coordinates, for any $\varphi \in \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}))$ and $x \in \Gamma(U,\mathcal{E})$

$$(\underline{\partial}^{\langle \underline{k} \rangle}\varphi)(x) = \sum_{\underline{i}\leq \underline{k}} (-1)^{|\underline{i}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k}-\underline{i} \rangle}(\varphi(\underline{\partial}^{\langle \underline{i} \rangle} x)). \tag{2.1.4.1.1}$$

(b) There exists on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{M})$ a unique structure of right $\mathcal{D}_X^{(m)}$-module such that for any open set $U \subset X$ such that U/S has coordinates, $x \in \Gamma(U,\mathcal{E})$, $\varphi \colon \mathcal{E}|_U \to \mathcal{M}|_U$ we have

$$(\varphi\underline{\partial}^{\langle \underline{k} \rangle})(x) = \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \varphi(\underline{\partial}^{\langle \underline{h} \rangle} x)\underline{\partial}^{\langle \underline{k}-\underline{h} \rangle}. \tag{2.1.4.1.2}$$

(c) There exists on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})$ a unique structure of left $\mathcal{D}_X^{(m)}$-module such that for any open set $U \subset X$ such that U/S has coordinates, $z \in \Gamma(U,\mathcal{N})$, $\psi \colon \mathcal{N}|_U \to \mathcal{M}|_U$ we have

$$(\underline{\partial}^{\langle \underline{k} \rangle}\psi)(z) = \sum_{\underline{h}\leq \underline{k}} (-1)^{|\underline{k}-\underline{h}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \psi(z\underline{\partial}^{\langle \underline{h} \rangle})\underline{\partial}^{\langle \underline{k}-\underline{h} \rangle}. \tag{2.1.4.1.3}$$

(d) Let $\mathcal{D}$ be a sheaf of rings on X. If the structure of left $\mathcal{D}_{X/S}^{(m)}$-module of $\mathcal{F}$ (resp. $\mathcal{E}$) extends to a structure of $(\mathcal{D}_{X/S}^{(m)},\mathcal{D})$-bimodule then the structure of left $\mathcal{D}_{X/S}^{(m)}$-module of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ extends to a structure of (resp. left) $(\mathcal{D}_{X/S}^{(m)},\mathcal{D})$-bimodule. The left structure is the internal homomorphism structure. Moreover, if the structure of left (resp. right) $\mathcal{D}_{X/S}^{(m)}$-module of $\mathcal{E}$ (resp. $\mathcal{M}$) extends to a structure of $(\mathcal{D}_{X/S}^{(m)},\mathcal{D})$-bimodule (resp. $(\mathcal{D},\mathcal{D}_{X/S}^{(m)})$-bimodule), then the structure of right $\mathcal{D}_{X/S}^{(m)}$-module of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{M})$ extends to a structure of right $(\mathcal{D},\mathcal{D}_{X/S}^{(m)})$-bimodule (resp. $(\mathcal{D},\mathcal{D}_{X/S}^{(m)})$-bimodule). Moreover, if the structure of right $\mathcal{D}_{X/S}^{(m)}$-module of $\mathcal{N}$ (resp. of $\mathcal{M}$) extends to a structure of $(\mathcal{D},\mathcal{D}_{X/S}^{(m)})$-bimodule, then the structure of left $\mathcal{D}_{X/S}^{(m)}$-module of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})$ extends to a structure of (resp. left) $(\mathcal{D}_{X/S}^{(m)},\mathcal{D})$-bimodule.

In the same way, these structures of $\mathcal{D}_{X/S}^{(m)}$-module do not depend on the level m.

Proof. (a) i) We construct the isomorphisms $\varepsilon_n^{\mathcal{G}}$. Since $p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)}$ is a locally free $\mathcal{O}_X$-module, then the canonical morphism

$$p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{F})$$

is an isomorphism. This yields that the canonical morphism of $\mathcal{P}^n_{X/S(m)}$-modules

$$\alpha_i \colon p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \to \mathcal{H}om_{\mathcal{P}^n_{X/S(m)}}(p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{E}, p^n_{i,(m)*}\mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{F}) \tag{2.1.4.1.4}$$

given by $\tau \otimes \varphi \mapsto (\tau' \otimes x \mapsto \tau\tau' \otimes \varphi(x))$, is an isomorphism. Setting $\mathcal{G} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$, we denote by $\varepsilon_n^{\mathcal{G}}$ the isomorphism of $\mathcal{P}^n_{X/S(m)}$-modules making commutative the diagram

$$\begin{array}{ccc} \mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow[\sim]{\alpha_1} & \mathcal{H}om_{\mathcal{P}^n_{X/S(m)}}(\mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{F}) \\ \downarrow{\scriptstyle \varepsilon_n^{\mathcal{G}}} & & \downarrow{\scriptstyle \mathcal{H}om_{\mathcal{P}^n_{X/S(m)}}((\varepsilon_n^{\mathcal{E}})^{-1}, \epsilon_n^{\mathcal{F}})} \\ \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S(m)} & \xrightarrow[\sim]{\alpha_0} & \mathcal{H}om_{\mathcal{P}^n_{X/S(m)}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S(m)}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S(m)}). \end{array} \tag{2.1.4.1.5}$$

ii) Let us check the formula 2.1.4.1.1. Suppose X/S has coordinates. Let $\varphi \colon \mathcal{E} \to \mathcal{F}$ be a section of $\mathcal{G}$ and $\psi := \alpha_1(1 \otimes \varphi) \colon \mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{P}^n_{X/S(m)} \otimes_{\mathcal{O}_X} \mathcal{F}$ be the image of $1 \otimes \varphi$ by the canonical isomorphism

α_1 of 2.1.4.1.4. For any $\underline{k} \in \mathbb{N}^d$, for any $n \geq |\underline{k}|$, let us denote by $\underline{\partial}^{\langle \underline{k} \rangle}\varphi$ or by $\underline{\partial}^{\langle \underline{k} \rangle}(\varphi)$ the section of $\mathcal{G}$ such that we get

$$\varepsilon_n^{\mathcal{G}}(1 \otimes \varphi) = \sum_{|\underline{k}| \leq n} \underline{\partial}^{\langle \underline{k} \rangle}\varphi \otimes \underline{\tau}^{\{\underline{k}\}}.$$

The map $\alpha_0(\epsilon_n^{\mathcal{G}}(1 \otimes \varphi))$ is the unique $\mathcal{P}^n_{X/S(m)}$-linear morphism $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S(m)} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S(m)}$ given by

$$x \otimes 1 \mapsto \sum_{|\underline{k}| \leq n} \underline{\partial}^{\langle \underline{k} \rangle}\varphi \otimes \underline{\tau}^{\{\underline{k}\}},$$

for any section x of $\mathcal{E}$. On the other hand, $\alpha_0(\epsilon_n^{\mathcal{G}}(1 \otimes \varphi)) = \epsilon_n^{\mathcal{F}} \circ \psi \circ (\epsilon_n^{\mathcal{E}})^{-1}$ by definition (see 2.1.4.1.5). We compute

$$\epsilon_n^{\mathcal{F}} \circ \psi \circ (\varepsilon_n^{\mathcal{E}})^{-1}(x \otimes 1) \overset{2.1.\underline{1}.5.5}{=} \epsilon_n^{\mathcal{F}} \circ \psi \left(\sum_{|\underline{i}| \leq n} (-1)^{|\underline{i}|} \underline{\tau}^{\{\underline{i}\}} \otimes \underline{\partial}^{\langle \underline{i} \rangle} x \right) = \epsilon_n^{\mathcal{F}} \left(\sum_{|\underline{i}| \leq n} (-1)^{|\underline{i}|} \underline{\tau}^{\{\underline{i}\}} \otimes \varphi(\underline{\partial}^{\langle \underline{i} \rangle} x) \right)$$

$$\overset{2.1.\underline{1}.5.4}{=} \sum_{|\underline{i}| \leq n} \sum_{|\underline{j}| \leq n} (-1)^{|\underline{i}|} \underline{\partial}^{\langle \underline{j} \rangle}(\varphi(\underline{\partial}^{\langle \underline{i} \rangle} x)) \otimes \underline{\tau}^{\{\underline{j}\}} \underline{\tau}^{\{\underline{i}\}} = \sum_{|\underline{k}| \leq n} \left(\sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} (-1)^{|\underline{i}|} \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle}(\varphi(\underline{\partial}^{\langle \underline{i} \rangle} x)) \right) \otimes \underline{\tau}^{\{\underline{k}\}}$$

Hence, we get the formula 2.1.4.1.1.

iii) Finally the fact that the cocycle condition holds for the family $(\varepsilon_n^{\mathcal{G}})$ (an then is an m-PD-stratification relative to X/S) is ether a consequence by construction of that of the families $(\varepsilon_n^{\mathcal{E}})$ and $(\varepsilon_n^{\mathcal{F}})$ or since this is local is an easy computation using the formula 2.1.4.1.1 and the remark 2.1.1.6.

(b) i) We have the canonical isomorphisms

$$\beta_i \colon (p_i^n)^\flat(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})) \overset{4.6.3.9}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{O}_X}((p_i^n)^*(\mathcal{E}), \mathcal{M}) \overset{2.1.2.1.1}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_i^n)^*(\mathcal{E}), (p_i^n)^\flat(\mathcal{M})\right). \tag{2.1.4.1.6}$$

We set $\mathcal{H} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$ and we denote by $\epsilon_n^{\mathcal{H}}$ the $\mathcal{P}^n_{X/S,(m)}$-linear isomorphism making commutative the following diagram

$$\begin{array}{ccc} (p_0^n)^\flat(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})) & \underset{\beta_0}{\xrightarrow{\sim}} & \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_0^n)^*(\mathcal{E}), (p_0^n)^\flat(\mathcal{M})\right) \\ \downarrow{\scriptstyle \epsilon_n^{\mathcal{H}}} & & \downarrow{\scriptstyle \mathcal{H}om(\epsilon_n^{\mathcal{E}}, \epsilon_n^{\mathcal{M}})} \\ (p_1^n)^\flat(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})) & \underset{\beta_1}{\xrightarrow{\sim}} & \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_1^n)^*(\mathcal{E}), (p_1^n)^\flat(\mathcal{M})\right). \end{array} \tag{2.1.4.1.7}$$

ii) Let us check the formula 2.1.4.1.2. Let $\varphi \colon \mathcal{E} \to \mathcal{M}$ be a section of $\mathcal{H}$, $\underline{k} \in \mathbb{N}^r$ and $\underline{\partial}^{\langle \underline{k} \rangle} \otimes \varphi$ be the section of $(p_0^n)^\flat(\mathcal{H})$ (via its identification with $\mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S,n}$). We compute the morphism $\beta_0(\varphi \otimes \underline{\partial}^{\langle \underline{k} \rangle}) \colon (p_0^n)^*(\mathcal{E}) \to (p_0^n)^\flat(\mathcal{M})$ is the morphism given by

$$x \otimes \underline{\tau}^{\{\underline{i}\}} \mapsto (\underline{\tau}^{\{\underline{j}\}} \mapsto \underline{\partial}^{\langle \underline{k} \rangle}(\underline{\tau}^{\{\underline{i}\}} \underline{\tau}^{\{\underline{j}\}})\varphi(x)). \tag{2.1.4.1.8}$$

Hence, modulo the identification $(p_0^n)^\flat(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S,n}$, we get

$$\beta_0(\varphi \otimes \underline{\partial}^{\langle \underline{k} \rangle})(x \otimes \underline{\tau}^{\{\underline{i}\}}) = \begin{cases} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \varphi(x) \otimes \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle} & \text{if } \underline{i} \leq \underline{k} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.4.1.9}$$

Hence, we compute

$$\epsilon_n^{\mathcal{M}} \circ \beta_0(\varphi \otimes \underline{\partial}^{\langle \underline{k}\rangle}) \circ \epsilon_n^{\mathcal{E}}(\underline{\tau}^{\{\underline{i}\}} \otimes x) \overset{2.1.\underline{1}.5.4}{=} \epsilon_n^{\mathcal{M}} \circ \beta_0(\varphi \otimes \underline{\partial}^{\langle \underline{k}\rangle}) \left(\sum_{|\underline{h}|\leq n} \underline{\partial}^{\langle \underline{h}\rangle} x \otimes \underline{\tau}^{\{\underline{h}\}}\underline{\tau}^{\{\underline{i}\}} \right)$$

$$\overset{2.1.\underline{4}.1.13}{=} \epsilon_n^{\mathcal{M}} \left(\sum_{\underline{h}\leq \underline{k}-\underline{i}} \left\{ \begin{matrix} \underline{i}+\underline{h} \\ \underline{i} \end{matrix} \right\} \left\{ \begin{matrix} \underline{k} \\ \underline{i} \end{matrix} + \underline{h} \right\} \varphi(\underline{\partial}^{\langle \underline{h}\rangle} x) \otimes \underline{\partial}^{\langle \underline{k}-\underline{i}-\underline{h}\rangle} \right)$$

$$\overset{2.1.\underline{2}.9.2}{=} \sum_{\underline{h}\leq \underline{k}-\underline{i}} \sum_{\underline{j}\leq \underline{k}-\underline{i}-\underline{h}} \left\{ \begin{matrix} \underline{i}+\underline{h} \\ \underline{i} \end{matrix} \right\} \left\{ \begin{matrix} \underline{k} \\ \underline{i} \end{matrix} + \underline{h} \right\} \left\{ \begin{matrix} \underline{k}-\underline{i}-\underline{h} \\ \underline{j} \end{matrix} \right\} \underline{\partial}^{\star\langle \underline{j}\rangle} \otimes \varphi(\underline{\partial}^{\langle \underline{h}\rangle} x) \underline{\partial}^{\langle \underline{k}-\underline{i}-\underline{h}-\underline{j}\rangle}.$$

We have $\epsilon_n^{\mathcal{H}}(\varphi \otimes \underline{\partial}^{\langle \underline{k}\rangle}) = \beta_1^{-1}(\epsilon_n^{\mathcal{M}} \circ \beta_0(\varphi \otimes \underline{\partial}^{\langle \underline{k}\rangle}) \circ \epsilon_n^{\mathcal{E}})$ by definition (see 2.1.4.1.12). Moreover, we have the commutative diagram

$$\begin{array}{ccc} (p_1^n)^\flat(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{M})) & \xrightarrow[\beta_1]{\sim} & \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_1^n)^*(\mathcal{E}), (p_1^n)^\flat(\mathcal{M})\right) \\ \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \sim} \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{M}) & \xleftarrow[\mathrm{ev}_1]{} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},(p_1^n)^\flat(\mathcal{M})) \end{array} \tag{2.1.4.1.10}$$

where the right isomorphism is the canonical one. Hence, via the commutativity of 2.1.4.1.13, we get

$$x \cdot \underline{\partial}^{\langle \underline{k}\rangle} := \mathrm{ev}_1 \circ \varepsilon_n^{\mathcal{H}}(x \otimes \underline{\partial}^{\langle \underline{k}\rangle}) = \mathrm{ev}_1 \left(\epsilon_n^{\mathcal{M}} \circ \beta_0(\varphi \otimes \underline{\partial}^{\langle \underline{k}\rangle}) \circ \epsilon_n^{\mathcal{E}}(1 \otimes x) \right)$$

$$= \mathrm{ev}_1 \left(\sum_{\underline{h}\leq \underline{k}} \sum_{\underline{j}\leq \underline{k}-\underline{h}} \left\{ \begin{matrix} \underline{k} \\ \underline{h} \end{matrix} \right\} \left\{ \begin{matrix} \underline{k}-\underline{h} \\ \underline{j} \end{matrix} \right\} \underline{\partial}^{\star\langle \underline{j}\rangle} \otimes \varphi(\underline{\partial}^{\langle \underline{h}\rangle} x) \underline{\partial}^{\langle \underline{k}-\underline{h}-\underline{j}\rangle} \right) = \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{matrix} \underline{k} \\ \underline{h} \end{matrix} \right\} \varphi(\underline{\partial}^{\langle \underline{h}\rangle} x) \underline{\partial}^{\langle \underline{k}-\underline{h}\rangle}.$$

iii) Finally the fact that the cocycle condition holds for the family $(\varepsilon_n^{\mathcal{H}})$ (an then is an m-PD-stratification relative to X/S) is ether a consequence by construction of that of the families $(\varepsilon_n^{\mathcal{E}})$ and $(\varepsilon_n^{\mathcal{M}})$ or since this is local is an easy computation using the formula 2.1.4.1.2 and the remark 2.1.2.10.

(c) i) Set $\mathcal{K} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})$. By functoriality and extension, we get the canonical $\mathcal{P}^n_{X/S,(m)}$-linear homomorphisms

$$\gamma_i \colon (p_i^n)^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})) \to \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_i^n)^\flat(\mathcal{N}), (p_i^n)^\flat(\mathcal{M})\right), \tag{2.1.4.1.11}$$

given by $\phi \otimes \tau \mapsto (u \mapsto (\tau' \mapsto \phi \circ u(\tau\tau')))$ with u a section of $(p_i^n)^\flat(\mathcal{N})$, τ and τ' of $\mathcal{P}^n_{X/S,(m)}$. By compositing 2.1.4.1.11 with the isomorphisms

$$\mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_i^n)^\flat(\mathcal{N}), (p_i^n)^\flat(\mathcal{M})\right) \overset{2.1.2.1.1}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{O}_X}\left((p_i^n)^\flat(\mathcal{N}), \mathcal{M}\right)$$
$$\overset{2.1.2.6}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{M}\right)$$

we get the map given by $\phi \otimes \tau \mapsto ((P \otimes x) \mapsto \phi(P(\tau)x))$. This composition is equal to the composition of the isomorphisms

$$p_{i*}^n\mathcal{P}^n_{X,(m)} \otimes_{\mathcal{O}_X} (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{N}, \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X), \mathcal{M}\right)\right)$$
$$\overset{4.6.3.9}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{M}\right)$$

where first map is given by $\phi \otimes \tau \mapsto (x \mapsto (P \mapsto P(\tau)\phi(x)))$, for any $\tau \in \mathcal{P}^n_{X,(m)}$, $\phi \in \mathcal{H}$, $x \in \mathcal{N}$, $P \in \mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X)$ and is an isomorphism because the homomorphism $p_{i*}\mathcal{P}^n_{X,(m)} \to \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{H}om_{\mathcal{O}_X}(p_{i*}\mathcal{P}^n_{X,(m)}, \mathcal{O}_X), \mathcal{O}_X\right)$ is an isomorphism, we get that the canonical map 2.1.4.1.11 is an isomorphism.

We denote by $\epsilon_n^{\mathcal{K}}$ the $\mathcal{P}^n_{X/S,(m)}$-linear isomorphism making commutative the following diagram

$$\begin{array}{ccc} (p_1^n)^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})) & \xrightarrow[\sim]{2.1.4.1.12} & \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_1^n)^\flat(\mathcal{N}),(p_1^n)^\flat(\mathcal{M})\right) \\ \Big\downarrow \epsilon_n^{\mathcal{K}} & & \Big\downarrow \mathcal{H}om(\epsilon_n^{\mathcal{N}},(\epsilon_n^{\mathcal{M}})^{-1}) \\ (p_0^n)^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})) & \xrightarrow[\sim]{2.1.4.1.12} & \mathcal{H}om_{\mathcal{P}^n_{X/S,(m)}}\left((p_0^n)^\flat(\mathcal{N}),(p_0^n)^\flat(\mathcal{M})\right) \end{array} \tag{2.1.4.1.12}$$

ii) As above in (a) or (b), it is sufficient to check the formula 2.1.4.1.3. Suppose X/S has coordinates.

1) Let $\underline{k}\in\mathbb{N}^r$ and let $\psi_{\underline{k}}$ be a section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})$). Modulo the canonical isomorphism $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n}\xrightarrow{\sim}p^{n\flat}_{0,(m)}(\mathcal{M})$ of 2.1.2.6.1, with notation 2.1.4.1.11 the map $\gamma_0(\psi_{\underline{k}}\otimes\underline{\tau}^{\{\underline{k}\}})$ is the $\mathcal{P}^n_{X/S,(m)}$-linear homomorphism of the form $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n}\to p^{n\flat}_{0,(m)}(\mathcal{N})$ defined by the formula

$$z\otimes\underline{\partial}^{\langle\underline{i}\rangle}\mapsto(\underline{\tau}^{\{\underline{k}\}}\mapsto\psi_{\underline{k}}(z)\otimes\underline{\partial}^{\langle\underline{i}\rangle}(\underline{\tau}^{\{\underline{j}\}}\underline{\tau}^{\{\underline{k}\}})).$$

Hence, modulo the canonical isomorphism $\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X,n}\xrightarrow{\sim}p^{n\flat}_{0,(m)}(\mathcal{N})$, with 1.2.4.5.3 we compute

$$\gamma_0(\psi_{\underline{k}}\otimes\underline{\tau}^{\{\underline{k}\}})(z\otimes\underline{\partial}^{\langle\underline{i}\rangle})=\begin{cases}\left\{\begin{matrix}\underline{i}\\ \underline{k}\end{matrix}\right\}\psi_{\underline{k}}(z)\otimes\underline{\partial}^{\langle\underline{i}-\underline{k}\rangle} & \text{if } \underline{i}\geq\underline{k}\\ 0 & \text{otherwise.}\end{cases} \tag{2.1.4.1.13}$$

2) Let ψ be a section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})$). We have the formula

$$\gamma_0(\varepsilon_n^{\mathcal{K}}(1\otimes\psi))(z\otimes\underline{\partial}^{\langle\underline{i}\rangle})\overset{2.1.1.5.4}{=}\sum_{|\underline{k}|\leq n}\gamma_0(\underline{\partial}^{\langle\underline{k}\rangle}\psi\otimes\underline{\tau}^{\{\underline{k}\}})(z\otimes\underline{\partial}^{\langle\underline{i}\rangle})\overset{2.1.4.1.13}{=}\sum_{\underline{k}\leq\underline{i}}\left\{\begin{matrix}\underline{i}\\ \underline{k}\end{matrix}\right\}(\underline{\partial}^{\langle\underline{k}\rangle}\psi)(z)\otimes\underline{\partial}^{\langle\underline{i}-\underline{k}\rangle}. \tag{2.1.4.1.14}$$

Modulo the identifications of 2.1.2.6.2, we compute

$$\gamma_1(1\otimes\psi)=id\otimes\psi\colon\mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^n_{X,(m)},\mathcal{O}_X)\otimes_{\mathcal{O}_X}\mathcal{M}\to\mathcal{H}om_{\mathcal{O}_X}(p_{1*}\mathcal{P}^n_{X,(m)},\mathcal{O}_X)\otimes_{\mathcal{O}_X}\mathcal{N}.$$

Via the commutative diagram 2.1.4.1.12, $\gamma_0(\varepsilon_n^{\mathcal{K}}(1\otimes\psi))=(\epsilon_n^{\mathcal{M}})^{-1}\circ(id\otimes\psi)\circ\epsilon_n^{\mathcal{N}}$. We compute

$$(\epsilon_n^{\mathcal{M}})^{-1}\circ(id\otimes\psi)\circ\epsilon_n^{\mathcal{N}}(z\otimes\underline{\partial}^{\langle\underline{i}\rangle})\overset{2.1.2.9.2}{=}(\epsilon_n^{\mathcal{M}})^{-1}\circ(id\otimes\psi)(z\otimes\underline{\partial}^{\langle\underline{i}\rangle})=(\epsilon_n^{\mathcal{M}})^{-1}\circ(id\otimes\psi)(\sum_{\underline{h}\leq\underline{i}}\left\{\begin{matrix}\underline{i}\\ \underline{h}\end{matrix}\right\}\underline{\partial}^{\star\langle\underline{i}-\underline{h}\rangle}\otimes z\underline{\partial}^{\langle\underline{h}\rangle})$$

$$=\sum_{\underline{h}\leq\underline{i}}\left\{\begin{matrix}\underline{i}\\ \underline{h}\end{matrix}\right\}(\epsilon_n^{\mathcal{M}})^{-1}(\underline{\partial}^{\star\langle\underline{i}-\underline{h}\rangle}\otimes\psi(z\underline{\partial}^{\langle\underline{h}\rangle}))\overset{2.1.2.9.3}{=}\sum_{\underline{h}\leq\underline{i}}\sum_{0\leq\underline{i}-\underline{k}\leq\underline{i}-\underline{h}}(-1)^{|\underline{k}-\underline{h}|}\left\{\begin{matrix}\underline{i}\\ \underline{h}\end{matrix}\right\}\left\{\begin{matrix}\underline{i}-\underline{h}\\ \underline{i}-\underline{k}\end{matrix}\right\}(\psi(z\underline{\partial}^{\langle\underline{h}\rangle}))\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}\otimes\underline{\partial}^{\langle\underline{i}-\underline{k}\rangle}$$

$$=\sum_{\underline{k}\leq\underline{i}}\left\{\begin{matrix}\underline{i}\\ \underline{k}\end{matrix}\right\}\left(\sum_{\underline{h}\leq\underline{k}}(-1)^{|\underline{k}-\underline{h}|}\left\{\begin{matrix}\underline{k}\\ \underline{h}\end{matrix}\right\}(\psi(z\underline{\partial}^{\langle\underline{h}\rangle}))\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}\right)\otimes\underline{\partial}^{\langle\underline{i}-\underline{k}\rangle}. \tag{2.1.4.1.15}$$

It follows from 2.1.4.1.13 and 2.1.4.1.15 the equality 2.1.4.1.3.

(d) The functoriality in $\mathcal{E}$, $\mathcal{F}$, $\mathcal{M}$ or $\mathcal{N}$ can be checked by using these formulas. It comes from this functoriality (or again thanks to 2.1.4.1.1, 2.1.4.1.2 and 2.1.4.1.3) the part (d) of the proposition holds.

□

2.2 Exchanging right and left $\mathcal{D}$-modules

2.2.1 Structure of right $\mathcal{D}$-modules on ω_X

Let d be the relative dimension of the smooth morphism $f\colon X\to S$. We set $\omega_{X/S}:=\Omega^d_{X/S}$, which is often written as ω_X when S is understood. The sheaf ω_X plays an important role in $\mathcal{D}$-module theory - it is used to switch left and right modules, and therefore to define $\mathcal{D}$-linear duality.

2.2.1.1 (ω_X is a right $\mathcal{D}_X$-module: first case)**.** First, in order to be able to use the extraordinary inverse functor $g^!$ by an embeddable morphism g in the context of complexes with quasi-coherent cohomology (see [Har66, III Thm 8.7]), suppose S is locally noetherian. Beware, that this functor $g^!$ has not to be confused with the extraordinary inverse functor as complexes of $\mathcal{D}$-modules that we will define later. Since $f\colon X \to S$ is smooth, by construction of the functor $f^! = f^\sharp$, we get $\omega_{X/S} = f^!(\mathcal{O}_S[-d])$. Since $f \circ p^n_{0,(m)} = f \circ p^n_{1,(m)}$, by transitivity of the extraordinary inverse functor (as defined in [Har66, III Thm 8.7]), we get the canonical isomorphisms

$$\varepsilon_n \colon p^{n\flat}_{0,(m)}(\omega_X) \xrightarrow{\sim} p^{n!}_{0,(m)}(\omega_X) \xrightarrow{\sim} p^{n!}_{0,(m)}(f^!(\mathcal{O}_S[-d])) \xrightarrow{\sim} p^{n!}_{1,(m)}(f^!(\mathcal{O}_S[-d])) \xrightarrow{\sim} p^{n!}_{1,(m)}(\omega_X) = p^{n\flat}_{1,(m)}(\omega_X).$$

By using again the transitivity of the functor $f^!$, we check that the cocycle conditions are satisfied and therefore that the isomorphisms ε_n is a costratification. From Proposition 2.1.2.8, this gives ω_X the structure of a right $\mathcal{D}^{(m)}_X$-module.

Definition 2.2.1.2 (Adjoint operator)**.** Suppose X/S has coordinates $t_1, \dots, t_d$. For $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle} \in \Gamma(X, \mathcal{D}^{(m)}_X)$ with $a_{\underline{k}} \in \Gamma(X, \mathcal{O}_X)$, the *adjoint* of P is defined to be

$${}^t P = \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}^{\langle \underline{k} \rangle} a_{\underline{k}}.$$

Lemma 2.2.1.3. *Suppose X/S has coordinates $t_1, \dots, t_d$. Let P, Q be two differential operators of $\Gamma(X, \mathcal{D}^{(m)}_X)$. With notation 2.2.1.2, the following equalities hold: ${}^{\mathrm{t}}({}^{\mathrm{t}}P) = P$, ${}^{\mathrm{t}}(PQ) = {}^{\mathrm{t}}Q {}^{\mathrm{t}}P$.*

Proof. 1) Let us check ${}^{\mathrm{t}}({}^{\mathrm{t}}P) = P$. By additivity of $P \mapsto {}^{\mathrm{t}}P$, we reduce to the case where $P = a\underline{\partial}^{\langle \underline{k} \rangle}$ with $a \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k} \in \mathbb{N}^d$. We compute:

$$\begin{aligned}
{}^{\mathrm{t}}({}^{\mathrm{t}}P) &\overset{1.4.2.7.1}{=} {}^{\mathrm{t}}\left((-1)^{|\underline{k}|} \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k}-\underline{i} \rangle}(a) \underline{\partial}^{\langle \underline{i} \rangle} \right) = \sum_{\underline{i} \leq \underline{k}} (-1)^{|\underline{i}|+|\underline{k}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{i} \rangle} \cdot (\underline{\partial}^{\langle \underline{k}-\underline{i} \rangle}(a)) \\
&\overset{1.4.2.7.1}{=} \sum_{\underline{i} \leq \underline{k}} \sum_{\underline{j} \leq \underline{i}} (-1)^{|\underline{i}|+|\underline{k}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{i} \\ \underline{j} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{i}-\underline{j} \rangle}(\underline{\partial}^{\langle \underline{k}-\underline{i} \rangle}(a)) \underline{\partial}^{\langle \underline{j} \rangle} \\
&\overset{1.4.2.7.c}{=} \sum_{\underline{j} \leq \underline{k}} \sum_{\underline{j} \leq \underline{i} \leq \underline{k}} (-1)^{|\underline{i}|+|\underline{k}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{i} \\ \underline{j} \end{smallmatrix} \right\} \left\langle \begin{matrix} \underline{k}-\underline{j} \\ \underline{k}-\underline{i} \end{matrix} \right\rangle \underline{\partial}^{\langle \underline{k}-\underline{j} \rangle}(a) \underline{\partial}^{\langle \underline{j} \rangle} \\
&= \sum_{\underline{j} \leq \underline{k}} n_{\underline{k},\underline{j}} \underline{\partial}^{\langle \underline{k}-\underline{j} \rangle}(a) \underline{\partial}^{\langle \underline{j} \rangle} \sum_{\underline{j} \leq \underline{i} \leq \underline{k}} (-1)^{|\underline{k}|-|\underline{i}|} \binom{\underline{k}-\underline{j}}{\underline{k}-\underline{i}} = a\underline{\partial}^{\langle \underline{k} \rangle}. \qquad (2.2.1.3.1)
\end{aligned}$$

where $n_{\underline{k},\underline{j}}$ is a constant only depending on $\underline{k}$ and $\underline{j}$ such that $n_{\underline{k},\underline{k}} = 1$.

2) Let us check ${}^{\mathrm{t}}(PQ) = {}^{\mathrm{t}}Q {}^{\mathrm{t}}P$. i) When $P \in \Gamma(X, \mathcal{O}_X)$ the formula is obvious. Moreover, if follows from the involution of the transposition that we have

$${}^t\left(\sum_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle} a_{\underline{k}} \right) = \sum_{\underline{k}} (-1)^{|\underline{k}|} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle}.$$

Hence, the formula ${}^{\mathrm{t}}(PQ) = {}^{\mathrm{t}}Q {}^{\mathrm{t}}P$ is clear when $Q \in \Gamma(X, \mathcal{O}_X)$. ii) By additivity, we reduce to the case where $P = a\underline{\partial}^{\langle \underline{k} \rangle}$ and $Q = \underline{\partial}^{\langle \underline{l} \rangle} b$ with $a, b \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k}, \underline{l} \in \mathbb{N}^d$. From the step i), we can suppose moreover $a = 1$ and $b = 1$. Then we can conclude thanks to 1.4.2.7.c.

□

Theorem 2.2.1.4. *Suppose S is locally noetherian. Suppose X/S has coordinates $t_1, \dots, t_d$. For the structure of right $\mathcal{D}_X$-module on ω_X as defined in 2.2.1.1, for any $a \in \Gamma(U, \mathcal{O}_X)$, $P \in \Gamma(U, \mathcal{D}_X)$ we have therefore*

$$(adt_1 \wedge \cdots \wedge dt_d) \cdot P = ({}^t P \cdot a) dt_1 \wedge \cdots \wedge dt_d.$$

Proof. This is proved by Berthelot at [Ber00, 1.2.3]. □

Corollary 2.2.1.5. *There exists a unique structure of right $\mathcal{D}_X$-module on ω_X such that for any open set U of X such that U/S has coordinates $t_1, \dots, t_d$, for any $a \in \Gamma(U, \mathcal{O}_X)$, $P \in \Gamma(U, \mathcal{D}_X)$ we have*

$$(adt_1 \wedge \cdots \wedge dt_d) \cdot P = ({}^tP \cdot a)dt_1 \wedge \cdots \wedge dt_d. \tag{2.2.1.5.1}$$

Proof. By uniqueness, the corollary is local and we can suppose X and S affine. Let $S' \to S$ be a morphism of schemes and $X' := X \times_S S'$ and $\varpi \colon X' \to X$ be the projection. Since $\varpi^* \mathcal{D}_{X/S} = \mathcal{D}_{X'/S'}$, then if $\omega_{X/S}$ has a structure of right $\mathcal{D}_{X/S}$-module satisfying the formula 2.2.1.5.1, then $\omega_{X'/S'}$ has also a structure of right $\mathcal{D}_{X'/S'}$-module satisfying the formula 2.2.1.5.1. Hence, by passing to the limits we reduce to the case where S is noetherian, which was already checked in 2.2.1.4. □

2.2.1.6 (Left and right $\mathcal{D}$-modules.)**.** We describe here how to switch between right and left $\mathcal{D}_X^{(m)}$-modules. Let $\mathcal{E}$ be a left $\mathcal{D}_X^{(m)}$-module and $\mathcal{M}$ be a right $\mathcal{D}_X^{(m)}$-module.

(a) Since ω_X is a right $\mathcal{D}_X^{(m)}$-module and $\mathcal{E}$ is a left $\mathcal{D}_X^{(m)}$-module then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is endowed with a canonical structure of right $\mathcal{D}_X^{(m)}$-module extending its structure of $\mathcal{O}_X$-module (see 2.1.3.1.b).

Let U be an open subset of X such that U/S has coordinates $t_1, \dots, t_d$. It follows from the formulas 2.1.3.1.2 and 2.2.1.5 that for any $P \in \Gamma(U, \mathcal{D}_X^{(m)})$, $x \in \Gamma(U, \mathcal{E})$, denoting by $\omega_0 := dt_1 \wedge \cdots \wedge dt_d$ a basis of the free $\mathcal{O}_U$-module of rank one $\omega_{U/S}$, the right action of P on $\omega_0 \otimes x \in \Gamma(U, \omega_X \otimes_{\mathcal{O}_X} \mathcal{E})$ is given by

$$(\omega_0 \otimes x) \cdot P = \omega_0 \otimes ({}^tP \cdot x).$$

In other words, by making the (non-canonical) identification of $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ with $\mathcal{E}$ via the ω_0, the action of right $\mathcal{D}_X^{(m)}$-module on $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is given by the transposition (which definition depends on the choice of coordinates). We say that this structure of right $\mathcal{D}_X^{(m)}$-module of $\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}$ is its "twisted structure".

(b) Via the canonical isomorphism of $\mathcal{O}_X$-modules $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{M})$, since the right term as a canonical structure of left $\mathcal{D}_X^{(m)}$-module (see 2.1.4.1.c), we get on $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ a canonical structure of left $\mathcal{D}_X^{(m)}$-module.

Let U be an open subset of X such that U/S has coordinates $t_1, \dots, t_d$. Let $\omega_0^\vee = (dt_1 \wedge \cdots \wedge dt_d)^\vee$ be a basis of the free $\mathcal{O}_U$-module of rank one $\omega_{U/S}^{-1}$. It follows from the formulas 2.1.4.1.3 and 2.2.1.5 that for any $P \in \Gamma(U, \mathcal{D}_X^{(m)})$, $x \in \Gamma(U, \mathcal{M})$,

$$P \cdot (x \otimes \omega_0^\vee) = (x \cdot {}^tP) \otimes \omega_0^\vee.$$

In other words, by making the (non-canonical) identification $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ with $\mathcal{M}$ via the $\omega_0^\vee$, the action of let $\mathcal{D}_X^{(m)}$-module on $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ is given by the transposition (which definition depends on the choice of coordinates). We say that this structure of left $\mathcal{D}_X^{(m)}$-module of $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ is its "twisted structure".

(c) From the local description, since the transposition is an involution, we compute that the canonical isomorphism of $\mathcal{O}_X$-modules

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{O}_X} \omega_X^{-1} \xrightarrow{\sim} \mathcal{E}, \tag{2.2.1.6.1}$$

$$\omega_X \otimes_{\mathcal{O}_X} (\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \xrightarrow{\sim} \mathcal{M} \tag{2.2.1.6.2}$$

are $\mathcal{D}_X^{(m)}$-linear. This implies that the functor $\omega_X \otimes_{\mathcal{O}_X} -$ from the category of left $\mathcal{D}_X^{(m)}$-modules to the category of right $\mathcal{D}_X^{(m)}$-modules is an equivalence of categories with quasi-inverse $- \otimes_{\mathcal{O}_X} \omega_X^{-1}$. We refer to this as the side-changing operation.

2.2.2 Transposition isomorphisms

Proposition 2.2.2.1. *Let $\mathcal{E}$ be a left $\mathcal{D}_X^{(m)}$-module. Since $\mathcal{D}_X^{(m)}$ is a $\mathcal{D}_X^{(m)}$-bimodule, then it follows from 2.1.3.1.c that we get a structure of $\mathcal{D}_X^{(m)}$-bimodule on $\mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ (resp. $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)}$), where the tensor product is computed with the structure of right $\mathcal{D}_X^{(m)}$-module (resp. left $\mathcal{D}_X^{(m)}$-module) of $\mathcal{D}_X^{(m)}$.*

Let $\mathcal{M}$ be a right $\mathcal{D}_X^{(m)}$-module. It follows from 2.1.3.1.c that we get a structure of right $\mathcal{D}_X^{(m)}$-bimodule on $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}$, where the tensor product is computed with the structure of left $\mathcal{D}_X^{(m)}$-module of $\mathcal{D}_X^{(m)}$. The structure of right $\mathcal{D}_X^{(m)}$-module on $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}$ given by the tensor product is said to be the right one, the other one is the left structure.

(a) There is a unique $\mathcal{D}_X^{(m)}$-bimodule isomorphism

$$\gamma_{\mathcal{E}}\colon \mathcal{D}_X^{(m)}\otimes_{\mathcal{O}_X}\mathcal{E}\xrightarrow{\sim}\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)} \tag{2.2.2.1.1}$$

such that $\gamma_{\mathcal{E}}(1\otimes e)=e\otimes 1$ for any section e of $\mathcal{E}$. In local coordinates we have

$$\gamma_{\mathcal{E}}(\underline{\partial}^{\langle\underline{k}\rangle}\otimes e)=\sum_{\underline{h}\le\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} y\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}e\otimes\underline{\partial}^{\langle\underline{h}\rangle}.$$

(b) There exists a unique involution

$$\delta_{\mathcal{M}}\colon \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}\to\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}$$

exchanging both structures of right $\mathcal{D}_X^{(m)}$-modules such that $\delta_{\mathcal{M}}(x\otimes 1)=x\otimes 1$ for any section x of $\mathcal{M}$. In local coordinates we have

$$\delta_{\mathcal{M}}(x\otimes\underline{\partial}^{\langle\underline{k}\rangle})=\sum_{\underline{h}\le\underline{k}}(-1)^{|\underline{h}|}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} x\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}\otimes\underline{\partial}^{\langle\underline{h}\rangle}.$$

The map $\gamma_{\mathcal{E}}$ (resp. $\delta_{\mathcal{M}}$) is said to be the transposition isomorphism of $\mathcal{E}$ (resp. $\mathcal{M}$).

Proof. (a) The canonical $\mathcal{O}_X$-linear homomorphism $\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}$ given by $e\mapsto e\otimes 1$ induces by extension $\gamma_{\mathcal{E}}$ which is a homomorphism of left $\mathcal{D}_X^{(m)}$-modules. Next we need to see that $\gamma_{\mathcal{E}}$ is right $\mathcal{D}_X^{(m)}$-linear, i.e. $\gamma_{\mathcal{E}}((1\otimes e)Q)=\gamma_{\mathcal{E}}(1\otimes e)Q$ for any section Q of $\mathcal{D}_X^{(m)}$. Since this is clear by $\mathcal{O}_X$-linearity of $\gamma_{\mathcal{E}}$ when $Q\in\mathcal{O}_X$, then we reduce to the case where $Q=\underline{\partial}^{\langle\underline{k}\rangle}$. We do a calculation:

$$\begin{aligned}
&\gamma_{\mathcal{E}}((1\otimes e)\underline{\partial}^{\langle\underline{k}\rangle})\\
&=\gamma_{\mathcal{E}}\left(\sum_{\underline{h}\le\underline{k}}(-1)^{|\underline{k}-\underline{h}|}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\underline{\partial}^{\langle\underline{h}\rangle}\otimes\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}e\right)\ \text{by (2.1.3.1.2)}\\
&=\sum_{\underline{h}\le\underline{k}}(-1)^{|\underline{k}-\underline{h}|}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\underline{\partial}^{\langle\underline{h}\rangle}(\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}e\otimes 1)\ \text{by left } \mathcal{D}_X^{(m)} \text{ linearity of } \gamma_{\mathcal{E}}\\
&=\sum_{\underline{h}\le\underline{k}}(-1)^{|\underline{k}-\underline{h}|}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\left(\sum_{\underline{i}\le\underline{h}}\left\{\begin{matrix}\underline{h}\\ \underline{i}\end{matrix}\right\}\underline{\partial}^{\langle\underline{h}-\underline{i}\rangle}\underline{\partial}^{\langle\underline{k}-\underline{h}\rangle}e\otimes\underline{\partial}^{\langle\underline{i}\rangle}\right)\ \text{by (2.1.3.1.1)}\\
&=\sum_{\underline{i}\le\underline{k}}\left(\sum_{\underline{i}\le\underline{h}\le\underline{k}}(-1)^{|\underline{k}-\underline{h}|}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\left\{\begin{matrix}\underline{h}\\ \underline{i}\end{matrix}\right\}\left\langle\begin{matrix}\underline{k}-\underline{i}\\ \underline{h}-\underline{i}\end{matrix}\right\rangle\right)\underline{\partial}^{\langle\underline{k}-\underline{i}\rangle}e\otimes\underline{\partial}^{\langle\underline{i}\rangle}\ \text{by proposition 1.4.2.7}\\
&=e\otimes\underline{\partial}^{\langle\underline{k}\rangle}\ \text{similarly to the computation 2.2.1.3.1.}
\end{aligned}$$

By extension from the $\mathcal{O}_X$-linear homomorphism $\mathcal{E}\to\mathcal{D}_X^{(m)}\otimes_{\mathcal{O}_X}\mathcal{E}$ given by $e\mapsto 1\otimes e$ the homomorphism of right $\mathcal{D}_X^{(m)}$-modules $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}\to\mathcal{D}_X^{(m)}\otimes_{\mathcal{O}_X}\mathcal{E}$ given by $e\otimes P\mapsto(1\otimes e)P$. This map gives an inverse of $\gamma_{\mathcal{E}}$, in particular this is an isomorphism.

(b) Similarly, the canonical $\mathcal{O}_X$-linear (with the $\mathcal{O}_X$-module structure on the target coming from the left structure of right $\mathcal{D}_X^{(m)}$-module) homomorphism $\mathcal{M}\to\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}$ given by $x\mapsto x\otimes 1$ induces the right $\mathcal{D}_X^{(m)}$-linear homomorphism $\delta_{\mathcal{M}}\colon\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}\to\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_X^{(m)}$ (for the right structure for the source and the left structure for the target). By doing a similar computation, we check the second right $\mathcal{D}_X^{(m)}$-linearity (i.e. for the left structure for the source and the right structure for the target) and the local formula. □

Example 2.2.2.2. From proposition 2.2.2.1.b we have the canonical isomorphism of right $\mathcal{D}_X^{(m)}$-bimodules

$$\delta_X : \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)} \xrightarrow{\sim} \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)} \tag{2.2.2.2.1}$$

which is an involution exchanging the two right $\mathcal{D}_X^{(m)}$-module structures. By tensoring with $\otimes_{\mathcal{O}_X} \omega_X^{-1}$ for the right (resp. left) structure of right $\mathcal{D}_X^{(m)}$ of the source (resp. target) of δ_X, we get the isomorphism of $\mathcal{D}_X^{(m)}$-bimodules

$$\alpha_X : \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \omega_X^{-1} \xrightarrow{\sim} \mathcal{D}_X^{(m)}. \tag{2.2.2.2.2}$$

By tensoring α_X with $\otimes_{\mathcal{O}_X} \omega_X^{-1}$, this yields the isomorphism of left $\mathcal{D}_X^{(m)}$-bimodules

$$\beta_X : \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \omega_X^{-1} \xrightarrow{\sim} \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \omega_X^{-1}$$

which is an involution exchanging the two left $\mathcal{D}_X^{(m)}$-module structures.

2.3 Level 0 case

2.3.1 Stratifications of order ≤ 1 and connections

Definition 2.3.1.1. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. With notations 1.4.1.2 and 1.4.1.14, an *m-PD stratification relative to X/S of order ≤ 1* is the data of a $\mathcal{P}^1_{X/S,(m)}$-linear isomorphism $\varepsilon_1 \colon \mathcal{P}^1_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S,(m)}$ such that the following diagram

$$\begin{array}{ccccc}
\psi^{1,0*}_{X/S,(m)}(\mathcal{P}^1_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) & \xrightarrow{\sim} & \mathcal{P}^0_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\sim} & \mathcal{E}\ , \\
\downarrow{\scriptstyle \psi^{1,0*}_{X/S,(m)}(\varepsilon_1)} & & \downarrow{\scriptstyle \sim} & & \| \\
\psi^{n',n*}_{X/S,(m)}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S,(m)}) & \xrightarrow{\sim} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^0_{X/S,(m)} & \xrightarrow{\sim} & \mathcal{E}
\end{array} \tag{2.3.1.1.1}$$

whose horizontal isomorphisms are the canonical ones, is commutative. When $m = \infty$, we simply say *stratification relative to X/S of order ≤ 1*.

Remark 2.3.1.2. We have a canonical map from the set of m-PD stratifications relative to X/S of $\mathcal{E}$ (see 2.1.1.1) to that of m-PD stratifications relative to X/S of order ≤ 1 given by $(\varepsilon_n)_{n\in\mathbb{N}} \mapsto \varepsilon_1$.

Proposition 2.3.1.3. *Given an $\mathcal{O}_X$-module $\mathcal{E}$. The following are equivalent.*

(a) An $\mathcal{O}_X$-linear homomorphism $\theta_1 \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(m)})$ (the $\mathcal{O}_X$-module structure of this latter is induced by the right structure of $\mathcal{P}^1_{X,(m)}$) making commutative the diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\sim} & \mathcal{E} \otimes \mathcal{P}^0_{X/S,(m)} \\
& {\scriptstyle \theta_1} \searrow & \uparrow{\scriptstyle \mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S,(m)}} \\
& & \mathcal{E} \otimes \mathcal{P}^1_{X/S,(m)}
\end{array} \tag{2.3.1.3.1}$$

whose top isomorphism is the canonical one (recall $\mathcal{P}^0_{X/S,(m)} = \mathcal{O}_X$).

(b) An m-PD stratification $\varepsilon = (\varepsilon_n^{\mathcal{E}})$ of order ≤ 1 on $\mathcal{E}$.

The bijection between both data is given by $\epsilon_1 \mapsto \epsilon_1 \circ p^1_{1,\mathcal{E}} = \theta_1$.

Proof. This comes from the proof of 2.1.1.5. □

2.3.1.4. When m is an integer, it follows from the local description of 1.1.2.3 and 1.4.1.5 that the canonical homomorphism $\psi^1_m \colon \mathcal{P}^1_{X/S,(m)} \to \mathcal{P}^1_{X/S}$ of 1.4.1.11 is an isomorphism. Hence, the data of a stratification relative to X/S of order ≤ 1 on $\mathcal{E}$ is equivalent to that of an m-PD stratification relative to X/S of order ≤ 1 on $\mathcal{E}$. In other words, the integer m has no importance and by default we simply consider stratifications relative to X/S of order ≤ 1.

Definition 2.3.1.5. A *connection relative to X/S* on an $\mathcal{O}_X$-module $\mathcal{E}$ is an additive map $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ such that for any open $U \subseteq X$, for any $x \in \mathcal{E}(U)$, $a \in \mathcal{O}_X(U)$ we have

$$\nabla(ax) = a\nabla x + x \otimes da$$

where $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$ is the constant $\mathcal{O}_S$-derivation.

Proposition 2.3.1.6. *Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. The map $\epsilon_1 \mapsto (\mathrm{id} \otimes \varpi) \circ \epsilon_1 \circ p^1_{1,\mathcal{E}}$, where $p^1_{1,\mathcal{E}}$ is defined at 1.1.3.1, gives a bijection between the set of connections relative to X/S on $\mathcal{E}$ to that of stratifications relative to X/S of order ≤ 1 on $\mathcal{E}$.*

Proof. 1) Let ϵ_1 be a stratification relative to X/S of order ≤ 1 on $\mathcal{E}$. Following 2.3.1.3, it corresponds to ε_1 a homomorphism $\theta_1 := \epsilon_1 \circ p^1_{1,\mathcal{E}} \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S})$ (the $\mathcal{O}_X$-module structure of this latter is induced by the right structure of $\mathcal{P}^1_{X/S}$) making commutative the diagram 2.3.1.3.1.

2) To simplify notation, we do not write the canonical isomorphisms of the form $\mathcal{E} \xrightarrow{\sim} \mathcal{P}^0_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ or $\mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^0_{X/S,(m)}$. i) Let $\theta_1 \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S,(m)}$ be an additive map making commutative the diagram 2.3.1.3.1. We get the additive map $\nabla := (\mathrm{id}_{\mathcal{E}} \otimes \varpi) \circ \theta_1$. Since $\varpi = \mathrm{id} - p^1_0 \circ \psi^{1,0}_{X/S}$, then we compute $\nabla = \theta_1 - (\mathrm{id}_{\mathcal{E}} \otimes p^1_0 \circ \psi^{1,0}_{X/S}) \circ \theta_1 \overset{2.3.1.3.1}{=} \theta_1 - \mathrm{id}_{\mathcal{E}} \otimes p^1_0$. Hence

$$(\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S}) \circ \nabla = (\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S}) \circ \theta_1 - (\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S}) \circ (\mathrm{id}_{\mathcal{E}} \otimes p^1_0) \overset{2.3.1.3.1}{=} \mathrm{id}_{\mathcal{E}} - (\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S} \circ p^1_0) = \mathrm{id}_{\mathcal{E}} - \mathrm{id}_{\mathcal{E}} = 0.$$

Hence, ∇ factors through $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S}$.

ii) Conversely let $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ be an additive map. Set $\theta_1 := \nabla + \mathrm{id}_{\mathcal{E}} \otimes p^1_0$. Since $\Omega^1_{X/S} = \ker \psi^{1,0}_{X/S}$, then we compute

$$(\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S}) \circ \theta_1 = (\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S}) \circ (\mathrm{id}_{\mathcal{E}} \otimes p^1_0) = (\mathrm{id}_{\mathcal{E}} \otimes \psi^{1,0}_{X/S} \circ p^1_0) = \mathrm{id}_{\mathcal{E}},$$

which means that the diagram 2.3.1.3.1 is commutative.

iii) Hence we get a bijection between the set of additive maps $\theta_1 \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S,(m)}$ making commutative the diagram 2.3.1.3.1 and that of additive maps $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$. It remains to check that θ_1 is $\mathcal{O}_X$-linear (the $\mathcal{O}_X$-module structure of this latter is induced by the right structure of $\mathcal{P}^1_{X/S}$) if and only if ∇ is a connection, which is checked in the following last step.

3) Let a be a section of $\mathcal{O}_X$ and x of $\mathcal{E}$. Denoting by $\mathbb{1}$ the unit element of $\mathcal{P}^1_{X/S}$, we compute in $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S}$:

$$\nabla(ax) = \theta_1(ax) - ax \otimes \mathbb{1}. \tag{2.3.1.6.1}$$

On the other hand, we get two sections of $\mathcal{P}^1_{X/S}$ by setting $a \otimes 1 := p^1_0(a)$ and $1 \otimes a := p^1_1(a)$. Since $\nabla(x) \in \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$, then $a\nabla(x) = (1 \otimes a)\nabla(x)$. Hence, we compute in $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S}$:

$$a\nabla(x) + x \otimes da = (1 \otimes a)\,(\theta_1(x) - x \otimes \mathbb{1}) + x \otimes (1 \otimes a - a \otimes 1) = (1 \otimes a)\theta_1(x) - ax \otimes \mathbb{1}. \tag{2.3.1.6.2}$$

Hence, with 2.3.1.6.1 and 2.3.1.6.2, we get that ∇ is a connection if and only if θ_1 is $\mathcal{O}_X$-linear (for the right structure of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S,(m)}$).

□

2.3.2 Integrable connections and $\mathcal{D}$-modules of level 0

2.3.2.1. Let $\Omega^i_{X/S} = \wedge^i_{\mathcal{O}_X} \Omega_{X/S}$ for $i \geq 0$ be the ith exterior power. For any integers i, j, denote by $c_{i,j} \colon \Omega^i_{X/S} \otimes_{\mathcal{O}_X} \Omega^j_{X/S} \to \Omega^{i+j}_{X/S}$ the canonical projection. Recall the constant $\mathcal{O}_S$-derivation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$ (see 1.1.2.6). Moreover, (e.g. see [Gro67, IV.16.6.2]), for any $i \geq 1$, there is a unique homomorphism of $f^{-1}\mathcal{O}_S$-modules $d^i \colon \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ such that $d^i(a_0 da_1 \wedge \cdots \wedge da_i) = da_0 \wedge da_1 \wedge \cdots \wedge da_i$ where $a_0, \ldots, a_i$ are local sections of $\mathcal{O}_X$. In degree 0, remark $d^0 = d$, i.e. we retrieve the constant $\mathcal{O}_S$-derivation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$. We get the complex of sheaves of $f^{-1}\mathcal{O}_S$-modules

$$0 \to \Omega^0_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

which is called the de Rham complex of X/S.

We compute $d^i(aw) - ad^i(\omega) = da \wedge \omega$ for any section a of $\mathcal{O}_X$ and section $\omega = a_0 da_1 \wedge \cdots \wedge da_i$ of $\Omega^i_{X/S}$. Hence, $\omega \mapsto d^i(aw) - ad^i(\omega)$ is linear, i.e. the map d^i is a differential operator of order 1 (see 1.1.3.4.1). By definition, there exists a unique $\mathcal{O}_X$-linear homomorphism $\varpi^i \colon \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ whose composition with $p^1_{1,\Omega^i_{X/S}} \colon \Omega^i_{X/S} \to \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ is equal to d^i. We compute

$$\varpi^i(da \otimes \omega) = \varpi^i(\mathbb{1} \otimes a\omega) - a\varpi^i(\mathbb{1} \otimes \omega) = d^i(aw) - ad^i(\omega) = da \wedge \omega, \qquad (2.3.2.1.1)$$

for any section a of $\mathcal{O}_X$ and section ω of $\Omega^i_{X/S}$. The formula 2.3.2.1.1 means that the restriction of ϖ^i on $\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ is the canonical map $c_{i,i+1}$.

Notation 2.3.2.2. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module and $i, j \geq 0$ be two integers. Let y (resp. z) be a section of $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ (resp. $\Omega^i_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$) and ω be a section of $\Omega^j_{X/S}$. Then we denote by $y \wedge \omega$ (resp. $\omega \wedge z$) the image of $y \otimes \omega$ (resp. $\omega \otimes z$) via $\mathrm{id}_{\mathcal{E}} \otimes c_{i,j}$ (resp. $c_{i,j} \otimes \mathrm{id}_{\mathcal{E}}$). Denote by $y \mapsto {}^t y$ the canonical isomorphism $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \xrightarrow{\sim} \Omega^i_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$. Beware ${}^t(y \wedge \omega) = (-1)^{ij} \omega \wedge {}^t y$.

2.3.2.3 (Integrable connection, de Rham complex)**.** Let $\mathcal{E}$ be an $\mathcal{O}_X$-module endowed with a connection. Denote by ϵ_1 the associated relative to X/S stratification of order ≤ 1 on $\mathcal{E}$ (see 2.3.1.6). Following 2.3.1.3, let $\theta_1 \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X$ be the corresponding $\mathcal{O}_X$-linear homomorphism making commutative the diagram 2.3.1.3.1. We get the map $\nabla^i \colon \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}$ from θ_1 and d^i by composition as follows

$$\begin{array}{ccccc}
\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} & \xrightarrow{\theta_1 \otimes \mathrm{id}_{\Omega^i_{X/S}}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} & \xrightarrow{\mathrm{id}_{\mathcal{E}} \otimes \varpi^i} & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}, \\
& \searrow^{p^1_{1,\mathcal{E}\otimes\Omega^i}} & \uparrow{\scriptstyle \epsilon_1 \otimes \mathrm{id}} & & \\
& & \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} & &
\end{array} \qquad (2.3.2.3.1)$$

where the triangle is commutative. Beware that the terms $\mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ have two structures of $\mathcal{O}_X$-modules: the left one and the right one. The morphisms $p^1_{1,\mathcal{E}\otimes\Omega^i}$ and $\theta_1 \otimes \mathrm{id}_{\Omega^i_{X/S}}$ are $\mathcal{O}_X$-linear for the right one, whereas $\mathrm{id}_{\mathcal{E}} \otimes \varpi^i$ is $\mathcal{O}_X$-linear for the left one. The composition ∇^i is not $\mathcal{O}_X$-linear but since $\epsilon_1 \otimes \mathrm{id}$ is $\mathcal{O}_X$-linear for both structure (and in particular for the left structure), then $\nabla^i \colon \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}$ is a differential operator of order 1.

For any sections y of $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ and ω of $\Omega^i_{X/S}$, it follows from the computation of 2.3.2.1 and the notation 2.3.2.2 that we have $\mathrm{id}_{\mathcal{E}} \otimes \varpi^i(y \otimes \omega) = y \wedge \omega$. Since $\theta_1(x) = \nabla(x) + x \otimes \mathbb{1}$ for any section x of $\mathcal{E}$ and ω of $\Omega^i_{X/S}$, then we compute

$$\nabla^i(x \otimes \omega) = \mathrm{id}_{\mathcal{E}} \otimes \varpi^i(\theta_1(x) \otimes \omega) = \mathrm{id}_{\mathcal{E}} \otimes \varpi^i(\nabla(x) \otimes \omega) + \mathrm{id}_{\mathcal{E}} \otimes \varpi^i(x \otimes \mathbb{1} \otimes \omega) = \nabla(x) \wedge \omega + x \otimes d^i(\omega). \qquad (2.3.2.3.2)$$

More generally, for any sections y_i of $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ and ω_j of $\Omega^j_{X/S}$, with notation 2.3.2.2, it follows from 2.3.2.3.2 the formula:

$$\nabla^{i+j}(y_i \wedge \omega_j) = \nabla^i(y_i) \wedge \omega_j + (-1)^i y_i \wedge d^j(\omega_j). \qquad (2.3.2.3.3)$$

Indeed, by additivity, we can suppose y_i is of the form $y_i = x \otimes \omega_i$, where x is a section of $\mathcal{O}_X$ and ω_i is a section of $\Omega^i_{X/S}$. We compute

$$\nabla^{i+j}(y_i \wedge \omega_j) = \nabla^{i+j}(x \otimes (\omega_i \wedge \omega_j)) \overset{2.3.2.3.2}{=} \nabla(x) \wedge (\omega_i \wedge \omega_j) + x \otimes d^{i+j}(\omega_i \wedge \omega_j)$$
$$= (\nabla(x) \wedge \omega_i) \wedge \omega_j + x \otimes (d^i(\omega_i) \wedge \omega_j + (-1)^i \omega_i \wedge d^j(\omega_j)) \overset{2.3.2.3.2}{=} \nabla^i(y_i) \wedge \omega_j + (-1)^i y_i \wedge d^j(\omega_j).$$

We say that the connection ∇ of $\mathcal{E}$ is integrable if $\nabla^1 \circ \nabla = 0$. In that case $\nabla^{i+1} \circ \nabla^i = 0$ for any i and we get the complex

$$0 \to \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/S} \to \cdots \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^d_{X/S} \to 0, \qquad (2.3.2.3.4)$$

which is called the de Rham complex.

2.3.2.4 (de Rham complex: exchanging the position)**.** Let $\mathcal{E}$ be an $\mathcal{O}_X$-module endowed with a connection. To define the map $\nabla^i : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}$ (see 2.3.2.3.1), it was more natural to put $\mathcal{E}$ on the left. However, when $\mathcal{E} = \mathcal{D}^{(0)}_{X/S}$ for instance, it is much better (to avoid confusion with the right structure of $\mathcal{O}_X$-module of $\mathcal{D}^{(0)}_{X/S}$) to put $\mathcal{E}$ on the right (see 2.3.3.10.2). Moreover, when the connection is integral, since $\mathcal{E}$ can be viewed as a left $\mathcal{D}^{(0)}_{X/S}$-module (see later 2.3.2.6), one might prefer to put $\mathcal{E}$ on the right. In other words, we define $\nabla^i : \Omega^i_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{i+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ to be the $\mathcal{O}_S$-linear map making commutative the diagram

$$\begin{array}{ccc} \Omega^i_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\nabla^i} & \Omega^{i+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S} & \xrightarrow[2.3.2.3.1]{\nabla^i} & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/S}, \end{array} \tag{2.3.2.4.1}$$

where the vertical isomorphism are the canonical ones. Via the formulas given at 2.3.2.2 and 2.3.2.3.3, for any sections z_i of $\Omega^i_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ and ω_j of $\Omega^j_{X/S}$, with notation 2.3.2.2, we get

$$\nabla^{i+j}(\omega_j \wedge z_i) = (-1)^j \omega_j \wedge \nabla^i(z_i) + d^j(\omega_j) \wedge z_i. \tag{2.3.2.4.2}$$

When the connection is integrable, by convention we denote by $\mathrm{DR}(\mathcal{E})$ the de Rham complex of $\mathcal{E}$, with $\mathcal{E}$ on the right side.

2.3.2.5. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module endowed with a connection. Suppose X/S has coordinates $t_1, \dots, t_d$. For any section x of $\mathcal{E}$, we denote by $\partial_1(x), \dots, \partial_d(x)$ the elements of $\mathcal{E}$ such that

$$\nabla(x) = \sum_{i=1}^d \partial_i(x) \otimes dt_i. \tag{2.3.2.5.1}$$

Using 2.3.2.3.2, we compute:

$$\nabla^1 \circ \nabla(x) = \nabla^1 (\sum_{i=1}^d \partial_i(x) \otimes dt_i) = \sum_{j=1}^d \sum_{i=1}^d \partial_j(\partial_i(x)) \otimes dt_j \wedge dt_i.$$

Hence, the connection is integral is equivalent to saying that the maps $\partial_i \in \mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$ commute two by two, i.e., for any $i, j \in \{1, \dots, d\}$, for any section x of $\mathcal{E}$ we have the equality:

$$\partial_j(\partial_i(x)) = \partial_i(\partial_j(x)).$$

For any $\underline{k} \in \mathbb{N}^d$, we get the $\mathcal{O}_S$-linear map $\underline{\partial}^{\underline{k}} : \mathcal{E} \to \mathcal{E}$ by setting $\underline{\partial}^{\underline{k}} = \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}$. When, the connection is integrable, for any $\underline{k}, \underline{l} \in \mathbb{N}^d$, we get the equality

$$\underline{\partial}^{\underline{l}} \circ \underline{\partial}^{\underline{k}} = \underline{\partial}^{\underline{k}} \circ \underline{\partial}^{\underline{l}} = \underline{\partial}^{\underline{k}+\underline{l}}. \tag{2.3.2.5.2}$$

Theorem 2.3.2.6. *Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. The following are equivalent.*

(a) An integrable connection relative to X/S on $\mathcal{E}$.

(b) A structure of left $\mathcal{D}^{(0)}_{X/S}$-module on $\mathcal{E}$ extending its structure of $\mathcal{O}_X$-module.

Proof. 0) Recall the data of a structure of left $\mathcal{D}^{(0)}_{X/S}$-module on $\mathcal{E}$ extending its structure of $\mathcal{O}_X$-module is equivalent to that of a 0-PD-stratification relative to X/S on $\mathcal{E}$.

1) Let $(\epsilon_n)_{n \in \mathbb{N}}$ be a 0-PD-stratification relative to X/S on $\mathcal{E}$. We check that the connection $\nabla := (\mathrm{id} \otimes \varpi) \circ \epsilon_1 \circ p^1_{1,\mathcal{E}}$ where $p^1_{1,\mathcal{E}}$ is defined at 2.3.1.3 is integrable as follows. Since this is local, we can suppose X/S has coordinates $t_1, \dots, t_d$. Let $\theta_1 := \epsilon_1 \circ p^1_{1,\mathcal{E}}$. Following the proof of 2.3.1.6, we have $\nabla = \theta_1 - \mathrm{id}_{\mathcal{E}} \otimes p^1_0$. Following 2.1.1.5.4, we get $\theta_1(x) = \sum_{i=1}^d \partial_i(x) \otimes \tau_i + x \otimes \mathbb{1}$. Since $\tau_i = dt_i$ in $\mathcal{P}^1_{X/S}$, then we get $\nabla(x) = \sum_{i=1}^d \partial_i(x) \otimes dt_i$, which justifies the notation of 2.3.2.5.1 concerning the definition of

$\partial_i(x)$ from the connection. Following 2.3.2.5, ∇ is integrable if and only if $\partial_j(\partial_i(x)) = \partial_i(\partial_j(x))$, which is obvious since $\partial_i\partial_j = \partial_j\partial_i$ in $\mathcal{D}^{(0)}_{X/S}$.

2) From the step 1), we get the map ψ sending a integrable connection relative to X/S on $\mathcal{E}$ to a 0-PD-stratification relative to X/S on $\mathcal{E}$ given by $\psi((\epsilon_n)_{n\in\mathbb{N}}) = (\mathrm{id}\otimes\varpi)\circ\epsilon_1\circ p^1_{1,\mathcal{E}}$. It remain check that ψ is bijective. Since this is local, then we can suppose X/S has coordinates $t_1,\dots,t_d$. Since the connection determines the action of $\partial_1,\dots,\partial_d$, since the ring $\mathcal{D}^{(0)}_{X/S}$ is generated as $\mathcal{O}_S$-algebra by $\mathcal{O}_X$ and by $\partial_1,\dots,\partial_d$, then we get the injectivity of ψ. Let us now check the surjectivity. Let ∇ be an integrable connection. With notation 2.3.2.5, for any $\underline{k}\in\mathbb{N}^d$, we get the $\mathcal{O}_S$-linear map $\underline{\partial}^{\underline{k}}\colon\mathcal{E}\to\mathcal{E}$. For any $P\in\mathcal{D}^{(0)}_{X/S}$, we define the $\mathcal{O}_S$-linear map $P\colon\mathcal{E}\to\mathcal{E}$ as follows. We can uniquely write P as a finite sum of the form $P=\sum_{\underline{k}} a_{\underline{k}}\underline{\partial}^{\underline{k}}$, where $a_{\underline{k}}$ is a section of $\mathcal{O}_X$. Hence, we set $P(x) := \sum_{\underline{k}} a_{\underline{k}}(\underline{\partial}^{\underline{k}}(x))$.

Let $P,Q\in\mathcal{D}^{(0)}_{X/S}$. We have to check that $P(Q(x)) = PQ(x)$ for any section x of $\mathcal{E}$. By additivity, we reduce to the case where $P = a\underline{\partial}^{\underline{k}}$ and $Q = b\underline{\partial}^{\underline{l}}$ where a and b are sections of $\mathcal{O}_X$ and $\underline{k},\underline{l}\in\mathbb{N}^d$. By construction of the maps of the form $P\colon\mathcal{E}\to\mathcal{E}$, we can suppose $a=1$.

For any global section y of $\mathcal{E}$, by using the formula 2.3.2.5.1, the equality $\nabla(by) = b\nabla y + y\otimes db$ is translated by the formulas

$$\partial_i(by) = b\partial_i(y) + \partial_i(b)y, \qquad (2.3.2.6.1)$$

for any $i=1,\dots,d$. Let x be a section of $\mathcal{E}$. Using 2.3.2.6.1 in the case where $y=\underline{\partial}^{\underline{l}}(x)$, we get

$$\partial_i(Q(x)) \overset{2.3.2.6.1}{=} b\partial_i(\underline{\partial}^{\underline{l}}(x)) + \partial_i(b)(\underline{\partial}^{\underline{l}}(x)) \overset{2.3.2.5.2}{=} b\partial_i\underline{\partial}^{\underline{l}}(x) + \partial_i(b)\underline{\partial}^{\underline{l}}(x) = (b\partial_i\underline{\partial}^{\underline{l}} + \partial_i(b)\underline{\partial}^{\underline{l}})(x) = (\partial_i Q)(x)$$

the last equality coming from $\partial_i b = b\partial_i + \partial_i(b)$ (see 1.4.2.7.1). This yields by induction on $|\underline{k}|$ the formula $\underline{\partial}^{\underline{k}}(Q(x)) = (\underline{\partial}^{\underline{k}}Q)(x)$ and we are done.

□

The following proposition states that the equivalence of Theorem 2.3.2.6 is in fact an equivalence of categories.

Proposition 2.3.2.7. *Let $\mathcal{E},\mathcal{F}$ be two left $\mathcal{D}^{(0)}_{X/S}$-modules and $f\colon\mathcal{E}\to\mathcal{F}$ be an $\mathcal{O}_X$-linear homomorphism. The following are equivalent.*

(a) The morphism f is $\mathcal{D}^{(0)}_{X/S}$-linear.

(b) The square below is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla} & \mathcal{E}\otimes_{\mathcal{O}_X}\Omega^1_{X/S} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes\mathrm{id}} \\ \mathcal{F} & \xrightarrow{\nabla} & \mathcal{F}\otimes_{\mathcal{O}_X}\Omega^1_{X/S}, \end{array} \qquad (2.3.2.7.1)$$

where the connections are the ones associated via 2.3.2.6 with the left $\mathcal{D}^{(0)}_{X/S}$-module structures.

Proof. Since the proposition is local, then we can suppose X/S has coordinates $t_1,\dots,t_d$. By using the formula 2.3.2.5.1, we compute that the square 2.3.2.7.1 is commutative if and only if $\partial_i(f(x)) = f(\partial_i(x))$ for any $i=1,\dots,r$. Since the ring $\mathcal{D}^{(0)}_{X/S}$ is generated as $\mathcal{O}_S$-algebra by $\mathcal{O}_X$ and by $\partial_1,\dots,\partial_d$, then we are done. □

2.3.2.8. Let $f\colon\mathcal{E}\to\mathcal{F}$ be a homomorphism of left $\mathcal{D}^{(0)}_{X/S}$-modules. It follows from 2.3.2.4.2 that the commutativity of 2.3.2.7.1 implies that of

$$\begin{array}{ccc} \mathcal{E}\otimes_{\mathcal{O}_X}\Omega^i_{X/S} & \xrightarrow{\nabla^i} & \mathcal{E}\otimes_{\mathcal{O}_X}\Omega^{i+1}_{X/S} \\ \downarrow{\scriptstyle f\otimes\mathrm{id}} & & \downarrow{\scriptstyle f\otimes\mathrm{id}} \\ \mathcal{F}\otimes_{\mathcal{O}_X}\Omega^i_{X/S} & \xrightarrow{\nabla^i} & \mathcal{F}\otimes_{\mathcal{O}_X}\Omega^{i+1}_{X/S}. \end{array} \qquad (2.3.2.8.1)$$

Hence, we get the morphism of $C(\mathcal{O}_S)$ of the form $\mathrm{DR}(\mathcal{E})\to\mathrm{DR}(\mathcal{F})$ (see notation 2.3.2.4).

2.3.2.9. Let $\mathcal{E}$ be a left $\mathcal{D}^{(0)}_{X/S}$-module. We get the integrable connection $\nabla\colon \mathcal{E} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ (see 2.3.2.6 and 2.3.2.4). This yields (see the construction of 2.3.2.3 and 2.3.2.4) the $\mathcal{O}_S$-linear map

$$\nabla^{n-1}\colon \Omega^{n-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}. \tag{2.3.2.9.1}$$

Suppose now X/S has coordinates $t_1, \ldots, t_d$. Let $\{i_1, \ldots, i_{n-1}\}$ be $n-1$ elements of $\{1, \ldots, d\}$. Let $\{j_1, \ldots, j_{d-n+1}\}$ be the complementary. For any section x of $\mathcal{E}$, the formula 2.3.2.4.2 yields

$$\begin{aligned}\nabla^{n-1}((dt_{i_1} \wedge \cdots \wedge dt_{i_{n-1}}) \otimes x) &= (-1)^{n-1} dt_{i_1} \wedge \cdots \wedge dt_{i_{n-1}} \wedge \nabla(x)\\ &= (-1)^{n-1} \sum_{a=1}^{d-n+1} (dt_{i_1} \wedge \cdots \wedge dt_{i_{n-1}} \wedge dt_{j_a}) \otimes \partial_{j_a}(x). \end{aligned}\tag{2.3.2.9.2}$$

2.3.2.10. Viewing $\mathcal{D}^{(0)}_{X/S}$ as a left $\mathcal{D}^{(0)}_{X/S}$-module, we get from 2.3.2.9.1 the $\mathcal{O}_S$-linear map:

$$\nabla^{n-1}\colon \Omega^{n-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S} \to \Omega^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S}. \tag{2.3.2.10.1}$$

In fact, since $\mathcal{D}^{(0)}_{X/S}$ as a $\mathcal{D}^{(0)}_{X/S}$-bimodule then we get by functoriality (see 2.3.2.7) that 2.3.2.10.1 is a homomorphism of right $\mathcal{D}^{(0)}_{X/S}$-modules. When X/S has coordinates $t_1, \ldots, t_d$, following 2.3.2.9.2 and with its notation we have

$$\nabla^{n-1}((dt_{i_1} \wedge \cdots \wedge dt_{i_{n-1}}) \otimes 1) = (-1)^{n-1} \sum_{a=1}^{d-n+1} (dt_{i_1} \wedge \cdots \wedge dt_{i_{n-1}} \wedge dt_{j_a}) \otimes \partial_{j_a}. \tag{2.3.2.10.2}$$

Let $\mathcal{E}$ be a left $\mathcal{D}^{(0)}_{X/S}$-module. Using the formulas 2.3.2.9.2 and 2.3.2.10.2, we get the commutative diagram:

$$\begin{array}{ccc} (\Omega^{n-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S}) \otimes_{\mathcal{D}^{(0)}_{X/S}} \mathcal{E} & \xrightarrow{\nabla^{n-1}\otimes \mathrm{id}} & (\Omega^{n-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S}) \otimes_{\mathcal{D}^{(0)}_{X/S}} \mathcal{E} \\ \downarrow\wr & & \downarrow\wr \\ \Omega^{n-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\nabla^{n-1}} & \Omega^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}, \end{array}\tag{2.3.2.10.3}$$

where the vertical isomorphisms are the canonical ones. Hence we have the canonical isomorphism of $C({}^{\mathrm{r}}\mathcal{D}^{(0)}_{X/S})$ of the form $\mathrm{DR}(\mathcal{E}) \xrightarrow{\sim} \mathrm{DR}(\mathcal{D}^{(0)}_{X/S}) \otimes_{\mathcal{D}^{(0)}_{X/S}} \mathcal{E}$.

2.3.3 Tangent sheaf, homological dimension, Spencer resolutions

2.3.3.1 (Tangent sheaf)**.** We set $\mathcal{T}_{X/S} := \mathcal{H}om_{\mathcal{B}_X}(\Omega^1_{X/S}, \mathcal{B}_X)$, the tangent sheaf relative to X/S. From $\Omega^1_{X/S} \hookrightarrow \mathcal{P}^1_{X/S}$, we obtain by duality the canonical epimorphism $\mathcal{D}_{X/S,1} \twoheadrightarrow \mathcal{T}_{X/S}$ whose kernel is $\mathcal{D}_{X/S,0} = \mathcal{O}_X$. From the canonical epimorphism morphism $\varpi\colon p_{0*}\mathcal{P}^1_{X/S} \twoheadrightarrow \Omega^1_{X/S}$ (see 1.1.2.6.3), we get by duality the $\mathcal{O}_X$-linear monomorphism $\varpi^\vee\colon \mathcal{T}_{X/S} \hookrightarrow \mathcal{D}_{X/S,1}$ (for the left structure of $\mathcal{D}_{X/S,1}$).

The morphisms $\mathcal{P}^1_{X/S} \to \mathcal{P}^1_{X/S,(m)}$ and $\mathcal{D}^{(m)}_{X/S,1} \to \mathcal{D}_{X/S,1}$ are isomorphisms for any $m \in \mathbb{N}$. This yields $\mathrm{gr}_1\, \mathcal{D}^{(m)}_{X/S} \xrightarrow{\sim} \mathcal{T}_{X/S}$. Moreover, we get the $\mathcal{O}_X$-linear monomorphism

$$\iota^{(m)}\colon \mathcal{T}_{X/S} \overset{\varpi^\vee}{\hookrightarrow} \mathcal{D}_{X/S,1} \xleftarrow{\sim} \mathcal{D}^{(m)}_{X/S,1} \hookrightarrow \mathcal{D}^{(m)}_{X/S}. \tag{2.3.3.1.1}$$

Suppose X/S has coordinates $(t_\lambda)_{\lambda=1,\ldots,d}$. Following the local description given at 1.1.2.6, $\iota^{(m)}(\mathcal{T}_{X/S})$ is equal to the free $\mathcal{O}_X$-submodule (for the left or the right structure) of $\mathcal{D}^{(m)}_{X/S}$ generated by elements $\partial_1, \ldots, \partial_d$.

Proposition 2.3.3.2. *The canonical homomorphism $\iota^{(0)}\colon \mathcal{T}_{X/S} \to \mathcal{D}^{(0)}_{X/S}$ induces an isomorphism*

$$\mathbb{S}(\mathcal{T}_{X/S}) \xrightarrow{\sim} \mathrm{gr}\, \mathcal{D}^{(0)}_{X/S}.$$

Proof. Since this is local, we can suppose X/S has coordinates $(t_\lambda)_{\lambda=1,\dots,d}$. Following 2.3.3.1, $\iota^{(0)}$ identifies $\mathcal{T}_{X/S}$ with the free $\mathcal{O}_X$-submodule of $\mathcal{D}^{(0)}_{X/S}$ generated by elements $\partial_1,\dots,\partial_d$. Denote by ξ_i the image of ∂_i in $\mathrm{gr}_1\,\mathcal{D}^{(0)}_{X/S}$. Then it follows from 1.4.2.7.1 that $\mathrm{gr}\,\mathcal{D}^{(0)}_{X/S}$ is equal to the commutative polynomial $\mathcal{O}_X$-algebra with the variable $\xi_1,\dots,\xi_d$.

□

We suppose from now $f\colon X\to S$ has pure relative dimension d.

Notation 2.3.3.3. We sometimes remove the canonical inclusion $\iota^{(0)}$ in the notation, i.e. we might canonically identify $\mathcal{T}_{X/S}$ with a sub-$\mathcal{O}_X$-module of $\mathcal{D}^{(0)}_{X/S}$. For any sections v_1,v_2 of $\mathcal{T}_{X/S}$, we denote by $[v_1,v_2]$ the section of $\mathcal{T}_{X/S}$ which corresponds to the section $v_1v_2-v_2v_1$ (we use the ring structure of $\mathcal{D}^{(0)}_{X/S}$ and we remark $v_1v_2-v_2v_1\in\mathcal{T}_{X/S}$). The $f^{-1}\mathcal{O}_S$-bilinear map $[-,-]\colon\mathcal{T}_{X/S}\times\mathcal{T}_{X/S}\to\mathcal{T}_{X/S}$ satisfies the Jacobi identity, i.e. we get a Lie bracket on the tangent space.

Definition 2.3.3.4. Let $\mathcal{E}=(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a filtered left $\mathcal{D}^{(0)}_{X/S}$-module, i.e a left $\mathcal{D}^{(0)}_{X/S}$-module $\mathcal{E}$ endowed with an increasing exhaustive filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ by $\mathcal{O}_X$-submodules so that $\mathcal{D}^{(0)}_{X/S,n'}\cdot\mathcal{E}_n\subset\mathcal{E}_{n+n'}$.

For any sections v_1,v_2 of $\mathcal{T}_{X/S}$ and b of $\mathcal{O}_X$, we compute $v_1b-bv_1=v_1(b)$ in $\mathcal{D}^{(0)}_{X/S}$ and $[bv_1,v_2]=b[v_1,v_2]-v_2(b)v_1$ in $\mathcal{T}_{X/S}$. Hence, similarly to [Kas95, 1.6], we can check that the morphism of left $\mathcal{D}^{(0)}_{X/S}$-modules

$$\delta\colon\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^i\mathcal{T}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}_{j-1}\to\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^{i-1}\mathcal{T}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}_j \tag{2.3.3.4.1}$$

given by

$$\begin{aligned}\delta\left(P\otimes(v_1\wedge\cdots\wedge v_i)\otimes u\right)=&\sum_{a=1}^{i}(-1)^{a-1}Pv_a\otimes(v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge v_i)\otimes u\\&-\sum_{a=1}^{i}(-1)^{a-1}P\otimes(v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge v_i)\otimes v_au\\&+\sum_{1\le a<b\le i}(-1)^{a+b}P\otimes([v_a,v_b]\wedge v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge\widehat{v_b}\wedge\cdots\wedge v_i)\otimes u\end{aligned}$$

is well defined. Moreover, we compute that we get the following complex of left $\mathcal{D}^{(0)}_{X/S}$-modules

$$0\to\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^d\mathcal{T}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}_{n-d}\cdots\xrightarrow[\delta]{}\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge\mathcal{T}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}_{n-1}\xrightarrow[\delta]{}\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}_n\to\mathcal{E}\to 0. \tag{2.3.3.4.2}$$

We call 2.3.3.4.2 the first Spencer sequence of degree n of $\mathcal{E}$ and denote it by $\mathrm{Sp}_{n,\mathcal{D}^{(0)}_{X/S}}(\mathcal{E})$ or $\mathrm{Sp}_n(\mathcal{E})$.

Theorem 2.3.3.5. *With the notations of 2.3.3.4, let us suppose moreover that the filtration of $\mathcal{E}$ is good. Hence, for s large enough, $\mathrm{Sp}^\bullet_{s,\mathcal{D}^{(0)}_{X/S}}(\mathcal{E})$ is exact.*

Proof. This is checked similarly to [Kas95, 1.6.1] (see also the similar proof of 4.7.3.6). □

2.3.3.6. In particular, taking the trivial filtration of $\mathcal{O}_X$, we get the exact sequence of left $\mathcal{D}^{(0)}_{X/S}$-modules

$$0\to\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^d\mathcal{T}_{X/S}\cdots\xrightarrow[\delta_d]{}\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge\mathcal{T}_{X/S}\xrightarrow[\delta_1]{}\mathcal{D}^{(0)}_{X/S}\to\mathcal{O}_X\to 0 \tag{2.3.3.6.1}$$

where the map

$$\delta_i\colon\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^i\mathcal{T}_{X/S}\to\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^{i-1}\mathcal{T}_{X/S} \tag{2.3.3.6.2}$$

is given by the formula

$$\begin{aligned}\delta_i\left(P\otimes(v_1\wedge\cdots\wedge v_i)\right)=&\sum_{a=1}^{i}(-1)^{a-1}Pv_a\otimes(v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge v_i)\\&+\sum_{1\le a<b\le i}(-1)^{a+b}P\otimes([v_a,v_b]\wedge v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge\widehat{v_b}\wedge\cdots\wedge v_i).\end{aligned} \tag{2.3.3.6.3}$$

2.3.3.7. Let $\mathcal{A}$ be a commutative algebra on a topological space X. Let $\mathcal{E}$ be a $\mathcal{A}$-module. We denote by $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{A})$ the $\mathcal{A}$-linear dual. For any integer $n \geq 0$, we have a canonical $\mathcal{A}$-linear morphism

$$\wedge^n (\mathcal{E}^\vee) \to (\wedge^n(\mathcal{E}))^\vee. \tag{2.3.3.7.1}$$

The morphism is constructed as follows. By composing the morphisms

$$(\mathcal{E}^\vee)^n = \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{A}^n) \xrightarrow{\wedge^n} \mathcal{H}om_{\mathcal{A}}(\wedge^n(\mathcal{E}),\wedge^n(\mathcal{A}^n)) \underset{\sim}{\overset{\det}{\longrightarrow}} (\wedge^n(\mathcal{E}))^\vee \tag{2.3.3.7.2}$$

we get an alternating n-linear mapping. By universal property, this yields the $\mathcal{A}$-linear map 2.3.3.7.1 given by $\phi_1 \wedge \cdots \wedge \phi_n \mapsto ((x_1 \wedge \cdots \wedge x_n) \mapsto \det(\phi_i(x_j)))$.

From now, suppose $\mathcal{E}$ is locally free of rank N. Then 2.3.3.7.1 is an isomorphism. More precisely, suppose $\mathcal{E}$ has the basis $e_1,\dots,e_N$. We denote by $e_1^*,\dots,e_N^*$ the dual basis of $\mathcal{E}^\vee$ and by $(e_{i_1}\wedge\cdots\wedge e_{i_n})^*$ the dual basis of $(\wedge^n(\mathcal{E}))^\vee$ associated with the basis given by the elements $e_{i_1}\wedge\cdots\wedge e_{i_n}$. Then, the map 2.3.3.7.1 sends $e_{i_1}^*\wedge\cdots\wedge e_n^*$ to $(e_{i_1}\wedge\cdots\wedge e_{i_n})^*$.

Moreover, we have the canonical morphism

$$\wedge^{N-n}\mathcal{E} \to \mathcal{H}om_{\mathcal{A}}(\wedge^n\mathcal{E},\wedge^N\mathcal{E}), \tag{2.3.3.7.3}$$

given by $\omega \mapsto (\omega' \mapsto \omega\wedge\omega')$. Reducing to the case where $\mathcal{E}$ is free, we compute that this is an isomorphism. Since $\wedge^N\mathcal{E}$ is locally free, by using the isomorphism 2.3.3.7.1 then the canonical morphisms

$$\wedge^n (\mathcal{E}^\vee)\otimes_{\mathcal{A}}\wedge^N\mathcal{E} \to (\wedge^n\mathcal{E})^\vee\otimes_{\mathcal{A}}\wedge^N\mathcal{E} \to \mathcal{H}om_{\mathcal{A}}(\wedge^n\mathcal{E},\wedge^N\mathcal{E}) \tag{2.3.3.7.4}$$

are both isomorphisms. Hence, by composing 2.3.3.7.3 and the inverse of 2.3.3.7.4, we get the canonical isomorphism

$$\wedge^{N-n}\mathcal{E} \xrightarrow{\sim} \wedge^n(\mathcal{E}^\vee)\otimes_{\mathcal{A}}\wedge^N\mathcal{E}. \tag{2.3.3.7.5}$$

2.3.3.8. We can construct the map 2.3.2.10.1 in a second way by duality from the Spencer morphisms as follows. Since $\mathcal{D}^{(0)}_{X/S}$ is a locally free $\mathcal{O}_X$-module for its left structure (in particular), since $\Omega^i_{X/S}$ is locally free of finite type for any integer $0\leq i\leq d$, then we get the canonical isomorphism of $(\mathcal{O}_X,\mathcal{D}^{(0)}_{X/S})$-bimodules:

$$\begin{aligned}\Omega^i_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S} &\xrightarrow{\sim} \wedge^i(\mathcal{T}^\vee_{X/S})\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S} \underset{2.3.3.7.1}{\xrightarrow{\sim}} \wedge^i(\mathcal{T}_{X/S})^\vee\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S} \xrightarrow{\sim} \\ &\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}\left(\wedge^i\mathcal{T}_{X/S},\mathcal{D}^{(0)}_{X/S}\right) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}^{(0)}_{X/S}}\left(\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^i\mathcal{T}_{X/S},\mathcal{D}^{(0)}_{X/S}\right).\end{aligned} \tag{2.3.3.8.1}$$

Hence, for any integer $0\leq n\leq d$, we get the morphism of right $\mathcal{D}^{(0)}_{X/S}$-modules $\delta^\star_n$ making commutative the diagram

$$\begin{array}{ccc} \Omega^{n-1}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S} & \xrightarrow{\delta^\star_n} & \Omega^n_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S} \\ \sim\downarrow 2.3.3.8.1 & & \sim\downarrow 2.3.3.8.1 \\ \mathcal{H}om_{\mathcal{D}^{(0)}_{X/S}}\left(\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^{n-1}\mathcal{T}_{X/S},\mathcal{D}^{(0)}_{X/S}\right) & \xrightarrow{\delta^\vee_n} & \mathcal{H}om_{\mathcal{D}^{(0)}_{X/S}}\left(\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^{n}\mathcal{T}_{X/S},\mathcal{D}^{(0)}_{X/S}\right), \end{array} \tag{2.3.3.8.2}$$

where $\delta^\vee_n$ is the image under the functor $\mathcal{H}om_{\mathcal{D}^{(0)}_{X/S}}(-,\mathcal{D}^{(0)}_{X/S})$ of the Spencer morphism of left $\mathcal{D}^{(0)}_{X/S}$-modules 2.3.3.6.2.

Lemma 2.3.3.9. *Both maps 2.3.2.10.1 and 2.3.3.8.2 are equal, i.e.* $\delta^\star_n = \nabla^{n-1}$.

Proof. Since this is local, we can suppose X/S has coordinates $t_1,\dots,t_d$. Let $\{i_1,\dots,i_{n-1}\}$ be $n-1$ elements of $\{1,\dots,d\}$. Let $\{j_1,\dots,j_{d-n+1}\}$ be the complementary.

0) For any $1\leq i\leq d$, let us denote by α_i the isomorphism 2.3.3.8.1. For any i elements $\{l_1,\dots,l_i\}$ of $\{1,\dots,d\}$, we compute $\alpha_i((dt_{l_1}\wedge\cdots\wedge dt_{l_i})\otimes 1) = (1\otimes(\partial_{l_1}\wedge\cdots\wedge\partial_{l_i}))^*$ where the elements $1\otimes(\partial_{l_1}\wedge\cdots\wedge\partial_{l_i})$ form a basis of $\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^i\mathcal{T}_{X/S}$ as left $\mathcal{D}^{(0)}_{X/S}$-module and where the star symbol means

the dual basis as right $\mathcal{D}^{(0)}_{X/S}$-module. By right $\mathcal{D}^{(0)}_{X/S}$-linearity of α_i, for any section P of $\mathcal{D}^{(0)}_{X/S}$, we get $\alpha_i((dt_{l_1}\wedge\cdots\wedge dt_{l_i})\otimes P)=(1\otimes(\partial_{l_1}\wedge\cdots\wedge\partial_{l_i}))^*P$.

1) For any integers $1\leq k_1,\ldots,k_n\leq d$, since $\delta_n(1\otimes(\partial_{k_1}\wedge\cdots\wedge\partial_{k_n}))=\sum_{a=1}^{i}(-1)^{a-1}\partial_{k_a}\otimes(\partial_{k_1}\wedge\cdots\wedge\widehat{\partial_{k_a}}\wedge\cdots\wedge\partial_{k_n})$ (see 2.3.3.6.3), then we compute

$$\delta_n^\vee\left((1\otimes(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}}))^*\right)(1\otimes(\partial_{k_1}\wedge\cdots\wedge\partial_{k_n}))$$
$$=\sum_{a=1}^{i}(-1)^{a-1}\partial_{k_a}(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}})^*(\partial_{k_1}\wedge\cdots\wedge\widehat{\partial_{k_a}}\wedge\cdots\wedge\partial_{k_n}). \tag{2.3.3.9.1}$$

Hence, if $\{i_1,\ldots,i_{n-1}\}\not\subset\{k_1,\ldots,k_n\}$, then $\delta_n^\vee(1\otimes(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}}))^*(1\otimes(\partial_{k_1}\wedge\cdots\wedge\partial_{k_n}))=0$. Moreover, for any integer $1\leq a\leq d-n+1$, we get $\delta_n^\vee((1\otimes(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}}))^*)(1\otimes(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}}\wedge\partial_{j_a}))=(-1)^{n-1}\partial_{j_a}$. With the step 0), this yields

$$\delta_n^\vee\circ\alpha_{n-1}\left((dt_{i_1}\wedge\cdots\wedge dt_{i_{n-1}})\otimes 1\right)=\delta_n^\vee\left((1\otimes(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}}))^*\right)$$
$$=(-1)^{n-1}\sum_{a=1}^{d-n+1}(1\otimes(\partial_{i_1}\wedge\cdots\wedge\partial_{i_{n-1}}\wedge\partial_{j_a}))^*\partial_{j_a}$$
$$=\alpha_n\left((-1)^{n-1}\sum_{a=1}^{d-n+1}(dt_{i_1}\wedge\cdots\wedge dt_{i_{n-1}}\wedge dt_{j_a})\otimes\partial_{j_a}\right)\overset{2.3.2.10.2}{=}\alpha_n\circ\nabla^{n-1}\left((dt_{i_1}\wedge\cdots\wedge dt_{i_{n-1}})\otimes 1\right).$$

□

2.3.3.10. By applying the functor $\mathcal{H}om_{\mathcal{D}^{(0)}_{X/S}}(-,\mathcal{D}^{(0)}_{X/S})$ to the complex of left $\mathcal{D}^{(0)}_{X/S}$-modules

$$\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^d\mathcal{T}_{X/S}\underset{\delta_d}{\longrightarrow}\cdots\underset{\delta}{\longrightarrow}\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge\mathcal{T}_{X/S}\underset{\delta_1}{\longrightarrow}\mathcal{D}^{(0)}_{X/S} \tag{2.3.3.10.1}$$

via the canonical isomorphisms 2.3.3.8.1, we get the complex of right $\mathcal{D}^{(0)}_{X/S}$-modules

$$\mathcal{D}^{(0)}_{X/S}\underset{\nabla^0}{\longrightarrow}\Omega^1_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S}\underset{\nabla^1}{\longrightarrow}\cdots\underset{\nabla}{\longrightarrow}\Omega^{d-1}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S}\underset{\nabla^{d-1}}{\longrightarrow}\omega_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S}. \tag{2.3.3.10.2}$$

It follows from 2.3.3.9 that the complex of $C({}^{\mathrm{r}}\mathcal{D}^{(0)}_{X/S})$ defined at 2.3.3.10.2 is the de Rham complex of $\mathcal{D}^{(0)}_{X/S}$ and is denoted by $\mathrm{DR}(\mathcal{D}^{(0)}_{X/S})$ (see 2.3.2.3).

If $\mathcal{E}\in D({}^{\mathrm{l}}\mathcal{D}^{(0)}_{X/S})$, we define the de Rham complex of $\mathcal{E}$ to be the complex

$$\mathrm{DR}(\mathcal{E}):=\mathrm{DR}(\mathcal{D}^{(0)}_{X/S})\otimes_{\mathcal{D}^{(0)}_{X/S}}\mathcal{E}.$$

Remark, when $\mathcal{E}$ is a left $\mathcal{D}^{(0)}_{X/S}$-module, following 2.3.2.10 we retrieve the usual de Rham complex (up to canonical isomorphisms) as defined at 2.3.2.3.

2.3.3.11. By applying the functor $\omega_{X/S}\otimes_{\mathcal{O}_X}-$ to the morphism of left $\mathcal{D}^{(0)}_{X/S}$-modules 2.3.3.6.2, we get the morphism of right $\mathcal{D}^{(0)}_{X/S}$-modules:

$$\omega_{X/S}\otimes_{\mathcal{O}_X}(\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^n\mathcal{T}_{X/S})\to\omega_{X/S}\otimes_{\mathcal{O}_X}(\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^{n-1}\mathcal{T}_{X/S}). \tag{2.3.3.11.1}$$

We have moreover the isomorphism of right $\mathcal{D}^{(0)}_{X/S}$-modules

$$\Omega^{d-n}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S}\underset{2.3.3.7.5}{\overset{\sim}{\longrightarrow}}(\wedge^n\mathcal{T}_{X/S}\otimes_{\mathcal{O}_X}\omega_{X/S})\otimes_{\mathcal{O}_X}\mathcal{D}^{(0)}_{X/S}$$
$$\underset{\delta_X\otimes\mathrm{id}_{\wedge^n\mathcal{T}_{X/S}}}{\overset{\sim}{\longrightarrow}}\omega_{X/S}\otimes_{\mathcal{O}_X}(\mathcal{D}^{(0)}_{X/S}\otimes_{\mathcal{O}_X}\wedge^n\mathcal{T}_{X/S}), \tag{2.3.3.11.2}$$

where δ_X is the transposition isomorphism (see 2.2.2.2.1).

Lemma 2.3.3.12. *For any integer $0 \leq n \leq d$, the following square of right $\mathcal{D}^{(0)}_{X/S}$-modules*

$$\begin{array}{ccc} \Omega^{d-n}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S} & \xrightarrow[2.3.3.11.2]{\sim} & \omega_{X/S} \otimes_{\mathcal{O}_X} (\mathcal{D}^{(0)}_{X/S} \otimes_{\mathcal{O}_X} \wedge^n \mathcal{T}_{X/S}) \\ {\scriptstyle (-1)^{d-n+1}\nabla^{d-n}} \downarrow {\scriptstyle 2.3.2.10.1} & & {\scriptstyle 2.3.3.11.1} \downarrow {\scriptstyle \mathrm{id} \otimes \delta_n} \\ \Omega^{d-n+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S} & \xrightarrow[2.3.3.11.2]{\sim} & \omega_{X/S} \otimes_{\mathcal{O}_X} (\mathcal{D}^{(0)}_{X/S} \otimes_{\mathcal{O}_X} \wedge^{n-1} \mathcal{T}_{X/S}), \end{array} \tag{2.3.3.12.1}$$

is commutative.

Proof. Since this is local, we can suppose X/S has coordinates $t_1, \ldots, t_d$. Choose a subset $\{i_1, \ldots, i_{d-n}\}$ of $d-n$ elements of $\{1, \ldots, d\}$. Let $\{j_1, \ldots, j_n\}$ be the complementary. Set $\omega_{d-n} := dt_{i_1} \wedge \cdots \wedge dt_{i_{d-n}}$, $\omega_n := dt_{j_1} \wedge \cdots \wedge dt_{j_n}$, $\omega_n^* := \partial_{j_1} \wedge \cdots \wedge \partial_{j_n}$ and $\omega_d := \omega_{d-n} \wedge \omega_n$. We compute the isomorphism $\Omega^{d-n}_{X/S} \xrightarrow{\sim} \wedge^n \mathcal{T}_{X/S} \otimes_{\mathcal{O}_X} \omega_{X/S}$ of 2.3.3.7.5 sends ω_{d-n} to $\omega_n^* \otimes \omega_d$. Hence, the top isomorphism of the diagram 2.3.3.12 sends $\omega_{d-n} \otimes 1$ to $\delta_X \otimes \mathrm{id}(\omega_n^* \otimes \omega_d \otimes 1) = \delta_X(\omega_d \otimes 1) \otimes \omega_n^* = \omega_d \otimes 1 \otimes \omega_n^*$ (see the formulas of 2.2.2.1 satisfied by the transposition isomorphism).

For any $a \in \{j_1, \ldots, j_n\}$, set $\omega_{n,\widehat{a}} := dt_{j_1} \wedge \cdots \wedge \widehat{dt_{j_a}} \wedge \cdots \wedge dt_{j_n}$ and $\omega^*_{n,\widehat{a}} := \partial_{j_1} \wedge \cdots \wedge \widehat{\partial_{j_a}} \wedge \cdots \wedge \partial_{j_n}$. Following 2.3.3.6.3, we compute $\delta_n(1 \otimes \omega_n^*) = \sum_{a=1}^n (-1)^{a-1} \partial_{j_a} \otimes \omega^*_{n,\widehat{a}}$. Hence, the right morphism of the diagram 2.3.3.12 sends $\omega_d \otimes 1 \otimes \omega_n^*$ to $\sum_{a=1}^i (-1)^{a-1} \omega_d \otimes \partial_{j_a} \otimes \omega^*_{n,\widehat{a}}$.

On the other hand, the left morphism of the diagram 2.3.3.12 sends $\omega_{d-n} \otimes 1$ to $-\sum_{a=1}^n (\omega_{d-n} \wedge dt_a) \otimes \partial_{j_a}$ (see the formula 2.3.2.10.2). Set $\omega_{d-n,a} := \omega_{d-n} \wedge dt_{j_a}$. We compute $\omega_{d-n,a} \wedge \omega_{n,\widehat{a}} = (-1)^{a-1} \omega_d$. Hence, the isomorphism $\Omega^{d-n+1}_{X/S} \xrightarrow{\sim} \wedge^{n-1} \mathcal{T}_{X/S} \otimes_{\mathcal{O}_X} \omega_{X/S}$ of 2.3.3.7.5 sends $\omega_{d-n,a}$ to $(-1)^{a-1} \omega^*_{n,\widehat{a}} \otimes \omega_d$. Hence, the bottom isomorphism of 2.3.3.12 sends $-\sum_{a=1}^n \omega_{d-n,a} \otimes \partial_{j_a}$ to $-\sum_{a=1}^n \delta_X \otimes \mathrm{id}((-1)^{a-1} \omega^*_{n,\widehat{a}} \otimes \omega_d \otimes \partial_{j_a}) = \sum_{a=1}^n (-1)^{a-1} \omega_d \otimes \partial_{j_a} \otimes \omega^*_{n,\widehat{a}}$ (see the formulas of 2.2.2.1 satisfied by the transposition isomorphism). □

Proposition 2.3.3.13. *We have the following assertions.*

(i) The map $\omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S} \xrightarrow{\beta} \omega_{X/S}$ given by the structure of a right $\mathcal{D}^{(0)}_{X/S}$-module on $\omega_{X/S}$ induces a $\mathcal{D}^{(0)}_{X/S}$-linear resolution $\mathrm{DR}(\mathcal{D}^{(0)}_{X/S})[d] \xrightarrow{\sim} \omega_{X/S}$ of $\omega_{X/S}$.

(ii) $\mathcal{E}xt^i_{\mathcal{D}^{(0)}_{X/S}}(\mathcal{O}_X, \mathcal{D}^{(0)}_{X/S}) = 0$ for $i \neq d$. There is a canonical isomorphism of right $\mathcal{D}^{(0)}_{X/S}$-modules

$$\mathcal{E}xt^d_{\mathcal{D}^{(0)}_{X/S}}(\mathcal{O}_X, \mathcal{D}^{(0)}_{X/S}) \xrightarrow{\sim} \omega_{X/S}.$$

Proof. By applying the functor $\omega_{X/S} \otimes_{\mathcal{O}_X} -$ to the exact sequence 2.3.3.6.1, with 2.3.3.12, we get that $\mathrm{DR}(\mathcal{D}^{(0)}_{X/S})[d]$ is a resolution of $\omega_{X/S}$. Moreover, we compute that the map $\omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(0)}_{X/S} \xrightarrow{\beta} \omega_{X/S}$ of the resolution $\mathrm{DR}(\mathcal{D}^{(0)}_{X/S})[d] \xrightarrow{\sim} \omega_{X/S}$ is given by the structure of a right $\mathcal{D}^{(0)}_{X/S}$-module on $\omega_{X/S}$. □

2.3.4 Homological global dimension

2.3.4.1 (Left and right global Dimension)**.** Let D be a ring. We recall few facts on the definition of the homological/global dimension (see [BouAlgX] §8; [Wei94] Ch 4). We denote by $i.\dim(E)$ (resp. $p.\dim(E)$, resp. $f.\dim(E)$) the injective dimension (resp. projective dimension, resp. flat dimension) of a left or right D-module E. The following numbers are the same:

i) $\sup\{i.\dim(F)$: F is a left D-module$\}$;

ii) $\sup\{p.\dim(E)$: E is a left D-module$\}$;

ii') $\sup\{p.\dim(E)$: E is a monogeneous left D-module$\}$;

iii) $\sup\{n$: $\mathrm{Ext}^n_D(E, F) \neq 0$ for some left D-modules $E, F\}$;

iii') $\sup\{n\colon \operatorname{Ext}^n_D(E,F)\neq 0$ for some monogeneous left D-module E and left D-module $F\}$.

This common number (possibly ∞) is called the *left global dimension of* D and is denoted by $\mathrm{l.gl.dim}(D)$. Bourbaki [BX] calls it the *homological dimension of* D. One might sometimes say homological global dimension of D. One may define the right global dimension $\mathrm{r.gl.dim}(D)$ similarly.

When $\mathrm{r.gl.dim}(D) = \mathrm{l.gl.dim}(D)$, we simply write them $\mathrm{gl.dim}(D)$ and it is called the global dimension of D, or the homological global dimension of D.

When D is coherent, this notion is very similar to the tor dimension (see 1.4.3.30) but not equal a priori (beware in the condition iii') that the monogeneous left D-module is not coherent).

2.3.4.2 (Comparison with the tor-dimension)**.** We have obviously $\mathrm{tor.dim}(D) \leq r.\mathrm{gl.dim}(D)$ and $\mathrm{tor.dim}(D) \leq \mathrm{l.gl.dim}(D)$. If moreover D is left (resp. right) noetherian, then $\mathrm{tor.dim}(D) = r.\mathrm{gl.dim}(D)$ (resp. $\mathrm{tor.dim}(D) = \mathrm{l.gl.dim}(D)$ (see [Wei94, 4.1.5]). Hence, if D is both left and right noetherian, then $r.\mathrm{gl.dim}(D) = \mathrm{l.gl.dim}(D)$. We have almost the equality by considering inductive limits of noetherings rings : see below 2.3.4.3.

Proposition 2.3.4.3. *Let $(D^{(m)})_{m\in\mathbb{N}}$ be an inductive system indexed by $\mathbb{N}$ of noetherian rings such that the ring homomorphisms $D^{(m)}\to D^{(m+1)}$ are injective and left (resp. right) flat. Set $D^\dagger := \varinjlim_m D_m$. We have the inequality*

$$\mathrm{tor.dim}\, D^\dagger \leq \mathrm{l.gl.dim}\, D^\dagger \leq \mathrm{tor.dim}\, D^\dagger + 1 \qquad \textit{(resp. } \mathrm{tor.dim}\, D^\dagger \leq \mathrm{r.gl.dim}\, D^\dagger \leq \mathrm{tor.dim}\, D^\dagger + 1\textit{)}.$$

Proof. We have nothing to prove if the tor dimension is infinite. Suppose $\mathrm{tor.dim}\, D^\dagger$ is an integer d. We have to prove the vanishing

$$\operatorname{Ext}^i_{D^\dagger}(E,M)=0$$

for all cyclic $D^\dagger$-module E, all $D^\dagger$-module M and all $i > d+1$ (see 2.3.4.1). Let $E \simeq D^\dagger/I$ be a presentation of E and for any $m\in\mathbb{N}$ put

$$I_m = D^\dagger(I\cap D^{(m)}), \qquad E_m = D^\dagger/I_m.$$

Then $I=\cup_m I_m$, $E\simeq \varinjlim_m E_m$, and for any M,

$$\operatorname{Hom}_{D^\dagger}(E,M) \xrightarrow{\sim} \varprojlim_m \operatorname{Hom}_{D^\dagger}(E_m,M),$$

from which a spectral sequence follows

$$E_2^{i,j} = R^i\varprojlim_m \operatorname{Ext}^j_{D^\dagger}(E_m,M) \Rightarrow E^n = \operatorname{Ext}^n_{D^\dagger}(E,M).$$

Now for all m, since $D^{(m)}$ is noetherian then I_m is left ideal of finite type of $D^\dagger$. Thus E_m is coherent on $D^\dagger$. By using 1.4.3.32, this yields $\operatorname{Ext}^j_{D^\dagger}(E_m,M)$ are zero for $j>d$ and all m. As $R^i\varprojlim_m$ are zero for $i>1$, the assertion follows. □

Remark 2.3.4.4. Let $\mathcal{D}$ be a left coherent sheaf of rings on a topological space X. Suppose there exist an integer n and an open basis $\mathfrak{B}$ of X such that for any $U\in\mathfrak{B}$, $\mathrm{l.gl.dim}(\mathcal{D}(U))\leq n$ (one might define this property by saying that $\mathrm{l.gl.dim}(\mathcal{D})\leq n$). Then $\mathcal{D}$ has finite tor-dimension $\leq n$ (use 1.4.3.12). This yields the equality $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}) = D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D})$ (see 1.4.3.29).

Proposition 2.3.4.5 (Homological global dimension)**.** *Suppose X is affine, S is affine and regular. Let $r := \sup_{s\in f(X)} \dim \mathcal{O}_{S,s}$ which is suppose to be finite. Then the ring $D^{(0)}_{X/S} := \Gamma(X,\mathcal{D}^{(0)}_{X/S})$ has homological global dimension equal to $2d+r$.*

Proof. Since $D^{(0)}_{X/S}$ is noetherian, then $\mathrm{tor.dim}(D^{(0)}_{X/S}) = \mathrm{gl.dim}(D^{(0)}_{X/S})$ (see 2.3.4.2). This yields that the corollary is local in X (see 1.4.3.26) and we can therefore suppose that $X\to S$ has coordinates $t_1,\dots,t_d$. Following 2.3.3.2, $\mathrm{gr}\, D^{(0)}_{X/S}$ is a polynomial ring over $\Gamma(X,\mathcal{O}_X)$ with d variables. Hence, $\dim\mathrm{gr}\, D^{(0)}_{X/S} = 2d+r$. Since we have the inequality $\mathrm{gl.dim}\, D^{(0)}_{X/S} \leq \mathrm{gl.dim}\,\mathrm{gr}\, D^{(0)}_{X/S}$ (see [LvO96, I.7.2 Corollary 2]), this yields $\mathrm{gl.dim}\, D^{(0)}_{X/S}\leq 2d+r$. It remains to exhibit a left $D^{(0)}_{X/S}$-module E

such that $\mathrm{Ext}^{2d+r}_{D^{(0)}_{X/S}}(E, D^{(0)}_{X/S}) \neq 0$. Let $s \in f(X)$ such that $\dim \mathcal{O}_{S,s} = r$, $s_1, \ldots, s_r \in \mathfrak{m}_{S,x} \subset \mathcal{O}_{S,s}$ be a regular sequence of generators. Since $s_1, \ldots, s_r, t_1^p, \ldots, t_d^p$ are in the center of $\mathcal{D}^{(0)}_{X/S}$ then the sub-$\mathcal{O}_X$-module of $\mathcal{O}_X$ generated by $(s_1, \ldots, s_r, t_1^p, \ldots, t_d^p)$ is a sub-$\mathcal{D}^{(0)}_{X/S}$-module of $\mathcal{O}_X$. Hence, we get a left $\mathcal{D}^{(0)}_{X/S}$-module (resp. a left $D^{(0)}_{X/S}$-module) by setting $\mathcal{E} := \mathcal{O}_X/(s_1, \ldots, s_r, t_1^p, \ldots, t_d^p)$ (resp. $E = \Gamma(X, \mathcal{E})$). Let $K_\bullet := K_\bullet(s_1, \ldots, s_r, t_1^p, \ldots, t_d^p)$ be the Koszul complex of $\Gamma(X, \mathcal{O}_X)$-free modules given by the sequence of global sections $s_1, \ldots, s_r, t_1^p, \ldots, t_d^p$ of $\Gamma(X, \mathcal{O}_X)$ (e.g. see the description in 5.2.5.4). Since $s_1, \ldots, s_r, t_1^p, \ldots, t_d^p$ are in the center of $\mathcal{D}^{(0)}_{X/S}$ then $K_\bullet$ is also a complex of left $\mathcal{D}^{(0)}_{X/S}$-modules (where $\mathcal{O}_X$ is endowed with its constant structure of left $\mathcal{D}^{(0)}_{X/S}$-module). By using the spectral sequence $E_1^{i,j} = \mathrm{Ext}^j_{D^{(0)}_{X/S}}(K_i, D^{(0)}_{X/S}) \Rightarrow \mathrm{Ext}^n_{D^{(0)}_{X/S}}(E, D^{(0)}_{X/S})$, since $E_1^{i,j} = 0$ if $j \neq d$ (see 2.3.3.13), then $\mathrm{Ext}^{2d+r}_{D^{(0)}_{X/S}}(E, D^{(0)}_{X/S})$ is the cokernel of the map $E_1^{d+r-1,d} \to E_1^{d+r,d}$ which is isomorphic to the map $\Gamma(X, \omega_{X/S})^{d+r} \to \Gamma(X, \omega_{X/S})$ whose jth component is the multiplication by $(-1)^j u_j$ where u_j is the jth component of $(s_1, \ldots, s_r, t_1^p, \ldots, t_d^p)$. Hence, $\mathrm{Ext}^{2d+r}_{D^{(0)}_{X/S}}(E, D^{(0)}_{X/S}) \neq 0$. □

Chapter 3

Logarithmic differential modules

We refer the readers to [Ogu18], for more details on log schemes.

3.1 Sheaf of differential operators of infinite level and finite order on logarithmic schemes

We will keep the following notation. If $X^\sharp$ is a log scheme, we denote by $\alpha_{X^\sharp}: M_{X^\sharp} \to \mathcal{O}_X$ the underlying log structure of $X^\sharp$. If $f: X^\sharp \to Y^\sharp$ is a morphism of fine log schemes, then we denote by X and Y the underlying schemes, by $\underline{f}: X \to Y$ the underlying morphism of schemes, by $f^\sharp: \mathcal{O}_Y \to f_*\mathcal{O}_X$ (or sometimes $f^\sharp: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$) and $f^\flat: M_{Y^\sharp} \to f_* M_{X^\sharp}$ (or sometimes $f^\flat: f^{-1}M_{Y^\sharp} \to M_{X^\sharp}$). By abuse of notation, we might sometimes still denote by f instead of $\underline{f}$ or of $f^\sharp$. We work in the category of fine log schemes or p-adic fine log formal schemes. Hence, fiber products are computed in these categories.

3.1.1 nth infinitesimal neighborhood

Let i be an integer and $S^\sharp$ be a fine log scheme over the scheme $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$. First, let us introduce the notion of nice log-schemes that we will need only from 3.4.

Definition 3.1.1.1. We say that $S^\sharp$ is a "nice" fine log scheme if there exists a scheme Z over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ such that $S^\sharp$ is a log flat Z-log-scheme (see the definition [Ogu18, IV.4.1.1] of log-flatness, which we can also simply be called flatness). Following Lemma [Ogu18, IV.4.1.3] (i.e. we can use the zero monoid for Z), this means that fppf locally on $S^\sharp$ and Z, there exists a flat and strict morphism of Z-log-schemes of the form $\alpha: S^\sharp \to Z \times A_P$ where P is an integral monoid and the product is computed in the category of log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$.

Remark 3.1.1.2. Remark that it follows from [Ogu18, IV.4.1.2.4] that if $Y^\sharp$ is nice and $X^\sharp \to Y^\sharp$ is log flat (e.g log smooth), then $X^\sharp$ is also nice.

Lemma 3.1.1.3. *Let $U := (S^\sharp)^*$ be the open in $S^\sharp$ subscheme with trivial log-structure (see [Ogu18, III.1.2.8]) and $j: U \to S^\sharp$ be the canonical inclusion. If $S^\sharp$ is a nice fine log scheme (see definition 3.1.1.1), then the canonical morphism $\mathcal{O}_{S^\sharp} \to j_*\mathcal{O}_U$ is injective.*

Proof. Let Z be a scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ such that $S^\sharp$ is a log flat Z-log-scheme. Since the lemma is fppf local, we can suppose there exists a flat and strict morphism of Z-log-schemes $\alpha: S^\sharp \to Z \times A_P$ with P an integral monoid. Since α and the projection $Z \times A_P \to A_P$ are strict, then it follows from [Ogu18, III.1.2.11] that we have $U = \alpha^{-1}((Z \times A_P)^*)$ and $(Z \times A_P)^* = Z \times A_P^*$. Since $A_P^* = A_{P^{\mathrm{gr}}}$ (see [Ogu18, III.1.2.10]), then $U = \alpha^{-1}(Z \times A_{P^{\mathrm{gr}}})$. Let $\iota: Z \times A_{P^{\mathrm{gr}}} \to Z \times A_P$ be the canonical open immersion. Since $\mathcal{O}_{Z \times A_P} \to \iota_*\mathcal{O}_{Z \times A_{P^{\mathrm{gr}}}}$ is injective and α is flat, then we can conclude. □

Remark 3.1.1.4. With the notation of 3.1.1.3, it is not clear that the canonical morphism $j_*\mathcal{O}_U \to \mathcal{O}_{S^\sharp}$ is injective in general. Let $X^\sharp$ is a log smooth $S^\sharp$-log scheme. When $S^\sharp$ is a nice fine log scheme, we will check (by using this injectivity) that $\omega_{X^\sharp/S^\sharp}$ has a canonical right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure (see Lemma 3.4.5.1). There exist some extensions of this latter result (when $m = 0$ or when the log scheme is the reduction modulo π^{i+1} of some log formal scheme: see [CV17, Section 3.6]). Since the context of nice log schemes is largely sufficient for us, this justifies why we only consider the case of nice log schemes from the subsection 3.4.

Definition 3.1.1.5. Let $f\colon X^\sharp \to Y^\sharp$ be a morphism of log schemes and $\mathcal{B}_X$ be an $\mathcal{O}_X$-algebra. We have the following quasi-flatness notions which will be useful to define derived duality or derived tensor products over the ring of differential operators (see the example 4.6.3.3).

1. We say that f is quasi-flat if there exists a morphism of schemes $g\colon Y \to Z$ such that the morphism of schemes $g \circ f\colon X \to Z$ is flat.
2. We say that $\mathcal{B}_X$ is a quasi-flat $f^{-1}\mathcal{O}_Y$-algebra if there exists a morphism of schemes $g\colon Y \to Z$ such that the (induced by $g \circ f$) morphism of ringed spaces $(X, \mathcal{B}_X) \to Z$ is flat.

Remark 3.1.1.6. We will give later a notion of quasi-flatness in the context of "ringed logarithmic schemes" (see 4.4.1.3). With the notation of the last statement of Definition 3.1.1.5, $(X^\sharp, \mathcal{B}_X) \to Y^\sharp$ is a quasi-flat morphism of ringed logarithmic schemes means that $\mathcal{B}_X$ is a quasi-flat $f^{-1}\mathcal{O}_Y$-algebra (see 4.4.1.3.a). We will moreover gives a notion of quasi-flatness for morphism of relative ringed logarithmic schemes (see 4.4.1.3).c).

Remark 3.1.1.7. Let $f\colon X^\sharp \to Y^\sharp$ be log smooth (e.g log flat) morphism of fine log schemes (resp. of locally noetherian fine log schemes). We suppose that for any $y \in Y$, the monoid $(M_{Y^\sharp}/\mathcal{O}_Y^*)_y$ is generated by one element. Then $X^\sharp \to Y^\sharp$ is integral (see [Kat89, 4.4] or [Ogu18, III.2.5.3.3]). Hence, it follows from [Kat89, 4.5] (resp. [Ogu18, IV.4.3.5.1]) that the underlying morphism of schemes $X \to Y$ is flat.

Definition 3.1.1.8. Let $u\colon Z^\sharp \to X^\sharp$ be a morphism of log-schemes.

(a) According to [Ogu18, II.1.1.12], we say that u is an immersion (resp. closed immersion) if $\underline{u}$ is an immersion (resp. a closed immersion) of schemes and if $u^*M_{X^\sharp} \to M_{Z^\sharp}$ is surjective (here u^* means the pullback of log structures [Kat89, 1.4]). An immersion (resp. a closed immersion) is exact if and only if $u^*M_{X^\sharp} \to M_{Z^\sharp}$ is an isomorphism.

(b) We say that u is an open immersion if u is an exact immersion such that $\underline{u}$ is an open immersion of schemes.

(c) Let n be an integer. A "log thickening of order (n)" (resp. "log thickening of order n") is an *exact* closed immersion $u\colon U^\sharp \hookrightarrow T^\sharp$ such that $\mathcal{I}^{(n)} = 0$ (resp. such that $\mathcal{I}^{n+1} = 0$), where $\mathcal{I}$ is the ideal associated with the closed immersion $\underline{u}$. The convention of the respective case is that of [Ogu18, IV.2.1.1] and is convenient when we are dealing with n-infinitesimal neighborhood.

(d) Let $a \in \mathbb{N}$. A "(p)-nilpotent log thickening of order a" is a log thickening of order (p^{a+1}). A "(p)-nilpotent log thickening" is a (p)-nilpotent log thickening of order a for some $a \in \mathbb{N}$ large enough.

An $S^\sharp$-immersion (resp. $S^\sharp$-log-thickening) is an immersion (resp. log thickening) which is an $S^\sharp$-morphism.

Remark 3.1.1.9. (a) If $u\colon Z^\sharp \to X^\sharp$ and $f\colon X^\sharp \to Y^\sharp$ are two $S^\sharp$-morphisms of log schemes such that $f \circ u$ is an $S^\sharp$-immersion, then so is u (use [Gro60, 5.3.13]).

(b) If $u\colon Z^\sharp \to X^\sharp$ and $f\colon X^\sharp \to Y^\sharp$ are two $S^\sharp$-morphisms of log schemes such that $f \circ u$ is a closed $S^\sharp$-immersion and $\underline{f}$ is separated, then u is a closed immersion (use [Gro60, 5.4.4]).

(c) We can decompose a (strict) $S^\sharp$-immersion u into $u = u_1 \circ u_2$, where u_1 is an open $S^\sharp$-immersion and u_2 is a (strict) closed $S^\sharp$-immersion.

(d) Let $u\colon U^\sharp \hookrightarrow T^\sharp$ be an $S^\sharp$-log-thickening of order (p^a) for some integer a. Since p is nilpotent in $\mathcal{O}_T$, by applying finitely many times the functor $\mathcal{I} \mapsto \mathcal{I}^{(p)}$ to the ideal defined by $\underline{u}$ we obtain the zero ideal, which justifies the definition of "(p)-nilpotent log thickening". This also implies that u is the composition of several $S^\sharp$-log-thickenings of order (p).

Definition 3.1.1.10. (a) We denote by $\mathscr{C}_\sharp$ the category whose objects are $S^\sharp$-immersions of fine log-schemes and whose morphisms $u' \to u$ are commutative diagrams of the form

$$\begin{array}{ccc} X'^\sharp & \xrightarrow{f} & X^\sharp \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ Z'^\sharp & \longrightarrow & Z^\sharp . \end{array} \qquad (3.1.1.10.1)$$

We say that $u' \to u$ is strict (resp. flat, resp. cartesian) if f is strict (resp. $\underline{f}$ is flat, resp. the square 3.1.1.10.1 is cartesian).

(b) Let $n \in \mathbb{N}$. We denote by $\mathscr{C}_{\sharp n}$ (resp. $\mathscr{T}hick_{(p)}$) the full subcategory of $\mathscr{C}_\sharp$ whose objects are $S^\sharp$-log-thickening of order n (resp. (p)-nilpotent $S^\sharp$-log-thickenings).

(c) Let u be an object of $\mathscr{C}_\sharp$. A "log thickening of order n induced by u" is an object u' of $\mathscr{C}_{\sharp n}$ endowed with a morphism $u' \to u$ of $\mathscr{C}_\sharp$ satisfying the following universal property: for any object u'' of $\mathscr{C}_{\sharp n}$ endowed with a morphism $f : u'' \to u$ of $\mathscr{C}_\sharp$ there exists a unique morphism $u'' \to u'$ of $\mathscr{C}_{\sharp n}$ whose composition with $u' \to u$ is f. The unicity up to canonical isomorphism of the log thickening of order n induced by u is obvious. We will denote by $P^{\sharp n}(u)$ the log thickening of order n induced by u. We also say that $P^{\sharp n}(u)$ is the "nth infinitesimal neighbourhood of u" (see [Kat89, 5.8]). The existence is checked below (see 3.1.1.17).

Remark 3.1.1.11. (a) If $u' \to u$ is a strict cartesian morphism of $\mathscr{C}_\sharp$ with $u \in \mathscr{C}_{\sharp n}$ (resp. $u \in \mathscr{T}hick_{(p)}$), then $u' \in \mathscr{C}_{\sharp n}$ (resp. $u' \in \mathscr{T}hick_{(p)}$). Indeed, the corresponding square of the form 3.1.1.10.1 of $u' \to u$ remains cartesian after applying the forgetful functor from the category of fine log schemes to the category of schemes (to check this fact, we need a priori the strictness of $u' \to u$).

(b) The category $\mathscr{C}_\sharp$ has fibered products. More precisely, let $u : Z^\sharp \hookrightarrow X^\sharp$, $u' : Z'^\sharp \hookrightarrow X'^\sharp$, $u'' : Z^{\sharp\prime\prime} \hookrightarrow X^{\sharp\prime\prime}$ be some objects of $\mathscr{C}_\sharp$; let $u' \to u$ and $u'' \to u$ be two morphisms of $\mathscr{C}_\sharp$. Then $u' \times_u u''$ is the immersion $Z'^\sharp \times_{Z^\sharp} Z^{\sharp\prime\prime} \hookrightarrow X'^\sharp \times_{X^\sharp} X^{\sharp\prime\prime}$. If $u' \to u$ is moreover cartesian, then so is the projection $u' \times_u u'' \to u''$.

In order to be precise, let us clarify the standard definitions.

Definition 3.1.1.12. Let $f : X^\sharp \to Y^\sharp$ be an $S^\sharp$-morphism of fine log schemes.

(a) We say that f is "fine formally log étale" (resp. "fine formally log unramified") if it satisfies the following property: for any commutative diagram of fine log schemes of the form

$$\begin{array}{ccc} U^\sharp & \xrightarrow{u_0} & X^\sharp \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle f} \\ T^\sharp & \xrightarrow{v} & Y^\sharp \end{array} \qquad (3.1.1.12.1)$$

such that ι is an object of $\mathscr{C}_{\sharp 1}$, there exists a unique morphism (resp. there exists at most one morphism) $u : T^\sharp \to X^\sharp$ such that $u \circ \iota = u_0$ and $f \circ u = v$.

(b) We say that f is "log étale" if f is fine formally log étale and if $\underline{f}$ is of finite presentation.

(c) We say that f is "étale" if f is log étale and strict (which is equivalent to saying that $\underline{f}$ is étale and f is strict).

Remark 3.1.1.13. (a) The definitions appearing in 3.1.1.12 do not depend on the choice of the fine log scheme $S^\sharp$.

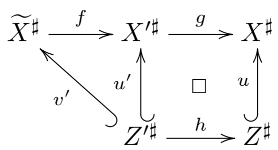
(b) Let $f : X^\sharp \to Y^\sharp$ be an $S^\sharp$-morphism of fine log schemes. The notion of etaleness of Kato appearing in [Kat89, 3.3] is what we have defined as "log etaleness". We distinguish by definition "log etale" from "etale" morphisms in order to avoid confusion when we say for instance "etale locally".

(c) There exists in the literature a notion of etale morphism of log schemes with coherent log structures (see in Ogus's book at [Ogu18, IV.3.1.1]). This notion is compatible with Kato's notion of etale morphism of fine log schemes. Indeed, both notions have the same characterization when we focus on morphisms of fine log schemes (see respectively Theorem [Kat89, 3.5] and Theorem [Ogu18, IV.3.3.1]).

To avoid confusion with the etale notion in the classical sense, we will call such a morphism a "log etale" morphism of log schemes with coherent log structures (instead of "etale morphism").

Moreover, let $f\colon X^\sharp \to Y^\sharp$ be a morphism of log schemes with coherent log structures. From [Ogu18, IV.3.1.11], $X^{\sharp\mathrm{int}} \to X^\sharp$ and $Y^{\sharp\mathrm{int}} \to Y^\sharp$ are log etale (see [Ogu18, III.2.1.5.1] concerning the functor $X^\sharp \mapsto X^{\sharp\mathrm{int}}$). Hence, using Remark [Ogu18, IV.3.1.2], we can check that f is log etale if and only if f^{int} is log etale.

3.1.1.14. We recall in the paragraph how we can exactify an immersion. Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an $S^\sharp$-immersion of fine log-schemes. Let $\overline{z}$ be a geometric point of $Z^\sharp$. Using the proof of [Kat89, 4.10.1] and Proposition [Gro67, IV.18.1.1], we can check that there exists a commutative diagram of the form



such that the square is cartesian, f is log étale, $\underline{f}$ is affine, g is étale, v' is an exact closed $S^\sharp$-immersion and h is an étale neighborhood of $\overline{z}$ in $Z^\sharp$.

Lemma 3.1.1.15. *Let $u' \to u$ be a strict cartesian morphism of $\mathscr{C}_\sharp$. Suppose that $P^{\sharp n}(u)$, the log thickening of order n induced by u, exists. Then the log thickening of order n induced by u' exists and we have $P^{\sharp n}(u') = P^{\sharp n}(u) \times_u u'$.*

Proof. Using the remarks of 3.1.1.11, since $u' \to u$ is strict and cartesian, then so is the projection $P^{\sharp n}(u) \times_u u' \to P^{\sharp n}(u)$ and then $P^{\sharp n}(u) \times_u u' \in \mathscr{C}_{\sharp n}$. Hence, we can check easily that $P^{\sharp n}(u) \times_u u'$ endowed with the projection $P^{\sharp n}(u) \times_u u' \to u'$ satisfies the corresponding universal property of $P^{\sharp n}(u')$. We can check similarly the respective case. □

Lemma 3.1.1.16. *Let $n \in \mathbb{N}$, $f\colon X^\sharp \to Y^\sharp$ be a fine formally log étale morphism of fine log $S^\sharp$-schemes, $u\colon Z^\sharp \hookrightarrow X^\sharp$ and $v\colon Z^\sharp \hookrightarrow Y^\sharp$ be two $S^\sharp$-immersions of fine log schemes such that $v = f \circ u$. If $P^{\sharp n}(u)$ exists, then $P^{\sharp n}(v)$ exists and we have $P^{\sharp n}(u) = P^{\sharp n}(v)$.*

Proof. Abstract nonsense. □

Proposition 3.1.1.17. *We have the following properties.*

(a) For any integer n, the inclusion functor $For_{\sharp n}\colon \mathscr{C}_{\sharp n} \to \mathscr{C}_\sharp$ has a right adjoint functor which we will denote by $P^{\sharp n}\colon \mathscr{C}_\sharp \to \mathscr{C}_{\sharp n}$. Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an object of $\mathscr{C}_\sharp$. Then $Z^\sharp$ is also the source of $P^{\sharp n}(u)$.

(b) Moreover, let $U^\sharp$ be an open of $X^\sharp$ containing $Z^\sharp$ such that the induced immersion $v\colon Z^\sharp \hookrightarrow U^\sharp$ is closed. Denoting abusively by $P^{\sharp n}(u)$ the target of the arrow $P^{\sharp n}(u)$, the underlying morphism of schemes of $P^{\sharp n}(u) \to U^\sharp$ is affine. We denote by $\mathcal{P}^n(u)$ the quasi-coherent $\mathcal{O}_U$-algebra so that $P^n(u) = \mathrm{Spec}(\mathcal{P}^n(u))$ where $P^n(u)$ is the underline scheme of $P^{\sharp n}(u)$. The sheaf $\mathcal{P}^n(u)$ has his support in Z and we also denote by $\mathcal{P}^n(u)$ the sheaf $v^{-1}\mathcal{P}^n(u)$ which is independent on the choice of U.

(c) If X is noetherian, then so is $P^n(u)$.

Proof. Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an $S^\sharp$-immersion of fine log-schemes. If u is an open immersion, then $P^{\sharp n}(u)$ is the identity of $Z^\sharp$. Similarly, we reduce to the case where u is a closed immersion.

Using 3.1.1.15, the existence of $P^{\sharp n}(u)$ (and then the whole proposition) is étale local on $X^\sharp$ (i.e. following our convention, this is local for the Zariski topology and we can proceed by descent of a finite covering with étale quasi-compact morphisms). Hence, by 3.1.1.14, we may thus assume that there exists a commutative diagram of the form

$$\begin{array}{ccc} \widetilde{X}^\sharp & \xrightarrow{f} & X^\sharp \\ & \nwarrow^{\widetilde{u}} & \uparrow{\scriptstyle u} \\ & & Z^\sharp \end{array}$$

such that f is log étale, $\underline{f}$ is affine and $\widetilde{u}$ is an exact closed $S^\sharp$-immersion. Let $\mathcal{I}$ be the ideal given by $\widetilde{u}$. Let $P^{\sharp n} \hookrightarrow \widetilde{X}^\sharp$ be the exact closed immersion which is induced by $\mathcal{I}^{n+1}$. Using 3.1.1.16, we can check that $P^{\sharp n}(u)$ is the exact closed immersion $Z^\sharp \hookrightarrow P^{\sharp n}$. When X is noetherian, then so are $\widetilde{X}^\sharp$ and P^n. □

Lemma 3.1.1.18. *Let $u \to v$ be a morphism of $\mathscr{C}_\sharp$. Let $w := P^{\sharp n}(v) \times_v u$ (this is the product in $\mathscr{C}_\sharp$). Then $P^{\sharp n}(w)$ and $P^{\sharp n}(u)$ are isomorphic in $\mathscr{C}_{\sharp n}$.*

Proof. We can easily check that the composition $P^{\sharp n}(w) \to w \to u$ satisfies the universal property of $P^{\sharp n}(u) \to u$. Hence, we are done. □

Notation 3.1.1.19. Let P be a monoid. We denote by $A_P := (\mathrm{Spec}(\mathbb{Z}[P]), M_P)$ the log formal scheme whose underlying scheme is $\mathrm{Spec}(\mathbb{Z}[P])$ and whose log structure is the log structure associated with the pre-log structure induced canonically by $P \to \mathbb{Z}[P]$.

Definition 3.1.1.20. Let $f : X^\sharp \to Y^\sharp$ be a morphism of fine $S^\sharp$-log schemes.

(a) We say that a finite set $u_1, \ldots, u_d$ of elements of $\Gamma(X, M_{X^\sharp})$ are "formal logarithmic coordinates of f" if the corresponding $Y^\sharp$-morphism $X^\sharp \to Y^\sharp \times_{\mathrm{Spec}\,\mathbb{Z}} A_{\mathbb{N}^d}$ is formally log étale (see notation 3.1.1.19).

(b) We say that a finite set $u_1, \ldots, u_d$ of elements of $\Gamma(X, M_{X^\sharp})$ are "logarithmic coordinates of f" if the corresponding $Y^\sharp$-morphism $X^\sharp \to Y^\sharp \times_{\mathrm{Spec}\,\mathbb{Z}} A_{\mathbb{N}^d}$ is log étale (see notation 3.1.1.19).

(c) We say that f is "weakly log smooth" if, étale locally on $X^\sharp$, f has formal logarithmic coordinates. Notice that this notion of weak log smoothness is étale local on Y.

Recall that f is "log smooth" if, étale locally on $X^\sharp$, f has logarithmic coordinates. Notice that this notion of weak log smoothness is étale local on $Y^\sharp$.

Definition 3.1.1.21. Let $f : X^\sharp \to Y^\sharp$ be a *strict* morphism of fine $S^\sharp$-log schemes.

(a) We say that a finite set $t_1, \ldots, t_d$ of elements of $\Gamma(X, \mathcal{O}_X)$ are "formal coordinates of f" if the corresponding $Y^\sharp$-morphism $X^\sharp \to Y^\sharp \times_{\mathrm{Spec}\,\mathbb{Z}} A_{\mathbb{N}^d}$ is (formally) log étale.

(b) We say that a finite set $t_1, \ldots, t_d$ of elements of $\Gamma(X, \mathcal{O}_X)$ are "coordinates of f" if the corresponding $Y^\sharp$-morphism $X^\sharp \to Y^\sharp \times_{\mathrm{Spec}\,\mathbb{Z}} A_{\mathbb{N}^d}$ is (log) étale.

(c) We say that f is "weakly smooth" if f is weakly log smooth morphism.

Remark, this is compatible with Definition 1.1.1.4.

Remark 3.1.1.22. To simplify the terminology here, the definitions 3.1.1.20 and 3.1.1.21 are different from that of [CV17] where we had distinguished in the terminology both cases where the morphism $X^\sharp \to Y^\sharp \times_{\mathrm{Spec}\,\mathbb{Z}} A_{\mathbb{N}^d}$ is strict or not (what is called log coordinates was called log basis).

Notation 3.1.1.23. Let $P^\sharp = (P, M_{P^\sharp})$ be a log scheme. We denote by $\alpha_P : (M_{P^\sharp}, \cdot) \to (\mathcal{O}_P, \times)$ or simply α the underlying monoid morphism and we can identity $M^*_{P^\sharp}$ and $\mathcal{O}^*_P$ via α.

Lemma 3.1.1.24. *Let $i : X^\sharp \hookrightarrow P^\sharp$ be an exact closed $S^\sharp$-immersion of fine log schemes.*

(a) We have the equality $\ker(i^{-1}\mathcal{O}^*_P \to \mathcal{O}^*_X) = \ker(i^{-1}M^{\mathrm{gr}}_{P^\sharp} \to M^{\mathrm{gr}}_{X^\sharp})$. In particular, $\ker(\mathcal{O}^*_{P,\overline{x}} \to \mathcal{O}^*_{X,\overline{x}}) = \ker(M^{\mathrm{gr}}_{P^\sharp,\overline{x}} \to M^{\mathrm{gr}}_{X^\sharp,\overline{x}})$ for any geometric point $\overline{x}$ of $X^\sharp$.

(b) When the ideal $\mathcal{I}$ of $\mathcal{O}_P$ given by i is moreover an nil ideal then $\ker(i^{-1}M^{\mathrm{gr}}_{P^\sharp} \to M^{\mathrm{gr}}_{X^\sharp}) = 1 + i^{-1}\mathcal{I}$.

Proof. The last statement is [Ogu18, IV.2.1.2]. Let us now check (a). Let $\overline{x}$ be a geometric point of $X^\sharp$. Since i is an exact closed immersion, we have $(M_{P^\sharp}/\mathcal{O}_P^*)_{\overline{x}} = (M_{X^\sharp}/\mathcal{O}_X^*)_{\overline{x}}$ (use [Kat89, 1.4.1]) and thus $(M_{P^\sharp}^{\mathrm{gp}}/\mathcal{O}_P^*)_{\overline{x}} = (M_{X^\sharp}^{\mathrm{gp}}/\mathcal{O}_X^*)_{\overline{x}}$ (because the functor $M \mapsto M^{\mathrm{gr}}$ commutes with inductive limits). Hence, the inclusion $\ker(\mathcal{O}_{P,\overline{x}}^* \to \mathcal{O}_{X,\overline{x}}^*) \subset \ker(M_{P^\sharp,\overline{x}}^{\mathrm{gr}} \to M_{X^\sharp,\overline{x}}^{\mathrm{gr}})$ is in fact an equality. Since we have the canonical inclusion $\ker(i^{-1}\mathcal{O}_P^* \to \mathcal{O}_X^*) \subset \ker(i^{-1}M_{P^\sharp}^{\mathrm{gr}} \to M_{X^\sharp}^{\mathrm{gr}})$, this yields that this latter inclusion is an equality. □

Proposition 3.1.1.25. *Let $f\colon X^\sharp \to Y^\sharp$ be an $S^\sharp$-morphism of fine log-schemes endowed with formal logarithmic coordinates $(b_\lambda)_{\lambda=1,\dots,r}$ (see definition 3.1.1.20). Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ and $v\colon Z^\sharp \hookrightarrow Y^\sharp$ be two $S^\sharp$-immersions of fine log schemes such that $v = f \circ u$. Suppose given $y_\lambda \in \Gamma(Y, M_{Y^\sharp})$ whose images in $\Gamma(Z, M_{Z^\sharp})$ coincide with the images of b_λ. Let $Z^\sharp \hookrightarrow D'_n$ and $Z^\sharp \hookrightarrow D_n$ be the nth infinitesimal neighborhood of order n of u and v respectively (see Definition 3.1.1.10.c and recall following Proposition 3.1.1.17, the source is indeed $Z^\sharp$). Let $\alpha\colon D'_n \to X^\sharp$ be the canonical morphism. Using multiplicative notation, put $u_\lambda := \frac{\alpha^*(b_\lambda)}{\alpha^*(f^*(y_\lambda))} \in \ker(\Gamma(D'_n, M^{\mathrm{gr}}_{D'_n}) \to \Gamma(Z, M^{\mathrm{gr}}_{Z^\sharp})) = \ker(\Gamma(D'_n, \mathcal{O}^*_{D'_n}) \to \Gamma(Z, \mathcal{O}^*_Z))$ (see 3.1.1.24). We set $\mathcal{O}_{D_n}[T_1,\dots,T_r]_n := \mathcal{O}_{D_n}[T_1,\dots,T_r]/\left(I_{D_n} + (T_1,\dots,T_r)\right)^{n+1}$, where I_{D_n} is the ideal defined by the closed immersion $Z^\sharp \hookrightarrow D_n$. Then, we have the isomorphism of $\mathcal{O}_{D_n}$-algebras*

$$\begin{aligned} \mathcal{O}_{D_n}[T_1,\dots,T_r]_n &\xrightarrow{\sim} \mathcal{O}_{D'_n} \\ T_\lambda &\mapsto u_\lambda - 1. \end{aligned} \tag{3.1.1.25.1}$$

Proof. By using Lemma 3.1.1.16, we reduce to the case where $X^\sharp = Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{N}^r}$, $f\colon X^\sharp \to Y^\sharp$ is the first projection, and that the family $(b_\lambda)_{\lambda=1,\dots,r}$ of elements of $\Gamma(X, M_{X^\sharp})$ is given by the canonical basis $(e_\lambda)_{\lambda=1\dots r}$ of $\mathbb{N}^r$. Using lemma 3.1.1.18, we may furthermore assume that $Y^\sharp = S^\sharp$, $Z^\sharp \hookrightarrow Y^\sharp$ is an exact closed immersion of order n. In particular, we get $D_n = Y^\sharp$.

Let $i\colon Y^\sharp \to Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{Z}^r}$ be the exact closed $Y^\sharp$-immersion defined by $e_\lambda \mapsto 1 \in \Gamma(Y, M_{Y^\sharp})$. By copying the parts 1), 2) and 3) of the proof of 3.2.1.17 (we replace the use of Lemma 3.2.1.7 by the use of Lemma 3.1.1.16), we reduce to the case where $u = i \circ v$, $X^\sharp = Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{Z}^r}$, $(b_\lambda)_{\lambda=1,\dots,r}$ are the elements of $\Gamma(X, M_{X^\sharp})$ corresponding to the canonical basis of $\mathbb{Z}^r$, and $(y_\lambda)_{\lambda=1,\dots,r}$ are equal to 1. This case is obvious.

□

Remark 3.1.1.26. Suppose we are in the situation of 3.2.1.17: let $f\colon X^\sharp \to Y^\sharp$ be an $S^\sharp$-morphism of fine log-schemes, $(b_\lambda)_{\lambda=1,\dots,r}$ be some elements of $\Gamma(X, M_{X^\sharp})$. Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ and $v\colon Z^\sharp \hookrightarrow Y^\sharp$ be two $S^\sharp$-immersions of fine log schemes such that $v = f \circ u$. Then, since $v^* M_{Y^\sharp} \to M_{Z^\sharp}$ is surjective (for the étale topology), then étale locally on $Y^\sharp$, there exist $y_\lambda \in \Gamma(Y, M_{Y^\sharp})$ whose images in $\Gamma(Z, M_{Z^\sharp})$ coincide with the images of b_λ.

3.1.2 Sheaf of principal parts relative to a weakly log smooth morphism

Let i be an integer and $S^\sharp$ be a fine log scheme over the scheme $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$. Let $f\colon X^\sharp \to S^\sharp$ be a weakly log smooth morphism of fine log-schemes. We write $X^\sharp = (X, M_{X^\sharp})$.

Notation 3.1.2.1. Let $\Delta_{X^\sharp/S^\sharp}(r)\colon X^\sharp \hookrightarrow X^{\sharp r+1}_{/S^\sharp}$ denote the diagonal immersion. With notation 3.1.1.17, we set

$$\Delta^n_{X^\sharp/S^\sharp}(r) := P^{\sharp n}(\Delta_{X^\sharp/S^\sharp}(r)).$$

Recall the source of $\Delta^n_{X^\sharp/S^\sharp}$ is $X^\sharp$. For any $n' \geq n$, we denote by

$$\psi^{n',n}_{X^\sharp/S^\sharp}(r)\colon \Delta^n_{X^\sharp/S^\sharp}(r) \hookrightarrow \Delta^{n'}_{X^\sharp/S^\sharp}(r) \tag{3.1.2.1.1}$$

the canonical exact closed immersion. Moreover, we can denote abusively the underlying scheme of the target of $\Delta^n_{X^\sharp/S^\sharp}$ by $\Delta^n_{X^\sharp/S^\sharp}$.

Let us write $\mathcal{P}^n_{X^\sharp/S^\sharp}(r) := \mathcal{P}^n(\Delta_{X^\sharp/S^\sharp}(r))$ (see Notation 3.1.1.17). By default $\mathcal{P}^n_{X^\sharp/S^\sharp}(r)$ is viewed as a sheaf of rings on X: let $U^\sharp$ be an open of $X^{\sharp r+1}_{/S^\sharp}$ containing the image of $\Delta_{X^\sharp/S^\sharp}(r)$ such that the immersion $v\colon X^\sharp \to U^\sharp$ induced by $\Delta^n_{X^\sharp/S^\sharp}(r)$ is closed. Recall, by definition, $\mathcal{P}^n_{X^\sharp/S^\sharp}(r)$ is a sheaf on X

such that $v_*\mathcal{P}^n_{X^\sharp/S^\sharp}(r)$ is a quasi-coherent $\mathcal{O}_U$-algebra and $\mathrm{Spec}(v_*\mathcal{P}^n_{X^\sharp/S^\sharp}(r)) = \Delta^n_{X^\sharp/S^\sharp}(r)$ (see Notation 3.1.1.17).

For $i = 0, \ldots, r$, let $p_i \colon X^{\sharp r+1}_{/S^\sharp} \to X^\sharp$ be the projections. Let $p_i^{U^\sharp} \colon U^\sharp \hookrightarrow X^{\sharp r+1}_{/S^\sharp} \to X^\sharp$ be the morphism induced by the projection p_i. Since $p_i^{U^\sharp} \circ v = \mathrm{id}$ then $\mathcal{P}^n_{X^\sharp/S^\sharp}(r) = p^{U^\sharp}_{i*}(v_*\mathcal{P}^n_{X^\sharp/S^\sharp}(r))$. Since $v_*\mathcal{P}^n_{X^\sharp/S^\sharp}(r)$ is endowed with a structure of quasi-coherent $\mathcal{O}_U$-algebra, we get $r+1$-structures of quasi-coherent $\mathcal{O}_{X^\sharp}$-algebras on $\mathcal{P}^n_{X^\sharp/S^\sharp}(r)$.

To clarify which $\mathcal{O}_{X^\sharp}$-algebra structure we consider, we set $p_{i*}\mathcal{P}^n_{X^\sharp/S^\sharp}(r) := p^{U^\sharp}_{i*}\mathcal{P}^n(\mathcal{I}(r))$. By composing $p_i^{U^\sharp}$ with the canonical morphism $\Delta^n_{X^\sharp/S^\sharp}(r) \to U^\sharp$, we get the projection $p_i^n \colon \Delta^n_{X^\sharp/S^\sharp}(r) \to X^\sharp$. This yields the ring homomorphisms $p_i^n(r) \colon \mathcal{O}_{X^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp}(r)$ and the monoid homomorphisms $p_i^n(r) \colon M_{X^\sharp} \to M_{\Delta^n_{X^\sharp/S^\sharp}(r)}$.

When $r = 1$, we simply write $\mathcal{I}$, $\Delta^n_{X^\sharp/S^\sharp}$, $\Delta_{X^\sharp/S^\sharp}$, $\mathcal{P}^n_{X^\sharp/S^\sharp}$, $\mathcal{P}_{X^\sharp/S^\sharp}$, p_i^n. The left (resp. right) structure of $\mathcal{O}_{X^\sharp}$-algebra on $\mathcal{P}^n_{X^\sharp/S^\sharp}$ is by definition the one given by $p_{0*}\mathcal{P}^n_{X^\sharp/S^\sharp}$ (resp. $p_{1*}\mathcal{P}^n_{X^\sharp/S^\sharp}$).

We denote by $M^n_{X^\sharp/S^\sharp}$ the log structure of $\Delta^n_{X^\sharp/S^\sharp}$. If $a \in M_{X^\sharp}$, we denote by $\mu^n(a)$ the unique section (see 3.1.1.24) of $\ker(\mathcal{P}^{n*}_{X^\sharp/S^\sharp} \to \mathcal{O}^*_X)$ such that we get in $M^n_{X^\sharp/S^\sharp}$ the equality

$$p_1^n(a) = p_0^n(a)\mu^n(a). \tag{3.1.2.1.2}$$

We get the monoid morphism $\mu^n \colon M_{X^\sharp} \to \ker(\mathcal{P}^{n*}_{X^\sharp/S^\sharp} \to \mathcal{O}^*_X)$ given by $a \mapsto \mu^n(a)$.

Lemma 3.1.2.2. *The morphisms p_1^n and p_0^n are strict.*

Proof. Let $\iota^n \colon X^\sharp \hookrightarrow \Delta^n_{X^\sharp/S^\sharp}$ be the structural morphism. Since $\iota^{-1} = \mathrm{id}$, then from [Kat89, 1.4.1] we get the isomorphisms $p_i^{n*}(M_{X^\sharp})/\mathcal{P}^{n*}_{X^\sharp/S^\sharp} \xrightarrow{\sim} M_{X^\sharp}/\mathcal{O}^*_X$ and $M^n_{X^\sharp/S^\sharp}/\mathcal{P}^{n*}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \iota^{n*}(M^n_{X^\sharp/S^\sharp})/\mathcal{O}^*_X \xrightarrow{\sim} M_{X^\sharp}/\mathcal{O}^*_X$ (the last isomorphism is a consequence of the exactness of ι^n). Hence, $p_i^{n*}(M_{X^\sharp})/\mathcal{P}^{n*}_{X^\sharp/S^\sharp} \xrightarrow{\sim} M^n_{X^\sharp/S^\sharp}/\mathcal{P}^{n*}_{X^\sharp/S^\sharp}$. This implies that the canonical morphism $p_i^{n*}(M_{X^\sharp}) \to M^n_{X^\sharp/S^\sharp}$ is an isomorphism. □

Proposition 3.1.2.3 (Local description of $\mathcal{P}^n_{X^\sharp/S^\sharp}$)**.** *Let $(u_\lambda)_{\lambda=1,\ldots,r}$ be logarithmic coordinates of f. Put $\tau_{\sharp\lambda,n} := \mu^n(u_\lambda) - 1$. We have the following isomorphism of $\mathcal{O}_X$-algebras:*

$$\begin{aligned} \mathcal{O}_X[T_1, \ldots, T_r]_n &\xrightarrow{\sim} \mathcal{P}^n_{X^\sharp/S^\sharp} \\ T_\lambda &\mapsto \tau_{\sharp\lambda,n}, \end{aligned} \tag{3.1.2.3.1}$$

where the structure of $\mathcal{O}_X$-module of $\mathcal{P}^n_{X^\sharp/S^\sharp}$ is given by p_1^n or p_0^n.

Proof. Since the case of p_1^n is checked symmetrically, let us compute the case where the $\mathcal{O}_X$-module of $\mathcal{P}^n_{X^\sharp/S^\sharp}$ is given by p_0^n. Consider the commutative diagram

$$\begin{array}{ccccc} A_{\mathbb{N}^r} & \xleftarrow{p_1} & X^\sharp \times_{S^\sharp} A_{\mathbb{N}^r} & & \\ \uparrow u & \Box & \uparrow u\times\mathrm{id} & \searrow q & \\ X^\sharp & \xleftarrow{p_1} & X^\sharp \times_{S^\sharp} X^\sharp & \xrightarrow{p_0} & X^\sharp, \end{array} \tag{3.1.2.3.2}$$

where p_0, p_1 means respectively the left and right projection, where q is the projection, where u is the $S^\sharp$-morphism induced by $u_1, \ldots, u_r$, where $u \times \mathrm{id}$ is the $X^\sharp$-morphism induced by $p_1^*(u_1), \ldots, p_1^*(u_r)$. Since $(p_1^*(u_\lambda))_{\lambda=1,\ldots,r}$ are logarithmic coordinates of p_0 (because the square of the diagram 3.1.2.3.2 is cartesian), we can apply Proposition 3.1.1.25 in the case where f is p_0, u is $\Delta_{X^\sharp/S^\sharp}$, b_λ is $p_1^*(u_\lambda)$, and y_λ is u_λ. □

Remark 3.1.2.4. From the local description of 3.1.2.3, we get that the morphisms p_1^n and p_0^n are finite (i.e. the underlying morphism of schemes is finite).

3.1.3 Sheaf of relative logarithmic differentials

We keep notation 3.1.2.

Definition 3.1.3.1. Let $Y^\sharp \to T^\sharp$ be a morphism of log schemes.

(a) A log derivation of $Y^\sharp/T^\sharp$ with values in a sheaf of $\mathcal{O}_Y$-modules $\mathcal{E}$ is a pair (d, δ), where $d\colon \mathcal{O}_Y \to \mathcal{E}$ is a homomorphism of abelian sheaves and $\delta\colon M_{Y^\sharp} \to \mathcal{E}$ is a homomorphism of sheaves of monoids such that the following conditions holds:

   (i) $d(\alpha_{Y^\sharp}(m)) = \alpha_{Y^\sharp}(m)\delta(m)$ for every local section m of $M_{Y^\sharp}$;

   (ii) $\delta(f^\flat(n)) = 0$ for every local section n of $f^{-1}(M_{T^\sharp})$;

   (iii) $d(ab) = ad(b) + bd(a)$ for every pari of local sections a, b of $\mathcal{O}_Y$;

   (iv) $d(f^\sharp(c)) = 0$ for every local section c of $f^{-1}(\mathcal{O}_T)$.

   We denote by $\mathrm{Der}_{Y^\sharp/T^\sharp}(\mathcal{E})$ the set of all such derivations.

(b) We denote by $(d_{Y^\sharp/T^\sharp}, d\log_{Y^\sharp/T^\sharp})\colon \mathcal{O}_Y \to \Omega^1_{Y^\sharp/T^\sharp}$ the log derivation which is a universal object representing the functor $\mathcal{E} \mapsto \mathrm{Der}_{Y^\sharp/T^\sharp}(\mathcal{E})$ from the category of $\mathcal{O}_Y$-modules (see [Ogu18, Theorem IV.1.2.4]). We can also simply write $(d, d\log)$. The map $(d_{Y^\sharp/T^\sharp}, d\log_{Y^\sharp/T^\sharp})$ is called the constant logarithmic derivation relative to $Y^\sharp/T^\sharp$ and $\Omega^1_{Y^\sharp/T^\sharp}$ is called the "sheaf of relative logarithmic differentials relative to $Y^\sharp/T^\sharp$".

3.1.3.2. We have the following proprieties making some links between the sheaf of relative differentials and the sheaf of principal parts of order ≤ 1 of $X^\sharp/S^\sharp$.

(a) We denote by $\mathcal{I}^1_{X^\sharp/S^\sharp}$ the ideal of the closed immersion $\Delta^1_{X^\sharp/S^\sharp}$. Then, following [Ogu18, IV.3.4.5], we have $\Omega^1_{X^\sharp/S^\sharp} = (\Delta^1_{X^\sharp/S^\sharp})^{-1}(\mathcal{I}^1_{X^\sharp/S^\sharp})$. In other words, $\Omega^1_{X^\sharp/S^\sharp}$ is the kernel of the canonical morphism $\psi^{1,0}_{X^\sharp/S^\sharp}\colon \mathcal{P}^1_{X^\sharp/S^\sharp} \to \mathcal{P}^0_{X^\sharp/S^\sharp} = \mathcal{O}_X$. Since $\Omega^1_{X^\sharp/S^\sharp}$ is an ideal of $\mathcal{P}^1_{X/S}$ of order 2 then the left and right structure of $\mathcal{O}_X$-module of $\Omega^1_{X^\sharp/S^\sharp}$ (recall $\psi^{1,0}_{X^\sharp/S^\sharp}$ is $\mathcal{O}_X$-linear for both structures) are in fact identical.

(b) We have the exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_{X^\sharp/S^\sharp} \xrightarrow{j} p_{0*}\mathcal{P}^1_{X^\sharp/S^\sharp} \xrightarrow{\psi^{1,0}_{X^\sharp/S^\sharp}} \mathcal{O}_X \to 0, \tag{3.1.3.2.1}$$

where j is the canonical inclusion (recall p_{0*} means that $\mathcal{P}^1_{X^\sharp/S^\sharp}$ is considered as an $\mathcal{O}_X$-algebra for its left structure). The exact sequence 3.1.3.2.1 splits via the section $p^1_0\colon \mathcal{O}_X \to p_{0*}\mathcal{P}^1_{X^\sharp/S^\sharp}$, which yields the isomorphism of $\mathcal{O}_X$-modules

$$(p^1_0, j)\colon \mathcal{O}_X \oplus \Omega^1_{X^\sharp/S^\sharp} \xrightarrow{\sim} p_{0*}\mathcal{P}^1_{X^\sharp/S^\sharp}. \tag{3.1.3.2.2}$$

In particular, since $p_{0*}\mathcal{P}^1_{X^\sharp/S^\sharp}$ is a locally free $\mathcal{O}_X$-module, then so is $\Omega^1_{X^\sharp/S^\sharp}$.

(c) Via the isomorphism 3.1.3.2.2, we get the $\mathcal{O}_X$-linear epimorphism

$$\varpi_{X^\sharp/S^\sharp}\colon p_{0*}\mathcal{P}^1_{X^\sharp/S^\sharp} \twoheadrightarrow \Omega^1_{X^\sharp/S^\sharp} \tag{3.1.3.2.3}$$

which is a left inverse of the inclusion $\Omega^1_{X^\sharp/S^\sharp} \subset p_{0*}\mathcal{P}^1_{X^\sharp/S^\sharp}$. We compute $\varpi_{X^\sharp/S^\sharp} = \mathrm{id} - p^1_0 \circ \psi^{1,0}_{X^\sharp/S^\sharp}$.

(d) We have the exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_{X^\sharp/S^\sharp} \xrightarrow{j} p_{1*}\mathcal{P}^1_{X^\sharp/S^\sharp} \xrightarrow{\psi^{1,0}_{X^\sharp/S^\sharp}} \mathcal{O}_X \to 0, \tag{3.1.3.2.4}$$

The exact sequence 3.1.3.2.4 splits via the section $p^1_1\colon \mathcal{O}_X \to p_{1*}\mathcal{P}^1_{X^\sharp/S^\sharp}$, which yields the isomorphism of $\mathcal{O}_X$-modules

$$(p^1_1, j)\colon \mathcal{O}_X \oplus \Omega^1_{X^\sharp/S^\sharp} \xrightarrow{\sim} p_{1*}\mathcal{P}^1_{X^\sharp/S^\sharp}. \tag{3.1.3.2.5}$$

Definition 3.1.3.3. We can extend the definition 1.1.3.2 as follows: Let $\mathcal{E}, \mathcal{F}$ be two $\mathcal{O}_X$-modules, $n \geq 0$ be an integer. We say that a $f^{-1}\mathcal{O}_S$-linear homomorphism $D\colon \mathcal{E} \to \mathcal{F}$ is a "differential operator of order $\leq n$ (relatively to $X^\sharp/S^\sharp$)" if there exists a homomorphism of $\mathcal{O}_X$-modules $u\colon p^n_{0*}(\mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}) \to \mathcal{F}$ such that $D = u \circ p^n_{1,\mathcal{E}}$ (see notation 3.2.2.13). Beware that this is not clear that such a u is unique (hence one might prefer to call such a u as a differential operator of order $\leq n$ relatively to $X^\sharp/S^\sharp$). However, when $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, then by using 3.1.1.3 to check the unicity, we can reduce to the case where the log structures are trivial, which has already been checked.

3.1.3.4. We have the following description of $(d_{X^\sharp/S^\sharp}, d\log_{X^\sharp/S^\sharp})$.

(a) The map $d_{X^\sharp/S^\sharp}\colon \mathcal{O}_X \to \Omega^1_{X^\sharp/S^\sharp}$ is given by $a \mapsto p^1_1(a) - p^1_0(a)$ for any local section a of $\mathcal{O}_X$. Indeed, let $v\colon X^\sharp \to X$ be the canonical morphism. Since the construction of $\Omega^1_{X^\sharp/S^\sharp}$, the sheaf of relative differentials of $X^\sharp/S^\sharp$ (see notation 3.1.3.2), is functorial then for any $i = 0, 1$, we have the commutative diagram:

$$\begin{array}{ccccc} X^\sharp & \hookrightarrow & \Delta^1_{X^\sharp/S^\sharp} & \xrightarrow{p^1_i} & X^\sharp \\ \downarrow{\scriptstyle v} & & \downarrow & & \downarrow{\scriptstyle v} \\ X & \hookrightarrow & \Delta^1_{X/S} & \xrightarrow{p^1_i} & X. \end{array} \tag{3.1.3.4.1}$$

By construction (see the proof of [Ogu18, IV.1.2.4]), the composition of the constant $\mathcal{O}_X$-derivation $d_{X/S}\colon \mathcal{O}_X \to \Omega^1_{X/S}$ (see 1.1.2.6) with the canonical map $\Omega^1_{X/S} \to \Omega^1_{X^\sharp/S^\sharp}$ is $d_{X^\sharp/S^\sharp}$. Hence, by using the commutativity of 3.1.3.4.1 we are done.

(b) Moreover, $d\log_{X^\sharp/S^\sharp}\colon M_{X^\sharp} \to \Omega^1_{X^\sharp/S^\sharp}$ is given by $u \mapsto \mu^1(u) - 1$ (see notation 3.1.2.1.2). Indeed, we compute

$$\alpha_{X^\sharp}(u)(\mu^1(u) - 1) = p^1_0(u)(\mu^1(u) - 1) = p^1_1(u) - p^1_0(u) = p^1_1(\alpha_{X^\sharp}(u)) - p^1_0(\alpha_{X^\sharp}(u)) = d_{X^\sharp/S^\sharp}(\alpha_{X^\sharp}(u)).$$

Since $\alpha_{X^\sharp}(u) d\log_{X^\sharp/S^\sharp} = d_{X^\sharp/S^\sharp}(\alpha_{X^\sharp}(u))$ (see 3.1.3.1.ai), then $\alpha_{X^\sharp}(u)(\mu^1(u)-1) = \alpha_{X^\sharp}(u) d\log_{X^\sharp/S^\sharp}$. Since $\Omega^1_{X^\sharp/S^\sharp}$ is locally free, then we conclude by using 3.1.1.3.

(c) The composition of $\varpi_{X^\sharp/S^\sharp}$ with $p^1_1\colon \mathcal{O}_X \to \mathcal{P}^1_{X^\sharp/S^\sharp}$ is the constant $\mathcal{O}_S$-derivation $d_{X^\sharp/S^\sharp}$. (Indeed, since $\psi^{1,0}_{X^\sharp/S^\sharp} \circ p^1_1 = \mathrm{id}$, then $\varpi_{X^\sharp/S^\sharp} \circ p^1_1 = p^1_1 - p^1_0 = d_{X^\sharp/S^\sharp}$.) Since $\varpi_{X^\sharp/S^\sharp}$ is $\mathcal{O}_X$-linear, this means that $d_{X^\sharp/S^\sharp}$ (or $\varpi_{X^\sharp/S^\sharp}$) is a differential operator of order ≤ 1 relatively to $X^\sharp/S^\sharp$ (see Definition 3.1.3.3).

3.1.3.5 (Local computation)**.** Suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. Let $\tau_{\sharp\lambda,1} := \mu^1(u_\lambda) - 1$ in $\mathcal{P}^1_{X^\sharp/S^\sharp}$ for $\lambda = 1, \dots, d$. It follows from 3.1.3.4.b that $\tau_{\sharp\lambda,1} = d\log\, u_\lambda$. Moreover, following 3.1.2.3, $\mathcal{P}^1_{X^\sharp/S^\sharp}$ is $\mathcal{O}_X$-free with the basis $1, \tau_{\sharp 1,1}, \dots, \tau_{\sharp d,1}$. Since $\psi^{1,0}_{X^\sharp/S^\sharp}(\tau_{\sharp\lambda,1}) = 0$ and $\psi^{1,0}_{X^\sharp/S^\sharp}(1) = 1$, then $\Omega^1_{X^\sharp/S^\sharp}$ is $\mathcal{O}_X$-free with the basis $\tau_{\sharp 1,1}, \dots, \tau_{\sharp d,1}$, i.e. $d\log\, u_1, \dots, d\log\, u_d$. Moreover, $\varpi_{X^\sharp/S^\sharp}(1) = 0$ (and $\ker \varpi_{X^\sharp/S^\sharp} = \mathcal{O}_X$) and $\varpi_{X^\sharp/S^\sharp}(\tau_{\sharp,\lambda,1}) = \tau_{\sharp\lambda,1}$.

We complete this subsection by checking that our notion of fine formally log unramified is the same than that of formally log unramified appearing in the literature.

Proposition 3.1.3.6. *Let $f\colon X^\sharp \to Y^\sharp$ be an $S^\sharp$-morphism of fine log schemes and $\Delta_{X^\sharp/Y^\sharp}\colon X^\sharp \hookrightarrow X^\sharp \times_{Y^\sharp} X^\sharp$ (as always the product is taken in the category of fine log schemes) be the diagonal $S^\sharp$-immersion. The following assertions are equivalent:*

(a) the morphism $P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})$ is an isomorphism ;

(b) the morphism f is fine formally log unramified ;

(c) the morphism f is formally log unramified (this notion is defined at [Ogu18, IV.3.1.1]).

Proof. Following [Ogu18, IV.3.1.3], the first and the last assertions are equivalent. Moreover, by definition, if f is formally log unramified then f is fine formally log unramified. It follows from 3.1.3.2.a that the property $\Omega_{X^\sharp/Y^\sharp} = 0$ is equivalent to saying that $P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})$ is an isomorphism. Copying the proof of "if f is formally log unramified then $\Omega_{X^\sharp/Y^\sharp} = 0$" of [Ogu18, IV.3.1.3] we can check in the same way that if f is fine formally log unramified then $\Omega_{X^\sharp/Y^\sharp} = 0$ (indeed, since $X^\sharp$ fine, then the log scheme $T := X^\sharp \oplus \Omega_{X^\sharp/Y^\sharp}$ is fine because its log structure is $M_{X^\sharp} \oplus \Omega_{X^\sharp/Y^\sharp}$: see [Ogu18, IV.2.1.5]). □

3.1.4 Sheaf of differential operators relative to weakly log smooth schemes

We keep notation 3.1.2.

3.1.4.1. The exact closed immersions $\Delta^n_{X^\sharp/S^\sharp}$ and $\Delta^{n'}_{X^\sharp/S^\sharp}$ induce $\Delta^{n,n'}_{X^\sharp/S^\sharp} := (\Delta^n_{X^\sharp/S^\sharp}, \Delta^{n'}_{X^\sharp/S^\sharp}) \colon X^\sharp \hookrightarrow \Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp}$. Since the morphisms p^n_1 and p^n_0 are strict (see 3.1.2.2), then $\Delta^{n,n'}_{X^\sharp/S^\sharp}$ is also an exact closed immersion. We get $\Delta^{n,n'}_{X^\sharp/S^\sharp} \in \mathscr{C}_{\sharp n+n'}$. Using the universal property of the $n+n'$ infinitesimal neighborhood of $\Delta_{X^\sharp/S^\sharp}$, we get a unique morphism $p^{n,n'}_{02} \colon \Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to \Delta^{n+n'}_{X^\sharp/S^\sharp}$ of $\mathscr{C}_{\sharp n+n'}$ inducing the commutative diagram

$$\begin{array}{ccccc} X^\sharp & \hookrightarrow & \Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} & \longrightarrow & X^\sharp \times_{S^\sharp} X^\sharp \times_{S^\sharp} X^\sharp \\ \| & & \downarrow{\scriptstyle p^{n,n'}_{02}} & & \downarrow{\scriptstyle p_{02}} \\ X^\sharp & \hookrightarrow & \Delta^{n+n'}_{X^\sharp/S^\sharp} & \longrightarrow & X^\sharp \times_{S^\sharp} X^\sharp. \end{array} \tag{3.1.4.1.1}$$

We denote by $\delta^{n,n'} \colon \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp}$ the ring homomorphism induced by $p^{n,n'}_{02}$. By composing the canonical morphism $\Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to X^\sharp \times_{S^\sharp} X^\sharp \times_{S^\sharp} X^\sharp$ with the three projections $p_i \colon X^\sharp \times_{S^\sharp} X^\sharp \times_{S^\sharp} X^\sharp \to X^\sharp$ for $i = 0, 1, 2$, we get three projections

$$p^{n,n'}_i \colon \Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to X^\sharp. \tag{3.1.4.1.2}$$

This yields three structures of locally free of finite type $\mathcal{O}_X$-algebra on $\mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp}$: the left one, the middle one (equal the structure given by the tensor product) and the right one. Since $p_0 \circ p_{02} = p_0$ and $p_1 \circ p_{02} = p_2$, then we remark that $\delta^{n,n'}$ is also a homomorphism of $\mathcal{O}_X$-algebra for respectively the left structure and for the right structure.

By replacing p_{02} by p_{01} (resp. p_{12}), we get a unique morphism $p^{n,n'}_{01}$ (resp. $p^{n,n'}_{12}$ of the form $\Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to \Delta^{n+n'}_{X^\sharp/S^\sharp}$ making commutative the diagram 3.2.2.14.1. We notice that $p^{n,n'}_{01}$ is the composition $\Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to \Delta^n_{X^\sharp/S^\sharp} \xrightarrow{\psi^{n+n',n}_{X^\sharp/S^\sharp}} \Delta^{n+n'}_{X^\sharp/S^\sharp}$, where the first morphism is given by the left projection. Similarly, $p^{n,n'}_{12}$ is the composition $\Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to \Delta^{n'}_{X^\sharp/S^\sharp} \to \Delta^{n+n'}_{X^\sharp/S^\sharp}$, where the first morphism is given by the right projection. We denote by $q^{n,n'}_0 \colon \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp}$ (resp. $q^{n,n'}_1 \colon \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp}$) the corresponding morphism (or simply $q^{n,n'}_0$ or q_0). The morphism $q^{n,n'}_0$ is equal to the composition $q^{n,n'}_0 \colon \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp}$ (the last morphism is $\tau \mapsto \tau \otimes 1$). Moreover, $q^{n,n'}_1$ is equal to the composition $q^{n,n'}_1 \colon \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^{n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp}$ (the last morphism is $\tau \mapsto 1 \otimes \tau$).

Lemma 3.1.4.2. *For any $a \in M_{X^\sharp}$, for any integers $n, n' \in \mathbb{N}$, we have $\delta^{n,n'}(\mu^{n+n'}(a)) = \mu^n(a) \otimes \mu^{n'}(a)$.*

Proof. Removing some (m), we copy word by word the proof of Lemma 3.2.2.17. □

Definition 3.1.4.3. The sheaf of differential operators of order $\leq n$ of f is defined by putting $\mathcal{D}_{X^\sharp/S^\sharp,n} := \mathcal{H}om_{\mathcal{O}_X}(p^n_{0*}\mathcal{P}^n_{X^\sharp/S^\sharp}, \mathcal{O}_X)$. For any $n' \geq n$, from the canonical epimorphisms $\psi^{n',n*}_{X^\sharp/S^\sharp} \colon \mathcal{P}^{n'}_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp}$ (see notation 3.1.2.1.1), we get the inclusion

$$\mathcal{D}_{X^\sharp/S^\sharp,n} \hookrightarrow \mathcal{D}_{X^\sharp/S^\sharp,n'}.$$

The sheaf of differential operators of f is defined by putting $\mathcal{D}_{X^\sharp/S^\sharp} := \cup_{n\in\mathbb{N}}\mathcal{D}_{X^\sharp/S^\sharp,n}$.

Let $P \in \mathcal{D}_{X^\sharp/S^\sharp,n}$, $P' \in \mathcal{D}_{X^\sharp/S^\sharp,n'}$. We define the product $PP' \in \mathcal{D}_{X^\sharp/S^\sharp,n+n'}$ to be the composition

$$PP' \colon \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} \xrightarrow{\delta^{n,n'}} \mathcal{P}^{n}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp} \xrightarrow{\mathrm{id}\otimes P'} \mathcal{P}^{n}_{X^\sharp/S^\sharp} \xrightarrow{P} \mathcal{O}_X. \tag{3.1.4.3.1}$$

Suppose f has the formal logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. Put $\tau_{\sharp\lambda} := \mu^n(u_\lambda)-1$. For any $\underline{k} \in \mathbb{N}^r$, set $\underline{\tau}_\sharp^{\underline{k}} := \prod_{i=1}^r \tau_{\sharp i}^{k_i}$. Following 3.1.2.3, the elements $\{\underline{\tau}_\sharp^{\underline{k}}\}_{|\underline{k}|\leq n}$ form a basis of $\mathcal{P}^n_{X^\sharp/S^\sharp}$. The corresponding dual basis of $\mathcal{D}_{X^\sharp/S^\sharp,n}$ will be denoted by $\{\underline{\partial}^{[\underline{k}]},\ |\underline{k}|\leq n\}$. Hence, $\mathcal{D}_{X^\sharp/S^\sharp}$ is a free $\mathcal{O}_X$-module (for both structures) with the basis $\{\underline{\partial}_\sharp^{[\underline{k}]},\ \underline{k}\in\mathbb{N}^r\}$.

Proposition 3.1.4.4. *The sheaf $\mathcal{D}_{X^\sharp/S^\sharp}$ is a sheaf of rings with the product as defined in 3.1.4.3.1.*

Proof. Using Lemma 3.1.4.2 (instead of Lemma 3.2.2.17), we can check the proposition 3.1.4.4 similarly to the proposition 3.2.3.3. □

3.1.4.5. We would like to endow the sheaf $\mathcal{O}_X$ with a canonical structure of left $\mathcal{D}_{X^\sharp/S^\sharp}$-module so that we retrieve the canonical left $\mathcal{D}_{X^\sharp/S^\sharp}$-module of 1.1.4.1.2 when log structure are trivial. By definition, the action of $P \in \mathcal{D}_{X^\sharp/S^\sharp,n}$ on $f \in \mathcal{O}_X$ is denoted by $P(f)$ (in general we avoid to write $P\cdot f$ since this can be confused with the multiplication in $\mathcal{D}_{X^\sharp/S^\sharp}$) and is defined by setting

$$P(f) := P \circ p_1^n(f). \tag{3.1.4.5.1}$$

Contrary to the non-logarithmic case, we need to check that this gives a structure of left $\mathcal{D}_{X^\sharp/S^\sharp}$-module: Let $P \in \mathcal{D}_{X^\sharp/S^\sharp,n}$, $P' \in \mathcal{D}_{X^\sharp/S^\sharp,n'}$. Consider the following diagram

$$\begin{array}{ccccccc}
\mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} & \xrightarrow{\delta^{n,n'}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp} & \xrightarrow{\mathrm{id}\otimes P'} & \mathcal{P}^{n}_{X^\sharp/S^\sharp} & \xrightarrow{P} & \mathcal{O}_X \\
{\scriptstyle p_1^{n+n'}}\uparrow & & \uparrow & & \uparrow{\scriptstyle p_1^{n}} & & \\
\mathcal{O}_X & \xrightarrow{p_1^{n'}} & \mathcal{P}^{n'}_{X^\sharp/S^\sharp} & \xrightarrow{P'} & \mathcal{O}_X, & &
\end{array} \tag{3.1.4.5.2}$$

where the middle vertical map is given by the projection $\Delta^n_{X^\sharp/S^\sharp} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp} \to \Delta^{n'}_{X^\sharp/S^\sharp}$. Since $\delta^{n,n'}_{(m)}$ is a homomorphism of $\mathcal{O}_X$-algebras for the right structure, then the left square of 3.1.4.5.2 is commutative. Since the right square of the diagram 3.1.4.5.2 is commutative by functoriality, then 3.1.4.5.2 is commutative. Hence, by definition of the product given at 3.1.4.3.1 this means that $PP'(x) = P(P'(x))$.

3.2 Sheaf of differential operators of finite level m and finite order on logarithmic schemes

We keep notation of the section 3.1.

3.2.1 Log m-PD envelope

Let $m, i \in \mathbb{N}$ be two integers and $S^\sharp$ be a fine log scheme over the scheme $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$. Let (I_S, J_S, γ) be a quasi-coherent m-PD-ideal of $\mathcal{O}_S$. Let us fix some definitions.

Definition 3.2.1.1. Let $n \geq 1$ be an integer.

(a) Let $\mathscr{C}^{(m)}_{\sharp\gamma}$ (resp. $\mathscr{C}^{(m)}_{\sharp\gamma,n}$) be the category whose objects are pairs (u,δ) where u is an exact closed $S^\sharp$-immersion $Z^\sharp \hookrightarrow X^\sharp$ of fine log-schemes and δ is an m-PD-structure on the ideal $\mathcal{I}$ of $\mathcal{O}_X$ given by u which is compatible (see definition 1.2.4.3) with γ (resp. and such that $\mathcal{I}^{\{n+1\}_{(m)}} = 0$), where $\mathcal{I}^{\{n+1\}_{(m)}}$ is defined in the appendix of [Ber00]) ; whose morphisms $(u',\delta') \to (u,\delta)$ are commutative diagrams of the form

$$\begin{array}{ccc}
X'^\sharp & \xrightarrow{f} & X^\sharp \\
{\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\
Z'^\sharp & \longrightarrow & Z^\sharp
\end{array} \tag{3.2.1.1.1}$$

such that f is an m-PD-morphism with respect to the m-PD-structures δ and δ' (i.e., denoting by $\mathcal{I}'$ the sheaf of ideals of $\mathcal{O}_{X'}$ defined by u', for any affine open sets U' of $X'^\sharp$ and U of $X^\sharp$ such that $f(U') \subset U$, the morphism f induces the m-PD-morphism $(\mathcal{O}_X(U), \mathcal{I}(U), \delta) \to (\mathcal{O}_{X'}(U'), \mathcal{I}'(U'), \delta')$). Beware that theses categories depend on $S^\sharp$ and also on the quasi-coherent m-PD-ideal (I_S, J_S, γ). The objects of $\mathscr{C}^{(m)}_{\sharp\gamma}$ (resp. $\mathscr{C}^{(m)}_{\sharp\gamma,n}$) are called m-PD-$S^\sharp$-immersions compatible with γ (resp. m-PD-$S^\sharp$-immersions of order n compatible with γ). We remark that we have the inclusions $\mathscr{C}^{(m)}_{\sharp\gamma} \subset \mathscr{C}^{(m')}_{\sharp\gamma}$ for any integer $m' \geq m$ (recall an m-PD-structure is also an m'-PD-structure).

We say that a morphism $(u', \delta') \to (u, \delta)$ of $\mathscr{C}^{(m)}_{\sharp\gamma}$ (resp. $\mathscr{C}^{(m)}_{\sharp\gamma,n}$) is strict (resp. flat, resp. cartesian) if f is strict (resp. $\underline{f}$ is flat, resp. the square 3.2.1.1.1 is cartesian).

(b) Let u be an object of $\mathscr{C}_\sharp$ (see the notation 3.1.1.10). An "m-PD-envelope compatible with γ of u" is an object (u', δ') of $\mathscr{C}^{(m)}_{\sharp\gamma}$ endowed with a morphism $u' \to u$ in $\mathscr{C}_\sharp$ satisfying the following universal property: for any object (u'', δ'') of $\mathscr{C}^{(m)}_{\sharp\gamma}$ endowed with a morphism $f: u'' \to u$ of $\mathscr{C}_\sharp$ there exists a unique morphism $(u'', \delta'') \to (u', \delta')$ of $\mathscr{C}^{(m)}_{\sharp\gamma}$ whose composition with $u' \to u$ is f. The unicity up to canonical isomorphism of the m-PD-envelope compatible with γ of u is obvious. We will denote by $P^\sharp_{(m),\gamma}(u)$ the m-PD-envelope compatible with γ of u. By abuse of notation we also denote by $P^\sharp_{(m),\gamma}(u)$ the underlying exact closed immersion or its target. The existence is checked below (see 3.2.1.9).

(c) Let u be an object of $\mathscr{C}_\sharp$. An "m-PD-envelope of order n compatible with γ of u" is an object (u', δ') of $\mathscr{C}^{(m)}_{\sharp\gamma,n}$ endowed with a morphism $u' \to u$ in $\mathscr{C}_\sharp$ satisfying the following universal property: for any object (u'', δ'') of $\mathscr{C}^{(m)}_{\sharp\gamma,n}$ endowed with a morphism $f: u'' \to u$ of $\mathscr{C}_\sharp$ there exists a unique morphism $(u'', \delta'') \to (u', \delta')$ of $\mathscr{C}^{(m)}_{\sharp\gamma,n}$ whose composition with $u' \to u$ is f. The unicity up to canonical isomorphism of the m-PD-envelope of order n compatible with γ of u is obvious. We will denote by $P^{\sharp n}_{(m),\gamma}(u)$ the m-PD-envelope of order n compatible with γ of u. By abuse of notation we also denote by $P^{\sharp n}_{(m),\gamma}(u)$ the underlying exact closed immersion or its target. The existence is checked below (see 3.2.1.9).

(d) Since p is nilpotent in $S^\sharp$, we get the forgetful functor $For^{(m)}_\sharp: \mathscr{C}^{(m)}_{\sharp\gamma} \to \mathscr{T}hick_{(p)}$ (resp. $For^{(m)}_{\sharp n}: \mathscr{C}^{(m)}_{\sharp\gamma,n} \to \mathscr{T}hick_{(p)}$) given by $(u, \delta) \mapsto u$. We denote by $\mathscr{C}'^{(m)}_{\sharp\gamma}$ (resp. $\mathscr{C}'^{(m)}_{\sharp\gamma,n}$) the image of $For^{(m)}_\sharp$ (resp. $For^{(m)}_{\sharp n}$).

Notation 3.2.1.2. In this paragraph, suppose $J_S = p\mathcal{O}_S$. Then, there is a unique PD-structure on J_S which we will denote by $\gamma_\emptyset$. Let $u: Z^\sharp \hookrightarrow X^\sharp$ be an exact closed $S^\sharp$-immersion of fine log-schemes and δ be an m-PD-structure on the ideal $\mathcal{I}$ of $\mathcal{O}_X$ defined by u. It follows from Lemma 1.2.3.4 that the m-PD-structure δ of $\mathcal{I}$ is always compatible with $\gamma_\emptyset$. Hence, in the description of $\mathscr{C}^{(m)}_{\sharp\gamma_\emptyset}$ (resp. $\mathscr{C}^{(m)}_{\sharp\gamma_\emptyset,n}$) we can remove "compatible with $\gamma_\emptyset$" without changing the respective categories. For this reason, we put $\mathscr{C}^{(m)}_\sharp := \mathscr{C}^{(m)}_{\sharp\gamma_\emptyset}$ (resp. $\mathscr{C}^{(m)}_{\sharp n} := \mathscr{C}^{(m)}_{\sharp\gamma_\emptyset,n}$). But, recall these categories depend on $S^\sharp$ even if this is not written in the notation. Finally, for any quasi-coherent m-PD-ideal (I_S, J_S, γ) of $\mathcal{O}_S$, we have the inclusions

$$\mathscr{C}^{(m)}_{\sharp\gamma} \subset \mathscr{C}^{(m)}_\sharp \text{ and } \mathscr{C}^{(m)}_{\sharp\gamma,n} \subset \mathscr{C}^{(m)}_{\sharp n}. \tag{3.2.1.2.1}$$

The following proposition gives the link between our categories.

Proposition 3.2.1.3. *Suppose that $J_S + p\mathcal{O}_S$ is locally principal.*

(a) We have the inclusion $\mathscr{C}_{\sharp 1} \subset \mathscr{C}'^{(m)}_{\sharp\gamma,1}$.

(b) For any $n, m \in \mathbb{N}$ such that $n + 1 \leq p^m$, we have the inclusion $\mathscr{C}_{\sharp n} \subset \mathscr{C}'^{(m)}_{\sharp\gamma,n}$;

(c) We have the equality $\cup_{m\in\mathbb{N}} \mathscr{C}'^{(m)}_{\sharp\gamma} = \mathscr{T}hick_{(p)}$.

Proof. Let us check the first two assertions. Let $u: U^\sharp \hookrightarrow T^\sharp$ a $S^\sharp$-log-thickening of order n, let $\mathcal{I}$ be the ideal defined by the closed immersion $\underline{u}$. When $\mathcal{I}^2 = 0$, we get a PD-structure γ on $\mathcal{I}$ defined by putting $\gamma_n = 0$ for any integer $n \geq 2$. Since $J_S + p\mathcal{O}_S$ is locally principal, then from [Ber96c, 1.3.2.(i).b)]

γ extends to T. Hence, $\mathscr{C}_{\sharp 1} \subset \mathscr{C}^{\sharp \prime (0)}_{\gamma,1}$, which yields the first inclusion to prove. Suppose now $\mathcal{I}^{n+1} = 0$ and $n+1 \leq p^m$. In that case, $\mathcal{I}^{(p^m)} = 0$. Hence, $(0, \delta)$ is an m-PD-structure of $\mathcal{I}$ (where δ is the unique PD-structure on 0). Let us check that the m-PD structure $(0, \delta)$ of $\mathcal{I}$ is compatible with γ. By definition, we have to check two properties (see 1.2.4.3). Since γ extends to T, then the property 1.2.4.3.(iia) is satisfied (see Definition [Ber96c, 1.2.2]). The second one 1.2.4.3.(iib) is a straightforward consequence of Lemma [Ber96c, 1.2.4.(i)]. Hence, $(u, \delta) \in \mathscr{C}^{(m)}_{\sharp\gamma}$. Since $\mathcal{I}^{n+1} = 0$, we have in fact $(u, \delta) \in \mathscr{C}^{(m)}_{\sharp\gamma,n}$. By definition, this yields $u \in \mathscr{C}'^{(m)}_{\sharp\gamma,n}$.

Let us check the last statement. The inclusion $\cup_{m\in\mathbb{N}} \mathscr{C}'^{(m)}_{\sharp\gamma} \subset \mathscr{T}hick_{(p)}$ is tautologic. Conversely, let $u\colon U^\sharp \hookrightarrow T^\sharp$ be an $S^\sharp$-log-thickening of order (p^m), let $\mathcal{I}$ be the ideal defined by the closed immersion $\underline{u}$. Since $\mathcal{I}^{(p^m)} = 0$, then following the first part of the proof, we get that the m-PD structure $(0, \delta)$ is compatible with γ of $\mathcal{I}$. Hence, $u \in \mathscr{C}'^{(m)}_{\sharp\gamma}$, which concludes the proof of the last statement. □

3.2.1.4. Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an exact $S^\sharp$-immersion of fine log-schemes. Set $(\underline{v}, \delta) := P_{(m),\gamma}(\underline{u})$ (see 1.3.3.4). Let (v, δ) be the object of $\mathscr{C}^{(m)}_{\sharp\gamma}$ whose underlying object of $\mathscr{C}^{(m)}_{\gamma}$ is $(\underline{v}, \delta)$ and v is defined so that the morphism $v \to u$ of $\mathscr{C}_\sharp$ is strict (see the definition 3.1.1.10). Then (v, δ) is the m-PD-envelope compatible with γ of u.

Remark 3.2.1.5. Let $\alpha\colon (u', \delta') \to (u, \delta)$ be a strict cartesian morphism of $\mathscr{C}^{(m)}_{\sharp\gamma}$. Let (u'', δ'') be an object of $\mathscr{C}^{(m)}_{\sharp\gamma}$ and $\beta\colon u'' \to u'$ be a morphism of $\mathscr{C}_\sharp$. We remark that if $For^{(m)}_\sharp(\alpha) \circ \beta$ is in the image of $For^{(m)}_\sharp$ then so is β. Indeed, the morphism α is defined by a cartesian diagram of the form 3.2.1.1.1. Since α is moreover strict, then we remark that $Z'^\sharp = Z^\sharp \times_{X^\sharp} X'^\sharp$, i.e. the diagram 3.2.1.1.1 remains cartesian after applying the forgetful functor from the category of fine log schemes to the category of schemes. Hence, we can conclude.

3.2.1.6. Let $u' \to u$ be a strict, flat, cartesian morphism of $\mathscr{C}_\sharp$, i.e. let

$$\begin{array}{ccc} X'^\sharp & \xrightarrow{g} & X^\sharp \\ {\scriptstyle u'}\uparrow & \square & \uparrow{\scriptstyle u} \\ Z'^\sharp & \longrightarrow & Z^\sharp \end{array}$$

be a cartesian square whose morphism g is strict and $\underline{g}$ is flat. Suppose that the m-PD-envelope compatible with γ of u exists (in fact, this existence will be proved later in 3.2.1.9). Let (v, δ) be this m-PD-envelope. Set $v' := v \times_u u'$ and let $g'\colon v' \to v$ be the projection. Since $\underline{g}$ is flat and g is strict, then g' is strict and $\underline{g}'$ is flat. From [Ber96c, 1.3.2.(i)], there exists a canonical m-PD-structure δ' compatible with γ on the ideal defined by $v' := v \times_u u'$ such that the projection $g'\colon v' \to v$ induces a strict cartesian morphism of $\mathscr{C}^{(m)}_{\sharp\gamma}$ of the form $(v', \delta') \to (v, \delta)$. With the remark 3.2.1.5, we can check that (v', δ') is an m-PD-envelope compatible with γ of u'.

Lemma 3.2.1.7. *Let $f\colon X^\sharp \to Y^\sharp$ be a log étale $S^\sharp$-morphism, $u\colon Z^\sharp \hookrightarrow X^\sharp$ and $v\colon Z^\sharp \hookrightarrow Y^\sharp$ be two $S^\sharp$-immersions of fine log schemes such that $v = f \circ u$. If the m-PD envelope of order n exists then it is also an m-PD envelope of order n of v.*

Proof. Let $(P(u), \delta)$ be the m-PD envelope (resp. of order n) of u. Let us check that the composition of the canonical morphism $P(u) \to u$ with the morphism $u \to v$ (induced by f) satisfies the universal property of the m-PD envelope (resp. of order n). Let (v', δ') be an object of $\mathscr{C}^{(m)}_{\sharp\gamma}$ (resp. $\mathscr{C}^{(m)}_{\sharp\gamma,n}$) and $g\colon v' \to v$ be a morphism of $\mathscr{C}_\sharp$. Using the universal property of log étaleness, we get a unique morphism $h\colon v' \to u$ of $\mathscr{C}_\sharp$ whose composition with $u \to v$ gives g. Using the universal property of the m-PD-envelope of u compatible with γ that there exists a unique morphism $(v', \delta') \to (P(u), \delta)$ of $\mathscr{C}^{(m)}_{\sharp\gamma}$ (resp. $\mathscr{C}^{(m)}_{\sharp\gamma,n}$) such that the composition of $v' \to P(u)$ with $P(u) \to u$ is h.

□

Lemma 3.2.1.8. *The inclusion functor $For_{\sharp n}\colon \mathscr{C}^{(m)}_{\sharp\gamma,n} \to \mathscr{C}^{(m)}_{\sharp\gamma}$ has a right adjoint. We denote by $Q^{\sharp n}_{(m),\gamma}\colon \mathscr{C}^{(m)}_{\sharp\gamma} \to \mathscr{C}^{(m)}_{\sharp\gamma,n}$ this right adjoint functor. The functor $Q^{\sharp n}_{(m),\gamma}$ preserves the sources.*

Proof. Let (u, δ) be an object of $\mathscr{C}^{(m)}_{\sharp\gamma}$ and $\mathcal{I}$ be the ideal defined by the exact closed immersion $u \colon Z^\sharp \hookrightarrow X^\sharp$. Let $Q^n \hookrightarrow X^\sharp$ be the exact closed immersion which is defined by $\mathcal{I}^{\{n+1\}_{(m)}}$. It follows from 1.3.1.4 that $Q^{\sharp n}_{(m),\gamma}(u)$ exists and is equal to the exact closed immersion $Z^\sharp \hookrightarrow Q^n$. □

Proposition 3.2.1.9. *Let $u \colon Z^\sharp \hookrightarrow X^\sharp$ be an object of $\mathscr{C}_\sharp$.*

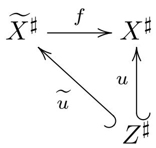
(a) The m-PD-envelope compatible with γ of u exists. In other words, the canonical functor $For^{(m)}_\sharp \colon \mathscr{C}^{(m)}_{\sharp\gamma} \to \mathscr{C}_\sharp$ has a right adjoint. We denote by $P^\sharp_{(m),\gamma} \colon \mathscr{C}_\sharp \to \mathscr{C}^{(m)}_{\sharp\gamma}$ this right adjoint functor. Similarly, the m-PD-envelope of order n compatible with γ of u exists, i.e. we get the right adjoint functor $P^{\sharp n}_{(m),\gamma} \colon \mathscr{C}_\sharp \to \mathscr{C}^{(m)}_{\sharp\gamma,n}$ of the canonical functor $For^{(m)}_{\sharp n} \colon \mathscr{C}^{(m)}_{\sharp\gamma,n} \to \mathscr{C}_\sharp$. We have the relation $P^{\sharp n}_{(m),\gamma} = Q^{\sharp n}_{(m),\gamma} \circ P^\sharp_{(m),\gamma}$.

(b) If γ extends to $Z^\sharp$ then the source of $P^\sharp_{(m),\gamma}(u)$ is $Z^\sharp$.

(c) By an abuse of notation we let $P^\sharp_{(m),\alpha}(u)$ (resp. $P^{\sharp n}_{(m),\alpha}(u)$) denote the target of the arrow $P^\sharp_{(m),\alpha}(u)$ (resp. $P^{\sharp n}_{(m),\alpha}(u)$). Let $U^\sharp$ be an open of $X^\sharp$ containing $Z^\sharp$ such that the induced immersion $v \colon Z^\sharp \hookrightarrow U^\sharp$ is closed. Then $P^\sharp_{(m),\alpha}(u) = P^\sharp_{(m),\alpha}(v)$ and the underling morphism of schemes of $P^\sharp_{(m),\alpha}(u) \to U$ (resp. $P^{\sharp n}_{(m),\alpha}(u) \to U$) is affine. We denote by $\mathcal{P}_{(m),\gamma}(u)$ (resp. $\mathcal{P}^n_{(m),\gamma}(u)$) the quasi-coherent $\mathcal{O}_U$-algebra so that $P_{(m),\gamma}(u) = \mathrm{Spec}(\mathcal{P}_{(m),\gamma}(u))$ (resp. $P^n_{(m),\gamma}(u) = \mathrm{Spec}(\mathcal{P}^n_{(m),\gamma}(u))$) where $P_{(m),\gamma}(u)$ (resp. $P^n_{(m),\gamma}(u)$) is the underlying scheme of $P^\sharp_{(m),\gamma}(u)$ (resp. $P^{\sharp n}_{(m),\gamma}(u)$). The m-PD ideal of $\mathcal{P}_{(m),\gamma}(u)$ will be denoted by $(\mathscr{I}_{(m),\gamma}(u), \mathscr{J}_{(m),\gamma}(u), ^{[\,]})$. Moreover, the sheaf $\mathcal{P}_{(m),\gamma}(u)$ has his support in $v(Z)$ and we can simply denote $v^{-1}\mathcal{P}_{(m),\gamma}(u)$ by $\mathcal{P}_{(m),\gamma}(u)$.

(d) Suppose that $J_S + p\mathcal{O}_S$ is locally principal and that X is noetherian. Then $P^n_{(m),\gamma}(u)$ is a noetherian scheme.

Proof. 1) We can suppose u is a closed immersion. First, let us check the proposition concerning the existence of $P^\sharp_{(m),\gamma}(u)$ and its properties (i.e. the second part of the proposition and the affinity of the morphism $P^\sharp_{(m),\gamma}(u) \to X^\sharp$). Using 3.2.1.6, the existence of $P^\sharp_{(m),\gamma}(u)$ and its properties are étale local on $X^\sharp$. Hence, by 3.1.1.14, we may thus assume that there exists a commutative diagram of the form



such that f is log étale, $\underline{f}$ is affine and $\widetilde{u}$ is an exact closed $S^\sharp$-immersion. In that case, following 3.2.1.4 the m-PD-envelope compatible with γ of $\widetilde{u}$ exists and the induced object of $\mathscr{C}^{(m)}_\gamma$ is $P_{(m),\gamma}(\underline{\widetilde{u}})$. Following 3.2.1.7, the m-PD-envelope compatible with γ of u exists and is isomorphic to that of $\widetilde{u}$. Concerning the second statement, when γ extends to $Z^\sharp$, following [Ber96c, 2.1.1] (or [Ber96c, 1.4.5] for the affine version), the source of the immersion $P_{(m),\gamma}(\underline{\widetilde{u}})$ is $Z^\sharp$. Since $P^\sharp_{(m),\gamma}(\widetilde{u})$, $\widetilde{u}$ are exact closed immersion, since the morphism $P^\sharp_{(m),\gamma}(\widetilde{u}) \to \widetilde{u}$ is strict (see 3.2.1.4), then so is the morphism of sources induced by $P^\sharp_{(m),\gamma}(\widetilde{u}) \to \widetilde{u}$. Hence, we get the second statement. We can check the third statement by recalling that the target of $P_{(m),\gamma}(\underline{\widetilde{u}})$ is affine over $\widetilde{X}$ (see 1.3.3.4) and that $P_{(m),\gamma}(\underline{\widetilde{u}}) \to \widetilde{u}$ is strict. Concerning the noetherianity, if X is noetherian then so is $\widetilde{X}$. Hence, using [Ber96c, 1.4.4] and the description of the m-PD filtration given in the proof of [Ber96c, A.2], we get that $P^n_{(m),\gamma}(\underline{\widetilde{u}})$ is noetherian (but not $P_{(m),\gamma}(\underline{\widetilde{u}})$).

2) From Lemma 3.2.1.8, we can check that the functor $Q^{\sharp n}_{(m),\gamma} \circ P^\sharp_{(m),\gamma}$ is a right adjoint of $For^{(m)}_{\sharp n} \colon \mathscr{C}^{(m)}_{\sharp\gamma,n} \to \mathscr{C}_\sharp$. Moreover, with the description of the functor $Q^{\sharp n}_{(m),\gamma}$ given in the proof of 3.2.1.8, we can check the other properties concerning $P^{\sharp n}_{(m),\gamma}$ from that of $P^\sharp_{(m),\gamma}$.

□

Remark 3.2.1.10. Let (u, δ) be an object of $\mathscr{C}^{(m)}_{\sharp\gamma}$. Then $P^{\sharp}_{(m),\delta}(u) = (u, \delta)$. But, beware that $P_{(m),\gamma}(u) \neq (u, \delta)$ in general.

3.2.1.11 (The case of an exact closed immersion)**.** Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an exact closed $S^\sharp$-immersion of fine log-schemes and $\mathcal{I}$ be the ideal defined by u. We denote by $u^{(m)}\colon Z^{\sharp(m)} \hookrightarrow X^\sharp$ the exact closed $S^\sharp$-immersion of fine log-schemes so that $\mathcal{I}^{(p^m)}$ is the ideal defined by $u^{(m)}$. Since the closed immersion u is exact, in the proof of 3.2.1.9, we can skip the part concerning the exactification of u (i.e. we can suppose $f = \mathrm{id}$ or equivalently $\widetilde{u} = u$). Hence, we remark that, as in the proof of [Ber96c, 1.4.1], we get the equality

$$P^{\sharp}_{(m),\gamma}(u) = P^{\sharp}_{(0),\gamma}(u^{(m)}). \tag{3.2.1.11.1}$$

We have also the same construction as in the proof of [Ber96c, 1.4.1] (too technical to be described here in few words) of the m-PD ideal $(\mathscr{I}_{(m),\gamma}(u), \mathscr{J}_{(m),\gamma}(u), ^{[\,]})$ of $\mathcal{P}_{(m),\gamma}(u)$ directly from the level 0 case. For the detailed descriptions, see the proof of [Ber96c, 1.4.1]. These descriptions, in particular 3.2.1.11.1, are useful to check the Frobenius descent for arithmetic $\mathcal{D}$-modules (see [Ber00, 2.3.6]).

Lemma 3.2.1.12. *We have the equality* $P^{\sharp n}_{(m),\gamma} \circ For_{\sharp n} \circ P^{\sharp n} = P^{\sharp n}_{(m),\gamma}$*, where* $For_{\sharp n}\colon \mathscr{C}_{\sharp n} \to \mathscr{C}_{\sharp}$ *is the canonical functor and* $P^{\sharp n}\colon \mathscr{C}_{\sharp} \to \mathscr{C}_{\sharp n}$ *is its right adjoint (see 3.1.1.17).*

Proof. Let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an object of $\mathscr{C}_\sharp$. Looking at the construction of P^n and $P^{\sharp n}_{(m),\gamma}$ (see the proof of 3.1.1.17 and 3.2.1.9), we reduce to the case where u is an exact closed immersion. In that case, the Lemma is a reformulation of 1.3.2.8.2. □

The following proposition will not be useful later but it allows us to extend 3.1.3.6 is some particular case.

Proposition 3.2.1.13. *Suppose that* $J_S + p\mathcal{O}_S$ *is locally principal. Let* $f\colon X^\sharp \to Y^\sharp$ *be an* $S^\sharp$*-morphism of fine log schemes and* $\Delta_{X^\sharp/Y^\sharp}\colon X^\sharp \hookrightarrow X^\sharp \times_{Y^\sharp} X^\sharp$ *(as always the product is taken in the category of fine log schemes) be the diagonal* $S^\sharp$*-immersion. The following assertions are equivalent:*

1. *the morphism* f *is fine formally log unramified ;*
2. *the morphism* $P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})$ *is an isomorphism ;*
3. *the morphism* $P^{\sharp 1}_{(m),\gamma}(\Delta_{X^\sharp/Y^\sharp})$ *is an isomorphism.*

Proof. The equivalence between 1) and 2) has already been checked (see 3.1.3.6). Following 3.2.1.3, since $J_S + p\mathcal{O}_S$ is locally principal, then $\mathscr{C}_{\sharp 1} \subset \mathscr{C}'^{(m)}_{\sharp\gamma,1}$. Hence, we have 3) $\Rightarrow$ 1). It follows from 3.2.1.12 that $P^{\sharp 1}_{(m),\gamma}(P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})) = P^{\sharp 1}_{(m),\gamma}(\Delta_{X^\sharp/Y^\sharp})$. If $P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})$ is an isomorphism, then $P^{\sharp 1}_{(m),\gamma}(P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})) = P^{\sharp 1}(\Delta_{X^\sharp/Y^\sharp})$. Hence, we get the implication 2) $\Rightarrow$ 4). It remains to check 4) $\Rightarrow$ 3). Suppose $P^{\sharp 1}_{(m),\gamma}(\Delta_{X^\sharp/Y^\sharp})$ is an isomorphism and let $(\iota, \delta) \in \mathscr{C}^{(m)}_{\sharp,\gamma,1}$ and let

$$\begin{array}{ccc} U^\sharp & \xrightarrow{u_0} & X^\sharp \\ {\scriptstyle \iota}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T^\sharp & \xrightarrow{v} & Y^\sharp \end{array}$$

be a commutative diagram of fine log schemes. Suppose there exist a morphism $u\colon T^\sharp \to X^\sharp$ such that $u \circ \iota = u_0$ and $f \circ u = v$, and a morphism $u'\colon T^\sharp \to X^\sharp$ such that $u' \circ \iota = u_0$ and $f \circ u' = v$. We get the morphism $(u, u')\colon T^\sharp \to X^\sharp \times_{Y^\sharp} X^\sharp$. We denote by and $\phi\colon \iota \to \Delta_{X^\sharp/Y^\sharp}$ be a morphism of $\mathscr{C}_\sharp$ induced by (u', u) and u_0. Using the universal property of the m-PD envelope of order 1, there exists a unique morphism $\psi\colon (\iota, \delta) \to P^{\sharp 1}_{(m),\gamma}(\Delta_{X^\sharp/Y^\sharp})$ of $\mathscr{C}^{(m)}_{\sharp,\gamma,1}$ such that the composition of $For^{(m)}_{\sharp n}(\psi)$ with the canonical map $P^{\sharp 1}_{(m),\gamma}(\Delta_{X^\sharp/Y^\sharp}) \to \Delta_{X^\sharp/Y^\sharp}$ is ϕ. Since $P^{\sharp 1}_{(m),\gamma}(\Delta_{X^\sharp/Y^\sharp})$ is an isomorphism, this yields that $(u, u')\colon T^\sharp \to X^\sharp \times_{Y^\sharp} X^\sharp$ is the composition of a morphism of the form $T^\sharp \to X^\sharp$ with $\Delta_{X^\sharp/Y^\sharp}$. Hence, $u = u'$. □

Lemma 3.2.1.14. *Let $u \to v$ be a morphism of $\mathscr{C}_\sharp$. Let δ be the m-PD-structure of $P^\sharp_{(m),\gamma}(v)$ and $w := P^\sharp_{(m),\gamma}(v) \times_v u$ (this is the product in $\mathscr{C}_\sharp$). We denote by $P^\sharp_{(m),\delta}(w)$ the m-PD-envelope of w compatible with δ. Then $P^\sharp_{(m),\delta}(w)$ and $P^\sharp_{(m),\gamma}(u)$ are isomorphic in $\mathscr{C}^{(m)}_{\sharp\gamma}$. Moreover, $P^{\sharp n}_{(m),\delta}(w)$ and $P^{\sharp n}_{(m),\gamma}(u)$ are isomorphic in $\mathscr{C}^{(m)}_{\sharp\gamma,n}$.*

Proof. Since the second assertion is checked in the same way, let us prove the first one. Let us check that the composition $P^\sharp_{(m),\delta}(w) \to w \to u$ satisfies the universal property of $P^\sharp_{(m),\gamma}(u) \to u$. Let $(u', \delta') \in \mathscr{C}^{(m)}_{\sharp\gamma}$ and $f : u' \to u$ be a morphism of $\mathscr{C}_\sharp$. First let us check the existence. Composing f with $u \to v$ we get a morphism $g : u' \to v$. Using the universal property of the m-PD envelope, there exists a morphism $\phi : (u', \delta') \to (P_{(m),\gamma}(v), \delta)$ of $\mathscr{C}^{(m)}_{\sharp\gamma}$ such that the composition $u' \to P^\sharp_{(m),\gamma}(v) \to v$ is g. Hence, we get the morphism $(\phi, f) : u' \to w$. Using the universal property of $P^\sharp_{(m),\delta}(w)$, we get a morphism $u' \to P^\sharp_{(m),\delta}(w)$ of $\mathscr{C}^{(m)}_{\sharp\delta}$ (and then of $\mathscr{C}^{(m)}_{\sharp\gamma}$) whose composition with $P^\sharp_{(m),\delta}(w) \to w \to u$ is f. Let us check the unicity. Let $\alpha : u' \to P^\sharp_{(m),\delta}(w)$ be a morphism of $\mathscr{C}^{(m)}_{\sharp\gamma}$ whose composition with $P^\sharp_{(m),\delta}(w) \to w \to u$ is f. This implies that the composition of α with $P^\sharp_{(m),\delta}(w) \to w \to P^\sharp_{(m),\gamma}(v) \to v$ is g. Since the composition $P^\sharp_{(m),\delta}(w) \to w \to P^\sharp_{(m),\gamma}(v)$ is a morphism of $\mathscr{C}^{(m)}_{\sharp\delta}$, then so is the composition of α with $P^\sharp_{(m),\delta}(w) \to w \to P^\sharp_{(m),\gamma}(v)$ (in particular, this implies that $\alpha \in \mathscr{C}^{(m)}_{\sharp\delta}$). Using the universal property of $P^\sharp_{(m),\gamma}(v)$, this latter composition morphism is uniquely determined by g. Hence, the composition of α with $P^\sharp_{(m),\delta}(w) \to w$ is a morphism of $\mathscr{C}_\sharp$ uniquely determined by f. Since α is a morphism of $\mathscr{C}^{(m)}_{\sharp\delta}$, we conclude using the universal property of $P^\sharp_{(m),\delta}(w)$. □

Lemma 3.2.1.15. *Let $r \geq 0$ be an integer, $(v, \delta) \in \mathscr{C}^{(m)}_{\sharp\gamma}$ where $v : T^\sharp \hookrightarrow D^\sharp$ is an exact closed $S^\sharp$-immersion of fine log-schemes and $(\widetilde{\mathcal{K}}, \delta)$ is an m-PD-structure compatible with γ on the ideal $\mathcal{K}$ of $\mathcal{O}_D$ defined by v. Let $(e_\lambda)_{\lambda=1\dots r}$ be the canonical basis of $\mathbb{Z}^r$. Let $i : D^\sharp \to D^\sharp \times_{\mathbb{Z}} A_{\mathbb{Z}^r}$ be the exact closed D-immersion defined by $e_\lambda \mapsto 1 \in \Gamma(D, M_{D^\sharp})$. With notation 3.2.1.9 and 1.3.3.6, we have the following properties.*

(a) The homomorphism of rings

$$\mathcal{O}_D\langle T_1, \dots, T_r\rangle_{(m)} \to \mathcal{P}_{(m),\delta}(i \circ v)$$

given by $T_\lambda \mapsto e_\lambda - 1$ is an isomorphism.

(b) The structural m-PD ideal $(\mathscr{I}_{(m),\gamma}(i \circ v), \mathscr{J}_{(m),\gamma}(i \circ v), {}^{[\,]})$ of $\mathcal{P}_{(m),\delta}(i \circ v)$ is given by

$$\mathscr{I}_{(m),\gamma}(i \circ v) = \mathscr{I}_{D,(m),r} + \mathcal{K}\mathcal{P}_{(m),\delta}(i \circ v),$$
$$\mathscr{J}_{(m),\gamma}(i \circ v) = \mathscr{J}_{D,(m),r} + \widetilde{\mathcal{K}}\mathcal{P}_{(m),\delta}(i \circ v).$$

Proof. By using the remark 1.3.2.9 and 3.2.1.4, we can suppose that $v = \mathrm{id}$. Since the ideal of the exact closed immersion i is generated by the regular sequence $(e_\lambda - 1)_{\lambda=1,\dots r}$, using 1.3.2.6 and 3.2.1.4 we can check that the morphism of $\mathcal{O}_D$-algebras $\mathcal{O}_D\langle T_1, \dots, T_r\rangle_{(m)} \to \mathcal{P}_{(m),\delta}(i)$ given by $T_\lambda \mapsto e_\lambda - 1$ is an isomorphism. □

Notation 3.2.1.16. With notation 3.2.1.15, we set $\mathcal{O}_{(v,\delta)}\langle T_1, \dots, T_r\rangle_{(m)} := \mathcal{P}_{(m),\delta}(i \circ v)$ and $\mathcal{O}_{(v,\delta)}\langle T_1, \dots, T_r\rangle_{(m),n} := \mathcal{P}^n_{(m),\delta}(i \circ v)$.

Proposition 3.2.1.17. *Let $f : X^\sharp \to Y^\sharp$ be an $S^\sharp$-morphism of fine log-schemes, $(b_\lambda)_{\lambda=1,\dots,r}$ be some elements of $\Gamma(X, M_{X^\sharp})$ such that $(b_\lambda)_{\lambda=1,\dots,r}$ are formal logarithmic coordinates of f.*

Let $u : Z^\sharp \hookrightarrow X^\sharp$ and $v : Z^\sharp \hookrightarrow Y^\sharp$ be two $S^\sharp$-immersions of fine log schemes such that $v = f \circ u$. Suppose given $y_\lambda \in \Gamma(Y, M_{Y^\sharp})$ whose images in $\Gamma(Z, M_{Z^\sharp})$ coincide with the images of b_λ.

Let $P^\sharp_{(m),\gamma}(u) = (T'^\sharp \hookrightarrow D'^\sharp, \delta')$, $P^\sharp_{(m),\gamma}(v) = (T^\sharp \hookrightarrow D^\sharp, \delta)$, and $\alpha : D'^\sharp \to X^\sharp$ be the canonical morphism. Using multiplicative notation, put $u_\lambda := \frac{\alpha^*(b_\lambda)}{\alpha^*(f^*(y_\lambda))} \in \ker(\Gamma(D', M^{\mathrm{gr}}_{D'^\sharp}) \to \Gamma(T', M^{\mathrm{gr}}_{T'^\sharp})) = \ker(\Gamma(D', \mathcal{O}^*_{D'}) \to \Gamma(T', \mathcal{O}^*_{T'}))$ (see 3.1.1.24). Let $P^{\sharp n}_{(m),\gamma}(u) = (T'^\sharp \hookrightarrow D'^\sharp_n, \delta'_n)$, and $u_{\lambda,n}$ be the image of u_λ in $\ker(\Gamma(D'_n, \mathcal{O}^*_{D'_n}) \to \Gamma(T', \mathcal{O}^*_{T'}))$.

(a) By using notation 3.2.1.16, we have the isomorphism of m-PD-$\mathcal{O}_D$-algebras

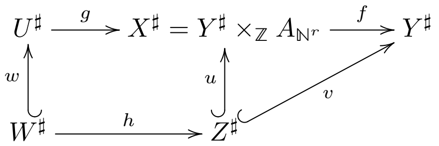
$$\mathcal{O}_{P_{(m),\gamma}(v)}\langle T_1,\dots,T_r\rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}^n_{(m),\gamma}(u)$$
$$T_\lambda \mapsto u_{\lambda,n} - 1. \qquad (3.2.1.17.1)$$

(b) If $(b_\lambda)_{\lambda=1,\dots,r}$ are moreover logarithmic coordinates then we have the isomorphism of m-PD-$\mathcal{O}_D$-algebras

$$\mathcal{O}_{P_{(m),\gamma}(v)}\langle T_1,\dots,T_r\rangle_{(m)} \xrightarrow{\sim} \mathcal{P}_{(m),\gamma}(u)$$
$$T_\lambda \mapsto u_\lambda - 1. \qquad (3.2.1.17.2)$$

Proof. In order to check 3.2.1.17.1 (resp. 3.2.1.17.2), using the first part of lemma 3.2.1.7 (resp. the second part of lemma 3.2.1.7) and using the first remark of 3.1.1.9, we may assume that $X^\sharp = Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{N}^r}$, $f\colon X^\sharp \to Y^\sharp$ is the first projection, and that the family $(b_\lambda)_{\lambda=1,\dots,r}$ are the elements of $\Gamma(X, M_{X^\sharp})$ corresponding to the canonical basis $(e_\lambda)_{\lambda=1\dots r}$ of $\mathbb{N}^r$. Using lemma 3.2.1.14, we may furthermore assume that $Y^\sharp = S^\sharp$, $Z^\sharp \hookrightarrow Y^\sharp$ is the exact closed immersion whose ideal of definition is I_S. In particular, we get $D^\sharp = Y^\sharp$ and γ is the canonical m-PD structure of $D^\sharp$.

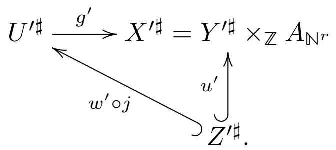
Let $\overline{z}$ be a geometric point of $Z^\sharp$. From 3.1.1.14, there exists a commutative diagram of the form



(3.2.1.17.3)

where g is log étale, h is an étale neighborhood of $\overline{z}$ in $Z^\sharp$, and w is an exact closed $S^\sharp$-immersion. We set $v_\lambda := \frac{g^*(b_\lambda)}{(f\circ g)^*(y_\lambda)} \in \mathrm{Ker}(\Gamma(U, M^{\mathrm{gp}}_{U^\sharp}) \to \Gamma(W, M^{\mathrm{gp}}_{W^\sharp}))$. Since w is an exact closed immersion, using 3.1.1.24, shrinking $U^\sharp$ if necessary we may thus assume that $v_\lambda \in \Gamma(U, \mathcal{O}^*_U)$.

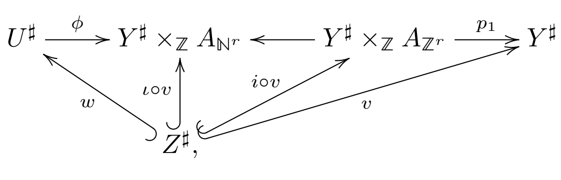
1) In this step, we reduce to the case where $h = \mathrm{id}$. According to [Gro67, IV.18.1.1], there exist an étale neighborhood $Y'^\sharp \to Y^\sharp$ of $\overline{z}$ in $Y^\sharp$ and an open $Z^\sharp$-immersion (see the definition 3.1.1.8) $\rho\colon Z'^\sharp := Z^\sharp \times_{Y^\sharp} Y'^\sharp \to W^\sharp$ which is a morphism of étale neighborhoods of $\overline{z}$ in $Z^\sharp$ (in particular $h\circ\rho\colon Z^\sharp \times_{Y^\sharp} Y'^\sharp \to Z^\sharp$ is the canonical projection). Let us use the prime symbol to denote the base change by $Y'^\sharp \to Y^\sharp$ of a $Y^\sharp$-log scheme or a morphism of $Y^\sharp$-log schemes. Set $j := (\rho, v')\colon {}'^\sharp \to W^\sharp \times_{Y^\sharp} Y'^\sharp = W'^\sharp$. Since $h\circ\rho\colon Z'^\sharp \to Z^\sharp$ is the canonical projection, then we compute that $h'\circ j = \mathrm{id}$. Since *id* is an immersion, then j is an immersion (see the first remark of 3.1.1.9). Since h' and *id* are etale then so is j. Hence, j is an open $Y^\sharp$-immersion. Using $h'\circ j = \mathrm{id}$, we get the commutative diagram over $Y'^\sharp$



Using 3.1.1.9.c we may assume (shrinking $U^\sharp$ if necessary) that the exact $Y^\sharp$-immersion $w'\circ j$ is closed. Since the proposition is étale local on $Y^\sharp$, we can drop the primes, i.e. we can suppose $h = \mathrm{id}$.

2) Consider the $Y^\sharp$-morphism $\phi\colon U^\sharp \to Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ defined by the v_λ's. Since the $(d\log g^*(b_1),\dots,d\log g^*(b_r))$ forms a basis of $\Omega_{U^\sharp/Y^\sharp}$ (because g is log étale), then so does $(d\log v_1,\dots,d\log v_r)$. This implies that the canonical map $\phi^*\Omega_{X^\sharp/Y^\sharp} \to \Omega_{U^\sharp/Y^\sharp}$ induced by ϕ is an isomorphism. Since $U^\sharp/Y^\sharp$ is log smooth we get that ϕ is log étale (use [Kat89, 3.12]).

Let $\iota\colon Y^\sharp \to Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ be the $Y^\sharp$-morphism defined by $e_\lambda \mapsto 1 \in \Gamma(Y, M_{Y^\sharp})$, and $i\colon Y^\sharp \to Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{Z}^r}$ be the exact closed $Y^\sharp$-immersion defined by $e_\lambda \mapsto 1 \in \Gamma(Y, M_{Y^\sharp})$. We compute that the diagram of morphisms of log schemes



(3.2.1.17.4)

where p_1 is the first projection, is commutative.

3) In this step, we reduce to the case where $u = i \circ v$, $X^\sharp = Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{Z}^r}$, $(b_\lambda)_{\lambda=1,\dots,r}$ are the elements of $\Gamma(X, M_{X^\sharp})$ corresponding to the canonical basis of $\mathbb{Z}^r$, and $(y_\lambda)_{\lambda=1,\dots,r}$ are equal to 1.

By using the commutativity of 3.2.1.17.3 (in the case where $h = \mathrm{id}$ thanks to the step 1), and using Lemma 3.2.1.7, since g is log etale, then the m-PD envelope compatible with γ of w is equal to $T'^\sharp \hookrightarrow D'^\sharp$. Again, by using the commutativity of 3.2.1.17.4, and using Lemma 3.2.1.7, since ϕ is log etale, then the m-PD envelope compatible with γ of w (equal to $T'^\sharp \hookrightarrow D'^\sharp$) is equal to the m-PD envelope compatible with γ of $\iota \circ v$. More precisely, following the proof of Lemma 3.2.1.7, the composition of the structural morphism $(T'^\sharp \hookrightarrow D'^\sharp) \to w$ with ϕ is equal to the structural morphism $(T'^\sharp \hookrightarrow D'^\sharp) \to \iota \circ v)$. Hence, we compute the image of b_λ via the structural morphism $(T'^\sharp \hookrightarrow D'^\sharp) \to \iota \circ v)$ is u_λ. Again, since $Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{Z}^r} \to Y^\sharp \times_{\mathbb{Z}} A_{\mathbb{N}^r}$ is log etale, then the m-PD envelope compatible with γ of $i \circ v$ is equal to $T'^\sharp \hookrightarrow D'^\sharp$. Hence, the image of b_λ via the structural morphism $(T'^\sharp \hookrightarrow D'^\sharp) \to i \circ v)$ is still u_λ.

4) By using Lemma 3.2.1.15, we conclude. □

3.2.2 Sheaf of principal parts of level m

Let $m, i \in \mathbb{N}$ be two integers and $S^\sharp$ be a fine log scheme over the scheme $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$. Let $f\colon X^\sharp \to S^\sharp$ be a log smooth morphism of fine log-schemes. Let (I_S, J_S, γ) be a quasi-coherent m-PD-ideal of $\mathcal{O}_S$ such that γ extends to $X^\sharp$ (e.g. from [Ber96c, 1.3.2.(i).c)] when $J_S + p\mathcal{O}_S$ is locally principal, or with the remark 3.1.1.7 when the log structure on $S^\sharp$ is trivial). Let $m \geq 0$ be an integer.

Remark 3.2.2.1. Since γ extends to $X^\sharp$, then the m-PD envelope compatible with γ (of order n) of the identity of $X^\sharp$ is the identity of $X^\sharp$. Indeed, using the arguments given in the proof of 3.2.1.3, we can check that the ideal 0 of $\mathcal{O}_X$ is endowed with a (unique) m-PD structure compatible with γ.

Notation 3.2.2.2. Let $\Delta_{X^\sharp/S^\sharp}(r)\colon X^\sharp \hookrightarrow X^{\sharp r+1}_{/S^\sharp}$ denote the diagonal immersion. With notation 3.2.1.9, we set

$$\Delta^n_{X^\sharp/S^\sharp,(m),\gamma}(r) := P^{\sharp n}_{(m),\gamma}(\Delta_{X^\sharp/S^\sharp}(r)), \qquad \Delta_{X^\sharp/S^\sharp,(m),\gamma}(r) := P^\sharp_{(m),\gamma}(\Delta_{X^\sharp/S^\sharp}(r)).$$

For any $n' \geq n$, we denote by

$$\psi^{n',n}_{X^\sharp/S^\sharp,(m),\gamma}(r)\colon \Delta^n_{X^\sharp/S^\sharp,(m),\gamma}(r) \hookrightarrow \Delta^{n'}_{X^\sharp/S^\sharp,(m),\gamma}(r) \tag{3.2.2.2.1}$$

the canonical exact closed immersion. We denote by $M_{X^\sharp/S^\sharp,(m),\gamma}(r)$ (resp. $M^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$) the log structure of $\Delta_{X^\sharp/S^\sharp,(m),\gamma}(r)$ (resp. $\Delta^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$). Since γ extends to $X^\sharp$, the source of $\Delta^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$ is $X^\sharp$. We can denote abusively the underlying scheme of the target of $\Delta_{X^\sharp/S^\sharp,(m),\gamma}(r)$ by $\Delta_{X^\sharp/S^\sharp,(m),\gamma}(r)$.

Let us write $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r) := \mathcal{P}^n_{(m),\gamma}(\Delta_{X^\sharp/S^\sharp}(r))$ (see Notation 3.2.1.9). By default, we only consider $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$ as a sheaf on X: Let $U^\sharp$ be an open of $X^{\sharp r+1}_{/S^\sharp}$ containing the image of $\Delta_{X^\sharp/S^\sharp}(r)$ such that the immersion $v\colon X^\sharp \to U^\sharp$ induced by $\Delta_{X^\sharp/S^\sharp}(r)$ is closed. Recall, by definition, $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$ is the sheaf of rings on X such that $v_*\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$ is a quasi-coherent $\mathcal{O}_U$-algebra satisfying $\mathrm{Spec}(v_*\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)) = \Delta^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$ (see Notation 3.2.1.9).

For $i = 0, \dots, r$, let $p_i\colon X^{\sharp r+1}_{/S^\sharp} \to X^\sharp$ be the projections. Let $p_i^{U^\sharp}\colon U^\sharp \hookrightarrow X^{\sharp r+1}_{/S^\sharp} \to X^\sharp$ be the morphism induced by the projection p_i. Since $p_i^{U^\sharp} \circ v = \mathrm{id}$ then $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r) = p^{U^\sharp}_{i*}(v_*(\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)))$. Since $v_*(\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r))$ is endowed with a structure of quasi-coherent $\mathcal{O}_U$-algebra, this yields $r+1$-structures of quasi-coherent $\mathcal{O}_{X^\sharp}$-algebras on $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$. To clarify which $\mathcal{O}_{X^\sharp}$-algebra structure we consider, we write $p_{i*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}(r)$ to mean we use the structure induced by the ith projection. By composing $p_i^{U^\sharp}$ with the canonical morphism $\Delta^n_{X^\sharp/S^\sharp,(m)}(r) \to U^\sharp$, we get the projection $p^n_{i(m)}\colon \Delta^n_{X^\sharp/S^\sharp,(m),\gamma}(r) \to X^\sharp$ and $p_{i(m)}\colon \Delta_{X^\sharp/S^\sharp,(m),\gamma}(r) \to X^\sharp$, for $i = 0, \dots, r$. We get the ring homomorphisms $p^n_{i(m)}\colon \mathcal{O}_{X^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}(r)$ and $p_{i(m)}\colon \mathcal{O}_{X^\sharp} \to \mathcal{P}_{X^\sharp/S^\sharp,(m)}(r)$. We can simply denote $p^n_{i(m)}$ by p^n_i and $p_{i(m)}$ by p_i. The m-PD-ideal of $\mathcal{P}_{X^\sharp/S^\sharp,(m)}(r)$ will be denoted by $(\mathcal{I}_{X^\sharp/S^\sharp,(m)}(r), \mathcal{J}_{X^\sharp/S^\sharp,(m)}(r), ^{[\,]})$. The m-PD-ideal of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}(r)$ will be denoted by $(\mathcal{I}^n_{X^\sharp/S^\sharp,(m)}(r), \mathcal{J}^n_{X^\sharp/S^\sharp,(m)}(r), ^{[\,]})$.

When $r = 1$, we remove "(r)" and we simply write $\psi^{n',n}_{X^\sharp/S^\sharp,(m),\gamma}$, $\Delta^n_{X^\sharp/S^\sharp,(m),\gamma}$, $\Delta_{X^\sharp/S^\sharp,(m),\gamma}$, $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$, $\mathcal{P}_{X^\sharp/S^\sharp,(m),\gamma}$ etc. The left (resp. right) structure of $\mathcal{O}_{X^\sharp}$-algebra on $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$ is by definition the one given by $p_{0*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$ (resp. $p_{1*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$). As in 3.1.2.2, we can check that $p^n_{1,(m)}, p^n_{0,(m)} \colon \Delta^n_{X^\sharp/S^\sharp,(m),\gamma} \to X^\sharp$ are strict morphisms.

If $a \in M_{X^\sharp}$, using 3.1.1.24, we denote by $\mu_{(m),\gamma}(a)$ the unique section of $\ker(\mathcal{P}^*_{X^\sharp/S^\sharp,(m),\gamma} \to \mathcal{O}^*_X)$ such that we get in $M_{X^\sharp/S^\sharp,(m),\gamma}$ the equality

$$p^*_{1,(m)}(a) = p^*_{0,(m)}(a)\mu_{(m),\gamma}(a). \tag{3.2.2.2.2}$$

We get the monoid homomorphism $\mu_{(m),\gamma} \colon M_{X^\sharp/S^\sharp,(m),\gamma} \to \ker(\mathcal{P}^*_{X^\sharp/S^\sharp,(m),\gamma} \to \mathcal{O}^*_X) = 1 + \mathcal{I}^n_{X^\sharp/S^\sharp,(m)}$ given by $a \mapsto \mu_{(m),\gamma}(a)$. Similarly we have the monoid homomorphism

$$\mu^n_{(m),\gamma} \colon M^n_{X^\sharp/S^\sharp,(m),\gamma} \to \ker(\mathcal{P}^{n*}_{X^\sharp/S^\sharp,(m),\gamma} \to \mathcal{O}^*_X) \tag{3.2.2.2.3}$$

given by the formula

$$p^{n*}_{1,(m)}(a) = p^{n*}_{0,(m)}(a)\mu^n_{(m),\gamma}(a). \tag{3.2.2.2.4}$$

3.2.2.3. Suppose $X^\sharp/S^\sharp$ is strict. Then X/S is smooth and we get $\mathcal{P}^n_{X/S,(m)} = \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$.

Proposition 3.2.2.4 (Local description of $\mathcal{P}_{X^\sharp/S^\sharp,(m),\gamma}$)**.** *Suppose f is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. Put $\tau_{\sharp\lambda(m),\gamma} := \mu_{(m),\gamma}(u_\lambda) - 1 \in \mathcal{I}_{X^\sharp/S^\sharp,(m)}$, and $\tau_{\sharp\lambda(m),\gamma,n} := \mu^n_{(m),\gamma}(u_\lambda) - 1$.*

(a) We have the following $\mathcal{O}_X$-m-PD isomorphism

$$\begin{aligned} \mathcal{O}_X\langle T_1,\dots,T_r\rangle_{(m),n} &\xrightarrow{\sim} \mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma} \\ T_\lambda &\mapsto \tau_{\sharp\lambda,(m),\gamma,n}, \end{aligned} \tag{3.2.2.4.1}$$

where the structure of $\mathcal{O}_X$-module of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$ is given by p^n_1 or p^n_0.

(b) We have the following $\mathcal{O}_X$-m-PD isomorphism

$$\begin{aligned} \mathcal{O}_X\langle T_1,\dots,T_r\rangle_{(m)} &\xrightarrow{\sim} \mathcal{P}_{X^\sharp/S^\sharp,(m),\gamma} \\ T_\lambda &\mapsto \tau_{\sharp\lambda,(m),\gamma}, \end{aligned} \tag{3.2.2.4.2}$$

where the structure of $\mathcal{O}_X$-module of $\mathcal{P}_{X^\sharp/S^\sharp,(m),\gamma}$ is given by p_1 or p_0.

Proof. By symmetry, we can focus on the case where the structure of $\mathcal{O}_X$-module of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$ (resp. $\mathcal{P}_{X^\sharp/S^\sharp,(m),\gamma}$) is given by p^n_0 (resp. p_0). In the first assertion (resp. the second one), we are in the situation to use formula 3.2.1.17.1 (resp. 3.2.1.17.2) in the case where $u = \Delta$ and f is the left projection $p_0 \colon X^\sharp \times_{S^\sharp} X^\sharp \to X^\sharp$. Indeed, we first remark that $(p^*_1(u_\lambda))_{\lambda=1,\dots,r}$ are logarithmic coordinates of p_0. Indeed, log étaleness is stable under base change. Since the m-PD envelope compatible with γ of order n (resp. m-PD envelope compatible with γ) of the identity of $X^\sharp$ is $X^\sharp$ (see remark 3.2.2.1), proposition 3.2.1.17 yields the result. □

Notation 3.2.2.5. With the following remarks, we can lighten the notation.

(a) From the local description 3.2.2.4.1, we get that $\mathcal{P}^n_{X^\sharp/S^\sharp,(m),\gamma}$ does not depend on the m-PD-structure (satisfying the conditions of the subsection). Hence, from now, we reduce to the case where $\gamma = \gamma_\emptyset$ (see Notation 3.2.1.2) and we remove γ in the notation: we simply write $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$, $\Delta^n_{X^\sharp/S^\sharp,(m)}$, $M^n_{X^\sharp/S^\sharp,(m)}$, $\mu^n_{(m)}$, and $\tau_{\sharp\lambda(m),n}$.

(b) From 3.2.2.4.2, $\mathcal{P}_{X^\sharp/S^\sharp,(m),\gamma}$ does not depend on the m-PD-structure (satisfying the conditions of the subsection). Hence, we can remove γ in the corresponding notation.

Remark 3.2.2.6. From the local description of 3.2.2.4, we get that the morphisms $p^n_{1,(m)}$ and $p^n_{0,(m)}$ are finite (i.e. the underlying morphism of schemes is finite). Moreover, since the composition the closed nil-immersion $X^\sharp \hookrightarrow \Delta^n_{X^\sharp/S^\sharp,(m),\gamma}$ with $p^n_{1,(m)}$ or $p^n_{0,(m)}$ is the identity, then $p^n_{1,(m)}$ or $p^n_{0,(m)}$ are homeomorphisms and $p^n_{1,(m)*} = p^n_{0,(m)*}$.

Notation 3.2.2.7. Suppose f is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,r}$. With notation 3.2.2.4, for any integer $m' \geq m$ and $n' \geq n$, we remark that the canonical map $\mathcal{P}^n_{X^\sharp/S^\sharp,(m')} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ (resp. $\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$) sends $\tau_{\sharp\lambda(m'),n}$ to $\tau_{\sharp\lambda(m),n}$ (resp. $\tau_{\sharp\lambda(m),n'}$ to $\tau_{\sharp\lambda(m),n}$). Hence, this will be harmless to denote abusively $\tau_{\sharp\lambda(m),n}$ by $\tau_{\sharp\lambda}$. For any $\underline{k} \in \mathbb{N}^r$, we put

$$\underline{\tau}_\sharp^{\underline{k}} := \prod_{i=1}^r \tau_{\sharp i}^{k_i}, \qquad \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}} := \prod_{i=1}^r \tau_{\sharp i}^{\{k_i\}_{(m)}}. \tag{3.2.2.7.1}$$

Remark 3.2.2.8. Noticing that the main Theorem [Ogu18, IV.3.2.6] on log smoothness is valid for coherent log structures and not only fine log structures, one might wonder why we are focusing on fine log structures. The first reason we have in mind is that the important tool consisting of exactifying closed immersions (see 3.1.1.14) needs fine log structures. One might refute that in the first chapter we might replace in the definition of $\mathscr{C}_\sharp$ (see 3.1.1.10) the word fine by the word coherent (but in the other categories, e.g. $\mathscr{C}^{(m)}_{\sharp\gamma}$ we keep fine log structures). But, if we replace in 3.2.2.4 fine log structures by coherent log structures, the isomorphism 3.2.2.4.1 is not any more true: instead we have $\mathcal{O}_{X^{\mathrm{int}}}\langle T_1,\ldots,T_r\rangle_{(m),n} \xrightarrow{\sim} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$. Recall that since $X^\sharp$ is only coherent and not fine then we have in general $\mathcal{O}_{X^{\mathrm{int}}} \neq \mathcal{O}_X$.

3.2.2.9. Let $g\colon S^{\sharp\prime} \to S^\sharp$ be a morphism of fine log schemes over $\mathbb{Z}_{(p)}$, let $(I_{S'}, J_{S'}, \gamma')$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_{S'}$ such that g becomes an m-PD-morphism. Put $X'^\sharp := X^\sharp \times_{S^\sharp} S'^\sharp$. We suppose that γ' extends to $X'^\sharp$. Then, the m-PD-morphism $\Delta_{X'^\sharp/S'^\sharp,(m)} \to \Delta_{X^\sharp/S^\sharp,(m)}$ induces the isomorphism $\Delta_{X'^\sharp/S'^\sharp,(m)} \xrightarrow{\sim} \Delta_{X^\sharp/S^\sharp,(m)} \times_{S^\sharp} S'^\sharp$. Indeed, since the morphisms $p_0\colon \Delta_{X^\sharp/S^\sharp,(m)} \to X^\sharp$ and $p_0\colon \Delta_{X'^\sharp/S'^\sharp,(m)} \to X'^\sharp$ are strict, then the morphism $\Delta_{X'^\sharp/S'^\sharp,(m)} \to \Delta_{X^\sharp/S^\sharp,(m)} \times_{S^\sharp} S'^\sharp$ is strict. Hence, this is sufficient to check that the morphism $g^*\mathcal{P}_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}_{X'^\sharp/S'^\sharp,(m)}$ is an isomorphism. This can be checked by using the local description of 3.2.2.4.1.

3.2.2.10. Let $m' \geq m$ be two integers. Since $\mathscr{C}'^{(m)}_{\sharp n} \subset \mathscr{C}'^{(m')}_{\sharp n}$, then by using the universal property defining $\Delta^n_{X^\sharp/S^\sharp,(m')}$ we get a morphism $\psi^n_{m,m'}\colon \Delta^n_{X^\sharp/S^\sharp,(m)} \to \Delta^n_{X^\sharp/S^\sharp,(m')}$ and then the homomorphism $\psi^n_{m,m'}\colon \mathcal{P}^n_{X^\sharp/S^\sharp,(m')} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$.

From 3.2.1.12, we get $P^{\sharp n}_{(m)}(P^{\sharp n}(\Delta_{X^\sharp/S^\sharp})) = P^{\sharp n}_{(m)}(\Delta_{X^\sharp/S^\sharp})$. Hence, we get a canonical map $\psi^n_m\colon \Delta^n_{X^\sharp/S^\sharp,(m)} \to \Delta^n_{X^\sharp/S^\sharp}$ and then the homomorphism $\psi^n_m\colon \mathcal{P}^n_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$.

Now, suppose that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,r}$. With the notation of 1.2.1.3, 3.2.2.4 and 3.2.2.7, we have

$$\psi^n_{m,m'}(\underline{\tau}_\sharp^{\{\underline{k}\}_{(m')}}) = \frac{q^{(m)}_{\underline{k}}!}{q^{(m')}_{\underline{k}}!}\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}, \qquad \psi^n_m(\underline{\tau}_\sharp^{\underline{k}}) = q^{(m)}_{\underline{k}}!\,\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}. \tag{3.2.2.10.1}$$

3.2.2.11. By composing the canonical morphism $\Delta_{X^\sharp/S^\sharp,(m)}(r) \to X^{\sharp r+1}_{/S^\sharp}$ and $\Delta^n_{X^\sharp/S^\sharp,(m)}(r) \to X^{\sharp r+1}_{/S^\sharp}$ with the ith projection $p_i\colon X^{\sharp r+1}_{/S^\sharp} \to X^\sharp$ for $i = 0,\ldots,r$, we get

$$p_{i,(m)}(r)\colon \Delta^n_{X^\sharp/S^\sharp,(m)}(r) \to X^\sharp, \qquad p^n_{i,(m)}(r)\colon \Delta^n_{X^\sharp/S^\sharp,(m)}(r) \to X^\sharp. \tag{3.2.2.11.1}$$

If there are no risk of confusion, we can simply write p_i.

We denote by $\Delta_{X^\sharp/S^\sharp,(m)}(r) \times_{p_i,X^\sharp,p_{i'}} \Delta_{X^\sharp/S^\sharp,(m)}(r')$ the base change of $p_{i,(m)}(r)\colon \Delta_{X^\sharp/S^\sharp,(m)}(r) \to X^\sharp$ by $p_{i',(m)}(r')\colon \Delta_{X^\sharp/S^\sharp,(m)}(r') \to X^\sharp$. Since $p_{i,(m)}(r)$ and $p_{i',(m)}(r')$ are strict, then we can check the immersion $X^\sharp \hookrightarrow \Delta_{X^\sharp/S^\sharp,(m)}(r) \times_{p_i,X^\sharp,p_{i'}} \Delta_{X^\sharp/S^\sharp,(m)}(r')$ induced by $X^\sharp \hookrightarrow \Delta_{X^\sharp/S^\sharp,(m)}(r)$ and $X^\sharp \hookrightarrow \Delta_{X^\sharp/S^\sharp,(m)}(r')$ is an exact closed immersion. Using 3.2.2.4, we easily check this is an m-PD closed immersion (for more details, see [Ber96c, 2.1.3.(i)]). Moreover, thanks to the universal property of m-PD-envelopes, we get the m-PD-morphism $q_{(m)}(r,r')$ making commutative the diagram:

$$\begin{array}{ccccccc}
X^\sharp & \hookrightarrow & \Delta_{X^\sharp/S^\sharp,(m)}(r) \times_{p_i,X^\sharp,p_{i'}} \Delta_{X^\sharp/S^\sharp,(m)}(r') & \longrightarrow & X^{\sharp r+1}_{/S^\sharp} \times_{p_i,X^\sharp,p_{i'}} X^{\sharp r'+1}_{/S^\sharp} & \xrightarrow{p_i\times p_{i'}} & X^\sharp \times_{X^\sharp} X^\sharp \\
\| & & \downarrow{\scriptstyle q_{(m)}(r,r')} & & \downarrow\wr & & \downarrow\wr \\
X^\sharp & \hookrightarrow & \Delta_{X^\sharp/S^\sharp,(m)}(r+r') & \longrightarrow & X^{\sharp r+r'+1}_{/S^\sharp} & \xrightarrow{p_i} & X^\sharp.
\end{array} \tag{3.2.2.11.2}$$

By using again 3.2.2.4, we check that this arrow $q_{(m)}(r,r')$ is in fact an m-PD-isomorphism. Similarly, the immersion $X^\sharp \hookrightarrow \Delta^n_{X^\sharp/S^\sharp,(m)}(r) \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp,(m)}(r')$ induced by $X^\sharp \hookrightarrow \Delta^n_{X^\sharp/S^\sharp,(m)}(r)$ and $X^\sharp \hookrightarrow \Delta^{n'}_{X^\sharp/S^\sharp,(m)}(r')$ is an exact closed immersion endowed with a canonical m-PD structure of order $n+n'$ and we have the m-PD-morphism $q^{n,n'}_{(m)}(r,r')$ making commutative the diagram

$$\begin{array}{ccccccc} X^\sharp & \hookrightarrow & \Delta^n_{X^\sharp/S^\sharp,(m)}(r) \times_{p_i,X,p_{i'}} \Delta^{n'}_{X^\sharp/S^\sharp,(m)}(r') & \longrightarrow & X^{\sharp r+1}_{/S^\sharp} \times_{p_i,X,p_{i'}} X^{\sharp r'+1}_{/S^\sharp} & \xrightarrow{p_i \times p_{i'}} & X^\sharp \times_{X^\sharp} X^\sharp \\ \| & & \Big\downarrow{\scriptstyle q^{n,n'}_{(m)}(r,r')} & & \Big\downarrow \wr & & \Big\downarrow \wr \\ X^\sharp & \hookrightarrow & \Delta^{n+n'}_{X^\sharp/S^\sharp,(m)}(r+r') & \longrightarrow & X^{\sharp r+r'+1}_{/S^\sharp} & \xrightarrow{p_i} & X^\sharp . \end{array} \tag{3.2.2.11.3}$$

When $r=r'=1$, we simply write $q_{(m)}$ and $q^{n,n'}_{(m)}$.

Notation 3.2.2.12. For any integer n and any integers $0 \le i < j \le 2$, it follows from the universal property of m-PD-envelopes of order n that we get a unique m-PD-morphism $q^n_{ij,(m)} \colon \Delta^n_{X^\sharp/S^\sharp,(m)}(2) \to \Delta^n_{X^\sharp/S^\sharp,(m)}$ making commutative the diagram

$$\begin{array}{ccccccc} X^\sharp & \hookrightarrow & \Delta^n_{X^\sharp/S^\sharp,(m)}(2) & \longrightarrow & X^\sharp \times_{S^\sharp} X^\sharp \times_{S^\sharp} X^\sharp & \overset{p_i}{\underset{p_j}{\rightrightarrows}} & X^\sharp \\ \| & & \Big\downarrow{\scriptstyle q^n_{ij,(m)}} & & \Big\downarrow{\scriptstyle p_{ij}} & & \| \\ X^\sharp & \hookrightarrow & \Delta^n_{X^\sharp/S^\sharp,(m)} & \longrightarrow & X^\sharp \times_{S^\sharp} X^\sharp & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & X^\sharp \end{array} \tag{3.2.2.12.1}$$

We still denote by $q^n_{ij,(m)} \colon p_{0*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \to p_{i*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}(2)$ or $q^n_{ij,(m)} \colon p_{1*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \to p_{j*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}(2)$ the corresponding homomorphism of m-PD-$\mathcal{O}_X$-algebras. We can simply write the m-PD-morphism $q^n_{ij,(m)} \colon \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}(2)$ and recall that this homomorphism is also a homomorphism of m-PD-$\mathcal{O}_X$-algebras for two structures. Similarly we denote by $q_{ij,(m)} \colon \Delta_{X^\sharp/S^\sharp,(m)}(2) \to \Delta_{X^\sharp/S^\sharp,(m)}$ making commutative the diagram 3.2.2.12.1 without the order n condition.

Notation 3.2.2.13. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. By convention, $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ means $p^n_{1*}(\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}) \otimes_{\mathcal{O}_X} \mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ means $\mathcal{E} \otimes_{\mathcal{O}_X} p^n_{0*}(\mathcal{P}^n_{X^\sharp/S^\sharp,(m)})$. For instance, $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}$ is $p^n_{1*}(\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}) \otimes_{\mathcal{O}_X} p^{n'}_{0*}(\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)})$.

We have two structures of $\mathcal{O}_X$-module on the sheaf $\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$: the "left structure" given by functoriality from the left structure of $\mathcal{P}^n_{X/S,(m)}$ and the "right structure" given by the internal tensor product. We denote by $p_{0*}(\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E})$ (resp. $p_{1*}(\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E})$) to clarify we are considering the left structure (resp. right structure).

Similarly, we denote by $p_{0*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)})$ (resp. $p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)})$) the $\mathcal{O}_X$-module given by the internal tensor product (resp. by functoriality from the right $\mathcal{O}_X$-module structure of $\mathcal{P}^n_{X/S,(m)}$) which is called the left (resp. right) structure.

We denote by $p^n_{0,\mathcal{E}} \colon \mathcal{E} \to p_{0*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)})$ the canonical $\mathcal{O}_X$-linear map given by $x \mapsto x \otimes \mathbb{1}$, i.e. is the composition of $\mathrm{id}_{\mathcal{E}} \otimes p^n_0$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^0_{X/S,(m)}$. We denote by $p^n_{1,\mathcal{E}} \colon \mathcal{E} \to p_{1*}(\mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E})$ the canonical map given by $x \mapsto \mathbb{1} \otimes x$, i.e. is the composition of $p^n_1 \otimes \mathrm{id}_{\mathcal{E}}$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{P}^0_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$.

Notation 3.2.2.14. We simply denote by $\Delta^n_{X^\sharp/S^\sharp,(m)} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp,(m)}$ the base change of $p^{n'}_{0,(m)} \colon \Delta^{n'}_{X^\sharp/S^\sharp,(m)} \to X^\sharp$ by $p^n_{1,(m)} \colon \Delta^n_{X^\sharp/S^\sharp,(m)} \to X^\sharp$. Similarly, $X^{\sharp 2}_{/S^\sharp} \times_{p_1,X^\sharp,p_0} X^{\sharp 2}_{/S^\sharp}$ is simply denoted by $X^{\sharp 2}_{/S^\sharp} \times_{X^\sharp} X^{\sharp 2}_{/S^\sharp}$.

By composition of 3.2.2.11.3 and 3.2.2.12.1, we get

$$
\begin{array}{ccccccc}
X^{\sharp} & \hookrightarrow & \Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \times_{X^{\sharp}} \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} & \longrightarrow \; X^{\sharp 2}_{/S^{\sharp}} \times_{X^{\sharp}} X^{\sharp 2}_{/S^{\sharp}} \xrightarrow{\sim} & X^{\sharp} \times_{S^{\sharp}} X^{\sharp} \times_{S^{\sharp}} X^{\sharp} & & \\
\| & & \downarrow{\scriptstyle q^{n,n'}_{(m)}} & & \| & & \\
X^{\sharp} & \hookrightarrow & \Delta^{n+n'}_{X^{\sharp}/S^{\sharp},(m)}(2) & \longrightarrow & X^{\sharp} \times_{S^{\sharp}} X^{\sharp} \times_{S^{\sharp}} X^{\sharp} & \overset{p_i}{\underset{p_j}{\rightrightarrows}} & X^{\sharp} \\
\| & & \downarrow{\scriptstyle q^{n+n'}_{ij,(m)}} & & \downarrow{\scriptstyle p_{ij}} & & \| \\
X^{\sharp} & \hookrightarrow & \Delta^{n+n'}_{X^{\sharp}/S^{\sharp},(m)} & \longrightarrow & X^{\sharp} \times_{S^{\sharp}} X^{\sharp} & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & X^{\sharp}.
\end{array}
\tag{3.2.2.14.1}
$$

We get the morphism $p^{n,n'}_{ij,(m)} := q^{n+n'}_{ij,(m)} \circ q^{n,n'}_{(m)} : \Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \times_{X^{\sharp}} \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to \Delta^{n+n'}_{X^{\sharp}/S^{\sharp},(m)}$ (which satisfies also a universal property). The morphism $p^{n,n'}_{01,(m)}$ is the composition

$$p^{n,n'}_{01,(m)} : \Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \times_{X^{\sharp}} \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to \Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \overset{\psi^{n+n',n}_{X^{\sharp}/S^{\sharp},(m)}}{\hookrightarrow} \Delta^{n+n'}_{X^{\sharp}/S^{\sharp},(m)},$$

where the first morphism is given by the left projection. Similarly, $p^{n,n'}_{12,(m)}$ is the composition $\Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \times_{X^{\sharp}} \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to \Delta^{n+n'}_{X^{\sharp}/S^{\sharp},(m)}$, where the first morphism is given by the right projection.

By composing the canonical morphism $\Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \times_{X^{\sharp}} \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to X^{\sharp} \times_{S^{\sharp}} X^{\sharp} \times_{S^{\sharp}} X^{\sharp}$ with the ith projection $p_i : X^{\sharp} \times_{S^{\sharp}} X^{\sharp} \times_{S^{\sharp}} X^{\sharp} \to X^{\sharp}$ for $i = 0, 1, 2$, we get

$$p^{n,n'}_{i,(m)} : \Delta^{n}_{X^{\sharp}/S^{\sharp},(m)} \times_{X^{\sharp}} \Delta^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to X^{\sharp}. \tag{3.2.2.14.2}$$

This yields three ring homomorphisms $p^{n,n'}_{i,(m)} : \mathcal{O}_X \to \mathcal{P}^{n}_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X,(m)}$. When $i = 0$ (resp. $i = 1$, resp. $i = 2$), this is said to be the left (resp. middle, resp right) $\mathcal{O}_X$-algebra structure of $\mathcal{P}^{n}_{X,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X,(m)}$ and this is equal to the $\mathcal{O}_X$-algebra structure given by the left structure of $\mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)}$ (resp. the tensor product, resp. the right structure of $\mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$). Using the associated universal properties, we have the equalities $p^{n,n'}_{i,(m)} = p^{n+n'}_{i,(m)}(2) \circ p^{n,n'}_{ij,(m)}$ for any $i, j = 0, 1, 2$.

We denote by $\delta^{n,n'}_{(m)} : \mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$ (resp. $q^{n,n'}_{0(m)} : \mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$, resp. $q^{n,n'}_{1(m)} : \mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$) the morphism of m-PD-algebras associated to the morphism $p^{n,n'}_{02,(m)}$ (resp. $p^{n,n'}_{01,(m)}$, resp. $p^{n,n'}_{12,(m)}$). If there is no doubt on the level, we can also simply write $\delta^{n,n'}$, $q^{n,n'}_0$, $q^{n,n'}_1$.

The morphism $q^{n,n'}_{0(m)}$ is equal to the composition $q^{n,n'}_{0(m)} : \mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$ (the last morphism is $\tau \mapsto \tau \otimes 1$). Moreover, $q^{n,n'}_{1(m)}$ is equal to the composition $q^{n,n'}_{1(m)} : \mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$ (the last morphism is $\tau \mapsto 1 \otimes \tau$). In order words, we have the relation $q^{n,n'}_{0(m)} = \varpi^{n,n'}_{i(m)} \circ \psi^{n+n',n}_{X^{\sharp}/S^{\sharp},(m)}$ and $q^{n,n'}_{1(m)} = \varpi^{n,n'}_{i(m)} \circ \psi^{n+n',n'}_{X^{\sharp}/S^{\sharp},(m)}$, where $\varpi^{n,n'}_{0(m)} : \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$ and $\varpi^{n,n'}_{1(m)} : \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)} \to \mathcal{P}^{n}_{X^{\sharp}/S^{\sharp},(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp},(m)}$ are the homomorphisms associated with the projections. The morphism $q^{n,n'}_{0(m)}$ is $\mathcal{O}_X$-linear for the left (resp. right) structure of $\mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp}(m)}$ and the left structure (resp. of the center) of $\mathcal{P}^{n}_{X^{\sharp}/S^{\sharp}(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp}(m)}$. Finally, the morphism $q^{n,n'}_{0(m)}$ is $\mathcal{O}_X$-linear for the left (resp. right) structure of $\mathcal{P}^{n+n'}_{X^{\sharp}/S^{\sharp}(m)}$ and the structure of the center (resp. right) of $\mathcal{P}^{n}_{X^{\sharp}/S^{\sharp}(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^{\sharp}/S^{\sharp}(m)}$.

Using the commutativity of the diagram 3.2.2.14.1, we see that $\delta^{n,n'}_{(m)}$ is also an $\mathcal{O}_X$-algebra homomorphism for the respective left structures and for the respective right structures, i.e., $\delta^{n,n'}_{(m)} \circ p^{n+n'}_{0,(m)} = p^{n,n'}_{0,(m)}$ and $\delta^{n,n'}_{(m)} \circ p^{n+n'}_{1,(m)} = p^{n,n'}_{2,(m)}$. Using the commutativity of the diagram 3.2.2.14.1, we see that the morphism

$q^{n,n'}_{0(m)}$ is $\mathcal{O}_X$-linear for the left (resp. right) structure of $\mathcal{P}^{n+n'}_{X^\sharp/S^\sharp(m)}$ and the left (resp. middle) structure of $\mathcal{P}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}$.

By using 1.4.1.11 and the universal property of the m-PD envelopes, these morphisms are compatible with the change of level and we have the commutative diagram:

$$\begin{array}{ccccc}
\mathcal{P}^{n+n'}_{X^\sharp/S^\sharp} & \longrightarrow & \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m')} & \xrightarrow{\psi^{n+n'}_{m,m'}} & \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m)} \\
\downarrow{\scriptstyle \delta^{n,n'}} & & \downarrow{\scriptstyle \delta^{n,n'}_{(m')}} & & \downarrow{\scriptstyle \delta^{n,n'}_{(m)}} \\
\mathcal{P}^{n}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp} & \longrightarrow & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m')} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m')} & \xrightarrow{\psi^{n}_{m,m'}\otimes\psi^{n'}_{m,m'}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}.
\end{array} \tag{3.2.2.14.3}$$

Remark 3.2.2.15. The canonical m-PD structure on $\Delta^{n}_{X^\sharp/S^\sharp,(m)} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp,(m)}$ is characterized by the following property: the projections $q^{n,n'}_{0(m)} \colon \Delta^{n}_{X^\sharp/S^\sharp,(m)} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp,(m)} \to \Delta^{n}_{X^\sharp/S^\sharp,(m)}$ and $q^{n,n'}_{1(m)} \colon \Delta^{n}_{X^\sharp/S^\sharp,(m)} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp,(m)} \to \Delta^{n'}_{X^\sharp/S^\sharp,(m)}$ are morphisms of $\mathscr{C}^{(m)}_{\sharp n+n'}$.

Notation 3.2.2.16. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. With notation 3.2.2.14), we get the following three $\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}$-modules

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} := \mathcal{E} \otimes_{\mathcal{O}_X} \left(\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}\right) := p^{n,n'*}_{0(m)}(\mathcal{E}) \tag{3.2.2.16.1}$$

$$\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} := p^{n,n'*}_{1(m)}(\mathcal{E}) \tag{3.2.2.16.2}$$

$$\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} := \left(\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}\right) \otimes_{\mathcal{O}_X} \mathcal{E} := p^{n,n'*}_{2(m)}(\mathcal{E}). \tag{3.2.2.16.3}$$

Lemma 3.2.2.17. *For any $a \in M_{X^\sharp}$, for any integers $n, n' \in \mathbb{N}$, we have $\delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(a)) = \mu^{n}_{(m)}(a) \otimes \mu^{n'}_{(m)}(a)$.*

Proof. We fix the integers $n, n' \in \mathbb{N}$. We denote by $M^{n,n'}_{X^\sharp/S^\sharp,(m)}$ the log structure of $\Delta^{n}_{X^\sharp/S^\sharp,(m)} \times_{X^\sharp} \Delta^{n'}_{X^\sharp/S^\sharp,(m)} = \mathrm{Spec}(\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)})$. Let $a \in M_{X^\sharp}$. We denote by $\mu^{n,n'}_{i,j(m)}(a)$ the unique section of $\ker((\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)})^* \to \mathcal{O}^*_X)$ such that we get in $M^{n,n'}_{X^\sharp/S^\sharp,(m)}$ the equality $p^{n,n'}_{j,(m)}(a) = p^{n,n'}_{i,(m)}(a)\mu^{n,n'}_{i,j(m)}(a)$ (use Lemma 3.1.1.24) for any integers $i, j \in \{0, 1, 2\}$. We get

$$\mu^{n,n'}_{i,j(m)} \colon M_{X^\sharp/S^\sharp,(m),\gamma} \to \ker((\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)})^* \to \mathcal{O}^*_X)$$

given by $a \mapsto \mu^{n,n'}_{i,j(m)}(a)$. For any $i, j, k \in \{0, 1, 2\}$ we have,

$$\mu^{n,n'}_{i,j(m)}(a)\mu^{n,n'}_{j,k(m)}(a) = \mu^{n,n'}_{i,k(m)}(a). \tag{3.2.2.17.1}$$

With notation 3.2.2.2.3, 3.2.2.14.1 and 3.2.2.14.2, since $p^{n,n'}_{i,(m)} = p^{n+n'}_{0(m)} \circ p^{n,n'}_{ij,(m)}$ and $p^{n,n'}_{j,(m)} = p^{n+n'}_{1(m)} \circ p^{n,n'*}_{ij,(m)}$ then

$$\begin{aligned}
p^{n,n'*}_{j,(m)}(a) &= p^{n,n'*}_{ij,(m)}(p^{n+n'*}_1(a)) \\
&= p^{n,n'*}_{ij,(m)}(p^{n+n'*}_0(a) \cdot \mu^{n+n'}_{(m)}(a)) \\
&= p^{n,n'*}_{i,(m)}(a) \cdot p^{n,n'*}_{ij,(m)}(\mu^{n+n'}_{(m)}(a))
\end{aligned}$$

We deduce by uniqueness that

$$p^{n,n'*}_{ij,(m)}(\mu^{n+n'}_{(m)}(a)) = \mu^{n,n'}_{i,j(m)}. \tag{3.2.2.17.2}$$

Since $p^{n,n'*}_{02,(m)}(\mu^{n+n'}_{(m)}(a)) = \delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(a))$, since $p^{n,n'*}_{01,(m)}(\mu^{n+n'}_{(m)}(a)) = (\mu^{n+n'}_{(m)}(a)) = q^{n,n'}_{0(m)}(\mu^{n+n'}_{(m)}(a)) = \mu^{n}_{(m)}(a) \otimes 1$, since $p^{n,n'*}_{12,(m)}(\mu^{n+n'}_{(m)}(a)) = (\mu^{n+n'}_{(m)}(a)) = q^{n,n'}_{1(m)}(\mu^{n+n'}_{(m)}(a)) = 1 \otimes \mu^{n'}_{(m)}(a)$, then it follows from equations 3.2.2.17.1 and 3.2.2.17.2 that

$$\begin{aligned}\delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(a)) &= p^{n,n'*}_{01,(m)}(\mu^{n+n'}_{(m)}(a))p^{n,n'*}_{12,(m)}(\mu^{n+n'}_{(m)}(a)),\\ &= (\mu^{n}_{(m)}(a)\otimes 1)(1\otimes \mu^{n'}_{(m)}(a)),\\ &= \mu^{n}_{(m)}(a)\otimes \mu^{n'}_{(m)}(a).\end{aligned}$$

□

Proposition 3.2.2.18. *We have the commutative diagram:*

$$\begin{array}{ccc} \mathcal{P}^{n+n'+n''}_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\delta^{n,n'+n''}_{(m)}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'+n''}_{X^\sharp/S^\sharp,(m)} \\ \downarrow{\scriptstyle \delta^{n+n',n''}_{(m)}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\delta^{n',n''}_{(m)}} \\ \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n''}_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\delta^{n,n'}_{(m)}\otimes\mathrm{id}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n''}_{X^\sharp/S^\sharp,(m)} \end{array} \quad (3.2.2.18.1)$$

Proof. Since this is local, we can suppose that f has logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. Using Lemma 3.2.2.17, we compute that the images of $\mu^{n+n'+n''}_{(m)}(u_\lambda)$ by both maps $\mathcal{P}^{n+n'+n''}_{X^\sharp/S^\sharp,(m)}\to\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n''}_{X^\sharp/S^\sharp,(m)}$ are the same and equal to $\mu^{n}_{(m)}(u_\lambda)\otimes\mu^{n'}_{(m)}(u_\lambda)\otimes\mu^{n''}_{(m)}(u)$. Since $\tau_{\sharp\lambda}:=\mu^{n}_{(m)}(u_\lambda)-1$, then so is $\tau_{\sharp\lambda}$. Since all maps of the diagram 3.2.2.18.1 are m-PD-morphisms (see 3.2.2.19 for the m-PD-structure), then by using the isomorphism 3.2.2.4.2, we get the desired commutativity. □

The following Lemma will be useful to check the associativity of the product law of the sheaf of differential operator:

Lemma 3.2.2.19. *We denote by $\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}$ the base change of $p^{n'}_0\circ q^{n',n''}_0\colon\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}\to X^\sharp$ by $p^n_1\colon\Delta^{n}_{X^\sharp/S^\sharp,(m)}\to X^\sharp$. The exact closed immersion $X^\sharp\hookrightarrow\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}$ induced by $X^\sharp\hookrightarrow\Delta^{n}_{X^\sharp/S^\sharp,(m)}$, $X^\sharp\hookrightarrow\Delta^{n'}_{X^\sharp/S^\sharp,(m)}$ and $X^\sharp\hookrightarrow\Delta^{n''}_{X^\sharp/S^\sharp,(m)}$ is endowed with a canonical m-PD structure. By abuse of notation, we denote by $\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}$ this object of $\mathscr{C}^{(m)}_{\sharp,n+n'+n''}$. This m-PD structure on $\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}$ is characterized by the following property: the projections $\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}\to\Delta^{n}_{X^\sharp/S^\sharp,(m)}$, $\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}\to\Delta^{n'}_{X^\sharp/S^\sharp,(m)}$, and $\Delta^{n}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n'}_{X^\sharp/S^\sharp,(m)}\times_{X^\sharp}\Delta^{n''}_{X^\sharp/S^\sharp,(m)}\to\Delta^{n''}_{X^\sharp/S^\sharp,(m)}$ are morphisms of $\mathscr{C}^{(m)}_{\sharp,n+n'+n''}$.*

Proof. This is checked similarly to 3.2.2.14. □

3.2.3 Sheaf of logarithmic differential operators of level m and finite order on log smooth schemes

We keep notation 3.2.2.

Definition 3.2.3.1. The sheaf of differential operators of level m and order $\le n$ of f is defined by putting $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}:=\mathcal{H}om_{\mathcal{O}_X}(p^n_{0,(m)*}\mathcal{P}^n_{X^\sharp/S^\sharp,(m)},\mathcal{O}_X)$. Following notation 3.2.2.2.1, for any $n'\ge n$, we have the canonical projection $\psi^{n',n*}_{X^\sharp/S^\sharp,(m)}\colon\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}\to\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)}$. By duality, this yields the monomorphisms

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}\hookrightarrow\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n'}.$$

The sheaf of differential operators of level m of f is defined by putting $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}:=\cup_{n\in\mathbb{N}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$.

Let $P\in\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$, $P'\in\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n'}$. We define the product $PP'\in\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n+n'}$ to be the composition

$$PP'\colon\mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m)}\xrightarrow{\delta^{n,n'}_{(m)}}\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}\xrightarrow{\mathrm{id}\otimes P'}\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)}\xrightarrow{P}\mathcal{O}_X. \quad (3.2.3.1.1)$$

Example 3.2.3.2. Suppose $X^\sharp/S^\sharp$ is strict. Then X/S is smooth and we get $\mathcal{P}^n_{X/S,(m)} = \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ (see 3.2.2.3). This yields $\mathcal{D}^{(m)}_{X/S} = \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$, where $\mathcal{D}^{(m)}_{X/S}$ is the sheaf of differential operators defined by Berthelot in 1.4.2.

Proposition 3.2.3.3. *The sheaf $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a sheaf of rings with the product as defined in 3.2.3.1.1.*

Proof. We have to check the product as defined in 3.2.3.1.1 is associative. One checks the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathcal{P}^{n+n'+n''}_{X^\sharp/S^\sharp,(m)} & = & \mathcal{P}^{n+n'+n''}_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\delta^{n,n'+n''}_{(m)}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'+n''}_{X^\sharp/S^\sharp,(m)} & = & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'+n''}_{X^\sharp/S^\sharp,(m)} \\
 & & \downarrow \delta^{n+n',n''}_{(m)} & & \downarrow \mathrm{id} \otimes \delta^{n',n''}_{(m)} & & \\
 & & \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\delta^{n,n'}_{(m)} \otimes \mathrm{id}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n''}_{X^\sharp/S^\sharp,(m)} & & \\
\downarrow (PP')P'' & & \downarrow \mathrm{id} \otimes P'' & & \downarrow \mathrm{id} \otimes \mathrm{id} \otimes P'' & & \downarrow \mathrm{id} \otimes P'P'' \\
 & & \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\delta^{n,n'}_{(m)}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} & & \\
 & & \downarrow PP' & & \downarrow \mathrm{id} \otimes P' & & \\
\mathcal{O}_X & = & \mathcal{O}_X & \xleftarrow{P} & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} & = & \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}.
\end{array} \tag{3.2.3.3.1}$$

Indeed, the commutativity of the top square of the middle was already proved (see 3.2.2.18.1). Since the commutativity of the other squares are obvious, we conclude the proof. □

Notation 3.2.3.4 (Description in logarithmic coordinates)**.** Suppose that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. With the notation 3.2.2.7, the elements $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}, |\underline{k}| \leq n\}$ form a basis of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$. The corresponding dual basis of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$ will be denoted by $\{\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}, |\underline{k}| \leq n\}$. This yields the basis (as $\mathcal{O}_X$-module for both structures) $\{\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}, \underline{k} \in \mathbb{N}^r\}$ of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

Let $\underline{\epsilon}_1, \dots, \underline{\epsilon}_r$ be the canonical basis of $\mathbb{N}^r$, i.e. the coordinates of $\underline{\epsilon}_i$ are 0 except for the ith term which is 1. We put $\partial_{\sharp i} := \underline{\partial}_\sharp{}^{\langle \underline{\epsilon}_i\rangle_{(m)}}$ and $\partial_{\sharp i}^{\langle k\rangle_{(m)}} := \underline{\partial}_\sharp{}^{\langle k\underline{\epsilon}_i\rangle_{(m)}}$ for any $k \in \mathbb{N}$.

3.2.3.5. For any $m' \geq m$, from the homomorphisms $\psi^n_{m,m'} \colon \mathcal{P}^n_{X^\sharp/S^\sharp,(m')} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ and $\psi^n_m \colon \mathcal{P}^n_{X^\sharp/S^\sharp} \to \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ of 3.2.2.10, we get by duality, the maps

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \overset{\rho_{m',m}}{\longrightarrow} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \overset{\rho_{m'}}{\longrightarrow} \mathcal{D}_{X^\sharp/S^\sharp}. \tag{3.2.3.5.1}$$

It follows from 3.1.4.3.1, 3.2.3.1.1 and the log version of 3.2.2.14.3 that the maps of 3.2.3.5.1 are homomorphisms of filtered rings.

When $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$, with notation 3.2.3.4 dualizing 3.2.2.10.1, we get the formula

$$\rho_{m',m}(\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}) = \frac{q^{(m)}_{\underline{k}}!}{q^{(m')}_{\underline{k}}!} \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m')}} \text{ and } \rho_m(\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}) = q^{(m)}_{\underline{k}}! \underline{\partial}_\sharp^{[\underline{k}]}. \tag{3.2.3.5.2}$$

Since this is local, by using the formula 3.2.3.5.2, we can check that the filtered ring morphism

$$\rho \colon \varinjlim_m \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{D}_{X^\sharp/S^\sharp}. \tag{3.2.3.5.3}$$

induced by taking the inductive limit of ρ_m is an isomorphism. However, beware that the homomorphisms 3.2.3.5.1 are not necessarily injective.

Since $\mathcal{O}_X$ is endowed with a canonical structure of left $\mathcal{D}_{X^\sharp/S^\sharp}$-module (see 3.1.4.5), then this induces a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{O}_X$. It follows from 3.1.4.5.1 and 3.2.3.5.1 that the structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{O}_X$ is given via the formula

$$P(f) := P \circ p^n_{1(m)}(f). \tag{3.2.3.5.4}$$

Notation 3.2.3.6. For $P \in \Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n})$, with notation 3.2.2.2.3 define $P \colon M^n_{X^\sharp/S^\sharp,(m),\gamma} \times \mathcal{O}_X \to \mathcal{O}_X$ by:

$$P(a,x) = P(\mu^n_{(m)}(a) {p^n_{1,(m)}}^*(x)). \tag{3.2.3.6.1}$$

For any $n' \geq n$, we can also view P as an element of $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n'})$ but we remark that the formula 3.2.3.6.1 does not depend on n such that $P \in \Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n})$.

For any $a \in M$ and any $x \in \mathcal{O}_X$, we write simply $P(x) = P(0,x)$ and $P_a(x) = P(a,x)$. Following 3.2.3.5.4 that $\mathcal{O}_X$ is a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and that $P(x)$ corresponds to the action of P on x, i.e. the notation are compatible.

Proposition 3.2.3.7. *Suppose f is endowed with logarithmic coordinates $(u_1, \ldots, u_r)$. With notations 1.2.1.3 and 3.2.3.4, we have the following properties.*

(a) For any $x \in \Gamma(X, \mathcal{O}_X)$, $s \in \Gamma(X, M_{X^\sharp})$, we have the following Taylor formulas:

$$p^n_{1,(m)}(x) = \sum_{|\underline{k}| \leq n} p^n_{0,(m)}(\underline{\partial}^{\langle \underline{k} \rangle}_\sharp(x)) \underline{\tau}^{\{\underline{k}\}}_\sharp, \qquad \mu^n_{(m)}(s) = \sum_{|\underline{k}| \leq n} \underline{\partial}^{\langle \underline{k} \rangle}(s,1) \underline{\tau}^{\{\underline{k}\}}_\sharp. \tag{3.2.3.7.1}$$

(b) For any $x \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k} \in \mathbb{N}^r$, we have in $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$

$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_\sharp x = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k} - \underline{i} \rangle_{(m)}}_\sharp (x) \underline{\partial}^{\langle \underline{i} \rangle_{(m)}}_\sharp. \tag{3.2.3.7.2}$$

Proof. a) Both Taylor formulas 3.2.3.7.1 come from the fact that $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_\sharp, |\underline{k}| \leq n\}$ is the dual basis of $\{\underline{\tau}^{\{\underline{k}\}_{(m)}}_\sharp, |\underline{k}| \leq n\}$ (recall $\underline{\partial}^{\langle \underline{k} \rangle}_\sharp(s,1) = \underline{\partial}^{\langle \underline{k} \rangle}_\sharp(\mu^n_{(m)}(s))$).

b) We do the same computation than in 1.4.2.7.b. □

Lemma 3.2.3.8. *Let $\Phi \in \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$ and $\Psi \in \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n'}$, $a, a' \in M_{X^\sharp}$ and $x \in \mathcal{O}_X$. Then:*

(a) $\Phi(a', \alpha_X(a)x) = \alpha_X(a)\Phi(a \cdot a', x)$,

(b) $(\Phi \circ \Psi)(a,x) = \Phi(a, \Psi(a,x))$.

Proof. With notation 3.1.1.23, since, $p^{n*}_{i,(m)}(\alpha_X(a)) = \alpha(p^{n*}_{i,(m)}(a))$ (recall, $p^n_{i,(m)}$ is a morphism of log schemes), then it follows from 3.2.2.2.4 that $p^{n*}_{1,(m)}(\alpha_X(a)) = p^{n*}_{0,(m)}(\alpha_X(a))\mu^n_{(m)}(a)$. Since the morphism $\mu^n_{(m)}$ of 3.2.2.2.3 is a monoid morphism, by $\mathcal{O}_X$-linearity of Φ, then we check a) as follows:

$$\begin{aligned} \Phi(a', \alpha_X(a)x) &= \Phi\left(\mu^n_{(m)}(a') p^{n*}_{1,(m)}(\alpha_X(a)x)\right) \\ &= \Phi\left(\mu^n_{(m)}(a') p^{n*}_{1,(m)}(\alpha_X(a)) p^{n*}_{1,(m)}(x)\right) \\ &= \Phi\left(\mu^n_{(m)}(a') p^{n*}_{0,(m)}(\alpha_X(a)) \mu^n_{(m)}(a) p^{n*}_{1,(m)}(x)\right) \\ &= \alpha_X(a)\Phi\left(\mu^n_{(m)}(a \cdot a') p^{n*}_{1,(m)}(x)\right) \\ &= \alpha_X(a)\Phi(a \cdot a', x). \end{aligned}$$

b) Fix $a \in M_{X^\sharp}$. Omit $S^\sharp$ in the notations. Consider the following diagram

$$\begin{array}{ccccccc} \mathcal{O}_X & \xrightarrow{\mu^{n'}_{(m)}(a) p^{n'*}_{1,(m)}} & \mathcal{P}^{n'}_{X^\sharp,(m)} & \xrightarrow{\Psi} & \mathcal{O}_X & & \\ \downarrow{\scriptstyle \mu^{n+n'}_{(m)}(a) p^{n+n'*}_{1,(m)}} & & \downarrow{\scriptstyle \mu^n_{(m)}(a) \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu^n_{(m)}(a) p^{n*}_{1,(m)}} & \searrow^{\Phi_a} & \\ \mathcal{P}^{n+n'}_{X^\sharp,(m)} & \xrightarrow[\delta^{n,n'}_{(m)}]{} & \mathcal{P}^n_{X^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp,(m)} & \xrightarrow[\mathrm{id} \otimes \Psi]{} & \mathcal{P}^n_{X^\sharp,(m)} & \xrightarrow[\Phi]{} & \mathcal{O}_X \end{array} \quad , \tag{3.2.3.8.1}$$

where $\mu^n_{(m)}(a) \otimes \mathrm{id}$ is the morphism given by $e' \mapsto \mu^n_{(m)}(a) \otimes e'$, where $id \otimes \Psi$ is the morphism given by $e \otimes e' \mapsto e p^{n*}_{1,(m)}(\Psi(e'))$. The right triangle commutes by definition of Φ_a. Via a straightforward computation, we get that the middle square commutes. It remains to check the commutativity of the left square. On one hand,

$$(\mu^n_{(m)}(a)\otimes id)\circ(\mu^{n'}_{(m)}(a)p^{n'*}_{1,(m)})(x)=\mu^n_{(m)}(a)\otimes\mu^{n'}_{(m)}(a)p^{n'*}_{1,(m)}(x).$$

Since $\delta^{n,n'}_{(m)}$ is a homomorphism of $\mathcal{O}_X$-algebras for the right structures (more precisely, see 3.2.2.14), then we get $\delta^{n,n'}_{(m)}(p^{n+n'*}_{1,(m)}(x))=1\otimes p^{n'*}_{1,(m)}(x)$. With the formula 3.2.2.17, then we get

$$\begin{aligned}\delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(a)p^{n+n'*}_{1,(m)}(x))&=\delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(a))\delta^{n,n'}_{(m)}(p^{n+n'*}_{1,(m)}(x))\\&=(\mu^n_{(m)}(a)\otimes\mu^{n'}_{(m)}(a))\cdot(1\otimes p^{n'*}_{1,(m)}(x))\\&=\mu^n_{(m)}(a)\otimes\mu^{n'}_{(m)}(a)p^{n'*}_{1,(m)}(x).\end{aligned}$$

Hence, the diagram 3.2.3.8.1 is commutative. The composition above of 3.2.3.8.1 is $\Phi_a\circ\Psi_a$ and that below is $(\Phi\circ\Psi)_a$. This proves b). □

3.2.3.9 (Comparison of the local description of differential operators with or without logarithmic structure)**.** Suppose $X^\sharp/S^\sharp$ is strict and there exists coordinates $(t_\lambda)_{\lambda=1,\dots,r}$ of $X^\sharp/S^\sharp$ (see definition 3.1.1.21).

(a) Copying the proof of 1.3.3.11, we get the following isomorphism of m-PD-$\mathcal{O}_X$-algebras

$$\begin{aligned}\mathcal{O}_X\langle T_1,\dots,T_r\rangle_{(m),n}&\xrightarrow{\sim}\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}\\T_\lambda&\mapsto\tau_\lambda,\end{aligned}\qquad(3.2.3.9.1)$$

where $\tau_\lambda:=p_1^*(t_\lambda)-p_0^*(t_\lambda)$. The elements $\{\underline{\tau}^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}|\leq n}$ form a basis of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$. The corresponding dual basis of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$ will be denoted by $\{\underline{\partial}^{\langle\underline{k}\rangle_{(m)}}\}_{|\underline{k}|\leq n}$.

(b) Consider the following diagram

$$\begin{array}{ccc}Y^\sharp & \xrightarrow{u} & A_{\mathbb{N}^r}\times T^\sharp\\ \downarrow{\scriptstyle f} & & \downarrow\\ X^\sharp & \xrightarrow{t} & \mathbb{A}^r\times S^\sharp\end{array}\qquad(3.2.3.9.2)$$

where the right arrow is induced by a morphism of fine log schemes of the form $T^\sharp\to S^\sharp$ and by the canonical morphism $A_{\mathbb{N}^r}\to\mathbb{A}^r$, the bottom arrow is induced by the coordinates $(t_\lambda)_{\lambda=1,\dots,r}$ and where the top arrow is induced by the logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. Let $\underline{\tau}_\sharp$ (resp. $\underline{\partial}_\sharp{}^{\langle\underline{k}\rangle_{(m)}}$) be the element constructed from $(u_\lambda)_{\lambda=1,\dots,r}$ (resp. $(t_\lambda)_{\lambda=1,\dots,r}$) as defined in 3.2.2.4 (resp. 3.2.3.4) and according to notation 3.2.2.7. Then the functorial morphism $f^*\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}\to\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)}$ (see 3.2.3.17) is explicitly described by

$$\underline{\tau}^{\{\underline{k}\}_{(m)}}\mapsto\underline{t}^{\underline{k}}\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}},\qquad(3.2.3.9.3)$$

where the action of $\underline{t}^{\underline{k}}$ is induced by the left structure of $\mathcal{O}_Y$-algebra of $\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)}$. Indeed, since $f^*\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}\to\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)}$ is an m-PD-morphism, we reduce to compute the image of τ_i. We compute the image of τ_i is $1\otimes u_i-u_i\otimes 1=u_i\tau_{i\sharp}=t_i\tau_{i\sharp}$ (recall definition 3.2.2.4 of $\tau_{i\sharp,(m)}$). Moreover, by duality, we get $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}\to f^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ (see 3.2.3.17) is explicitly described by

$$\underline{\partial}_\sharp{}^{\langle\underline{k}\rangle_{(m)}}\mapsto\underline{t}^{\underline{k}}\underline{\partial}^{\langle\underline{k}\rangle_{(m)}}.\qquad(3.2.3.9.4)$$

(c) Suppose now that the t_λ's lie in $\Gamma(X,\mathcal{O}_X^*)$. Then $(t_\lambda)_{\lambda=1,\dots,r}$ are also logarithmic coordinates of $X^\sharp/S^\sharp$ (see definition 3.1.1.20). We have

$$\underline{\tau}^{\{\underline{k}\}_{(m)}}=p_0^*(\underline{t}^{\underline{k}})\underline{\tau}_{\sharp(m)}^{\{\underline{k}\}_{(m)}}\quad\text{and}\quad\underline{\partial}_\sharp{}^{\langle\underline{k}\rangle_{(m)}}=\underline{t}^{\underline{k}}\underline{\partial}^{\langle\underline{k}\rangle_{(m)}},\qquad(3.2.3.9.5)$$

where $\underline{\tau}_{\sharp(m)}$ (resp. $\underline{\partial}_\sharp{}^{\langle\underline{k}\rangle_{(m)}}$) is defined in 3.2.2.4 (resp. 3.2.3.4).

Lemma 3.2.3.10. *Suppose that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,r}$. With notation 3.2.3.4, we have the formula*

$$\delta^{n,n'}_{(m)}(\tau_{\sharp i}) = \tau_{\sharp i} \otimes \tau_{\sharp i} + \tau_{\sharp i} \otimes 1 + 1 \otimes \tau_{\sharp i}. \tag{3.2.3.10.1}$$

Proof. This is a consequence of 3.2.2.17:

$$\begin{aligned}\delta^{n,n'}_{(m)}(\tau_{\sharp i}) &= \delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(u_i) - 1) = \delta^{n,n'}_{(m)}(\mu^{n+n'}_{(m)}(u_i)) - 1 \\ &\overset{3.2.2.17}{=} \mu^{n}_{(m)}(u_i) \otimes \mu^{n'}_{(m)}(u_i) - 1 = (\tau_{\sharp i} + 1) \otimes (1 + \tau_{\sharp i}) - 1 = \tau_{\sharp i} \otimes \tau_{\sharp i} + \tau_{\sharp i} \otimes 1 + 1 \otimes \tau_{\sharp i}.\end{aligned}$$

□

Lemma 3.2.3.11. *Suppose f is endowed with logarithmic coordinates $(u_1, \ldots, u_r)$. We set $t_\lambda := \alpha_{X^\sharp}(u_\lambda)$, for any $\lambda = 1, \ldots, r$. Let $\underline{n}, \underline{k} \in \mathbb{N}^r$. We write $\underline{u}^{\underline{n}} := \prod_{i=1}^r u_i^{n_i}$, $\underline{t}^{\underline{n}} := \prod_{i=1}^r t_i^{n_i}$. With notation 3.2.3.4 we have the following formulas.*

(a) If $\underline{k} \leq \underline{n}$ (see convention 1.2.1.3), then $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(\underline{u}^{\underline{n}}, 1) = \underline{q}_{\underline{k}}^{(m)}!\binom{\underline{n}}{\underline{k}}$. If $\underline{k} \not\leq \underline{n}$ then $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(\underline{u}^{\underline{n}}, 1) = 0$.

(b) If $\underline{k} \leq \underline{n}$, then $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(\underline{t}^{\underline{n}}) = \underline{q}_{\underline{k}}^{(m)}!\binom{\underline{n}}{\underline{k}}\underline{t}^{\underline{n}-\underline{k}}$. Otherwise, $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(\underline{t}^{\underline{n}}) = 0$.

(c) We have the formula $\frac{\underline{k}!}{\underline{q}_{\underline{k}}^{(m)}!}\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} = \prod_{0 \leq \underline{k}' < \underline{k}}(\underline{\partial}_\sharp - \underline{k}')$,

where $\underline{k}' < \underline{k}$ means $k'_i < k_i$ for any i.

Proof. a) By definition (see 3.2.3.6.1), we compute:

$$\begin{aligned}\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(\underline{u}^{\underline{n}}, 1) &\overset{3.2.3.6.1}{=} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}\left(\mu^{|\underline{k}|}_{(m)}(\underline{u}^{\underline{n}})\right) = \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}\left(\prod_{j=1}^r \mu^{|\underline{k}|}_{(m)}(u_j)^{n_j}\right) \\ &= \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}\left(\prod_{j=1}^r (\tau_{\sharp j} + 1)^{n_j}\right) = \underline{\partial}_\sharp^{\langle \underline{k} \rangle}\left(\prod_{j=1}^r \sum_{l_j=0}^{n_j} \binom{n_j}{l_j} \tau_{\sharp j}^{l_j}\right) \\ &= \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}\left(\prod_{j=1}^r \sum_{l_j=0}^{n_j} \binom{n_j}{l_j} q_{l_j}^{(m)}!\tau_{\sharp j}^{\{l_j\}_{(m)}}\right) \\ &= \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}\left(\sum_{\underline{l}=\underline{0}}^{\underline{n}} \binom{\underline{n}}{\underline{l}} \underline{q}_{\underline{l}}^{(m)}!\underline{\tau}_\sharp^{\{\underline{l}\}_{(m)}}\right).\end{aligned}$$

When $\underline{k} \leq \underline{n}$, this yields

$$\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}\left(\sum_{\underline{l}=\underline{0}}^{\underline{n}} \binom{\underline{n}}{\underline{l}} \underline{q}_{\underline{l}}^{(m)}!\underline{\tau}_\sharp^{\{\underline{l}\}_{(m)}}\right) = \underline{q}_{\underline{k}}^{(m)}!\binom{\underline{n}}{\underline{k}}, \tag{3.2.3.11.1}$$

and otherwise this is null. Hence we are done.

b) By using the formula 3.2.3.8.(a) in the case where $a = \underline{u}^{\underline{n}}$, $a' = 0$ and $x = 1$ and next 3.2.3.11.a, since $\alpha(\underline{u}^{\underline{n}}) = \underline{t}^{\underline{n}}$, then we get $\underline{\partial}_\sharp^{\langle \underline{k} \rangle}(0, \underline{t}^{\underline{n}}) = \underline{t}^{\underline{n}}\underline{\partial}_\sharp^{\langle \underline{k} \rangle}(\underline{u}^{\underline{n}}, 1) = \underline{t}^{\underline{n}}\underline{q}_{\underline{k}}^{(m)}!\binom{\underline{n}}{\underline{k}}$.

c) We calculate $\delta^{n,n'}_{(m)}(\underline{\tau}_\sharp^{\{\underline{k}\}})$ as follows. Since $q^{n,n'}_{1(m)} \colon \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}$ is an m-PD-morphism (see 3.2.2.14), then by using 1.2.4.5.c, we have

$$(\tau_{\sharp i} \otimes \tau_{\sharp i})^{\{\alpha\}} = ((\tau_{\sharp i} \otimes 1) \cdot (1 \otimes \tau_{\sharp i}))^{\{\alpha\}} = (\tau_{\sharp i}^{\alpha} \otimes 1) \cdot (1 \otimes \tau_{\sharp i})^{\{\alpha\}} = (\tau_{\sharp i}^{\alpha} \otimes 1) \cdot (1 \otimes \tau_{\sharp i}^{\{\alpha\}}) = q_\alpha^{(m)}!\tau_{\sharp i}^{\{\alpha\}} \otimes \tau_{\sharp i}^{\{\alpha\}}.$$

Since $\delta^{n,n'}_{(m)}$, $q^{n,n'}_{0(m)}$ and $q^{n,n'}_{1(m)}$ are m-PD morphisms, with the formulas of 1.2.4.5 (e.g. 1.2.4.5.2 and 1.2.4.5.3), then we have

$$\begin{aligned}
\delta^{n,n'}_{(m)}(\tau_{\sharp i}^{\{k_i\}}) &= \delta^{n,n'}_{(m)}(\tau_{\sharp i})^{\{k_i\}} \overset{3.2.3.10.1}{=} (\tau_{\sharp i}\otimes\tau_{\sharp i}+\tau_{\sharp i}\otimes 1+1\otimes\tau_{\sharp i})^{\{k_i\}}\\
&= \sum_{\alpha=0}^{k_i}\left\langle {k_i \atop \alpha}\right\rangle (\tau_{\sharp i}\otimes\tau_{\sharp i})^{\{\alpha\}}(\tau_{\sharp i}\otimes 1+1\otimes\tau_{\sharp i})^{\{k_i-\alpha\}}\\
&= \sum_{\alpha=0}^{k_i}\left\langle {k_i \atop \alpha}\right\rangle q^{(m)}_{\alpha}!\tau_{\sharp i}^{\{\alpha\}}\otimes\tau_{\sharp i}^{\{\alpha\}}\left(\sum_{\beta+\gamma=k_i-\alpha}\left\langle {k_i-\alpha \atop \beta}\right\rangle (1\otimes\tau_{\sharp i}^{\{\beta\}})(\tau_{\sharp i}^{\{\gamma\}}\otimes 1)\right)\\
&= \sum_{\alpha+\beta+\gamma=k_i}\left\langle {k_i \atop \alpha}\right\rangle\left\langle {k_i-\alpha \atop \beta}\right\rangle q^{(m)}_{\alpha}!(\tau_{\sharp i}^{\{\alpha\}}\otimes\tau_{\sharp i}^{\{\alpha\}})(1\otimes\tau_{\sharp i}^{\{\beta\}})(\tau_{\sharp i}^{\{\gamma\}}\otimes 1)\\
&= \sum_{\alpha+\beta+\gamma=k_i}\left\langle {k_i \atop \alpha}\right\rangle\left\langle {k_i-\alpha \atop \beta}\right\rangle q^{(m)}_{\alpha}!\tau_{\sharp i}^{\{\alpha\}}\tau_{\sharp i}^{\{\gamma\}}\otimes\tau_{\sharp i}^{\{\alpha\}}\tau_{\sharp i}^{\{\beta\}}\\
&= \sum_{\alpha+\beta+\gamma=k_i}\left\langle {k_i \atop \alpha}\right\rangle\left\langle {k_i-\alpha \atop \beta}\right\rangle q^{(m)}_{\alpha}!\left\{ {\alpha+\gamma \atop \alpha}\right\}\left\{ {\alpha+\beta \atop \alpha}\right\}\tau_{\sharp i}^{\{\alpha+\gamma\}}\otimes\tau_{\sharp i}^{\{\alpha+\beta\}}\\
&= \sum_{\alpha+\beta+\gamma=k_i}\binom{k_i}{\alpha}\binom{k_i-\alpha}{\beta}\frac{q^{(m)}_{\alpha+\gamma}!q^{(m)}_{\alpha+\beta}!}{q^{(m)}_{k_i}!}\tau_{\sharp i}^{\{\alpha+\gamma\}}\otimes\tau_{\sharp i}^{\{\alpha+\beta\}}.
\end{aligned}$$

Hence we deduce that

$$\delta^{n,n'}_{(m)}(\underline{\tau}_{\sharp}^{\{\underline{k}\}}) = \sum_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}=\underline{k}}\binom{\underline{k}}{\underline{\alpha}}\binom{\underline{k}-\underline{\alpha}}{\underline{\beta}}\frac{q^{(m)}_{\underline{\alpha}+\underline{\gamma}}!q_{\underline{\alpha}+\underline{\beta}}!}{q^{(m)}_{\underline{k}}!}\underline{\tau}_{\sharp}^{\{\underline{\alpha}+\underline{\gamma}\}}\otimes\underline{\tau}_{\sharp}^{\{\underline{\alpha}+\underline{\beta}\}}. \tag{3.2.3.11.2}$$

Since $\underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}\partial_{\sharp i,} = \underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}\circ(id\otimes\partial_{\sharp i})\circ\delta^{n,n'}_{(m)}$, then it follows from 3.2.3.11.2 that $\underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}\partial_{\sharp i,}(\underline{\tau}_{\sharp}^{\{\underline{k}\}})\neq 0$ if and only if there exists $\underline{\alpha},\underline{\beta},\underline{\gamma}\in\mathbb{N}^r$ such that $\underline{\alpha}+\underline{\beta}+\underline{\gamma}=\underline{k}$, $\underline{\alpha}+\underline{\gamma}=l$ and $\underline{\alpha}+\underline{\beta}=\underline{\varepsilon}_i$. We have two cases: either $\underline{\alpha}=\underline{\varepsilon}_i$, $\underline{\beta}=\underline{0}$, $\underline{\gamma}=\underline{l}-\underline{\varepsilon}_i$ and $\underline{k}=\underline{l}$ or $\underline{\alpha}=\underline{0}$, $\underline{\beta}=\underline{\varepsilon}_i$, $\underline{\gamma}=\underline{l}$ and $\underline{k}=\underline{l}+\underline{\varepsilon}_i$. Hence, we compute:

$$\underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}\partial_{\sharp i,} = l_i\underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}+(l_i+1)\frac{q^{(m)}_{l_i}!}{q^{(m)}_{l_i+1}!}\underline{\partial}_{\sharp}^{\langle\underline{l}+\underline{\varepsilon}_i\rangle} = l_i\underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}+(l_i+1)\frac{q^{(m)}_{\underline{l}}!}{q^{(m)}_{\underline{l}+\underline{\varepsilon}_i}!}\underline{\partial}_{\sharp}^{\langle\underline{l}+\underline{\varepsilon}_i\rangle}. \tag{3.2.3.11.3}$$

Since $(l_i+1)=\frac{(\underline{l}+\underline{\varepsilon}_i)!}{\underline{l}!}$, then we get the first equality from 3.2.3.11.3:

$$\frac{(\underline{l}+\underline{\varepsilon}_i)!}{q^{(m)}_{\underline{l}+\underline{\varepsilon}_i}!}\underline{\partial}_{\sharp}^{\langle\underline{l}+\underline{\varepsilon}_i\rangle} = \frac{\underline{l}!}{q^{(m)}_{\underline{l}}!}\underline{\partial}_{\sharp}^{\langle\underline{l}\rangle}(\partial_{\sharp i,}-l_i).$$

By induction on $|\underline{k}|$, this yields

$$\frac{(\underline{l}+\underline{\varepsilon}_i)!}{q^{(m)}_{\underline{l}+\underline{\varepsilon}_i}!}\underline{\partial}_{\sharp}^{\langle\underline{l}+\underline{\varepsilon}_i\rangle} = \left(\prod_{0\leq\underline{n}<\underline{l}}(\partial_{\sharp i}-n_i)\right)(\partial_{\sharp i,}-l_i) = \prod_{0\leq\underline{n}<\underline{l}+\underline{\varepsilon}_i}(\partial_{\sharp i}-n_i).$$

□

Remark 3.2.3.12. It follows from part c) of the above lemma that in characteristic p we have

$$(\underline{\partial}_{\sharp}^{\langle\underline{\varepsilon}_i\rangle_{(1)}})^p = \underline{\partial}^{\langle\underline{\varepsilon}_i\rangle_{(1)}}.$$

Indeed, according to the lemma,

$$(\underline{\partial}_{\sharp}^{\langle\underline{\varepsilon}_i\rangle_{(1)}})^p - \underline{\partial}^{\langle\underline{\varepsilon}_i\rangle_{(1)}} = \prod_{j=0}^{p-1}(\underline{\partial}_{\langle\underline{\varepsilon}_i\rangle}-j) = p!\underline{\partial}_{\langle p\underline{\varepsilon}_i\rangle} = 0.$$

From this we deduce that the p-curvature of the canonical connection associated to a $\mathcal{D}^{(1)}_X$-module $\mathcal{E}$ is zero. In fact the p-curvature of ∇ is $\psi(\nabla)(\partial) = \nabla(\partial)^p - \nabla(\partial^{(p)})$. Then $\nabla(\partial_{\sharp i}) = (id \otimes \partial_{\sharp i}) \circ \nabla = \underline{\partial}_{\sharp i}$ and $\partial^{(p)}_{\sharp i} = \partial_{\sharp i}$. Thus $\psi(\nabla)(\partial_{\sharp i}) = 0$.

Lemma 3.2.3.13. *Suppose f is endowed with logarithmic coordinates $(u_1, \dots, u_r)$. With notations 1.2.1.3 and 3.2.3.4, for any $\underline{k}'$, $\underline{k}'' \in \mathbb{N}^r$, we have the formula*

$$\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle_{(m)}} \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle_{(m)}} = \sum_{\underline{k}=\sup\{\underline{k}',\underline{k}''\}}^{\underline{k}'+\underline{k}''} \frac{\underline{k}!}{(\underline{k}'+\underline{k}''-\underline{k})!(\underline{k}-\underline{k}')!(\underline{k}-\underline{k}'')!} \frac{q^{(m)}_{\underline{k}'}!q^{(m)}_{\underline{k}''}!}{q^{(m)}_{\underline{k}}!} \underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m)}}, \qquad (3.2.3.13.1)$$

with $\frac{\underline{k}!}{(\underline{k}'+\underline{k}''-\underline{k})!(\underline{k}-\underline{k}')!(\underline{k}-\underline{k}'')!} \frac{q^{(m)}_{\underline{k}'}!q^{(m)}_{\underline{k}''}!}{q^{(m)}_{\underline{k}}!} \in \mathbb{Z}_{(p)}$.

Proof. Let $\underline{k}$, $\underline{k}'$, $\underline{k}'' \in \mathbb{N}^r$. If $\underline{k}'' \leq \underline{k}$, then it follows from 3.2.3.11.2 that we have

$$(id \otimes \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle}) \circ \delta^{n,n'}_{(m)}(\underline{\tau}_{\sharp}^{\{\underline{k}\}}) = \sum_{\underline{\alpha}=0}^{\underline{k}''} \binom{\underline{k}}{\underline{\alpha}} \binom{\underline{k}-\underline{\alpha}}{\underline{k}''-\underline{\alpha}} \frac{q^{(m)}_{\underline{k}-\underline{k}''+\underline{\alpha}}!q^{(m)}_{\underline{k}''}!}{q^{(m)}_{\underline{k}}!} \underline{\tau}_{\sharp}^{\{\underline{k}-\underline{k}''+\underline{\alpha}\}}. \qquad (3.2.3.13.2)$$

When $\underline{k}'' \not\leq \underline{k}$ the left hand side of 3.2.3.13.2 is 0. Since we have

$$\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle}(\underline{\tau}_{\sharp}^{\{\underline{k}\}}) := \underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \circ (id \otimes \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle}) \circ \delta^{n,n'}_{(m)}(\underline{\tau}_{\sharp}^{\{\underline{k}\}}), \qquad (3.2.3.13.3)$$

then from 3.2.3.13.2, the term 3.2.3.13.3 is not null if and only if there exists $0 \leq \underline{\alpha} \leq \underline{k}''$ such that $\underline{k} - \underline{k}'' + \underline{\alpha} = \underline{k}'$, i.e. $\underline{k} = (\underline{k}'' - \underline{\alpha}) + \underline{k}'$. This is equivalent to saying that $\sup\{\underline{k}', \underline{k}''\} \leq \underline{k} \leq \underline{k}' + \underline{k}''$. In that case, we compute

$$\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle}(\underline{\tau}_{\sharp}^{\{\underline{k}\}}) := \underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \circ (id \otimes \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle}) \circ \delta^{n,n'}_{(m)}(\underline{\tau}_{\sharp}^{\{\underline{k}\}}) = \frac{\underline{k}!}{(\underline{k}'+\underline{k}''-\underline{k})!(\underline{k}-\underline{k}')!(\underline{k}-\underline{k}'')!} \frac{q^{(m)}_{\underline{k}'}!q^{(m)}_{\underline{k}''}!}{q^{(m)}_{\underline{k}}!}$$

and the lemma follows.

It follows from 1.2.1.5.a that we have the equalities in $\mathbb{Z}_{(p)}$:

$$\left\langle \begin{matrix} \underline{\alpha}+\underline{\beta}+\underline{\gamma} \\ \underline{\alpha} \end{matrix} \right\rangle \left\langle \begin{matrix} \underline{\beta}+\underline{\gamma} \\ \underline{\beta} \end{matrix} \right\rangle q^{(m)}_{\underline{\alpha}}! \left\{ \begin{matrix} \underline{\alpha}+\underline{\gamma} \\ \underline{\alpha} \end{matrix} \right\} \left\{ \begin{matrix} \underline{\alpha}+\underline{\beta} \\ \underline{\alpha} \end{matrix} \right\} = \binom{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}{\underline{\alpha}} \binom{\underline{\beta}+\underline{\gamma}}{\underline{\beta}} \frac{q^{(m)}_{\underline{\alpha}+\underline{\gamma}}!q_{\underline{\alpha}+\underline{\beta}}!}{q^{(m)}_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}!}$$
$$= \frac{(\underline{\alpha}+\underline{\beta}+\underline{\gamma})!}{\underline{\alpha}!\underline{\beta}!\underline{\gamma}!} \frac{q^{(m)}_{\underline{\alpha}+\underline{\gamma}}!q_{\underline{\alpha}+\underline{\beta}}!}{q^{(m)}_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}!}. \qquad (3.2.3.13.4)$$

Replacing $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ by respectively $\underline{k}' + \underline{k}'' - \underline{k}$, $\underline{k} - \underline{k}'$, $\underline{k} - \underline{k}''$ we are done. ☐

Remark 3.2.3.14. With its notation, it follows from the formula 3.2.3.13, the following assertions.

(a) We have $\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \circ \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle} = \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle} \circ \underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle}$.

(b) In general, contrary to the non-logarithmic formula (see 1.4.2.7.c) the equality $\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle} = \left\langle \begin{matrix} \underline{k}'+\underline{k}'' \\ \underline{k}' \end{matrix} \right\rangle_{(m)} \underline{\partial}_{\sharp}^{\langle \underline{k}'+\underline{k}'' \rangle}$ is false but it becomes true modulo the operator of less order i.e.

$$\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle} - \left\langle \begin{matrix} \underline{k}' - \underline{k}'' \\ \underline{k}' \end{matrix} \right\rangle_{(m)} \underline{\partial}_{\sharp}^{\langle \underline{k}'+\underline{k}'' \rangle} \in \oplus_{\sup\{\underline{k}',\underline{k}''\} \leq \underline{k} < \underline{k}'+\underline{k}''} \mathbb{Z}_{(p),X} \underline{\partial}_{\sharp}^{\langle \underline{k} \rangle} \subset \oplus_{|\underline{k}| < |\underline{k}'+\underline{k}''|-1} \mathbb{Z}_{(p),X} \underline{\partial}_{\sharp}^{\langle \underline{k} \rangle}.$$
(3.2.3.14.1)

(c) In the particular case where $\underline{k}'$, $\underline{k}'' \in \mathbb{N}^r$ have disjoint supports, we get $\underline{\partial}_{\sharp}^{\langle \underline{k}' \rangle} \circ \underline{\partial}_{\sharp}^{\langle \underline{k}'' \rangle} = \underline{\partial}_{\sharp}^{\langle \underline{k}'+\underline{k}'' \rangle}$. For any $\underline{k} \in \mathbb{N}^r$, this yields the formula

$$\underline{\partial}_{\sharp}^{\langle \underline{k} \rangle} = \partial_{\sharp,1}^{\langle k_1 \rangle} \circ \partial_{\sharp,2}^{\langle k_2 \rangle} \cdots \partial_{\sharp,r}^{\langle k_r \rangle}. \qquad (3.2.3.14.2)$$

(d) Suppose m is finite. If $\underline{k} = p^m \underline{q} + \underline{r}$ with $0 \leq r_i < p^m$, if $r_i = \sum_{j=0}^{m-1} a_{i,j} p^j$ with $0 \leq a_{i,j} < p$, then by using 3.2.3.14.1, following the computations of 1.4.2.8.3 we get

$$\underline{\partial}_\sharp^{\langle \underline{k} \rangle} - u_{\underline{k}} \prod_{i=1}^{p} \left(\prod_{j=0}^{m-1} (\partial_{\sharp,i}^{\langle p^j \rangle})^{a_{i,j}} \right) (\partial_{\sharp,i}^{[p^m]})^{q_i} \in \oplus_{|\underline{l}|<|\underline{k}|-1} \mathbb{Z}_{(p),X} \underline{\partial}_\sharp^{\langle \underline{l} \rangle}. \tag{3.2.3.14.3}$$

for some invertible elements $u_{\underline{k}}$ in $\mathbb{Z}_{(p)}^*$.

Proposition 3.2.3.15. *Suppose m is finite. Suppose f is endowed with logarithmic coordinates $(u_1, \ldots, u_r)$. With notation3.2.3.4, the sheaf $\mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$ is generated as $\mathcal{O}_S$-algebra by $\mathcal{O}_X$ and the two by two commuting operators $\partial_{\sharp,i}, \partial_{\sharp,i}^{\langle p \rangle}, \cdots, \partial_{\sharp,i}^{\langle p^m \rangle}$, $1 \leq i \leq r$.*

More precisely, fix $1 \leq i \leq r$. Then, for any $0 \leq j \leq m$, for any $0 \leq k < p^j$, the operator $\partial_{\sharp,i}^{\langle k \rangle}$ belongs to the sub $\mathbb{Z}_{(p)}$-algebra of $\mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$ generated by $\partial_{\sharp,i}, \partial_{\sharp,i}^{\langle p \rangle}, \cdots, \partial_{\sharp,i}^{\langle p^{j-1} \rangle}$. Moreover, for any $k \in \mathbb{N}$, $\partial_{\sharp,i}^{\langle k \rangle}$ belongs to the sub $\mathbb{Z}_{(p)}$-algebra of $\mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$ generated by $\partial_{\sharp,i}, \partial_{\sharp,i}^{\langle p \rangle} \cdots, \partial_{\sharp,i}^{\langle p^m \rangle}$.

Proof. It follows from 3.2.3.14.2 that we reduce to check the proposition without multi-indices. By doing an induction on $|\underline{k}|$ and by using 3.2.3.14.3, we conclude. □

Proposition 3.2.3.16. *Suppose m is finite. We have the following properties.*

(a) The graded ring gr $\mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$ associated to the order filtration $(\mathcal{D}_{X^\sharp/S^\sharp,n}^{(m)})_{n\in\mathbb{N}}$ is a commutative ring. If $X^\sharp/S^\sharp$ is endowed with logarithmic coordinates, the relation $\underline{\partial}_\sharp^{\langle \underline{k} \rangle} \underline{\partial}_\sharp^{\langle \underline{h} \rangle} = \left\langle \begin{smallmatrix} \underline{k}+\underline{h} \\ \underline{k} \end{smallmatrix} \right\rangle \underline{\partial}_\sharp^{\langle \underline{k}+\underline{h} \rangle}$ becomes exact in $\operatorname{gr} \mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$.

(b) Suppose S be a locally noetherian scheme. Let U be an affine open of X (resp. $x \in X$). Then $\Gamma(U, \operatorname{gr}\ \mathcal{D}_{X^\sharp/S^\sharp}^{(m)})$ (resp. $D_U^{(m)} := \Gamma(U, \mathcal{D}_{X^\sharp/S^\sharp}^{(m)})$, resp. $\operatorname{gr}\ \mathcal{D}_{X^\sharp/S^\sharp,x}^{(m)}$, resp. $\mathcal{D}_{X^\sharp/S^\sharp,x}^{(m)}$) is left and right noetherian.

(c) For any inclusion of affine opens of X of the form $V \subset U$, the canonical morphism $\Gamma(U, \mathcal{D}_{X^\sharp/S^\sharp}^{(m)}) \to \Gamma(V, \mathcal{D}_{X^\sharp/S^\sharp}^{(m)})$ is flat.

(d) Suppose S be a locally noetherian scheme. Then the sheaf of rings $\mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$ is right and left coherent.

Proof. The first statement is a consequence of 3.2.3.14.1 and of 3.2.3.7.2. By replacing Proposition 1.4.2.10 by 3.2.3.15, to check the second assertion we can simply copy the proof of 1.4.2.11.(b). Finally, we get the last two statements by copying the proof of 1.4.5.3. □

3.2.3.17. Let $g\colon S'^\sharp \to S^\sharp$ be a morphism of fine log schemes over $\mathbb{Z}_{(p)}$. Consider the commutative diagram

$$\begin{array}{ccc} X'^\sharp & \xrightarrow{f} & X^\sharp \\ \downarrow{\scriptstyle \pi_{X'^\sharp}} & & \downarrow{\scriptstyle \pi_{X^\sharp}} \\ S'^\sharp & \xrightarrow{g} & S^\sharp \end{array} \tag{3.2.3.17.1}$$

such that $\pi_{X^\sharp}$ and $\pi_{X'^\sharp}$ are formal log smooth of level m. Using the universal property of the m-PD envelope, we get the m-PD-morphism $f^*\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}^n_{X'^\sharp/S'^\sharp,(m)}$. This yields the morphism $\mathcal{D}^{(m)}_{X'^\sharp/S'^\sharp,n} \to f^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$ and then $\mathcal{D}^{(m)}_{X'^\sharp/S'^\sharp} \to f^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

When the diagram 3.2.3.17.1 is cartesian (in the category of fine log schemes), the morphism $f^*\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}^n_{X'^\sharp/S'^\sharp,(m)}$ is in fact an isomorphism of rings and so is $\mathcal{D}^{(m)}_{X'^\sharp/S'^\sharp} \to f^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

When $g = \mathrm{id}$ and f is formally log étale of level m, then the morphism $f^*\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \to \mathcal{P}^n_{X'^\sharp/S^\sharp,(m)}$ is in fact an isomorphism and so is $\mathcal{D}^{(m)}_{X'^\sharp/S^\sharp} \to f^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

3.2.4 Increasing the level: finiteness of the tor-dimension

Let $S^\sharp$ be a nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$, where $i \geq 0$ is an integer (see definition 3.1.1.1). Moreover, let $X^\sharp \to S^\sharp$ be a log smooth morphism of log schemes. Let $m \in \mathbb{N}$ be an integer.

Proposition 3.2.4.1. Suppose S is of characteristic $p > 0$. Suppose that $X^\sharp$ have logarithmic coordinates $t_1, \ldots, t_d$.

(a) For any integer q, we have $\left(\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}\right)^q = v_{m,q}\partial_{\sharp,i}^{\langle p^{m+1}q\rangle_{(m)}}$, with $v_{m,q} \in \mathbb{Z}_{(p)}^*$.

(b) The center of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ contains the polynomial algebra with coefficients in $\mathcal{O}_{X^{(m+1)}}$ in the operators $\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}$.

(c) For any integer $q \geq 0$, $0 \leq r \leq p^{m+1} - 1$, by setting $k := p^{m+1}q + r$, we have $\partial_{\sharp,i}^{\langle k\rangle_{(m)}} = u_{m,k}\left(\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}\right)^q \partial_{\sharp,i}^{\langle r\rangle_{(m)}}$, for $i = 1, \ldots, d$, with $u_{m,k} \in \mathbb{Z}_{(p)}^*$.

(d) Let $\mathcal{K}_{(m)}$ be the set whose elements are the finite sums of the form $\sum_{\underline{k}\not< p^{m+1}} a_{\underline{k}}\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$, with $a_{\underline{k}} \in \mathcal{O}_X$ and $\underline{k} \not< p^{m+1}$ meaning that $k_i \geq p^{m+1}$ for at least one $1 \leq i \leq d$. Then $\mathcal{K}_{(m)}$ is the two-sided ideal of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ generated by the operators $\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}$.

Proof. Following 1.4.4.1, both $t_i^{p^{m+1}}$ and $\partial_i^{\langle p^{m+1}\rangle_{(m)}}$ are in the center of $\mathcal{D}^{(m)}_{U/S^\sharp}$. Since $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to j_{U*}\mathcal{D}^{(m)}_{U/S^\sharp}$ is an injective morphism of rings then $\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}} = t_i^{p^{m+1}}\partial_i^{\langle p^{m+1}\rangle_{(m)}}$ is in the center of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. Hence, a and b is a consequence of 1.4.4.1.a and 1.4.4.1.d. Moreover, it follows from 1.4.4.1 that $\partial_i^{\langle k\rangle_{(m)}} = u_{m,k}\left(\partial_i^{\langle p^{m+1}\rangle_{(m)}}\right)^q \partial_i^{\langle r\rangle_{(m)}}$, for $i = 1, \ldots, d$, with $u_{m,k} \in \mathbb{Z}_{(p)}^*$. This yields

$$\partial_{\sharp,i}^{\langle k\rangle_{(m)}} = t_i^k\partial_i^{\langle k\rangle_{(m)}} = u_{m,k}t_i^r\left(\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}\right)^q\partial_i^{\langle r\rangle_{(m)}} = u_{m,k}\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}\partial_{\sharp,i}^{\langle r\rangle_{(m)}},$$

the last equality being a consequence of the fact that $\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}$ is in the center of $\mathcal{D}^{(m)}_{U/S^\sharp}$. Finally, c implies d. □

The following proposition is the logarithmic version of 1.4.4.2.

Proposition 3.2.4.2. Suppose that $X^\sharp$ have logarithmic coordinates $t_1, \ldots, t_d$. Let $m' \geq m+1$ be an integer.

(a) The kernel of the canonical homomorphism $\rho^{(m,m')} \colon \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ is $\mathcal{K}_{(m)}$, the two-sided ideal of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ generated by the operators $\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}}$.

(b) Moreover, $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ is free on the image of $\rho^{(m,m')}$, with the operators of the form $\underline{\partial}_\sharp^{\langle p^{m+1}\underline{n}\rangle_{(m')}}$, $\underline{n} \in \mathbb{N}^d$ as a basis.

Proof. Suppose that $X^\sharp$ have logarithmic coordinates $t_1, \ldots, t_d$. It follows from the computations of the proof of 1.4.4.2 (in particular, the three last lignes) that the kernel of $\rho^{(m,m')}$ is By using 3.2.4.1.c, this yields that $\partial_{\sharp,i}^{\langle p^{m+1}\rangle_{(m)}} = t_i^{p^{m+1}}\partial_i^{\langle p^{m+1}\rangle_{(m)}}$ generates $\partial_{\sharp,i}^{\langle k\rangle_{(m)}} = t_i^k\partial_i^{\langle k\rangle_{(m)}}$ for $k \geq p^{m+1}$. This yields the result for the kernel. This implies, that the image of $\rho^{(m,m')}$ is the set of finite sums of the form $\sum_{\underline{k}<p^{m+1}} a_{\underline{k}}\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m')}}$, where $\underline{k} < p^{m+1}$ means $k_i < p^{m+1}$ for any $i = 1, \ldots, d$ (we use that $\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}} = \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m')}}$ for $\underline{k} < p^{m+1}$). For $\underline{r} < p^{m+1}$, $\underline{q} \geq 0$ (i.e. $q_i \geq 0$ for any $i = 1, \ldots, d$) and $\underline{k} = p^{m+1}\underline{q} + \underline{r}$, by using 1.4.2.7.(c) we have $\underline{\partial}^{\langle \underline{k}\rangle_{(m')}} = \underline{\partial}^{\langle \underline{r}\rangle_{(m')}} \cdot \underline{\partial}^{\langle p^{m+1}\underline{q}\rangle_{(m')}}$. Since $t_i^{p^{m+1}}$ is in the center of $\mathcal{D}^{(m)}_P$, then $t_i^{p^{m+1}}$ is in the center of the image of $\rho^{(m,m')}$. Since $\underline{\partial}^{\langle \underline{r}\rangle_{(m')}}$ is in this image, this yields that $\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m')}} = \underline{\partial}_\sharp^{\langle \underline{r}\rangle_{(m')}} \cdot \underline{\partial}_\sharp^{\langle p^{m+1}\underline{q}\rangle_{(m')}}$. Hence we are done. □

Corollary 3.2.4.3. *Suppose p is locally nilpotent on S. For any $m' > m$, $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ has left and right tor-dimension d over $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.*

Proof. Similarly to 1.4.4.4, this follows from the proposition 3.2.4.2. □

3.3 Sheaf of differential operators of finite order on logarithmic formal schemes

Let $\mathcal{V}$ be complete discrete valuation ring of mixed characteristic $(0, p)$. We denote by $\mathfrak{S} := \mathrm{Spf}\ \mathcal{V}$ be the p-adic formal scheme, i.e. such that $p\mathcal{V}$ is an ideal of definition. A formal $\mathcal{V}$-scheme (or formal $\mathfrak{S}$-scheme) is by definition an adic morphism of the form $\mathfrak{X} \to \mathfrak{S}$ (in the sense of [Gro60, 10.12], i.e. we assume that a $\mathcal{V}$-formal scheme is always locally noetherian). We recall that Shiho introduced the notion of $\mathcal{V}$-log formal schemes (see [Shi00, 2.1.1.(4)]) as follows: A $\mathcal{V}$-log formal scheme $\mathfrak{X}^\sharp$ is a formal $\mathcal{V}$-scheme $\mathfrak{X}$ endowed with a logarithmic structure $\alpha\colon M_{\mathfrak{X}^\sharp} \to \mathcal{O}_{\mathfrak{X}}$ (this means that α is a logarithmic morphism of sheaves of monoids for the étale topology over $\mathfrak{X}^\sharp$, i.e. α is such that $\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}}) \to \mathcal{O}^*_{\mathfrak{X}}$ is an isomorphism). When $M_{\mathfrak{X}^\sharp}$ is fine as sheaf for the étale topology over the special fiber of $\mathfrak{X}^\sharp$ (i.e. when $M_{\mathfrak{X}^\sharp}$ is integral and $M_{\mathfrak{X}^\sharp}$ is coherent in the sense defined at the end of the remark [Ogu18, II.2.1.5]), we say that the logarithmic structure $M_{\mathfrak{X}^\sharp}$ is fine. We say that $\mathfrak{X}^\sharp$ is a "fine $\mathcal{V}$-log formal scheme" if $M_{\mathfrak{X}^\sharp}$ is fine.

Let $i \geq 0$ be an integer. We set $S_i := \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$. If $\mathfrak{X}^\sharp$ is a fine $\mathcal{V}$-log formal scheme and $i \in \mathbb{N}$, then we denote by $X^\sharp_i$ the fine log S_i-scheme so that $X_i := \mathfrak{X} \times_{\mathfrak{S}} S_i$ and the morphism $X^\sharp_i \to \mathfrak{X}^\sharp$ is strict. For $i = 0$, we can simply denote $X^\sharp_0$ by $X^\sharp$. If $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ is a morphism of fine $\mathcal{V}$-log formal schemes, then we denote by $f_i\colon X^\sharp_i \to Y^\sharp_i$ the induced morphism of fine S_i-log schemes. We remark that if $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ is a morphism of fine $\mathfrak{S}$-log formal schemes, then $f_i\colon X^\sharp_i \to Y^\sharp_i$ is a morphism of fine log S_i-schemes.

3.3.1 From log schemes to formal log schemes

3.3.1.1 (Charts for $\mathfrak{S}$ log formal schemes)**.** Let P be a fine monoid and $\mathcal{V}\{P\}$ be the p-adic completion of $\mathcal{V}[P]$. Since $\mathcal{V}$ is fixed, we denote by $\mathfrak{A}_P$ the fine $\mathcal{V}$-log formal scheme whose underlying formal $\mathcal{V}$-scheme is $\mathrm{Spf}(\mathcal{V}\{P\})$ and whose log structure is the log structure associated with the pre-log structure induced canonically by $P \to \mathcal{V}\{P\}$.

Let $\mathfrak{X}^\sharp$ be a fine $\mathfrak{S}$-log formal scheme. We denote by $P_{\mathfrak{X}^\sharp}$ the sheaf associated to the constant presheaf of P over $\mathfrak{X}^\sharp$. Following Shiho's definition of [Shi00, 2.1.7], a chart of $\mathfrak{X}^\sharp$ is a morphism of monoids $\alpha\colon P_{\mathfrak{X}^\sharp} \to \mathcal{O}_{\mathfrak{X}}$ whose associated log structure is isomorphic to $M_{\mathfrak{X}^\sharp} \to \mathcal{O}_{\mathfrak{X}}$. A chart of $\mathfrak{X}^\sharp$ is equivalent to the data of a strict morphism of the form $\mathfrak{X}^\sharp \to \mathfrak{A}_P$.

Lemma 3.3.1.2. *Let $\mathfrak{X}^\sharp$ be a fine $\mathfrak{S}$-log formal scheme. Let $i \geq 0$ be an integer. Then, the morphisms $\mathcal{O}^*_{\mathfrak{X}} \to \mathcal{O}^*_{X^\sharp_i}$ and $M_{\mathfrak{X}^\sharp} \to M_{X^\sharp_i}$ are surjective.*

Proof. The fact that $\mathcal{O}^*_{\mathfrak{X}} \to \mathcal{O}^*_{X^\sharp_i}$ is surjective comes from the fact that $\mathcal{O}_{\mathfrak{X}}$ is complete for the p-adic topology. The fact that $M_{\mathfrak{X}^\sharp} \to M_{X^\sharp_i}$ is surjective is étale local on $\mathfrak{X}^\sharp$. Hence, we can suppose there exists a fine monoid P and a morphism of sheaves of monoids $\alpha\colon P_{\mathfrak{X}^\sharp} \to \mathcal{O}_{\mathfrak{X}}$ (here $P_{\mathfrak{X}^\sharp}$ means the sheaf associated to the constant presheaf of P over $\mathfrak{X}^\sharp$) which induces the isomorphism of sheaves of monoids $P_{\mathfrak{X}^\sharp} \oplus_{\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})} \mathcal{O}^*_{\mathfrak{X}} \xrightarrow{\sim} M_{\mathfrak{X}^\sharp}$ and the isomorphism $P_{X^\sharp_i} \oplus_{\alpha_i^{-1}(\mathcal{O}^*_{X^\sharp_i})} \mathcal{O}^*_{X^\sharp_i} \xrightarrow{\sim} M_{X^\sharp_i}$. Since $\mathcal{O}^*_{\mathfrak{X}} \to \mathcal{O}^*_{X^\sharp_i}$ is surjective, we conclude.

□

Proposition 3.3.1.3. Let $\mathfrak{X}^\sharp$ be a fine $\mathfrak{S}$-log formal scheme. Then, in the category of fine $\mathfrak{S}$-log formal schemes, $\mathfrak{X}^\sharp$ is the inductive limit of the system $(X^\sharp_i)_i$.

Proof. From [Gro60, I.10.6.1], $\underline{\mathfrak{X}}$ is the inductive limit of the system $(\underline{X}_i)_i$. It remains to check that the canonical morphism of sheaves of monoids $M_{\mathfrak{X}^\sharp} \to \varprojlim_i M_{X^\sharp_i}$ is an isomorphism. Since this is étale local on $\mathfrak{X}^\sharp$ and since $\mathfrak{X}^\sharp$ is fine then we can suppose there exists a fine monoid P and a morphism of sheaves of monoids $\alpha\colon P_{\mathfrak{X}^\sharp} \to \mathcal{O}_{\mathfrak{X}}$ which induces the isomorphism of sheaves of monoids $P_{\mathfrak{X}^\sharp} \oplus_{\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})} \mathcal{O}^*_{\mathfrak{X}} \xrightarrow{\sim} M_{\mathfrak{X}^\sharp}$. Let $i \geq 0$ be an integer. We get the morphism of sheaves of monoids $\alpha_i\colon P_{X_i^\sharp} \to \mathcal{O}_{X_i^\sharp}$ which induces the isomorphism $P_{X_i^\sharp} \oplus_{\alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})} \mathcal{O}^*_{X_i^\sharp} \xrightarrow{\sim} M_{X_i^\sharp}$. Hence, we reduce to prove that the canonical map $P_{\mathfrak{X}^\sharp} \oplus_{\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})} \mathcal{O}^*_{\mathfrak{X}} \to \varprojlim_i P_{X_i^\sharp} \oplus_{\alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})} \mathcal{O}^*_{X_i^\sharp}$ is an isomorphism. We put $\mathcal{F}_i := P_{X_i^\sharp}\text{“}\oplus\text{”}_{\alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})}\mathcal{O}^*_{X_i^\sharp}$ where “ $\oplus$ ” means that the amalgamated sum is computed in the category of presheaves. We put $\mathcal{E}_i := P_{X_i^\sharp} \oplus \mathcal{O}^*_{X_i^\sharp}$, $\theta_i\colon \mathcal{E}_i \to \mathcal{F}_i$ the canonical surjective morphism, $\mathcal{G}_i := P_{X_i^\sharp} \oplus_{\alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})} \mathcal{O}^*_{X_i^\sharp}$ and $\epsilon_i\colon \mathcal{F}_i \to \mathcal{G}_i$ the canonical morphism from a presheaf to its associated sheaf. We put $\phi_i := \epsilon_i \circ \theta_i$. We denote by $\pi_i\colon \mathcal{O}^*_{X_{i+1}^\sharp} \to \mathcal{O}^*_{X_i^\sharp}$, $\pi_i\colon \mathcal{E}_{i+1} \to \mathcal{E}_i$ $\pi_i\colon \mathcal{F}_{i+1} \to \mathcal{F}_i$, $\pi_i\colon \mathcal{G}_{i+1} \to \mathcal{G}_i$ the canonical projections. Let $\mathfrak{U} \to \mathfrak{X}$ be an étale map such that $\mathfrak{U}$ is connected.

1) Let $s_{i+1} \in \mathcal{F}_{i+1}(U_{i+1})$ and $s_i := \pi_i(s_{i+1}) \in \mathcal{F}_i(U_i)$. Then the canonical map $\pi_i\colon \theta_{i+1}^{-1}(s_{i+1}) \to \theta_i^{-1}(s_i)$ induced by $\pi_i\colon \mathcal{E}_{i+1}(U_{i+1}) \to \mathcal{E}_i(U_i)$ is a bijection.

a) Let us check the injectivity. Let $(x,a),(x',a') \in \theta_{i+1}^{-1}(s_{i+1})$ such that $\pi_i(x,a) = \pi_i(x',a')$ (where $x, x' \in P$ and $a, a' \in \mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1})$). The latter equality yields $x = x'$. Since P is integral, $\theta_{i+1}(x,a) = \theta_{i+1}(x,a')$ implies $a = a'$ (for the computation, use the remark of [Kat89, 1.3]).

b) Let us check the surjectivity. Let $(y,b) \in \theta_i^{-1}(s_i)$. We remark that $\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})(\mathfrak{U}) = \alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})(U_i)$ and we denote it by Q. Since θ_{i+1} is an epimorphism (in the category of presheaves) then there exists $(x,a) \in \theta_{i+1}^{-1}(s_{i+1})$. Since $\pi_i(x,a) = (x, \pi_i(a)) \in \theta_i^{-1}(s_i)$, there exists $q, q' \in Q(U_i)$ such that $\pi_i(a)\alpha_i(q) = b\alpha_i(q')$ and $xq' = yq$ (see the remark of [Kat89, 1.3]). Set $a' := a\alpha_{i+1}(q)\alpha_{i+1}(q')^{-1}$. Then $\pi_i(a') = b$ and $\theta_{i+1}(x,a) = \theta_{i+1}(y,a')$, i.e. $\pi_i(y,a') = (y,b)$ and $(y,a') \in \theta_{i+1}^{-1}(s_{i+1})$.

2) Let $t_{i+1} \in \mathcal{G}_{i+1}(U_{i+1})$ and $t_i := \pi_i(t_{i+1}) \in \mathcal{G}_i(U_i)$. Then the canonical map $\pi_i\colon \phi_{i+1}^{-1}(t_{i+1}) \to \phi_i^{-1}(t_i)$ is a bijection.

a) Let us check the injectivity. Let $r, r' \in \phi_{i+1}^{-1}(t_{i+1})$ such that $\pi_i(r) = \pi_i(r')$. There exists an étale covering $(\mathfrak{U}_\lambda \to \mathfrak{U})_\lambda$ of $\mathfrak{U}$ such that $\theta_{i+1}(r)|U_\lambda = \theta(r')_{i+1}|U_\lambda$. From 1) (applied for $\mathfrak{U}_\lambda$ instead of $\mathfrak{U}$), this yields $r|U_\lambda = r'|U_\lambda$. Hence, $r = r'$.

b) Let us check the surjectivity. Let $r \in \phi_i^{-1}(t_i)$. Put $s := \theta_i(r)$. There exist an étale covering $(\mathfrak{U}_\lambda \to \mathfrak{U})_\lambda$ of $\mathfrak{U}$ and sections $s_\lambda \in \mathcal{F}_{i+1}(U_\lambda)$ such that $\epsilon_{i+1}(s_\lambda) = t_{i+1}|U_\lambda$ and $\pi_i(s_\lambda) = s|U_\lambda$. From 1.b), there exists $r_\lambda \in \mathcal{E}_{i+1}(U_\lambda)$ such that $\pi_i(r_\lambda) = r|U_\lambda$ and $\theta_{i+1}(r_\lambda) = s_\lambda$. Hence, $\pi_i(r_\lambda) = r|U_\lambda$ and $\phi_{i+1}(r_\lambda) = t_{i+1}|U_\lambda$. From 2.a), this yields that $(r_\lambda)_\lambda$ come from a section of $\mathcal{E}_{i+1}(U_{i+1})$.

3) Now, let us check that the canonical map $P_{\mathfrak{X}^\sharp} \oplus_{\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})} \mathcal{O}^*_{\mathfrak{X}} \to \varprojlim_i P_{X_i^\sharp} \oplus_{\alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})} \mathcal{O}^*_{X_i^\sharp}$ is an isomorphism. First, we start with the injectivity. As above, put $\mathcal{E} := P_{\mathfrak{X}^\sharp} \oplus \mathcal{O}^*_{\mathfrak{X}}$, $\mathcal{F} := P_{\mathfrak{X}^\sharp}\text{“}\oplus\text{”}_{\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})}\mathcal{O}^*_{\mathfrak{X}}$, $\mathcal{G} := P_{\mathfrak{X}^\sharp} \oplus_{\alpha^{-1}(\mathcal{O}^*_{\mathfrak{X}})} \mathcal{O}^*_{\mathfrak{X}}$, $\theta\colon \mathcal{E} \to \mathcal{F}$, $\epsilon\colon \mathcal{F} \to \mathcal{G}$, $\phi := \epsilon \circ \theta$. Let $(x,a),(y,b) \in \mathcal{E}(\mathfrak{U})$ such that the image of $\phi(x,a)$ and $\phi(y,b)$ in $\varprojlim_i P_{X_i^\sharp} \oplus_{\alpha_i^{-1}(\mathcal{O}^*_{X_i^\sharp})} \mathcal{O}^*_{X_i^\sharp}(\mathfrak{U})$ are equal (where $x, y \in P$, and $a, b \in \mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U})$). Since the injectivity is locally etale, we reduce to check that $\phi(x,a) = \phi(y,b)$. Denote by $(x,a_i),(y,b_i) \in \mathcal{E}_i$ the image of $(x,a),(y,b)$. Shrinking $\mathfrak{U}$ if necessary, we can suppose that $\theta_0(x,a_0) = \theta_0(y,b_0)$. Doing the same computation as in 1.b), we can check there exists $c \in \mathcal{O}^*_{\mathfrak{X}}$ such that $\theta(x,a) = \theta(y,c)$. Moreover, since P is integral, we can check that $\phi_i(y,c_i) = \phi_i(y,b_i)$ if and only if $c_i = b_i$ (since this is etale local, we reduce to check $\theta_i(y,c_i) = \theta_i(y,b_i)$ if and only if $c_i = b_i$). Hence, $b = c$, which implies $\phi(x,a) = \phi(y,b)$. Hence, we have checked the injectivity. The surjectivity is an easy consequence of 2). □

Définition 3.3.1.4. We define the category of strict inductive systems of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$ as follows. A strict inductive system of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$ is the data, for any integer $i \in \mathbb{N}$, of a noetherian fine S_i-log scheme $X_i^\sharp$, of an exact closed S_i immersion $X_i^\sharp \hookrightarrow X_{i+1}^\sharp$ such that the induced morphism $X_i^\sharp \to X_{i+1}^\sharp \times_{S_{i+1}} S_i$ is an isomorphism. A morphism $(X_i^\sharp)_{i\in\mathbb{N}} \to (Y_i^\sharp)_{i\in\mathbb{N}}$ of strict inductive systems of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$ is a family of S_i-morphism $X_i^\sharp \to Y_i^\sharp$ making commutative the diagram

$$\begin{array}{ccc} X_i^\sharp & \longrightarrow & X_{i+1}^\sharp \\ \downarrow & & \downarrow \\ Y_i^\sharp & \longrightarrow & Y_{i+1}^\sharp. \end{array}$$

Définition 3.3.1.5. Let X be a scheme. Let M be a sheaf (for the étale topology) of monoids over X

1. Following [Ogu18, II.1.1.3], we say that M is integral if for every $x \in X$, M_x is integral.
2. We say that M is coherent if there exists an open covering $\mathfrak{U}$ such that the restriction of M to each U in $\mathfrak{U}$ admits a chart subordinate to a finitely generated monoid (see definition [Ogu18, II.2.1.5]).
3. We say that M is fine if M is integral and coherent.
4. Let $f\colon M \to N$ be a morphism of sheaves (for the étale topology) of monoids over X. We say that f is local if the induced morphism $M^* \to M \times_N N^*$ is an isomorphism (see Definitions [Ogu18, I.4.1.1] for monoids and we have taken a similar to [Ogu18, II.1.1.4] definition for sheaves of monoids).

Lemma 3.3.1.6. *Let X be a scheme. Let $M \to N$ be a local morphism of sheaves (for the étale topology) of monoids over X. Suppose M integral, N fine. Then M is fine.*

Proof. Let us fix some notation. Let $\overline{x}$ be a geometric point of X. Since N is fine, using [Ogu18, I.1.3.3 and II.1.8.1], we can check that $\overline{N}_{\overline{x}}$ is fine. Since $\overline{M} = \overline{N}$, we get that $\overline{M}_{\overline{x}}$ is fine. Hence, there exist a free $\mathbb{Z}$-module of finite type L endowed with a morphism $\alpha\colon L \to M^{\mathrm{gr}}_{\overline{x}}$ such that the composition of α with the projection $M^{\mathrm{gr}}_{\overline{x}} \to \overline{M}^{\mathrm{gr}}_{\overline{x}}$ is surjective (following Ogus's terminology appearing in [Ogu18, II.3.3], this means $\alpha\colon L \to M^{\mathrm{gr}}_{\overline{x}}$ is a markup of $M_{\overline{x}}$). We put $P := L \times_{M^{\mathrm{gr}}_{\overline{x}}} M_{\overline{x}}$. Since $M_{\overline{x}}$ is integral, then following [Ogu18, I.4.2.1] the homomorphism of monoids $M_{\overline{x}} \to \overline{M}_{\overline{x}}$ is exact. Hence, we get the equality $P := L \times_{M^{\mathrm{gr}}_{\overline{x}}} M_{\overline{x}} = L \times_{\overline{M}^{\mathrm{gr}}_{\overline{x}}} \overline{M}_{\overline{x}}$. Following [Ogu18, I.2.1.17.6], since L and $\overline{M}_{\overline{x}}$ are fine and since $\overline{M}^{\mathrm{gr}}_{\overline{x}}$ is integral, then $P = L \times_{\overline{M}^{\mathrm{gr}}_{\overline{x}}} \overline{M}_{\overline{x}}$ is fine.

Let $P \to M_{\overline{x}}$ be the projection. Using [Ogu18, II.2.2.4], there exist an étale neighborhood $u\colon U \to X$ of $\overline{x}$ and a morphism of monoids $P \to M(U)$ inducing $P \to M_{\overline{x}}$. Let $\beta\colon P_U \to u^*M$ be the corresponding morphism. We get the factorization of β of the form $P_U \to P^{\beta}_U \xrightarrow{\beta^{\mathrm{a}}} u^*M$, where β^{a} is the log structure associated to β i.e. the homomorphism of monoids β^{a} is logarithmic and is universal for such a factorization (see [Ogu18, II.1.1.5]).

We prove that, shrinking U is necessary, the morphism β^{a} is an isomorphism (and then M is coherent). Following [Ogu18, I.4.1.2], for any geometric point $\overline{y}$ of U, since $\beta^{\mathrm{a}}_{\overline{y}}$ is sharp and since $M_{\overline{y}}$ is quasi-integral, we can check that the morphism $\beta^{\mathrm{a}}_{\overline{y}}\colon (P^{\beta}_U)_{\overline{y}} \to M_{\overline{y}}$ is an isomorphism if and only if $\overline{\beta^{\mathrm{a}}}_{\overline{y}}\colon : \overline{(P^{\beta}_U)}_{\overline{y}} \to \overline{M}_{\overline{y}}$ is an isomorphism (recall $\overline{M}_{\overline{y}} = \overline{M_{\overline{y}}}$). Using [Ogu18, II.1.8.1], we can check that the canonical morphism $P/\beta^{-1}_{\overline{y}}(M^*_{\overline{y}}) \to \overline{(P^{\beta}_U)}_{\overline{y}}$ is an isomorphism. Hence, $\beta^{\mathrm{a}}_{\overline{y}}$ is an isomorphism if and only if the canonical morphism $P/\beta^{-1}_{\overline{y}}(M^*_{\overline{y}}) \to \overline{M}_{\overline{y}}$ is an isomorphism.

Let β_0 be the composition of β with $u^*M \to u^*N$. We get the factorization of β_0 of the form $P_U \to P^{\beta_0}_U \xrightarrow{\beta^{\mathrm{a}}_0} u^*N$, where β^{a}_0 is the sharp localisation of β_0. Since $P := L \times_{M^{\mathrm{gr}}_{\overline{x}}} M_{\overline{x}} = L \times_{\overline{M}^{\mathrm{gr}}_{\overline{x}}} \overline{M}_{\overline{x}} = L \times_{\overline{N}^{\mathrm{gr}}_{\overline{x}}} \overline{N}_{\overline{x}} = L \times_{N^{\mathrm{gr}}_{\overline{x}}} N_{\overline{x}}$, since N is fine, following [Ogu18, II.3.4] (which is checked similarly than [Kat89, 2.10]), replacing U if necessary, we can suppose that β^{a}_0 is an isomorphism. For any geometric point $\overline{y}$ of U, this yields that the morphism $\beta^{\mathrm{a}}_{0,\overline{y}}\colon (P^{\beta_0}_U)_{\overline{y}} \to N_{\overline{y}}$ is an isomorphism. Hence so is $\overline{\beta^{\mathrm{a}}_0}_{\overline{y}}\colon \overline{(P^{\beta_0}_U)}_{\overline{y}} \to \overline{N}_{\overline{y}}$, i.e., $P/\beta^{-1}_{0,\overline{y}}(N^*_{\overline{y}}) \to \overline{N}_{\overline{y}}$ is an isomorphism.

Since the morphism $M \to N$ is local, the induced morphism $M^* \to M \times_N N^*$ is an isomorphism. Hence, we get $M^*_{\overline{y}} \to M_{\overline{y}} \times_{N_{\overline{y}}} N^*_{\overline{y}}$, i.e. the morphism $M_{\overline{y}} \to N_{\overline{y}}$ is local. Hence, we get $\beta^{-1}_{0,\overline{y}}(N^*_{\overline{y}}) = \beta^{-1}_{\overline{y}}(M^*_{\overline{y}})$. Recalling that $\overline{M}_{\overline{y}} = \overline{N}_{\overline{y}}$, this implies that $P/\beta^{-1}_{\overline{y}}(M^*_{\overline{y}}) \to \overline{M}_{\overline{y}}$ is an isomorphism. Hence, we are done.

□

Proposition 3.3.1.7. Let $(X^{\sharp}_i)_{i \in \mathbb{N}}$ be a strict inductive systems of noetherian fine log schemes over $(S_i)_{i \in \mathbb{N}}$. Then $\varinjlim_i X^{\sharp}_i$ is a fine $\mathfrak{S}$-log formal scheme. Moreover, the canonical morphism $X^{\sharp}_i \to (\varinjlim_i X^{\sharp}_i) \times_{\mathfrak{S}} S_i$ is an isomorphism of fine log schemes.

Proof. We already know that $\mathfrak{X} := \varinjlim_i X_i$ is a formal $\mathcal{V}$-scheme such that $X_i \xrightarrow{\sim} \underline{\mathfrak{X}} \times_{\mathfrak{S}} S_i$. We have $\varinjlim_i X^{\sharp}_i = (\varinjlim_i X_i, \varprojlim_i M_{X^{\sharp}_i})$. Put $M := \varprojlim_i M_{X^{\sharp}_i}$, $\mathfrak{X}^{\sharp} := \varinjlim_i X^{\sharp}_i$. It remains to check that M is fine log structure of $\mathfrak{X}$. This is checked in the step I).

I)1) The canonical map $\theta\colon M \to \mathcal{O}_{\mathfrak{X}}$, canonically induced by the structural morphisms $\theta_i\colon M_{X_i^\sharp} \to \mathcal{O}_{X_i^\sharp}$, is a log structure. Indeed, we compute $M^*(\mathfrak{U}) = (\varprojlim_i M_{X_i^\sharp}(U_i))^* = \varprojlim_i (M_{X_i^\sharp}(U_i))^* = \varprojlim_i M^*_{X_i^\sharp}(U_i) = \varprojlim_i \mathcal{O}^*_{X_i^\sharp}(U_i) = \mathcal{O}^*_{\mathfrak{X}^\sharp}(\mathfrak{U})$, for any etale morphism $\mathfrak{U} \to \mathfrak{X}$. Hence, $M^* = \mathcal{O}^*_{\mathfrak{X}^\sharp}$. It remains to check that the morphism $M^* \to M \times_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}^*_{\mathfrak{X}}$ is an isomorphism, i.e. $M^*(\mathfrak{U}) \to M(\mathfrak{U}) \times_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})^*$ is an isomorphism. Since $M(\mathfrak{U}) \times_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})^* \subset M(\mathfrak{U})$, the injectivity is obvious. Let us check the surjectivity. Let $(a_i)_{i\in\mathbb{N}} \in \varprojlim_i M_{X_i^\sharp}(U_i)$ such that $(\theta_i(a_i))_{i\in\mathbb{N}} \in \varprojlim_i \mathcal{O}^*_{X_i^\sharp}(U_i)$. Since $M_{X_i^\sharp}$ is a log structure of $X_i^\sharp$, we get $a_i \in \mathcal{O}^*_{X_i^\sharp}(U_i)$, hence $(a_i)_{i\in\mathbb{N}} \in M^*(\mathfrak{U})$.

2) Let $\mathfrak{U}^\sharp \to \mathfrak{X}^\sharp$ be an etale morphism. Suppose $\mathfrak{U}$ affine. We prove in this step that the canonical morphisms $M^{\mathrm{gr}}_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) \to M^{\mathrm{gr}}_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i)$ and $M_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) \to M_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i)$ are isomorphisms. First, we remark that since $(\pi^i\mathcal{O}_{X_{i+1}^\sharp})^2 = 0$, then we have the canonical isomorphism of groups $(1+\pi^i\mathcal{O}_{X_{i+1}^\sharp}, \times) \xrightarrow{\sim} (\pi^i\mathcal{O}_{X_{i+1}^\sharp}, +)$. Since U_{i+1} is affine and $\pi^i\mathcal{O}_{X_{i+1}^\sharp}$ is quasi-coherent, this yields $H^1(U_{i+1}, 1+\pi^i\mathcal{O}_{X_{i+1}^\sharp}) = 0$. Hence, we get the commutative diagram (we use in the proof multiplicative notation)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \to & \mathcal{O}^*_{X_i^\sharp}(U_i) & \to & M^{\mathrm{gr}}_{X_i^\sharp}(U_i) & \to & M^{\mathrm{gr}}_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i) & \to & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \to & \mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) & \to & M^{\mathrm{gr}}_{X_{i+1}^\sharp}(U_{i+1}) & \to & M^{\mathrm{gr}}_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) & \to & 1 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1+\pi^i\mathcal{O}_{X_{i+1}^\sharp}(U_{i+1}) & = & 1+\pi^i\mathcal{O}_{X_{i+1}^\sharp}(U_{i+1}) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{3.3.1.7.1}$$

whose two rows and two columns are exact. Hence, the morphism $M^{\mathrm{gr}}_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) \to M^{\mathrm{gr}}_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i)$ is an isomorphism. Since $M_{X_{i+1}^\sharp} \to M_{X_i^\sharp}$ is are exact closed immersion, then following this is a log thickening of finite order (see definition [Ogu18, IV.2.1.1]). Following [Ogu18, IV.2.1.2.4], this yields that we have the surjective projection $M_{X_{i+1}^\sharp}(U_{i+1}) = M^{\mathrm{gr}}_{X_{i+1}^\sharp}(U_{i+1}) \times_{M^{\mathrm{gr}}_{X_i^\sharp}(U_i)} M_{X_i^\sharp}(U_i) \to M_{X_i^\sharp}(U_i)$. Hence, $M_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) \to M_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i)$ is surjective. Since $M_{X_i^\sharp}(U_i)$ and $M_{X_{i+1}^\sharp}(U_{i+1})$ is integral, from [Ogu18, I.1.3.3], the horizontal morphisms of the commutative diagram

$$\begin{array}{ccc}
M_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) & \hookrightarrow & M^{\mathrm{gr}}_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) \\
\downarrow & & \downarrow\wr \\
M_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i) & \hookrightarrow & M^{\mathrm{gr}}_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i)
\end{array}$$

are injective. Hence, $M_{X_{i+1}^\sharp}(U_{i+1})/\mathcal{O}^*_{X_{i+1}^\sharp}(U_{i+1}) \to M_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i)$ is an isomorphism.

3) Using Mittag-Leffler condition, we get the exact sequence

$$1 \to \varprojlim_i \mathcal{O}^*_{X_i^\sharp}(U_i) \to \varprojlim_i M^{\mathrm{gr}}_{X_i^\sharp}(U_i) \to \varprojlim_i M^{\mathrm{gr}}_{X_i^\sharp}(U_i)/\mathcal{O}^*_{X_i^\sharp}(U_i) \to 1.$$

Since $\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) = \varprojlim_i \mathcal{O}^*_{X_i^\sharp}(U_i)$, using the step 2) we obtain $(\varprojlim_i M^{\mathrm{gr}}_{X_i^\sharp}(U_i))/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) \xrightarrow{\sim} M^{\mathrm{gr}}_{X_0^\sharp}(U_0)/\mathcal{O}^*_{X_0^\sharp}(U_0)$.

By considering the commutative diagram

$$\begin{array}{ccc} (\varprojlim_i M^{\mathrm{gr}}_{X_i^\sharp}(U_i))/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) & \xrightarrow{\sim} & M^{\mathrm{gr}}_{X_0^\sharp}(U_0)/\mathcal{O}^*_{X_0^\sharp}(U_0) \\ \uparrow & & \uparrow \\ M(\mathfrak{U})/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) = (\varprojlim_i M_{X_i^\sharp}(U_i))/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) & \longrightarrow & M_{X_0^\sharp}(U_0)/\mathcal{O}^*_{X_0^\sharp}(U_0) \end{array}$$

we get the injectivity of the map $M(\mathfrak{U})/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) \to M_{X_0^\sharp}(U_0)/\mathcal{O}^*_{X_0^\sharp}(U_0)$. Since the maps $M_{X_{i+1}^\sharp}(U_{i+1}) \to M_{X_i^\sharp}(U_i)$ are surjective (this is checked in the step 2), we get that $M(\mathfrak{U}) \to M_{X_0^\sharp}(U_0)$ is surjective and then so is $M(\mathfrak{U})/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) \to M_{X_0^\sharp}(U_0)/\mathcal{O}^*_{X_0^\sharp}(U_0)$. Hence, the canonical morphism $M(\mathfrak{U})/\mathcal{O}^*_{\mathfrak{X}}(\mathfrak{U}) \to M_{X_0^\sharp}(U_0)/\mathcal{O}^*_{X_0^\sharp}(U_0)$ is an isomorphism. This yields $\overline{M} = M/M^* = M_{X_0^\sharp}/M^*_{X_0^\sharp} = \overline{M_{X_0^\sharp}}$.

4) We compute that the induced morphism $M^* \to M \times_{M_{X_0^\sharp}} M^*_{X_0^\sharp}$ is an isomorphism, i.e. that the morphism $M \to M_{X_0^\sharp}$ is local. Hence, since $\overline{M} = \overline{M_{X_0^\sharp}}$ (see step 3), since $M_{X_0^\sharp}$ is fine and M is integral, using Lemma 3.3.1.6, we get that M is fine.

II) In this last step, we establish that $X_i^\sharp \to \mathfrak{X}^\sharp \times_{\mathfrak{S}} S_i$ is an isomorphism of fine log schemes. Let $u_i \colon X_i^\sharp \to \mathfrak{X}^\sharp$ be the canonical morphism. We already know that $\underline{X}_i \xrightarrow{\sim} \mathfrak{X} \times_{\mathfrak{S}} S_i$. It remains to check that the morphism $u_i^* M \to M_{X_i^\sharp}$ is an isomorphism. Since this is a morphism of fine log structures, then using [Ogu18, I.4.1.2], this is equivalent to check the isomorphism $\overline{u_i^* M} \xrightarrow{\sim} \overline{M_{X_i^\sharp}}$. Following [Kat89, 1.4.1], $\overline{u_i^* M} = \overline{M}$. From the step I).3), we have $\overline{M} \xrightarrow{\sim} \overline{M_{X_i^\sharp}}$ and we are done. □

Theorem 3.3.1.8. The functors $\mathfrak{X}^\sharp \mapsto (X_i^\sharp)_{i\in\mathbb{N}}$ and $(X_i^\sharp)_{i\in\mathbb{N}} \mapsto \varinjlim_i X_i^\sharp$ are quasi-inverse equivalences of categories between the category of fine $\mathfrak{S}$-log formal schemes to that of strict inductive systems of noetherian fine log schemes over $(S_i)_{i\in\mathbb{N}}$.

Proof. This is a consequence of Propositions 3.3.1.3 and 3.3.1.7. □

Lemma 3.3.1.9. *Let $f \colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathfrak{S}$-log formal schemes. Then f is strict if and only if, for any $i \in \mathbb{N}$, f_i is strict.*

Proof. If f is strict then f_i, the base change of f by $S_i \hookrightarrow \mathfrak{S}$ is strict. Conversely, suppose that for any $i \in \mathbb{N}$, f_i is strict. Let $\mathfrak{Z}^\sharp$ be the fine $\mathfrak{S}$-log formal scheme whose underlying fine formal $\mathfrak{S}$-scheme is $\mathfrak{X}$ and whose log structure is $f^*(M^\sharp_{\mathfrak{Y}})$. Then $Z_i^\sharp \to Y_i^\sharp$ is strict and $Z_i = X_i^\sharp$. Hence, $Z_i^\sharp = X_i^\sharp$. Using 3.3.1.3, this yields that $\mathfrak{X}^\sharp = \mathfrak{Z}^\sharp$, i.e. f is strict. □

Let us finish the section by the following definition which will be useful from 3.4. The "very nice" notion will only be useful from 9.2.1.

Definition 3.3.1.10. Let $\mathfrak{S}^\sharp$ be a fine $\mathcal{V}$-log formal scheme.

(a) We say that $\mathfrak{S}^\sharp$ is a "nice" fine $\mathcal{V}$-log formal scheme if there exists a $\mathcal{V}$-formal scheme $\mathfrak{T}$ such that $\mathfrak{S}^\sharp$ is a log flat $\mathfrak{T}$-log formal scheme, i.e. for any integer $i \geq 0$ $S_i^\sharp$ is a log flat T_i-log scheme (see definition [Ogu18, IV.4.1.1]). Following 3.1.1.1, if $\mathfrak{S}^\sharp$ is a nice fine $\mathcal{V}$-log formal scheme, then $S_i^\sharp$ is a nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ for any integer $i \geq 0$.

(b) We say that $\mathfrak{S}^\sharp$ is a "very nice" fine $\mathcal{V}$-log formal scheme if there exists a p-torsion free Noetherian of finite Krull dimension $\mathcal{V}$-formal scheme $\mathfrak{T}$ such that $\mathfrak{S}^\sharp$ is a log flat of finite type $\mathfrak{T}$-log formal scheme. Remark that since $\mathfrak{S}^\sharp \to \mathfrak{T}$ is a log flat and integral (because the log structure of $\mathfrak{T}$ is trivial: see [Ogu18, III.2.5.3.3]) then following [Ogu18, IV.4.3.5] the underlying morphism $\mathfrak{S} \to \mathfrak{T}$ is flat. In particular $\mathfrak{S}$ is also p-torsion free.

Definition 3.3.1.11. Let $f \colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a (adic) morphism of fine $\mathcal{V}$-log formal schemes and $\mathcal{B}_{\mathfrak{X}}$ be an $\mathcal{O}_{\mathfrak{X}}$-algebra. We have the following quasi-flatness notions which will be useful in 4.6.3.3 to define duality or tensor products over the ring of differential operators.

(a) We say that f is quasi-flat if there exists a morphism of $\mathcal{V}$-formal schemes $g \colon \mathfrak{Y} \to \mathfrak{Z}$ such that the morphism of schemes $g \circ f \colon \mathfrak{X} \to \mathfrak{Z}$ is flat.

(b) We say that $\mathcal{B}_{\mathfrak{X}}$ is a quasi-flat $f^{-1}\mathcal{O}_{\mathfrak{Y}}$-algebra if there exists a morphism of $\mathcal{V}$-formal schemes $g\colon \mathfrak{Y} \to \mathfrak{Z}$ such that the (induced by $g \circ f$) morphism of ringed spaces $(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}) \to \mathfrak{Z}$ is flat.

Remark 3.3.1.12. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathcal{V}$-log formal schemes. We suppose that for any $y \in \mathfrak{Y}$, the monoid $(M_{\mathfrak{Y}^\sharp}/\mathcal{O}^*_{\mathfrak{Y}})_y$ is generated by one element. It follows from 3.1.1.7 that the morphism of schemes $X_i^\sharp \to Y_i^\sharp$ is flat. Since f is of finite type, then f is flat.

3.3.2 Around log etaleness

Définition 3.3.2.1. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathfrak{S}$-log formal schemes. We say that f is "log étale" (resp. "log smooth, resp. "formally log étale") if for any integer $i \in \mathbb{N}$ the morphism f_i is log étale (resp. log smooth, resp. fine formally log étale).

When the morphism is strict, we remove the word "log". For instance, f is étale means that f is strict and f is log étale.

Remark 3.3.2.2. Our definition of log étaleness was named by Shiho formal log étaleness (see [Shi00, 2.2.2]). We hope there will be no confusion.

Proposition 3.3.2.3. Let $\mathfrak{Y}^\sharp$ be a fine $\mathfrak{S}$-log formal schemes. Let $f_0\colon X_0^\sharp \to Y_0^\sharp$ be a log smooth morphism of fine log S_0-schemes such that X_0 is affine. Then there exists a log smooth morphism of fine $\mathfrak{S}$-log formal schemes of the form $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ whose reduction modulo π is f_0. We say that such morphism f is a log smooth lifting of f_0.

Proof. From [Kat89, 3.14.(1)], there exists a unique up to isomorphism log smooth morphism of fine log S_i-schemes $f_i\colon X_i^\sharp \to Y_i^\sharp$ endowed with an isomorphism $X_0^\sharp \xrightarrow{\sim} X_i^\sharp \times_{Y_i^\sharp} Y_0^\sharp$. Put $\mathfrak{Y}^\sharp := \varinjlim_i Y_i^\sharp$. Let $f\colon \mathfrak{Y}^\sharp \to \mathfrak{X}^\sharp$ be the induced morphism. Following Theorem 3.3.1.8, $\mathfrak{Y}^\sharp$ is a fine $\mathfrak{S}$-log formal schemes. By construction, f is log smooth since f_i is log smooth for any $i \in \mathbb{N}$. □

Proposition 3.3.2.4. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathfrak{S}$-log formal schemes. The morphism f is log étale if and only if f is formally log étale and f_0 is log étale.

Proof. If f_0 is log étale then $\underline{f_0}$ is of finite type. This yields that $\underline{f_i}$ is of finite type, which proves the non-respective case. □

Lemma 3.3.2.5. *Let A be a commutative $\mathcal{V}$-algebra, $\phi\colon M' \to M$ be a morphism of A-modules. Suppose that M is π-torsion free and ϕ induces the isomorphism $\overline{\phi}\colon M'/\pi M' \xrightarrow{\sim} M/\pi M$.*

The p-adic completion of ϕ, $\widehat{M'} \to \widehat{M}$, is therefore an isomorphism.

Proof. The fact that $\overline{\phi}$ is injective means that we have the inclusion $\phi^{-1}(\pi M) \subset \pi M'$. SInce M is π-torsion free, we get by induction on n the inclusion $\phi^{-1}(\pi^n M) \subset \pi^n M'$.

The fact tha $\overline{\phi}$ is surjective can be translated by the relation: $M = \phi(M') + \pi M$. This yields by induction on $n \geq 1$ the equality: $M = \phi(M') + \pi^n M$. Hence, ϕ induces the isomorphisms $\overline{\phi}_n\colon M'/\pi^n M' \xrightarrow{\sim} M/\pi^n M$. We are done. □

3.3.2.6. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathfrak{S}$-log formal schemes. We set $\Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} := \varprojlim_i \Omega^1_{X_i^\sharp/Y_i^\sharp}$. When f is log smooth, then from [Kat89, 3.10] the $\mathcal{O}_{\mathfrak{X}}$-module $\Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ is locally free of finite type.

Lemma 3.3.2.7. *Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$, $g\colon \mathfrak{Y}^\sharp \to \mathfrak{Z}^\sharp$ be two morphisms of fine $\mathfrak{S}$-log formal schemes such that $\mathcal{O}_{\mathfrak{X}}$ is p-torsion free, the morphism $g \circ f$ is log smooth and $f_0\colon X_0^\sharp \to Y_0^\sharp$ is log étale. Then f is log étale.*

Proof. We construct by p-adic completion the morphism $\phi\colon f^*\Omega^1_{\mathfrak{Y}^\sharp/\mathfrak{Z}^\sharp} \to \Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Z}^\sharp}$, where we put $f^*\Omega^1_{\mathfrak{Y}^\sharp/\mathfrak{Z}^\sharp} := \varprojlim_i f_i^*\Omega^1_{Y_i^\sharp/Z_i^\sharp}$. Since $g\circ f\colon \mathfrak{X}^\sharp \to \mathfrak{Z}^\sharp$ is log smooth, the $\mathcal{O}_{\mathfrak{X}}$-module $\Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Z}^\sharp}$ is locally free of finite type (see 3.3.2.6). In particular, $\Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Z}^\sharp}$ is p-torsion free. The reduction of ϕ modulo π is canonically isomorphic to $f_0^*\Omega^1_{Y_0^\sharp/Z_0} \to \Omega^1_{X_0/Z_0}$. Since f_0 is log étale, this latter homomorphism is an isomorphism. Since $\Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Z}^\sharp}$ is p-torsion free, this yields that ϕ is an isomorphism (e.g. use Lemma 3.3.2.5). This implies that the canonical morphism $f_i^*\Omega^1_{Y_i^\sharp/Z_i^\sharp} \to \Omega^1_{X_i^\sharp/Z_i^\sharp}$ is an isomorphism. Since $X_i^\sharp \to Z_i^\sharp$ is log smooth, from [Kat89, 3.12], we conclude that f_i is log-étale. □

Proposition 3.3.2.8. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathfrak{S}$-log formal schemes such that $\mathcal{O}_{\mathfrak{X}}$ is p-torsion free. The morphism f is log smooth if and only if, étale locally on $\mathfrak{X}^\sharp$ there exists a log étale $\mathfrak{Y}^\sharp$-morphism of the form $\mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp \times_{\mathfrak{S}} \mathfrak{A}_{\mathbb{N}^r}$.

Proof. Suppose f is log smooth. Since f_0 is log smooth, when can suppose there exists a morphism $X_0^\sharp \to A_{\mathbb{N}^r}$ such that the induced $Y_0^\sharp$-morphism $X_0^\sharp \to Y_0^\sharp \times A_{\mathbb{N}^r}$ is log-étale.Using 3.3.1.2, we can suppose that $X_0^\sharp \to A_{\mathbb{N}^r}$ has the lifting of the form $\mathfrak{X}^\sharp \to \mathfrak{A}_{\mathbb{N}^r}$. We get the $\mathfrak{Y}^\sharp$-morphism $\mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp \times_{\mathfrak{S}} \mathfrak{A}_{\mathbb{N}^r}$. We conclude by applying Lemma 3.3.2.7 that this latter morphism is log-étale. □

3.3.3 Sheaf of differential operators of infinite level and finite order over weakly log smooth $\mathfrak{S}$-log formal scheme

Définition 3.3.3.1. As in 3.1.1.10 we define the category $\mathfrak{C}_\sharp$ of $\mathfrak{S}$-immersions of fine $\mathfrak{S}$-log formal schemes. For any integer n, we denote by $\mathfrak{C}_n$ the full subcategory of $\mathfrak{C}$ whose objects are exact closed immersions of order n.

3.3.3.2. By using [Kat89, 3.111 and 3.14] we easily get the following formal version of 3.1.1.14 of the exactification of an immersion: Let $u\colon \mathfrak{Z}^\sharp \hookrightarrow \mathfrak{X}^\sharp$ be an $\mathfrak{S}^\sharp$-immersion of fine log formal schemes. Let $\overline{z}$ be a geometric point of $\mathfrak{Z}^\sharp$. There exists a commutative diagram of the form

$$\begin{array}{ccccc} \widetilde{\mathfrak{X}}^\sharp & \xrightarrow{f} & \mathfrak{X}'^\sharp & \xrightarrow{g} & \mathfrak{X}^\sharp \\ & {}^{v'}\nwarrow & \uparrow{\scriptstyle u'} & \square & \uparrow{\scriptstyle u} \\ & & \mathfrak{Z}'^\sharp & \xrightarrow{h} & \mathfrak{Z}^\sharp \end{array}$$

such that the square is cartesian, f is log étale, $\underline{f}$ is affine, g is étale, v' is an exact closed $\mathfrak{S}^\sharp$-immersion and h is an étale neighborhood of $\overline{z}$ in $\mathfrak{Z}^\sharp$.

Lemma 3.3.3.3. *The inclusion functor $\mathfrak{C}_{\sharp n} \to \mathfrak{C}_\sharp$ has a right adjoint functor which we will denote by $P^{\sharp n}\colon \mathfrak{C}_\sharp \to \mathfrak{C}_{\sharp n}$. Let $u\colon \mathfrak{Z}^\sharp \hookrightarrow \mathfrak{X}^\sharp$ be an object of $\mathfrak{C}_\sharp$. Then $\mathfrak{Z}^\sharp$ is also the source of $P^{\sharp n}(u)$.*

Proof. Let $u\colon \mathfrak{Z}^\sharp \hookrightarrow \mathfrak{X}^\sharp$ be an object of $\mathfrak{C}_\sharp$. Since $u_i\colon Z_i^\sharp \hookrightarrow X_i^\sharp$ is an object of $\mathscr{C}_\sharp$, from 3.1.1.17, we get the object $P^{\sharp n}(u_i)\colon Z_i^\sharp \hookrightarrow P^{\sharp n}(u_i)$ of $\mathscr{C}_{\sharp n}$ such that $P^{\sharp n}(u_i) \to X_i^\sharp$ is affine and $P^{\sharp n}(u_i)$ is noetherian. Hence, using Theorem 3.3.1.8, we get that $\varinjlim_i P^{\sharp n}(u_i)$ satisfies the universal property of $P^{\sharp n}(u)$. □

Définition 3.3.3.4. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of fine $\mathfrak{S}$-log formal schemes.

1. We say that a finite set $(b_\lambda)_{\lambda=1,\ldots,r}$ of elements of $\Gamma(\mathfrak{X}, M_{\mathfrak{X}^\sharp})$ are " formal logarithmic coordinates of f" if the induced $\mathfrak{Y}^\sharp$-morphism $\mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp \times_{\mathfrak{S}} \mathfrak{A}_{\mathbb{N}^r}$ is formally log étale (concerning $\mathfrak{A}_{\mathbb{N}^r}$, see the notation of 3.3.1.1).
2. We say that f is "weakly log smooth" if, étale locally on $\mathfrak{X}^\sharp$, f has formal log coordinates. Notice that this notion of weak log smoothness is étale local on $\mathfrak{Y}^\sharp$. We also say that $\mathfrak{X}^\sharp$ is a "weakly log smooth $\mathfrak{Y}^\sharp$-log formal scheme" (following our terminology, the log structure of such an $\mathfrak{X}^\sharp$ is understood to be fine).
3. When f is strict we remove "log" in the terminology, e.g. we get the notion of "étale" morphisms, "smooth" morphisms or "weakly smooth" morphisms.

Remark 3.3.3.5. Following Proposition 3.3.2.8, a log smooth morphism is weakly log smooth which justifies the terminology.

3.3.3.6 (nth infinitesimal neighborhood)**.** Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{T}^\sharp$ be a weakly log smooth morphism of fine $\mathfrak{S}$-log formal schemes. Let $\Delta_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}\colon \mathfrak{X}^\sharp \hookrightarrow \mathfrak{X}^\sharp \times_{\mathfrak{T}^\sharp} \mathfrak{X}^\sharp$ be the diagonal immersion. Since $\Delta_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$ is an object of $\mathfrak{C}_\sharp$, then we can put $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} := P^n(\Delta_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp})$. From 3.1.1.15, we have $\Delta^n_{X_i^\sharp/S_i} = \Delta^n_{X_{i+1}^\sharp/T_{i+1}^\sharp} \times_{T_{i+1}^\sharp} T_i^\sharp$. Hence, using Theorem 3.3.1.8, we get the equality of fine $\mathfrak{S}$-log formal schemes $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} = \varinjlim_i \Delta^n_{X_i^\sharp/T_i^\sharp}$. Taking the inductive limits to the strict morphisms of fine log schemes $p_0^n\colon \Delta^n_{X_i^\sharp/T_i^\sharp} \to X_i^\sharp$ (resp. $p_1^n\colon \Delta^n_{X_i^\sharp/T_i^\sharp} \to$

$X_i^\sharp$), using Lemma 3.3.1.9 we get the strict morphism of fine $\mathfrak{T}^\sharp$-log formal schemes $p_0^n \colon \Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \to \mathfrak{X}^\sharp$ (resp. $p_1^n \colon \Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \to \mathfrak{X}^\sharp$). Using the remark 3.1.2.4, we can check that the underlying morphism of formal $\mathfrak{S}$-schemes of $p_0^n \colon \Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \to \mathfrak{X}^\sharp$ and $p_1^n \colon \Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \to \mathfrak{X}^\sharp$ are finite (more precisely, we can check the local description 3.3.3.7.1). Hence, we denote by $\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$ the coherent $\mathcal{O}_{\mathfrak{X}}$-algebra such that $\operatorname{Spf} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} = \underline{\Delta}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$.

If $a \in M_{\mathfrak{X}^\sharp}$, we denote by $\mu_{(m)}(a)$ the unique section of $\ker(\mathcal{O}^*_{\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}} \to \mathcal{O}^*_{\mathfrak{X}})$ such that we get in $M^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$ the equality $p_1^{n*}(a) = p_0^{n*}(a)\mu^n(a)$ (see 3.1.1.24). We get $\mu^n \colon M_{\mathfrak{X}^\sharp} \to \ker(\mathcal{O}^*_{\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}} \to \mathcal{O}^*_{\mathfrak{X}})$ given by $a \mapsto \mu^n(a)$.

Proposition 3.3.3.7 (Local description of $\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$)**.** With notation 3.3.3.6, suppose given $(u_\lambda)_{\lambda=1,\dots,r}$ some formal log coordinates of f. Put $\tau_{\sharp\lambda,n} := \mu^n(u_\lambda) - 1$. We have the following isomorphism of $\mathcal{O}_{\mathfrak{X}}$-algebras:

$$\mathcal{O}_{\mathfrak{X}}[T_1, \dots, T_r]_n \xrightarrow{\sim} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$$
$$T_\lambda \mapsto \tau_{\sharp\lambda,n}. \tag{3.3.3.7.1}$$

Proof. This is a consequence of 3.1.2.3. □

Définition 3.3.3.8. With the notation 3.3.3.6, the sheaf of differential operators of order $\leq n$ of f is defined by putting $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,n} := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(p^n_{0*}\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}, \mathcal{O}_{\mathfrak{X}})$. The sheaf of differential operators of $f$is defined by putting $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} := \cup_{n\in\mathbb{N}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,n}$.

Let $P \in \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,n}$, $P' \in \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,n'}$. We define the product $PP' \in \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,n+n'}$ to be the composition

$$PP' \colon \mathcal{P}^{n+n'}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \xrightarrow{\delta^{n,n'}} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^{n'}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \xrightarrow{\mathrm{id}\otimes P'} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp} \xrightarrow{P} \mathcal{O}_{\mathfrak{X}}. \tag{3.3.3.8.1}$$

Similarly to 3.1.4.4, we can check that the sheaf $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$ is a sheaf of rings with the product as defined in 3.3.3.8.1

3.3.4 Sheaf of differential operators of finite level m and order over log smooth $\mathfrak{S}$-log formal schemes

Let $m \geq 0$ be an integer. The principal ideal (p) of $\mathcal{V}$ is endowed with a canonical m-PD-structure, which we will denote by $\gamma_\emptyset$.

Définition 3.3.4.1. As in 3.2.1.1, we define the categories $\mathfrak{C}^{(m)}_{\sharp n}$ whose objects are pairs (u, δ) where u is an exact closed $\mathfrak{S}$-immersion of fine log $\mathfrak{S}$-schemes and δ is an m-PD-structure on the ideal $\mathcal{I}$ defined by u (which is compatible with $\gamma_\emptyset$) and such that $\mathcal{I}^{\{n+1\}_{(m)}} = 0$ and whose morphisms $(u', \delta') \to (u, \delta)$ are morphisms $u' \to u$ of $\mathfrak{C}_\sharp$ which are compatible with the m-PD-structures δ and δ'.

Proposition 3.3.4.2.

1. The canonical functor $\mathfrak{C}^{(m)}_{\sharp n} \to \mathfrak{C}_\sharp$ has a right adjoint, which we will denote by $P^{\sharp n}_{(m)} \colon \mathfrak{C}_\sharp \to \mathfrak{C}^{(m)}_{\sharp n}$.
2. Let u be an object of $\mathfrak{C}_\sharp$. The source of $P^{\sharp n}_{(m)}(u)$ is the source of u.

Proof. The first assertion is a consequence of 3.3.1.8 and 3.2.1.9 (we need in particular the 3.2.1.9.4). Since $\gamma_\emptyset$ extends to any $\mathfrak{S}$-log formal schemes (because the ideal of the m-PD-structure $\gamma_\emptyset$ is locally principal: see [Ber96c, 1.3.2.c)]), we get the second assertion. □

3.3.4.3. Let u be an object of $\mathfrak{C}_\sharp$. We call $P^{\sharp n}_{(m)}(u)$ the m-PD-envelope compatible of order n of u. We sometimes denote abusively by $P^{\sharp n}_{(m)}(u)$ the target of the arrow $P^{\sharp n}_{(m)}(u)$.

3.3.4.4. Let $f \colon \mathfrak{X}^\sharp \to \mathfrak{T}^\sharp$ be a log smooth morphism of fine $\mathcal{V}$-log formal schemes. Using 3.2.2.6, we can check the underlying scheme of $\Delta^n_{X_i^\sharp/T_i^\sharp,(m)}$ is noetherian. Moreover, from the local description 3.2.2.4.1, we get $\Delta^n_{X_i^\sharp/T_i^\sharp,(m)} \xrightarrow{\sim} \Delta^n_{X_{i+1}^\sharp/T_{i+1}^\sharp,(m)} \times_{T_{i+1}^\sharp} T_i^\sharp$ (recall also that $p_0^n \colon \Delta^n_{X_i^\sharp/T_i^\sharp,(m)} \to$

$X_i^\sharp$ is strict). Using Theorem 3.3.1.8, we get the fine $\mathfrak{T}^\sharp$-log formal schemes $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$ by putting $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)} := \varinjlim_i \Delta^n_{X_i^\sharp/T_i^\sharp,(m)}$. Let $p_1^n,\, p_0^n \colon \Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)} \to \mathfrak{X}^\sharp$ be the morphisms induced respectively by $p_1^n,\, p_0^n \colon \Delta^n_{X_i^\sharp/T_i^\sharp,(m)} \to X_i^\sharp$. From 3.2.2.2, 3.3.1.9 and 3.2.2.6, the morphisms $p_1^n,\, p_0^n \colon \Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)} \to \mathfrak{X}^\sharp$ are strict and finite (more precisely concerning the finiteness, we have the local description 3.3.4.5.1).

We denote by $M^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$ the log structure of $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$. We denote by $\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$ the coherent $\mathcal{O}_{\mathfrak{X}}$-algebra corresponding to the underlying formal $\mathcal{V}$-scheme of $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$. Hence, $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$ is an exact closed immersion of the form $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)} \colon \mathfrak{X}^\sharp \hookrightarrow (\operatorname{Spf} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}, M^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)})$. We sometimes denote abusively by $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$ the target of the arrow $\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$.

As in paragraph 3.2.3.1, we can define $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}$, the sheaf of differential operator on $\mathfrak{X}^\sharp$ of level m.

3.3.4.5 (Local description)**.** Suppose in this paragraph that $\mathfrak{X}^\sharp \to \mathfrak{T}^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$ of f. Put $\tau_{\sharp\lambda(m)} := \mu^n_{(m)}(u_\lambda) - 1$ (or simply $\tau_{\sharp\lambda}$), where $\mu^n_{(m)}(a)$ is the unique section of $\ker(\mathcal{O}^*_{\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}} \to \mathcal{O}^*_{\mathfrak{X}})$ such that we get in $M^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$ the equality$p_1^{n*}(a) = p_0^{n*}(a)\mu^n_{(m)}(a)$. Taking the limits to 3.2.2.4, we get the isomorphism of m-PD-$\mathcal{O}_{\mathfrak{X}}$-algebras

$$\begin{aligned} \mathcal{O}_{\mathfrak{X}}\langle T_1,\dots,T_r\rangle_{(m),n} &\xrightarrow{\sim} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)} \\ T_\lambda &\mapsto \tau_{\sharp\lambda,(m)} \end{aligned} \tag{3.3.4.5.1}$$

where the first term is defined as in 1.3.3.6. In particular, the elements $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}|\leq n}$ form an $\mathcal{O}_{\mathfrak{X}}$-basis of $\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,(m)}$. The corresponding dual basis of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp,n}$ will be denoted by $\{\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}\}_{|\underline{k}|\leq n}$. Let $\epsilon_1,\dots,\epsilon_r$ be the canonical basis of $\mathbb{N}^r$, i.e. the coordinates of ϵ_i are 0 except for the ith term which is 1. We put $\partial_{\sharp i} := \underline{\partial}_\sharp^{\langle \epsilon_i\rangle_{(m)}}$. We can define the logarithmic adjoint operator as in 3.4.1.2 and we can check that the properties analogous to the subsection 3.4.1 are still satisfied of the formal context.

3.4 Logarithmic differential modules

Let $m \in \mathbb{N}\cup\{+\infty\}$. When $m = +\infty$, we remove (m) in the notation. We consider here simultaneously the following both cases.

(i) Algebraic case: $S^\sharp$ is a nice fine log scheme over $\operatorname{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$, where i is an integer (see definition 3.1.1.1). Moreover, $X^\sharp \to S^\sharp$ is a log smooth morphism of log schemes.

(ii) Formal case: $S^\sharp$ is a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, $X^\sharp \to S^\sharp$ is a log smooth morphism of log formal schemes.

Recall notation 3.2.2.5 and the fact that we can simply consider the case where the m-PD-structure $\gamma_\emptyset$ on the base is given by the unique PD-structure on $p\mathcal{O}_S$. Let $Y := X^{\sharp *}$ be the open of X where $M_{X^\sharp}$ is trivial and $j\colon Y \hookrightarrow X^\sharp$ be the canonical open immersion. Remark $\omega_{Y/S^\sharp} = \omega_{Y/(S^\sharp)^*}$ and $\mathcal{D}^{(m)}_{Y/S^\sharp} = \mathcal{D}^{(m)}_{Y/(S^\sharp)^*}$, i.e. over Y we come back to the non-logarithmic case. Since $X^\sharp$ is nice (see 3.1.1.2), then the canonical morphisms $\omega_{X^\sharp/S^\sharp} \to j_*\omega_{Y/S^\sharp}$ and $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to j_*\mathcal{D}^{(m)}_{Y/S^\sharp}$ are injective (this is a consequence of 3.1.1.3).

3.4.1 Logarithmic adjoint operators

We suppose that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,n}$. Let $(t_\lambda)_{\lambda=1,\dots,n}$ be the induced coordinates of Y/S, $\underline{\tau}_{\sharp(m)}$ (resp. $\underline{\partial}_\sharp{}^{\langle \underline{k}\rangle_{(m)}}$) be the element constructed from $(u_\lambda)_{\lambda=1,\dots,n}$ (resp. $(t_\lambda)_{\lambda=1,\dots,n}$) as defined in 3.2.2.4 (resp. 3.2.3.4).

3.4.1.1. Following 3.2.3.9.5, via the inclusion $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to j_*\mathcal{D}^{(m)}_{Y/S^\sharp}$ we have the equality

$$\underline{\partial}_\sharp{}^{\langle \underline{k}\rangle_{(m)}} = \underline{t}^{\underline{k}}\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}. \tag{3.4.1.1.1}$$

Let $Q \in \Gamma(Y, \mathcal{D}^{(m)}_{Y/S^\sharp})$ and by using 3.4.1.1.1, Q can be written of the form $Q = \sum_{\underline{k}} b_{\underline{k}}\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}$, with $b_{\underline{k}} \in \Gamma(Y, \mathcal{O}_Y)$. The adjoint operator of Q is by definition ${}^{\mathrm{t}}Q := \sum_{\underline{k}} (-1)^{|\underline{k}|}\underline{\partial}^{\langle \underline{k}\rangle_{(m)}} b_{\underline{k}}$.

Notation 3.4.1.2. Let us fix some notation concerning the logarithmic adjoint operator.

(a) We set $\alpha_{0,0} := 1$. Let $j \geq 1$ be an integer. We set $\alpha_{0,j} = 0$. For any $1 \leq i \leq j$, we set $\alpha_{i,j} := (-1)^j \left\{ {j \atop i} \right\}_{(m)} q^{(m)}_{j-i}! \left({j-1 \atop j-i} \right)$ where $q^{(m)}_{j-i}$ means the quotient of the Euclidian division of $j-i$ by p^m.

For any integer $\lambda = 1, \ldots, n$, for any $k \geq 0$, we set

$$\widetilde{\partial}^{\langle k \rangle_{(m)}}_{\sharp\lambda} := \sum_{0 \leq i \leq k} \alpha_{i,k} \partial^{\langle i \rangle_{(m)}}_{\sharp\lambda}. \tag{3.4.1.2.1}$$

In particular, $\widetilde{\partial}^{\langle 0 \rangle_{(m)}}_{\sharp\lambda} = 1$. Remark that when $k \geq 1$, we have in fact, $\widetilde{\partial}^{\langle k \rangle_{(m)}}_{\sharp\lambda} = \sum_{1 \leq i \leq k} \alpha_{i,k} \partial^{\langle i \rangle_{(m)}}_{\sharp\lambda}$.

(b) Let $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$. We set $\widetilde{\underline{\partial}}^{\langle \underline{k} \rangle_{(m)}}_{\sharp} := \prod_{\lambda=1}^n \widetilde{\partial}^{\langle k_\lambda \rangle_{(m)}}_{\sharp\lambda}$. For any $\underline{i} \leq \underline{k}$, we put $\alpha_{\underline{i},\underline{k}} := \prod_{\lambda=1}^n \alpha_{i_\lambda, k_\lambda} \in \mathbb{Z}$.

$$\widetilde{\underline{\partial}}^{\langle \underline{k} \rangle_{(m)}}_{\sharp} := \prod_{\lambda=1}^n \widetilde{\partial}^{\langle k_\lambda \rangle_{(m)}}_{\sharp\lambda} = \sum_{\underline{i} \leq \underline{k}} \alpha_{\underline{i},\underline{k}} \underline{\partial}^{\langle \underline{i} \rangle_{(m)}}_{\sharp}. \tag{3.4.1.2.2}$$

(c) Let $P \in \Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$ be differential operator. We can uniquely write P of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{\sharp}$ with $a_{\underline{k}} \in \Gamma(X, \mathcal{O}_X)$. We set

$$\widetilde{P} := \sum_{\underline{k}} \widetilde{\underline{\partial}}^{\langle \underline{k} \rangle_{(m)}}_{\sharp} a_{\underline{k}}. \tag{3.4.1.2.3}$$

We say that $\widetilde{P}$ is the "logarithmic adjoint operator" of P. We can sometimes write it by ${}^{t_{\log}}P$ instead of $\widetilde{P}$.

(d) For any differential operators $P, Q \in \Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$, for any $a \in \Gamma(X, \mathcal{O}_X)$, we can easily check $\widetilde{a} = a$, $\widetilde{P+Q} = \widetilde{P} + \widetilde{Q}$ and $\widetilde{aP} = \widetilde{P}a$.

Proposition 3.4.1.3 (Comparison between adjoint operator with or without logarithmic structure)**.** *Suppose $u_1, \ldots, u_n \in \mathcal{O}_X^*$. Let P be a differential operator of $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$. We have the equality*

$$\widetilde{P} = \underline{t}\ {}^{\mathrm{t}}P\ \frac{1}{\underline{t}}, \tag{3.4.1.3.1}$$

where $\underline{t} = \underline{t}^{\underline{1}} = t_1 \cdots t_n$ and where ${}^tP$ means the adjoint operator of P as an object of $\Gamma(Y, \mathcal{D}^{(m)}_{Y/S^\sharp})$ via the canonical inclusion $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \subset \Gamma(Y, \mathcal{D}^{(m)}_{Y/S^\sharp})$.

Proof. By additivity of the functors $P \mapsto \widetilde{P}$ and of $P \mapsto {}^tP$, since for any $a \in \Gamma(X, \mathcal{O}_X)$ and differential operator $P \in \Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$, we have $\widetilde{aP} = \widetilde{P}a$ and ${}^t(aP) = {}^t(P)a$, then we reduce to check the formula 3.4.1.3.1 in the case where $P = \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{\sharp}$. Following 3.4.1.1.1, we have $\underline{\partial}_{\sharp}{}^{\langle \underline{k} \rangle_{(m)}} = \underline{t}^{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$. We compute

$$\begin{aligned} \underline{t}\ {}^{\mathrm{t}}(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{\sharp})\ \frac{1}{\underline{t}} &= \underline{t}\ {}^{\mathrm{t}}(\underline{t}^{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}})\ \frac{1}{\underline{t}} = \underline{t}(-1)^{|\underline{k}|} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \underline{t}^{\underline{k}-\underline{1}} \\ &= (-1)^{|\underline{k}|} \underline{t}^{\underline{1}-\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{\sharp} \underline{t}^{\underline{k}-\underline{1}} = \prod_{\lambda=1}^n (-1)^{k_\lambda} t_\lambda^{1-k_\lambda} \partial^{\langle k_\lambda \rangle_{(m)}}_{\sharp\lambda} t_\lambda^{k_\lambda - 1}, \end{aligned}$$

where $\underline{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$. Hence, we reduce to check $\widetilde{\partial}^{\langle k \rangle_{(m)}}_{\sharp\lambda} = (-1)^k t_\lambda^{1-k} \partial^{\langle k \rangle_{(m)}}_{\sharp\lambda} t_\lambda^{k-1}$, for any integer $k \geq 0$. When $k = 0$, the equality is obvious since $\partial^{\langle k \rangle_{(m)}}_{\sharp\lambda} = 1$ and $\widetilde{\partial}^{\langle k \rangle_{(m)}}_{\sharp\lambda} = 1$. Let us suppose $k \geq 1$. From 3.2.3.7.2, we have $\partial^{\langle k \rangle_{(m)}}_{\sharp\lambda} t_\lambda^{k-1} = \sum_{i \leq k} \left\{ {k \atop i} \right\} \partial^{\langle k-i \rangle_{(m)}}_{\sharp\lambda} (t_\lambda^{k-1}) \partial^{\langle i \rangle_{(m)}}_{\sharp\lambda}$. Following the formula 3.2.3.11, we have $\partial^{\langle k-i \rangle_{(m)}}_{\sharp\lambda} (t_\lambda^{k-1}) = t_\lambda^{k-1} q_{k-i}! \left({k-1 \atop k-i} \right)$ if $i \geq 1$ and 0 if $i = 0$. Hence,

$$(-1)^k t_\lambda^{1-k} \partial^{\langle k \rangle_{(m)}}_{\sharp\lambda} t_\lambda^{k-1} = (-1)^k \sum_{1 \leq i \leq k} \left\{ {k \atop i} \right\} q_{k-i}! \left({k-1 \atop k-i} \right) \partial^{\langle i \rangle_{(m)}}_{\sharp\lambda} = \widetilde{\partial}^{\langle k \rangle_{(m)}}_{\sharp\lambda}.$$

□

Remark 3.4.1.4. With notation 3.4.1.4, the definition of 3.4.1.2.3 was precisely introduced to get $\widetilde{\underline{\partial}_\sharp}^{\langle \underline{k}\rangle_{(m)}} = \underline{t}\ {}^{\mathrm{t}}\underline{\partial}_\sharp{}^{\langle \underline{k}\rangle_{(m)}}\ \frac{1}{\underline{t}}$. One reason to introduce the logarithmic adjoint operator is the formula 3.4.5.1.1.

Proposition 3.4.1.5. *For any differential operators P and Q of $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$, we have the equalities $\widetilde{\widetilde{P}} = P$ and $\widetilde{PQ} = \widetilde{Q}\widetilde{P}$.*

Proof. Via the canonical inclusion $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \subset \Gamma(Y, \mathcal{D}^{(m)}_{Y/S^\sharp})$, we reduce to check the equalities in $\Gamma(Y, \mathcal{D}^{(m)}_{Y/S^\sharp})$. Hence, we can suppose $u_1, \ldots, u_n \in \mathcal{O}^*_X$. The proposition is therefore a consequence of 3.4.1.3. □

Remark 3.4.1.6 (Logarithmic adjoint operator at the level 0). The definition of $\widetilde{\underline{\partial}_\sharp}^{\langle \underline{k}\rangle_{(m)}}$ given at 3.4.1.2.2 looks very complicated compared to the non-logarithmic adjoint operator. At the level 0, this is possible to get a definition as simple as in the non-logarithmic case as follows. Any differential operator of $\mathcal{D}^{(0)}_{X^\sharp/S^\sharp}$ can be written uniquely of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}_\sharp{}^{\underline{k}}$, where $\underline{\partial}_\sharp{}^{\underline{k}} = \prod_{\lambda=1}^n \partial_{\sharp\lambda}^{k_\lambda}$ with $\partial_{\sharp\lambda} = \partial_{\sharp\lambda}^{\langle 1\rangle_{(0)}}$. Beware that $\underline{\partial}_\sharp{}^{\underline{k}} \neq \underline{\partial}_\sharp{}^{\langle \underline{k}\rangle_{(0)}}$. Since $\widetilde{\underline{\partial}_\sharp} = -\underline{\partial}_\sharp$, we get $\widetilde{P} = \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}_\sharp{}^{\underline{k}} a_{\underline{k}}$, which looks like the non-logarithmic adjoint operator definition.

Remark 3.4.1.7. Recall that from 3.2.3.7.2, for any $a \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k} \in \mathbb{N}^r$, we have in $\Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$, $\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}} a = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i}\rangle_{(m)}}(a) \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}}$. Using 3.4.1.5, this yields

$$a\widetilde{\underline{\partial}_\sharp}^{\langle \underline{k}\rangle_{(m)}} = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \widetilde{\underline{\partial}_\sharp}^{\langle \underline{i}\rangle_{(m)}} \times \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i}\rangle_{(m)}}(a). \tag{3.4.1.7.1}$$

of the formula 3.4.1.7.1, beware that we can not replace $\widetilde{\underline{\partial}_\sharp}$ by $\underline{\partial}_\sharp$.

3.4.1.8. The logarithmic adjoint operator commutes with the canonical morphism $\rho_{m+1,m} \colon \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{D}^{(m+1)}_{X^\sharp/S^\sharp}$, i.e., for any $P \in \Gamma(X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$, we have the formula $\rho_{m+1,m}(\widetilde{P}) = (\rho_{m+1,m}(P))^\sim$. Indeed, by additivity of the logarithmic adjoint operator, we reduce to check the case where $P = a\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$, with $a \in \Gamma(X, \mathcal{O}_X)$ and $\underline{k} \in \mathbb{N}^d$. By $\mathcal{O}_X$-bilinearity of $\rho_{m+1,m}$, we come down to the case where $P = \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$. For any $\underline{i} \in \mathbb{N}^d$, we have $\rho_{m+1,m}(\underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}}) = \gamma_{\underline{i}} \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m+1)}}$, with $\gamma_{\underline{i}} \in \mathbb{Z}$. Hence, using 3.4.1.2.2, we can check that both terms $\rho_{m+1,m}(\widetilde{\underline{\partial}_\sharp}^{\langle \underline{k}\rangle_{(m)}})$ and $(\rho_{m+1,m}(\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}))\sim$ are of the form $\sum_{\underline{i} \leq \underline{k}} \alpha_{\underline{i}} \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}}$, with $\alpha_{\underline{i}} \in \mathbb{Z}$. Hence, using the same arguments of the part 1.iii) of the proof, we reduce to the case where $u_1, \ldots, u_n \in \mathcal{O}^*_X$, i.e., we can use the formula 3.4.1.3.1. Via the formula 3.4.1.3.1, we reduce to check $\rho_{m+1,m}({}^{\mathrm{t}}\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}) = {}^{\mathrm{t}}(\rho_{m+1,m}(\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}))$, which is obvious.

3.4.2 PD stratification of level m

We keep notation 3.2.2 (resp. 3.3.3 and 3.3.4). Even if one might consider the étale topology, an $\mathcal{O}_X$-module will mean an $\mathcal{O}_X$-module for the Zariski topology.

Definition 3.4.2.1. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. With notations 3.2.2.13 and 3.2.2.16, an m-PD-stratification (or a PD-stratification of level m) relatively to $X^\sharp/S^\sharp$ is the data of a family indexed by $n \in \mathbb{N}$ of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-linear homomorphisms

$$\varepsilon^{\mathcal{E}}_n \colon \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$$

satisfying the following conditions:

(a) $\varepsilon^{\mathcal{E}}_0 = \mathrm{id}_{\mathcal{E}}$ and the family is compatible with respect to the exact closed immersion $\psi^{n+1,n}_{X^\sharp/S^\sharp,(m)}$ (see 3.2.2.2.1), i.e. for any $n' \geq n$ in $\mathbb{N}$ we have the commutative diagram:

$$\begin{array}{ccc} \psi^{n',n*}_{X^\sharp/S^\sharp,(m)}(\mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) & \xrightarrow{\psi^{n',n*}_{X^\sharp/S^\sharp,(m)}(\varepsilon^{\mathcal{E}}_{n'})} & \psi^{n',n*}_{X^\sharp/S^\sharp,(m)}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}) \ ; \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\varepsilon^{\mathcal{E}}_n} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \end{array}$$

(b) for any n, n' in $\mathbb{N}$, the diagram of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}$-modules:

$$\begin{array}{ccc}
\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\delta^{n,n'*}_{(m)}(\varepsilon^{\mathcal{E}}_{n+n'})} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \\
{\scriptstyle q^{n,n'*}_{1(m)}(\varepsilon^{\mathcal{E}}_{n+n'})} \searrow & & \nearrow {\scriptstyle q^{n,n'*}_{0(m)}(\varepsilon^{\mathcal{E}}_{n+n'})} \\
& \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} &
\end{array}$$

is commutative.

Say an $\mathcal{O}_X$-linear homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ between modules equipped with m-PD stratifications relatively to $X^\sharp/S^\sharp$ is *horizontal* if it commutes with all ε_n.

3.4.2.2. Similarly to 2.1.1.2, with notation 3.2.2.12, this yields that the above condition 3.4.2.1.b is equivalent to

$$\forall\, n \in \mathbb{N}, \qquad q^{n*}_{02,(m)}(\varepsilon_n) = q^{n*}_{01,(m)}(\varepsilon_n) \circ q^{n*}_{12,(m)}(\varepsilon_n). \tag{3.4.2.2.1}$$

Proposition 3.4.2.3. *Let $\mathcal{E}$ be an $\mathcal{O}_X$-module together with an m-PD stratification $(\varepsilon^{\mathcal{E}}_n)$ relative to $X^\sharp/S^\sharp$. Then the homomorphisms $\varepsilon^{\mathcal{E}}_n$ are $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-linear isomorphisms.*

Proof. We can copy the proof of 2.1.1.3. □

Example 3.4.2.4. For example, $\mathcal{O}_X$ is endowed with the m-PD-stratification of $\mathcal{O}_X$ given by the isomorphisms $\varepsilon^{\mathcal{O}_X}_n$ making commutative the diagram

$$\begin{array}{ccc}
\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X & \underset{\sim}{\xrightarrow{\varepsilon^{\mathcal{O}_X}_n}} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \\
\sim \searrow & & \swarrow \sim \\
& \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}, &
\end{array} \tag{3.4.2.4.1}$$

where the oblique arrows are the canonical ones. Indeed, we can check the cocycle conditions with the following diagram

$$\begin{array}{ccccc}
\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X & & \underset{\sim}{\xrightarrow{\delta^{n,n'*}_{(m)}(\epsilon^{\mathcal{O}_X}_{n+n'})}} & & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \\
\| & \sim \searrow & & \swarrow \sim & \uparrow {\scriptstyle \sim\ q^{n,n'*}_{0(m)}(\epsilon^{\mathcal{O}_X}_{n+n'})} \\
\| & & \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} & & \\
\| & \sim \nearrow & & \nwarrow \sim & \\
\mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X & & \underset{q^{n,n'*}_{1(m)}(\epsilon^{\mathcal{O}_X}_{n+n'})}{\xrightarrow{\sim}} & & \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)}
\end{array} \tag{3.4.2.4.2}$$

where the oblique arrows are the canonical ones and where the triangles (except the left one) are the images under $\delta^{n,n'*}_{(m)}$, $q^{n,n'*}_{0(m)}$ and $q^{n,n'*}_{1(m)}$ of the triangle 3.4.2.4.1.

Proposition 3.4.2.5. *We have the following properties.*

(I) Given an $\mathcal{O}_X$-module $\mathcal{E}$. The following are equivalent.

(a) A left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{E}$ extending its $\mathcal{O}_X$-module structure.

(b) A family of $\mathcal{O}_X$-linear homomorphisms $\theta_n \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)})$ (the $\mathcal{O}_X$-module structure of this latter is induced by the right structure of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$) satisfying

(i) $\theta_0 = \mathrm{id}_{\mathcal{E}}$ *and for any* $n, n' \in \mathbb{N}$, *the diagram*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\theta_n} & \mathcal{E} \otimes \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \\ \| & & \uparrow {\scriptstyle \mathrm{id} \otimes \psi^{n+n',n}_{X^\sharp/S^\sharp,(m)}} \\ \mathcal{E} & \xrightarrow[\theta_{n+n'}]{} & \mathcal{E} \otimes \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp,(m)} \end{array} \tag{3.4.2.5.1}$$

is commutative.

(ii) *for all* n, n' *we have commutative diagrams (cocycle condition)*

$$\begin{array}{ccc} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\mathrm{id} \otimes \delta^{n,n'}_{(m)}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \\ {\scriptstyle \theta^{\mathcal{E}}_{n+n'}} \uparrow & & \uparrow {\scriptstyle \theta^{\mathcal{E}}_n \otimes \mathrm{id}} \\ \mathcal{E} & \xrightarrow{\theta^{\mathcal{E}}_{n'}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp,(m)} \end{array} \tag{3.4.2.5.2}$$

(c) *An* m*-PD stratification* $\varepsilon = (\varepsilon^{\mathcal{E}}_n)$ *on* $\mathcal{E}$.

(II) *Let* $\mathcal{E}$ *be left* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-module and let* $\theta^{\mathcal{E}} = (\theta^{\mathcal{E}}_n)$, $\varepsilon^{\mathcal{E}} = (\varepsilon^{\mathcal{E}}_n)$ *be the associated family or* m*-PD stratification.*

(a) *We retrieve from* $\theta^{\mathcal{E}}$ *(resp.* $\varepsilon^{\mathcal{E}}$*) the action by a section* P *of* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ *on* $\mathcal{E}$ *via the following composition of the bottom (resp. top) horizontal morphisms of the commutative diagram:*

$$\begin{array}{ccccccc} \mathcal{E} & \xrightarrow[3.2.2.13]{p^n_{1(m),\mathcal{E}}} & \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\varepsilon^{\mathcal{E}}_n} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\mathrm{id} \otimes P} & \mathcal{E} \\ \| & & & & \| & & \| \\ \mathcal{E} & & \xrightarrow{\theta^{\mathcal{E}}_n} & & p_{1*}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}) & \xrightarrow{\mathrm{id} \otimes P} & \mathcal{E}. \end{array} \tag{3.4.2.5.3}$$

(b) *Suppose* $X^\sharp \to S^\sharp$ *is endowed with logarithmic coordinates* $(u_\lambda)_{\lambda=1,\dots,d}$. *Conversely, with notation 3.2.3.4, for any* $x \in \mathcal{E}$, *we have the Taylor expansion formula*

$$\theta^{\mathcal{E}}_n(x) = \varepsilon^{\mathcal{E}}_n(1 \otimes x) = \sum_{|\underline{k}| \leq n} \underline{\partial}^{\langle \underline{k} \rangle}_\sharp x \otimes \underline{\tau}^{\{\underline{k}\}}_\sharp. \tag{3.4.2.5.4}$$

(III) *An* $\mathcal{O}_X$*-linear morphism* $\phi \colon \mathcal{E} \to \mathcal{F}$ *between two left* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-modules is* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-linear if and only if* ϕ *is horizontal.*

Proof. The proof is identical to that of 2.1.1.5 except that the inverse of $\varepsilon^{\mathcal{E}}_n$ is given by the formula 3.4.5.3.1, formula which is computed later. □

Example 3.4.2.6. For example, $\mathcal{O}_X$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure via its canonical m-PD-stratification defined at 3.4.2.4. Using 3.4.2.5.3, the action by a section P of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ on $\mathcal{O}_X$ is given by the composition:

$$\mathcal{O}_X \xrightarrow{p^n_{1(m)}} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \xrightarrow{P} \mathcal{O}_X. \tag{3.4.2.6.1}$$

The following proposition will be generalized with coefficients (see 4.2.3.1.1). However, in order to define what is a coefficient and its properties (see Lemma 4.1.1.2), we need at least the case of a tensor product of two left $\mathcal{D}$-modules.

Proposition 3.4.2.7. *Let $\mathcal{E}$, $\mathcal{F}$ be two left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules. There exists on the tensor product $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}$ a unique structure of left $\widetilde{\mathcal{D}}^{(m)}_X$-module functorial in $\mathcal{E}$ and $\mathcal{F}$ such that, for any system of logarithmic coordinates on an open subset $U\subset X$, and any $\underline{k}\in\mathbb{N}^d$, $x\in\Gamma(U,\mathcal{E})$, $y\in\Gamma(U,\mathcal{F})$, we have*

$$\underline{\partial}^{\langle \underline{k}\rangle}_\sharp(x\otimes y)=\sum_{\underline{i}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{i}\end{smallmatrix}\right\}\underline{\partial}^{\langle \underline{i}\rangle}_\sharp(x)\otimes\underline{\partial}^{\langle \underline{k}-\underline{i}\rangle}_\sharp(y). \tag{3.4.2.7.1}$$

Moreover, the following formula holds

$$\widetilde{\underline{\partial}}^{\langle \underline{k}\rangle}_\sharp(x\otimes y)=\sum_{\underline{i}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{i}\end{smallmatrix}\right\}\widetilde{\underline{\partial}}^{\langle \underline{i}\rangle}_\sharp x\otimes\widetilde{\underline{\partial}}^{\langle \underline{k}-\underline{i}\rangle}_\sharp y. \tag{3.4.2.7.2}$$

Proof. The proof is similar to that of Proposition 2.1.3.1.a: We endow the sheaf $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}$ with the m-PD stratification with coefficients in $\mathcal{O}_X$: $\varepsilon^{\mathcal{E}\otimes\mathcal{F}}_n:=\varepsilon^{\mathcal{E}}_n\otimes_{\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}}\varepsilon^{\mathcal{F}}_n$. By using the formulas 3.4.2.5.4, we get 3.4.2.7.1.

By replacing the use of the formula 3.4.2.5.4 by that of 3.4.5.3.1, instead of 3.4.2.7.1 we get the formula 3.4.2.7.2 by symmetrical computations. More precisely, since $(\varepsilon^{\mathcal{E}\otimes\mathcal{F}}_n)^{-1}:=(\varepsilon^{\mathcal{E}}_n)^{-1}\otimes(\varepsilon^{\mathcal{F}}_n)^{-1}$, then we compute

$$(\varepsilon^{\mathcal{E}\otimes\mathcal{F}}_n)^{-1}((x\otimes y)\otimes 1)=\left(\sum_{|\underline{i}|\leq n}\underline{\tau}^{\{\underline{i}\}}_\sharp\otimes\widetilde{\underline{\partial}}^{\langle \underline{i}\rangle}_\sharp x\right)\otimes\left(\sum_{|\underline{j}|\leq n}\underline{\tau}^{\{\underline{j}\}}_\sharp\otimes\widetilde{\underline{\partial}}^{\langle \underline{j}\rangle}_\sharp y\right)=\sum_{|\underline{k}|\leq n}\underline{\tau}^{\{\underline{k}\}}_\sharp\otimes\left(\sum_{\underline{i}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{i}\end{smallmatrix}\right\}\widetilde{\underline{\partial}}^{\langle \underline{k}-\underline{i}\rangle}_\sharp y\otimes\widetilde{\underline{\partial}}^{\langle \underline{i}\rangle}_\sharp x\right),$$

Using again 3.4.5.3.1, we get 3.4.2.7.2. □

Remark 3.4.2.8. With notation 3.4.2.7, if $\mathcal{F}$ is moreover a $\widetilde{\mathcal{D}}^{(m)}_X$-bimodule, then $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}$ has a unique structure of $\widetilde{\mathcal{D}}^{(m)}_X$-bimodule induced by functoriality from 3.4.2.7.

3.4.3 PD-costratifications of level m

Definition 3.4.3.1. An m-PD-*costratification on $\mathcal{M}$ relatively to $X^\sharp/S^\sharp$* on an $\mathcal{O}_X$-module $\mathcal{M}$ is the data of a family of isomorphisms $\mathcal{P}_{X^\sharp/S^\sharp(m)}$-linear

$$\varepsilon_n\colon \mathcal{H}om_{\mathcal{O}_X}(p^n_{0*}\mathcal{P}_{X^\sharp/S^\sharp(m)},\mathcal{M})\to\mathcal{H}om_{\mathcal{O}_X}(p^n_{1*}\mathcal{P}_{X^\sharp/S^\sharp(m)},\mathcal{M}),$$

this ones satisfying the following conditions :

(a) $\varepsilon_0=\mathrm{id}_{\mathcal{M}}$ and for any $n'\geq n$ in $\mathbb{N}$, ε_n and $\psi^{n',n\flat}_{X/S,(m)}(\varepsilon_{n'})$ are canonically isomorphic, i.e. the following diagram

$$\begin{array}{ccc}\psi^{n',n\flat}_{X^\sharp/S^\sharp,(m)}(p^{n'\flat}_{0,(m)}(\mathcal{M})) & \xrightarrow{\psi^{n',n\flat}_{X^\sharp/S^\sharp,(m)}(\varepsilon_{n'})} & \psi^{n',n\flat}_{X^\sharp/S^\sharp,(m)}(p^{n'\flat}_{1,(m)}(\mathcal{M}))\\ \downarrow\wr & & \downarrow\wr\\ p^{n\flat}_{0,(m)}(\mathcal{M}) & \xrightarrow{\varepsilon_n} & p^{n\flat}_{1,(m)}(\mathcal{M})\end{array} \tag{3.4.3.1.1}$$

whose vertical isomorphisms are the canonical ones, is commutative ;

(b) For any n, n', the diagram

$$\begin{array}{ccc}\mathcal{H}om_{\mathcal{O}_X}(p^{n,n'}_{0*}(\mathcal{P}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}),\mathcal{M}) & \xrightarrow{\delta^{n,n'\flat}_{(m)}(\varepsilon_{n+n'})} & \mathcal{H}om_{\mathcal{O}_X}(p^{n,n'}_{2*}(\mathcal{P}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}),\mathcal{M})\\ {\scriptstyle q^{n,n'\flat}_0(\varepsilon_{n+n'})}\searrow & & \nearrow{\scriptstyle q^{n,n'\flat}_1(\varepsilon_{n+n'})}\\ & \mathcal{H}om_{\mathcal{O}_X}(p^{n,n'}_{1*}(\mathcal{P}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{O}_X}\mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}),\mathcal{M}) & \end{array} \tag{3.4.3.1.2}$$

is commutative.

Say an $\mathcal{O}_X$-linear homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ between $\mathcal{O}_X$-modules equipped with m-PD costratifications relatively to $X^\sharp/S^\sharp$ is *horizontal* if it commutes with all ε_n.

3.4.3.2. With notation 3.2.2.12, similarly to 2.1.1.2, we check that the above condition 3.4.3.1.2 is equivalent to

$$\forall\, n, \qquad q^{n\flat}_{02,(m)}(\varepsilon_n) = q^{n\flat}_{01,(m)}(\varepsilon_n) \circ q^{n\flat}_{12,(m)}(\varepsilon_n). \tag{3.4.3.2.1}$$

Proposition 3.4.3.3. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module together with an m-PD costratification $(\varepsilon^{\mathcal{M}}_n)$ relative to $X^\sharp/S^\sharp$. Then the homomorphisms $\varepsilon^{\mathcal{M}}_n$ are $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-linear isomorphisms.*

Proof. Copy the proof of 2.1.1.3. □

Proposition 3.4.3.4. *For any $\mathcal{O}_X$-module $\mathcal{M}$, there is equivalence between the following data :*

(a) A structure of right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{M}$ extending its structure of $\mathcal{O}_X$-module ;

(b) An m-PD-costratification $(\varepsilon^{\mathcal{M}}_n)$ relatively to $X^\sharp/S^\sharp$ on $\mathcal{M}$.

A $\mathcal{O}_X$-linear homomorphism between two right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear if and only if it is horizontal. Finally, via the identification 2.1.2.6.1 and 2.1.2.6.2, in logarithmic coordinates, for each section x of $\mathcal{M}$, we have the formula :

$$\varepsilon^{\mathcal{M}}_n(x \otimes \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}) = \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\star\langle \underline{h}\rangle} \otimes x \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h}\rangle}, \tag{3.4.3.4.1}$$

where $\{\underline{\partial}_\sharp^{\star\langle \underline{h}\rangle},\ |\underline{h}| \leq n\}$ is the basis of $\mathcal{H}om_{\mathcal{O}_X}(p^n_{1*}\mathcal{P}^n_{X^\sharp/S^\sharp(m)}, \mathcal{O}_X)$ which is the dual basis of the basis $\{\underline{\tau}_\sharp^{\{\underline{h}\}},\ |\underline{h}| \leq n\}$ of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$.

Proof. By copying 2.1.2.8 and 2.1.2.9, we obtain the proposition. □

3.4.4 On the preservation of $\mathcal{D}$-module structures under pullbacks, base change

Let

$$\begin{array}{ccc} X^\sharp & \xrightarrow{\quad f \quad} & Y^\sharp \\ \downarrow & & \downarrow \\ (S^\sharp, \mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) & \xrightarrow{\ \phi\ } & (T^\sharp, \mathfrak{a}_T, \mathfrak{b}_T, \alpha_T), \end{array} \tag{3.4.4.0.1}$$

be a commutative diagram where $S^\sharp$ and $T^\sharp$ are nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ as defined in 3.1.1.1 (resp. $S^\sharp$ and $T^\sharp$ are nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10 endowed with quasi-coherent (resp. coherent) m-PD-ideals, where the bottom arrow is an m-PD-morphism and where $X^\sharp$ is a log smooth $S^\sharp$-log scheme (resp. log smooth $S^\sharp$-log formal scheme) and $Y^\sharp$ is a log smooth $T^\sharp$-log scheme (resp. log smooth $T^\sharp$-log formal scheme).

3.4.4.1. Let us now construct the inverse image of a $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module We denote by $(f,\phi)\colon X^\sharp/S^\sharp \to Y^\sharp/T^\sharp$ the morphism of relative logarithmic schemes induced by the diagram 3.4.4.0.1. When $\phi\colon S^\sharp \to T^\sharp$ in understood, by abuse of notation, we also simply denote (f,ϕ) by f. According to notation 3.2.2.2, we denote by $p^n_{0,(m)}$ and $p^n_{1,(m)}\colon \Delta^n_{X^\sharp/S^\sharp(m)} \to X$ the left and right projections. According to notation 3.2.2.12, for any integer n and any integers $0 \leq i < j \leq 2$, it follows from the universal property of m-PD-envelopes of order n (see 3.2.1.1) that we get a unique m-PD-morphism $q^n_{ij,(m)}\colon \Delta^n_{X^\sharp/S^\sharp,(m)}(2) \to \Delta^n_{X^\sharp/S^\sharp,(m)}$ making commutative the diagram 3.2.2.12.1. For any integer $n \geq 0$ by using the universal property of the m-PD-envelopes of order n, we have the m-PD-morphisms $f^n_{(m)}\colon \Delta^n_{X^\sharp/S^\sharp(m)} \to \Delta^n_{Y^\sharp/T^\sharp(m)}$

and $f^n_{(m)}(2)\colon \Delta^n_{X^\sharp/S^\sharp(m)}(2) \to \Delta^n_{Y^\sharp/T^\sharp(m)}(2)$ making commutative the diagram of logarithmic schemes for any integers $0 \le i < j \le 2$ and $0 \le k \le 1$:

$$\begin{array}{ccccc} \Delta^n_{X^\sharp/S^\sharp,(m)}(2) & \xrightarrow{q^n_{ij,(m)}} & \Delta^n_{X^\sharp/S^\sharp,(m)} & \xrightarrow{p^n_{k,(m)}} & X^\sharp \\ \downarrow {\scriptstyle f^n_{(m)}(2)} & & \downarrow {\scriptstyle f^n_{(m)}} & & \| \\ \Delta^n_{Y^\sharp/S^\sharp,(m)}(2) & \xrightarrow{q^n_{ij,(m)}} & \Delta^n_{Y^\sharp/T^\sharp,(m)} & \xrightarrow{p^n_{k,(m)}} & Y^\sharp. \end{array} \tag{3.4.4.1.1}$$

These homomorphisms are compatible when n varies, i.e., with notation 3.2.2.2.1, we have the equality $\psi^{n',n}_{Y^\sharp/T^\sharp,(m),\gamma}(r) \circ f^n_{(m)}(r) = f^{n'}_{(m)}(r) \circ \psi^{n',n}_{X^\sharp/S^\sharp,(m),\gamma}(r)$, for any $n' \ge n$. Let $n \in \mathbb{N}$. This yields the m-PD ring homomorphism

$$f^{-1}(\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}(r)) \to \mathcal{P}^n_{X^\sharp/S(m)}(r). \tag{3.4.4.1.2}$$

which is $f^{-1}\mathcal{O}_Y$-linear for any structures (via the canonical ring homomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ for $\mathcal{P}^n_{X^\sharp/S(m)}$). We get the $\mathcal{O}_X$-module $f^*(p^n_{0,(m)*}\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}) := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(p^n_{0,(m)*}\mathcal{P}^n_{Y^\sharp/T^\sharp(m)})$ and the homomorphism of $\mathcal{O}_X$-modules:

$$f^n_{(m)}\colon f^*(p^n_{0,(m)*}\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}) \to p^n_{0,(m)*}\mathcal{P}^n_{X^\sharp/S(m)}. \tag{3.4.4.1.3}$$

3.4.4.2. Let $\mathcal{E}$ be a left $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module and $(\varepsilon^{\mathcal{E}}_n)$ its m-PD-stratification. The $\mathcal{O}_X$-module $f^*\mathcal{E}$ has a canonical structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. More precisely, the isomorphisms $\varepsilon^{f^*\mathcal{E}}_n := f^{n*}_{(m)}(\varepsilon^{\mathcal{E}}_n)$ endow $f^*\mathcal{E}$ with an m-PD-stratification (for the cocycle conditions, we use the commutativity of 3.4.4.1.1).

Let $\mathcal{D}$ be a sheaf of rings. When $\mathcal{E}$ is a $(\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{D})$-bimodule, then by functoriality $f^*(\mathcal{E})$ is a $(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{D})$-bimodule. For instance we get the $(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp})$-bimodule $f^*(\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp})$ that we will denote by $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}$.

3.4.4.3. Dualizing the morphisms 3.4.4.1.3, we get

$$f^{n\vee}_{(m)}\colon \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X^\sharp/S(m)}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(f^*(\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}, \mathcal{O}_X) \xrightarrow[\iota]{\sim} f^*\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp,n}, \tag{3.4.4.3.1}$$

where ι is the canonical morphism. Passing to limit we get the canonical homomorphisms

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to f^*\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}. \tag{3.4.4.3.2}$$

Proposition 3.4.4.4. *The canonical homomorphism 3.4.4.3.2 sends* 1 *to* $1 \otimes 1$ *and is* $(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{O}_Y)$*-bilinear.*

Proof. Let $P \in \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$ for some integer n. Since the lemma is local, we can suppose that $Y^\sharp \to T^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,r}$. Put $\tau_{\sharp\lambda} := \mu^n_{(m),\gamma}(u_\lambda) - 1$, where $\mu^n_{(m),\gamma}\colon M^n_{Y^\sharp/T^\sharp,(m),\gamma} \to \ker(\mathcal{P}^{n*}_{Y^\sharp/T^\sharp,(m),\gamma} \to \mathcal{O}^*_Y)$ is the morphism defined as in 3.2.2.2.3. The elements $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}, |\underline{k}| \le n\}$ form a basis of $\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)}$ as $\mathcal{O}_Y$-module. The corresponding dual basis of $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp,n}$ will be denoted by $\{\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}, |\underline{k}| \le n\}$. This induces the basis $\{1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}, |\underline{k}| \le n\}$ of $f^*\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)}$ as $\mathcal{O}_X$-module and the basis $\{1 \otimes \underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}, |\underline{k}| \le n\}$ of $f^*\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp,n}$ as $\mathcal{O}_X$-module.

By definition (see 3.4.4.3.1), $f^{n\vee}_{(m)}(P) = \iota(P \circ f^n_{(m)})$. Hence,

$$f^{n\vee}_{(m)}(P) = \sum_{|\underline{k}| \le n} P \circ f^n_{(m)}(1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}) \otimes (1 \otimes \underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}). \tag{3.4.4.4.1}$$

Let $(\varepsilon^{\mathcal{D}}_n)$ be the m-PD-stratification associated with $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$. Following 3.4.2.5.4, we have, $\varepsilon^{\mathcal{D}}_n(1 \otimes 1) = \sum_{|\underline{k}| \le n} \underline{\partial}_\sharp^{\langle\underline{k}\rangle} \otimes \underline{\tau}_\sharp^{\{\underline{k}\}}$. This yields: $f^{n*}_{(m)}(\varepsilon^{\mathcal{D}}_n)(1 \otimes (1 \otimes 1)) = \sum_{|\underline{k}| \le n} (1 \otimes \underline{\partial}_\sharp^{\langle\underline{k}\rangle}) \otimes f^n_{(m)}(1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}})$. Then following, 3.4.2.5.3, the action of P on $1 \otimes 1 \in f^*\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp,n}$ is

$$P(1 \otimes 1) = (id \otimes P) \circ f^{n*}_{(m)}(\varepsilon^{\mathcal{D}}_n)(1 \otimes (1 \otimes 1)) = \sum_{|\underline{k}| \le n} P \circ f^n_{(m)}(1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}})(1 \otimes \underline{\partial}_\sharp^{\langle\underline{k}\rangle}). \tag{3.4.4.4.2}$$

It follows from the equalities 3.4.4.4.1 and 3.4.4.4.2 that $P(1 \otimes 1) = f^{n\vee}_{(m)}(P)$. Since $1(1 \otimes 1) = 1 \otimes 1$ then the canonical homomorphism 3.4.4.3.2 sends 1 to $1 \otimes 1$ and is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. This yields the $(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{O}_Y)$-bilinearity by an easy computation. □

Proposition 3.4.4.5. *We have the canonical isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules:*

$$f^*\mathcal{E} \xrightarrow{\sim} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{E}. \tag{3.4.4.5.1}$$

Proof. We have the isomorphism of $(\mathcal{O}_X, f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules:

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \xrightarrow{\sim} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}.$$

This yields the $\mathcal{O}_X$-linear isomorphism of the form 3.4.4.5.1 given by $a \otimes x \mapsto (a \otimes 1) \otimes x$ for any section a of $\mathcal{O}_X$ and section x of $f^{-1}\mathcal{E}$. It remains to check its $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linearity.

Let $\rho\colon \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp} \to \mathcal{E}$ be a $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-linear morphism and let x be the image of 1 via ρ. By applying the functor f^*, this yields the $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear morphism $f^*(\rho)\colon \mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \to f^*\mathcal{E}$, which is given by $a \otimes Q \mapsto a \otimes (Q \cdot x)$ for any section a of $\mathcal{O}_X$ and section Q of $f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$. Hence, $f^*(\rho)(P \cdot (a \otimes 1)) = P \cdot (a \otimes x)$ for any section P of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and any section a of $\mathcal{O}_X$.

By composing $f^*(\rho)$ with 3.4.4.5.1, we get the morphism $\mathcal{D}_{X^\sharp/S^\sharp} \to \mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{E}$ given by $a \otimes Q \mapsto (a \otimes 1) \otimes (Q \cdot x) = (a \otimes Q) \otimes x$, which is the image of ρ by the functor $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}(-)$ and therefore is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Hence, the map 3.4.4.5.1 sends $P \cdot (a \otimes x)$ to $P \cdot ((a \otimes 1) \otimes x)$ for any section P of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and we are done. □

3.4.4.6. Let $\mathcal{B}_Y$ be a commutative $\mathcal{O}_Y$-algebra endowed with of a compatible structure of left $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module. Then the action of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $f^*\mathcal{B}_Y$ is compatible with its structure of $\mathcal{O}_X$-algebra.

3.4.5 Canonical right $\mathcal{D}_{X^\sharp/S^\sharp}$-module structure on $\omega_{X^\sharp/S^\sharp}$ and an inverse formula

Lemma 3.4.5.1. *The sheaf $\omega_{X^\sharp/S^\sharp}$ is a right $\mathcal{D}_{X^\sharp/S^\sharp}$-submodule of $j_*\omega_{Y/S^\sharp}$. Suppose there exists logarithmic coordinates $u_1, \dots, u_d \in M_{X^\sharp}$. The action of $P \in \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ on the section $a\, d\log u_1 \wedge \cdots \wedge d\log u_d$, where a is section of $\mathcal{O}_X$ is given by the formula*

$$(a\ d\log u_1 \wedge \cdots \wedge d\log u_d) \cdot P = \widetilde{P}(a) d\log u_1 \wedge \cdots \wedge d\log u_d. \tag{3.4.5.1.1}$$

Proof. Since this can been checked locally, we can suppose there exists logarithmic coordinates $u_1, \dots, u_d \in M_{X^\sharp}$. Let $(t_\lambda)_{\lambda=1,\dots,r}$ be the induced coordinates of Y/S. Then $\omega_{X^\sharp/S^\sharp}$ is a free $\mathcal{O}_X$-module of rank 1 and a basis is given by $d\log u_1 \wedge \cdots \wedge d\log u_d$. The map $\omega_{X^\sharp/S^\sharp} \to j_*\omega_{Y/S^\sharp}$ sends $d\log u_1 \wedge \cdots \wedge d\log u_d$ to $\frac{1}{t_1 \cdots t_d} dt_1 \wedge \cdots \wedge dt_d$. Then this is an easy computation from 2.2.1.5 and 3.4.1.3.1. □

Lemma 3.4.5.2. *We suppose that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,n}$. If $\underline{k} \neq 0$ then we have the formula:*

$$0 = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{h} \rangle_{(m)}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} - \underline{h} \rangle}. \tag{3.4.5.2.1}$$

Proof. Consider the diagram

$$\begin{array}{ccc} p_0^{n\flat}(\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) & \xrightarrow{\epsilon_n} & p_1^{n\flat}(\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \\ \downarrow \simeq & & \downarrow \simeq \\ p_0^{n\flat}(\omega_{X^\sharp/S^\sharp}) \otimes_{\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}} p_0^{n*}(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) & \xrightarrow{\epsilon_n} & p_1^{n\flat}(\omega_{X^\sharp/S^\sharp}) \otimes_{\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}} p_1^{n*}(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \end{array} \tag{3.4.5.2.2}$$

Let $e_0 := d\log t_1 \wedge \cdots \wedge d\log t_d$. It follows from 3.4.3.4.1 the formula

$$\varepsilon_n((e_0 \otimes P) \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) = \sum_{\underline{j} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{j} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\star \langle \underline{j} \rangle} \otimes (e_0 \otimes P) \underline{\partial}_\sharp^{\langle \underline{k} - \underline{j} \rangle}. \tag{3.4.5.2.3}$$

The element $(e_0 \otimes P) \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}$ is sent via the left arrow of the diagram 3.4.5.2.2 to $(e_0 \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) \otimes (P \otimes 1)$. This latter is sent via the bottom arrow to

$$\epsilon_n(e_0 \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) \otimes \epsilon_n^{-1}(P \otimes 1) \overset{3.4.3.4.1}{=} \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left(\underline{\partial}_\sharp^{\star \langle \underline{h} \rangle} \otimes e_0 \underline{\partial}_\sharp^{\langle \underline{k} - \underline{h} \rangle} \right) \otimes \epsilon_n^{-1}(P \otimes 1).$$

Since $e_0 \underline{\partial}_\sharp^{\langle \underline{i} \rangle} = \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle}(1) e_0 = 0$ if $\underline{i} \neq \underline{0}$, then we get

$$\sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left(\underline{\partial}_\sharp^{\star \langle \underline{h} \rangle} \otimes e_0 \underline{\partial}_\sharp^{\langle \underline{k} - \underline{h} \rangle} \right) = \underline{\partial}_\sharp^{\star \langle \underline{k} \rangle} \otimes e_0.$$

Hence,

$$\epsilon_n(e_0 \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) \otimes \epsilon_n^{-1}(P \otimes 1) = \underline{\partial}_\sharp^{\star \langle \underline{k} \rangle} \otimes e_0 \otimes \epsilon_n^{-1}(P \otimes 1). \tag{3.4.5.2.4}$$

Let us write $\epsilon_n^{-1}(P \otimes 1) = \sum_{|\underline{i}| \leq n} \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes a_{\underline{i}}$, with $a_{\underline{i}} \in \mathcal{D}_{X^\sharp}^{(m)}$. This yields

$$\epsilon_n(e_0 \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) \otimes \epsilon_n^{-1}(P \otimes 1) = \sum_{|\underline{i}| \leq n} \underline{\partial}_\sharp^{\star \langle \underline{k} \rangle} \otimes e_0 \otimes \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes a_{\underline{i}}, \tag{3.4.5.2.5}$$

which is sent via the right arrow of the diagram 3.4.5.2.2 to

$$\underline{\tau}_\sharp^{\{\underline{j}\}} \mapsto \sum_{|\underline{i}| \leq n} \underline{\partial}_\sharp^{\star \langle \underline{k} \rangle} (\underline{\tau}_\sharp^{\{\underline{j}\}} \underline{\tau}_\sharp^{\{\underline{i}\}}) e_0 \otimes a_{\underline{i}} = \left\{ \begin{smallmatrix} \underline{k} \\ \underline{j} \end{smallmatrix} \right\} e_0 \otimes a_{\underline{k} - \underline{j}}, \tag{3.4.5.2.6}$$

i.e. to $\sum_{\underline{j} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{j} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\star \langle \underline{j} \rangle} \otimes (e_0 \otimes a_{\underline{k} - \underline{j}})$. By using the commutativity of the diagram 3.4.5.2.2, and the formula 3.4.5.2.3, this yields $e_0 \otimes a_{\underline{i}} = (e_0 \otimes P) \underline{\partial}_\sharp^{\langle \underline{i} \rangle} \overset{3.4.5.1.1}{=} e_0 \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} P$. Hence,

$$\epsilon_n^{-1}(P \otimes 1) = \sum_{|\underline{i}| \leq n} \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} P. \tag{3.4.5.2.7}$$

From 3.4.2.5.4, this yields

$$\begin{aligned} P \otimes 1 = \epsilon_n \circ \epsilon_n^{-1}(P \otimes 1) &= \epsilon_n \Big(\sum_{|\underline{i}| \leq n} \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} P \Big) = \sum_{|\underline{i}| \leq n} \underline{\tau}_\sharp^{\{\underline{i}\}} \epsilon_n (1 \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} P) \\ = \sum_{|\underline{i}| \leq n} \underline{\tau}_\sharp^{\{\underline{i}\}} \sum_{|\underline{j}| \leq n} \underline{\partial}_\sharp^{\langle \underline{j} \rangle_{(m)}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} P \otimes \underline{\tau}_{\sharp (m)}^{\{\underline{j}\}} &= \sum_{|\underline{i}| \leq n} \sum_{|\underline{j}| \leq n} \left\{ \begin{smallmatrix} \underline{i} + \underline{j} \\ \underline{j} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{j} \rangle_{(m)}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} P \otimes \underline{\tau}_{\sharp (m)}^{\{\underline{j} + \underline{i}\}} \\ &= \sum_{|\underline{k}| \leq n} \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{h} \rangle_{(m)}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} - \underline{h} \rangle} P \otimes \underline{\tau}_{\sharp (m)}^{\{\underline{k}\}} \end{aligned} \tag{3.4.5.2.8}$$

This yields, if $\underline{k} \neq 0$,

$$0 = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{h} \rangle_{(m)}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} - \underline{h} \rangle}. \tag{3.4.5.2.9}$$

□

Proposition 3.4.5.3. *Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\mathcal{D}_{X^\sharp/S^\sharp}^{(m)}$-module. We suppose that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,n}$.*

(a) Then we have the inverse formula of the Taylor formula 3.4.2.5.4 satisfied by $\mathcal{E}$:

$$(\epsilon_n^{\mathcal{E}})^{-1}(x \otimes 1) = \sum_{|\underline{i}| \leq n} \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{i} \rangle} \cdot x. \tag{3.4.5.3.1}$$

(b) We have the inverse formula of the Taylor formula 3.4.3.4.1 satisfied by $\mathcal{M}$:

$$(\varepsilon_n^{\mathcal{M}})^{-1}(\underline{\partial}^{\star\langle\underline{k}\rangle}\otimes x)=\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}x\widetilde{\underline{\partial}}_\sharp^{\langle\underline{k}-\underline{h}\rangle}\otimes\underline{\partial}_\sharp^{\langle\underline{h}\rangle}. \tag{3.4.5.3.2}$$

Proof. a) Since, we have $\epsilon_n^{\mathcal{E}}(1\otimes y)=\sum_{|\underline{i}|\leq n}\underline{\partial}_\sharp^{\langle\underline{i}\rangle}\cdot y\otimes\underline{\tau}_\sharp^{\{\underline{i}\}}$, then the formula 3.4.5.3.1 is a consequence of 3.4.5.2.1 and of the computation 3.4.5.2.8.

b) Let us now check that the morphism $\zeta_n^{\mathcal{M}}$ defined by the formula 3.4.5.3.2 is the inverse of the morphism $\varepsilon_n^{\mathcal{M}}$. We compute

$$\varepsilon_n^{\mathcal{M}}\circ\zeta_n^{\mathcal{M}}(\underline{\partial}^{\star\langle\underline{k}\rangle}\otimes x)=\varepsilon_n^{\mathcal{M}}\left(\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}x\widetilde{\underline{\partial}}_\sharp^{\langle\underline{h}\rangle}\otimes\underline{\partial}_\sharp^{\langle\underline{k}-\underline{h}\rangle}\right)$$
$$=\sum_{\underline{h}\leq\underline{k}}\sum_{\underline{i}\leq\underline{k}-\underline{h}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\underline{k}-\underline{h}\\ \underline{i}\end{smallmatrix}\right\}\underline{\partial}_\sharp^{\star\langle\underline{k}-\underline{h}-\underline{i}\rangle}\otimes x\widetilde{\underline{\partial}}_\sharp^{\langle\underline{h}\rangle}\underline{\partial}_\sharp^{\langle\underline{i}\rangle}=\sum_{\underline{j}\leq\underline{k}}\sum_{\underline{i}\leq\underline{j}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{j}-\underline{i}\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\underline{k}-\underline{j}+\underline{i}\\ \underline{i}\end{smallmatrix}\right\}\underline{\partial}_\sharp^{\star\langle\underline{k}-\underline{j}\rangle}\otimes x\widetilde{\underline{\partial}}_\sharp^{\langle\underline{j}-\underline{i}\rangle}\underline{\partial}_\sharp^{\langle\underline{i}\rangle}$$
$$\sum_{\underline{j}\leq\underline{k}}\sum_{\underline{i}\leq\underline{j}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{j}\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\underline{j}\\ \underline{i}\end{smallmatrix}\right\}\underline{\partial}_\sharp^{\star\langle\underline{k}-\underline{j}\rangle}\otimes x\widetilde{\underline{\partial}}_\sharp^{\langle\underline{j}-\underline{i}\rangle}\underline{\partial}_\sharp^{\langle\underline{i}\rangle}\overset{3.4.5.2.1}{=}\underline{\partial}^{\star\langle\underline{k}\rangle}\otimes x. \tag{3.4.5.3.3}$$

Similarly, we compute $\zeta_n^{\mathcal{M}}\circ\varepsilon_n^{\mathcal{M}}(x\otimes\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}})=x\otimes\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}$. □

Chapter 4

Logarithmic differential modules with coefficients

4.1 Sheaf of logarithmic differential operators of finite order with coefficients

We keep notation 3.4.

4.1.1 Coefficients

We will need later to add overconvergent singularities to the sheaf of differential operators (see 8.7.3.1). To do so, let us introduce the notion of coefficients of the sheaves of differential operators.

Definition 4.1.1.1. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra. We say that a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{B}_X$ is compatible with its $\mathcal{O}_X$-algebra structure, if

(a) the $\mathcal{O}_X$-module structure coming from the left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure via the canonical inclusion $\mathcal{O}_X \subset \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ agrees with the one coming from its $\mathcal{O}_X$-algebra structure ;

(b) the isomorphisms $\varepsilon^{\mathcal{B}_X}_n : \mathcal{P}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ of the m-PD-stratification coming from the $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure of $\mathcal{B}$ are isomorphisms of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-algebras.

Such an $\mathcal{O}_X$ algebra can be viewed as an extended version of $\mathcal{O}_X$ which is the coefficient of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

Lemma 4.1.1.2. *Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra endowed with a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure such that the condition (a) of 4.1.1.1 holds. We endow $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{B}_X$ with its canonical left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module induced by that of $\mathcal{B}_X$ (see 3.4.2.7). Then the following condition are equivalent:*

(b) The condition (b) of 4.1.1.1 holds.

(c) For any open $U^\sharp$ such that $U^\sharp/S^\sharp$ has logarithmic coordinates, for any $f, g \in \Gamma(U, \mathcal{B}_X)$ we have the Leibniz formula:

$$\underline{\partial}_\sharp^{\langle \underline{k} \rangle}(fg) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{h} \rangle}(f) \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle}(g). \tag{4.1.1.2.1}$$

(d) The multiplication map $\mu \colon \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{B}_X \to \mathcal{B}_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. Since the assertions are local, then we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates. For any $f, g \in \Gamma(X, \mathcal{B}_X)$, for any $x, y \in \Gamma(X, \mathcal{P}^n_{X^\sharp/S^\sharp,(m)})$, we compute

$$\varepsilon_n^{\mathcal{E}}(x \otimes f)\varepsilon_n^{\mathcal{E}}(y \otimes g) \overset{3.4.2.5.4}{=} \left(\sum_{|\underline{i}|\leq n} \underline{\partial}_\sharp^{\langle \underline{i}\rangle}(f) \otimes \underline{\tau}_\sharp^{\{\underline{i}\}} x\right)\left(\sum_{|\underline{j}|\leq n} \underline{\partial}_\sharp^{\langle \underline{j}\rangle}(g) \otimes \underline{\tau}_\sharp^{\{\underline{j}\}} y\right)$$

$$\overset{1.2.4.5.3}{=} \sum_{|\underline{k}|\leq n} \sum_{\underline{i}+\underline{j}=\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{i}\rangle}(f)\underline{\partial}_\sharp^{\langle \underline{j}\rangle}(g) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}} xy \qquad (4.1.1.2.2)$$

Hence, we get the equivalence between (b) and (c) conditions. Finally, the equivalence between (c) and (d) conditions follows from 3.4.2.7.1. □

Example 4.1.1.3. For example, the canonical left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{O}_X$ (see 3.4.2.4) is compatible with its $\mathcal{O}_X$-algebra structure. Indeed, since the oblique arrows of the diagram 3.4.2.4.1 are isomorphisms of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-algebras, then so are the isomorphisms $\varepsilon_n^{\mathcal{O}_X}$ of the canonical m-PD-stratification of $\mathcal{O}_X$.

4.1.1.4. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure which is compatible with its algebra structure. Let $\rho\colon \mathcal{O}_X \to \mathcal{B}_X$ be the structure algebra homomorphism. Then ρ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Indeed, since $\varepsilon_n^{\mathcal{B}_X}$ is a $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-algebra homomorphism, then for any section a of $\mathcal{O}_X$ we compute in $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$ the equalities $\varepsilon_n^{\mathcal{B}_X}(1 \otimes \rho(a)) = \varepsilon_n^{\mathcal{B}_X}(p^n_{1(m)}(a) \otimes 1) = (1 \otimes 1)p^n_{1(m)}(a) = (\rho \otimes \mathrm{id})(1 \otimes p^n_{1(m)}(a)) = (\rho \otimes \mathrm{id})(\varepsilon_n^{\mathcal{B}_X}(1 \otimes a))$.

Remark 4.1.1.5. Let $\rho\colon \mathcal{B}_X \to \mathcal{B}'_X$ be a commutative algebra homomorphism. We suppose $\mathcal{B}_X$ and $\mathcal{B}'_X$ are endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module such that ρ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Since ρ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear, then $\mathcal{C}_X := \operatorname{Im}\rho$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module so that the epimorphism $\mathcal{B}_X \twoheadrightarrow \mathcal{C}_X$ and the monomorphism $\mathcal{C}_X \hookrightarrow \mathcal{B}'_X$ are $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. We easily see (e.g. use 4.1.1.2.1) that this is a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{C}_X$ which is compatible with its $\mathcal{O}_X$-algebra structure.

Proposition 4.1.1.6. *Let $\mathcal{B}_X$ (resp. $\mathcal{C}_X$) be a commutative $\mathcal{O}_X$ algebra endowed with a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure which is compatible with its $\mathcal{O}_X$-algebra structure (see definition 4.1.1.1). Then the natural left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module on the tensor product $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{C}_X$ (see 3.4.2.7) is compatible with its $\mathcal{O}_X$-algebra structure.*

Proof. By construction, the m-PD-stratification is given by $\epsilon_n^{\mathcal{B}\otimes\mathcal{C}} := \varepsilon_n^{\mathcal{B}} \otimes \varepsilon_n^{\mathcal{C}}$ and is indeed an isomorphism of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-algebras. □

4.1.2 Sheaf of logarithmic differential operators of finite order with coefficients

Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure which is compatible with its algebra structure.

4.1.2.1. The canonical morphisms of $\mathcal{B}_X$-modules $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X)$ given by $b \otimes P \mapsto (\tau \mapsto P(\tau)b)$ is an isomorphism for any $n \in \mathbb{N}$. Via this identification, the product $P \cdot P' \in \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n+n'}$ of the operators $P \in \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$ and $P' \in \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n'}$ is by definition the composition

$$\mathcal{P}^{n+n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{\delta^{n,n'}_{(m)}} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{\mathrm{id}\otimes P'} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\epsilon_n^{\mathcal{B}_X}} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \xrightarrow{\mathrm{id}\otimes P} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\mu} \mathcal{B}_X, \qquad (4.1.2.1.1)$$

where μ is the canonical multiplication of $\mathcal{B}_X$.

Proposition 4.1.2.2. *The sheaf $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a sheaf of rings with the product as defined in 4.1.2.1.1. This is the unique ring structure on $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ such that*

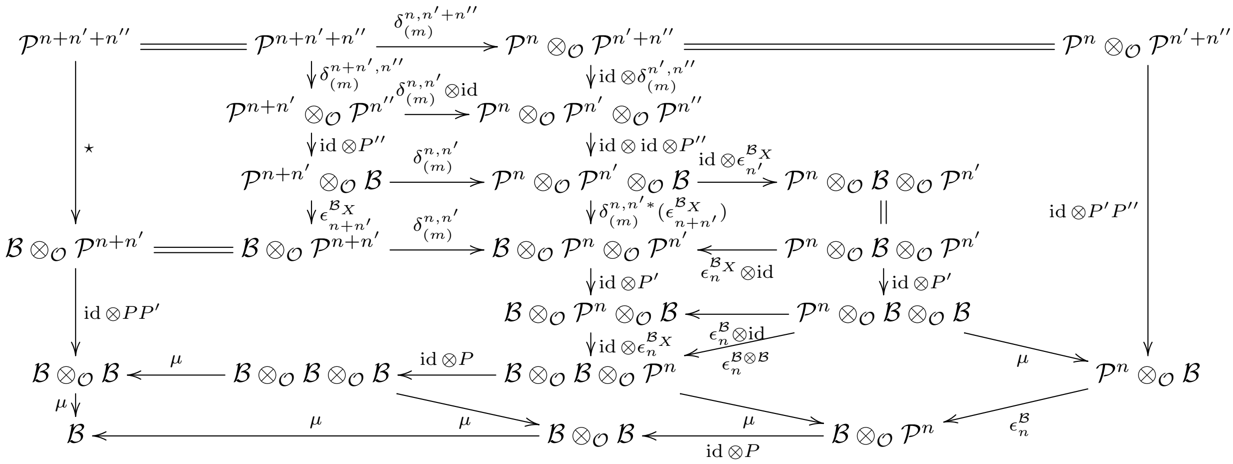
$$\mathcal{B}_X \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} : b \mapsto b \otimes 1, \tag{4.1.2.2.1}$$

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} : P \mapsto 1 \otimes P \tag{4.1.2.2.2}$$

are ring homomorphisms satisfying the formula $(b\otimes 1)(1\otimes P) = b\otimes P$ and for any logarithmic coordinates

$$(1 \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle})(b \otimes 1) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle}(b) \otimes \underline{\partial}_\sharp^{\langle \underline{h} \rangle}. \tag{4.1.2.2.3}$$

Proof. i) Let us check in this step that the sheaf $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a sheaf of rings with the product as defined in 4.1.2.1.1. Let $P \in \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$, $P' \in \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n'}$ and $P'' \in \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n''}$ be three operators. Consider the following diagram



(4.1.2.2.4)

where the $\star$ morphism is the one making commutative the top left rectangle. Following 3.2.2.18.1, the top square (in the middle) is commutative. The commutativity of the right square of the third row comes from the cocycle condition of the m-PD-stratification of $\mathcal{B}$. The triangle is commutative by definition of the m-PD-stratification of $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{B}$ (see the proof of 3.4.2.7). The commutativity of the right rhombus follows from the $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linearity of μ (see 4.1.1.2). The other squares or rhombus are commutative by functoriality. The other parts of the diagram are commutative by definition. Hence, the diagram 4.1.2.2.4 is commutative. The composition of the left vertical arrows of the contour is equal to $(PP')P''$, whereas the composition of the top,right bottom morphisms of the contour gives $P(P'P'')$. Hence, we are done.

ii) Via the canonical isomorphisms $\mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,0} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^0_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{B}_X)$, by using the case where $n = 0$ and $n' = 0$ in the definition of the product, we check that the canonical map $\mathcal{B}_X \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is an injective ring homomorphism. Moreover, to check that $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a ring homomomorphism, either we use 3.2.3.1.1 or we use the local computations 3.2.3.7.2 and 4.1.2.2.3 (use also the fact that following 4.1.1.4 the canonical homorphism $\rho\colon \mathcal{O}_X \to \mathcal{B}_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear). □

Remark 4.1.2.3. When the log structure of $X^\sharp$ and $S^\sharp$ are trivial, then replacing logarithmic coordinates the non-logarithmic version of 4.1.2.2.3 holds, i.e. we have

$$(1 \otimes \underline{\partial}^{\langle \underline{k} \rangle})(b \otimes 1) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k}-\underline{h} \rangle}(b) \otimes \underline{\partial}^{\langle \underline{h} \rangle}. \tag{4.1.2.3.1}$$

4.1.2.4. Let $\rho\colon \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ be the canonical morphism (see 4.1.2.2.2).

(a) The $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure on $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ induced by ρ is equal to the $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure induced by the tensor product from the left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure of $\mathcal{B}_X$ and the canonical $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ (see 3.4.2.8).

Indeed, the first equality of 4.1.2.2 implies $(b\otimes P)(1\otimes P') = b\otimes PP'$, i.e. $(b\otimes P)\cdot P' = (b\otimes P)(1\otimes P')$, where the action of P' on $b \otimes P$ is that given by the right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure induced by functoriality of the tensor product (see 3.4.2.8). Finally, via the formulas 4.1.2.2.3 and 3.4.2.7.1, we get $(1 \otimes \underline{\partial}^{\langle \underline{k}\rangle}_\sharp)(b \otimes 1) = \underline{\partial}^{\langle \underline{k}\rangle}_\sharp \cdot (b \otimes 1)$ where the action of $\underline{\partial}^{\langle \underline{k}\rangle}_\sharp$ on $b \otimes 1$ is that given by the tensor product.

(b) It follows from (a) and 3.4.2.7.2 the formula:

$$\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle}_\sharp \times b = \underline{\widetilde{\partial}}^{\langle \underline{k}\rangle}_\sharp (b \otimes 1) = \sum_{\underline{i}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}^{\langle \underline{i}\rangle}_\sharp (b) \otimes \underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{i}\rangle}_\sharp = \sum_{\underline{i}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{i}\rangle}_\sharp (b) \times \underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{i}\rangle}_\sharp, \tag{4.1.2.4.1}$$

where $\times$ is the product given by the ring structure of $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

Notation 4.1.2.5. We define a sheaf of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-algebras by setting $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$. The canonical morphism $\widetilde{p}^n_{0(m)} \colon \mathcal{B}_X \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$ endows the sheaf $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ with a structure of $\mathcal{B}_X$-algebra, that we will call *left* structure. Moreover, the morphism of $\mathcal{B}_X$-algebras

$$\widetilde{p}^n_{1(m)} \colon \mathcal{B}_X \to \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\epsilon^{\mathcal{B}_X}_n} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$$

induces a second structure of $\mathcal{B}_X$-algebra on $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$, that we call the *right* structure. If there are no risk of confusion, we can simply write $\widetilde{p}^n_0$ and $\widetilde{p}^n_1$ instead of $\widetilde{p}^n_{0(m)}$ and $\widetilde{p}^n_{1(m)}$. To sum-up $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ is canonically a $(\mathcal{B}_X, \mathcal{B}_X)$-algebra (i.e. a $\mathcal{B}_X \otimes_{\mathbb{Z}} \mathcal{B}_X$-algebra).

Lemma 4.1.2.6. *The canonical map*

$$\rho_n \colon \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \tag{4.1.2.6.1}$$

given by $\tau \mapsto 1 \otimes \tau$ is a homomorphism of $(\mathcal{O}_X, \mathcal{O}_X)$-algebras making cocartesian for any $j = 0, 1$ the diagrams of algebras

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{p^n_{j(m)}} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \\ \downarrow & & \downarrow{\scriptstyle \rho_n} \\ \mathcal{B}_X & \xrightarrow{\widetilde{p}^n_{j(m)}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}, \end{array} \tag{4.1.2.6.2}$$

where the homomorphisms $p^n_{j(m)}$ (resp. $\widetilde{p}^n_{j(m)}$) are defined at 3.2.2.2 (resp. 4.1.2.5).

Proof. The commutativity and cocartesianity of 4.1.2.6.2 are clear for the case $j = 0$. Moreover, as $\mathcal{O}_X \to \mathcal{B}_X$ is horizontal (see 4.1.1.4), then we obtain the commutative diagram which concludes the proof:

$$\begin{array}{ccccc} \mathcal{B}_X & \longrightarrow & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X & \xrightarrow[\sim]{\epsilon^{\mathcal{B}_X}_n} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_X & \longrightarrow & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X & \xrightarrow[\sim]{\epsilon^{\mathcal{O}_X}_n} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \\ \| & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathcal{O}_X & \xrightarrow{p^n_{1(m)}} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} & \overline{\overline{\qquad\qquad}} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.1.2.6.3}$$

□

Notation 4.1.2.7. Let $\mathcal{E}$ be a $\mathcal{B}_X$-module. By convention, $\widetilde{\mathcal{P}}^n_{X/S,(m)} \otimes_{\mathcal{B}_X} \mathcal{E}$ means $p^n_{1*}(\widetilde{\mathcal{P}}^n_{X/S,(m)}) \otimes_{\mathcal{B}_X} \mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X/S,(m)}$ means $\mathcal{E} \otimes_{\mathcal{B}_X} p^n_{0*}(\widetilde{\mathcal{P}}^n_{X/S,(m)})$. For instance, $\widetilde{\mathcal{P}}^n_{X/S,(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X/S,(m)}$ is $p^n_{1*}(\widetilde{\mathcal{P}}^n_{X/S,(m)}) \otimes_{\mathcal{B}_X} p^{n'}_{0*}(\widetilde{\mathcal{P}}^{n'}_{X/S,(m)})$.

We have two structures of $\mathcal{B}_X$-module on the sheaf $\widetilde{\mathcal{P}}^n_{X/S,(m)} \otimes_{\mathcal{B}_X} \mathcal{E}$: the "left structure" given by functoriality from the left structure of $\widetilde{\mathcal{P}}^n_{X/S,(m)}$ and the "right structure" given by the internal tensor product. We denote by $p_{0*}(\widetilde{\mathcal{P}}^n_{X/S,(m)} \otimes_{\mathcal{B}_X} \mathcal{E})$ (resp. $p_{1*}(\widetilde{\mathcal{P}}^n_{X/S,(m)} \otimes_{\mathcal{B}_X} \mathcal{E})$) to clarify we are considering the left structure (resp. right structure).

Similarly, we denote by $p_{0*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X/S,(m)})$ (resp. $p_{1*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X/S,(m)})$) the $\mathcal{B}_X$-module given by the internal tensor product (resp. by functoriality from the right $\mathcal{B}_X$-module structure of $\widetilde{\mathcal{P}}^n_{X/S,(m)}$) which is called the left (resp. right) structure.

We denote by $\widetilde{p}^n_{0,\mathcal{E}} \colon \mathcal{E} \to p_{0*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X/S,(m)})$ the canonical $\mathcal{B}_X$-linear map given by $x \mapsto x \otimes \mathbb{1}$, i.e. is the composition of $\mathrm{id}_{\mathcal{E}} \otimes p^n_0$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^0_{X/S,(m)}$. We denote by $\widetilde{p}^n_{1,\mathcal{E}} \colon \mathcal{E} \to p_{1*}(\widetilde{\mathcal{P}}^n_{X/S,(m)} \otimes_{\mathcal{B}_X} \mathcal{E})$ the canonical map given by $x \mapsto \mathbb{1} \otimes x$, i.e. is the composition of $\widetilde{p}^n_1 \otimes \mathrm{id}_{\mathcal{E}}$ with the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \widetilde{\mathcal{P}}^0_{X/S,(m)} \otimes_{\mathcal{B}_X} \mathcal{E}$.

Notation 4.1.2.8. We denote by $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$ the tensor product of $\mathcal{B}_X$-algebras with the right structure of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ and the left structure of $\widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$. We endow $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$ with three structures of locally free of finite type $\mathcal{B}_X$-algebra: the left one given by the left structure of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$, the middle one given by the tensor product and the right one given by the right structure of $\widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)}$. This endows $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$ with a structure of $(\mathcal{B}_X, \mathcal{B}_X, \mathcal{B}_X)$-algebra, i.e. of $(\mathcal{B}_X \otimes_{\mathbb{Z}} \mathcal{B}_X \otimes_{\mathbb{Z}} \mathcal{B}_X)$-algebra. When $\mathcal{B}_X = \mathcal{O}_X$, we retrieve the usual three structures (defined at 3.2.2.14). With 4.1.2.6, we get the morphism of $(\mathcal{O}_X, \mathcal{O}_X, \mathcal{O}_X)$-algebras

$$\rho_{n,n'} := \rho_n \otimes \rho_{n'} \colon \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}. \tag{4.1.2.8.1}$$

4.1.2.9. With notation 3.2.2.16 we can endow $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}$ with three canonical structures of $\mathcal{B}_X$-algebras as follows First, the *left* structure is induced by the canonical morphism $\widetilde{p}^{n,n'}_{0(m)} \colon \mathcal{B}_X \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}$ given by $b \mapsto b \otimes 1 \otimes 1$. Next, we define the *middle* structure via the morphism of $\mathcal{B}_X$-algebras

$$\widetilde{p}^{n,n'}_{1(m)} \colon \mathcal{B}_X \to \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{q_0^{n,n'*}(\epsilon^{\mathcal{B}_X}_{n+n'})} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)},$$

where the first morphism is given by $b \mapsto 1 \otimes b \otimes 1$. Finally, the *right* structure can be built as follows :

$$\widetilde{p}^{n,n'}_{2(m)} \colon \mathcal{B}_X \to \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\delta^{n,n'*}_{(m)}(\epsilon^{\mathcal{B}_X}_{n+n'})} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}, \tag{4.1.2.9.1}$$

where the first morphism is given by $b \mapsto 1 \otimes 1 \otimes b$.

Proposition 4.1.2.10. *a) The canonical map*

$$\mathcal{B}_X \otimes_{\mathcal{O}_X} (\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}) \xrightarrow{\sim} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}, \tag{4.1.2.10.1}$$

given by $b \otimes \tau \otimes \tau' \mapsto (b \otimes \tau) \otimes (1 \otimes \tau')$, is an isomorphism of $\mathcal{B}_X$-algebras for the left (resp. middle, resp. right) structures as defined respectively in 4.1.2.5 and 4.1.2.9. Hence, we still denote therefore by $\widetilde{p}^{n,n'}_{j(m)}$ for $j = 0, 1, 2$ the corresponding ring homomorphisms $\mathcal{B}_X \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$.

b) Moreover, for any $j = 0, 1, 2$, we have the following canonical cocartesian diagram

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{p^{n,n'}_{j(m)}} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \\ \downarrow & & \downarrow {\scriptstyle \rho_{n,n'}\ 4.1.2.8.1} \\ \mathcal{B}_X & \xrightarrow{\widetilde{p}^{n,n'}_{j(m)}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.1.2.10.2}$$

Proof. a) i) By using the $(\mathcal{O}_X, \mathcal{O}_X)$-linearity of the morphism ρ_n and $\rho_{n'}$ of 4.1.2.6.1, we get the well defined isomorphism

$$\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{\sim} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}) \otimes_{\mathcal{B}_X} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}) \quad (4.1.2.10.3)$$

given by $b \otimes \tau \otimes \tau' \mapsto (b \otimes \tau) \otimes (1 \otimes \tau')$, which is the isomorphism 4.1.2.10.1.

ii) It remains to check the linearity. The $\mathcal{B}_X$-linearity for the respective left structures is obvious. By using $\epsilon^{\mathcal{B}}_n \otimes \mathrm{id} = q_0^{n,n'*}(\epsilon^{\mathcal{B}_X}_{n+n'})$, we get that 4.1.2.10.3 is $\mathcal{B}_X$-linear for the middle structures thanks to the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{B}_X & \longrightarrow & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} & \underset{\sim}{\xrightarrow{\epsilon^{\mathcal{B}}_n \otimes \mathrm{id}}} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} & \overset{4.1.2.10.3}{\underset{\sim}{\searrow}} & \\
\| & & \uparrow{\scriptstyle \mathrm{id} \otimes \mathrm{id} \otimes p^{n'}_{0(m)}} & & \uparrow{\scriptstyle \mathrm{id} \otimes \mathrm{id} \otimes p^{n'}_{0(m)}} & & \\
\mathcal{B}_X & \longrightarrow & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X & \underset{\sim}{\xrightarrow{\epsilon^{\mathcal{B}}_n \otimes \mathrm{id}}} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} & \xrightarrow{\widetilde{p}^{n'}_{0(m)}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}
\end{array}$$

where the left top arrow is given by $b \mapsto 1 \otimes b \otimes 1$, the left bottom arrow is given by $b \mapsto 1 \otimes b$, the homomorphisms $p^{n'}_{0(m)}$ (resp. $\widetilde{p}^{n'}_{0(m)}$) is defined at 3.2.2.2 (resp. 4.1.2.5).

Now, let us check the morphism 4.1.2.10.1 is a morphism of $\mathcal{B}_X$-algebras for the right structures. In the following commutative diagram,

$$\begin{array}{ccccccccc}
\mathcal{B} & \longrightarrow & \mathcal{P}^{n'} \otimes_{\mathcal{O}} \mathcal{B} & \underset{\sim}{\xrightarrow{\epsilon^{\mathcal{B}}_{n'}}} & \mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^{n'} & \longrightarrow & (\mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{B}) \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^{n'}) & \underset{\sim}{\xrightarrow{\epsilon^{\mathcal{B}}_n \otimes \mathrm{id}}} & (\mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^n) \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^{n'}) \\
\| & & \downarrow & & \downarrow & \underset{\sim}{\nearrow} & & \underset{\sim}{\nearrow} & \\
\mathcal{B} & \rightarrow & \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{P}^{n'} \otimes_{\mathcal{O}} \mathcal{B} & \underset{\sim}{\xrightarrow{\mathrm{id} \otimes \epsilon^{\mathcal{B}}_{n'}}} & \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^{n'} & \underset{\sim}{\xrightarrow{\epsilon^{\mathcal{B}}_n \otimes \mathrm{id}}} & \mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{P}^{n'} & & \\
\| & & & & & & \| & & \\
\mathcal{B} & \rightarrow & \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{P}^{n'} \otimes_{\mathcal{O}} \mathcal{B} & & \underset{\sim}{\xrightarrow{\delta^{n,n'*}_{(m)}(\epsilon^{\mathcal{B}_X}_{n+n'})}} & & \mathcal{B} \otimes_{\mathcal{O}} \mathcal{P}^n \otimes_{\mathcal{O}} \mathcal{P}^{n'}, & &
\end{array}$$

(4.1.2.10.4)

where we omitted to indicate the indices "$X^\sharp$" or "$X^\sharp/S^\sharp(m)$", that of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$ is equal the composite morphism of the top, whereas that of $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}$ is the composite morphism of the bottom. By using the cocycle condition 3.4.2.1 (noticing that $\mathrm{id} \otimes \epsilon^{\mathcal{B}}_{n'} = q_1^{n,n'*}(\epsilon^{\mathcal{B}_X}_{n+n'})$ and $\epsilon^{\mathcal{B}}_n \otimes \mathrm{id} = q_0^{n,n'*}(\epsilon^{\mathcal{B}_X}_{n+n'})$), $\widetilde{p}_2^{n,n'}$ correspond to the path of 4.1.2.10.4 going through by the bottom. Hence we are done.

b) Since the other cases are treated similarly, let us only prove the cocartesian diagram 4.1.2.10.2 when $j = 2$. We have the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_X & \xrightarrow{p^{n,n'}_{2(m)}} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} & = & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \\
\| & & \uparrow\wr & & \uparrow\wr \\
\mathcal{O}_X & \longrightarrow & \left(\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}\right) \otimes_{\mathcal{O}_X} \mathcal{O}_X & \xrightarrow{\delta^{n,n'*}_{(m)}(\epsilon^{\mathcal{O}_X}_{n+n'})} & \mathcal{O}_X \otimes_{\mathcal{O}_X} \left(\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{B}_X & \longrightarrow & \left(\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}\right) \otimes_{\mathcal{O}_X} \mathcal{B}_X & \xrightarrow{\delta^{n,n'*}_{(m)}(\epsilon^{\mathcal{B}_X}_{n+n'})} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \left(\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}\right)
\end{array}$$

where the left bottom square and the bottom rectangle are cocartesian (and therefore so is the right bottom square). Since the bottom composite morphism is $\widetilde{p}^{n,n'}_{2(m)}$, then we are done. □

4.1.2.11. We introduce the following $\mathcal{B}_X$-algebra homomorphisms

$$\widetilde{\delta}^{n,n'}_{(m)} \colon \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{\mathrm{id} \otimes \delta^{n,n'}_{(m)}} \mathcal{B}_X \otimes_{\mathcal{O}_X} (\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}) \underset{\sim}{\xrightarrow{4.1.2.10.1}} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} ,$$

$$\widetilde{q_0}^{n,n'} \colon \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} \longrightarrow \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \longrightarrow \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)},$$

$$\widetilde{q_1}^{n,n'} \colon \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} \longrightarrow \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \longrightarrow \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}.$$

We have the canonical cocartesian diagram :

$$\begin{array}{ccc} \mathcal{P}^{n+n'}_{X^\sharp/S^\sharp(m)} & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}, \end{array} \tag{4.1.2.11.1}$$

where the top (resp. bottom) triple arrows are the morphisms $q_0^{n,n'}$, $q_1^{n,n'}$, $\delta^{n,n'}_{(m)}$ (resp. $\widetilde{q_0}^{n,n'}$, $\widetilde{q_1}^{n,n'}$, $\widetilde{\delta}^{n,n'}_{(m)}$). Moreover, these homomorphisms satisfy the analogous properties without the tildes. For instance, the morphism $\widetilde{\delta}^{n,n'}_{(m)}$ is $\mathcal{B}_X$-linear for the respective right structures. Indeed, this is by construction a consequence of 4.1.2.9.1.

By applying the functor $\mathcal{B}_X \otimes_{\mathcal{O}_X} -$ to the commutative diagram 3.2.2.18.1, we get (up to canonical isomorphism) the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{P}}^{n+n'+n''}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\widetilde{\delta}^{n+n',n''}_{(m)}} & \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n''}_{X^\sharp/S^\sharp(m)} \\ \downarrow{\scriptstyle \widetilde{\delta}^{n,n'+n''}_{(m)}} & & \downarrow{\scriptstyle \widetilde{\delta}^{n,n'}_{(m)} \otimes \mathrm{id}} \\ \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'+n''}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\mathrm{id} \otimes \widetilde{\delta}^{n',n''}_{(m)}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n''}_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.1.2.11.2}$$

To finish, we remark that the diagram similar to 4.1.2.11.2, with "$\widetilde{\delta}_{(m)}$" replaced by "$\widetilde{q_0}$" (resp. "$\widetilde{q_1}$"), is not commutative.

4.1.2.12. We will denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n} := \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X)$ the $\mathcal{B}_X$-linear dual for the left structure of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$. For $n' \geq n$, from the canonical projection $\psi^{n',n}_{X^\sharp/S^\sharp,(m)} \colon \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} \to \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$ (see 3.2.2.2.1), we get the canonical projections

$$\widetilde{\psi}^{n',n}_{X^\sharp/S^\sharp,(m)} := \mathrm{id}_{\mathcal{B}_X} \otimes \psi^{n',n}_{X^\sharp/S^\sharp,(m)} \colon \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}. \tag{4.1.2.12.1}$$

By construction, the map 4.1.2.12.1 is $\mathcal{B}_X$-linear for the left structure. In fact, since the m-PD stratification of $\mathcal{B}_X$ are compatible with respect to the morphism $\psi^{n',n}_{X^\sharp/S^\sharp,(m)}$ (see 3.4.2.1.a) then we get that 4.1.2.12.1 is $\mathcal{B}_X$-linear for the right structure. By taking the duality of 4.1.2.12.1 with respect to the left structures, this gives the monomorphisms

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n} \hookrightarrow \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n'}.$$

We endow the sheaf $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} := \cup_{n\in\mathbb{N}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n}$ with a ring structure by using the pairings

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n} \times \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n'} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n+n'} \tag{4.1.2.12.2}$$

defined as follows: if $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n}$, $P' \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n'}$, then $P \cdot P'$ is the composition homomorphism

$$\widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{\widetilde{\delta}^{n,n'}_{(m)}} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \xrightarrow{\mathrm{id} \otimes P'} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \xrightarrow{P} \mathcal{B}_X. \tag{4.1.2.12.3}$$

Similarly to 3.2.3.3 (e.g. use the commutative diagram 4.1.2.11.2), we can check this law of composition is associative and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is therefore a ring.

Remark 4.1.2.13. With the notations 4.1.2.11 and 4.1.2.12.1, we have the relation $\widetilde{q}^{n,n'}_{i(m)} = \widetilde{\varpi}^{n,n'}_{i(m)} \circ \widetilde{\psi}^{n+n',n}_{X^\sharp/S^\sharp,(m)}$ for $i = 0, 1$, where $\widetilde{\varpi}^{n,n'}_{0(m)} \colon \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp,(m)} \to \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)}$ and $\widetilde{\varpi}^{n,n'}_{1(m)} \colon \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)} \to \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)}$ are the homomorphisms corresponding to the projections.

4.1.2.14. From the canonical isomorphisms of $\mathcal{B}_X$-algebras

$$\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,\,n} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{n}_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X) = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\,n}, \qquad (4.1.2.14.1)$$

we get $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. The proposition below means that these two structures of $\mathcal{B}_X$-algebra are compatible.

Proposition 4.1.2.15. *The canonical isomorphism $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ (see 4.1.2.14), is an isomorphism of rings where the right (resp. left) term is endowed with the ring structure of 4.1.2.2 (resp. 4.1.2.12).*

Proof. Let $P \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{n}_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X) = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n}$ and of $P' \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X) = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X,n'}$. Let us denote by Q (resp. Q') the operator of $\mathcal{H}om_{\mathcal{B}_X}(\widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X)$ (resp. $\mathcal{H}om_{\mathcal{B}_X}(\widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X)$) associated to P (resp. P') via the isomorphisms 4.1.2.14.1.

Since $Q = \mu \circ (\mathrm{id} \otimes P)$, we reduce to check the commutativity of the following diagram.

$$\begin{array}{ccccccc}
\mathcal{P}^{n+n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\delta^{n,n'}_{(m)}} & \mathcal{P}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\mathrm{id}\otimes P'} & \mathcal{P}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X & \xrightarrow{\epsilon^{\mathcal{B}_X}_n} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X^\sharp/S^\sharp(m)} \\
\downarrow & & \downarrow & & & & \| \\
\widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\widetilde{\delta}^{n,n'}_{(m)}} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\mathrm{id}\otimes Q'} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{B}_X & \xrightarrow[\iota]{\sim} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}
\end{array} \qquad (4.1.2.15.1)$$

where the isomorphism ι is the canonical one. It follows from 4.1.2.11.1 that the left square is commutative.

Let $\tau \in \mathcal{P}^{n}_{X^\sharp/S^\sharp(m)}$ and $\tau \in \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}$. Since $\epsilon^{\mathcal{B}_X}_n$ is a morphism of $\mathcal{P}^{n}_{X^\sharp/S^\sharp(m)}$-algebras,

$$\epsilon^{\mathcal{B}_X}_n \circ (\mathrm{id} \otimes P')(\tau \otimes \tau') = \epsilon^{\mathcal{B}_X}_n(\tau \otimes P'(\tau')) = (1 \otimes \tau) \times \epsilon^{\mathcal{B}_X}_n(1 \otimes P'(\tau')).$$

We put $\widetilde{\tau} := 1 \otimes \tau \in \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}$, $\widetilde{\tau}' := 1 \otimes \tau' \in \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}$. We have the relation

$$\iota \circ (\mathrm{id} \otimes Q')(\widetilde{\tau} \otimes \widetilde{\tau}') = \widetilde{\tau} \cdot P'(\tau'),$$

the term $P'(\tau')$ acting on $\widetilde{\tau}'$ for the right structure of $\mathcal{B}_X$-algebra of $\widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}$. By definition, then we have

$$\widetilde{\tau} \cdot P'(\tau') = \widetilde{\tau} \times \epsilon^{\mathcal{B}_X}_n(1 \otimes P'(\tau')).$$

This yields the commutativity of the right rectangle of 4.1.2.15.1. □

4.1.2.16 (Local description and notation)**.** Suppose in this paragraph that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,r}$.

(a) For any $\lambda = 1, \ldots, r$, put $\tau_{\sharp\lambda(m)} := \mu^{n}_{(m)}(u_\lambda) - 1$ (or simply $\tau_{\sharp\lambda}$), where for any $a \in M_{X^\sharp}$ $\mu^{n}_{(m)}(a)$ is the unique section of $\ker(\mathcal{O}^{*}_{\Delta^{n}_{X^\sharp/S^\sharp,(m)}} \to \mathcal{O}^{*}_{X})$ such that we get in $M^{n}_{X^\sharp/S^\sharp,(m)}$ the equality $p^{n*}_1(a) = p^{n*}_0(a)\mu^{n}_{(m)}(a)$. We still denote by $\tau_{\sharp\lambda(m)}$ its image via the canonical morphism $\mathcal{P}^{n}_{X^\sharp/S^\sharp,(m)} \to \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp,(m)}$. Taking the limits to 3.2.2.4, we get the isomorphism of m-PD-$\mathcal{B}_X$-algebras

$$\begin{aligned} \mathcal{B}_X\langle T_1, \ldots, T_r\rangle_{(m),n} &\xrightarrow{\sim} \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp,(m)} \\ T_\lambda &\mapsto \tau_{\sharp\lambda,(m)}, \end{aligned} \qquad (4.1.2.16.1)$$

where the first term is defined as in 1.3.3.6. In particular, the elements $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}|\le n}$ form a $\mathcal{B}_X$-basis of $\widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp,(m)}$. The corresponding dual basis of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}$ will still be denoted by $\{\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}\}_{|\underline{k}|\le n}$.

Let $\epsilon_1, \dots, \epsilon_r$ be the canonical basis of $\mathbb{N}^r$, i.e. the coordinates of ϵ_i are 0 except for the ith term which is 1. We put $\partial_{\sharp i} := \underline{\partial}_{\sharp}^{\langle \epsilon_i \rangle_{(m)}}$. A section $P \in \Gamma(\mathfrak{X}, \mathcal{B}_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$ can uniquely be written of the form

$$P = \sum_{\underline{k} \in \mathbb{N}^d} b_{\underline{k}} \underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m)}}, \tag{4.1.2.16.2}$$

where $b_{\underline{k}} \in B_X$ and the sum is finite.

(b) Let $(t_\lambda)_{\lambda=1,\dots,r}$ be the coordinates of Y/S induced by $(u_\lambda)_{\lambda=1,\dots,r}$. Set $\mathcal{B}_Y := \mathcal{B}_X|Y$. For any $\lambda = 1, \dots, r$, set $\tau_\lambda := p_1^*(t_\lambda) - p_0^*(t_\lambda)$. We still denote by τ_λ its image in $\mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y/S,(m)}$. It follows from 1.3.3.11, the elements $\{\underline{\tau}^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}| \leq n}$ form a basis of the $\mathcal{B}_Y$-module $\mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y/S,(m)}$. The corresponding dual basis of $\mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y/S,n}$ is denoted by $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}\}_{|\underline{k}| \leq n}$. A section $P \in \Gamma(\mathfrak{X}, \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y/S})$ can uniquely be written of the form

$$P = \sum_{\underline{k} \in \mathbb{N}^d} b_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}, \tag{4.1.2.16.3}$$

where $b_{\underline{k}} \in B_Y$ and the sum is finite.

Proposition 4.1.2.17. *Suppose m is finite. We have the following properties.*

(a) The graded ring (associated to the order filtration) $\operatorname{gr} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a commutative ring. If $X^\sharp/S^\sharp$ is endowed with logarithmic coordinates, the relation $\underline{\partial}_{\sharp}^{\langle \underline{k} \rangle} \underline{\partial}_{\sharp}^{\langle \underline{h} \rangle} = \left\langle \begin{smallmatrix} \underline{k}+\underline{h} \\ \underline{k} \end{smallmatrix} \right\rangle \underline{\partial}_{\sharp}^{\langle \underline{k}+\underline{h} \rangle}$ becomes exact in $\operatorname{gr} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$.

(b) Suppose $X^\sharp$ is endowed with logarithmic coordinates. Then for any integers $1 \leq i \leq d$ and $k \in \mathbb{N}$, the operator $\partial_{\sharp i}^{\langle k \rangle_{(m)}}$ belongs to the $\mathbb{Z}_{(p)}$-algebra generated by the $m+1$ operators $\partial_{\sharp i}^{[p^j]} = \partial_{\sharp i}^{\langle p^j \rangle_{(m)}}$ where $0 \leq j \leq m$. The operators $\partial_{\sharp i}^{\langle k \rangle_{(m)}}$ and $\partial_{\sharp i'}^{\langle k' \rangle_{(m)}}$ commutes and the sheaf $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is generated as $\mathbb{Z}_{(p)}$-algebra by $\mathcal{B}_X$ and by the operators $\partial_{\sharp i}^{\langle p^j \rangle_{(m)}}$, where $1 \leq i, i' \leq d$ and $0 \leq j \leq m$.

(c) For any affine open subscheme $U \subset X$, the canonical homomorphism

$$\Gamma(U, \mathcal{B}_X) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \to \Gamma(U, \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \tag{4.1.2.17.1}$$

is an isomorphism.

(d) Suppose there exist a basis of affine opens $\mathfrak{B}$ of X such that for any $U \in \mathfrak{B}$, the ring $\Gamma(U, \mathcal{B}_X)$ is noetherian. Then for any $U \in \mathfrak{B}$, the rings $\Gamma(U, \operatorname{gr} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\Gamma(U, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ (resp. $\operatorname{gr} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,x}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,x}$ for any $x \in X^\sharp$) are right and left noetherian.

(e) Suppose that there exists a basis of affine opens $\mathfrak{B}$ of X such that

(i) For any $U \in \mathfrak{B}$, the ring $\Gamma(U, \mathcal{B}_X)$ is noetherian ;

(ii) For any $U, V \in \mathfrak{B}$ such that $V \subset U$, the homomorphism $\Gamma(U, \mathcal{B}_X) \to \Gamma(V, \mathcal{B}_X)$ is flat.

Then for any $U, V \in \mathfrak{B}$ such that $V \subset U$, the canonical morphism $\Gamma(U, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to \Gamma(V, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ is flat. Moreover, the sheaf of rings $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is right and left coherent.

Proof. We check (a) (resp. (b)) as 3.2.3.16.(a) (resp. 3.2.3.15). Let us check (c). Since $\mathcal{D}^{(m)}_{U^\sharp/S^\sharp,n}$ is a locally free $\mathcal{O}_U$-module of finite type, since U is affine, then $\mathcal{D}^{(m)}_{U^\sharp/S^\sharp,n}$ is a direct summand of a free $\mathcal{O}_U$-module of finite type. Replacing in 4.1.2.17.1 the sheaf $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ by a free $\mathcal{O}_U$-module of finite type (and therefore by $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$), we get an isomorphism. Since U is affine, then U is a coherent topological space and $\Gamma(U, -)$ commutes with filtered inductive limits (see [SGA4.2, VI.5.3]). Hence, we get (c). Moreover, we can check (d) as 3.2.3.16.(b). Finally, by using (c), we get the flatness of the canonical morphism $\Gamma(U, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to \Gamma(V, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. With 1.4.5.2, we are done. □

4.1.3 Good filtrations, Theorems A and B for coherent $\mathcal{D}$-modules

We keep notation 4.1.2 but we suppose the level is finite, i.e. $m \in \mathbb{N}$. Let $\mathfrak{B}$ be the category of affine opens of X. We suppose

(i) For any $U \in \mathfrak{B}$, the ring $\Gamma(U, \mathcal{B}_X)$ is noetherian ;

(ii) For any $U, V \in \mathfrak{B}$ such that $V \subset U$, the homomorphism $\Gamma(U, \mathcal{B}_X) \to \Gamma(V, \mathcal{B}_X)$ is flat.

Recall that it follows from 1.4.5.2 that $\mathcal{B}_X$ is a coherent sheaf of rings. Moreover, we suppose that $\mathcal{B}_X$ satisfies theorems A and B for coherent modules in the sense 1.4.3.14 below (remark following 4.1.3.2 this is automatic in the algebraic case, i.e., when X is a scheme):

Remark 4.1.3.1. This subsection remains valid by replacing (left) modules by right modules. Moreover, we can extend the results of this subsection in the case where $\mathfrak{B}$ is replaced by a basis of affine opens of X: in that case we need to replace everywhere "affine open of X" by "object of $\mathfrak{B}$".

In the case of schemes, we have the well known following example.

Proposition 4.1.3.2 (Theorems A and B for quasi-coherent $\mathcal{D}$-module over schemes)**.** *Suppose X is a scheme. $\mathcal{D}$ be a sheaf of rings on X such that, for any affine open $U \subset X$, $\Gamma(U, \mathcal{D})$ is a noetherian ring. We suppose there is a morphism $\mathcal{O}_X \to \mathcal{D}$ such that the left multiplication by the sections of $\mathcal{O}_X$ makes $\mathcal{D}$ a quasi-coherent $\mathcal{O}_X$-module. We have therefore the following properties.*

(a) The sheaf $\mathcal{D}$ is a left coherent.

(b) For a left $\mathcal{D}$-module $\mathcal{M}$ to be coherent, it is necessary and sufficient that both conditions are fulfilled

(i) $\mathcal{M}$ is quasi-coherent and that

(ii) there exists a covering $(U_\lambda)_{\lambda \in \Lambda}$ of X by affine opens such that for any $\lambda \in \Lambda$, $\Gamma(U_\lambda, \mathcal{M})$ is a left $\Gamma(U_\lambda, \mathcal{D})$-module of finite type.

(c) Suppose X is affine, and $D := \Gamma(X, \mathcal{D})$.

(i) The functors $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ and $M \mapsto \mathcal{D} \otimes_D M$ are exact quasi-inverse equivalences of categories between coherent left $\mathcal{D}$-modules and left D-modules of finite type (resp. quasi-coherent left $\mathcal{D}$-modules and left D-modules).

(ii) For any quasi-coherent $\mathcal{D}$-module $\mathcal{M}$, we have $H^q(X, \mathcal{M}) = 0$ for all $q \geq 1$.

Proof. The assertion (a) is a consequence of 1.4.5.2. We know that any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ satisfyies $H^q(X, \mathcal{M}) = 0$ for all $q \geq 1$. Since $\mathcal{D}$ is quasi-coherent, then the functors $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ and $M \mapsto \mathcal{D} \otimes_D M$ are exact quasi-inverse equivalences of categories between quasi-coherent left $\mathcal{D}$-modules and left D-modules. Hence, by using the five lemma and their exactness, the functors $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ and $M \mapsto \mathcal{D} \otimes_D M$ are exact quasi-inverse equivalences of categories between left $\mathcal{D}$-modules having a global finite presentation and left D-modules of finite type.

Suppose $X = \operatorname{Spec} A$ is affine and let $\mathcal{M}$ be a quasi-coherent left $\mathcal{D}$-module such that there exists a covering $(U_\lambda)_{\lambda \in \Lambda}$ of X by affine opens such that for any $\lambda \in \Lambda$, $\Gamma(U_\lambda, \mathcal{M})$ is a left $\Gamma(U_\lambda, \mathcal{D})$-module of finite type. To finish the proof, it remains to check that $\Gamma(X, \mathcal{M})$ is a left D-modules of finite type, which is standard (e.g. compare with [Har77, II.3.2]) and is proved as follows. Since X is noetherian, we can suppose $\Lambda = \{1, 2, \ldots, r\}$ and that there exists $f_\lambda \in A$ such that $U_\lambda = D(f_\lambda)$. Set $M := \Gamma(X, \mathcal{M})$, $D_{f_\lambda} := A_{f_\lambda} \otimes_A D$, $M_{f_\lambda} := D_{f_\lambda} \otimes_D M$. By hypothesis, M_{f_λ} is a D_{f_λ}-module of finite type. If N is a sub D-module of M, then $N = M$ if and only if for any λ we have $M_{f_\lambda} = N_{f_\lambda}$. Since D_{f_λ} are noetherian, we conclude that M is noetherian.

□

4.1.3.3. Let $\mathcal{D}$ be a sheaf of rings on X which satisfies theorems A and B for coherent modules. Suppose X affine and set $D := \Gamma(X, \mathcal{D})$. Then the quasi-inverse equivalences of categories $\Gamma(X, -)$ and $\mathcal{D} \otimes_D -$ between the category of coherent $\mathcal{D}$-modules and that of coherent D-modules are exact. Then, we have the following (non-exhaustive) properties:

(a) Let $\phi\colon \mathcal{E}\to\mathcal{F}$ be a morphism of left coherent $\mathcal{D}$-modules. Denote by $D:=\Gamma(X,\mathcal{D})$, $E:=\Gamma(X,\mathcal{E})$, $F:=\Gamma(X,\mathcal{F})$ and by $f\colon E\to F$ the induced morphism of left D-modules. Then $\Gamma(X,\operatorname{Ker}\phi)=\operatorname{Ker} f$, $\Gamma(X,\operatorname{Coker}\phi)=\operatorname{Coker} f$. Moreover, f is a monomorphism (resp. epimorphism) if and only if ϕ is a monomorphism (resp. epimorphism).

(b) Let $\mathcal{E},\mathcal{F}$ be two coherent sub-$\mathcal{D}$-modules of a coherent $\mathcal{D}$-module $\mathcal{G}$. Then $\mathcal{E}+\mathcal{F}$ (resp. $\mathcal{E}\cap\mathcal{F}$, resp. $\mathcal{E}\cup\mathcal{F}$) is a coherent sub-$\mathcal{D}$-modules of $\mathcal{G}$ and $\Gamma(X,\mathcal{E}+\mathcal{F})=\Gamma(X,\mathcal{E})+\Gamma(X,\mathcal{F})$ (resp. $\Gamma(X,\mathcal{E}\cap\mathcal{F})=\Gamma(X,\mathcal{E})\cap\Gamma(X,\mathcal{F})$, resp. $\Gamma(X,\mathcal{E}\cup\mathcal{F})=\Gamma(X,\mathcal{E})\cup\Gamma(X,\mathcal{F})$).

(c) Let $\mathcal{E}$ be a $\mathcal{D}$-module, let $(\mathcal{E})_{n\in\mathbb{N}}$ be an exhaustive increasing filtration of $\mathcal{E}$ by coherent left $\mathcal{D}$-module. Then the canonical morphism $\mathcal{D}\otimes_D\Gamma(X,\mathcal{E}_n)\to\mathcal{E}_n$ are isomorphisms and the morphisms $\Gamma(X,\mathcal{E}_n)\to\Gamma(X,\mathcal{E}_{n+1})$ are injective. Since filtered inductive limits commute to the global section functor $\Gamma(X,-)$, we then obtain the canonical isomorphisms $\varinjlim_{n\in\mathbb{N}}\Gamma(X,\mathcal{E}_n)\xrightarrow{\sim}\Gamma(X,\mathcal{E})$ and the canonical morphism $\Gamma(X,\mathcal{E}_n)\to\Gamma(X,\mathcal{E})$ is therefore injective. Hence, $(\Gamma(X,\mathcal{E}_n))_{n\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{E}$.

(d) Suppose $\mathcal{D}$ is commutative and let $\mathcal{E},\mathcal{F}$ be two coherent $\mathcal{D}$-modules. Then $\mathcal{E}\otimes_{\mathcal{D}}\mathcal{F}$ is coherent $\mathcal{D}$-module and $\Gamma(X,\mathcal{E}\otimes_{\mathcal{D}}\mathcal{F})=\Gamma(X,\mathcal{E})\otimes_D\Gamma(X,\mathcal{F})$.

Notation 4.1.3.4. For any integers n and n', we put $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'}$ as the image via the $\mathcal{B}_X$-linear homomorphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'}\to\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n+n'}$. Since this is a morphism of coherent $\mathcal{B}_X$-modules, then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'}$ is a coherent $\mathcal{B}_X$-module.

Suppose X is affine. For any integers n,n', set $D^{(m)}_{X^\sharp/S^\sharp,n}:=\Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})$ and define similarly $D^{(m)}_{X^\sharp/S^\sharp,n}\cdot D^{(m)}_{X^\sharp/S^\sharp,n'}$. With 4.1.3.3 we get

$$\Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'})=\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n'}.\tag{4.1.3.4.1}$$

Proposition 4.1.3.5. For $n<0$, we set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}:=0$. For any couple $(r,s)\in\mathbb{N}^2$,

$$\sum_{j=0}^{p^m-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r-j}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,s+j}=\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r+s}.$$

Proof. Since this is local, we can suppose $\mathfrak{X}$ affine and endowed with logarithmic coordinates. Since the sheaves $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r-j}$, $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,s+j}$ et $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r+s}$ are $\mathcal{B}_X$-coherent, then so is the sheaf $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r-j}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,s+j}$. Hence, we have the inclusion of coherent $\mathcal{B}_X$-modules:

$$\sum_{j=0}^{p^m-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r-j}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,s+j}\subset\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r+s}.$$

Since Theorems A and B for coherent modules hold for $\mathcal{B}_X$, with 4.1.3.3, 4.1.3.4.1 and its notation, it remains to prove that an element of $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,r+s}$ belongs to $\sum_{j=0}^{p^m-1}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,r-j}\cdot\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,s+j}$. By $\mathcal{B}_X$-linearity and since for any $\underline{k}\in\mathbb{N}^d$ we have $\underline{\partial}_\sharp^{\langle\underline{k}\rangle}=\prod_{i=1,\dots,d}\partial_{\sharp,i}^{\langle k_i\rangle}$, we reduce to consider the differential operators of the form $\partial_{\sharp i}^{\langle k\rangle}$, for some i with $0\le k\le r+s$. We proceed by induction on $0\le k\le r+s$ to check that

$$\partial_{\sharp i}^{\langle k\rangle}\in\sum_{j=0}^{p^m-1}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,r-j}\cdot\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,s+j}.$$

When $k\le r$, this is a consequence of $\partial_{\sharp i}^{\langle k\rangle}\in\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,r}$ and $1\in\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,s}$. Hence, we reduce to suppose $r\le k\le r+s$. Set $k=\sum_{l=0}^{m-1}a_lp^l+ap^m$, with $0\le a_l<p$. By using the induction hypothesis, it follows from the formula 3.2.3.14.3 that we reduce to check

$$\left(\prod_{l=0}^{m-1}(\partial_{\sharp i}^{[p^l]})^{a_l}\right)(\partial_{\sharp i}^{[p^m]})^a\in\sum_{j=0}^{p^m-1}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,r-j}\cdot\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,s+j}.\tag{4.1.3.5.1}$$

Let j be the rest of the euclidian division of r by p^m and q be its quotient. The element $(\partial_{\sharp i}^{[p^m]})^q$ belongs therefore to $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,r-j}$. Moreover, since $r \leq k \leq r+s$, then $s+j \geq k-(r-j) = \sum_{l=0}^{m-1} a_l p^l + (a-q)p^m$. The element $\left(\prod_{l=0}^{m-1}(\partial_{\sharp i}^{[p^l]})^{a_l}\right)(\partial_{\sharp i}^{[p^m]})^{a-q}$ belongs therefore to $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,s+j}$. Hence, we are done. $\square$

Notation 4.1.3.6. Let $\mathcal{M}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Let $\mathcal{M}'$ be a sub-$\mathcal{B}_X$-module of $\mathcal{M}$. For any integer r, we denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r} \cdot \mathcal{M}'$ is the image of the map $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r} \otimes_{\mathcal{B}_X} \mathcal{M}' \to \mathcal{M}$.

Définition 4.1.3.7. Let $\mathcal{M}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. A *filtration* of $\mathcal{M}$ (as $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module) is a family $(\mathcal{M}_r)_{r\in\mathbb{N}}$ of sub-$\mathcal{B}_X$-modules of $\mathcal{M}$ satisfying:

(a) For any $r,\ s \in \mathbb{N}$: $\mathcal{M}_r \subset \mathcal{M}_{r+1}$, and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r} \cdot \mathcal{M}_s \subset \mathcal{M}_{r+s}$,

(b) $\mathcal{M} = \cup_{r\in\mathbb{N}}\mathcal{M}_r$.

Définition 4.1.3.8. Let $\mathcal{M}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module endowed with a filtration $(\mathcal{M}_r)_{r\in\mathbb{N}}$. The filtration $(\mathcal{M}_r)_{r\in\mathbb{N}}$ is *good* (resp. *quasi-good*) if the following two conditions hold (resp. the following first condition holds):

(a) For any $r \in \mathbb{N}$, $\mathcal{M}_r$ is $\mathcal{B}_X$-coherent ;

(b) There exists an integer $r_1 \in \mathbb{N}$ such that for any integer $r \geq r_1$, we have

$$\mathcal{M}_r = \sum_{j=0}^{p^m-1} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r-r_1+j} \cdot \mathcal{M}_{r_1-j}.$$

Example 4.1.3.9. The family $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r})_{r\in\mathbb{N}}$ is a good filtration of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. Indeed, following proposition 4.1.3.5, the condition (b) is satisfied for any integer r_1. The condition (a) being also valid, this filtration is therefore good. We call it the *order filtration*.

Proposition 4.1.3.10. *Suppose X is quasi-compact. Let $\mathcal{M}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module endowed with two good filtrations $(\mathcal{M}_r)_{r\in\mathbb{N}}$ and $(\mathcal{M}'_r)_{r\in\mathbb{N}}$. Then there exists $k \in \mathbb{N}$ such that for any $r \geq k$ we have*

$$\mathcal{M}_{r-k} \subset \mathcal{M}'_r \subset \mathcal{M}_{r+k}. \tag{4.1.3.10.1}$$

Proof. Since the filtrations are good, there exists r_1 such that the condition (b) of 4.1.3.8 is satisfied for both filtrations. Hence, we reduce to check the inclusions 4.1.3.10.1 for $r \leq r_1 + p^m$ for some k large enough, which easily follows from the condition (a) of 4.1.3.8 and the quasi-compactness of X. $\square$

Proposition 4.1.3.11 (Theorem B)**.** *Suppose X affine and let $\mathcal{M}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module admitting a quasi-good filtration. We have the following properties.*

(a) For any $U, V \in \mathfrak{B}$ such that $V \subset U$, the homomorphism $\Gamma(U, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to \Gamma(V, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ is flat.

(b) For any integer $i > 0$, $H^i(X, \mathcal{M}) = 0$.

Proof. Since filtered inductive limits commute to the functors $H^i(X, -)$ for any $i \geq 0$, since $\mathcal{B}_X$ is a sheaf of rings on X which satisfies theorems A and B for coherent modules, then we are done. $\square$

We will need the following lemmas to check Proposition 4.1.3.16.

Lemma 4.1.3.12. *Let ϕ: $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^r \to (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear morphism. There exists an integer $n_0 \geq 0$ such that for any $l \in \mathbb{N}$ we have*

$$\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r) \subset (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l+n_0})^s. \tag{4.1.3.12.1}$$

Proof. Since this is local, we can suppose X is affine and $X^\sharp/S^\sharp$ is endowed with logarithmic coordinates. The data of a morphism $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^r \to (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s$ is equivalent to that of a family of global sections $(P_{i,j})_{i=1,\dots r; i=1,\dots,s}$ of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. The integer n_0 equal to the maximum order of the differential operators $(P_{i,j})_{i=1,\dots r; i=1,\dots,s}$ then fits. □

Lemma 4.1.3.13. *Let X be a coherent topological space. Let I be a small filtered category, let J be a finite category and $\mathcal{E}\colon I \times J^{op} \to \mathrm{Sh}(X)$ be a functor also denoted by $(i,j) \mapsto \mathcal{E}_{i,j}$. Then the canonical morphism*

$$\varinjlim_{i\in I}\varprojlim_{j\in J}\mathcal{E}_{i,j} \to \varprojlim_{j\in J}\varinjlim_{i\in I}\mathcal{E}_{i,j} \qquad (4.1.3.13.1)$$

is an isomorphism. In other words, filtered inductive limits of sheaves on X of sets commute to finite projective limits.

Proof. Let U be a coherent open subset of X. Let $\mathcal{E}_j := \varinjlim_{i\in I}\mathcal{E}_{i,j}$, $E_{i,j} := \Gamma(U, \mathcal{E}_{i,j})$ and $E_j := \varinjlim_{i\in I} E_{i,j}$. We have the canonical maps $\alpha_{ij}\colon \mathcal{E}_{i,j} \to \mathcal{E}_i$ and $\beta_{ij}\colon E_{i,j} \to E_j$. Recall the morphism 4.1.3.13.1 that we will denote by α is the one induced by the family of morphisms

$$\alpha_i := \varprojlim_{j\in J}\alpha_{ij}\colon \varprojlim_{j\in J}\mathcal{E}_{i,j} \to \varprojlim_{j\in J}\mathcal{E}_j$$

(use the universal property of the inductive limits indexed by I). Similarly, we set the family of morphisms $\beta_i := \varprojlim_{j\in J}\beta_{ij}\colon \varprojlim_{j\in J} E_{i,j} \to \varprojlim_{j\in J} E_j$ which induces $\beta\colon \varinjlim_{i\in I}\varprojlim_{j\in J} E_{i,j} \to \varprojlim_{j\in J}\varinjlim_{i\in I} E_{i,j}$. We have the commutative diagram

$$\begin{array}{ccc}
\Gamma(U, \varprojlim_{j\in J}\mathcal{E}_{ij}) & \xrightarrow{\Gamma(U,\alpha_i)} & \Gamma(U, \varprojlim_{j\in J}\mathcal{E}_j) \\
\downarrow\sim & & \downarrow\sim \\
\varprojlim_{j\in J}\Gamma(U, \mathcal{E}_{ij}) & \xrightarrow{\varprojlim_{j\in J}\Gamma(U,\alpha_{ij})} & \varprojlim_{j\in J}\Gamma(U, \mathcal{E}_j)
\end{array} \qquad (4.1.3.13.2)$$

Since U is coherent and I is filtered, then $\Gamma(U, -)$ and $\varinjlim_{i\in I}$ commutes (see [SGA4.2, VI.5.2]) and this yields the canonical commutative diagram

$$\begin{array}{ccc}
\Gamma(U, \mathcal{E}_{ij}) & \xrightarrow{\Gamma(U,\alpha_{ij})} & \Gamma(U, \mathcal{E}_j) \\
\| & & \downarrow\sim \\
E_{ij} & \xrightarrow{\beta_{ij}} & E_j.
\end{array} \qquad (4.1.3.13.3)$$

By applying the functor $\varprojlim_{j\in J}$ to the square 4.1.3.13.3, we get a square whose composition with 4.1.3.13.2 is the following diagram:

$$\begin{array}{ccc}
\Gamma(U, \varprojlim_{j\in J}\mathcal{E}_{ij}) & \xrightarrow{\Gamma(U,\alpha_i)} & \Gamma(U, \varprojlim_{j\in J}\mathcal{E}_j) \\
\downarrow\sim & & \downarrow\sim \\
\varprojlim_{j\in J} E_{ij} & \xrightarrow{\beta_i} & \varprojlim_{j\in J} E_j.
\end{array} \qquad (4.1.3.13.4)$$

Next, we obtain from 4.1.3.13.4 the commutative bottom square of the diagram

$$\begin{array}{ccc}
\Gamma(U, \varinjlim_{i\in I}\varprojlim_{j\in J}\mathcal{E}_{ij}) & \xrightarrow{\Gamma(U,\alpha)} & \Gamma(U, \varprojlim_{j\in J}\varinjlim_{i\in I}\mathcal{E}_{ij}) \\
\downarrow\sim & & \| \\
\varinjlim_{i\in I}\Gamma(U, \varprojlim_{j\in J}\mathcal{E}_{ij}) & \longrightarrow & \Gamma(U, \varprojlim_{j\in J}\mathcal{E}_j) \\
\downarrow\sim & & \downarrow\sim \\
\varinjlim_{i\in I}\varprojlim_{j\in J} E_{ij} & \xrightarrow{\beta} & \varprojlim_{j\in J}\varinjlim_{i\in I} E_{i,j},
\end{array} \qquad (4.1.3.13.5)$$

where the top commutative square comes from the functoriality of the commutation of $\Gamma(U,-)$ with $\varinjlim_{i\in I}$. Since the bottom morphism is an isomorphism (see [KS06, 3.1.6]), then we are done. $\square$

Lemma 4.1.3.14. *Let $\mathcal{M}$ be a coherent $\mathcal{B}_X$-module, and $(\mathcal{M}_l)_{l\in\mathbb{N}}$ an increasing sequence of coherent sub-$\mathcal{B}_X$-modules of $\mathcal{M}$. Then the sequence $\mathcal{M}_l$ is stationary.*

Proof. Since X is quasi-compact, the proposition is local on X. We can therefore assume that X is affine. The lemma then follows from the theorem A for coherent $\mathcal{B}_X$-modules and of the fact that the ring $\Gamma(X,\mathcal{B}_X)$ is noetherian. $\square$

Lemma 4.1.3.15. *Let $\phi\colon \mathcal{E}\to\mathcal{F}$ be a morphism of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. Let $\mathcal{E}'$ be a sub-$\mathcal{B}_X$-module of $\mathcal{E}$. Then we have the equality of $\mathcal{B}_X$-modules:*

$$\phi\left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r}\cdot\mathcal{E}'\right)=\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r}\cdot\phi(\mathcal{E}') \tag{4.1.3.15.1}$$

Proof. Since ϕ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear, we get the commutative square

$$\begin{array}{ccccc}
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r}\otimes_{\mathcal{B}_X}\mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}\\
\downarrow & & & & \|\\
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,r}\otimes_{\mathcal{B}_X}\phi(\mathcal{E}') & & \longrightarrow & & \mathcal{F}
\end{array}$$

whose right arrow is surjective. Hence, we are done by definition (see notation 4.1.3.6). $\square$

Proposition 4.1.3.16. A $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module having a global finite presentation admits a good filtration.

Proof. Choose a finite presentation of $\mathcal{E}$ of the form

$$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^r\xrightarrow{\phi}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s\xrightarrow{\psi}\mathcal{E}\to 0.$$

Set $\mathcal{E}_n:=\psi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s)$. By $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity of ψ, by using 4.1.3.15.1 we get the first equality:

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\psi\left((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'})^s\right)=\psi\left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\left((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'})^s\right)\right)=\psi\left(\left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\cdot\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n'}\right)^s\right).$$

With 4.1.3.5, this yields that the family $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is a filtration of $\mathcal{E}$ satisfying the condition (b) of the definition 4.1.3.8 for any integer r_1. As for condition (a), this is equivalent to saying that for any integer n, the $\mathcal{B}_X$-modules $\mathrm{Im}(\phi)\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s$ are coherent. Let $n_0\geq 0$ be an integer satisfying the inclusion 4.1.3.12.1 for any $l\in\mathbb{N}$. Since $\mathcal{B}_X$ is coherent, then the image of a $\mathcal{B}_X$-linear morphism between two coherent $\mathcal{B}_X$-module is coherent. Hence, $\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r)$ is $\mathcal{B}_X$-coherent. Since $\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r)$ and $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s$ are coherent sub-$\mathcal{B}_X$-modules of the coherent $\mathcal{B}_X$-module $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l+n+n_0})^s$, then the $\mathcal{B}_X$-module $\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r)\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s$ is coherent. Since $\mathrm{Im}\phi=\cup_{l\in\mathbb{N}}\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r)$ and that filtered inductive limits of sheaves on X commute to finite projective limits (see 4.1.3.13), we get

$$\mathrm{Im}\phi\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s=\bigcup_{l\in\mathbb{N}}\left(\phi\left((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r\right)\cap\left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\right)^s\right). \tag{4.1.3.16.1}$$

By applying the lemma 4.1.3.14 to $\mathcal{M}=(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s$ and $\mathcal{M}_l=\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r)\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s$, we obtain an integer l_0 such that

$$\bigcup_{l\in\mathbb{N}}\left(\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l})^r)\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s\right)=\phi((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l_0})^r)\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s.$$

With 4.1.3.16.1, this implies that $\mathrm{Im}\phi\cap(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^s$ is $\mathcal{B}_X$-coherent. Hence we are done. $\square$

In the algebraic case, we can improve 4.1.3.16:

Proposition 4.1.3.17. *Suppose we are in the algebraic case of 3.4. A coherent $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\mathcal{F}$ admits a good filtration.*

Proof. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then it is the inductive limit of its coherent $\mathcal{O}_X$-submodules (see [Gro60, 9.4.9]), there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{G}$ of $\mathcal{F}$ such that the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear map $\varpi\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{G} \twoheadrightarrow \mathcal{F}$ is surjective. This yields the filtration $(\mathcal{F}_l)_{l\in\mathbb{N}}$ by setting $\mathcal{F}_l = \varpi(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,l} \otimes_{\mathcal{O}_X} \mathcal{G})$. It follows from 4.1.3.5 that this is a good filtration. □

4.1.3.18. Suppose X is affine. Set $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} := \Gamma(X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n} := \Gamma(X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})$, $B_X := \Gamma(X, \mathcal{B}_X)$. Since the ring sheaf $\mathcal{B}_X$ verifies theorem A and B for coherent modules, with 4.1.3.3, the sequence $(\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n})_{n\in\mathbb{N}}$ endowed $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$ with a filtration (i.e., in the definition 4.1.3.7 we replace $\mathcal{D}$ by D). Moreover, the canonical morphism $\Gamma(X, \mathcal{B}_X \otimes_B \widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n}) \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}$ is an isomorphism. The commutation of filtered inductive limits with $\Gamma(X,-)$ and tensor products then gives us the isomorphisms:

$$\mathcal{B}_X \otimes_{B_X} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} \mathcal{B}_X \otimes_{B_X} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n} \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.1.3.18.1}$$

Theorem 4.1.3.19 (Theorem A)**.** *Suppose X is affine. Then the functors $\mathcal{E} \mapsto \Gamma(X,\mathcal{E})$ and $E \mapsto \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} E$ are canonical quasi-inverse equivalences between the category of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules globally of finite presentation and the category of $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-modules of finite type.*

Proof. 1) Let E be a $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module of finite type. Since $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$ is a Noetherian ring (see 4.1.2.17), then E is a $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module of finite presentation. The $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} E$ is therefore globally of finite presentation, i.e., the second functor of the proposition is therefore well defined.

2) Let $(\widetilde{D}^{(m)}_{X^\sharp/S^\sharp})^r \xrightarrow{\epsilon} E$ be a surjective morphism of left $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-modules. We now show that the canonical morphism

$$E \to \Gamma(X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} E).$$

is an isomorphism. Let us then note E_n the image of $(\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n})^r$ by ϵ and $B := \Gamma(X, \mathcal{B}_X)$. The module E is therefore equal to the union of the B-modules E_n which are of finite type. As the ring sheaf $\mathcal{B}_X$ verifies theorem A, then the functorial in E_n canonical morphism $E_n \to \Gamma(X, \mathcal{B}_X \otimes_B E_n)$ is an isomorphism. Commutation of filtered inductive limits to tensor products and to the functor $\Gamma(X,-)$ then gives us the canonical isomorphisms

$$\Gamma(X, \mathcal{B}_X \otimes_B E) \xrightarrow{\sim} \Gamma(X, \varinjlim_{n\in\mathbb{N}} \mathcal{B}_X \otimes_B E_n) \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} \Gamma(X, \mathcal{B}_X \otimes_B E_n) \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} E_n \xrightarrow{\sim} E. \tag{4.1.3.19.1}$$

We deduce from the isomorphisms 4.1.3.18.1 than the canonical morphism

$$\mathcal{B}_X \otimes_B E \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} E \tag{4.1.3.19.2}$$

is an isomorphism. Using 4.1.3.19.1, this yields that the canonical morphism

$$E \to \Gamma(X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} E)$$

is an isomorphism.

3) Let $\mathcal{E}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module endowed with a finite presentation

$$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^r \xrightarrow{\phi} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s \xrightarrow{\epsilon} \mathcal{E} \to 0. \tag{4.1.3.19.3}$$

i) Let's start by showing that $\Gamma(X,\mathcal{E})$ is a $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module of finite type. According to the lemma 4.1.3.12, there exists an integer n_0 such that for any integer n, the composite morphism $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^r \hookrightarrow$

$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^r \xrightarrow{\phi} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s$ factorizes into a morphism ϕ_n: $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})^r \to (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n+n_0})^s$. The $\mathcal{B}_X$-modules $\mathcal{M}_n := \mathrm{Im}(\phi_n)$ are then coherent. The family of $\mathcal{B}_X$-modules $(\mathcal{M}_n)_{n\in\mathbb{N}}$ endows therefore $\mathcal{M} := \mathrm{Im}(\phi)$ with an exhaustive filtration by coherent $\mathcal{B}_X$-modules.

Moreover, it follows from 4.1.3.19.3 the exact sequence:

$$0 \to \mathcal{M} \to (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s \xrightarrow{\epsilon} \mathcal{E} \to 0, \tag{4.1.3.19.4}$$

where the morphism $\mathcal{M} \to (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^s$ is the canonical injection. Since the functor $H^1(X,-)$ commutes to filtered inductive limits, since $\mathcal{B}_X$ satisfies theorems A and B for coherent modules, then $H^1(X,\mathcal{M}) \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} H^1(X,\mathcal{M}_n) \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} 0 \xrightarrow{\sim} 0$. By applying the functor $\Gamma(X,-)$ to the exact sequence 4.1.3.19.4, we therefore get that the morphism $\Gamma(X,\epsilon)$: $(\widetilde{D}^{(m)}_{X^\sharp/S^\sharp})^s \to \Gamma(X,\mathcal{E})$ is surjective. We have therefore shown that the functor $\Gamma(X,-)$ of the proposition is well defined.

ii) It remains to show that the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} \Gamma(X,\ E) \to \mathcal{E}$ is an isomorphism. Thanks to proposition 4.1.3.16, there is a good filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$. Since the $\mathcal{B}_X$-modules $\mathcal{E}_n$ are coherent, since $\mathcal{B}_X$ satisfies the theorems A and B for coherent modules, then the canonical morphism $\mathcal{B}_X \otimes_B \Gamma(X,\mathcal{E}_n) \to \mathcal{E}_n$ are isomorphisms. Since filtered inductive limits commute to the global section functors and tensor products, we then obtain the canonical isomorphisms

$$\mathcal{B}_X \otimes_B \Gamma(X,\mathcal{E}) \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} \mathcal{B}_X \otimes_B \Gamma(X,\mathcal{E}_n) \xrightarrow{\sim} \varinjlim_{n\in\mathbb{N}} \mathcal{E}_n \xrightarrow{\sim} \mathcal{E}. \tag{4.1.3.19.5}$$

The canonical isomorphism $\mathcal{B}_X \otimes_B \Gamma(X,\mathcal{E}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} \Gamma(X,\mathcal{E})$ (see 4.1.3.19.2) and 4.1.3.19.5 then provide us with the expected canonical isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} \Gamma(X,\mathcal{E}) \xrightarrow{\sim} \mathcal{E}.$$

□

Remark 4.1.3.20. Suppose $X^\sharp$ is a log-scheme and not a formal log-scheme, i.e. we consider the algebraic case of 3.4. Suppose moreover X affine and $\mathcal{B}_X$ is quasi-coherent. Let $\mathcal{E}$ be a coherent $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $\mathcal{E}$ are quasi-coherent $\mathcal{O}_X$-modules and the canonical morphisms $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,\mathcal{E}) \to \mathcal{E}$ are isomorphisms. The first one induces that the canonical morphism $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,\mathcal{E}) \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})} \Gamma(X,\mathcal{E})$ is an isomorphism. Hence, so is the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})} \Gamma(X,\mathcal{E}) \to \mathcal{E}$ is an isomorphism. This yields that $\mathcal{E}$ is globally of finite presentation. Hence, 4.1.3.19, remain valid by replacing the hypothesis "globally of finite presentation" by "coherent". With 4.1.3.16 and 4.1.3.11, this implies that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ satisfies theorems A and B for coherent modules in the sense of 1.4.3.14.

Theorem 4.1.3.21. *Let $\mathcal{E}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Then $\mathcal{E}$ is coherent if and only if it admits good filtrations locally.*

Proof. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is coherent (see 4.1.2.17), then it follows from the proposition 4.1.3.16, that if $\mathcal{E}$ is a coherent $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module then it locally admits good filtrations. Let us show the converse.

The assertion being local, we are reduced to the case where X is an affine formal scheme above which there is a good filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$. By hypothesis, there therefore exists an integer $r_1 \in \mathbb{N}$ such that for any integer $n \geq r_1$, we have $\mathcal{E}_n = \sum_{j=0}^{p^m-1} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n-r_1+j} \cdot \mathcal{E}_{r_1-j}$. Since the sheaf of rings $\mathcal{B}_X$ verifies theorems A and B for coherent modules, then we get from 4.1.3.3 the equality

$$\Gamma(X,\mathcal{E}_n) = \sum_{j=0}^{p^m-1} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n-r_1+j} \Gamma(X,\mathcal{E}_{n_1-j}).$$

This yields that $\Gamma(X,\mathcal{E})$ is generated as $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module by $\Gamma(X,\mathcal{E}_{n_1})$. It is thus a $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module of finite type.

Since is $\mathcal{E}$ is equipped with a filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ by coherent $\mathcal{B}_X$-modules, then it follows from 4.1.3.19.5 (to check this isomorphism, we only need a filtration by coherent $\mathcal{B}_X$-modules) that we have the canonical isomorphism:

$$\mathcal{B}_X \otimes_{\Gamma(X,\mathcal{B}_X)} \Gamma(X,\mathcal{E}) \xrightarrow{\sim} \mathcal{E}. \tag{4.1.3.21.1}$$

Now, the proposition 4.1.3.19 gives us the isomorphism

$$\mathcal{B}_X \otimes_{\Gamma(X,\mathcal{B}_X)} \Gamma(X,\mathcal{E}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})} \Gamma(X,\mathcal{E}). \tag{4.1.3.21.2}$$

By using 4.1.3.21.1 and 4.1.3.21.2, we obtain that the canonical morphism

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\Gamma(X,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})} \Gamma(X,\mathcal{E}) \to \mathcal{E}.$$

is an isomorphism. The theorem 4.1.3.19 allows us to conclude. □

4.1.3.22. The canonical order filtration $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n})_{n\in\mathbb{N}}$ of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ gives us a sheaf of rings

$$\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} := \oplus_{n\in\mathbb{N}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n-1}.$$

Also, the graduated ring associated with this filtration $\operatorname{gr}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} := \oplus_{n\in\mathbb{N}}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n}/\widetilde{D}^{(m)}_{X^\sharp/S^\sharp,n-1}$ is isomorphic to $\Gamma(X,\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ (this results from the commutation of $\Gamma(X,-)$ to filtered inductive limits and from the fact that $\mathcal{B}_X$ satisfies the theorems A and B for coherent -modules). Similarly to 4.1.3.18.1, we have the canonical isomorphism

$$\mathcal{B}_X \otimes_{B_X} \operatorname{gr}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \left(\oplus_{n\in\mathbb{N}}\mathcal{B}_X \otimes_{B_X} \operatorname{gr}_n \widetilde{D}^{(m)}_{X^\sharp/S^\sharp}\right) \xrightarrow{\sim} \oplus_{n\in\mathbb{N}} \operatorname{gr}_n \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.1.3.22.1}$$

This yields that for any $\mathcal{B}_X \otimes_{B_X} \operatorname{gr}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module G, the canonical morphism

$$\mathcal{B}_X \otimes_{B_X} G \to \operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\operatorname{gr}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} G \tag{4.1.3.22.2}$$

is an isomorphism.

When a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\mathcal{E}$ is endowed with a filtration $(\mathcal{E}_r)_{r\in\mathbb{N}}$, if we take the convention that $\mathcal{E}_r = 0$ for any integer $r \leq -1$ and then writing for $r \geq 0$, $\operatorname{gr}_r \mathcal{E} := \mathcal{E}_r/\mathcal{E}_{r-1}$ we can then associate its graded ring with him:

$$\operatorname{gr}(\mathcal{E}) := \oplus_{r\in\mathbb{N}} \operatorname{gr}_r \mathcal{E}.$$

The sheaf $\operatorname{gr}(\mathcal{E})$ is then equipped with a canonical structure of $\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.

Following 4.1.2.17.a, the sheaf of rings $\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is commutative. Moreover, the canonical ring morphism $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ (see 4.1.2.2) induces a commutative ring homomorphism $\operatorname{gr}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. Also, as with all $r \in \mathbb{N}$, $\operatorname{gr}_r \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a locally free $\mathcal{B}_X$-module, we verify that the canonical morphism $\mathcal{B}_X \otimes_{\mathcal{O}_{X^\sharp/S^\sharp}} \operatorname{gr}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a ring isomorphism.

Proposition 4.1.3.23 (Theorem A)**.** *Suppose X is affine. Then the functors $\mathcal{G} \mapsto \Gamma(X,\mathcal{G})$ and $G \mapsto \operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\operatorname{gr}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} G$ are quasi-inverse category equivalences between the category of left $\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module globally of finite presentation and the category of $\operatorname{gr}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-modules of finite type.*

Proof. The proof is analogous to that of the theorem 4.1.3.19. □

Remark 4.1.3.24. Suppose $X^\sharp$ is a log-scheme and not a formal log-scheme, i.e. we consider the algebraic case of 3.4. Suppose moreover X affine and $\mathcal{B}_X$ is quasi-coherent. Similarly to 4.1.3.20, we can check that a coherent $\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module is globally of finite presentation.

The following theorem gives us a characterization of the good filtrations among the filtrations.

Theorem 4.1.3.25. *Let $\mathcal{E}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and $(\mathcal{E}_n)_{n\in\mathbb{N}}$ a filtration of $\mathcal{E}$. The following two assertions are equivalent:*

(a) The filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is good.

(b) The $\mathrm{gr}\,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\mathrm{gr}\,\mathcal{E}$ is coherent.

Proof. The proposition being of local nature, we are therefore reduced to assuming that X is affine and that $X^\sharp/S^\sharp$ is endowed with logarithmic coordinates.

1) Let us prove that $(a) \Rightarrow (b)$. So let's assume that the filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is good. By hypothesis, there therefore exists an integer $n_1 \in \mathbb{N}$ such that for any integer $n \geq n_1$, we have

$$\mathcal{E}_n = \sum_{j=0}^{p^m-1} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp, n-n_1+j}\mathcal{E}_{n_1-j}.$$

We deduce that the canonical morphism of coherent $\mathcal{B}_X$-modules

$$\oplus_{j=0}^{p^m-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp, n-n_1+j} \otimes_{\mathcal{B}_X} \mathcal{E}_{n_1-j} \to \mathcal{E}_n,$$

is surjective. Since the sheaf of rings $\mathcal{B}_X$ satisfies the theorem A and B for coherent modules, then it follows from 4.1.3.3 that the canonical morphism of $B := \Gamma(X, \mathcal{B}_X)$-modules

$$\Gamma(X, \oplus_{j=0}^{p^m-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp, n-n_1+j} \otimes_{\mathcal{B}_X} \mathcal{E}_{n_1-j}) = \oplus_{j=0}^{p^m-1}\widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n-n_1+j} \otimes_B \Gamma(X, \mathcal{E}_{n_1-j}) \to \Gamma(X, \mathcal{E}_n),$$

is surjective. We therefore deduce the equality $\Gamma(X, \mathcal{E}_n) = \sum_{j=0}^{p^m-1} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n-n_1+j}\Gamma(X, \mathcal{E}_{n_1-j})$. Moreover, since the $\mathcal{B}_X$-modules $\mathcal{E}_n$ are coherent, using again 4.1.3.3, we obtain the equality $\Gamma(X, \mathrm{gr}\,\mathcal{E}) = \oplus_{n=0}^{\infty}\Gamma(X, \mathcal{E}_n)/\Gamma(X, \mathcal{E}_{n-1})$. From these two formulas, we conclude that $\Gamma(X, \mathrm{gr}\,\mathcal{E})$ is generated as $\mathrm{gr}\,\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module by $\oplus_{n=0}^{n=n_1}\Gamma(X, \mathrm{gr}\,\mathcal{E}_n)$. Since $\oplus_{n=0}^{n=n_1}\Gamma(X, \mathrm{gr}\,\mathcal{E}_n)$ is a $\Gamma(X, \mathcal{B}_X)$-module of finite type, $\Gamma(X, \mathrm{gr}\,\mathcal{E})$ is therefore a $\mathrm{gr}\,\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module of finite type. Since $\mathrm{gr}\,\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$ is noetherian, then it is of finite presentation.

Since $\mathcal{B}_X$ satisfies theorems A and B for coherent modules, then the canonical morphisms $\mathcal{B}_X \otimes_B \Gamma(X, \mathrm{gr}_n\,\mathcal{E}) \to \mathrm{gr}_n\,\mathcal{E}$ are isomorphisms for any $n \in \mathbb{N}$. The tensor product commuting to direct sums, we deduce that the canonical morphism $\mathcal{B}_X \otimes_B \Gamma(X, \mathrm{gr}\,\mathcal{E}) \to \mathrm{gr}\,\mathcal{E}$ is an isomorphism. With 4.1.3.22.2, this yields that the canonical morphism $\mathrm{gr}\,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathrm{gr}\,\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}} \Gamma(X, \mathrm{gr}\,\mathcal{E}) \to \mathrm{gr}\,\mathcal{E}$ is also an isomorphism.

We then deduce from the proposition 4.1.3.23 that $\mathrm{gr}\,\mathcal{E}$ is a $\mathrm{gr}\,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of finite presentation and therefore a coherent $\mathrm{gr}\,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.

2) Conversely, let us check $(b) \Rightarrow (a)$. So suppose $\mathrm{gr}\,\mathcal{E}$ is $\mathrm{gr}\,\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-coherent.

i) By induction on $n \in \mathbb{N}$, we find that the $\mathcal{B}_X$-coherence for all n of $\mathcal{E}_n$ is equivalent to the $\mathcal{B}_X$-coherence for all n of $\mathrm{gr}_n\,\mathcal{E}$. Let us show this last assertion. Now, according to the proposition 4.1.3.23, we have the canonical morphism:

$$\mathcal{B}_X \otimes_B \Gamma(X, \mathrm{gr}\,\mathcal{E}) \xrightarrow{\sim} \mathrm{gr}\,\mathcal{E}$$

is an isomorphism. Since the functor $\Gamma(X, -)$ commutes to filtered direct sums, then this yields that the canonical morphisms

$$\mathcal{B}_X \otimes_B \Gamma(X, \mathrm{gr}_n\,\mathcal{E}) \to \mathrm{gr}_n\,\mathcal{E}$$

are isomorphisms for all $n \in \mathbb{N}$. It only remains for us to show that for all n the B-module $\Gamma(X, \mathrm{gr}_n\,\mathcal{E})$ is of finite type. For this, according to the proposition 4.1.3.23, we use the fact that $\Gamma(X, \mathrm{gr}\,\mathcal{E})$ is a $\mathrm{gr}\,\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$-module of finite type. So there are homogeneous elements $x_1, ..., x_r$, of respective degree $n_1, ..., n_r$ generating $\Gamma(X, \mathrm{gr}\,\mathcal{E})$ and therefore we obtain by identification the equality for all $n \in \mathbb{N}$:

$$\Gamma(X, \mathrm{gr}_n\,\mathcal{E}) = \sum_{l=1}^{r} \mathrm{gr}_{n-n_l}\,\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \cdot x_l. \tag{4.1.3.25.1}$$

It therefore follows that $\Gamma(X, \mathrm{gr}_n\,\mathcal{E})$ is a B-module of finite type. We have therefore shown that for all n, $\mathrm{gr}\,\mathcal{E}_n$ is a coherent $\mathcal{B}_X$-module.

ii) Also, since $\Gamma(X, \operatorname{gr}_n \mathcal{E}) = \Gamma(X, \mathcal{E}_n)/\Gamma(X, \mathcal{E}_{n-1})$ and in setting $N_0 = \sup(n_1, ..., n_r)$, by using the equality 4.1.3.25.1, we can check by induction on the integer $n \geq -N_0$ the following equality:

$$\Gamma(X, \mathcal{E}_{N_0+n}) = \sum_{l=1}^{r} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n+N_0-n_l} x_l. \tag{4.1.3.25.2}$$

We further know that $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n+N_0-n_l} = \sum_{j=0}^{p^m-1} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n+j} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, N_0-n_l-j}$. By injecting this equality into 4.1.3.25.2, we then obtain

$$\Gamma(X, \mathcal{E}_{N_0+n}) = \sum_{j=0}^{p^m-1} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n+j} (\sum_{l=1}^{r} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, N_0-n_l-j} . x_l). \tag{4.1.3.25.3}$$

However, the formula 4.1.3.25.2 applied to n equal to $-j$ shows that the term placed between parentheses of the sum of 4.1.3.25.3 is none other than $\Gamma(X, \mathcal{E}_{N_0-j})$. We therefore obtain:

$$\Gamma(X, \mathcal{E}_{N_0+n}) = \sum_{j=0}^{p^m-1} \widetilde{D}^{(m)}_{X^\sharp/S^\sharp, n+j} \Gamma(X, \mathcal{E}_{N_0-j}),$$

which completes the demonstration. □

Corollary 4.1.3.26. *Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \xrightarrow{\epsilon} 0$ be an exact sequence of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-coherent modules, and $(\mathcal{E}_n)_{n\in\mathbb{N}}$ a good filtration of E. Then the filtrations of $\mathcal{E}'$, $\mathcal{E}''$ defined by: $\mathcal{E}'_n := \mathcal{E}_n \cap \mathcal{E}'$, $\mathcal{E}''_n = \epsilon(\mathcal{E}_n)$ are good.*

Proof. The filtrations induce the exact sequence $0 \to \operatorname{gr}\mathcal{E}' \to \operatorname{gr}\mathcal{E} \to \operatorname{gr}\mathcal{E}'' \to 0$. According to the theorem 4.1.3.25, it then suffices to show that $\operatorname{gr}\mathcal{E}'$ is $\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-coherent. Since the assertion is local, we are reduced to the case where X is affine.

As the functor $\Gamma(X, -)$ is left exact, $\Gamma(X, \operatorname{gr}\mathcal{E}')$ is a submodule of $\Gamma(X, \operatorname{gr}\mathcal{E})$. Since X is affine, then $\Gamma(X, \operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ is Noetherian and $\Gamma(X, \operatorname{gr}\mathcal{E}')$ is therefore of finite type. Show coherence of $\operatorname{gr}\mathcal{E}'$ then amounts to showing that the canonical morphism

$$\operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\Gamma(X, \operatorname{gr}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})} \Gamma(X, \operatorname{gr}\mathcal{E}') \to \operatorname{gr}\mathcal{E}',$$

is an isomorphism. By identifying the terms of the same degree, it is up to the same to show that, for all n, the induced canonical morphisms $\operatorname{gr}_n \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\Gamma(X, \operatorname{gr}_n \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})} \Gamma(X, \operatorname{gr}_n \mathcal{E}') \to \operatorname{gr}_n \mathcal{E}'$ are isomorphisms. It suffices to show that the $\mathcal{B}_X$-modules $\mathcal{E}_n$ are coherent for any integer n. Since $B := \Gamma(X, \mathcal{B}_X)$ is a Noetherian ring and we have a canonical inclusion $\Gamma(X, \mathcal{E}'_n) \subset \Gamma(X, \mathcal{E}_n)$, we deduce that $\Gamma(X, \mathcal{E}'_n)$ is of finite type. Also, the extension $B \to \mathcal{B}_X$ being flat, we then have

$$\mathcal{B}_X \otimes_B \Gamma(X, \mathcal{E}'_n) \xrightarrow{\sim} \mathcal{B}_X \otimes_B (\Gamma(X, \mathcal{E}_n) \cap \Gamma(X, \mathcal{E}')) \xrightarrow{\sim} (\mathcal{B}_X \otimes_B \Gamma(X, \mathcal{E}_n)) \cap (\mathcal{B}_X \otimes_B \Gamma(X, \mathcal{E}')) \xrightarrow{\sim} \mathcal{E}_n \cap \mathcal{E}',$$

the last isomorphism comes from the fact that $\mathcal{E}'$ is a coherent $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (use 4.1.3.19) and that $\mathcal{E}_n$ is a coherent $\mathcal{B}_X$-module. □

Remark 4.1.3.27. This subsection (definitions and properties) still holds replacing left modules by right modules. For instance, if $\mathcal{M}$ is a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module endowed with a filtration $(\mathcal{M}_r)_{r\in\mathbb{N}}$, the filtration $(\mathcal{M}_r)_{r\in\mathbb{N}}$ is said to be *good* (resp. *quasi-good*) if the following two conditions hold (resp. the following first condition holds):

(a) For any $r \in \mathbb{N}$, $\mathcal{M}_r$ is $\mathcal{B}_X$-coherent ;

(b) There exists an integer $r_1 \in \mathbb{N}$ such that for any integer $r \geq r_1$, we have

$$\mathcal{M}_r = \sum_{j=0}^{p^m-1} \mathcal{M}_{r_1-j} \cdot \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp, r-r_1+j}.$$

Besides, the equivalence of categories 4.3.5.4 are compatible with this notion of good filtration, i.e., if $\mathcal{E}$ is a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{M} := \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$ is the corresponding right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module then $(\mathcal{E}_r)_{r\in\mathbb{N}} \mapsto (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}_r)_{r\in\mathbb{N}}$ is a bijection between (quasi-)good filtrations of $\mathcal{E}$ and (quasi-)good filtrations of $\mathcal{M}$.

The following corollary will be useful later.

Corollary 4.1.3.28. *We suppose $\mathcal{B}_X = \mathcal{O}_X$. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. There exists $Y \subset X$ open dense on which $\mathcal{E}$ is $\mathcal{O}_Y$-locally free.*

Proof. Since this is local, we can suppose $\mathfrak{X}$ is affine and $\mathcal{E}$ has a global finite presentation. Hence, $\mathcal{E}$ admits a good filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ (see 4.1.3.16) and the induced $\operatorname{gr}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module $\operatorname{gr}\mathcal{E}$ is coherent (see 4.1.3.25). Hence, following theorem A of 4.1.3.19 and 4.1.3.23, since $\operatorname{gr}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a commutative $\mathcal{O}_X$-algebra of finite type, then it follows from [Gro65, IV.6.9.1] (in the noetherian case) that there exists a principal open subset Y of X such that $\operatorname{gr}\mathcal{E}|Y$ is $\mathcal{O}_Y$-free. Hence, for any $n \in \mathbb{N}$, $\operatorname{gr}_n \mathcal{E}|Y$ is a projective $\mathcal{O}_Y$-module and we get a section $\operatorname{gr}_n \mathcal{E}|Y \to \mathcal{E}_n|Y$ of the canonical projection $\mathcal{E}_n|Y \to \operatorname{gr}_n \mathcal{E}|Y$. Denote by $\alpha_n\colon \operatorname{gr}_n \mathcal{E}|Y \to \mathcal{E}_n|Y \subset \mathcal{E}|Y$ the composite morphism. By induction on n, we can check that the morphism $\oplus_{i=0}^n \alpha_i \colon \oplus_{i=0}^n \colon \oplus_{i=0}^n \operatorname{gr}_i \mathcal{E}|Y \to \mathcal{E}|Y$ in injective and its image is $\mathcal{E}_n$. This yields the isomorphism $\oplus_{i=0}^n \alpha_i \colon \oplus_{i=0}^\infty \colon \oplus_{i=0}^\infty \operatorname{gr}_i \mathcal{E}|Y \to \mathcal{E}|Y$. □

4.2 Tensor products and internal homomorphism of $\mathcal{D}$-modules

We keep notation 3.4. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure which is compatible with its algebra structure. We keep notation 4.1.2, in particular we set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

4.2.1 PD-stratifications of level m with coefficients, nilpotence

Definition 4.2.1.1. An *m-PD-stratification* (or *PD-stratification of level m*) ε *relatively to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$* on a $\mathcal{B}_X$-module $\mathcal{E}$ is the data of a family of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-linear homomorphisms

$$\varepsilon_n \colon \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)},$$

where the tensor products are taken respectively for the right and left structures of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$, these isomorphisms satisfying the following conditions:

(a) $\varepsilon_0^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ and the family is compatible with respect to the projections $\widetilde{\psi}^{n+1,n*}_{X^\sharp/S^\sharp,(m)}$ (see 4.1.2.12.1), i.e. for any $n' \geq n$ in $\mathbb{N}$ we have the commutative diagram:

$$\begin{array}{ccc} \widetilde{\psi}^{n',n*}_{X^\sharp/S^\sharp,(m)}(\widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \mathcal{E}) & \xrightarrow{\widetilde{\psi}^{n',n*}_{X^\sharp/S^\sharp,(m)}(\varepsilon_{n'})} & \widetilde{\psi}^{n',n*}_{X^\sharp/S^\sharp,(m)}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)}) \, ; \\ \downarrow \wr & & \downarrow \wr \\ \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \mathcal{E} & \xrightarrow{\varepsilon_n} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \end{array}$$

(b) The cocycle condition is satisfied, i.e., for any n, n', the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \otimes \mathcal{E} & \xrightarrow{\widetilde{\delta}^{n,n'*}_{(m)}(\varepsilon_{n+n'})} & \mathcal{E} \otimes \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \\ & \searrow^{\tilde{q}^{n,n'*}_{1(m)}(\varepsilon_{n+n'})} \quad \nearrow_{\tilde{q}^{n,n'*}_{0(m)}(\varepsilon_{n+n'})} & \\ & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes \mathcal{E} \otimes \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} & \end{array} \tag{4.2.1.1.1}$$

is commutative.

Say a $\mathcal{B}_X$-linear homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ between $\mathcal{B}_X$-modules equipped with m-PD stratifications relatively to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$ is *horizontal* if it commutes with all ε_n.

4.2.1.2. With notation 3.2.2.12 (add some tildes), similarly to 2.1.1.2, we check that the above condition 4.2.1.1.1 is equivalent to

$$\forall\, n, \qquad \tilde{q}^{n*}_{02,(m)}(\varepsilon_n) = \tilde{q}^{n*}_{01,(m)}(\varepsilon_n) \circ \tilde{q}^{n*}_{12,(m)}(\varepsilon_n). \tag{4.2.1.2.1}$$

Proposition 4.2.1.3. *Let $\mathcal{M}$ be a $\mathcal{B}_X$-module together with an m-PD stratification $(\varepsilon^{\mathcal{M}}_n)$ relative to $X^\sharp/S^\sharp$ with coefficient in $\mathcal{B}_X$. Then the homomorphisms $\varepsilon^{\mathcal{M}}_n$ are $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$-linear isomorphisms.*

Proof. Copy the proof of 2.1.1.3. □

Proposition 4.2.1.4. *Let $\mathcal{E}$ be an $\mathcal{O}_X$-module together with an m-PD stratification $(\varepsilon^{\mathcal{E}}_n)$ relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$. Then the homomorphisms $\varepsilon^{\mathcal{E}}_n$ are $\mathcal{P}^n_{X/S,(m)}$-linear isomorphisms.*

Proof. We can copy word by word the proof of 2.1.1.3. □

Proposition 4.2.1.5. *We have the following properties.*

(I) Given a $\mathcal{B}_X$-module $\mathcal{E}$. The following are equivalent.

(a) A left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{E}$ extending its $\mathcal{B}_X$-module structure.

(b) A family of $\mathcal{B}_X$-linear homomorphisms $\theta_n \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)})$ (the $\mathcal{B}_X$-module structure of this latter is induced by the right structure of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$) satisfying

(i) $\theta_0 = \mathrm{id}_{\mathcal{E}}$ and for any $n, n' \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\theta_n} & \mathcal{E} \otimes \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \\ \| & & \uparrow{\scriptstyle \mathrm{id} \otimes \widetilde{\psi}^{n+n',n}_{X^\sharp/S^\sharp,(m)}} \\ \mathcal{E} & \xrightarrow[\theta_{n+n'}]{} & \mathcal{E} \otimes \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp,(m)} \end{array} \tag{4.2.1.5.1}$$

is commutative.

(ii) for all n, n' we have the commutative diagrams (cocycle condition)

$$\begin{array}{ccc} \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\mathrm{id} \otimes \delta^{n,n'}_{(m)}} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)} \\ \uparrow{\scriptstyle \theta^{\mathcal{E}}_{n+n'}} & & \uparrow{\scriptstyle \theta^{\mathcal{E}}_n \otimes \mathrm{id}} \\ \mathcal{E} & \xrightarrow{\theta^{\mathcal{E}}_{n'}} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp,(m)} \end{array} \tag{4.2.1.5.2}$$

(c) An m-PD stratification $\varepsilon = (\varepsilon^{\mathcal{E}}_n)$ on $\mathcal{E}$.

(II) Let $\mathcal{E}$ be left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and let $\theta^{\mathcal{E}} = (\theta^{\mathcal{E}}_n)$, $\varepsilon^{\mathcal{E}} = (\varepsilon^{\mathcal{E}}_n)$ be the associated family or m-PD stratification.

(a) The action by a section P of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ on $\mathcal{E}$ can be retrieved from $\theta^{\mathcal{E}}$ (resp. $\varepsilon^{\mathcal{E}}$) via the following composition of the bottom (resp. top) horizontal morphisms of the commutative diagram:

$$\begin{array}{ccccc} \mathcal{E} \xrightarrow[4.1.2.7]{\widetilde{p}^n_{1,(m),\mathcal{E}}} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \mathcal{E} & \xrightarrow{\varepsilon^{\mathcal{E}}_n} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\mathrm{id} \otimes P} & \mathcal{E} \\ \| & & \| & & \| \\ \mathcal{E} & \xrightarrow{\theta^{\mathcal{E}}_n} & p_{1*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}) & \xrightarrow{\mathrm{id} \otimes P} & \mathcal{E}. \end{array} \tag{4.2.1.5.3}$$

(b) *Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. Conversely, with notation 4.1.2.16 for any $x \in \mathcal{E}$, we have the Taylor expansion formula*

$$\theta_n^{\mathcal{E}}(x) = \varepsilon_n^{\mathcal{E}}(1 \otimes x) = \sum_{|\underline{k}| \leq n} \underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}}, \tag{4.2.1.5.4}$$

where $\underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x)$ is the action of the operator $\underline{\partial}_\sharp^{\langle \underline{k} \rangle}$ on x. The inverse of 4.2.1.5.4 can be described via the formula:

$$(\varepsilon_n^{\mathcal{E}})^{-1}(x \otimes 1) = \sum_{|\underline{k}| \leq n} \underline{\tau}_\sharp^{\{\underline{k}\}} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} \rangle} x. \tag{4.2.1.5.5}$$

(III) *A $\mathcal{B}_X$-linear morphism $\phi \colon \mathcal{E} \to \mathcal{F}$ between two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear if and only if ϕ is horizontal.*

Proof. We can copy the proof of the proposition without coefficients, i.e. when $\mathcal{B}_X = \mathcal{O}_X$ (see 3.4.2.5). We check 4.2.1.5.5 similarly to 3.4.5.3.1 by using 3.4.5.2.1. □

Remark 4.2.1.6. Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(b_\lambda)_{\lambda=1,\dots,d}$. Let $\mathcal{E}$ be a $\mathcal{B}_X$-module endowed with a family (as n varies) of $\widetilde{\mathcal{P}}^n_{X,(m)}$-linear isomorphisms

$$\varepsilon_n \colon \widetilde{\mathcal{P}}^n_{X,(m)} \otimes_{\mathcal{B}_X} \mathcal{E} = p_{1(m)}^{n*}(\mathcal{E}) \xrightarrow{\sim} p_{0(m)}^{n*}(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X,(m)},$$

satisfying the condition (a) of 4.2.1.1. For any x of $\mathcal{E}$, for any $\underline{k} \in \mathbb{N}^d$, for any $n \geq |\underline{k}|$, let us denote by $\underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x)$ the section of $\mathcal{E}$ such that we get

$$\varepsilon_n^{\mathcal{E}}(1 \otimes x) = \sum_{|\underline{k}| \leq n} \underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}}.$$

First remark that these elements do not depend on the choice of n which justifies the notation. Moreover, the cocycle condition is equivalent to the condition that the formula

$$\sum_{\underline{k}=\sup\{\underline{i},\underline{j}\}}^{\underline{i}+\underline{j}} \frac{\underline{k}!}{(\underline{i}+\underline{j}-\underline{k})!(\underline{k}-\underline{i})!(\underline{k}-\underline{j})!} \frac{\underline{q}_{\underline{i}}^{(m)}! \underline{q}_{\underline{j}}^{(m)}!}{\underline{q}_{\underline{k}}^{(m)}!} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(x) = \underline{\partial}_\sharp^{\langle \underline{i} \rangle}(\underline{\partial}_\sharp^{\langle \underline{j} \rangle}(x)) \tag{4.2.1.6.1}$$

holds for any section x of $\mathcal{E}$, for any $\underline{i}, \underline{j} \in \mathbb{N}^d$ (to understand where this formula is coming from, recall the formula 3.2.3.13.1).

Indeed, this is a consequence of the following computation: since $\delta^{n,n'}_{(m)}(\tau_{\sharp i}) = \tau_{\sharp i} \otimes \tau_{\sharp i} + \tau_{\sharp i} \otimes 1 + 1 \otimes \tau_{\sharp i}$ (see 3.2.3.10.1) and $\delta^{n,n'}_{(m)}$ is an m-PD-morphism, then using the formula 1.2.4.5.2, we get

$$\widetilde{\delta}^{n,n'*}_{(m)}(\varepsilon_{n+n'})(1 \otimes 1 \otimes x) = \sum_{|\underline{k}| \leq n+n'} \underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x) \otimes \delta^{n,n'}_{(m)}(\underline{\tau}_\sharp^{\{\underline{k}\}}) = \sum_{|\underline{k}| \leq n+n'} \sum_{|\underline{i}| \leq n} \sum_{|\underline{j}| \leq n'} \alpha^{\underline{k}}_{\underline{i},\underline{j}} \underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x) \otimes \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes \underline{\tau}_\sharp^{\{\underline{j}\}} \tag{4.2.1.6.2}$$

where $\alpha^{\underline{k}}_{\underline{i},\underline{j}} \in \mathbb{N}$. On the other hand, we have

$$\tilde{q}^{n,n'*}_{0(m)}(\varepsilon_{n+n'}) \circ \tilde{q}^{n,n'*}_{1(m)}(\varepsilon_{n+n'})(1 \otimes 1 \otimes x) = \tilde{q}^{n,n'*}_{0(m)}(\varepsilon_{n+n'}) \left(\sum_{|\underline{j}| \leq n'} \underline{\partial}_\sharp^{\langle \underline{j} \rangle}(x) \otimes \underline{\tau}_\sharp^{\{\underline{j}\}} \right)$$
$$= \sum_{|\underline{j}| \leq n'} \sum_{|\underline{i}| \leq n} \underline{\partial}_\sharp^{\langle \underline{i} \rangle}(\underline{\partial}_\sharp^{\langle \underline{j} \rangle}(x)) \otimes \underline{\tau}_\sharp^{\{\underline{i}\}} \otimes \underline{\tau}_\sharp^{\{\underline{j}\}} \tag{4.2.1.6.3}$$

The cocycle condition is satisfied if and only if the last terms of respectively 4.2.1.6.2 and 4.2.1.6.3 are equal. When $\mathcal{E}' = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$, the cocycle condition is satisfied and since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a free $\mathcal{B}_X$-module, then in the case $x = 1$ we get the first equality

$$\sum_{|\underline{k}| \leq n+n'} \alpha^{\underline{k}}_{\underline{i},\underline{j}} \underline{\partial}_\sharp^{\langle \underline{k} \rangle} = \underline{\partial}_\sharp^{\langle \underline{i} \rangle} \underline{\partial}_\sharp^{\langle \underline{j} \rangle} \overset{3.2.3.13.1}{=} \sum_{\underline{k}=\sup\{\underline{i},\underline{j}\}}^{\underline{i}+\underline{j}} \frac{\underline{k}!}{(\underline{i}+\underline{j}-\underline{k})!(\underline{k}-\underline{i})!(\underline{k}-\underline{j})!} \frac{\underline{q}_{\underline{i}}^{(m)}! \underline{q}_{\underline{j}}^{(m)}!}{\underline{q}_{\underline{k}}^{(m)}!} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}.$$

Hence, we are done.

Example 4.2.1.7. We denote by $\varepsilon_n^{\mathcal{B}_X}$ the m-PD-stratification induced by the left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure of $\mathcal{B}_X$. We denote by $\varepsilon^{\mathcal{B}_X}_{n/\mathcal{B}_X}\colon \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_X \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ the isomorphism making commutative the diagram of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-algebras

$$\begin{array}{ccc} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} & \xleftarrow{\sim} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{B}_X \\ \| & & \downarrow{\scriptstyle \sim\ \varepsilon^{\mathcal{B}_X}_{n/\mathcal{B}_X}} \\ \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} & \xleftarrow{\sim} & \mathcal{B}_X \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \end{array} \tag{4.2.1.7.1}$$

where the horizontal isomorphisms are the canonical ones. Similarly to 3.4.2.4, we can check that this is an m-PD-stratification relatively to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$. The induced canonical structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{B}_X$ is then described as follows : if $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $b \in \mathcal{B}_X$, the action of P on b is the image of b via the composition morphism: $\mathcal{B}_X \xrightarrow{\widetilde{p}^n_1} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \xrightarrow{P} \mathcal{B}_X$. Since $\{\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}, |\underline{k}| \leq n\}$ is the dual basis of $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}, |\underline{k}| \leq n\}$ (for the left structure), this yields the Taylor formula:

$$\widetilde{p}^n_{1,(m)}(b) = \sum_{|\underline{k}|\leq n} \widetilde{p}^n_{0,(m)}(\underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b))\underline{\tau}_\sharp^{\{\underline{k}\}}. \tag{4.2.1.7.2}$$

Lemma 4.2.1.8. *Let $\rho\colon \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ be the canonical morphism (see 4.1.2.2.2). Then the left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module induced via ρ by this left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structure on $\mathcal{B}_X$ defined at 4.2.1.7 is equal to the given left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure of $\mathcal{B}_X$.*

Proof. Since this is local, we can suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(b_\lambda)_{\lambda=1,\ldots,d}$. Then by using the Taylor formula satisfied by $\varepsilon_n^{\mathcal{B}_X}$ (see 3.4.2.5.4) for any section b of $\mathcal{B}_X$ we compute in $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$

$$(1\otimes 1)\cdot \widetilde{p}^n_{1(m)}(b) = (1\otimes 1)\cdot \epsilon_n^{\mathcal{B}_X}(1\otimes b) = \sum_{|\underline{k}|\leq n} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}}$$

where $\widetilde{p}^n_{1(m)})$ is defined at 4.1.2.5. Since the top horizontal isomorphism of the diagram 4.2.1.7.1 sends $(1\otimes 1)\otimes b$ to $(1\otimes 1)\cdot \widetilde{p}^n_{1(m)}(b)$ and the bottom horizontal isomorphism sends $\sum_{|\underline{k}|\leq n} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b) \otimes (1\otimes \underline{\tau}_\sharp^{\{\underline{k}\}})$ to $\sum_{|\underline{k}|\leq n} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}}$, then we get

$$\varepsilon^{\mathcal{B}_X}_{n/\mathcal{B}_X}((1\otimes 1)\otimes b) = \sum_{|\underline{k}|\leq n} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b) \otimes (1\otimes \underline{\tau}_\sharp^{\{\underline{k}\}}).$$

Hence, we conclude thanks to the formula 4.2.1.5.4 (recall by abuse of notation $1\otimes \underline{\tau}_\sharp^{\{\underline{k}\}} = \underline{\tau}_\sharp^{\{\underline{k}\}}$). □

4.2.1.9. We define a sheaf of $\mathcal{P}_{X^\sharp/S^\sharp(m)}$-algebras by setting $\widetilde{\mathcal{P}}_{X^\sharp/S^\sharp(m)} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}_{X^\sharp/S^\sharp(m)}$.
We have the following $\mathcal{B}_X$-m-PD isomorphism

$$\begin{aligned} \mathcal{B}_X\langle T_1,\ldots,T_r\rangle_{(m)} &\xrightarrow{\sim} \widetilde{\mathcal{P}}_{X^\sharp/S^\sharp,(m)} \\ T_\lambda &\mapsto \tau_{\sharp\lambda}, \end{aligned} \tag{4.2.1.9.1}$$

where the structure of $\mathcal{B}_X$-module of $\widetilde{\mathcal{P}}_{X^\sharp/S^\sharp,(m),\gamma}$ is given by the left structure or the right one.

Proposition 4.2.1.10. *Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(b_\lambda)_{\lambda=1,\ldots,d}$. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. The following conditions are equivalent:*

(a) For any section e of $\mathcal{E}$, locally there exists an integer N such that $\underline{\partial}_\sharp^{\langle \underline{k}\rangle}(x) = 0$ for any $|\underline{k}| \geq N$.

(b) The condition (a) holds for any local system of logarithmic coordinates on any open of X.

(c) There exists an isomorphism

$$\varepsilon\colon \widetilde{\mathcal{P}}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}_{X^\sharp/S^\sharp(m)},$$

satisfying the cocycle condition, and inducing by extension via the epimorphisms $\widetilde{\mathcal{P}}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ *the m-PD-stratification of* $\mathcal{E}$*.*

Proof. By using the Taylor formula 4.2.1.5.4 and its inverse 4.2.1.5.5 (use the formula 3.4.1.2.2 to check the can be extended to an inverse of ε) and the description of 4.2.1.9.1, we can follow the proof of [Ber96c, 2.3.7]. □

Definition 4.2.1.11. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. We say that $\mathcal{E}$ is "quasi-nilpotent" is locally on $\mathfrak{X}$ the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\mathcal{E}$ satisfies the conditions of the proposition 4.2.1.10. If moreover we can choose such an integer N for any local section e of $\mathcal{E}$, we say that $\mathcal{E}$ is "nilpotent".

4.2.2 PD-costratifications of level m with coefficients

We define m-PD-*costratifications relatively to* $X^\sharp/S^\sharp$ *with coefficients in* $\mathcal{B}_X$ as follows.

Definition 4.2.2.1. An m-PD-*costratification relatively to* $X^\sharp/S^\sharp$ *with coefficients in* $\mathcal{B}_X$ on a $\mathcal{B}_X$-module $\mathcal{M}$ is the data of a family of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$-linear homomorphisms

$$\varepsilon_n\colon \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}, \mathcal{M}) \to \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}, \mathcal{M}),$$

this ones satisfying the following conditions :

(a) $\varepsilon_0 = \mathrm{id}_{\mathcal{M}}$ and for any $n' \geq n$ in $\mathbb{N}$, ε_n and $\widetilde{\psi}^{n',n\flat}_{X/S,(m)}(\varepsilon_{n'})$ are canonically isomorphic, i.e. the following diagram

$$\begin{array}{ccc} \widetilde{\psi}^{n',n\flat}_{X^\sharp/S^\sharp,(m)}(p^{n'\flat}_{0,(m)}(\mathcal{M})) & \xrightarrow{\widetilde{\psi}^{n',n\flat}_{X^\sharp/S^\sharp,(m)}(\varepsilon_{n'})} & \widetilde{\psi}^{n',n\flat}_{X^\sharp/S^\sharp,(m)}(p^{n'\flat}_{1,(m)}(\mathcal{M})) \\ \downarrow\wr & & \downarrow\wr \\ \widetilde{p}^{n\flat}_{0,(m)}(\mathcal{M}) & \xrightarrow{\varepsilon_n} & \widetilde{p}^{n\flat}_{1,(m)}(\mathcal{M}) \end{array} \quad (4.2.2.1.1)$$

whose vertical isomorphisms are the canonical ones, is commutative ;

(b) For any n, n', with notation 2.1.2.1 the diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^{n,n'}_{0*}(\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}), \mathcal{M}) & \xrightarrow{\widetilde{\delta}^{n,n'\flat}_{(m)}(\varepsilon_{n+n'})} & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^{n,n'}_{2*}(\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}), \mathcal{M}) \\ & \searrow^{\tilde{q}^{n,n'\flat}_0(\varepsilon_{n+n'})} \quad \nearrow_{\tilde{q}^{n,n'\flat}_1(\varepsilon_{n+n'})} & \\ & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^{n,n'}_{1*}(\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)}), \mathcal{M}) & \end{array}$$

(4.2.2.1.2)

is commutative.

Say a $\mathcal{B}_X$-linear homomorphism $\varphi\colon \mathcal{E} \to \mathcal{F}$ between $\mathcal{B}_X$-modules equipped with m-PD costratifications relatively to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$ is *horizontal* if it commutes with all ε_n.

4.2.2.2. With notation 3.2.2.12 (with some tildes), similarly to 2.1.1.2, we check that the above condition 4.2.2.1.2 is equivalent to

$$\forall\, n, \qquad \tilde{q}^{n\flat}_{02,(m)}(\varepsilon_n) = \tilde{q}^{n\flat}_{01,(m)}(\varepsilon_n) \circ \tilde{q}^{n\flat}_{12,(m)}(\varepsilon_n). \quad (4.2.2.2.1)$$

Proposition 4.2.2.3. *Let* $\mathcal{M}$ *be a* $\mathcal{B}_X$*-module together with an m-PD costratification* $(\varepsilon^{\mathcal{M}}_n)$ *relative to* $X^\sharp/S^\sharp$ *with coefficient in* $\mathcal{B}_X$*. Then the homomorphisms* $\varepsilon^{\mathcal{M}}_n$ *are* $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$*-linear isomorphisms.*

Proof. Copy the proof of 2.1.1.3. ☐

Notation 4.2.2.4. Let $\mathcal{M}$ be a $\mathcal{B}_X$-module. Since $p_{i*}\mathcal{P}^n_{X,(m)}$ is locally free as $\mathcal{B}_X$-modules, then the canonical homomorphism

$$\iota_n^{\mathcal{M}} \colon \mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X,n} = \mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)},\mathcal{B}_X) \to \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)},\mathcal{M}) = \widetilde{p}^{n\flat}_{0,(m)}(\mathcal{M}) \tag{4.2.2.4.1}$$

given by $x\otimes P\mapsto(\tau\mapsto xP(\tau))$ is an isomorphism. Similarly, we have the canonical isomorphism

$$\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)},\mathcal{B}_X)\otimes_{\mathcal{B}_X}\mathcal{M} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)},\mathcal{M}) = \widetilde{p}^{n\flat}_{1,(m)}(\mathcal{M}). \tag{4.2.2.4.2}$$

Proposition 4.2.2.5. *For any $\mathcal{B}_X$-module $\mathcal{M}$, there is an equivalence between the following data :*

(a) A structure of right $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{M}$ extending its structure of $\mathcal{B}_X$-module ;

(b) An m-PD-costratification $(\varepsilon_n^{\mathcal{M}})$ with coefficients in $\mathcal{B}_X$ on $\mathcal{M}$;

A $\mathcal{B}_X$-linear homomorphism between two right $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules is $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear if and only if it is horizontal.

Proof. Copy the proof of 2.1.2.8. ☐

Lemma 4.2.2.6. *Let $\mathcal{M}$ be a right $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. Let $(\varepsilon_n^{\mathcal{M}})$ be the m-PD-costratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{M}$.*

(a) With notation 4.2.2.4.1, the action of $P\in\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ on $x\in\mathcal{M}$ is given from the costratification by the formula

$$xP = \mathrm{ev}_1\circ\epsilon_n\circ\iota_n^{\mathcal{M}}(x\otimes P). \tag{4.2.2.6.1}$$

(b) Suppose $u_1,\dots,u_d$ are logarithmic coordinates of $X^\sharp/S^\sharp$. Let $\{\underline{\partial}_\sharp^{\star\langle\underline{h}\rangle},|\underline{h}|\le n\}$ be the dual basis of $\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)},\mathcal{B}_X)$ of the basis $\{\underline{\tau}_\sharp^{\{\underline{h}\}},|\underline{h}|\le n\}$ of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$. Conversely, via the identification 4.2.2.4.1 and 4.2.2.4.2, the costratification ε_n of $\mathcal{M}$ satisfies the following formula for any $x\in\mathcal{M}$ and any $\underline{k}\in\mathbb{N}^d$:

$$\varepsilon_n^{\mathcal{M}}(x\otimes\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}) = \sum_{\underline{h}\le\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\underline{\partial}_\sharp^{\star\langle\underline{h}\rangle}\otimes x\underline{\partial}_\sharp^{\langle\underline{k}-\underline{h}\rangle}, \tag{4.2.2.6.2}$$

We have the following formula for the inverse

$$(\varepsilon_n^{\mathcal{M}})^{-1}(\underline{\partial}^{\star\langle\underline{k}\rangle}\otimes x) = \sum_{\underline{h}\le\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}x\widetilde{\underline{\partial}}_\sharp^{\langle\underline{k}-\underline{h}\rangle}\otimes\underline{\partial}_\sharp^{\langle\underline{h}\rangle}. \tag{4.2.2.6.3}$$

Proof. Copy the proof of 2.1.2.9 and 3.4.5.3.2. ☐

Remark 4.2.2.7. Let $\mathcal{M}$ be a $\mathcal{B}_X$-module endowed with a family (as n varies) of $\mathcal{P}^n_{X,(m)}$-linear isomorphisms

$$\varepsilon_n\colon \widetilde{p}^{n\flat}_{0,(m)}(\mathcal{M})\to\widetilde{p}^{n\flat}_{1,(m)}(\mathcal{M}),$$

satisfying the condition (a) of 4.2.2.1. We remark that the cocycle condition is local. Let us give a local description of this condition. Suppose $X^\sharp\to S^\sharp$ is endowed with logarithmic coordinates $(b_\lambda)_{\lambda=1,\dots,d}$. For any x of $\mathcal{M}$, for any $\underline{k}\in\mathbb{N}^d$, for any $n\ge|\underline{k}|$, we set

$$x\cdot\underline{\partial}^{\langle\underline{k}\rangle} := \mathrm{ev}_1\circ\varepsilon_n(x\otimes\underline{\partial}^{\langle\underline{k}\rangle}). \tag{4.2.2.7.1}$$

First remark that these elements do not depend on the choice of n which justifies the notation. Moreover, the cocycle condition is equivalent to the condition that the formula

$$\sum_{\underline{k}=\sup\{\underline{i},\underline{j}\}}^{\underline{i}+\underline{j}}\frac{\underline{k}!}{(\underline{i}+\underline{j}-\underline{k})!(\underline{k}-\underline{i})!(\underline{k}-\underline{j})!}\frac{q_{\underline{i}}^{(m)}!q_{\underline{j}}^{(m)}!}{q_{\underline{k}}^{(m)}!}x\cdot\underline{\partial}^{\langle\underline{k}\rangle} = (x\cdot\underline{\partial}^{\langle\underline{i}\rangle})\cdot\underline{\partial}^{\langle\underline{j}\rangle} \tag{4.2.2.7.2}$$

holds for any section x of $\mathcal{E}$, for any $\underline{i}, \underline{j} \in \mathbb{N}^d$. Indeed, as for the proof of 2.1.2.8 the family (ε_n) is equivalent to the family of $\mathcal{B}_X$-linear homomorphisms of the form $\mu_n^{\mathcal{M}} \colon \widetilde{p}_{1*}(\mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{D}^{(m)}_{X,n}) \to \mathcal{M}$ such that $\mu_0^{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$ and the analogue (i.e. replace $\mathcal{O}_X$ by $\mathcal{B}_X$ and add some tildes) of the left square of 2.1.2.8.4 is commutative. Moreover, we have checked the cocycle condition is equivalent to the commutativity of the analogue (i.e. replace $\mathcal{O}_X$ by $\mathcal{B}_X$ and add some tildes) of the right square of 2.1.2.8.4, i.e. of the associativity of the definition 4.2.2.7.1 (recall by construction, $x \cdot \underline{\partial}^{\langle \underline{k} \rangle} := \mu_n(x \otimes \underline{\partial}^{\langle \underline{k} \rangle})$. By $\mathcal{B}_X$-linearity of $\mu_n^{\mathcal{M}}$, we reduce to the case where the operators are of the form $\underline{\partial}^{\langle \underline{i} \rangle}$ and we are done thanks to the formula 3.2.3.13.1

4.2.3 Internal tensor products and internal homomorphisms

Proposition 4.2.3.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.*

(a) There exists on the tensor product $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}$ a unique structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module functorial in $\mathcal{E}$ and $\mathcal{F}$ such that, for any system of logarithmic coordinates on an open subset $U \subset X$, and any $\underline{k} \in \mathbb{N}^d$, $x \in \Gamma(U, \mathcal{E})$, $y \in \Gamma(U, \mathcal{F})$, we have

$$\underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x \otimes y) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{i} \rangle} x \otimes \underline{\partial}_\sharp^{\langle \underline{k} - \underline{i} \rangle} y. \tag{4.2.3.1.1}$$

Moreover, the following formula holds

$$\underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} \rangle}(x \otimes y) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle} x \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} - \underline{i} \rangle} y. \tag{4.2.3.1.2}$$

(b) There exists on the sheaf $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F})$ a unique structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module functorial in $\mathcal{E}$ and $\mathcal{F}$ such that, for any system of logarithmic coordinates on a open subset $U \subset X$, and any $\underline{k} \in \mathbb{N}^d$, $x \in \Gamma(U, \mathcal{E})$, $\phi \colon \mathcal{E}_{|U} \to \mathcal{F}_{|U}$, we have

$$(\underline{\partial}_\sharp^{\langle \underline{k} \rangle} \phi)(x) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k} - \underline{i} \rangle}(\phi(\underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle} x)), \tag{4.2.3.1.3}$$

Moreover, the following formula holds

$$(\underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} \rangle} \phi)(x) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} - \underline{i} \rangle}(\phi(\underline{\partial}_\sharp^{\langle \underline{i} \rangle} x)). \tag{4.2.3.1.4}$$

(c) Let $\mathcal{D}$ be a sheaf of rings on X. Suppose that the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{E}$ or $\mathcal{F}$ extends to a structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \mathcal{D})$-bimodule. Then the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}$ extends to a structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \mathcal{D})$-bimodule where the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module is the tensor product structure . In the same way, the sheaf $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F})$ is endowed with a canonical structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \mathcal{D})$-bimodule where the left structure is the internal homomorphism structure.

Proof. (a) The proof of (a) is similar to that of 2.1.3.1.(a): modulo the canonical isomorphisms of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-modules

$$\widetilde{p}^n_{i,(m)*} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} (\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}) \xrightarrow{\sim} (\widetilde{p}^n_{i,(m)*} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}} (\widetilde{p}^n_{i,(m)*} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{F}), \tag{4.2.3.1.5}$$

we endow the sheaf $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}$ with the m-PD stratification with coefficients in $\mathcal{B}_X$ by setting

$$\epsilon_n^{\mathcal{E} \otimes \mathcal{F}} := \varepsilon_n^{\mathcal{E}} \otimes \varepsilon_n^{\mathcal{F}} \colon \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} (\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}) \xrightarrow{\sim} (\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}) \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}.$$

By using the formulas 4.2.1.5.4, we get 4.2.3.1.1 similarly to 2.1.3.1.(a). By replacing the use of the formula 4.2.1.5.4 by that of 4.2.1.5.5, instead of 4.2.3.1.1 similarly to 3.4.2.7.2, we get the formula 4.2.3.1.2 by symmetrical computations.

(b) Similarly to 2.1.4.1.a, modulo the canonical isomorphisms of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-modules

$$\widetilde{p}^n_{i,(m)*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F}) \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}}(\widetilde{p}^n_{i,(m)*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{B}_X}\mathcal{E}, \widetilde{p}^n_{i,(m)*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{B}_X}\mathcal{F}),$$

we endow $\mathcal{G} := \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})$ with a m-PD-structure with coefficients in $\mathcal{B}_X$ by setting

$$\epsilon^{\mathcal{G}}_n = \mathcal{H}om_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}}((\epsilon^{\mathcal{E}}_n)^{-1}, \epsilon^{\mathcal{F}}_n).$$

By copying (resp. doing symmetrical computations of that of) the proof of 2.1.4.1.a, we check then the formula 4.2.3.1.3 (resp. 4.2.3.1.4) by using 4.2.1.5.4 and 4.2.1.5.5 (resp. and the equality $(\epsilon^{\mathcal{G}}_n)^{-1} = \mathcal{H}om_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}}(\epsilon^{\mathcal{E}}_n, (\epsilon^{\mathcal{F}}_n)^{-1})$).

The functoriality in $\mathcal{E}$ and $\mathcal{F}$ of these structures of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules is a consequence for instance of formula 4.2.3.1.1 and 4.2.3.1.3. It follows from this functoriality that the part (c) is exact (we can also prove it directly via the formula 4.2.3.1.1 and 4.2.3.1.3). □

Remark 4.2.3.2. When the log structure of $X^\sharp$ and $S^\sharp$ are trivial, then replacing logarithmic coordinates by coordinates, the formulas of Propositions 2.1.4.1 and 2.1.3.1 still holds with coefficients. Indeed, we can either check it similarly or deduce them from 3.4.1.3.1.

Remark 4.2.3.3. Let $f\colon X^\sharp \to Y^\sharp$ be a morphism of fine log smooth $S^\sharp$-schemes (resp. fine log smooth formal $S^\sharp$-schemes). Similarly to 4.2.1.1, we define on a $f^{-1}\mathcal{B}_Y$-module $\mathcal{E}$ an m-PD-stratification as being the data of a compatible family of $f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp(m)}$-linear isomorphisms

$$\epsilon^{\mathcal{E}}_n\colon f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp(m)} \otimes_{f^{-1}\mathcal{B}_Y} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp(m)}$$

satisfying two similar to 4.2.1.1 conditions. As for 4.2.1.5, if $\mathcal{E}$ is a $f^{-1}\mathcal{B}_Y$-module, then there is an equivalence between the data of a structure of left $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}$-module extending its structure of $f^{-1}\mathcal{B}_Y$-module and the data of an m-PD-stratification. Moreover, if $\mathcal{E}$ and $\mathcal{F}$ are two left $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}$-modules, we endow the tensor product $\mathcal{E} \otimes_{f^{-1}\mathcal{B}_Y} \mathcal{F}$ (resp. $\mathcal{H}om_{f^{-1}\mathcal{B}_Y}(\mathcal{E},\mathcal{F})$) with a structure of left $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}$-module taking the stratification $\epsilon^{\mathcal{E}}_n \otimes_{f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp(m)}} \epsilon^{\mathcal{F}}_n$ (resp. $\mathcal{H}om_{f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp(m)}}((\epsilon^{\mathcal{E}}_n)^{-1}, \epsilon^{\mathcal{F}}_n)$).

Remark 4.2.3.4. The structures of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules defined in the corollary above, does not depend of the choice of the level. Let us clarify what it means. Let $m' \leq m$ be a couple of nonnegative integers and let us denote by then $\mathrm{forg}_{m',m}$ the forgetful functor from the category of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules to that of $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}$-modules. If $\mathcal{E}$ and $\mathcal{F}$ are two $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, then we have the equalities

$$\mathrm{forg}_{m',m}(\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}) = \mathrm{forg}_{m',m}(\mathcal{E}) \otimes_{\mathcal{B}_X} \mathrm{forg}_{m',m}(\mathcal{F}),$$
$$\mathrm{forg}_{m',m}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F}) = \mathcal{H}om_{\mathcal{B}_X}(\mathrm{forg}_{m',m}\mathcal{E}, \mathrm{forg}_{m',m}\mathcal{F}).$$

Indeed, this is a consequence of formula 4.2.3.1.1 and 4.2.3.1.3, as well as of the relation 1.4.2.5.2.

Proposition 4.2.3.5. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{M}$, $\mathcal{N}$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.*

(a) There exists on $\mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{E}$ (resp. $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{M})$) a unique structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module which is functorial in $\mathcal{M}$ and $\mathcal{E}$ such that, for any system of logarithmic coordinates we have open $U \subset X$, and any $\underline{k} \in \mathbb{N}^d$, $x \in \Gamma(U,\mathcal{E})$, $y \in \Gamma(U,\mathcal{M})$ (resp. $\phi\colon \mathcal{E}_{|U} \to \mathcal{M}_{|U}$), we have

$$(y \otimes x)\underline{\partial}_\sharp^{\langle \underline{k}\rangle} = \sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} y\underline{\partial}_\sharp^{\langle \underline{k}-\underline{h}\rangle} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{h}\rangle} x, \tag{4.2.3.5.1}$$

$$(\phi\underline{\partial}_\sharp^{\langle \underline{k}\rangle})(x) = \sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \phi(\underline{\partial}_\sharp^{\langle \underline{h}\rangle} x)\underline{\partial}_\sharp^{\langle \underline{k}-\underline{h}\rangle}. \tag{4.2.3.5.2}$$

Moreover, the following formulas hold

$$(y \otimes x)\widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}\rangle} = \sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} y\widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}-\underline{h}\rangle} \otimes \underline{\partial}_\sharp^{\langle \underline{h}\rangle} x, \tag{4.2.3.5.3}$$

$$(\phi\widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}\rangle})(x) = \sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \phi(\widetilde{\underline{\partial}}_\sharp^{\langle \underline{h}\rangle} x)\widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}-\underline{h}\rangle}. \tag{4.2.3.5.4}$$

(b) *There exists on* $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{N},\mathcal{M})$ *a unique structure of left* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module functorial in* $\mathcal{N}$ *and* $\mathcal{M}$ *such that, for any system of logarithmic coordinates we have open* $U\subset X$*, and any* $\underline{k}\in\mathbb{N}^d$*,* $z\in\Gamma(U,\mathcal{N})$*,* $\psi\colon\mathcal{N}_{|U}\to\mathcal{M}_{|U}$*, we have*

$$(\underline{\partial}_\sharp^{\langle \underline{k}\rangle}\psi)(z)=\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\psi(z\underline{\partial}_\sharp^{\langle \underline{h}\rangle})\underline{\widetilde{\partial}}_\sharp^{\langle \underline{k}-\underline{h}\rangle}. \tag{4.2.3.5.5}$$

Moreover, the following formula holds

$$(\underline{\widetilde{\partial}}_\sharp^{\langle \underline{k}\rangle}\psi)(z)=\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\psi(z\underline{\widetilde{\partial}}_\sharp^{\langle \underline{h}\rangle})\underline{\partial}_\sharp^{\langle \underline{k}-\underline{h}\rangle}. \tag{4.2.3.5.6}$$

(c) *Let* $\mathcal{D}$ *be a sheaf of rings on* X*. If the structure of right (resp. left)* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module of* $\mathcal{M}$ *(resp.* $\mathcal{E}$*) extends to a structure of* $(\mathcal{D},\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*-bimodule (resp.* $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp},\mathcal{D})$*-bimodule), then the structure of right* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module of* $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E}$ *extends to a structure of (resp. right)* $(\mathcal{D},\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*-bimodule and that of* $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{M})$ *extends to a structure of* $(\mathcal{D},\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp},\mathcal{D})$*-bimodule.*

If the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module of* $\mathcal{N}$ *(resp.* $\mathcal{M}$*) extends to a structure of* $(\mathcal{D},\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*-bimodule, then the structure of left* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module of* $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{N},\mathcal{M})$ *extends to a structure of (resp. left)* $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp},\mathcal{D})$*-bimodule.*

In the same way, these structures of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module do not depend on the level* m*.*

Proof. By copying the proof of 2.1.3.1 and 2.1.4.1, we get the formulas 4.2.3.5.1, 4.2.3.5.2 and 4.2.3.5.5 from by using 4.2.1.5.4 and 4.2.1.5.5. By doing some symmetrical computations (more precisely see the part (a) of the proof of 4.2.3.1), we get the symmetrical formulas 4.2.3.1.4 and 4.2.3.1.2. □

4.2.3.6. Let $\mathcal{E},\mathcal{F}$ be two quasi-nilpotent left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, $\mathcal{M}$, $\mathcal{N}$ be two quasi-nilpotent right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. Then $\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F}$ (resp. $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E}$, resp. $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})$, resp. $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{M})$, resp. $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{N})$) is a quasi-nilpotent left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Indeed, this is easily checked by using the local formulas of 4.2.3.1 and 4.2.3.5.

Remark 4.2.3.7. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{M}$ a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule. To avoid the risks of confusions, we will keep the following convention : the tensor product structure of $\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{M}$ comes from the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{M}$ whereas that of $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E}$ can be computed via the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{M}$.

Remark 4.2.3.8. Let $\mathcal{C}_X$ (resp. $\mathcal{C}'_X$) be a $\mathcal{B}_X$-algebra commutative endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module such that $\mathcal{B}_X\to\mathcal{C}_X$ (resp. $\mathcal{B}_X\to\mathcal{C}'_X$) is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. The tensor product structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on the $\mathcal{B}_X$-algebra $\mathcal{C}_X\otimes_{\mathcal{B}_X}\mathcal{C}'_X$ is compatible. We have moreover the morphisms of algebras $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear $\mathcal{C}_X\to\mathcal{C}_X\otimes_{\mathcal{B}_X}\mathcal{C}'_X$ and $\mathcal{C}'_X\to\mathcal{C}_X\otimes_{\mathcal{B}_X}\mathcal{C}'_X$.

Proposition 4.2.3.9. *Let* $\mathcal{E}$ *and* $\mathcal{G}$ *be two left* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-modules and* $\mathcal{F}$ *be a right or left* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module. The following canonical isomorphisms*

$$\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{G}\xrightarrow{\sim}\mathcal{G}\otimes_{\mathcal{B}_X}\mathcal{E},\ \mathcal{E}\otimes_{\mathcal{B}_X}(\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G})\xrightarrow{\sim}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}. \tag{4.2.3.9.1}$$

are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-linear.*

Proof. This is a consequence of the formulas 4.2.3.5.1 and 4.2.3.1.1. □

4.2.4 Cartan isomorphisms and other relations between tensor products and homomorphisms

Proposition 4.2.4.1. *Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be three left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and $\mathcal{M},\mathcal{N}$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. The following canonical isomorphisms are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear :*

$$(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\xrightarrow{\sim}\mathcal{E}\otimes_{\mathcal{B}_X}(\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G}),\tag{4.2.4.1.1}$$

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{B}_X}(\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G}),\ \mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F}\xrightarrow{\sim}\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{E}.\tag{4.2.4.1.2}$$

Proof. We prove that these isomorphisms are horizontal which is straightforward by definition. □

Notation 4.2.4.2. Let $\mathcal{E},\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.

(a) Let $\mathcal{K}$ be a (resp. left, resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule. In both respective cases, usually, one $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structure of $\mathcal{K}$ is called the "left structure" and the other one is called the "right one". In the non-respective case, the left (resp. right) structure is the structure of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.

By convention, when we write $\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{K}$ (resp. $\mathcal{K}\otimes_{\mathcal{B}_X}\mathcal{E}$), we use the left (resp. right) structure of $\mathcal{K}$ to compute the tensor product.

Moreover, $\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{K}$ is endowed with a structure of (resp. left, resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule as follow: the "left structure" is by convention the structure given by tensor product (in the sense of 4.2.3.5) from the left structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{K}$ and from the structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{E}$, the other one given by functoriality from the (resp. left, resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure of $\mathcal{K}$ is called the " right structure".

Similarly, $\mathcal{K}\otimes_{\mathcal{B}_X}\mathcal{E}$ is endowed with a structure of (resp. left, resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule: the left structure is given by functoriality and the right one is given by the tensor product.

Similarly, in the case where $\mathcal{K}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule, $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{K}$ (resp. $\mathcal{K}\otimes_{\mathcal{B}_X}\mathcal{M}$) is endowed with a structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule: the right (resp. left) structure is given by functoriality and the left (resp. right) one is given by the tensor product.

(b) Let $\mathcal{K}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule. To distinguish which structure is used, we write

$$(\mathcal{K})_r\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E},\qquad(\mathcal{K})_l\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}\tag{4.2.4.2.1}$$

where the symbole r (resp. l) means that the right (resp. left) structure is taken to compute the tensor product. We keep the same notation in similar situation when a choice is needed in the computation of tensor products. Remark that by bifunctoriality we have the canonical isomorphism:

$$((\mathcal{K})_r\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F}\xrightarrow{\sim}((\mathcal{K})_l\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}.\tag{4.2.4.2.2}$$

Proposition 4.2.4.3 (Associativity of the tensor product)**.** *Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{N}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule, $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. With notation 4.2.4.2, the canonical bijections*

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{N})_r\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{B}_X}(\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}),\tag{4.2.4.3.1}$$

$$\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{N}\otimes_{\mathcal{B}_X}\mathcal{E})\xrightarrow{\sim}(\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{N})\otimes_{\mathcal{B}_X}\mathcal{E}.\tag{4.2.4.3.2}$$

given by associativity of the tensor product are isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.

Similarly, when $\mathcal{E}$ and $\mathcal{M}$ are two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, $\mathcal{N}$ is a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule (resp. right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule), then 4.2.4.3.1 is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear. Similarly, when $\mathcal{M}$ right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{N}$ is a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule, and $\mathcal{E}$ is a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module then 4.2.4.3.2 is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. 1) Let us prove the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity of the map 4.2.4.3.1. Since the other cases can be checked similarly, we can only check the first case. a) By associativity of the tensor product, we have the canonical isomorphisms of $\mathcal{B}_X$-modules

$$\theta\colon (\mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E})\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E} \tag{4.2.4.3.3}$$

given by $(x\otimes P)\otimes y\mapsto x\otimes Py$ where x, y, P are local sections of respectively $\mathcal{M}$, $\mathcal{E}$, $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. Let us check θ is right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear. Since this is local, we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates. By additivity, we reduce to check $\theta(((x\otimes 1)\otimes y)\cdot\underline{\partial}_\sharp^{\langle\underline{k}\rangle})=(x\otimes y)\cdot\underline{\partial}_\sharp^{\langle\underline{k}\rangle}$. We compute

$$((x\otimes 1)\otimes y)\cdot\underline{\partial}_\sharp^{\langle\underline{k}\rangle}=\Big((x\otimes 1)\cdot_l\underline{\partial}_\sharp^{\langle\underline{k}\rangle}\Big)\otimes y\overset{4.2.3.5.1}{=}\left(\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}x\underline{\partial}_\sharp^{\langle\underline{k}-\underline{h}\rangle}\otimes\underline{\widetilde{\partial}}_\sharp^{\langle\underline{h}\rangle}\right)\otimes y$$

where $\cdot_l$ means that we take the left structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of the right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule $\mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. This yields

$$\theta(((x\otimes 1)\otimes y)\cdot\underline{\partial}_\sharp^{\langle\underline{k}\rangle})=\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}x\underline{\partial}_\sharp^{\langle\underline{k}-\underline{h}\rangle}\otimes\underline{\widetilde{\partial}}_\sharp^{\langle\underline{h}\rangle}y\overset{4.2.3.5.1}{=}(x\otimes y)\cdot\underline{\partial}_\sharp^{\langle\underline{k}\rangle}.$$

b) Replacing $\mathcal{E}$ by $\mathcal{N}$ in 4.2.4.3.3, we get by functoriality that the canonical map

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{N}\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{N} \tag{4.2.4.3.4}$$

is an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodules. Replacing $\mathcal{E}$ by $\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}$ in 4.2.4.3.3, we get the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E})\xrightarrow{\sim}\mathcal{M}\otimes_{\mathcal{B}_X}(\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}). \tag{4.2.4.3.5}$$

This yields the general case via the isomorphisms:

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{N})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}\underset{4.2.4.3.4}{\xleftarrow{\sim}}(\mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}\underset{4.2.4.3.5}{\xrightarrow{\sim}}\mathcal{M}\otimes_{\mathcal{B}_X}(\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}).$$

2) We proceed similarly to check the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity of 4.2.4.3.2. □

Proposition 4.2.4.4. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{F}$ be a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Let $\mathcal{M}$, $\mathcal{N}$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. The canonical isomorphism*

$$\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})\xrightarrow{\sim}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F}),\qquad \mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{N})\xrightarrow{\sim}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{M}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp},\mathcal{N}),$$

are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. Let us denote by θ the first isomorphism of 4.2.4.4. Let us consider the non-respective case. By construction, this isomorphism is $\mathcal{B}_X$-linear. Hence, it is sufficient to prove that for any $\underline{k}\in\mathbb{N}^d$, for any sections e of $\mathcal{E}$, P of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ we have

$$\theta(\underline{\partial}_\sharp^{\langle\underline{k}\rangle}\cdot\phi)(P\otimes e)=(\underline{\partial}_\sharp^{\langle\underline{k}\rangle}\cdot\theta(\phi))(P\otimes e).$$

We compute

$$\theta(\underline{\partial}_\sharp^{\langle\underline{k}\rangle}\cdot\phi)(P\otimes e)=P\cdot\Big((\underline{\partial}_\sharp^{\langle\underline{k}\rangle}\cdot\phi)(e)\Big)\overset{4.2.3.1.3}{=}P\cdot\sum_{\underline{i}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{i}\end{smallmatrix}\right\}\underline{\partial}_\sharp^{\langle\underline{k}-\underline{i}\rangle}(\phi(\underline{\widetilde{\partial}}_\sharp^{\langle\underline{i}\rangle}e)), \tag{4.2.4.4.1}$$

Since $(P\otimes e)\underline{\partial}_\sharp^{\langle k\rangle} \overset{4.2.3.5.1}{=} \sum_{\underline{i}\leq\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} P\underline{\partial}_\sharp^{\langle \underline{k}-\underline{i}\rangle} \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{i}\rangle} e$, then we get

$$(\underline{\partial}_\sharp^{\langle \underline{k}\rangle} \cdot \theta(\phi))(P\otimes e) = \sum_{\underline{i}\leq\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} P\underline{\partial}_\sharp^{\langle \underline{k}-\underline{i}\rangle} \cdot (\phi(\underline{\widetilde{\partial}}_\sharp^{\langle \underline{i}\rangle} e)).$$

Hence, we are done. The other and respective cases are checked similarly. □

Proposition 4.2.4.5. *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be three left or right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.*

(a) We have the canonical isomorphism:

$$\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F},\,\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\,\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\,\mathcal{G})). \tag{4.2.4.5.1}$$

(b) If the structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{E}$, $\mathcal{F}$ or $\mathcal{G}$ extends to a structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule or left bimodule, then the isomorphism 4.2.4.5.1 is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. The isomorphism 4.2.4.7.1 is equal to the composite $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism:

$$\begin{aligned}
\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F},\mathcal{G}) &\overset{4.2.4.4}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F}),\mathcal{G}) \\
&\overset{4.2.4.3}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}((\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{F}),\mathcal{G}) \\
&\overset{4.6.3.9}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{F},\mathcal{G})) \\
&\overset{4.2.4.4}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\mathcal{G})) \overset{4.2.4.4}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\mathcal{G})).
\end{aligned}$$

□

Corollary 4.2.4.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be two complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. If $\mathcal{F}$ is a K-flat as complex of $\mathcal{B}_X$-modules and that $\mathcal{G}$ is a K-injective object of $K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, then $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\mathcal{G})$ is a K-injective object of $K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.*

Proof. For any $\mathcal{E}\in K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, it follows from 4.2.4.5.1 that we have

$$\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F},\,\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\,\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\,\mathcal{G}))\,.$$

If $\mathcal{E}$ is acyclic, then it follows from the hypotheses on $\mathcal{F}$ and $\mathcal{G}$ that the left term is also acyclic. Hence, we are done. □

Proposition 4.2.4.7 (Cartan isomorphism bis)**.** *Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ three left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. The Cartan isomorphism*

$$\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\mathcal{G})), \tag{4.2.4.7.1}$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. When all modules are left modules, the isomorphism **??** is equal to the composite $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism:

$$\begin{aligned}
\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F},\mathcal{G}) &\overset{4.2.4.3}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{F}),\mathcal{G}) \\
&\overset{4.6.3.9}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{F},\mathcal{G})) \overset{4.2.4.4}{\xrightarrow{\sim}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\mathcal{G})).
\end{aligned}$$

□

Proposition 4.2.4.8. *Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be three left or right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules except the case where $\mathcal{E}$ is right module and $\mathcal{F}$ is a left module. The canonical morphism:*

$$\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\to\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G}) \tag{4.2.4.8.1}$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. When all modules are left modules, the morphism 4.2.4.8.1 is equal to the composite $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear morphism:

$$\begin{aligned}
&\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\xrightarrow[\sim]{4.2.4.4}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\\
&\xrightarrow[\sim]{4.2.4.3}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{G})\\
&\to\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{G}))\\
&\xrightarrow[\sim]{4.2.4.3}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G})\xrightarrow[\sim]{4.2.4.4}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G}).
\end{aligned}$$

The other cases are treated similarly. □

Proposition 4.2.4.9. *Let $\mathcal{E}$ and $\mathcal{G}$ be two left or right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and $\mathcal{F}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule. The canonical morphism*

$$\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\to\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G}), \tag{4.2.4.9.1}$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. Since the other cases are similar, we can suppose $\mathcal{E}$ and $\mathcal{G}$ are left modules. The inverse of the isomorphism 4.2.4.3.1 applied to $\mathcal{M}=\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F})$, $\mathcal{N}=\widetilde{\mathcal{D}}^{(m)}_X$ and $\mathcal{E}=\mathcal{G}$ is written

$$\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F})\otimes_{\mathcal{B}_X}\mathcal{G}\xrightarrow{\sim}\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_X}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{G}).$$

Moreover, we have the canonical morphism

$$\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{G})\to\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_X\otimes_{\mathcal{B}_X}\mathcal{G})).$$

Finally, by using 4.2.4.3.1, we obtain:

$$\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{G}))\xrightarrow{\sim}\mathcal{H}om_{1\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{G}).$$

By composing these three morphisms, we obtain 4.2.4.9.1. □

Proposition 4.2.4.10. *Let $\mathcal{E}$ and $\mathcal{G}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, $\mathcal{M}$ and $\mathcal{N}$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and $\mathcal{F}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule. The canonical morphism*

$$\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{G}\to\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{G}), \tag{4.2.4.10.1}$$

$$\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{F})\to\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{N}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F}), \tag{4.2.4.10.2}$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. The map 4.2.4.10.1 is built by composing the following morphisms:

$$\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{G}\xrightarrow[\sim]{4.2.4.4}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{G} \tag{4.2.4.10.3}$$

$$\to\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{G})\xrightarrow[\sim]{4.2.4.4}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{G}). \tag{4.2.4.10.4}$$

We proceed similarly for 4.2.4.10.2. □

Corollary 4.2.4.11. *Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be three left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and $\mathcal{M},\mathcal{N}$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. The following canonical morphisms are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear :*

$$\mathrm{ev}\colon \mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})\to\mathcal{F},\qquad \mathrm{ev}\colon \mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{N})\to\mathcal{N} \tag{4.2.4.11.1}$$

$$\mathcal{E}\to\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F},\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{E}),\qquad \mathcal{E}\to\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E}). \tag{4.2.4.11.2}$$

Proof. The first (resp. second) isomorphism of 4.2.4.11.1 corresponds via the Cartan isomorphism to the identity of $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{F})$ (resp. $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{N})$). Since the identity is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear then we conclude with 4.2.4.5.1. The first (resp. second) isomorphism of 4.2.4.11.2 correspond via the Cartan isomorphism to the identity of $\mathcal{F}\otimes_{\mathcal{B}_X}\mathcal{E}$ (resp. $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E}$). Since the identity is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear then we conclude with 4.2.4.5.1. □

4.2.5 Logarithmic transposition isomorphisms

We will need the following proposition to get the isomorphism 18.2.3.2.2 which will allow us to prove 18.2.3.12.

Proposition 4.2.5.1. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{E}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}$ the sheaves obtained by computing the tensor product via respectively the left and right structure of $\mathcal{B}_X$-algebra of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. There exists a unique isomorphism of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodules*

$$\gamma_{\mathcal{E}}\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}\xrightarrow{\sim}\mathcal{E}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \tag{4.2.5.1.1}$$

such that for each section e of $\mathcal{E}$, $\gamma_{\mathcal{E}}(1\otimes e)=e\otimes 1$. In logarithmic coordinates, we have the following formulas

$$\gamma_{\mathcal{E}}(\underline{\partial}_\sharp^{\langle \underline{k}\rangle}\otimes e)=\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\underline{\partial}_\sharp^{\langle \underline{h}\rangle}e\otimes\underline{\partial}_\sharp^{\langle \underline{k}-\underline{h}\rangle},\qquad \gamma_{\mathcal{E}}^{-1}(e\otimes\underline{\partial}_\sharp^{\langle \underline{k}\rangle})=\sum_{\underline{h}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}e\underline{\partial}_\sharp^{\langle \underline{k}-\underline{h}\rangle}\otimes\underline{\widetilde{\partial}}_\sharp^{\langle \underline{h}\rangle}. \tag{4.2.5.1.2}$$

Proof. 1) The unicity of such a homomorphism $\gamma_{\mathcal{E}}$ follows from its linearity as left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. By left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity of $\gamma_{\mathcal{E}}$ (resp. by right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity of $\gamma_{\mathcal{E}}^{-1}$), the left (resp. right) formula of 4.2.5.1.2 is a consequence of 4.2.3.1.1 (resp. 4.2.3.5.1).

2) i) We construct the map as follows $\gamma_{\mathcal{E}}$: from the homomorphism of $\mathcal{B}_X$-modules $\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ given by $x\mapsto x\otimes 1$ we get by extension the homomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules $\gamma_{\mathcal{E}}$. By construction, $\gamma_{\mathcal{E}}(P\otimes e)=P(e\otimes 1)$, where $P\in\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $e\in\mathcal{E}$.

ii) Let us check now the right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linearity of $\gamma_{\mathcal{E}}$. Since this is local, we can suppose $X^\sharp\to S^\sharp$ is endowed with logarithmic coordinates $u_1,\dots,u_r$ and keep notation 4.3.5.2. It is therefore sufficient to establish, for any $\underline{k}\in\mathbb{N}^r$,

$$\gamma_{\mathcal{E}}((1\otimes e)\underline{\widetilde{\partial}}_\sharp^{\langle \underline{k}\rangle})=(e\otimes 1)\underline{\widetilde{\partial}}_\sharp^{\langle \underline{k}\rangle}. \tag{4.2.5.1.3}$$

In regards to the right term of 4.2.5.1.3, we have $(e \otimes 1)\underline{\widetilde{\partial}}_{\sharp}^{\langle k \rangle} = e \otimes \underline{\widetilde{\partial}}_{\sharp}^{\langle k \rangle}$. As for the left term, we compute

$$\gamma_{\mathcal{E}}\left((1 \otimes e)\underline{\widetilde{\partial}}_{\sharp}^{\langle k \rangle}\right) \overset{4.2.3.5.3}{=} \gamma_{\mathcal{E}}\left(\sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_{\sharp}^{\langle h \rangle} \otimes \underline{\partial}_{\sharp}^{\langle k-h \rangle} e\right) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_{\sharp}^{\langle h \rangle} (\underline{\partial}_{\sharp}^{\langle k-h \rangle} e \otimes 1).$$

$$\overset{4.2.3.1.2}{=} \sum_{\underline{h} \leq \underline{k}} \sum_{\underline{i} \leq \underline{h}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{h} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_{\sharp}^{\langle h-i \rangle} \underline{\partial}_{\sharp}^{\langle k-h \rangle} e \otimes \underline{\widetilde{\partial}}_{\sharp}^{\langle i \rangle}$$

$$= \sum_{\underline{i} \leq \underline{k}} \left(\sum_{\underline{i} \leq \underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{h} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_{\sharp}^{\langle h-i \rangle} \underline{\partial}_{\sharp}^{\langle k-h \rangle} e \right) \otimes \underline{\widetilde{\partial}}_{\sharp}^{\langle i \rangle}.$$

To get the formula 4.2.5.1.3, it is therefore sufficient to establish the equality

$$\sum_{\underline{i} \leq \underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{h} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_{\sharp}^{\langle h-i \rangle} \underline{\partial}_{\sharp}^{\langle k-h \rangle} = \delta_{\underline{i},\underline{k}}, \qquad (4.2.5.1.4)$$

where in the sum $\underline{i}$ and $\underline{k}$ are fixed and $\delta_{\underline{i},\underline{k}}$ is the Kronecker symbol. For this purpose, let us proceed as follows.

Suppose $\mathcal{E}$ is equal to $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ (to avoid confusion between the two $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$, it is still written $\mathcal{E}$). We have the commutative diagram:

$$\begin{array}{ccc} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E} & \hookrightarrow & j_*(\mathcal{D}^{(m)}_{Y} \otimes_{\mathcal{O}_Y} \mathcal{E}|Y) \\ {\scriptstyle \gamma_{\mathcal{E}}} \downarrow {\scriptstyle 4.2.5.1.1} & & {\scriptstyle j_*(\gamma_{\mathcal{E}|Y})} \downarrow \wr \\ \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} & \hookrightarrow & j_*(\mathcal{E}|Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y}). \end{array} \qquad (4.2.5.1.5)$$

where $\gamma_{\mathcal{E}|Y}$ is the (non-logarithmic) transposition isomorphism of $\mathcal{E}|Y$ (see 2.2.2.1.1) and where horizontal morphisms are injective (this is an consequence of 3.1.1.3 and of the fact that $\mathcal{E}$ is free as $\mathcal{O}_X$-module). Indeed, horizontal morphisms are $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bilinear, vertical arrows are left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear and $1 \otimes e \mapsto e \otimes 1$ via both paths. Moreover, it follows from 2.2.2.1, that the morphism $j_*\gamma_{\mathcal{E}|Y}$ is right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Hence, $\gamma_{\mathcal{E}}$ is right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Since $\mathcal{E} = \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a free $\mathcal{O}_X$-module, then this implies that the desired formula 4.2.5.1.4 holds.

It remains to check that $\gamma_{\mathcal{E}}$ is an isomorphism. It is sufficient to construct similarly the unique morphism of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodules $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}$ which sends, for each section e of $\mathcal{E}$, $e \otimes 1$ on $1 \otimes e$. □

Example 4.2.5.2. We have the commutative diagram whose oblique isomorphisms are the canonical ones:

$$\begin{array}{ccc} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{B}_X & \xrightarrow{\gamma_{\mathcal{B}_X}} & \mathcal{B}_X \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \nwarrow \sim & & \sim \nearrow \\ & \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} & \end{array} \qquad (4.2.5.2.1)$$

4.2.5.3. For any $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, we have the isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X/S}$-bimodules

$$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}) \overset{\mathrm{l}}{\otimes}_{\mathcal{B}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \overset{\mathrm{r}}{\otimes}_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}), \qquad (4.2.5.3.1)$$

where the symbole "l" or "r" means that we use respectively the left and right structure of left $\widetilde{\mathcal{D}}^{(m)}_{X/S}$ of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}$ to compute the $\widetilde{\mathcal{D}}^{(m)}_{X/S}$ structure given by $- \otimes_{\mathcal{B}_X} -$. making commutative the following diagram commutative:

$$\begin{array}{ccc} (\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \otimes_{\mathcal{O}_X} \omega_X^{-1} & \xrightarrow[\sim]{} & \mathcal{E} \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \\ \sim \uparrow {\scriptstyle \gamma_{\mathcal{E}} \otimes_{\mathcal{O}_X} \omega_X^{-1}} & & \vdots \uparrow \\ (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes_{\mathcal{O}_X} \omega_X^{-1} & \xrightarrow[\sim]{} & (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes_{\mathcal{B}_X} \mathcal{E}. \end{array} \qquad (4.2.5.3.2)$$

In order to give a complement of the first remark of 4.3.2.7, let us give the following lemma.

Lemma 4.2.5.4. *Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $u_1, \ldots, u_r$. For any $\underline{k} \in \mathbb{N}^r$, we have the formula in $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$:*

$$b \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left(1 \otimes \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle}\right) \cdot \left(\widetilde{\underline{\partial}}_\sharp^{\langle \underline{h} \rangle}(b) \otimes 1\right), \tag{4.2.5.4.1}$$

$$b \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} \rangle_{(m)}} = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left(1 \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}-\underline{h} \rangle}\right) \cdot \left(\underline{\partial}_\sharp^{\langle \underline{h} \rangle}(b) \otimes 1\right). \tag{4.2.5.4.2}$$

When log structures are trivial, for any $\underline{k} \in \mathbb{N}^r$, we have the formula in $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S}$:

$$b \otimes \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} = \sum_{\underline{h} \leq \underline{k}} (-1)^{|\underline{h}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left(1 \otimes \underline{\partial}^{\langle \underline{k}-\underline{h} \rangle}\right) \cdot \left(\underline{\partial}^{\langle \underline{h} \rangle}(b) \otimes 1\right). \tag{4.2.5.4.3}$$

Proof. Since the formula 4.2.5.4.3 is checked similarly, let us focus on 4.2.5.4.1. It follows from (the version without coefficient of) 4.2.5.1 that we have the transposition isomorphism $\gamma_{\mathcal{B}_X} : \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. By right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linearity of $\gamma_{\mathcal{B}_X}$ we compute $\gamma_{\mathcal{B}_X}((1 \otimes b)\underline{\partial}_\sharp^{\langle \underline{k} \rangle}) = (b \otimes 1) \cdot \underline{\partial}_\sharp^{\langle \underline{k} \rangle} = b \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle}$. From (the version without coefficient of) 4.2.3.5.1, we get $(1 \otimes b)\underline{\partial}_\sharp^{\langle \underline{k} \rangle} = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{h} \rangle}(b)$. By left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linearity of $\gamma_{\mathcal{B}_X}$ this yields $\gamma_{\mathcal{B}_X}((1 \otimes b)\underline{\partial}_\sharp^{\langle \underline{k} \rangle}) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left(1 \otimes \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle}\right) \cdot \left(\widetilde{\underline{\partial}}_\sharp^{\langle \underline{h} \rangle}(b) \otimes 1\right)$. Hence, we are done. We get the formula 4.2.3.5.3 instead of 4.2.3.5.1. □

Proposition 4.2.5.5. *Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ the sheaf obtained by computing the tensor product via the left structure of $\mathcal{B}_X$-algebra of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. There exists a unique involution of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodules*

$$\delta_{\mathcal{M}} \colon \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \tag{4.2.5.5.1}$$

exchanging the two structures of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and such that, for each section m of $\mathcal{M}$, $\delta_{\mathcal{M}}(m \otimes 1) = m \otimes 1$. In logarithmic coordinates, we have the following formula

$$\delta_{\mathcal{M}}(m \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle}) = \sum_{\underline{h} \leq \underline{k}} m \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle} \otimes \widetilde{\underline{\partial}}_\sharp^{\langle \underline{h} \rangle}. \tag{4.2.5.5.2}$$

Proof. We have the canonical $\mathcal{B}_X$-linear morphism $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ given by $x \mapsto x \otimes 1$ where the structure of $\mathcal{B}_X$-module on $\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ comes from its right structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. This yields by extension the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear morphism

$$\delta_{\mathcal{M}} \colon \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \tag{4.2.5.5.3}$$

for the right (resp. left) structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of the left (resp. right) term. Such a morphism is the unique one so that $\delta_{\mathcal{M}}(x \otimes 1) = x \otimes 1$. By $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity, we get the formula 4.2.5.5.2 from 4.2.3.5.1.

By an easy computation, doing a similar to 4.2.5.1 computation, we can check the second right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity (i.e. for the left structure for the source and the right structure for the target) from 2.2.2.1 and the local formula. Hence, $\delta_{\mathcal{M}} \circ \delta_{\mathcal{M}} = \mathrm{id}$. □

4.2.5.6. We get from 4.2.5.5 the transposition isomorphism

$$\widetilde{\delta}_{X^\sharp/S^\sharp} := \delta_{\widetilde{\omega}_{X^\sharp/S^\sharp}} \colon \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.2.5.6.1}$$

exchanging left and right structure. By applying to this isomorphism the functor $- \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}$ to the right (resp. left) structure from the source (resp. target), we get the isomorphism

$$\widetilde{\alpha}_{X^\sharp/S^\sharp} : \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \tag{4.2.5.6.2}$$

where $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp} = (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \overset{\mathrm{r}}{\otimes}_{\mathcal{O}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp} = \widetilde{\omega}_{X^\sharp/S^\sharp} \overset{\mathrm{l}}{\otimes}_{\mathcal{O}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp})$, where the symbol l or r means we choose respectively the left or right structure of the corresponding (left or right) bimodule in order to compute the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structure given by the internal tensor product $- \otimes_{\mathcal{O}_X} -$ (see 4.2.3.1 or 4.2.3.5).

By applying to this isomorphism the functor $- \otimes_{\mathcal{B}_X} \widetilde{\omega}_{X^\sharp/S^\sharp}$ to the left (resp. right) structure from the source (resp. target), this yields the isomorphism

$$\widetilde{\beta}_{X^\sharp/S^\sharp} : \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}. \tag{4.2.5.6.3}$$

exchanging the two structures of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.

Proposition 4.2.5.7. *We endow the sheaf $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$ with a ring structure via the transposition isomorphism $\gamma_{\mathcal{B}_X} : \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. This is the unique ring structure on $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$ such that*

$$\mathcal{B}_X \to \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X : b \mapsto 1 \otimes b, \tag{4.2.5.7.1}$$

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X : P \mapsto P \otimes 1 \tag{4.2.5.7.2}$$

are ring homomorphisms such that for any $b \in \mathcal{B}_X$ and $P \in \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ we have the formula $(P \otimes 1)(1 \otimes b) = P \otimes b$ and for any logarithmic coordinates

$$(1 \otimes b)(\underline{\partial}^{\langle \underline{k} \rangle}_\sharp \otimes 1) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k} - \underline{h} \rangle}_\sharp \otimes \widetilde{\underline{\partial}}^{\langle \underline{h} \rangle}_\sharp b. \tag{4.2.5.7.3}$$

Proof. a) Since $\varepsilon^{\mathcal{B}_X}(1 \otimes b) = b \otimes 1$, then it follows from 4.1.2.2.1 that the map 4.2.5.7.1 is a ring homomorphism.

b) Since $\varepsilon^{\mathcal{B}_X}$ are isomorphisms of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$-algebras, then $\widetilde{\underline{\partial}}^{\langle \underline{h} \rangle}_\sharp (1_{\mathcal{B}_X}) = 0$ if $\underline{h} \neq 0$ (here $1_{\mathcal{B}_X}$ is the unity of $\mathcal{B}_X$, usually simply denoted 1). Hence, it follows from the left equality of 4.2.5.1.2 that we get $\varepsilon^{\mathcal{B}_X}(P \otimes 1) = 1 \otimes P$. By using 4.1.2.2.2, this yields that the map 4.2.5.7.2 is a ring homomorphism.

c) Since $\varepsilon^{\mathcal{B}_X}$ is a ring homomorphism, we get the equality $\varepsilon^{\mathcal{B}_X}((P \otimes 1)(1 \otimes b)) = (1 \otimes P)(b \otimes 1) = P \cdot (b \otimes 1)$ (see also 4.1.2.4) for the last one). Since $\varepsilon^{\mathcal{B}_X}$ is a homomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module, we get the equality $\varepsilon^{\mathcal{B}_X}(P \otimes b) = \varepsilon^{\mathcal{B}_X}(P \cdot (1 \otimes b)) = P \cdot (b \otimes 1)$. Hence, $P \otimes b = (P \otimes 1)(1 \otimes b)$.

d) Since $\gamma^{-1}_{\mathcal{B}_X}(b \otimes \underline{\partial}^{\langle \underline{k} \rangle}_\sharp) = \gamma^{-1}_{\mathcal{B}_X}((b \otimes 1)(1 \otimes \underline{\partial}^{\langle \underline{k} \rangle}_\sharp)) = (1 \otimes b)(\underline{\partial}^{\langle \underline{k} \rangle}_\sharp \otimes 1)$, then we get the formula 4.2.5.7.3 from the right formula of 4.2.5.1.2. □

4.2.5.8. The $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure on $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$ induced by the canonical ring homomorphism $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$ (see 4.2.5.7.2) is equal to the $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure induced by the tensor product from the left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure of $\mathcal{B}_X$ and the canonical $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-bimodule structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ (see 4.2.3.5). Indeed, this comes from the formulas 4.2.5.7.3 (compare with 4.2.3.5.1) and $(P' \otimes 1)(P \otimes b) = P'P \otimes b$.

4.2.6 An isomophism switching $\mathcal{B}$ and $\mathcal{D}$ in tensor products

We give an application of the logarithmic transposition isomorphism.

Proposition 4.2.6.1 (Switching $\mathcal{B}$ and $\widetilde{\mathcal{D}}$). *Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule), $\mathcal{E},\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. We have the functorial in $\mathcal{E}$, $\mathcal{F}$, $\mathcal{M}$ canonical isomorphism of $\mathcal{O}_S$-modules (resp. left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules):*

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F}\xrightarrow{\sim}\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F}), \tag{4.2.6.1.1}$$

which is given by $(m\otimes e)\otimes f\mapsto m\otimes(e\otimes f)$ for any local section e,f,m of $\mathcal{E},\mathcal{F},\mathcal{M}$.

Proof. We have the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism

$$\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E}\xrightarrow{\sim}(\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\otimes_{\mathcal{B}_X}\mathcal{E}\xrightarrow[4.2.4.3.2]{\sim}\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}) \tag{4.2.6.1.2}$$

We get from the transposition isomorphism:

$$(\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{E})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F}\xrightarrow[4.2.6.1.2]{\sim}\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F} \tag{4.2.6.1.3}$$

$$\xrightarrow[4.2.5.1.1]{\sim}\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{F}\xrightarrow{\sim}\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{F}). \tag{4.2.6.1.4}$$

□

4.3 Coefficients extension of the ring of differential operators

We keep notation 3.4. Let $\rho\colon\mathcal{B}_X\to\mathcal{B}'_X$ be a homomorphism of commutative $\mathcal{O}_X$-algebras. We suppose $\mathcal{B}_X$ and $\mathcal{B}'_X$ are endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module which is compatible to its underlying $\mathcal{O}_X$-algebra structure such that $\mathcal{B}_X\to\mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

4.3.1 Some homomorphisms of rings of differential operators

Before considering the case where the level is fixed, let us give some preliminary ring homomorphism constructions. Let $m'\geq m$ be a second integer. Let $\sigma\colon\mathcal{B}_X\to\mathcal{C}_X$ be a homomorphism of commutative $\mathcal{O}_X$-algebras. We suppose $\mathcal{C}_X$ is endowed with a structure of left $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$-module which is compatible to its underlying $\mathcal{O}_X$-algebra structure. We suppose moreover that $\mathcal{B}_X\to\mathcal{C}_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear where $\mathcal{C}_X$ is viewed as a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module via the canonical homomorphism $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\to\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$.

Proposition 4.3.1.1. *We generalize with some coefficients the homomorphisms of 3.2.3.5.1 as follows.*

(a) The canonical $\mathcal{B}_X$-linear morphism

$$\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\to\mathcal{C}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \tag{4.3.1.1.1}$$

given by $b\otimes P\mapsto\sigma(b)\otimes\rho_{m',m}(P)$ is a ring homomorphism.

(b) The canonical $\mathcal{B}_X$-linear morphism we have the canonical map

$$\mathcal{D}^{(m)}_{X^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X\to\mathcal{D}^{(m')}_{X^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X \tag{4.3.1.1.2}$$

given by $P\otimes b\mapsto\rho_{m',m}(P)\otimes\sigma(b)$ is a ring homomorphism. Moreover, we have the commutative diagram

$$\begin{array}{ccc}\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp} & \xrightarrow{4.3.1.1.1} & \mathcal{C}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m')}_{X^\sharp}\\ {\scriptstyle\gamma_{\mathcal{B}_X}}\uparrow{\scriptstyle 4.2.5.1.1} & & {\scriptstyle\gamma_{\mathcal{C}_X}}\uparrow{\scriptstyle 4.2.5.1.1}\\ \mathcal{D}^{(m)}_{X^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X & \xrightarrow{4.3.1.1.2} & \mathcal{D}^{(m')}_{X^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X.\end{array} \tag{4.3.1.1.3}$$

Proof. a) Removing $/S^\sharp$ in the notation, consider the diagram:

$$\begin{array}{ccccccccccc}
\mathcal{P}^{n+n'}_{X^\sharp(m')} & \xrightarrow{\delta^{n,n'}_{(m')}} & \mathcal{P}^{n}_{X^\sharp(m')} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp(m')} & \xrightarrow{\mathrm{id}\otimes P'} & \mathcal{P}^{n}_{X^\sharp(m')} \otimes_{\mathcal{O}_X} \mathcal{B}_X & \xrightarrow{\epsilon_n^{\mathcal{B}_X}} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X^\sharp(m')} & \xrightarrow{\mathrm{id}\otimes P} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{B}_X & \xrightarrow{\mu} & \mathcal{B}_X \\
\downarrow{\scriptstyle \psi^{n+n'}_{m,m'}} & & \downarrow{\scriptstyle \psi^{n}_{m,m'}\otimes\psi^{n'}_{m,m'}} & & \downarrow{\scriptstyle \psi^{n}_{m,m'}\otimes\sigma} & & \downarrow{\scriptstyle \sigma\otimes\psi^{n}_{m,m'}} & & \downarrow{\scriptstyle \sigma\otimes\sigma} & & \downarrow{\scriptstyle \sigma} \\
\mathcal{P}^{n+n'}_{X^\sharp(m)} & \xrightarrow{\delta^{n,n'}_{(m)}} & \mathcal{P}^{n}_{X^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp(m)} & \xrightarrow{\mathrm{id}\otimes P'} & \mathcal{P}^{n}_{X^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{C}_X & \xrightarrow{\epsilon_n^{\mathcal{C}_X}} & \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X^\sharp(m)} & \xrightarrow{\mathrm{id}\otimes P} & \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{C}_X & \xrightarrow{\nu} & \mathcal{C}_X
\end{array} \qquad (4.3.1.1.4)$$

where ν is the multiplication of $\mathcal{C}_X$, $\psi^n_{m,m'}$ are the canonical morphisms (see notation 3.2.2.10). Since σ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear then the middle square is commutative. Since so are the other square by functoriality, then the diagram 4.3.1.1.4 is commutative. Since the composition of the top or bottom horizontal arrows is by definition the product law (see 4.1.2.1.1), then the canonical $\mathcal{B}_X$-linear morphism $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ is a ring homomorphism.

b) Via an easy computation using the left formula of 4.2.5.1.2, we can check the commutativity of the diagram 4.3.1.1.3. Since 4.3.1.1.1 and the transposition isomorphisms are ring homomorphisms, then so is 4.3.1.1.2. □

Proposition 4.3.1.2. *Using respectively the ring homomorphisms of 4.3.1.1.1 and 4.3.1.1.2, we have the following isomorphisms of bimodules.*

(a) The canonical morphism of $(\mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp})$*-bimodules*

$$(\mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \otimes_{\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}) \to \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}, \qquad (4.3.1.2.1)$$

given by $(c \otimes P) \otimes (b \otimes P') \mapsto (c \otimes \rho_{m',m}(P))(\sigma(b) \otimes P')$*, is an isomorphism.*

(b) The canonical morphism of $(\mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{C}_X)$*-bimodules,*

$$(\mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X) \otimes_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X} (\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{C}_X) \to \mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{C}_X, \qquad (4.3.1.2.2)$$

given by $(P' \otimes b) \otimes (P \otimes c) \mapsto (P' \otimes \sigma(b))(\rho_{m',m}(P) \otimes c)$*, is an isomorphism.*

Proof. a) From the ring homomorphism $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \to \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ (see 4.3.1.1.1), we get by extension the homomorphism of $(\mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp})$-bimodules 4.3.1.2.1. Since the canonical homomorphism $\mathcal{C}_X \otimes_{\mathcal{B}_X} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \to \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ of $(\mathcal{C}_X, \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$-bimodules is an isomorphism, then the canonical homomorphism of $(\mathcal{C}_X, \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp})$-bimodules:

$$\mathcal{C}_X \otimes_{\mathcal{B}_X} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}) \to (\mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \otimes_{\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}). \qquad (4.3.1.2.3)$$

is an isomorphism. By composing 4.3.1.2.3 with 4.3.1.2.1, we get the canonical homomorphism

$$\mathcal{C}_X \otimes_{\mathcal{B}_X} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}) \to \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \qquad (4.3.1.2.4)$$

which is an isomorphism. Hence, so is 4.3.1.2.1.

b) We proceed similarly to check the map 4.3.1.2.2 is an isomorphism. □

Proposition 4.3.1.3. *We have the following isomorphisms of bimodules.*

(a) The canonical morphism of $(\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}, \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$*-bimodules*

$$(\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}) \otimes_{\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \to \mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}, \qquad (4.3.1.3.1)$$

given by $(b \otimes P') \otimes (c \otimes P) \mapsto (\sigma(b) \otimes P')(c \otimes \rho_{m',m}(P))$*, is an isomorphism.*

(b) *The canonical morphism of* $(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X, \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X)$*-bimodules,*

$$(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X)\otimes_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X}(\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X)\to\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X, \tag{4.3.1.3.2}$$

given by $(P\otimes c)\otimes(P'\otimes b)\mapsto(\rho_{m',m}(P)\otimes c)(P'\otimes\sigma(b))$*, is an isomorphism.*

Proof. Consider the following diagram:

$$\begin{array}{ccc}
(\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m')}_{X^\sharp/S^\sharp})\otimes_{\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{C}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) & \xrightarrow[4.3.1.3.1]{} & \mathcal{C}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m')}_{X^\sharp/S^\sharp} \\
\uparrow{\scriptstyle \gamma_{\mathcal{B}_X}\otimes\gamma_{\mathcal{C}_X}}\ \wr & & \uparrow{\scriptstyle\gamma_{\mathcal{C}_X}}\ \wr \\
(\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X)\otimes_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}_X}(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X) & \xrightarrow[4.3.1.2.2]{\sim} & \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{C}_X.
\end{array} \tag{4.3.1.3.3}$$

whose left vertical arrow is well defined thanks to 4.3.1.1.3. Since the arrows of the square 4.3.1.3.3 are left $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$-linear, then to check its commutativity, we reduce to compute that the image of the element $(1\otimes b)\otimes(1\otimes c)$ of the bottom left term of the square is sent via both maps to the same element of the top right term, which is the case (more precisely, we compute $(\sigma(b)c\otimes 1)$. Via the commutativity of 4.3.1.3.3, we obtain that the canonical morphism 4.3.1.3.1 is an isomorphism. Similarly, via the transposition isomorphisms, since 4.3.1.2.1 is an isomorphism, then so is 4.3.1.3.2. □

4.3.2 Semi-linear PD-stratifications of level m

Lemma 4.3.2.1. *We suppose* $\mathcal{B}'_X$ *is endowed with a compatible structure of left* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-module such that* $\mathcal{B}_X\to\mathcal{B}'_X$ *is* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-linear. We still denote by* $\rho\colon\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\to\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ *the canonical ring homomorphism induced by* ρ *(see 4.3.1.1.a). From 3.4.2.5, let* $(\varepsilon_n^{\mathcal{B}_X})$ *(resp.* $(\varepsilon_n^{\mathcal{B}'_X})$*) be the m-PD-stratification with coefficients in* $\mathcal{O}_X$ *of* $\mathcal{B}_X$ *(resp. of* $\mathcal{B}'_X$*). The left* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module structure of* $\mathcal{B}'_X$ *induced via* ρ *by its left* $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-module structure gives an m-PD-stratification with coefficients in* $\mathcal{B}_X$ *of* $\mathcal{B}'_X$ *that we denote by* $(\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X})$*.*

(a) *The isomorphism* $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$ *is the one making commutative the diagram of commutative* $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$*-algebras*

$$\begin{array}{ccccc}
\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{B}_X}\mathcal{B}'_X & \xleftarrow[\varepsilon_n^{\mathcal{B}_X}\otimes\mathrm{id}]{\sim} & (\mathcal{P}^n_{X^\sharp/S^\sharp(m)}\underset{\mathcal{O}_X}{\otimes}\mathcal{B}_X)\otimes_{\mathcal{B}_X}\mathcal{B}'_X & \xrightarrow[\sim]{\alpha} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)}\underset{\mathcal{O}_X}{\otimes}\mathcal{B}'_X \\
\wr\ \vdots\ {\scriptstyle\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}} & & & & \wr\downarrow{\scriptstyle\varepsilon_n^{\mathcal{B}'_X}} \\
\mathcal{B}'_X\otimes_{\mathcal{B}'_X}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} & & \xrightarrow[\sim]{\beta} & & \mathcal{B}'_X\underset{\mathcal{O}_X}{\otimes}\mathcal{P}^n_{X^\sharp/S^\sharp(m)},
\end{array} \tag{4.3.2.1.1}$$

where α *(resp.* β*) is given by* $(\tau\otimes b)\otimes b'\mapsto\tau\otimes\rho(b)b'$ *(resp. by* $b'\otimes(b\otimes\tau)\mapsto\rho(b)b'\otimes\tau$*).*

(b) *The map* $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$ *are isomorphisms of* $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$*-algebras.*

Proof. Since this is local, we can suppose $X^\sharp\to S^\sharp$ is endowed with logarithmic coordinates $(b_\lambda)_{\lambda=1,\dots,d}$. Since $\varepsilon_n^{\mathcal{B}_X}(1\otimes 1)=1\otimes 1$, then by using the Taylor formula satisfied by $\varepsilon_n^{\mathcal{B}'_X}$ (see 3.4.2.5.4) we compute the morphism $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$ making commutative the diagram 4.3.2.1.1 should satisfied in $\mathcal{B}'_X\otimes_{\mathcal{B}'_X}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ the formula

$$\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}((1\otimes 1)\otimes b')=\sum_{|\underline{k}|\le n}\underline{\partial}_\sharp^{\langle\underline{k}\rangle}(b')\otimes(1\otimes\underline{\tau}_\sharp^{\{\underline{k}\}}),$$

for any section b' of $\mathcal{B}'_X$. Hence, since 4.3.2.1.1 is a diagram of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-algebras, since $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}\otimes_{\mathcal{B}_X}\mathcal{B}'_X$ is generated as $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-module by the elements of the form $(1\otimes 1)\otimes b'$ with b' a section of $\mathcal{B}'_X$, then we conclude thanks to the formula 4.2.1.5.4 (where by abuse of notation $1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}} = \underline{\tau}_\sharp^{\{\underline{k}\}}$). Finally, via the commutative diagram 4.3.2.1.1, we get that the isomorphism $\varepsilon_{n/\mathcal{B}_X}^{\mathcal{B}'_X}$ are more precisely isomorphisms of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-algebras. □

Remark 4.3.2.2. Taking the case where $\mathcal{B}_X$ (resp. $\mathcal{B}'_X$) is replaced in 4.3.2.1 by $\mathcal{O}_X$ (resp. by $\mathcal{B}_X$) we retrieve the lemma 4.2.1.7 from 4.3.2.1.

Lemma 4.3.2.3. *Let $\rho\colon \mathcal{B}_X \to \mathcal{B}'_X$ be an algebra homomorphism. We suppose $\mathcal{B}'_X$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module which is compatible to its underlying $\mathcal{O}_X$-algebra structure. We suppose moreover that $\mathcal{B}_X \to \mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Let $\mathcal{E}'$ be a left $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. We denote by $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X})$ the m-PD-stratification with coefficients in $\mathcal{B}'_X$ associated with $\mathcal{E}'$ (see 4.2.1.5). Following 4.3.1.1, we get the ring homomorphism $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ given by $b \otimes P \mapsto \rho(b) \otimes P$ that we still denote by ρ. Let us denote by $\rho_*(\mathcal{E}')$ the sheaf $\mathcal{E}'$ viewed as a left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module via ρ_*. Let $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X})$ be the m-PD-stratification with coefficients in $\mathcal{B}_X$ associated with $\rho_*(\mathcal{E}')$ (see 4.2.1.5). Let us denote by $\widetilde{\mathcal{P}}^n := \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ and $\widetilde{\mathcal{P}}'^n := \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$. Then we get the commutative diagram of $\widetilde{\mathcal{P}}^n$-modules:*

$$\begin{array}{ccccc} \widetilde{\mathcal{P}}'^n \otimes_{\mathcal{B}'_X} \mathcal{E}' & \xleftarrow[\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X} \otimes \mathrm{id}]{\sim} & (\widetilde{\mathcal{P}}^n \underset{\mathcal{B}_X}{\otimes} \mathcal{B}'_X) \otimes_{\mathcal{B}'_X} \mathcal{E}' & \xrightarrow{\sim} & \widetilde{\mathcal{P}}^n \underset{\mathcal{B}_X}{\otimes} \mathcal{E}' \\ \downarrow{\scriptstyle \sim\ \varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}} & & & & \downarrow{\scriptstyle \sim\ \varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X}} \\ \mathcal{E}' \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}'^n & & \xrightarrow{\sim} & & \mathcal{E}' \underset{\mathcal{B}_X}{\otimes} \widetilde{\mathcal{P}}^n. \end{array} \tag{4.3.2.3.1}$$

Proof. Since this is local, we can suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,d}$. Since $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$ are isomorphisms of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-algebras, then $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}(1 \otimes 1) = 1 \otimes 1$. Hence, by using the Taylor formula satisfied by $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X}$ (see 3.4.2.5.4) we compute the morphism $\varepsilon\colon \widetilde{\mathcal{P}}'^n \otimes_{\mathcal{B}'_X} \mathcal{E}' \to \mathcal{E}' \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}'^n$ making commutative the diagram 4.3.2.3.1 (i.e. ε is the composition of the top, right, bottom morphisms of the diagram 4.3.2.3.1) should satisfied in $\mathcal{E}' \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}'^n$ the formula

$$\varepsilon((1 \otimes 1) \otimes x') = \sum_{|\underline{k}| \leq n} \underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x') \otimes (1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}}),$$

for any section x' of $\mathcal{E}'$. Hence, since the morphisms of the diagram 4.3.2.3.1 are $\widetilde{\mathcal{P}}^n$-linear, since $\widetilde{\mathcal{P}}'^n \otimes_{\mathcal{B}'_X} \mathcal{E}'$ is generated as $\widetilde{\mathcal{P}}^n$-module by the elements of the form $(1 \otimes 1) \otimes x'$, then thanks to the formula 4.2.1.5.4, we get $\varepsilon = \varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}$. □

Remark 4.3.2.4. Let $\rho\colon \mathcal{B}_X \to \mathcal{C}_X$ be a commutative algebra homomorphism. We suppose $\mathcal{C}_X$ is endowed with a structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module such that $\mathcal{B}_X \to \mathcal{C}_X$ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear. Following 4.3.2.3.1 (where we replace $\mathcal{B}'_X$ by $\mathcal{B}_X$, $\mathcal{B}_X$ by $\mathcal{O}_X$ and $\mathcal{E}'$ by $\mathcal{C}_X$), we have the commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{C}_X & \xleftarrow[\varepsilon^{\mathcal{B}_X}_n \otimes \mathrm{id}]{\sim} & (\mathcal{P}^n_{X^\sharp/S^\sharp} \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_X) \otimes_{\mathcal{B}_X} \mathcal{C}_X & \xrightarrow{\sim} & \mathcal{P}^n_{X^\sharp/S^\sharp} \underset{\mathcal{O}_X}{\otimes} \mathcal{C}_X \\ \downarrow{\scriptstyle \sim\ \varepsilon^{\mathcal{C}_X}_{n/\mathcal{B}_X}} & & & & \downarrow{\scriptstyle \sim\ \varepsilon^{\mathcal{C}_X}_n} \\ \mathcal{C}_X \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp} & & \xrightarrow{\sim} & & \mathcal{C}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{P}^n_{X^\sharp/S^\sharp}. \end{array} \tag{4.3.2.4.1}$$

Hence, the isomorphisms of the m-PD-stratification with coefficients in $\mathcal{O}_X$, denoted by $\epsilon^{\mathcal{C}_X}_n$, are isomorphisms of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-algebras if and only if the $\varepsilon^{\mathcal{C}_X}_{n/\mathcal{B}_X}$ are isomorphisms of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$-algebras. If we define the notion of $\mathcal{B}_X$*-algebras compatible with its structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module* by asking that the isomorphisms $\varepsilon^{\mathcal{C}_X}_{n/\mathcal{B}_X}$ are isomorphisms of $\widetilde{\mathcal{P}}^n$-algebras, it amounts to demanding that the structure of $\mathcal{O}_X$-algebra is compatible with its structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module.

Definition 4.3.2.5. Let $\rho \colon \mathcal{B}_X \to \mathcal{B}'_X$ be a commutative algebra homomorphism. We suppose $\mathcal{B}'_X$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module which is compatible to its underlying $\mathcal{O}_X$-algebra structure. We suppose moreover that $\mathcal{B}_X \to \mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. We keep notation 4.3.2.1. Let $\mathcal{E}'$ be a left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. Let us denote by $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X})$ the m-PD-stratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{E}'$. We suppose the underlying structure of $\mathcal{B}_X$-module of $\mathcal{E}'$ extends to a structure of $\mathcal{B}'_X$-module. Set $\widetilde{\mathcal{P}}^n := \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ and $\widetilde{\mathcal{P}}'^n := \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$. Let us denote by

$$\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X} \colon \widetilde{\mathcal{P}}'^n \otimes_{\mathcal{B}'_X} \mathcal{E}' \xrightarrow{\sim} \mathcal{E}' \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}'^n$$

the isomorphism making commutative the diagram of $\widetilde{\mathcal{P}}^n$-modules

$$\begin{array}{ccccc}
\widetilde{\mathcal{P}}'^n \otimes_{\mathcal{B}'_X} \mathcal{E}' & \xleftarrow[\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X} \otimes \mathrm{id}]{\sim} & (\widetilde{\mathcal{P}}^n \underset{\mathcal{B}_X}{\otimes} \mathcal{B}'_X) \otimes_{\mathcal{B}'_X} \mathcal{E}' & \xrightarrow{\sim} & \widetilde{\mathcal{P}}^n \underset{\mathcal{B}_X}{\otimes} \mathcal{E}' \\
\sim \Big\downarrow \varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X} & & & & \sim \Big\downarrow \varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X} \\
\mathcal{E}' \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}'^n & & \xrightarrow{\sim} & & \mathcal{E}' \underset{\mathcal{B}_X}{\otimes} \widetilde{\mathcal{P}}^n .
\end{array} \tag{4.3.2.5.1}$$

We say that the isomorphisms $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X}$ are "semi-linear with respect to $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$" if the isomorphisms $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}$ are $\widetilde{\mathcal{P}}'^n$-linear.

Proposition 4.3.2.6. *Let $\mathcal{B}'_X$ be a commutative $\mathcal{B}_X$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module such that $\mathcal{B}_X \to \mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Let $\mathcal{E}'$ be a left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. We suppose the underlying structure of $\mathcal{B}_X$-module of $\mathcal{E}'$ extends to a structure of $\mathcal{B}'_X$-module. Let $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X})$ be the m-PD-stratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{E}'$. The following assertions are equivalent.*

(a) Both structures of left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{B}'_X$-module of $\mathcal{E}'$ extend (uniquely) to a structure of left $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module.

(b) The isomorphisms $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X}$ are semi-linear with respect to the isomorphisms $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$ for any $n \in \mathbb{N}$.

Proof. $(a) \to (b)$. Suppose $\mathcal{E}'$ is endowed with a structure of left $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module extending both structures of $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{B}'_X$. Then following 4.3.2.3, since the isomorphisms $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X})$ are equal to the m-PD-stratification with coefficients in $\mathcal{B}'_X$ associated with $\mathcal{E}'$, then they are in particular $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-linear which means the isomorphisms $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X})$ are semi-linear with respect to $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$.

$(b) \to (a)$. Suppose the isomorphisms $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X}$ are semi-linear with respect to the isomorphisms $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}$. Let $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}$ be the isomorphism making commutative the diagram 4.3.2.5.1. By hypothesis, $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}$ are $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-linear. In order to check that the data $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X})$ is an m-PD-stratification with coefficients in $\mathcal{B}'_X$ of $\mathcal{E}'$, it remains to prove the cocycle condition. Following the remark 4.2.1.6, this is equivalent to check the formula 4.2.1.6.1. By using the commutative diagram 4.3.2.5.1, since $\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}(1) = 1$, then we have

$$\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}((1 \otimes 1) \otimes x') = \sum_{|\underline{k}| \leq n} \underline{\partial}^{\langle \underline{k} \rangle}_{\sharp}(x') \otimes (1 \otimes \underline{\tau}^{\{\underline{k}\}}_{\sharp})$$

for any section x' of $\mathcal{E}'$. Since $\mathcal{E}'$ is a left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module, this yields the formula 4.2.1.6.1 holds. Hence, we are done. $\square$

Remark 4.3.2.7. Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,d}$. With its notation, both conditions of 4.3.2.6 hold if and only if we have the formula

$$b' \cdot \left(\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{\sharp} \cdot x' \right) = \sum_{\underline{h} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{k}-\underline{h} \rangle}_{\sharp} \cdot \left(\widetilde{\underline{\partial}}^{\langle \underline{h} \rangle}_{\sharp}(b') \cdot x' \right). \tag{4.3.2.7.1}$$

for any $b' \in \mathcal{B}'_X$, $x' \in \mathcal{E}'$, $\underline{k} \in \mathbb{N}^d$.

Indeed, let $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}$ be the isomorphism making commutative the diagram 4.3.2.5.1. Since $\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}$ is $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-linear then it is $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-linear if and only if we have the formula $b'\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}((1\otimes 1)\otimes x') = \varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}((b'\otimes 1)\otimes x')$ for any sections b' of $\mathcal{B}'_X$ and x' of $\mathcal{E}'$. We compute

$$\left((\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X})^{-1}\otimes \mathrm{id}\right)((b'\otimes 1)\otimes x') \overset{4.2.1.5.5}{=} \sum_{|\underline{i}|\le n}\left(\underline{\tau}^{\{\underline{i}\}}_\sharp \otimes \underline{\widetilde{\partial}}^{\langle \underline{i}\rangle}_\sharp(b')\right)\otimes x';$$

$$\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X}\left(\sum_{|\underline{i}|\le n}\underline{\tau}^{\{\underline{i}\}}_\sharp \otimes \underline{\widetilde{\partial}}^{\langle \underline{i}\rangle}_\sharp(b')\cdot x'\right) \overset{4.2.1.5.4}{=} \sum_{|\underline{i}|\le n}\sum_{|\underline{j}|\le n}\underline{\partial}^{\langle \underline{j}\rangle}_\sharp(\underline{\widetilde{\partial}}^{\langle \underline{i}\rangle}_\sharp(b')\cdot x')\otimes \underline{\tau}^{\{\underline{j}\}}_\sharp\underline{\tau}^{\{\underline{i}\}}_\sharp$$

$$= \sum_{|\underline{k}|\le n}\left(\sum_{\underline{h}\le\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\underline{\partial}^{\langle \underline{k}-\underline{h}\rangle}_\sharp\cdot\left(\underline{\widetilde{\partial}}^{\langle \underline{h}\rangle}_\sharp(b')\cdot x'\right)\right)\otimes\underline{\tau}^{\{\underline{k}\}}.$$

This yields

$$\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}((b'\otimes 1)\otimes x') = \sum_{|\underline{k}|\le n}\left(\sum_{\underline{h}\le\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\}\underline{\partial}^{\langle \underline{k}-\underline{h}\rangle}_\sharp\cdot\left(\underline{\widetilde{\partial}}^{\langle \underline{h}\rangle}_\sharp(b')\cdot x'\right)\right)\otimes(1\otimes\underline{\tau}^{\{\underline{k}\}}).$$

On the other hand, we compute

$$b'\cdot\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}'_X}((1\otimes 1)\otimes x') = b'\left(\sum_{|\underline{k}|\le n}\underline{\partial}^{\langle \underline{k}\rangle}_\sharp(x')\otimes(1\otimes\underline{\tau}^{\{\underline{k}\}}_\sharp)\right) = \sum_{|\underline{k}|\le n} b'(\underline{\partial}^{\langle \underline{k}\rangle}_\sharp(x'))\otimes(1\otimes\underline{\tau}^{\{\underline{k}\}}_\sharp),$$

Hence, we are done.

4.3.3 Semi-linear PD-costratifications of level m

Proposition 4.3.3.1. *Let $\mathcal{M}'$ be a right $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. We denote by $(\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}'_X})$ the m-PD-costratification with coefficients in $\mathcal{B}'_X$ associated with $\mathcal{M}'$ (see 4.2.2.5). Following 4.3.1.1, we get the ring homomorphism $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\to\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ given by $b\otimes P\mapsto\rho(b)\otimes P$ that we still denote by ρ. We denote by $\rho_*(\mathcal{M}')$ the sheaf $\mathcal{M}'$ viewed as a left $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module via ρ_*. Let $(\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}_X})$ be the m-PD-costratification with coefficients in $\mathcal{B}_X$ associated with $\rho_*(\mathcal{M}')$ (see 4.2.2.5). Let us denote by $\widetilde{\mathcal{P}}^n:=\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ and $\widetilde{\mathcal{P}}'^n:=\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$. Then we get the commutative diagram of $\widetilde{\mathcal{P}}^n$-modules:*

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n,\mathcal{M}') & \xrightarrow[\sim]{\alpha} & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{0*}\widetilde{\mathcal{P}}'^n,\mathcal{M}') \\
\sim\downarrow\varepsilon^{\mathcal{M}'}_n & & \sim\downarrow\varepsilon'^{\mathcal{M}'}_n \\
\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n,\mathcal{M}') \xrightarrow[\sim]{\beta} \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{\mathcal{P}}^n\otimes_{\mathcal{B}_X}\mathcal{B}'_X,\mathcal{M}') & \xleftarrow[\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}]{\sim} & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{1*}\widetilde{\mathcal{P}}'^n,\mathcal{M}'),
\end{array}\tag{4.3.3.1.1}$$

where α and β are the canonical isomorphisms.

Proof. We easily compute that the following diagram

$$\begin{array}{ccc}
\mathcal{M}'\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n,\mathcal{B}_X) & \xrightarrow{4.2.2.4.1} & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n,\mathcal{M}') \\
\downarrow\sim & & \\
\mathcal{M}'\otimes_{\mathcal{B}'_X}\left(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n,\mathcal{B}_X)\right) & & \sim\downarrow\alpha \\
\downarrow\sim & & \\
\mathcal{M}'\otimes_{\mathcal{B}'_X}\mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{0*}\widetilde{\mathcal{P}}'^n,\mathcal{B}'_X) & \xrightarrow{4.2.2.4.1} & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{0*}\widetilde{\mathcal{P}}'^n,\mathcal{M}')
\end{array}\tag{4.3.3.1.2}$$

where the bottom arrow is constructed as for 4.2.2.4.1 with $\mathcal{B}_X$ replaced by $\mathcal{B}'_X$, where the vertical isomorphism is the canonical ones, is commutative. To distinguish them, let $\{\underline{\partial}_\sharp^{\langle \underline{h} \rangle}, |\underline{h}| \leq n\}$ (resp. $\{\underline{\partial}_\sharp'^{\langle \underline{h} \rangle}, |\underline{h}| \leq n\}$) be the dual basis of $\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X)$ (resp. $\mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{0*}\widetilde{\mathcal{P}}'^n, \mathcal{B}'_X)$) of the basis $\{1 \otimes \underline{\tau}_\sharp^{\{\underline{h}\}}, |\underline{h}| \leq n\}$ of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ (resp. $\widetilde{\mathcal{P}}'^n_{X^\sharp/S^\sharp(m)}$). We compute the composition of the left vertical isomorphisms of 4.3.3.1.2 is given by for any $x' \in \mathcal{M}'$ and any $\underline{k} \in \mathbb{N}^d$ by $x' \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} \mapsto x' \otimes \underline{\partial}_\sharp'^{\langle \underline{k} \rangle_{(m)}}$

Consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n, \mathcal{B}_X) \otimes_{\mathcal{B}_X} \mathcal{M}' & \xrightarrow{4.2.2.4.2} & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n, \mathcal{M}') \\
\downarrow \wr & & \wr \downarrow \\
\mathcal{H}om_{\mathcal{B}'_X}(\widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_X} \mathcal{B}'_X, \mathcal{B}'_X) \otimes_{\mathcal{B}'_X} \mathcal{M}' & & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_X} \mathcal{B}'_X, \mathcal{M}') \\
\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X} \uparrow \wr & & \varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X} \uparrow \wr \\
\mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{1*}\widetilde{\mathcal{P}}'^n, \mathcal{B}'_X) \otimes_{\mathcal{B}'_X} \mathcal{M}' & \xrightarrow{4.2.2.4.2} & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{1*}\widetilde{\mathcal{P}}'^n, \mathcal{M}')
\end{array} \tag{4.3.3.1.3}$$

where both top vertical isomorphism are the canonical ones. To distinguish them, let $\{\underline{\partial}_\sharp^{\star\langle \underline{h} \rangle}, |\underline{h}| \leq n\}$ (resp. $\{\underline{\partial}_\sharp'^{\star\langle \underline{h} \rangle}, |\underline{h}| \leq n\}$) be the dual basis of $\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}, \mathcal{B}_X)$ (resp. $\mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{1*}\widetilde{\mathcal{P}}'^n, \mathcal{B}'_X)$) of the basis $\{1 \otimes \underline{\tau}_\sharp^{\{\underline{h}\}}, |\underline{h}| \leq n\}$ of $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ (resp. $\widetilde{\mathcal{P}}'^n_{X^\sharp/S^\sharp(m)}$). We compute the composition of the left vertical isomorphisms of 4.3.3.1.3 is given by for any $x' \in \mathcal{M}'$ and any $\underline{k} \in \mathbb{N}^d$ by $\underline{\partial}^{\star\langle \underline{k} \rangle} \otimes x' \mapsto \underline{\partial}'^{\star\langle \underline{k} \rangle} \otimes x'$. Henc, we conclude by using the local formula 4.2.2.6.2. □

Definition 4.3.3.2. Let $\rho \colon \mathcal{B}_X \to \mathcal{B}'_X$ be a commutative algebra homomorphism. We suppose $\mathcal{B}'_X$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module which is compatible to its underlying $\mathcal{O}_X$-algebra structure. We suppose moreover that $\mathcal{B}_X \to \mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. We keep notation 4.3.2.1. Let $\mathcal{E}'$ be a left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. Let us denote by $(\varepsilon^{\mathcal{E}'}_{n/\mathcal{B}_X})$ the m-PD-stratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{E}'$. We suppose the underlying structure of $\mathcal{B}_X$-module of $\mathcal{E}'$ extends to a structure of $\mathcal{B}'_X$-module. Set $\widetilde{\mathcal{P}}^n := \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ and $\widetilde{\mathcal{P}}'^n := \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$. Let us denote by $\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}'_X} \colon \mathcal{H}om_{\mathcal{B}'}(\widetilde{p}'^n_{0*}\widetilde{\mathcal{P}}'^n, \mathcal{M}') \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}'}(\widetilde{p}'^n_{1*}\widetilde{\mathcal{P}}'^n, \mathcal{M}')$ the isomorphism making commutative the diagram of $\widetilde{\mathcal{P}}^n$-modules

$$\begin{array}{ccccc}
\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{0*}\widetilde{\mathcal{P}}^n, \mathcal{M}') & & \xrightarrow{\quad\sim\quad} & & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{0*}\widetilde{\mathcal{P}}'^n, \mathcal{M}') \\
\wr\downarrow \varepsilon^{\mathcal{M}'}_{n/\mathcal{B}_X} & & & & \wr \downarrow \varepsilon^{\mathcal{M}'}_{n/\mathcal{B}'_X} \\
\mathcal{H}om_{\mathcal{B}_X}(\widetilde{p}^n_{1*}\widetilde{\mathcal{P}}^n, \mathcal{M}') & \xrightarrow[\sim]{} & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_X} \mathcal{B}'_X, \mathcal{M}') & \xleftarrow[\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X}]{\sim} & \mathcal{H}om_{\mathcal{B}'_X}(\widetilde{p}'^n_{1*}\widetilde{\mathcal{P}}'^n, \mathcal{M}')
\end{array} \tag{4.3.3.2.1}$$

We say that the isomorphisms $\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}_X}$ are "semi-linear with respect to $(\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X})^{-1}$" if the isomorphisms $\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}'_X}$ are $\widetilde{\mathcal{P}}'^n$-linear.

Proposition 4.3.3.3. *Let $\mathcal{B}'_X$ be a commutative $\mathcal{B}_X$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module such that $\mathcal{B}_X \to \mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Let $\mathcal{M}'$ be a right $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. We suppose the underlying structure of $\mathcal{B}_X$-module of $\mathcal{M}'$ extends to a structure of $\mathcal{B}'_X$-module. Let $(\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}_X})$ be the m-PD-costratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{M}'$. The following assertions are equivalent.*

(a) Both structures of right $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{B}'_X$-module of $\mathcal{M}'$ extend (uniquely) to a structure of right $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module.

(b) The isomorphisms $\varepsilon^{\mathcal{M}'}_{n/\mathcal{B}_X}$ are semi-linear with respect to the isomorphisms $(\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X})^{-1}$ for any $n \in \mathbb{N}$.

Proof. The proof is analogous to that of 4.3.2.6: this is a consequence of the formula 4.2.2.6.2 used by composing it with the evaluation at 1 morphism (recall also the diagram 4.3.3.2.1 and the fact that $\varepsilon_{n/\mathcal{B}_X}^{\mathcal{B}'_X}$ sends 1 to 1). □

4.3.4 Coefficients extension

4.3.4.1. We will write $\widetilde{\mathcal{D}}_X^{\prime(m)} := \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X^\sharp}^{(m)}$ and $\widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} := \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S^\sharp(m)}$. Let us denote simply by $\rho\colon \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}$ the homomorphism $\rho \otimes \mathrm{id}$. Similarly to 4.1.2.5, the canonical morphism $\widetilde{p}^{\prime n}_{0(m)}\colon \mathcal{B}'_X \to \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^{n}_{X^\sharp/S^\sharp(m)}$ endows the sheaf $\widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)}$ with a structure of $\mathcal{B}'_X$-algebra, that we will call *left* structure. Moreover, the morphism of $\mathcal{B}'_X$-algebras

$$\widetilde{p}^{\prime n}_{1(m)}\colon \mathcal{B}'_X \to \mathcal{P}^{\prime n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}'_X \xrightarrow{\epsilon_n^{\mathcal{B}'_X}} \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{P}^{\prime n}_{X^\sharp/S^\sharp(m)}$$

induces a second structure of $\mathcal{B}'_X$-algebra on $\widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)}$, that we call the *right* structure. Let $j = 0$ or 1. Since ρ is horizontal, then we can check the commutativity of the diagram in the case $j = 1$ (the case $j = 0$ is obvious):

$$\begin{array}{ccc} \mathcal{B}_X & \xrightarrow{\widetilde{p}_j^n} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho^n_{(m)}} \\ \mathcal{B}'_X & \xrightarrow{\widetilde{p}_j^{\prime n}} & \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.3.4.1.1}$$

This means that $\rho^n_{(m)}\colon \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)}$ is $\mathcal{B}_X$-linear for both left and right structures. This yields the homomorphism

$$\rho^n_{(m)} \otimes \rho^{n'}_{(m)}\colon \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{\prime n'}_{X^\sharp/S^\sharp(m)} \tag{4.3.4.1.2}$$

Similarly to 4.1.2.11, we define the homomorphisms

$$\widetilde{\delta}^{\prime n,n'}_{(m)}, \widetilde{q_0}^{\prime n,n'}, \widetilde{q_1}^{\prime n,n'}\colon \widetilde{\mathcal{P}}^{\prime n+n'}_{X^\sharp/S^\sharp(m)} \to \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{\prime n'}_{X^\sharp/S^\sharp(m)}.$$

We can also write the m-PD-algebras homomorphism $\widetilde{\delta}^{n,n'}_{(m)}$ (resp. $\widetilde{q}^{n,n'}_{0(m)}$, resp. $\widetilde{q}^{n,n'}_{1(m)}$) by $\widetilde{p}^{n,n'}_{02,(m)}$ (resp. $\widetilde{p}^{n,n'}_{01,(m)}$, resp. $\widetilde{p}^{n,n'}_{12,(m)}$) ; and similarly with some primes (see the notation 3.2.2.14). Then for $0 \le i < j \le 2$, we have the commutative diagram of commutative rings:

$$\begin{array}{ccc} \widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\widetilde{p}^{n,n'}_{ij,(m)}} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} \\ \downarrow{\scriptstyle \rho^{n+n'}_{(m)}} & & \downarrow{\scriptstyle \rho^n_{(m)} \otimes \rho^{n'}_{(m)}} \\ \widetilde{\mathcal{P}}^{\prime n+n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\widetilde{p}^{\prime n,n'}_{ij,(m)}} & \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}^{\prime n'}_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.3.4.1.3}$$

4.3.4.2. We keep notation 4.3.4.2. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module. We will write $\mathcal{E}' := \mathcal{B}'_X \otimes_{\mathcal{B}_X} \mathcal{E}$. Let $(\varepsilon_n^{\mathcal{E}})$ be the m-PD-stratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{E}$. Let $\varepsilon_n^{\mathcal{E}'}$ be the $\widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)}$-linear homomorphism induced by extension from $\varepsilon_n^{\mathcal{E}}$, i.e. making commutative the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}'_X} \mathcal{E}' & \xrightarrow[\sim]{\alpha} & \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} \otimes_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}} (\widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \mathcal{E}) \\ \downarrow{\scriptstyle \varepsilon_n^{\mathcal{E}'}} & & \sim \downarrow{\scriptstyle \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp}} \varepsilon_n^{\mathcal{E}}} \\ \mathcal{E}' \otimes_{\mathcal{B}'_X} \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)} & \xleftarrow[\beta]{\sim} & (\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)}) \otimes_{\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}} \widetilde{\mathcal{P}}^{\prime n}_{X^\sharp/S^\sharp(m)}, \end{array} \tag{4.3.4.2.1}$$

so that α and β are the canonical isomorphisms.

Lemma 4.3.4.3. *We keep notation 4.3.4.2. Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $u_1, \ldots, u_r$. Let $b' \in \mathcal{B}'_X$, $x \in \mathcal{E}$. We have the formula*

$$\varepsilon_n^{\mathcal{E}'}((1\otimes 1)\otimes(b'\otimes x)) = \sum_{|\underline{k}|\leq n} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b'\otimes x)\otimes \underline{\tau}_\sharp^{\{\underline{k}\}}, \tag{4.3.4.3.1}$$

where (by abuse of notation) $\{\underline{\tau}_\sharp^{\{\underline{k}\}},\ |\underline{k}|\leq n\}$ with $|\underline{k}|\leq n$ is the basis of $\widetilde{\mathcal{P}}'^n_{X^\sharp/S^\sharp(m)}$ induced by the basis $\{\underline{\tau}_\sharp^{\{\underline{k}\}},\ |\underline{k}|\leq n\}$ of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$ and where $\underline{\partial}_\sharp^{\langle \underline{k}\rangle}(b'\otimes x)$ is given by the action of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}$ via the tensor product (see 4.2.3.1.(a)).

Proof. With the notation 4.3.4.2.1, α (resp. β) is the homomorphism given by $\tau'\otimes(b'\otimes x)\mapsto \tau'\widetilde{p}'^n_{1(m)}(b')\otimes((1\otimes 1)\otimes x)$ (resp. $(x\otimes\tau)\otimes\tau'\mapsto(1\otimes x)\otimes\rho(\tau)\tau'$) where $(1\otimes 1)$ is the unit of $\widetilde{\mathcal{P}}'^n_{X^\sharp/S^\sharp(m)}$, $b'\in\mathcal{B}'_X$, $x\in\mathcal{E}$, $\tau\in\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ and $\tau'\in\widetilde{\mathcal{P}}'^n_{X^\sharp/S^\sharp(m)}$. Hence, we compute

$$\begin{aligned}
\varepsilon_n^{\mathcal{E}'}((1\otimes 1)\otimes(b'\otimes x)) &= \beta\left(\varepsilon_n^{\mathcal{E}}((1\otimes 1)\otimes x)\otimes\widetilde{p}'^n_{1(m)}(b')\right) \overset{4.2.1.5.4}{=} \beta\left((\sum_{|\underline{i}|\leq n}(\underline{\partial}_\sharp^{\langle\underline{i}\rangle}x\otimes\underline{\tau}_\sharp^{\{\underline{i}\}})\otimes\widetilde{p}'^n_{1(m)}(b')\right)\\
&= \sum_{|\underline{i}|\leq n}(1\otimes\underline{\partial}_\sharp^{\langle\underline{i}\rangle}x)\otimes\underline{\tau}_\sharp^{\{\underline{i}\}}\widetilde{p}'^n_{1(m)}(b') \overset{4.2.1.7.2}{=} \sum_{|\underline{i}|\leq n}\sum_{|\underline{j}|\leq n}(1\otimes\underline{\partial}_\sharp^{\langle\underline{i}\rangle}x)\otimes\widetilde{p}'^n_{0,(m)}(\underline{\partial}_\sharp^{\langle\underline{j}\rangle}(b'))\underline{\tau}_\sharp^{\{\underline{j}\}}\underline{\tau}_\sharp^{\{\underline{i}\}}\\
&= \sum_{|\underline{i}|\leq n}\sum_{|\underline{j}|\leq n}(\underline{\partial}_\sharp^{\langle\underline{j}\rangle}(b')\otimes\underline{\partial}_\sharp^{\langle\underline{i}\rangle}x)\otimes\underline{\tau}_\sharp^{\{\underline{j}\}}\underline{\tau}_\sharp^{\{\underline{i}\}} \overset{1.2.4.5.3}{=} \sum_{|\underline{k}|\leq n}\sum_{\underline{i}\leq\underline{k}}\left\{\begin{smallmatrix}\underline{k}\\ \underline{i}\end{smallmatrix}\right\}(\underline{\partial}_\sharp^{\langle\underline{k}-\underline{i}\rangle}(b')\otimes\underline{\partial}_\sharp^{\langle\underline{i}\rangle}x)\otimes\underline{\tau}_\sharp^{\{\underline{k}\}}\\
&\overset{4.2.3.1.1}{=} \sum_{|\underline{k}|\leq n}\underline{\partial}_\sharp^{\langle\underline{k}\rangle}(b'\otimes x)\otimes\underline{\tau}_\sharp^{\{\underline{k}\}}.
\end{aligned}$$

□

Proposition 4.3.4.4. *The family $(\varepsilon_n^{\mathcal{E}'})$ is an m-PD-stratification with coefficients in $\mathcal{B}'_X$ on $\mathcal{E}'$. This yields a canonical structure of left $\widetilde{\mathcal{D}}'^{(m)}_X$-module on $\mathcal{E}'$, which is named the structure induced by extension from the structure of left $\widetilde{\mathcal{D}}^{(m)}_X$-module of $\mathcal{E}$.*

Proof. Since this is local, we can suppose $X^\sharp\to S^\sharp$ is endowed with logarithmic coordinates. It follows from the formula 4.3.4.3.1 that the formula 4.2.1.6.1 holds. Hence, following 4.2.1.6 the cocycle condition holds for $(\varepsilon_n^{\mathcal{E}'})$. □

Lemma 4.3.4.5. *We keep notation 4.3.4.2. We still denote by $\rho\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\to\widetilde{\mathcal{D}}'^{(m)}_{X^\sharp/S^\sharp}$ the canonical ring homomorphism induced by ρ (see 4.3.1.1.a). Let $\rho_*(\mathcal{E}')$ be the sheaf $\mathcal{E}'$ endowed with the left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structure induced via ρ by its left $\widetilde{\mathcal{D}}'^{(m)}_{X^\sharp/S^\sharp}$-module structure. Since $\mathcal{B}'_X$ and $\mathcal{E}$ are both left $\widetilde{\mathcal{D}}^{(m)}_X$-modules, then there exists on $\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}$ another structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module given by the tensor product (see 4.2.3.1.(a)). In fact, these two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structures on $\mathcal{E}'$ are equal.*

Proof. Since this is local, we can suppose $X^\sharp\to S^\sharp$ is endowed with logarithmic coordinates. Then this is a consequence of Taylor formula 4.2.1.5.4 and of the formula 4.3.4.3.1. □

Lemma 4.3.4.6. *If $\mathcal{E}$ is a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module. The canonical homomorphism*

$$\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}\to(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{D}^{(m)}_{X^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}}\mathcal{E}, \tag{4.3.4.6.1}$$

given by $b'\otimes x\mapsto(b'\otimes 1)\otimes x$, is an isomorphism of left $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules, where the structure of the left term $\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}$ is defined at 4.3.4.2.

Proof. Since the canonical homomorphism $\mathcal{B}'_X \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp} \to \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ induced by the ring homomorphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp} \to \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ (see 4.3.1.1.1) is an isomorphism, then by associativity of the tensor product the map 4.3.4.6.1 is therefore an isomorphism of $\mathcal{B}'_X$-modules. It remains to check 4.3.4.6.1 is $\mathcal{D}^{(m)}_{X^\sharp}$-linear. Since this is local, we can suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $u_1, \ldots, u_r$ and keep notation 4.3.5.2. Let $\underline{k} \in \mathbb{N}^r$. It follows from the formula 4.2.3.1.1 and Lemma 4.3.4.5 that we have

$$\underline{\partial}_\sharp^{\langle \underline{k} \rangle} \cdot (b' \otimes x) = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{i} \rangle} b' \otimes \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle} x.$$

Hence, $\underline{\partial}_\sharp^{\langle \underline{k} \rangle} \cdot (b' \otimes x)$ is sent via the homomorphism 4.3.4.6.1 to

$$\sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} (\underline{\partial}_\sharp^{\langle \underline{i} \rangle} b' \otimes 1) \otimes \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle} x = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} (\underline{\partial}_\sharp^{\langle \underline{i} \rangle} b' \otimes 1)(1 \otimes \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle}) \otimes x$$
$$= \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} (\underline{\partial}_\sharp^{\langle \underline{i} \rangle} b' \otimes \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle}) \otimes x \overset{4.1.2.2.3}{=} \Big((1 \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle})(b' \otimes 1) \Big) \otimes x = \underline{\partial}_\sharp^{\langle \underline{k} \rangle} \cdot ((b' \otimes 1) \otimes x)\,.$$

□

4.3.4.7. We keep notation 4.3.4.2. Let $\mathcal{M}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module. We will write $\mathcal{M}' := \rho^\flat(\mathcal{M}) = \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}'_X, \mathcal{M})$. Let $(\varepsilon_n^{\mathcal{M}})$ be the m-PD-costratification with coefficients in $\mathcal{B}_X$ associated with $\mathcal{M}$. Let $\varepsilon_n^{\mathcal{M}'}$ be the $\widetilde{\mathcal{P}}'^n_{X^\sharp/S^\sharp(m)}$-linear homomorphism induced by $\varepsilon_n^{\mathcal{M}}$, i.e. making commutative the diagram

$$\begin{array}{ccc} \widetilde{p}'^{n\flat}_{0(m)}(\mathcal{M}) & \xrightarrow[\sim]{\alpha} & \rho^{n\flat}_{(m)}(\widetilde{p}^{n\flat}_{0(m)}(\mathcal{M})) \\ \Big\downarrow{\scriptstyle \varepsilon_n^{\mathcal{M}'}} & & \sim \Big\downarrow{\scriptstyle \rho^{n\flat}_{(m)}(\varepsilon_n^{\mathcal{M}})} \\ \widetilde{p}'^{n\flat}_{1(m)}(\mathcal{M}) & \xleftarrow[\beta]{\sim} & \rho^{n\flat}_{(m)}(\widetilde{p}^{n\flat}_{1(m)}(\mathcal{M})), \end{array} \tag{4.3.4.7.1}$$

so that α and β are the canonical isomorphisms. The family $(\varepsilon_n^{\mathcal{M}'})$ is an m-PD-costratification with coefficients in $\mathcal{B}'_X$ on $\mathcal{M}'$. This yields a canonical structure of left $\widetilde{\mathcal{D}}'^{(m)}_X$-module on $\mathcal{M}'$, which is named the structure induced by extension from the structure of left $\widetilde{\mathcal{D}}^{(m)}_X$-module of $\mathcal{M}$. We still denote by $\rho\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}'^{(m)}_{X^\sharp/S^\sharp}$ the canonical ring homomorphism induced by ρ (see 4.3.1.1.a). Let $\rho_*(\mathcal{M}')$ be the sheaf $\mathcal{M}'$ endowed with the right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structure induced via ρ by its right $\widetilde{\mathcal{D}}'^{(m)}_{X^\sharp/S^\sharp}$-module structure. There exists on $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}'_X, \mathcal{M})$ another structure of right $\widetilde{\mathcal{D}}^{(m)}_X$-module given by the internal homomorphism structure (see 4.2.3.5). Similarly to 4.3.4.5, we can check that these two right $\widetilde{\mathcal{D}}^{(m)}_X$-module structures on $\mathcal{M}'$ are equal.

4.3.4.8. Let $\mathcal{M}$ be a right $\mathcal{D}^{(m)}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$-module. Via the transposition isomorphism $\gamma_{\mathcal{B}_X} : \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ (see 4.2.5.7), $\mathcal{M}$ can be viewed as a right $\mathcal{D}^{(m)}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$-module. By associativity of the tensor product, we can check that the canonical homomorphism

$$\mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{B}'_X \to \mathcal{M} \otimes_{\mathcal{D}^{(m)}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X} (\mathcal{D}^{(m)}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}'_X), \tag{4.3.4.8.1}$$

given by $x \otimes b' \mapsto x \otimes (1 \otimes b')$ is an isomorphism of $\mathcal{B}'_X$-modules. It follows from the commutative diagram 4.3.1.1.3 that we have the canonical isomorphism

$$\mathrm{id} \otimes_{\gamma_{\mathcal{B}_X}} \gamma_{\mathcal{B}'_X} : \mathcal{M} \otimes_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X} (\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}'_X) \to \mathcal{M} \otimes_{\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \tag{4.3.4.8.2}$$

By composing 4.3.4.8.1 and 4.3.4.8.2, we get the canonical homomorphism

$$\mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{B}'_X \to \mathcal{M} \otimes_{\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \tag{4.3.4.8.3}$$

given by $x\otimes b'\mapsto x\otimes(b'\otimes 1)$, which is therefore a $\mathcal{B}'_X$-linear isomorphism. Via the formulas 4.2.3.5.1 and 4.2.5.7.3, we compute that 4.3.4.8.3 is right $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp}$-linear, where the structure of the left term is given by 4.2.3.5. Hence both structures of right $\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{B}'_X$-module of $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{B}'_X$ extend to a structure of right $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. In other words, following 4.3.3.3, this means that the tensor product m-PD-costratification $\varepsilon^{\mathcal{M}}_n\otimes(\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X})^{-1}$ of $\mathcal{M}\otimes_{\mathcal{B}_X}\mathcal{B}'_X$ is semi-linear with respect to $(\varepsilon^{\mathcal{B}'_X}_{n/\mathcal{B}_X})^{-1}$.

Proposition 4.3.4.9. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{F}$ a left $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{E}\to\mathcal{F}$ be a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear morphism. The canonical morphism*

$$\rho\colon\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}\to\mathcal{F},$$

is $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. Since the morphism ρ is equal to the composition morphism

$$\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}\xrightarrow[4.3.4.6.1]{\sim}\left(\mathcal{B}'_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}\right)\otimes_{\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}\mathcal{E}\to\mathcal{F},$$

then ρ is $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. □

Corollary 4.3.4.10. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{F}$ be a left $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. There exists a canonical isomorphism*

$$\mathcal{H}om_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{F})\xrightarrow{\sim}\mathcal{H}om_{\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_X}(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{F}).\tag{4.3.4.10.1}$$

Proof. By using 6.3.2.4, we construct the homomorphism 4.3.4.10.1. Since the canonical morphism $\mathcal{E}\to\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E}$ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear, we construct the inverse homomorphism of 4.3.4.10.1 via this one. □

Proposition 4.3.4.11. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, $\mathcal{M},\mathcal{N}$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{G}$ be a left or a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.*

(a) The canonical isomomorphism

$$\mathcal{B}'_X\otimes_{\mathcal{B}_X}(\mathcal{E}\otimes_{\mathcal{B}_X}\mathcal{G})\xrightarrow{\sim}(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E})\otimes_{\mathcal{B}'_X}(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{G}),\tag{4.3.4.11.1}$$

is $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

(b) The canonical homomorphisms

$$\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E},\mathcal{G})\to\mathcal{H}om_{\mathcal{B}'_X}(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{E},\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{G})\tag{4.3.4.11.2}$$

$$\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M},\mathcal{N})\to\mathcal{H}om_{\mathcal{B}'_X}(\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{M},\mathcal{B}'_X\otimes_{\mathcal{B}_X}\mathcal{N})\tag{4.3.4.11.3}$$

are $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. By definition, we get the $\mathcal{B}'_X$-linearity of the isomorphisms. It remains to check the $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linearity. Since this is local we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates. Via the formulas of 4.2.3.1 and 4.2.3.5, we check the isomorphisms commute with the action of $\underline{\partial}_\sharp^{\langle\underline{k}\rangle}$ and then we are done. □

Proposition 4.3.4.12. *Let $\mathcal{E}$, $\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, $\mathcal{E}'$ be a left $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module, $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and $\mathcal{M}'$ a right $\mathcal{B}'_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module.*

(a) *The canonical isomorphisms*

$$(\mathcal{B}'_X \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes_{\mathcal{B}'_X} \mathcal{E}' \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{E}', \qquad \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{E}') \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}'_X}(\mathcal{B}'_X \otimes_{\mathcal{B}_X} \mathcal{E}, \mathcal{E}'), \quad (4.3.4.12.1)$$

are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear. Hence, by structure transport, we can endow $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{E}'$ and $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{E}')$ with a canonical structure of left $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module which extends its structure of left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. For instance, the canonical structure (see 4.2.3.1) of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{B}'_X \otimes_{\mathcal{B}_X} \mathcal{E}$ extends to a structure of left $\mathcal{B}'_X \otimes_{\mathcal{B}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module.

(b) *In the same way, we obtain a canonical structure of left (resp. right) $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M}, \mathcal{M}')$ (resp. $\mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{E}'$, $\mathcal{M}' \otimes_{\mathcal{B}_X} \mathcal{E}$, $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{M}')$) which extends its structure of left (resp. right) $\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module.*

Proof. Use the same arguments than for the proof of 4.3.4.11. □

4.3.5 Switch left to right $\mathcal{D}$-module structures, (log) adjoint operators

We keep notations and hypotheses of 4.2.

Notation 4.3.5.1. Let us denote by $\widetilde{\omega}_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \omega_{X^\sharp/S^\sharp}$, $\mathcal{B}_Y := \mathcal{B}_X|Y$, $\widetilde{\mathcal{D}}^{(m)}_Y := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y$ and $\widetilde{\omega}_Y := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \omega_{Y/S^\sharp}$. Recall that $\omega_{X^\sharp/S^\sharp}$ is a right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-submodule of $j_*\omega_Y$ (see 3.4.5.1). Hence, by using 4.3.4.8, this yields a canonical structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module on $\widetilde{\omega}_{X^\sharp/S^\sharp}$.

4.3.5.2 (Adjoint operator with or without logarithmic structure)**.** Suppose in this paragraph that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. Let $(t_\lambda)_{\lambda=1,\dots,r}$ be the induced coordinates of Y/S.

(a) Set $\widetilde{D}^{(m)}_Y := \Gamma(Y, \widetilde{\mathcal{D}}^{(m)}_Y)$. With notation 4.1.2.16.3, the adjoint operator 2.2.1.2 extends to a map $\widetilde{D}^{(m)}_Y \to \widetilde{D}^{(m)}_Y$ given by

$$P = \sum_{\underline{k} \in \mathbb{N}^d} b_{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} \in \widetilde{D}^{(m)}_Y \mapsto {}^t P = \sum_{\underline{k}} (-1)^{|\underline{k}|} \underline{\partial}^{\langle \underline{k} \rangle} b_{\underline{k}},$$

where $b_{\underline{k}}$ is a finite sequence of elements of $\Gamma(Y, \mathcal{B}_X)$.

(b) Set $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. With notation 4.1.2.16, via the local description 4.1.2.16.2, the logarithmic adjoint operator (see 3.4.1.2.3) extends to a map $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$ given by

$$P = \sum_{\underline{k} \in \mathbb{N}^d} b_{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} \mapsto \widetilde{P} := \sum_{\underline{k}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} \rangle_{(m)}} b_{\underline{k}},$$

where $b_{\underline{k}}$ is a finite sequence of elements of $\Gamma(X, \mathcal{B}_X)$.

Proposition 4.3.5.3 (Comparison between adjoint operator with or without logarithmic structure)**.** *Suppose $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $u_1, \dots, u_r$ and keep notation 4.3.5.2. Let P, Q be two differential operators of $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$. With notation 4.3.5.2, the following properties hold.*

(a) *We have ${}^t({}^t P) = P$, $\widetilde{\widetilde{P}} = P$, ${}^t(PQ) = {}^t Q {}^t P$ and $\widetilde{PQ} = \widetilde{Q}\widetilde{P}$.*

(b) *Suppose $u_1, \dots, u_r \in \mathcal{O}^*_X$. We have the equality*

$$\rho(\widetilde{P}) = \rho\left(\underline{t}\ {}^t P\ \frac{1}{\underline{t}}\right), \qquad (4.3.5.3.1)$$

where $\underline{t} = \underline{t}^{\underline{1}} = t_1 \cdots t_n$ and where ρ is the canonical map $\widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{D}^{(m)}_Y$.

Proof. (a) Let $b \in \Gamma(X, \mathcal{B}_X)$ and $\underline{k} \in \mathbb{N}^r$ and $P := b\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} \in \Gamma(X, \widetilde{D}^{(m)}_{X^\sharp/S^\sharp})$. Following 4.1.2.4.1, we get $\widetilde{P} = \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} \rangle} b = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle}(b) \times \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k}-\underline{i} \rangle}$. Since $\widetilde{\widetilde{\underline{\partial}}}_\sharp^{\langle \underline{k}-\underline{i} \rangle} = \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle}$, then we get

$$\widetilde{\widetilde{P}} = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle} \times \underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle}(b).$$

Hence, the equality $\widetilde{\widetilde{P}} = P$ follows from the computations:

$$\sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i} \rangle} \times \underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle}(b) \overset{3.2.3.7.2}{=} \sum_{\underline{i} \leq \underline{k}} \sum_{\underline{j} \leq \underline{k}-\underline{i}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{k}-\underline{i} \\ \underline{j} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i}-\underline{j} \rangle_{(m)}} (\underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle}(b)) \underline{\partial}_\sharp^{\langle \underline{j} \rangle_{(m)}}$$

$$= \sum_{\underline{j} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{j} \end{smallmatrix} \right\} \left(\sum_{\underline{i} \leq \underline{k}-\underline{j}} \left\{ \begin{smallmatrix} \underline{k}-\underline{j} \\ \underline{i} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{i}-\underline{j} \rangle_{(m)}} \underline{\widetilde{\partial}}_\sharp^{\langle \underline{i} \rangle}(b) \right) \underline{\partial}_\sharp^{\langle \underline{j} \rangle_{(m)}} \overset{3.4.5.2.1}{=} b\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}.$$

We compute similarly the equality $\widetilde{PQ} = \widetilde{Q}\widetilde{P}$, whereas the equalities involving the adjoint operators are obvious.

(b) By using 4.1.1.5 and replacing $\mathcal{B}_X$ by the image of the morphism $\mathcal{B}_X \to j_*\mathcal{B}_Y$ if necessary, to check the assertion (b) we can suppose that ρ is injective. Similar to 2.2.1.3, 3.4.1.3 and 3.4.1.5, we can therefore conclude the proposition. □

4.3.5.4. Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.

(a) It follows from 4.2.4.3.1 that we have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules:

$$(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_r \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}. \tag{4.3.5.4.1}$$

(b) Since $\widetilde{\omega}_{X^\sharp/S^\sharp}$ is locally free (of rank one), the canonical $\mathcal{B}_X$-linear morphism

$$\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \mathcal{M}) \xrightarrow{\sim} \mathcal{M} \tag{4.3.5.4.2}$$

is an isomorphism. Following 4.2.4.11.1, this isomorphism is moreover $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear. Similarly, it follows from 4.2.4.11.2 that we have the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism:

$$\mathcal{E} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}). \tag{4.3.5.4.3}$$

(c) Following 4.3.4.12, $\mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, -)$ and $\mathcal{H}om_{\mathcal{O}_X}(\omega_{X^\sharp/S^\sharp}, -)$ (resp. $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} -$ and $\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} -$) are canonically isomorphism on the category of right (resp. left) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.

(d) This yields that the functors $- \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp} = \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, -)$ and $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} -$ (resp. $- \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp} = \mathcal{H}om_{\mathcal{O}_X}(\omega_{X^\sharp/S^\sharp}, -)$ and $\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} -$) induce quasi-inverse equivalences between the category of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and that of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.

4.3.5.5 (Local description)**.** Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Suppose in this paragraph that $X^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,r}$. Let $(t_\lambda)_{\lambda=1,\ldots,r}$ be the induced coordinates of Y/S.

(a) The element $d\log u_1 \wedge \cdots \wedge d\log u_d$ is a basis of the free $\mathcal{B}_X$-module $\widetilde{\omega}_{X^\sharp/S^\sharp}$. For any section $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $e \in \mathcal{E}$, we have the formula in $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$:

$$(d\log u_1 \wedge \cdots \wedge d\log u_d \otimes e)P = d\log u_1 \wedge \cdots \wedge d\log u_d \otimes \widetilde{P}e. \tag{4.3.5.5.1}$$

Indeed, by $\mathcal{B}_X$-linearity, it is sufficient to check it for $P = \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} \rangle}$. Moreover, via 4.3.4.12.1, the canonical isomomorphism $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \xrightarrow{\sim} \omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. It follows from 3.4.5.1, $(d\log u_1 \wedge \cdots \wedge d\log u_d) \cdot \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} \rangle} = 0$ is $\underline{k} \neq 0$. By using the formula 4.2.3.5.3, this yields that 4.3.5.5.1 holds for $P = \underline{\widetilde{\partial}}_\sharp^{\langle \underline{k} \rangle}$. Hence, we are done.

(b) This yields that, by identifying as above $\widetilde{\omega}_{X^\sharp/S^\sharp}$ and $\mathcal{B}_X$ the left action of $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ on $m \in \mathcal{M}$ in $\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}$ is equal to $m \cdot \widetilde{P}$. These structures are called "twisted" structures.

(c) We get the analogous description for $\widetilde{\omega}_Y \otimes_{\mathcal{B}_Y} \mathcal{E}|_Y$ and $\mathcal{M}|_Y \otimes_{\mathcal{B}_Y} \widetilde{\omega}_Y^{-1}$ by exchanging $\widetilde{P}$ by ${}^{\mathrm{t}}P$, i.e. by replacing the logarithmic adjoint by the adjoint.

Lemma 4.3.5.6. *Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. We have the following isomorphism of $\mathcal{O}_S$-modules:*

$$\mathcal{M} \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} \mathcal{E} \xrightarrow{\sim} (\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} (\mathcal{M} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}). \tag{4.3.5.6.1}$$

Proof. We construct the isomorphism 4.3.5.6.1 as follows:

$$\begin{aligned}
\mathcal{M} \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} \mathcal{E} &\underset{4.3.5.4.2}{\xleftarrow{\sim}} \big(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \mathcal{M})\big) \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} \mathcal{E} \\
&\underset{4.3.5.4.1}{\xleftarrow{\sim}} \Big((\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_r \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \mathcal{M}) \Big) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E} \\
&\underset{4.2.4.2.2}{\xrightarrow{\sim}} \Big((\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_l \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E} \Big) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \mathcal{M}) \\
&\underset{4.2.5.6.1}{\xrightarrow{\sim}} \Big((\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_r \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E} \Big) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \mathcal{M}) \\
&\underset{4.3.5.4.1}{\xrightarrow{\sim}} \big(\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}\big) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, \mathcal{M}).
\end{aligned}$$

□

Lemma 4.3.5.7. *The functors $- \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp} = \mathcal{H}om_{\mathcal{B}_X}(\widetilde{\omega}_{X^\sharp/S^\sharp}, -)$ and $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} -$ are exact and induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules.*

Proof. The exactness is obvious. With regards to the coherence and the local projectivity of finite type, it is sufficient to use the descriptions of 4.3.5.5 on the twisted structures (and recalling that the arrows that associates the logarithmic adjoint to a differential operator are isomorphisms). Finally, for the flatness, this is a consequence of the following fact: for any left $\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}$-module $\mathcal{E}$ and right $\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}$-module $\mathcal{M}$, we have the canonical isomorphism 4.3.5.6.1. □

Proposition 4.3.5.8. *Let $\mathcal{E}$, $\mathcal{F}$ of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and $\mathcal{M}$, $\mathcal{N}$ of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. The following canonical isomorphisms*

$$\begin{gathered}
\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} (\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}) \xrightarrow{\sim} (\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{B}_X} \mathcal{F}, \\
(\mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp} \xrightarrow{\sim} (\mathcal{M} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}) \otimes_{\mathcal{B}_X} \mathcal{E} \\
(\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}) \otimes_{\mathcal{B}_X} (\mathcal{M} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{E}, \\
\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{F}), \\
\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{M}) \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{M} \otimes_{\mathcal{O}_X} \omega^{-1}_{X^\sharp/S^\sharp}), \\
\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}, \omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{F}).
\end{gathered}$$

are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. The $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linearity being local, we reduce to suppose X affine and $X^\sharp/S^\sharp$ endowed with logarithmic coordinates. The formulas 4.2.3.1.1, 4.2.3.5.1, 4.2.3.1.3, 4.2.3.5.2 and 4.2.3.5.5 allow us to conclude. □

Proposition 4.3.5.9. *The functor $\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} -$ of the category of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules in that of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules is canonically isomorphic to the functor $(\omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} -$.*

Proof. By taking $\mathcal{M} = \omega_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$ and $\mathcal{N} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$, this is a consequence of 4.2.4.3.2. □

4.3.5.10. Let $\mathcal{E}, \mathcal{F}$ (resp. $\mathcal{M}, \mathcal{N}$) be two left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. It follows from 4.3.5.6 and 4.3.5.7 that we have the following isomorphism of $\mathcal{O}_S$-modules:

$$(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{F}) \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} \mathcal{E} \xrightarrow{\sim} (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} \mathcal{F}, \tag{4.3.5.10.1}$$

$$\mathcal{M} \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} (\mathcal{N} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \mathcal{N} \otimes_{\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}} (\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}). \tag{4.3.5.10.2}$$

4.4 On the preservation of $\mathcal{D}$-module structures under pullbacks, base change

4.4.1 Relative ringed topoi

Definition 4.4.1.1. We will need the following definition.

(a) We define the category of "ringed logarithmic schemes" as follows: a ringed logarithmic scheme consists in pairs $(U^\sharp, \mathcal{B}_U)$ where $U^\sharp$ is a logarithmic scheme and $\mathcal{B}_U$ is a commutative $\mathcal{O}_U$-algebra. A "morphism of ringed logarithmic schemes" $\widetilde{\alpha}\colon (U^\sharp, \mathcal{B}_U) \to (V^\sharp, \mathcal{B}_V)$ is the data of a morphism of log schemes of the form $U^\sharp \to V^\sharp$ (denoted by α) and of a morphism of commutative $\mathcal{O}_U$-algebras $\alpha^*\mathcal{B}_V \to \mathcal{B}_U$ (denoted by ρ_α). We compose such morphisms in the obvious way.

(b) We define the category of "relative ringed logarithmic schemes" as follows: a "relative ringed logarithmic schemes" $(U^\sharp, \mathcal{B}_U)/(V^\sharp, \mathcal{B}_V)$ is a morphism of ringed logarithmic schemes $(U^\sharp, \mathcal{B}_U) \to (V^\sharp, \mathcal{B}_V)$. A morphism of relative ringed logarithmic schemes $(U^\sharp, \mathcal{B}_U)/(V^\sharp, \mathcal{B}_V) \to (U'^\sharp, \mathcal{B}_{U'})/(V'^\sharp, \mathcal{B}_{V'})$ is a commutative square of the form

$$\begin{array}{ccc} (U^\sharp, \mathcal{B}_U) & \xrightarrow{\widetilde{f}} & (U'^\sharp, \mathcal{B}_{U'}) \\ \downarrow{\scriptstyle \widetilde{\alpha'}} & & \downarrow{\scriptstyle \widetilde{\alpha'}} \\ (V^\sharp, \mathcal{B}_V) & \xrightarrow{\widetilde{g}} & (V'^\sharp, \mathcal{B}_{V'}), \end{array} \tag{4.4.1.1.1}$$

such that all the arrows are morphisms of ringed logarithmic schemes.

(c) Similarly, we define the categories of "(relative) ringed $\mathcal{V}$-log formal schemes" and (resp. of "(relative) ringed topoi", resp. "(relative) ringed spaces") by replacing logarithmic schemes by $\mathcal{V}$-log formal schemes (resp. topoi, resp. topological spaces).

Example 4.4.1.2. A scheme or log scheme can be viewed as a ringed logarithmic scheme via the faithfully flat functor given by $Z^\sharp \mapsto (Z^\sharp, \mathcal{O}_Z)$.

Definition 4.4.1.3. We complete the definitions of the categories of 4.4.1.1 by introducing the following notion of (strongly) quasi-flatness as follow.

(a) A morphism of ringed logarithmic schemes (resp. a morphism of ringed $\mathcal{V}$-log formal schemes) of the form $\widetilde{\phi}\colon (U^\sharp, \mathcal{B}_U) \to V^\sharp$ is said to be "quasi-flat" if $\mathcal{B}_U$ is a quasi-flat $\phi^{-1}\mathcal{O}_V$-algebra (see definition 3.1.1.5 or resp. 3.3.1.11).

(b) A morphism of ringed $\mathcal{V}$-log formal schemes $(\mathfrak{U}^\sharp, \mathcal{B}_{\mathfrak{U}})/\mathfrak{V}^\sharp$ is "strongly quasi-flat" if there exists a morphism $\mathfrak{V} \to \mathfrak{T}$ of $\mathcal{V}$-formal schemes such that the induced morphism of ringed spaces $(\mathfrak{U}, \mathcal{B}_{\mathfrak{U}}) \to \mathfrak{T}$ is flat and such that, denoting by $\mathcal{I}_{\mathfrak{T}}$ an ideal of definition of $\mathfrak{T}$, the sheaf $\mathcal{O}_{T_0}$ (resp. $gr^\bullet_{\mathcal{I}_{\mathfrak{T}}}\mathcal{O}_{\mathfrak{T}}$) has finite tor dimension on $\mathcal{O}_{\mathfrak{T}}$ (resp. $\mathcal{O}_{T_0}$).

(c) A morphism of relative ringed logarithmic schemes (resp. a morphism of relative ringed $\mathcal{V}$-log formal schemes) of the form $(U^\sharp, \mathcal{B}_U)/V^\sharp \to (U'^\sharp, \mathcal{B}_{U'})/V'^\sharp$ is said to be "quasi-flat" if there exists a morphism of schemes (resp. a morphism of $\mathcal{V}$-formal schemes) $V' \to T$ such that both induced morphisms of ringed spaces $(U', \mathcal{B}_{U'}) \to T$ and $(U, \mathcal{B}_U) \to T$ are flat. Remark that in both $(U^\sharp, \mathcal{B}_U)/V^\sharp$ and $(U'^\sharp, \mathcal{B}_{U'})/V'^\sharp$ are quasi-flat in the sense of (a).

(d) Let $\widetilde{\alpha} \colon (\mathfrak{U}^\sharp, \mathcal{B}_{\mathfrak{U}})/\mathfrak{V}^\sharp \to (\mathfrak{U}'^\sharp, \mathcal{B}_{\mathfrak{U}'})/\mathfrak{V}'^\sharp$ be a morphism of relative ringed $\mathcal{V}$-log formal schemes. We say that $\widetilde{\alpha}$ is "strongly quasi-flat" if there exists a morphism $\mathfrak{V}' \to \mathfrak{T}$ of $\mathcal{V}$-formal schemes such that both induced morphisms of ringed spaces $(\mathfrak{U}', \mathcal{B}_{\mathfrak{U}'}) \to \mathfrak{T}$ and $(\mathfrak{U}, \mathcal{B}_{\mathfrak{U}}) \to \mathfrak{T}$ are flat and such that denoting by $\mathcal{I}_{\mathfrak{T}}$ an ideal of definition of $\mathfrak{T}$ the sheaf $\mathcal{O}_{T_0}$ (resp. $gr^\bullet_{\mathcal{I}_{\mathfrak{T}}} \mathcal{O}_{\mathfrak{T}}$) has finite tor dimension on $\mathcal{O}_{\mathfrak{T}}$ (resp. $\mathcal{O}_{T_0}$). Remark that in both $(\mathfrak{U}^\sharp, \mathcal{B}_{\mathfrak{U}})/\mathfrak{V}^\sharp$ and $(\mathfrak{U}'^\sharp, \mathcal{B}_{\mathfrak{U}'})/\mathfrak{V}'^\sharp$ are strongly quasi-flat in the sense of (b).

Example 4.4.1.4. We will give later that in the case of overconvergent singularities, we do have strongly quasi-flat morphisms (7.3.2.11.b).

4.4.1.5. Let us explain a bit the purpose of these notions of (strongly) quasi-flatness of 4.4.1.3.

(a) The quasi-flatness will be useful to be able to define correctly extraordinary pullbacks or pushforwards (see 5.1.1.3).

(b) Let $(\mathfrak{U}^\sharp, \mathcal{B}_{\mathfrak{U}})/\mathfrak{V}^\sharp$ be a strongly quasi-flat morphism of ringed $\mathcal{V}$-log formal schemes. Then denoting by $\mathcal{I}$ an ideal of definition of $\mathfrak{U}$, the sheaf $\mathcal{O}_{U_0}$ (resp. $gr^\bullet_{\mathcal{I}} \mathcal{O}_{\mathfrak{U}}$) has finite tor dimension on $\mathcal{O}_{\mathfrak{U}}$ (resp. $\mathcal{O}_{U_0}$). Hence, we will be able to use 7.3.2.9, 7.3.2.10 and 7.3.3.3.

4.4.2 Pullbacks

Let

$$\begin{array}{ccc} X^\sharp & \xrightarrow{f} & Y^\sharp \\ \downarrow & & \downarrow \\ S^\sharp & \longrightarrow & T^\sharp, \end{array} \tag{4.4.2.0.1}$$

be a commutative diagram where $S^\sharp$ and $T^\sharp$ are nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ as defined in 3.1.1.1 where i is an integer, (resp. $S^\sharp$ and $T^\sharp$ are nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10) where $X^\sharp$ is a log smooth $S^\sharp$-log scheme (resp. log smooth $S^\sharp$-log formal scheme) and $Y^\sharp$ is a log smooth $T^\sharp$-log scheme (resp. log smooth $T^\sharp$-log formal scheme). Let $\mathcal{B}_X$ (resp. $\mathcal{B}_Y$) be a commutative $\mathcal{O}_X$-algebra (resp. $\mathcal{O}_Y$-algebra) endowed with a compatible structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module). Let us recall that $f^*\mathcal{B}_Y$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module (see 3.4.4.2) which is compatible with its structure of $\mathcal{O}_X$-algebra (see 3.4.4.6). We suppose finally that we have a morphism of algebras $f^*\mathcal{B}_Y \to \mathcal{B}_X$ which is moreover $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. We will denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$.

We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{Y}^\sharp$) the ringed logarithmic scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(Y^\sharp, \mathcal{B}_Y)$), and by $\tilde{f} \colon \widetilde{X}^\sharp/S^\sharp \to \widetilde{Y}^\sharp/T^\sharp$ the morphism of relative ringed logarithmic schemes induced by the diagram 4.4.2.0.1 and by $f^*\mathcal{B}_{\underset{\sim}{Y}} \to \mathcal{B}_{\underset{\sim}{X}}$. When $S^\sharp \to T^\sharp$ in understood, by abuse of notation, we also denote by $\tilde{f}$ the induced morphism $\widetilde{X}^\sharp \to \widetilde{Y}^\sharp$ of ringed logarithmic schemes.

Notation 4.4.2.1. We have the following notation that can be also used by replacing respectively S and $X^\sharp$ by T and $Y^\sharp$.

For $j = 0, 1$, recall the morphisms $p^n_{j,(m)} \colon \Delta^n_{X^\sharp/S^\sharp(m)} \to X$ are finite and are homeomorphisms. We get a sheaf of rings on $\Delta^n_{X^\sharp/S^\sharp(m)}$ by setting $\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}} := p^{n*}_{0,(m)}(\mathcal{B}_X)$, i.e. $p^n_{0,(m)*}\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S(m)} = \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$. We get the ringed logarithmic ($\mathcal{V}$-formal) scheme

$$\widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)} := (\Delta^n_{X^\sharp/S^\sharp(m)}, \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}).$$

We denote by $\tilde{p}^n_{0,(m)}\colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)} \to \widetilde{X}^\sharp$ the morphism induced by the morphism of log (formal) schemes $p^n_{0,(m)}$ and by the ring homomorphism $(p^n_{0,(m)})^{-1}(\mathcal{B}_X) \to \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}$. By applying $p^n_{0,(m)*}$ to $(p^n_{0,(m)})^{-1}(\mathcal{B}_X) \to \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}$ we get $\widetilde{p}^n_{0,(m)}\colon \mathcal{B}_X \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ (see notation 4.1.2.5).

We denote by $\tilde{p}^n_{1,(m)}\colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)} \to \widetilde{X}^\sharp$ the morphism induced by the morphism of log (formal) schemes $p^n_{1,(m)}$ and by the ring homomorphism $(p^n_{1,(m)})^{-1}(\mathcal{B}_X) \to p^{n*}_{1,(m)}(\mathcal{B}_X) \xrightarrow[\epsilon^{\mathcal{B}_X}_n]{\sim} p^{n*}_0(\mathcal{B}_X) = \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}$. Since $p^n_{1,(m)*} = p^n_{0,(m)*}$ (see remark 3.2.2.6), then by applying $p^n_{0,(m)*}$ to this latter map, we get the homomorphism $\widetilde{p}^n_{1,(m)}\colon \mathcal{B}_X \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ (see notation 4.1.2.5).

According to notation 3.2.2.12, for any integer n and any integers $0 \le i < j \le 2$, it follows from the universal property of m-PD-envelopes of order n (see 3.2.1.1) that we get a unique m-PD-morphism $q^n_{ij,(m)}\colon \Delta^n_{X^\sharp/S^\sharp,(m)}(2) \to \Delta^n_{X^\sharp/S^\sharp,(m)}$ making commutative the diagram 3.2.2.12.1. For any $0 \le i \le 2$, we denote by $p^n_{i,(m)}(2)\colon \Delta^n_{X^\sharp/S^\sharp(m)}(2) \to X^\sharp$ the homomorphism equal to the composition of $\Delta^n_{X^\sharp/S^\sharp(m)}(2) \to X^{\sharp 3}_{/S^\sharp}$ with the ith projection. Similarly to the remark 3.2.2.6, $p^n_{i,(m)}(2)$ are finite homeomorphisms such that $p^n_{i,(m)}(2)_* = p^n_{0,(m)}(2)_*$. We set $\widetilde{\mathcal{P}}^n_{X^\sharp/S(m)}(2) := \mathcal{B}_X \otimes_{\mathcal{O}_X} p^n_{0,(m)}(2)_* \mathcal{P}^n_{X^\sharp/S(m)}(2)$. We get a sheaf of rings on $\Delta^n_{X^\sharp/S^\sharp(m)}(2)$ by setting $\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}(2)} := (p^n_{0,(m)}(2))^*(\mathcal{B}_X)$, i.e. $(p^n_{0,(m)}(2))^n_*(\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}) = \widetilde{\mathcal{P}}^n_{X^\sharp/S(m)}(2)$. We get the ring homomorphism $\widetilde{p}^n_{0,(m)}(2)\colon (p^n_{0,(m)}(2))^{-1}(\mathcal{B}_X) \to (p^n_{0,(m)}(2))^*(\mathcal{B}_X)$.

For $j = 1, 2$, via the identifications $(q^n_{0j,(m)})^* \circ p^{n*}_{0,(m)} = (p^n_{0,(m)}(2))^*$ and $(q^n_{0j,(m)})^* \circ p^{n*}_{1,(m)} = (p^n_{j,(m)}(2))^*$, we get the ring isomorphism $(q^n_{0j,(m)})^*(\epsilon^{\mathcal{B}_X}_n)\colon (p^n_{j,(m)}(2))^*(\mathcal{B}_X) \xrightarrow{\sim} (p^n_{0,(m)}(2))^*(\mathcal{B}_X)$. By composing with $(p^n_{0,(m)}(2))^{-1}(\mathcal{B}_X) = (p^n_{j,(m)}(2))^{-1}(\mathcal{B}_X) \to (p^n_{j,(m)}(2))^*(\mathcal{B}_X)$, we get the ring homomorphism $\widetilde{p}^n_{j,(m)}(2)\colon (p^n_{0,(m)}(2))^{-1}(\mathcal{B}_X) \to (p^n_{0,(m)}(2))^*(\mathcal{B}_X)$.

We set the ringed logarithmic ($\mathcal{V}$-formal) scheme

$$\widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)}(2) := (\Delta^n_{X^\sharp/S^\sharp(m)}(2), \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}(2)}).$$

We denote by $\widetilde{p}^n_{j,(m)}(2)\colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)}(2) \to \widetilde{X}^\sharp$ the morphism induced by the morphism of log (formal) schemes $p^n_{j,(m)}(2)$ and by the ring homomorphism $\widetilde{p}^n_{j,(m)}(2)\colon (p^n_{0,(m)}(2))^{-1}(\mathcal{B}_X) \to (p^n_{j,(m)}(2))^*(\mathcal{B}_X)$.

For $j = 1, 2$, we set $\widetilde{q}^n_{0j,(m)}\colon (q^n_{0j,(m)})^{-1}(\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}) \to (q^n_{0j,(m)})^*(\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}) = \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}(2)}$. We have moreover, $\widetilde{q}^n_{12,(m)}\colon (q^n_{12,(m)})^{-1}(\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}) \to (q^n_{12,(m)})^*(\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}) = (p^n_{1,(m)}(2))^*(\mathcal{B}_X) = (q^n_{01,(m)})^* \circ p^{n*}_{1,(m)}(\mathcal{B}_X) \xrightarrow[\varepsilon^{\mathcal{B}_X}_n]{\sim} (q^n_{01,(m)})^* \circ p^{n*}_{0,(m)}(\mathcal{B}_X) = \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}(2)}$. For any integer n and any integers $0 \le i < j \le 2$, we denote by $\widetilde{q}^n_{ij,(m)}\colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)}(2) \to \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)}$ the morphism induced by the morphism of log (formal) schemes $q^n_{ij,(m)}$ and by the ring homomorphism $\widetilde{q}^n_{ij,(m)}\colon (q^n_{ij,(m)})^{-1}(\mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}}) \to (p^n_{0,(m)}(2))^{-1}(\mathcal{B}_X) \to \mathcal{B}_{\Delta^n_{X^\sharp/S^\sharp(m)}(2)}$. We have by construction the equality of morphism of ringed logarithmic ($\mathcal{V}$-formal) schemes:

$$\widetilde{p}^n_{0,(m)} \circ \widetilde{q}^n_{ij,(m)} = \widetilde{p}^n_{i,(m)}(2), \qquad \widetilde{p}^n_{1,(m)} \circ \widetilde{q}^n_{ij,(m)} = \widetilde{p}^n_{j,(m)}(2). \tag{4.4.2.1.1}$$

4.4.2.2. It follows from the ring homomorphisms of 3.4.4.1.2 and from $f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ that we get the ring homomorphisms

$$f^{-1}(\mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y^\sharp/T^\sharp(m)}) \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}; \tag{4.4.2.2.1}$$

$$f^{-1}(\mathcal{P}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{O}_Y} \mathcal{B}_Y) \to \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X. \tag{4.4.2.2.2}$$

The fact that the homomorphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} f^{-1}(\mathcal{P}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{O}_Y} \mathcal{B}_Y) & \xrightarrow{4.4.2.2.2} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X \\ {\scriptstyle f^{-1}(\varepsilon^{\mathcal{B}_Y}_n)}\downarrow\wr & & {\scriptstyle \varepsilon^{\mathcal{B}_X}_n}\downarrow\wr \\ f^{-1}(\mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y^\sharp/T^\sharp(m)}) & \xrightarrow{4.4.2.2.1} & \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.4.2.2.3}$$

With notation 4.1.2.5, it follows from the commutative diagram 4.4.2.2.3, the commutativity of the diagram in the case $j = 1$ (the case $j = 0$ is obvious):

$$\begin{array}{ccc} f^{-1}(\mathcal{B}_Y) & \xrightarrow{f^{-1}\widetilde{p}^n_{j,(m)}} & f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} \\ \downarrow & & \downarrow {\scriptstyle 4.4.2.2.1} \\ \mathcal{B}_X & \xrightarrow{\widetilde{p}^n_{j,(m)}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}. \end{array} \tag{4.4.2.2.4}$$

According to notation 3.4.4.1, we have the canonical morphism $f^n_{(m)} \colon \Delta^n_{X^\sharp/S^\sharp(m)} \to \Delta^n_{Y^\sharp/T^\sharp(m)}$. We denote by $\widetilde{f}^n_{(m)} \colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)} \to \widetilde{\Delta}^n_{Y^\sharp/T^\sharp(m)}$ the morphism of ringed logarithmic ($\mathcal{V}$-formal) schemes given by the morphism of log (formal) schemes $f^n_{(m)}$ and by 4.4.2.2.1. For any couple of integers $n \geq n'$, for any integer $r \geq 0$, we denote by

$$\widetilde{\psi}^{n,n'}_{X^\sharp/S^\sharp,(m)}(r) \colon \widetilde{\Delta}^{n'}_{X^\sharp/S^\sharp,(m)}(r) \to \widetilde{\Delta}^n_{X^\sharp/S^\sharp,(m)}(r) \tag{4.4.2.2.5}$$

the canonical exact closed immersion induced by 4.1.2.12.1. We get from the commutativity of the right square of 3.4.4.1.1 and of 4.4.2.2.4 that of

$$\begin{array}{ccccc} \widetilde{\Delta}^{n'}_{X^\sharp/S^\sharp(m)} & \xhookrightarrow{\widetilde{\psi}^{n,n'}_{X^\sharp/S^\sharp,(m)}} & \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)} & \xrightarrow{\tilde{p}^n_{j,(m)}} & \widetilde{X}^\sharp \\ \downarrow {\scriptstyle \widetilde{f}^{n'}_{(m)}} & & \downarrow {\scriptstyle \widetilde{f}^n_{(m)}} & & \downarrow {\scriptstyle \tilde{f}} \\ \widetilde{\Delta}^{n'}_{Y^\sharp/T^\sharp(m)} & \xhookrightarrow{\widetilde{\psi}^{n,n'}_{Y^\sharp/T^\sharp,(m)}} & \widetilde{\Delta}^n_{Y^\sharp/T^\sharp(m)} & \xrightarrow{\tilde{p}^n_{j,(m)}} & \widetilde{Y}^\sharp. \end{array} \tag{4.4.2.2.6}$$

4.4.2.3. It follows from the ring homomorphisms of 3.4.4.1.2 and from $f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ that we get the ring homomorphisms

$$f^{-1}(\mathcal{B}_Y \otimes_{\mathcal{O}_Y} p^n_{0,(m)}(2)_* \mathcal{P}^n_{Y^\sharp/T^\sharp(m)}(2) \to \mathcal{B}_X \otimes_{\mathcal{O}_X} p^n_{0,(m)}(2)_* \mathcal{P}^n_{X^\sharp/S^\sharp(m)}(2). \tag{4.4.2.3.1}$$

We denote by $\widetilde{f}^n_{(m)}(2) \colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)}(2) \to \widetilde{\Delta}^n_{Y^\sharp/T^\sharp(m)}(2)$ the morphism of ringed logarithmic ($\mathcal{V}$-formal) schemes given by the morphism of log (formal) schemes $f^n_{(m)}(2)$ and by 4.4.2.3.1. These homomorphisms are compatible when n varies, i.e., with notation 3.2.2.2.1, we have the equality $\widetilde{\psi}^{n',n}_{Y^\sharp/T^\sharp,(m)}(2) \circ \widetilde{f}^n_{(m)}(2) = \widetilde{f}^{n'}_{(m)}(2) \circ \widetilde{\psi}^{n',n}_{X^\sharp/S^\sharp,(m)}(2)$, for any $n' \geq n$. We have the commutative diagram:

$$\begin{array}{ccc} \widetilde{\Delta}^n_{X^\sharp/S^\sharp,(m)}(2) & \xrightarrow{\widetilde{q}^n_{ij,(m)}} & \widetilde{\Delta}^n_{X^\sharp/S^\sharp,(m)} \\ \downarrow {\scriptstyle \widetilde{f}^n_{(m)}(2)} & & \downarrow {\scriptstyle \widetilde{f}^n_{(m)}} \\ \widetilde{\Delta}^n_{Y^\sharp/T^\sharp,(m)}(2) & \xrightarrow{\widetilde{q}^n_{ij,(m)}} & \widetilde{\Delta}^n_{Y^\sharp/T^\sharp,(m)}. \end{array} \tag{4.4.2.3.2}$$

4.4.2.4. Let us now construct the inverse image of a left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module within the terminology of m-PD-stratifications with coefficients in $\mathcal{B}_Y$.

Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module and $(\varepsilon^{\mathcal{E}}_n)$ its m-PD-stratification with coefficients in $\mathcal{B}_Y$. The $\mathcal{B}_X$-module $\widetilde{f}^*\mathcal{E}$ has a canonical structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Indeed, it is a consequence of the commutative diagram 4.4.2.2.6 that the isomorphisms $\varepsilon^{\widetilde{f}^*\mathcal{E}}_n := \widetilde{f}^{n*}_{(m)}(\varepsilon^{\mathcal{E}}_n)$ endow $\widetilde{f}^*\mathcal{E}$ with an m-PD-stratification with coefficients in $\mathcal{B}_Y$ (for the cocycle conditions, we use the commutativity of 4.4.2.3.2 and the equalities 4.4.2.1.1). Let $\mathcal{D}$ be a sheaf of rings. When $\mathcal{E}$ is a $(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{D})$-bimodule, then by functoriality $\widetilde{f}^*(\mathcal{E})$ is a $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{D})$-bimodule. For instance we get the $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodule $\widetilde{f}^*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ that we will denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}$.

Example 4.4.2.5. Let us consider the morphism of ringed logarithmic ($\mathcal{V}$-formal) schemes : $(X^\sharp, \mathcal{B}_X) \xrightarrow{\widetilde{\mathrm{id}}} (X^\sharp, \mathcal{O}_X)$ (and $S^\sharp = T^\sharp$). If $\mathcal{E}$ is a left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module, it is a consequence of 4.3.4.2.1, that the structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module on $(\widetilde{\mathrm{id}})^*(\mathcal{E})$ defined by inverse image is equal to the structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module defined by extension.

4.4.2.6. We get the $\mathcal{B}_X$-algebra $\widetilde{f}^*(\widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) := \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}(\widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)})$. The morphism 4.4.2.2.1 factors throught (by abuse of notation): the homomorphism of $\mathcal{B}_X$-algebras:

$$\widetilde{f}^n_{(m)} \colon \widetilde{f}^*(\widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) \to \widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}. \tag{4.4.2.6.1}$$

Dualizing the formula 4.4.2.6.1, we obtain the homomorphism of $\mathcal{B}_X$-modules

$$\widetilde{f}^{n\vee}_{(m)} \colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp,n}. \tag{4.4.2.6.2}$$

Passing to the limit, we get the canonical homomorphism of $\mathcal{B}_X$-modules:

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}. \tag{4.4.2.6.3}$$

Example 4.4.2.7. Suppose $f = \mathrm{id}$. Then we get from 4.4.2.6.3

$$\rho^\star \colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to X^\sharp/T^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}, \tag{4.4.2.7.1}$$

which is by construction induced by the $\mathcal{B}_X$-linear dual of the natural maps $\rho_n := \widetilde{\mathrm{id}}^n_{(m)} \colon \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp(m)} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$. Then it follows from the commutativity of 4.4.2.3.2 that for any $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\, n}$, $P' \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\, n'}$, we get the commutative diagram

$$\begin{array}{ccccccc}
\widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/T^\sharp(m)} & \xrightarrow{\widetilde{\delta}^{n,n'}_{(m)}} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/T^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\mathrm{id} \otimes \rho^\star(P')} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/T^\sharp(m)} & \xrightarrow{\rho^\star(P)} & \mathcal{B}_X \\
\downarrow{\scriptstyle \rho_{n+n'}} & & \downarrow{\scriptstyle \rho_n \otimes \rho_{n'}} & & \downarrow{\scriptstyle \rho_n} & & \| \\
\widetilde{\mathcal{P}}^{n+n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\widetilde{\delta}^{n,n'}_{(m)}} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^{n'}_{X^\sharp/S^\sharp(m)} & \xrightarrow{\mathrm{id} \otimes P'} & \widetilde{\mathcal{P}}^{n}_{X^\sharp/S^\sharp(m)} & \xrightarrow{P} & \mathcal{B}_X
\end{array} \tag{4.4.2.7.2}$$

whose horizontal bottom (resp. top) morphisms is then $P \cdot P'$ (resp. $\rho^\star(P) \cdot \rho^\star(P')$). This yields that 4.4.2.7.1 is a ring homomorphism. Besides, the structure of left $\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp}$ on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to X^\sharp/T^\sharp}$ is given by this ring homomorphism. When $S^\sharp \to T^\sharp$ is log-smooth, we will make some local commutations (see below 5.3.1).

Proposition 4.4.2.8. *The canonical homomorphism 4.4.2.6.3 sends* 1 *to* $1 \otimes 1$ *and is* $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{B}_Y)$*-bilinear.*

Proof. We can copy 3.4.4.4. □

Proposition 4.4.2.9. *Let* $\mathcal{F}$ *be a left* $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$*-module. We have the canonical isomorphism of left* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-modules:*

$$\widetilde{f}^*\mathcal{F} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{F}. \tag{4.4.2.9.1}$$

Proof. We can copy the proof of 3.4.4.5.1. □

Corollary 4.4.2.10. *Let* $\mathcal{F}$ *be a left* $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$*-module. If* $\mathcal{F}$ *is quasi-nilpotent, then so is the* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-module* $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{F}$*.*

Proof. By functoriality of the m-PD-envelope, it follows from the characterization 4.2.1.10.(c) of the quasi-nilpotence that $\widetilde{f}^*\mathcal{F}$ is quasi-nilpotent. □

4.4.2.11. Denote by $\rho \colon f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ the given algebra homomorphism. Let $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}$, $Q \in \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp,n}$ such that $\widetilde{f}^{n\vee}_{(m)}(P) = 1 \otimes Q$ where $1 \otimes Q$ is the $\mathcal{B}_X$-linear map making commutative the top right square of the following diagram:

$$\begin{array}{ccccc} f^{-1}(\mathcal{B}_Y) & \xrightarrow{f^{-1}\widetilde{p}^n_{1,(m)}} & f^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} & \xrightarrow{f^{-1}Q} & f^{-1}(\mathcal{B}_Y) \\ & & \downarrow & & \downarrow \rho \\ \downarrow \rho & & \widetilde{f}^*(\widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) & \xrightarrow{1\otimes Q} & \mathcal{B}_X \\ & & \widetilde{f}^n_{(m)} \downarrow 4.4.2.6.1 & & \| \\ \mathcal{B}_X & \xrightarrow{\widetilde{p}^n_{1,(m)}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} & \xrightarrow{P} & \mathcal{B}_X. \end{array} \tag{4.4.2.11.1}$$

The equality $\widetilde{f}^{n\vee}_{(m)}(P) = 1\otimes Q$ means that the bottom right square of 4.4.2.11.1 is commutative. Following 4.4.2.2.4, the left rectangle is commutative. Hence, with 4.2.1.5.3, this commutativity of the whole diagram 4.4.2.11.1 implies that for any $b \in \mathcal{B}_Y$, the following formula holds:

$$\rho(Q \cdot b) = P \cdot \rho(b), \tag{4.4.2.11.2}$$

where $Q \cdot b$ means the action of Q on b given by the structure of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module on $\mathcal{B}_Y$ and $P \cdot \rho(b)$ means the action of P on $\rho(b)$ via the left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{B}_X$.

4.4.2.12. We can extend the formula 4.4.2.11.2 as follows. Let $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}$, $Q_1, \ldots, Q_r \in \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp,n}$ $a_1, \ldots, a_r \in \mathcal{O}_X$ such that $\widetilde{f}^{n\vee}_{(m)}(P) = \sum_{j=1}^r a_r \otimes Q_r$. Let $\mathcal{F}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module. The morphism 4.4.2.2.1 factors throught the homomorphism of $\mathcal{B}_X$-algebras:

$$\widetilde{f}^n_{(m),\mathrm{r}} \colon \widetilde{f}^*(\widetilde{p}^n_{1,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) \to \widetilde{p}^n_{1,(m)*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}, \tag{4.4.2.12.1}$$

where we add the symbol r to distinguish it from 4.4.2.6.1. Consider the diagram:

$$\begin{array}{ccccc} \widetilde{f}^{-1}(\widetilde{p}^n_{1,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{B}_Y} \mathcal{F}) & \xrightarrow{\widetilde{f}^{-1}\varepsilon^{\mathcal{F}}_n} & \widetilde{f}^{-1}(\mathcal{F} \otimes_{\mathcal{B}_Y} \widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) & \xrightarrow{\sum_{j=1}^r a_r Q_r} & \widetilde{f}^*\mathcal{F} \\ \downarrow & & \downarrow & & \| \\ \widetilde{f}^*(\widetilde{p}^n_{1,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) \otimes_{\mathcal{B}_X} \widetilde{f}^*\mathcal{F} & & \widetilde{f}^*\mathcal{F} \otimes_{\mathcal{B}_X} \widetilde{f}^*(\widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) & \xrightarrow{\mathrm{id} \otimes \sum_{j=1}^r a_r \otimes Q_r} & \widetilde{f}^*\mathcal{F} \\ \downarrow \widetilde{f}^{n*}_{(m)\mathrm{r}} & & \downarrow \widetilde{f}^{n*}_{(m)} & & \| \\ \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \widetilde{f}^*\mathcal{F} & \xrightarrow{\varepsilon^{\widetilde{f}^*\mathcal{F}}_n} & \widetilde{f}^*\mathcal{F} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\mathrm{id} \otimes P} & \widetilde{f}^*\mathcal{F} \end{array} \tag{4.4.2.12.2}$$

By definition of the stratification of the inverse image, the left rectangle is commutative. Moreover, the equality $\widetilde{f}^{n\vee}_{(m)}(P) = \sum_{j=1}^r a_r \otimes Q_r$ means that the right square is commutative. Hence, the diagram 4.4.2.12.2 is commutative. By composing 4.4.2.12.2 with the following commutative diagram

$$\begin{array}{ccc} \widetilde{f}^{-1}\mathcal{F} & \xrightarrow[4.1.2.7]{\widetilde{f}^{-1}\widetilde{p}^n_{1,(m),\mathcal{F}}} & \widetilde{f}^{-1}(\widetilde{p}^n_{1,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{B}_Y} \mathcal{F}) \\ \downarrow & & \downarrow \\ \widetilde{f}^*\mathcal{F} & \xrightarrow[4.1.2.7]{\widetilde{f}^*\widetilde{p}^n_{1,(m),\mathcal{F}}} & \widetilde{f}^*(\widetilde{p}^n_{1,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) \otimes_{\mathcal{B}_X} \widetilde{f}^*\mathcal{F} \\ \| & & \downarrow \widetilde{f}^{n*}_{(m)\mathrm{r}} \\ \widetilde{f}^*\mathcal{F} & \xrightarrow[4.1.2.7]{\widetilde{p}^n_{1,(m),\widetilde{f}^*\mathcal{F}}} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \widetilde{f}^*\mathcal{F} \end{array} \tag{4.4.2.12.3}$$

we get a commutative diagram which can be translated by the formula (recall 4.2.1.5.3): for any $x \in \mathcal{F}$, we have

$$P(1 \otimes x) = \sum_{j=1}^r a_j \otimes Q_j(x). \tag{4.4.2.12.4}$$

Suppose $Y^\sharp \to T^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. Then, it follows from 4.4.2.12.4 that for any $x \in \mathcal{F}$, we have the formula

$$P(1 \otimes x) = \sum_{|\underline{k}| \leq n} P \circ \widetilde{f}^n_{(m)}(\underline{\tau}_\sharp^{\{\underline{k}\}}) \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle}(x). \tag{4.4.2.12.5}$$

For instance, if $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,1}$ is exactly of order 1 (i.e. $P(1) = 0$), then we compute

$$P(1 \otimes x) = \sum_{i=1}^{d} P(\widetilde{f}^*(t_i)) \otimes \partial_{\sharp i}(x). \tag{4.4.2.12.6}$$

4.4.2.13 (What about right $\mathcal{D}$-modules ?)**.** Suppose f is finite. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module. Following 4.2.2.5, this means that $\mathcal{M}$ is endowed with a PD-costratification of level m with coefficient in $\mathcal{B}_Y$. Similarly to 4.4.2.4 (we just have to replace functors of the form $\widetilde{f}^*$ by functors of the form $\widetilde{f}^\flat$), by applying the functors of the form $\widetilde{f}^\flat$ to the PD-costratification of level m of $\mathcal{M}$, we get a PD-costratification of level m structure on $\widetilde{f}^\flat(\mathcal{M})$, i.e. $\widetilde{f}^\flat(\mathcal{M})$ is canonically endowed with a structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (we copy word by word 4.4.2.4). By functoriality, viewing $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$ as a $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-bimodule, we get a structure of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodule on $\widetilde{f}^\flat(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. When f is an exact closed immersion, this will be studied more thoroughly (see 5.2.5.1).

4.4.3 Log étale case

We keep notation 4.4.2. We suppose the morphism f is log étale and that the bottom arrow of 4.4.2.0.1 is the identity.

Lemma 4.4.3.1. *We have the following properties.*

(a) The following canonical homomorphism of $\mathcal{B}_X$-algebras (see 4.4.2.6.1)

$$\widetilde{f}^n_{(m)} \colon \widetilde{f}^*(\widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp(m)}) \to \widetilde{p}^n_{0,(m)*}\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \tag{4.4.3.1.1}$$

is an isomorphism.

(b) The canonical homomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules (see 4.4.2.6.3)

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/S^\sharp} \tag{4.4.3.1.2}$$

is an isomorphism.

(c) The canonical composition map

$$\rho_{\widetilde{f}} \colon f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \underset{4.4.3.1.2}{\overset{\sim}{\longleftarrow}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \tag{4.4.3.1.3}$$

is a ring homomorphism which fits into the commutative diagram

$$\begin{array}{ccc} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} & \xrightarrow{4.4.3.1.3} & \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{B}_Y & \longrightarrow & \mathcal{B}_X, \end{array} \tag{4.4.3.1.4}$$

where the vertical arrows are the canonical embeddings and the bottom arrow is given by the morphism $\widetilde{f}$.

(d) The canonical morphism 4.4.3.1.2 is in fact an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules, where the structure of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is given via 4.4.3.1.3.

Proof. 0) Since the lemma is local we can suppose that $Y^\sharp \to T^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. By composing it with f this yields the logarithmic coordinates $(u'_\lambda)_{\lambda=1,\dots,r}$ of $X^\sharp \to T^\sharp$. For any $\lambda = 1,\dots,r$, put $\tau_{\sharp\lambda} := \mu^n_{(m)}(u_\lambda) - 1$ where for any $a \in M_{Y^\sharp}$ $\mu^n_{(m)}(a)$ is the unique section of $\ker(\mathcal{O}^*_{\Delta^n_{Y^\sharp/T^\sharp,(m)}} \to \mathcal{O}^*_Y)$ such that we get in $M^n_{Y^\sharp/T^\sharp,(m)}$ the equality $p_1^{n*}(a) = p_0^{n*}(a)\mu^n_{(m)}(a)$. We still denote by $\tau_\sharp$ its image via the canonical morphism $\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp,(m)}$. Similarly, replacing $Y^\sharp/T^\sharp$ by $X^\sharp/S^\sharp$ we get the elements $\tau'_{\sharp\lambda} := \mu^n_{(m)}(u'_\lambda) - 1$ of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ or $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$.

a) Following 4.1.2.16.1, we get the isomorphism of m-PD-$\mathcal{B}_Y$-algebras

$$\begin{aligned}\mathcal{B}_Y\langle T_1,\dots,T_r\rangle_{(m),n} &\xrightarrow{\sim} \widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp,(m)}\\ T_\lambda &\mapsto \tau_{\sharp\lambda,(m)},\end{aligned} \tag{4.4.3.1.5}$$

where the first term is defined as in 1.3.3.6. Then $\widetilde{f}^n_{(m)}(1\otimes\tau_{\sharp\lambda}) = \tau'_{\sharp\lambda}$ for any $\lambda = 1,\dots,r$. Indeed, we reduce to check it on $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$, i.e that $f^n_{(m)}(1\otimes\tau_{\sharp\lambda}) = \tau'_{\sharp\lambda}$. Since $X^\sharp$ is nice (see 3.1.1.2), then the canonical morphism $\mathcal{P}^n_{X^\sharp/S^\sharp(m)} \to j_*\mathcal{P}^n_{(X^\sharp)^*/S^\sharp(m)}$ is injective, where $j\colon (X^\sharp)^* \hookrightarrow X^\sharp$ is the open immersion (this is a consequence of 3.1.1.3). Hence, we reduce to the case where the logarithmic structures are trivial. By using the left equality of 3.2.3.9.5 and with the corresponding notation, we reduce to check $f^n_{(m)}(1\otimes\tau_\lambda) = \tau'_\lambda$, which is easy (use the right commutative square of the diagram 3.4.4.1.1). Since 3.4.4.1.2 is an m-PD morphism, then $\widetilde{f}^n_{(m)}(1\otimes\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}) = \underline{\tau}'^{\{\underline{k}\}_{(m)}}_\sharp$. This yields that 4.4.3.1.1 is an isomorphism.

b) By duality and passing to the limit, we get the part b) form the part a).

c) From the part a) of the proof, with the similar to 4.1.2.16.2 notation, we get $\rho_{\widetilde{f}}(\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}) = \underline{\partial}'^{\langle\underline{k}\rangle_{(m)}}_\sharp$. Denote by $\rho\colon f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ the given algebra homomorphism. By $f^{-1}\mathcal{B}_Y$-linearity, we get for any section b of $f^{-1}\mathcal{B}_Y$, the equality $\rho_{\widetilde{f}}(b\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}) = \rho(b)\underline{\partial}'^{\langle\underline{k}\rangle_{(m)}}_\sharp$. Let Q_1, Q_2 be two sections of $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$. We have to check $\rho_{\widetilde{f}}(Q_1\cdot Q_2) = \rho_{\widetilde{f}}(Q_1)\cdot\rho_{\widetilde{f}}(Q_2)$. By additivity, it follows from the description 4.1.2.16.2 that we reduce to the case where Q_1 and Q_2 are of the form $b\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}$ with b a section of $f^{-1}\mathcal{B}_Y$. Following 4.4.2.11.2, $\rho(Q\cdot b) = \rho_{\widetilde{f}}(Q)\cdot\rho(b)$ for any section Q of $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$. In particular, $\rho(\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}(b)) = \underline{\partial}'^{\langle\underline{k}\rangle_{(m)}}_\sharp(\rho(b))$. Hence, we can conclude thanks to 3.2.3.13.1, 4.1.2.2.3. □

Proposition 4.4.3.2. *We have the canonical isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules:*

$$\widetilde{f}^*\mathcal{E} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{E}. \tag{4.4.3.2.1}$$

Proof. This is a consequence of 4.4.2.9.1 and 4.4.3.1.(d). □

Lemma 4.4.3.3. *The canonical homomorphism of $\mathcal{B}_X$-modules:*

$$\widetilde{f}^*(\widetilde{\omega}_{Y^\sharp/S^\sharp}) \to \widetilde{\omega}_{X^\sharp/S^\sharp} \tag{4.4.3.3.1}$$

is an isomorphism.

Let us denote by $\rho^\omega_{\widetilde{f}}\colon f^{-1}(\widetilde{\omega}_{Y^\sharp/S^\sharp}) \to \widetilde{\omega}_{X^\sharp/S^\sharp}$ the canonical homomorphism. With notation 4.4.3.1.3, let

$$f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp}\otimes_{\mathcal{B}_Y}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right) \to \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \tag{4.4.3.3.2}$$

be the map given by $\omega\otimes P \mapsto \rho^\omega_{\widetilde{f}}(\omega)\otimes\rho_{\widetilde{f}}(P)$. The map 4.4.3.3.2 is a homomorphism of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-bimodules, where the right structure (resp. the left structure i.e. the twisted one) of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module of $\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ comes from its right structure (resp. left structure) of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module via the ring homomorphism 4.4.3.1.3.

Proof. By construction, the map 4.4.3.3.2 is a homomorphism of right $(f^{-1}\mathcal{B}_Y, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$-bimodules. It remains to check the $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-linearity for the left structure. Since this is local we can suppose that $Y^\sharp \to S^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. By composing with f this yields logarithmic coordinates $(u'_\lambda)_{\lambda=1,\dots,r}$ of $X^\sharp \to S^\sharp$. Then $d\log u_1 \wedge \cdots \wedge d\log u_r$ is a basis of $\widetilde{\omega}_{Y^\sharp/S^\sharp}$ and $d\log u'_1 \wedge \cdots \wedge d\log u'_r$ is a basis of $\widetilde{\omega}_{X^\sharp/S^\sharp}$. We compute $\rho^\omega_{\widetilde{f}}(d\log u_1 \wedge \cdots \wedge d\log u_r) = d\log u'_1 \wedge \cdots \wedge d\log u'_r$. Moreover, since $\rho_{\widetilde{f}}(\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}) = \underline{\partial}_\sharp'^{\langle \underline{k}\rangle_{(m)}}$ (see the proof of 4.4.3.1), then by using the local description of the twisted structure (see 4.3.5.5) of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ and of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$, we compute that 4.4.3.3.2 is $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-linearity for the respective twisted structures. □

4.4.3.4. By using the same computation than that of the proof of 4.4.3.3, we can check the following diagram

$$\begin{array}{ccc} f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right) & \xrightarrow{4.4.3.3.2} & \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \downarrow{\scriptstyle 4.2.5.5.1} & & \downarrow{\scriptstyle 4.2.5.5.1} \\ f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right) & \xrightarrow{4.4.3.3.2} & \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \end{array} \tag{4.4.3.4.1}$$

where the vertical maps are the transposition isomorphisms, is commutative. This yields the isomorphism of right $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$-bimodules

$$f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right)_l \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \tag{4.4.3.4.2}$$

where the index l means that in the tensor product we use the left structure of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module, where the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module) of $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is its left structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. comes from its structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module via the ring homomorphism $\rho_{\widetilde{f}}$ of 4.4.4.1.2). For any left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module $\mathcal{E}$, we get the isomorphisms of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules:

$$\begin{aligned} & f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E}\right) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \xrightarrow[4.3.5.4.1]{\sim}\ & (f^{-1}(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})_r \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{E}) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \xrightarrow[4.2.4.2.2]{\sim}\ & (f^{-1}(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})_l \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{E} \\ \xrightarrow[4.4.3.4.2]{\sim}\ & (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_r \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \mathcal{E} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \mathcal{E}). \end{aligned} \tag{4.4.3.4.3}$$

4.4.3.5. Using 4.4.3.1.4, we get the canonical homomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}, \mathcal{B}_X)$-bimodules

$$f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \otimes_{f^{-1}\mathcal{B}_Y} \mathcal{B}_X \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.4.3.5.1}$$

We can check by making a local computation that 4.4.3.5.1 is an isomorphism. For any right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module $\mathcal{M}$, this yields that the canonical morphism of $\mathcal{B}_X$-modules

$$\widetilde{f}^*\mathcal{M} := f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{B}_Y} \mathcal{B}_X \to f^{-1}\mathcal{M} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \tag{4.4.3.5.2}$$

is an isomorphism. Hence, we can endow $\widetilde{f}^*\mathcal{M}$ with a structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module which extends its structure of $\mathcal{B}_X$-module by transport structure via 4.4.3.5.2. We can check by a local computation that the canonical isomophism

$$\widetilde{f}^*(\widetilde{\omega}_{Y^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \tag{4.4.3.5.3}$$

of 4.4.3.3.1 is an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. Moreover, modulo the canonical isomorphism 4.4.3.2.1 and 4.4.3.5.2, for any left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module $\mathcal{E}$, the isomorphism 4.4.3.4.3 can be rewritten of the form:

$$\widetilde{f}^*\left(\widetilde{\omega}_{Y^\sharp/S^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E}\right) \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{f}^*(\mathcal{E}).$$

4.4.4 Base change

We keep notation 4.4.2. We suppose in this subsection that the diagram 4.4.2.0.1 is cartesian and the morphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism. In that case, we say that $\widetilde{f}^*$ is the base change via $S \to T$.

Proposition 4.4.4.1 (Base change)**.** *We have the following properties.*

(a) The morphism $\widetilde{f}^n_{(m)} \colon \widetilde{f}^*(\widetilde{\mathcal{P}}^n_{Y^\sharp/T(m)}) \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)}$ of 4.4.2.6.1 is an isomorphism (and then the right square of the diagram 4.4.2.2.6 is cartesian).

(b) The canonical morphism (see 4.4.2.6.3)

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}. \tag{4.4.4.1.1}$$

is an isomorphism.

(c) The composite map

$$\rho_{\widetilde{f}} \colon f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \to f^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.4.4.1.2}$$

is a homomorphism of sheaves of rings which fits into the commutative diagram

$$\begin{array}{ccc} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} & \xrightarrow{4.4.4.1.2} & \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{B}_Y & \longrightarrow & \mathcal{B}_X. \end{array} \tag{4.4.4.1.3}$$

The canonical morphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules 4.4.4.1.1 is in fact an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules, where the structure of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is given via 4.4.4.1.2.

(d) For any left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module $\mathcal{F}$ we have a canonical isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{f}^*\mathcal{F}. \tag{4.4.4.1.4}$$

Proof. Let us check (a). Since this is local we can suppose there exist logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$ of $Y^\sharp/T^\sharp$. This induces the logarithmic coordinates $(u'_\lambda = \widetilde{f}^*(u_\lambda))_{\lambda=1,\dots,r}$ of $X^\sharp/S^\sharp$. For any $\lambda = 1, \dots, r$, put $\tau_{\sharp\lambda} := \mu^n_{(m)}(u_\lambda) - 1$ where for any $a \in M_{Y^\sharp}$ $\mu^n_{(m)}(a)$ is the unique section of $\ker(\mathcal{O}^*_{\Delta^n_{Y^\sharp/T^\sharp,(m)}} \to \mathcal{O}^*_Y)$ such that we get in $M^n_{Y^\sharp/T^\sharp,(m)}$ the equality $p_1^{n*}(a) = p_0^{n*}(a)\mu^n_{(m)}(a)$. We still denote by $\tau_\sharp$ its image via the canonical morphism $\mathcal{P}^n_{Y^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp,(m)}$. Similarly, replacing $Y^\sharp/T^\sharp$ by $X^\sharp/S^\sharp$ we get the elements $\tau'_{\sharp\lambda} := \mu^n_{(m)}(u'_\lambda) - 1$ of $\mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$ or $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)}$. Similarly to the proof of 4.4.3.1, we compute $\widetilde{f}^n_{(m)}(1 \otimes \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}) = \underline{\tau}_\sharp'^{\{\underline{k}\}_{(m)}}$. This yields that $\widetilde{f}^n_{(m)}$ is an isomorphism. By copying the proof of 4.4.3.1, we get (b) and (c). The statement (d) is a consequence of 4.4.2.9.1 and of (c). □

Lemma 4.4.4.2. *The canonical homomorphism of $\mathcal{B}_X$-modules:*

$$\widetilde{f}^*(\widetilde{\omega}_{Y^\sharp/T^\sharp}) \to \widetilde{\omega}_{X^\sharp/S^\sharp} \tag{4.4.4.2.1}$$

is an isomorphism. Let us denote by $\rho^\omega_{\widetilde{f}} \colon f^{-1}(\widetilde{\omega}_{Y^\sharp/T^\sharp}) \to \widetilde{\omega}_{X^\sharp/S^\sharp}$ the canonical homomorphism. With 4.4.4.1.2, this yields the map

$$f^{-1}\left(\widetilde{\omega}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}\right) \to \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \tag{4.4.4.2.2}$$

given by $\omega \otimes P \mapsto \rho^\omega_{\widetilde{f}}(\omega) \otimes \rho_{\widetilde{f}}(P)$. The map 4.4.4.2.2 is a homomorphism of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-bimodules, where the right structure (resp. the left structure i.e. the twisted one) of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module of $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ comes from its right structure (resp. left structure) of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module via the ring homomorphism 4.4.4.1.2.

Proof. The lemma follows from analogous to that of the proof of 4.4.3.3 local computations. □

4.4.4.3. Similarly to 4.4.3.4, we can check the following diagram

$$\begin{array}{ccc} f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp}\otimes_{\mathcal{B}_Y}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right) & \xrightarrow{4.4.4.2.2} & \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \\ \downarrow{\scriptstyle 4.2.5.5.1} & & \downarrow{\scriptstyle 4.2.5.5.1} \\ f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp}\otimes_{\mathcal{B}_Y}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right) & \xrightarrow{4.4.4.2.2} & \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \end{array} \tag{4.4.4.3.1}$$

where the vertical maps are the transposition isomorphisms, is commutative. This yields the isomorphism of right $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$-bimodules

$$f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp}\otimes_{\mathcal{B}_Y}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right)_l \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \tag{4.4.4.3.2}$$

where the index l means that in the tensor product we use the left structure of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module, where the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module) of $\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is its left structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. comes from its structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module via the ring homomorphism $\rho_{\widetilde{f}}$ of 4.4.4.1.2). Similarly to 4.4.3.4.3, for any left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module, this implies the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules:

$$f^{-1}\left(\widetilde{\omega}_{Y^\sharp/S^\sharp}\otimes_{\mathcal{B}_Y}\mathcal{E}\right)\otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}}\mathcal{E}). \tag{4.4.4.3.3}$$

4.4.4.4. Using 4.4.4.1.3, we get the canonical homomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}, \mathcal{B}_X)$-bimodules

$$f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\otimes_{f^{-1}\mathcal{B}_Y}\mathcal{B}_X \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.4.4.4.1}$$

We can check by making a local computation that it is an isomorphism. For any right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module $\mathcal{M}$, this yields that the canonical morphism of $\mathcal{B}_X$-modules

$$\widetilde{f}^*\mathcal{M} := f^{-1}\mathcal{M}\otimes_{f^{-1}\mathcal{B}_Y}\mathcal{B}_X \to f^{-1}\mathcal{M}\otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \tag{4.4.4.4.2}$$

is an isomorphism. Hence, we can endow $\widetilde{f}^*\mathcal{M}$ with a structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module which extends its structure of $\mathcal{B}_X$-module by transport structure via 4.4.4.4.2. Via a local computation we can check the canonical isomophism

$$\widetilde{f}^*(\widetilde{\omega}_{Y^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp} \tag{4.4.4.4.3}$$

of 4.4.4.2.1 is an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. Moreover, modulo the canonical isomorphisms 4.4.4.1.4 and 4.4.4.4.2, for any left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module $\mathcal{E}$, the isomorphism 4.4.4.3.3 can be rewritten of the form:

$$\widetilde{f}^*\left(\widetilde{\omega}_{Y^\sharp/S^\sharp}\otimes_{\mathcal{B}_Y}\mathcal{E}\right) \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{f}^*(\mathcal{E}). \tag{4.4.4.4.4}$$

4.4.4.5. Suppose $(u_\lambda)_{\lambda=1,\dots,r}$ are logarithmic coordinates of $Y^\sharp/T^\sharp$. This induces the logarithmic coordinates $(u'_\lambda = \widetilde{f}^*(u_\lambda))_{\lambda=1,\dots,r}$ of $X^\sharp/S^\sharp$. By using theses fixed logarithmic coordinates, we get the logarithmic adjoint operator ${}^{t_{\log}}\colon \widetilde{D}^{(m)}_{Y^\sharp/T^\sharp} \to \widetilde{D}^{(m)}_{Y^\sharp/T^\sharp}$ and $: \widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{D}^{(m)}_{X^\sharp/S^\sharp}$. It follows from the commutativity of 4.4.4.3.1 that the square

$$\begin{array}{ccc} \widetilde{D}^{(m)}_{Y^\sharp/T^\sharp} & \xrightarrow{t_{\log}} & \widetilde{D}^{(m)}_{Y^\sharp/T^\sharp} \\ \downarrow{\scriptstyle 4.4.4.1.2} & & \downarrow{\scriptstyle 4.4.4.1.2} \\ \widetilde{D}^{(m)}_{X^\sharp/S^\sharp} & \xrightarrow{t_{\log}} & \widetilde{D}^{(m)}_{X^\sharp/S^\sharp} \end{array} \tag{4.4.4.5.1}$$

is commutative.

4.4.5 Compatibility with composition, extension or forgetful of the coefficients, glueing, homorphisms, tensor products

We keep notation 4.4.2. We suppose that the bottom arrow of 4.4.2.0.1 is the identity.

Notation 4.4.5.1. We denote by "forg" the forgetful functor from the category of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-modules to that of left $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-modules ; and similarly by replacing Y by X.

Lemma 4.4.5.2. *We suppose that $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism. Let $\mathcal{F}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module and* $\mathrm{forg}(\mathcal{F})$ *(or simply $\mathcal{F}$) be the induced left $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module. Let $\mathcal{G} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.*

(a) The canonical $\mathcal{O}_X$-linear homomorphism

$$f^*(\mathrm{forg}(\mathcal{F})) \to \widetilde{f}^*(\mathcal{F}) \tag{4.4.5.2.1}$$

is an isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules.

(b) The canonical homomorphism

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F}, \tag{4.4.5.2.2}$$

is an isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules.

(c) If $f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_X$ are tor independent over $f^{-1}\mathcal{B}_Y$, then the canonical morphisms

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{G} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{G}, \tag{4.4.5.2.3}$$

is an isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

Proof. 1) Since $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism, then the canonical homomorphism 4.4.5.2.1 is an isomorphism. Let us now check that this canonical $\mathcal{O}_X$-linear isomorphism $f^*(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^*(\mathcal{F})$ is horizontal. Let $(\widetilde{\epsilon}^{\mathcal{F}}_n)_n$ be the m-PD-stratification with coefficient in $\mathcal{B}_Y$ of $\mathcal{F}$ associated to its structure of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module. Let $(\epsilon^{\mathcal{F}}_n)_n$ be the corresponding m-PD-stratification semi-linear with respect to $(\epsilon^{\mathcal{B}_Y}_n)$. Since $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ is a ring isomorphism, since $\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y^\sharp/T^\sharp(m)}$ and $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}$ (see 4.4.2.6.1), then the following square of commutative rings

$$\begin{array}{ccc} f^{-1}\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}) & \xrightarrow{3.4.4.1.2} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \\ \downarrow & & \downarrow \\ f^{-1}(\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}) & \xrightarrow{4.4.2.2.1} & \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \end{array} \tag{4.4.5.2.4}$$

is cocartesian. Set $f^{n*}_{(m)} := \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{f^{-1}\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}} f^{-1}(-)$ and $\widetilde{f}^{n*}_{(m)} := \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\tilde{f}^{-1}\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}} \tilde{f}^{-1}(-)$. Let forg_n be the forgetful functor from the category of $\widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)}$-modules to that of $\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}$-modules, and similarly for $X^\sharp/S^\sharp$ instead of $Y^\sharp/T^\sharp$. The cocartesianity of 4.4.5.2.4 means that the functors $\mathrm{forg}_n \circ \widetilde{f}^{n*}_{(m)}$ and $f^{n*}_{(m)} \circ \mathrm{forg}_n$ are canonically isomorphic. Recall, following 4.3.2.5.1, the diagram of $\mathcal{P}^n_{Y^\sharp/T^\sharp(m)}$-modules

$$\begin{array}{ccccc} \widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{B}_Y} \mathcal{F} & \xleftarrow[\varepsilon^{\mathcal{B}_Y}_n \otimes \mathrm{id}]{\sim} & (\mathcal{P}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{O}_Y} \mathcal{B}_Y) \otimes_{\mathcal{B}_Y} \mathcal{F} & \xrightarrow[\mathrm{can}]{\sim} & \mathcal{P}^n_{Y^\sharp/T^\sharp(m)} \otimes_{\mathcal{O}_Y} \mathcal{F} \\ \sim\downarrow \widetilde{\varepsilon}^{\mathcal{F}}_n & & & & \sim\downarrow \epsilon^{\mathcal{F}}_n \\ \mathcal{F} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{P}}^n_{Y^\sharp/T^\sharp(m)} & & \xleftarrow[\mathrm{can}]{\sim} & & \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y^\sharp/T^\sharp(m)} \end{array} \tag{4.4.5.2.5}$$

is commutative. By applying the functor $\mathrm{forg}_n \circ \widetilde{f}^{n*}_{(m)}$ (resp. $f^{n*}_{(m)}$) to the left arrow of 4.4.5.2.5 (resp. to the other ones), by using the commutative diagram 4.4.2.2.3 we get the commutative diagram of $\mathcal{P}^n_{X^\sharp/S^\sharp(m)}$-modules:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{B}_X} \widetilde{f}^*(\mathcal{F}) & \underset{\varepsilon_n^{\mathcal{B}_X} \otimes \mathrm{id}}{\overset{\sim}{\longleftarrow}} & (\mathcal{P}^n_{X^\sharp/T^\sharp(m)} \otimes_{\mathcal{O}_X} \mathcal{B}_X) \otimes_{\mathcal{B}_X} \widetilde{f}^*(\mathcal{F}) & \overset{\sim}{\longrightarrow} & \mathcal{P}^n_{X^\sharp/S^\sharp(m)} \otimes_{\mathcal{O}_X} f^*(\mathcal{F}) \\
{\scriptstyle \sim}\downarrow {\scriptstyle \widetilde{f}^{n*}_{(m)}(\widetilde{\varepsilon}_n)} & & & & {\scriptstyle \sim}\downarrow {\scriptstyle f^{n*}_{(m)}(\epsilon^{\mathcal{F}}_n)} \\
\widetilde{f}^*(\mathcal{F}) \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} & & \underset{can}{\overset{\sim}{\longleftarrow}} & & f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp(m)}
\end{array}
\tag{4.4.5.2.6}
$$

Hence, the m-PD-stratification $(f^{n*}_{(m)}(\epsilon^{\mathcal{F}}_n))$ is semi-linear with respect to the isomorphisms $(\epsilon^{\mathcal{B}_X}_n)$ and its corresponding m-PD-stratification with coefficients in $\mathcal{B}_X$ is $\widetilde{f}^{n*}_{(m)}(\widetilde{\varepsilon}_n)$. Hence we are done.

2) Suppose $f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_X$ are tor independent over $f^{-1}\mathcal{B}_Y$. Let $\mathcal{P}$ be a K-flat complex of $K({}^l\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ representing $\mathcal{G}$. Then $f^{-1}\mathcal{P}$ is both a K-flat complex of $K({}^l f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ and a complex of $K({}^l f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp})$ which is a K-flat complex of $K(f^{-1}\mathcal{O}_Y)$. This yields the canonical morphism of $\mathbb{L}f^*(\mathrm{forg}(\mathcal{G})) \to \mathbb{L}\widetilde{f}^*(\mathcal{G})$ is an isomorphism of $D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ from the part 1).

3) We conclude the Lemma by using 4.4.2.9. □

Proposition 4.4.5.3. *We only consider here the non-respective case, i.e. the non-formal case. Let* $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S)$ *be a quasi-coherent m-PD-ideal of* $\mathcal{O}_S$. *We denote by* $S_0 = V(\mathfrak{a}_S)$, *by* $X^\sharp_0 := X^\sharp \times_S S_0$ *and by* $f_0 \colon X^\sharp_0 \to Y^\sharp$ *the induced morphism. Let* $f' \colon X^\sharp \to Y^\sharp$ *be a second morphism of* $T^\sharp$*-log schemes inducing the same restriction* $f_0 \colon X^\sharp_0 \to Y^\sharp$. *Suppose the m-PD-ideal* $\mathfrak{a}_S$ *is m-PD-nilpotent.*

(a) Let $\mathcal{F}$ *be a left* $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$*-module. Then, we have a canonical isomorphism of left* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-modules of the form*

$$\tau_{f,f'} \colon f'^*(\mathcal{F}) \xrightarrow{\sim} f^*(\mathcal{F}) \tag{4.4.5.3.1}$$

such that $\tau_{f,f} = \mathrm{id}$, *and, for any third morphism* $f'' \colon X^\sharp \to Y^\sharp$ *inducing the same restriction* $f_0 \colon X^\sharp_0 \to Y^\sharp$, *we have* $\tau_{f,f''} = \tau_{f,f'} \circ \tau_{f',f''}$.

(b) Suppose that f *is finite. Let* $\mathcal{M}$ *be right* $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$*-module. Then, we have a canonical isomorphism of right* $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$*-modules of the form*

$$\sigma_{f,f'} \colon f'^\flat(\mathcal{M}) \xrightarrow{\sim} f^\flat(\mathcal{M}) \tag{4.4.5.3.2}$$

such that $\sigma_{f,f} = \mathrm{id}$, *and, for any third morphism* $f'' \colon X^\sharp \to Y^\sharp$ *inducing the same restriction* $f_0 \colon X^\sharp_0 \to Y^\sharp$, *we have* $\sigma_{f,f''} = \sigma_{f,f'} \circ \sigma_{f',f''}$.

Proof. Set $g := (f, f') \colon X^\sharp \to Y^\sharp \times_{T^\sharp} Y^\sharp$. Since the m-PD-ideal $\mathfrak{a}_S$ is m-PD-nilpotent, then for s large enough, there exists a factorisation $g^s_{(m)}$ making commutative the diagram:

$$
\begin{array}{ccccc}
X^\sharp & \underset{g^s_{(m)}}{\longrightarrow} & \Delta^s_{Y^\sharp/T^\sharp(m)} & \underset{\mathrm{can}}{\longrightarrow} & Y^\sharp \times_{T^\sharp} Y^\sharp \\
\uparrow & & & \nwarrow & \uparrow \\
X^\sharp_0 & & \longrightarrow & & Y^\sharp.
\end{array}
\qquad (g = \mathrm{can} \circ g^s_{(m)})
\tag{4.4.5.3.3}
$$

Since $p^n_{1,(m)} \circ g^s_{(m)} = f'$ and $p^n_{0,(m)} \circ g^s_{(m)} = f$, then we construct the isomorphism 4.4.5.3.1 by setting $\tau_{f,f'} := g^{s*}_{(m)}(\epsilon^{\mathcal{F}}_s)$. This isomorphism does not depend on the such of the large enough integer s. It follows from the cocycle condition that we have the transitivity formula $\tau_{f,f''} = \tau_{f,f'} \circ \tau_{f',f''}$.

Set $\mathcal{E}' := f'^*(\mathcal{F})$ and $\mathcal{E} := f^*(\mathcal{F})$. It remains to check that $\tau_{f,f'}$ is horizontal, i.e. that we have the commutative diagram

$$\begin{array}{ccc} p^{n*}_{1(m)}(\mathcal{E}') & \xrightarrow{p^{n*}_{1(m)}(\tau_{f,f'})} & p^{n*}_{1(m)}(\mathcal{E}) \\ \downarrow \epsilon^{\mathcal{E}'}_n & & \downarrow \epsilon^{\mathcal{E}}_n \\ p^{n*}_{0(m)}(\mathcal{E}') & \xrightarrow{p^{n*}_{0(m)}(\tau_{f,f'})} & p^{n*}_{0(m)}(\mathcal{E}), \end{array} \tag{4.4.5.3.4}$$

for any integer n. It follows from 3.4.2.1.a, that it is sufficient to check it for $n \geq s$. Since $\tau_{f,f'}$ does not depend on the choice of s, then we can suppose $n = s$.

Set $h := (f^2, f'^2)\colon X^{\sharp 2}_{/S^\sharp} \to Y^{\sharp 2}_{/T^\sharp} \times_{T^\sharp} Y^{\sharp 2}_{/T^\sharp} = Y^{\sharp 4}_{/T^\sharp}$ and for any $i, j \in \{0,1,2,3\}$ such that $i < j$, let $q_{ij}\colon Y^{\sharp 4}_{/T^\sharp} \to Y^{\sharp 2}_{/T^\sharp}$ be the projection on the ith and jth factors. It follows from 3.2.2.5.a that the m-PD structure of the exact closed immersion $X^\sharp \hookrightarrow \Delta^s_{X^\sharp/S^\sharp(m)}$ is compatible with α_S. This implies that we the closed immersion $X^\sharp_0 \hookrightarrow \Delta^s_{X^\sharp/S^\sharp(m)}$ is an m-PD-immersion. By universal property of the m-PD-envelope, there exist two factorisations $h^s_{(m)}$ and $q^s_{ij(m)}$ making commutative the diagram

$$\begin{array}{ccccc} X^{\sharp 2}_{/S^\sharp} & \xrightarrow{h} & Y^{\sharp 4}_{/T^\sharp} & \xrightarrow{q_{ij}} & Y^{\sharp 2}_{/T^\sharp} \\ \uparrow \text{can} & & \uparrow \text{can} & & \uparrow \text{can} \\ \Delta^s_{X^\sharp/S^\sharp(m)} & \xrightarrow{h^s_{(m)}} & \Delta^s_{Y^\sharp/T^\sharp(m)}(3) & \xrightarrow{q^s_{ij(m)}} & \Delta^s_{Y^\sharp/T^\sharp(m)} \\ \uparrow & & \uparrow & & \uparrow \\ X^\sharp_0 & \xrightarrow{f_0} & Y^\sharp & = & Y^\sharp. \end{array} \tag{4.4.5.3.5}$$

Since $q_{02} \circ h = g \circ p_0$ (resp. $q_{13} \circ h = g \circ p_1$), then by using the universal property of the m-PD-envelope, we get $q^s_{02(m)} \circ h^s_{(m)} = g^s_{(m)} \circ p^s_{0(m)}$ (resp. $q^s_{13(m)} \circ h^s_{(m)} = g^s_{(m)} \circ p^s_{1(m)}$). This yields

$$p^{s*}_{0(m)}(\tau_{f,f'}) = h^{s*}_{(m)} \circ q^{s*}_{02(m)}(\epsilon^{\mathcal{F}}_s), \qquad p^{s*}_{1(m)}(\tau_{f,f'}) = h^{s*}_{(m)} \circ q^{s*}_{13(m)}(\epsilon^{\mathcal{F}}_s).$$

Similarly, with notation 3.4.4.1, since $q_{01} \circ h = f \times f$ (resp. $q_{23} \circ h = f' \times f'$) then $q^s_{01(m)} \circ h^s_{(m)} = f^s_{(m)}$ (resp. $q^s_{23(m)} \circ h^s_{(m)} = f'^s_{(m)}$). This yields

$$\epsilon^{\mathcal{E}}_s := f^{s*}_{(m)}(\epsilon^{\mathcal{F}}_s) = h^{s*}_{(m)} \circ q^{s*}_{01(m)}(\epsilon^{\mathcal{F}}_s), \qquad \epsilon^{\mathcal{E}'}_s := f'^{s*}_{(m)}(\epsilon^{\mathcal{F}}_s) = h^{s*}_{(m)} \circ q^{s*}_{23(m)}(\epsilon^{\mathcal{F}}_s).$$

We conclude the proof of (a) by using the cocycle condition which yields the identifications:

$$q^{s*}_{01(m)}(\epsilon^{\mathcal{F}}_s) \circ q^{s*}_{13(m)}(\epsilon^{\mathcal{F}}_s) = q^{s*}_{03(m)}(\epsilon^{\mathcal{F}}_s) = q^{s*}_{02(m)}(\epsilon^{\mathcal{F}}_s) \circ q^{s*}_{23(m)}(\epsilon^{\mathcal{F}}_s).$$

For the second one, we can copy this proof above by replacing the functor $f \mapsto f^*$ by $f \mapsto f^\flat$ (and by replacing the use of m-PD-stratifications by that of m-PD-costratifications). ☐

4.4.5.4. We keep notation 4.4.5.3. We can add coefficients to Proposition 4.4.5.3 as follows. The isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules $\tau_{f,f'}\colon f^*(\mathcal{B}_Y) \xrightarrow{\sim} f'^*(\mathcal{B}_Y)$ of 4.4.5.3.1 is also a ring homomorphism (because so are the isomorphisms given by the stratification of $\mathcal{B}_Y$). Denote by $\rho_f\colon f^*\mathcal{B}_Y \to \mathcal{B}_X$ the given homomorphism and by $\rho_{f'}\colon f'^*\mathcal{B}_Y \to \mathcal{B}_X$ the map so that $\rho_{f'} \circ \tau_{f,f'} = \rho_f$. Then it follows from 4.3.1.1.1 that the map

$$\tau_{f,f'} \otimes \mathrm{id}\colon f'^*(\mathcal{B}_Y) \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} f^*(\mathcal{B}_Y) \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \tag{4.4.5.4.1}$$

is a ring isomorphism. We set $\widetilde{X}^\sharp_0 := (X^\sharp_0, f^*_0\mathcal{B}_Y)$, $\widetilde{Y}^\sharp := (Y^\sharp, \mathcal{B}_Y)$, $\widetilde{X}^\sharp := (X^\sharp, \mathcal{B}_X)$, $\widetilde{f}\colon \widetilde{X}^\sharp \to \widetilde{Y}^\sharp$ and $\widetilde{f}'\colon \widetilde{X}^\sharp \to \widetilde{Y}^\sharp$ the morphisms induced respectively by (f, ρ_f) and by $(f', \rho_{f'})$.

Set $\widetilde{g} := (\widetilde{f}, \widetilde{f}')\colon \widetilde{X}^\sharp \to \widetilde{Y}^\sharp \times_{T^\sharp} \widetilde{Y}^\sharp$. It follows from 4.4.5.3.3 that for s large enough, there exists a factorisation $\widetilde{g}^s_{(m)}$ making commutative the diagram:

$$\begin{array}{ccccc} & & \widetilde{g} & & \\ \widetilde{X}^\sharp & \xrightarrow{\widetilde{g}^s_{(m)}} & \widetilde{\Delta}^s_{Y^\sharp/T^\sharp(m)} & \xrightarrow{\mathrm{can}} & \widetilde{Y}^\sharp \times_{T^\sharp} \widetilde{Y}^\sharp \\ \uparrow & & & \nwarrow & \uparrow \\ \widetilde{X}^\sharp_0 & \longrightarrow & & & \widetilde{Y}^\sharp. \end{array} \tag{4.4.5.4.2}$$

Since $\widetilde{p}^n_{1,(m)} \circ \widetilde{g}^s_{(m)} = \widetilde{f}'$ and $\widetilde{p}^{n*}_{0,(m)} \circ \widetilde{g}^s_{(m)} = \widetilde{f}$, then we get the $\mathcal{B}_X$-linear isomorphism:

$$\tau_{\widetilde{f},\widetilde{f}'} := \widetilde{g}^s_{(m)}(\epsilon^{\mathcal{F}}_s)\colon \widetilde{f}'^*(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^*(\mathcal{F}). \tag{4.4.5.4.3}$$

This is clear that $\tau_{\widetilde{f},\widetilde{f}} = \mathrm{id}$. We prove via the following two steps that $\tau_{\widetilde{f},\widetilde{f}'}$ is an an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules that we have the transitivity formula $\tau_{\widetilde{f},\widetilde{f}''} = \tau_{\widetilde{f},\widetilde{f}'} \circ \tau_{\widetilde{f}',\widetilde{f}''}$ for any third morphism $f''\colon X^\sharp \to Y^\sharp$ inducing the same restriction $f_0\colon X^\sharp_0 \to Y^\sharp$.

1) Suppose $\rho_f = \mathrm{id}$. It follows from 4.4.5.2 and 4.4.5.3.1 that 4.4.5.4.3 is also an isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules such that, with notation 4.4.5.1, we have moreover $\mathrm{forg}(\tau_{\widetilde{f},\widetilde{f}'}) = \tau_{f,f'}$. Hence, 4.4.5.4.3 is an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. This implies the transitivity formula from 4.4.5.3.

2) In general, let $a_f\colon \widetilde{X} \to (X^\sharp, f^*\mathcal{B}_Y)$ be the morphism induced by ρ_f. The morphism f induces $\widetilde{a}\colon (X^\sharp, f^*\mathcal{B}_Y) \to \widetilde{Y}$. Let $\widetilde{a}'$ be the composition $\widetilde{a}'\colon (X^\sharp, f^*\mathcal{B}_Y) \xrightarrow{\sim} (X^\sharp, f'^*\mathcal{B}_Y) \to \widetilde{Y}^\sharp$, where the last map is induced by f' and the first map is given by the isomorphism $\tau_{f,f'}\colon f^*(\mathcal{B}_Y) \xrightarrow{\sim} f'^*(\mathcal{B}_Y)$. We have $a_f \circ \widetilde{a} = \widetilde{f}$, $a_f \circ \widetilde{a}' = \widetilde{f}'$ and $\tau_{\widetilde{f},\widetilde{f}'} = a_f^*(\tau_{\widetilde{a},\widetilde{a}'})$. This yields from 1) that $\tau_{\widetilde{f},\widetilde{f}'}$ is an an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and that the formula $\tau_{\widetilde{f},\widetilde{f}''} = \tau_{\widetilde{f},\widetilde{f}'} \circ \tau_{\widetilde{f}',\widetilde{f}''}$ holds.

4.4.5.5. To check that the inverse image behaves well by composition, let moreover be the commutative diagram

$$\begin{array}{ccc} Y^\sharp & \xrightarrow{g} & Z^\sharp \\ \downarrow & & \downarrow \\ T^\sharp & \longrightarrow & U^\sharp \end{array} \tag{4.4.5.5.1}$$

where $U^\sharp$ is nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ as defined in 3.1.1.1 (resp. $U^\sharp$ is nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10), where $Z^\sharp$ is a log smooth $U^\sharp$-log scheme (resp. log smooth $U^\sharp$-log formal scheme). Moreover, let $\mathcal{B}_Z$ be an $\mathcal{O}_Z$-algebra endowed with a compatible structure of $\mathcal{D}^{(m}_{Z^\sharp/U^\sharp}$-module and a morphism of algebras $g^*\mathcal{B}_Z \to \mathcal{B}_Y$ which is moreover $\mathcal{D}^{(m}_{Z^\sharp/U^\sharp}$-linear. Set $\widetilde{Z}^\sharp := (Z^\sharp, \mathcal{B}_Z)$. Let $\widetilde{g}\colon \widetilde{Y}^\sharp \to \widetilde{Z}^\sharp$ and $\widetilde{g \circ f}\colon : \widetilde{X}^\sharp \to \widetilde{Z}^\sharp$ be the induced morphisms. We set $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp} := \mathcal{B}_Z \otimes_{\mathcal{O}_Z} \mathcal{D}^{(m)}_{Z^\sharp/U^\sharp}$.

Proposition 4.4.5.6. *With notation 4.4.5.5, let $\mathcal{G}$ be a $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}$-module. The canonical isomorphism*

$$\tilde{f}^* \circ \tilde{g}^*(\mathcal{G}) \xrightarrow{\sim} \widetilde{g \circ f}^*(\mathcal{G})$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. Let h be the composition morphism $g \circ f$. According to notation 3.4.4.1, for any $r = 1, 2$, we denote by $f^n_{(m)}(r)\colon \Delta^n_{X^\sharp/S^\sharp(m)}(r) \to \Delta^n_{Y^\sharp/T^\sharp(m)}(r)$ the canonical m-PD-morphism given by using the universal property of m-PD-envelopes of order n (see 3.2.1.1). We get similarly $g^n_{(m)}(r)$ and $h^n_{(m)}(r)$. It follows from the universal property of m-PD-envelopes of order n the equality $g^n_{(m)}(r) \circ f^n_{(m)}(r) = h^n_{(m)}(r)$. With notation 4.4.2.2, by adding coefficients, we get the morphism $\widetilde{f}^n_{(m)}(r)\colon \widetilde{\Delta}^n_{X^\sharp/S^\sharp(m)}(r) \to \widetilde{\Delta}^n_{Y^\sharp/T^\sharp(m)}(r)$, and similarly $\widetilde{g}^n_{(m)}(r)$, $\widetilde{h}^n_{(m)}(r)$. We have $\widetilde{g}^n_{(m)}(r) \circ \widetilde{f}^n_{(m)}(r) = \widetilde{h}^n_{(m)}(r)$. Hence we are done. □

Corollary 4.4.5.7. *Let $\mathcal{E}$ be a $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module. The canonical isomorphism*

$$\mathcal{B}_X \otimes_{\mathcal{O}_X} f^*\mathcal{E} \to \widetilde{f}^*(\mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{E}),$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear.

Proof. This is a consequence of the example 4.4.2.5 and of the proposition 4.4.5.6. □

Proposition 4.4.5.8. *Let $\mathcal{E}$ and $\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-modules. The canonical isomorphism*

$$\tilde{f}^*(\mathcal{E} \otimes_{\mathcal{B}_Y} \mathcal{F}) \xrightarrow{\sim} \tilde{f}^*\mathcal{E} \otimes_{\mathcal{B}_X} \tilde{f}^*\mathcal{F} \tag{4.4.5.8.1}$$

is $\widetilde{\mathcal{D}}^{(m)}_X$-linear. Moreover we have the following diagram

$$\begin{array}{ccc} \widetilde{f}'^*(\mathcal{E} \otimes_{\mathcal{B}_Y} \mathcal{F}) & \xrightarrow[4.4.5.14.1]{\sim} & \widetilde{f}'^*\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{f}'^*\mathcal{F} \\ \downarrow{\scriptstyle \tau_{\widetilde{f},\widetilde{f}'}} & & \downarrow{\scriptstyle \tau_{\widetilde{f},\widetilde{f}'} \otimes \tau_{\widetilde{f},\widetilde{f}'}} \\ \tilde{f}^*(\mathcal{E} \otimes_{\mathcal{B}_Y} \mathcal{F}) & \xrightarrow[4.4.5.14.1]{\sim} & \tilde{f}^*\mathcal{E} \otimes_{\mathcal{B}_X} \tilde{f}^*\mathcal{F} \end{array} \tag{4.4.5.8.2}$$

where $\tau_{\widetilde{f},\widetilde{f}'}$ is the glueing isomorphism of 4.4.5.4.2, is commutative.

Proof. Left to the reader. □

The glueing isomorphisms are compatible with composition:

Proposition 4.4.5.9. *With notation 4.4.5.5, we only consider the non-respective case, i.e. the non-formal case. Let $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S)$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_S$, $(\mathfrak{a}_T, \mathfrak{b}_T, \alpha_T)$ be a quasi-coherent m-PD-ideal of $\mathcal{O}_T$ such that we get the m-PD-morphism $(S, \mathfrak{a}_S, \mathfrak{b}_T, \alpha_T) \to (T, \mathfrak{a}_T, \mathfrak{b}_T, \alpha_T)$. Suppose the m-PD-ideals $\mathfrak{a}_S$ and $\mathfrak{a}_T$ arem-PD-nilpotent.*

We denote by $S_0 = V(\mathfrak{a}_S)$, by $X^\sharp_0 := X^\sharp \times_S S_0$, and by $f_0 \colon X^\sharp_0 \to Y^\sharp$ the induced morphism ; by $T_0 = V(\mathfrak{a}_T)$, by $Y^\sharp_0 := Y^\sharp \times_T T_0$, and by $g_0 \colon Y^\sharp_0 \to Z^\sharp$ the induced morphism. Let $f' \colon X^\sharp \to Y^\sharp$ be a second morphism of $T^\sharp$-log schemes inducing the same restriction $f_0 \colon X^\sharp_0 \to Y^\sharp$. Let $g' \colon Y^\sharp \to Z^\sharp$ be a second morphism of $U^\sharp$-log schemes inducing the same restriction $g_0 \colon Y^\sharp_0 \to Z^\sharp$.

We suppose $\mathcal{B}_X = f^*(\mathcal{B}_Y)$ and $\mathcal{B}_Y = g^*(\mathcal{B}_Z)$. We set $\mathcal{B}'_X := f'^*(\mathcal{B}_Y)$, $\mathcal{B}'_Y := g'^*(\mathcal{B}_Z)$, $\widetilde{X}'^\sharp := (X^\sharp, \mathcal{B}'_X)$, $\widetilde{Y}'^\sharp := (Y^\sharp, \mathcal{B}'_Y)$, $\widetilde{f}' \colon \widetilde{X}'^\sharp \to \widetilde{Y}^\sharp$ and $\widetilde{g}' \colon \widetilde{Y}'^\sharp \to \widetilde{Z}^\sharp$ the morphisms induced respectively by f' and g'.

(a) We have a canonical isomorphism of functors from the category of left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}$-modules to that of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules of the form

$$\begin{array}{ccc} (\tilde{g} \circ \tilde{f}')^* & \xrightarrow[\sim]{\tau_{\tilde{g}\circ\tilde{f},\tilde{g}\circ\tilde{f}'}} & (\tilde{g} \circ \tilde{f})^* \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \tilde{f}'^* \circ \tilde{g}^* & \xrightarrow[\sim]{\tau_{\tilde{f},\tilde{f}'}\circ\tilde{g}^*} & \tilde{f}^* \circ \tilde{g}^* \end{array} \qquad \begin{array}{ccc} (\tilde{g}' \circ \tilde{f})^* & \xrightarrow[\sim]{\tau_{\tilde{g}\circ\tilde{f},\tilde{g}'\circ\tilde{f}}} & (\tilde{g} \circ \tilde{f})^* \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \tilde{f}^* \circ \tilde{g}^* & \xrightarrow[\sim]{\tilde{f}^*\circ\tau_{\tilde{g},\tilde{g}'}} & \tilde{f}^* \circ \tilde{g}'^* \end{array} \tag{4.4.5.9.1}$$

Modula the above canonical vertical isomorphisms, this can be written $\tau_{\tilde{g}\circ\tilde{f},\tilde{g}\circ\tilde{f}'} = \tau_{\tilde{f},\tilde{f}'} \circ \tilde{g}^*$ and $\tilde{f}^* \circ \tau_{\tilde{g},\tilde{g}'} = \tau_{\tilde{g}\circ\tilde{f},\tilde{g}'\circ\tilde{f}}$.

(b) Suppose that f and g are finite. Then we have the equality $\sigma_{\tilde{g}\circ\tilde{f},\tilde{g}\circ\tilde{f}'} = \sigma_{\tilde{f},\tilde{f}'} \circ \tilde{g}^*$ and $\tilde{f}^* \circ \sigma_{\tilde{g},\tilde{g}'} = \sigma_{\tilde{g}\circ\tilde{f},\tilde{g}'\circ\tilde{f}}$

Proof. We set $\widetilde{X}^\sharp_0 := (X^\sharp_0, f_0^*\mathcal{B}_Y)$, $\widetilde{Y}^\sharp_0 := (Y^\sharp_0, g_0^*\mathcal{B}_Z)$. We denote by $\tau_{f,f'} \colon \widetilde{X}^\sharp \to \widetilde{X}'^\sharp$ the morphism induced by $\tau_{f,f'}$ (and similarly for $\tau_{g,g'}$) The proposition is a consequence of the following commutative diagram for n large enough

$$
\begin{array}{ccccc}
\widetilde{X}_0^\sharp & \longrightarrow & \widetilde{X}^\sharp & & \\
\downarrow \tilde{f}_0 & & \downarrow \widetilde{f} & & \\
\widetilde{Y}_0^\sharp & \longrightarrow & \widetilde{Y}^\sharp & \overset{\widetilde{g}}{\underset{\widetilde{g}'\circ\tau_{g,g'}}{\rightrightarrows}} & \widetilde{Z}^\sharp \\
\downarrow & & \downarrow \delta^n_{\tilde{g},\tilde{g}'} & & \| \\
\widetilde{Z}^\sharp & \longrightarrow & \widetilde{\Delta}^n_{Z^\sharp/U^\sharp(m)} & \overset{\tilde{p}_1}{\underset{\tilde{p}_2}{\rightrightarrows}} & \widetilde{Z}^\sharp,
\end{array}
\qquad
\begin{array}{ccccc}
\widetilde{X}_0^\sharp & \longrightarrow & \widetilde{X}^\sharp & \overset{\widetilde{f}}{\underset{\widetilde{f}'\circ\tau_{f,f'}}{\rightrightarrows}} & \widetilde{Z}^\sharp \\
\downarrow & & \downarrow \delta^n_{\tilde{f},\tilde{f}'} & & \| \\
\widetilde{Y}^\sharp & \longrightarrow & \widetilde{\Delta}^n_{Y^\sharp/T^\sharp(m)} & \overset{\tilde{p}_1}{\underset{\tilde{p}_2}{\rightrightarrows}} & \widetilde{Y}^\sharp \\
\downarrow \widetilde{g} & & \downarrow & & \downarrow \widetilde{g} \\
\widetilde{Z}^\sharp & \longrightarrow & \widetilde{\Delta}^n_{Z^\sharp/U^\sharp(m)}, & \overset{\tilde{p}_1}{\underset{\tilde{p}_2}{\rightrightarrows}} & \widetilde{Z},
\end{array}
\tag{4.4.5.9.2}
$$

where we have denoted by $\delta^n_{\tilde{g},\tilde{g}'}$ and $\delta^n_{\tilde{f},\tilde{f}'}$ the morphisms making commutative the diagram (see 4.4.5.4.2).

□

4.4.5.10 (Non-lifted case notation)**.** To highlight the "crystalline nature" of the pullback operation, we consider here an extension of 4.4.2 to the case where the morphism f_0 is not the reduction of a morphism f as follows: Let $S^\sharp$ and $T^\sharp$ be nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ as defined in 3.1.1.1 where i is an integer (resp. $S^\sharp$ and $T^\sharp$ are nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10). Let $X^\sharp$ be a log smooth $S^\sharp$-log scheme (resp. log smooth $S^\sharp$-log formal scheme) and $Y^\sharp$ be a log smooth $T^\sharp$-log scheme (resp. log smooth $T^\sharp$-log formal scheme). We suppose that S has a quasi-coherent m-PD-nilpotent-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) \subset \mathcal{O}_S$. Let $S_0^\sharp$ the exact closed logarithmic subscheme of $S^\sharp$ defined by $\mathfrak{a}_S$, $X_0^\sharp$ the reduction of $X^\sharp$ to $S_0^\sharp$. Let $\mathcal{B}_X$ (resp. $\mathcal{B}_Y$) be a commutative $\mathcal{O}_X$-algebra (resp. $\mathcal{O}_Y$-algebra) endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. left $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module).

Let $f_0 \colon X_0^\sharp \to Y^\sharp$ be a $T^\sharp$-morphism, $\mathcal{F}$ a left $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module. Any point $x \in X^\sharp$ has a neighbourhood $V^\sharp$ and a $f \colon V^\sharp \to Y^\sharp$ such that its restriction to $X_0 \cap V$ equals to f_0 (use [CV17, 1.24]). By Proposition 4.4.5.3 the left $\mathcal{D}^{(m)}_{U^\sharp/S^\sharp}$-module $f^*\mathcal{F}$ is independent, up to canonical linear $\mathcal{D}^{(m)}_{U^\sharp/S^\sharp}$ isomorphism of the form $\tau_{f,f'}$, of the choice of f. Hence, we can glue the locally defined $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module and obtain a canonical $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module starting from the data of $\mathcal{F}$ and f_0. We, by abuse of notation, denote this module by $f_0^*\mathcal{F}$; when f_0 lifts to f we have a canonical isomorphism $f_0^*\mathcal{F} \xrightarrow{\sim} f^*\mathcal{F}$.

Since the isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules $\tau_{f,f'} \colon f^*(\mathcal{B}_Y) \xrightarrow{\sim} f'^*(\mathcal{B}_Y)$ is a ring isomorphism (see 4.4.5.4), then the left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module $f_0^*\mathcal{B}_Y$ is endowed with a compatible structure of $\mathcal{O}_X$-algebra.

We suppose moreover that we have a morphism of algebras $\rho \colon f_0^*\mathcal{B}_Y \to \mathcal{B}_X$ which is moreover $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. We will denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$. Any point $x \in X^\sharp$ has a neighbourhood $U^\sharp$ and a $f \colon U^\sharp \to Y^\sharp$ such that its restriction to $X_0^\sharp \cap U^\sharp$ equals to f_0, which yields (with ρ) the homomorphism of ringed logarithmic ($\mathcal{V}$-formal) schemes $\widetilde{f} \colon (U^\sharp, \mathcal{B}_X \cap U) \to (Y^\sharp, \mathcal{B}_Y)$. Following 4.4.5.4.2 the left $\widetilde{\mathcal{D}}^{(m)}_{U^\sharp/S^\sharp}$-module $\widetilde{f}^*\mathcal{F}$ is independent, up to canonical linear $\widetilde{\mathcal{D}}^{(m)}_{U^\sharp/S^\sharp}$ isomorphism of the form $\tau_{\widetilde{f},\widetilde{f}'}$, of the choice of f. Hence, we can glue the locally defined $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and obtain a canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module starting from the data of $\mathcal{F}$, f_0 and ρ. We, by abuse of notation, denote this module by $\widetilde{f}_0^*\mathcal{F}$; when f_0 lifts to f we have a canonical isomorphism $\widetilde{f}_0^*\mathcal{F} \xrightarrow{\sim} \widetilde{f}^*\mathcal{F}$.

Suppose f_0 finite. Similarly, by glueing (via isomorphisms of the form $\sigma_{f,f'} \colon f'^\flat$ of 4.4.5.3.2), we get the functor $\widetilde{f}_0^\flat$ from the category of right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-modules to that of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules ; when f_0 lifts to f we have a canonical isomorphism $\widetilde{f}_0^\flat \xrightarrow{\sim} \widetilde{f}^\flat$.

Notation 4.4.5.11. We keep notation and hypotheses of 4.4.5.10. By functoriality, we get a $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodule by setting

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} := \widetilde{f}_0^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}.$$

Then for any left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module $\mathcal{F}$ we again have a canonical isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-modules

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f_0^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{f}_0^*\mathcal{F}.$$

4.4.5.12. We suppose that S has a quasi-coherent m-PD-nilpotent-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) \subset \mathcal{O}_S$. Let $S_0^\sharp$ the exact closed logarithmic subscheme of $S^\sharp$ defined by $\mathfrak{a}_S$, $X_0^\sharp$ the reduction of $X^\sharp$ to $S_0^\sharp$. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module. Set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. It follows from 4.4.5.10 that the category of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules only depend, up to canonical equivalence, to quasi-coherent m-PD-nilpotent-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) \subset \mathcal{O}_S$.

4.4.5.13. We keep notation and hypotheses of 4.4.5.10. Let $U^\sharp$ be a nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ where i is an integer (resp. $S^\sharp$ and $T^\sharp$ are nice fine $\mathcal{V}$-log formal schemes). Let $Z^\sharp$ be a log smooth $U^\sharp$-log scheme (resp. log smooth $U^\sharp$-log formal scheme).

Suppose that T has a quasi-coherent m-PD-nilpotent-ideal $(\mathfrak{a}_T, \mathfrak{b}_T, \alpha_T)$ such that $S^\sharp \to T^\sharp$ is a m-PD-morphism. Let $T_0^\sharp$ be the exact closed log subscheme defined by $\mathfrak{a}_T$, $Y_0^\sharp$ the reduction of $Y^\sharp$ on T_0. Let $\mathcal{B}_Z$ be a commutative $\mathcal{O}_Z$-algebra endowed with a compatible structure of $\mathcal{D}^{(m)}_{Z^\sharp/U^\sharp}$-module. Set $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp} = \mathcal{B}_Z \otimes_{\mathcal{O}_Z} \mathcal{D}^{(m)}_{Z^\sharp/U^\sharp}$. Let $g_0 \colon Y_0^\sharp \to Z^\sharp$ be a $T^\sharp$-morphism and $\sigma\colon g_0^*\mathcal{B}_Z \to \mathcal{B}_Y$ be a morphism of algebras which is moreover $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-linear. We get the functor $\widetilde{g}_0^*\colon M(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}) \to M(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. Denoting $g_0 \circ f_0 \colon X_0^\sharp \to Z^\sharp$ the composition of g_0 with the morphism $X_0^\sharp \to Y_0^\sharp$ induced by f_0, we get the functor $\widetilde{g_0 \circ f_0}^* \colon \mathrm{Mod}(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}) \to M(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ induced by $g_0 \circ f_0$ and $\rho \circ \sigma$ as above. It follows from 4.4.5.9 that the functor $\widetilde{g_0 \circ f_0}^*$ is canonically isomorphic to $\widetilde{f_0}^* \circ \widetilde{g_0}^*$.

Proposition 4.4.5.14. *Let $\mathcal{E}$ and $\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-modules. The canonical morphism*

$$\tilde{f}^*(\mathcal{H}om_{\mathcal{B}_Y}(\mathcal{E}, \mathcal{F})) \to \mathcal{H}om_{\mathcal{B}_X}(\tilde{f}^*\mathcal{E}, \tilde{f}^*\mathcal{F}) \tag{4.4.5.14.1}$$

is $\widetilde{\mathcal{D}}^{(m)}_X$-linear.

Proof. We set $\widetilde{\mathcal{P}}^n := \widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp/(m)}$. By functoriality, the following diagram

$$\begin{array}{ccc} \widetilde{f}^n_{(m)}(\mathcal{H}om_{\widetilde{\mathcal{P}}^n}(\widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_Y} \mathcal{E}, \widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_Y} \mathcal{F})) & \longrightarrow & \mathcal{H}om_{\widetilde{\mathcal{P}}^n}(\widetilde{f}^n_{(m)}(\widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_Y} \mathcal{E}), \widetilde{f}^n_{(m)}(\widetilde{\mathcal{P}}^n \otimes_{\mathcal{B}_Y} \mathcal{F})) \\ \sim\downarrow & & \sim\downarrow \\ \widetilde{f}^n_{(m)}(\mathcal{H}om_{\widetilde{\mathcal{P}}^n}(\mathcal{E} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{P}}^n, \mathcal{F} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{P}}^n)) & \longrightarrow & \mathcal{H}om_{\widetilde{\mathcal{P}}^n}(\widetilde{f}^n_{(m)}(\mathcal{E} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{P}}^n), \tilde{f}^*_n(\mathcal{F} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{P}}^n)), \end{array} \tag{4.4.5.14.2}$$

where the horizontal arrows are the canonical homomorphisms and the vertical ones are induced by the isomorphisms $\varepsilon_n^{\mathcal{E}}$ and $\varepsilon_n^{\mathcal{F}}$, is commutative. Moreover, by definition, the left morphism of the diagram 4.4.5.14.2 is the morphism defining the m-PD-stratification of $\tilde{f}^*(\mathcal{H}om_{\mathcal{B}_Y}(\mathcal{E}, \mathcal{F}))$ whereas the right one comes from the m-PD-stratification of $\mathcal{H}om_{\mathcal{B}_X}(\tilde{f}^*\mathcal{E}, f^*\mathcal{F})$. The morphism 4.4.5.14.1 is therefore horizontal. □

4.5 $\mathcal{D}$-modules in the case of relative strict normal crossing divisors

4.5.1 Semi-logarithmic coordinates

Let $T^\sharp$ be a noetherian nice (see definition 3.1.1.1) fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ with i an integer (resp. $T^\sharp$ be a p-torsion free noetherian nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10). Let $X^\sharp$ be a log smooth $T^\sharp$-log schemes (resp. a p-torsion free log smooth $T^\sharp$-log formal schemes). Let $Y := X^{\sharp *}$ be the open of X where $M_{X^\sharp}$ is trivial and $j\colon Y \hookrightarrow X^\sharp$ be the canonical open immersion. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$-module structure which is compatible with its algebra structure. We set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$.

4.5.1.1 (Notations)**.** Let $d \geq r \geq 0$ be two integers. Let $A^{d,r}$ be the log-scheme whose underlying space is $\mathbb{A}^d_{\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}[t_1, \ldots, t_d]$ and such that a pre-log-structure is given by $\mathbb{N}^r \to \mathbb{Z}[t_1, \ldots, t_d]$ defined by

$e_i \mapsto t_i$ for $i = 1, \dots, r$. In other words, $A^{d,r} = A_{\mathbb{N}^r} \times \mathbb{A}^{d-r}_{\mathbb{Z}}$. We set $B^{d,r} = A_{\mathbb{Z}^r} \times \mathbb{A}^{d-r}_{\mathbb{Z}} = (A^{d,r})^*$. Let $\mathfrak{A}^{d,r}$ be the formal $\mathbb{Z}_p$-scheme equal to the p-adic completion of $A^{d,r}$. We get $\mathcal{D}^{(m)}_{A^{d,r}/\mathbb{Z}} = \mathcal{D}^{(m)}_{A_{\mathbb{N}^r}/\mathbb{Z}} \boxtimes \mathcal{D}^{(m)}_{\mathbb{A}^{d-r}_{\mathbb{Z}}/\mathbb{Z}}$ where by abuse of notation $\mathbb{Z}$ means $\operatorname{Spec} \mathbb{Z}$.

We set $A^{d,r}_{T^\sharp} := A^{d,r} \times_{\operatorname{Spec} \mathbb{Z}} T$. If $T^\sharp$ is a nice fine $\mathcal{V}$-log formal scheme, we set $A^{d,r}_{T^\sharp} := \mathfrak{A}^{d,r} \times_{\operatorname{Spf} \mathbb{Z}_p} T^\sharp$ or $\mathfrak{A}^{d,r}_{T^\sharp} := \mathfrak{A}^{d,r} \times_{\operatorname{Spf} \mathbb{Z}_p} T^\sharp$.

We have the canonical exact closed immersions: $A^{r,r}_{T^\sharp} \hookrightarrow A^{d,r}_{T^\sharp}$ given by $t_{r+1} = 0, \dots, t_d = 0$.

Let $m \in \mathbb{N} \cup \{\infty\}$. We have the inclusions $\mathcal{D}^{(m)}_{A^{d,0}_{T^\sharp}/T^\sharp} \subset \mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp} \subset \mathcal{D}^{(m)}_{A^{d,d}_{T^\sharp}/T^\sharp}$. Since $t_1, \dots, t_d$ are coordinates of $A^{d,0}_{T^\sharp/T^\sharp}$, then the sheaf of rings $\mathcal{D}^{(m)}_{A^{d,0}_{T^\sharp}/T^\sharp}$ is a free $\mathcal{O}_{\mathbb{A}^d_{T^\sharp}}$-module (for both right or left structures) with the basis $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}, \underline{k} \in \mathbb{N}^d\}$ (for both structures). Since $t_1, \dots, t_d$ are logarithmic coordinates of $A^{d,d}_{T^\sharp}/T^\sharp$, then the sheaf of rings $\mathcal{D}^{(m)}_{A^{d,d}_{T^\sharp}/T^\sharp}$ is is a free $\mathcal{O}_{\mathbb{A}^d_{T^\sharp}}$-module (for both right or left structures) with the basis $\{\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}, \underline{k} \in \mathbb{N}^d\}$ (for both structures). The canonical inclusion $\mathcal{D}^{(m)}_{A^{d,0}_{T^\sharp}/T^\sharp} \subset \mathcal{D}^{(m)}_{A^{d,d}_{T^\sharp}/T^\sharp}$ sends $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}$ to $\underline{t}^{\underline{k}} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$, which is abusively written by the relation $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} = \underline{t}^{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}$, where $\underline{t}^{\underline{k}} := t_1^{k_1} \cdots t_d^{k_d}$ for $(k_1, \dots, k_d) = \underline{k}$. For any $(k_1, \dots, k_d) = \underline{k} \in \mathbb{N}^d$, we get an element of $\mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp}$ by setting

$$\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} := \underline{\partial}_\sharp^{\langle (k_1, \dots, k_r, 0, \dots, 0) \rangle_{(m)}} \underline{\partial}^{\langle (0, \dots, 0, k_{r+1}, \dots, k_d) \rangle_{(m)}} = \underline{\partial}_\sharp^{\langle (\underline{i}, \underline{0}) \rangle_{(m)}} \underline{\partial}^{\langle (\underline{0}, \underline{j}) \rangle_{(m)}}. \tag{4.5.1.1.1}$$

Moreover, the $\mathcal{O}_{\mathbb{A}^d_{T^\sharp}}$-module (for both right or left structures) $\mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp}$ is free with the base $\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}}$, $\underline{k} \in \mathbb{N}^d$.

Let $\underline{k} \in \mathbb{N}^d$ with $\underline{i} \in \mathbb{N}^r$ and $\underline{j} \in \mathbb{N}^{d-r}$. By using the tautological formula 3.4.1.2.2, the logarithmic adjoint operator $\widetilde{\underline{\partial}}_\sharp^{\langle (k_1, \dots, k_r, 0, \dots, 0) \rangle_{(m)}} = \widetilde{\underline{\partial}}_\sharp^{\langle (\underline{i}, \underline{0}) \rangle_{(m)}}$ of $\underline{\partial}_\sharp^{\langle (\underline{i}, \underline{0}) \rangle_{(m)}}$ belongs to $D^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp}$. The adjoint operator ${}^t\underline{\partial}^{\langle (0, \dots, 0, k_{r+1}, \dots, k_d) \rangle_{(m)}}$ of $\underline{\partial}^{\langle (\underline{0}, \underline{j}) \rangle_{(m)}}$ is an element of $D^{(m)}_{A^{d,0}_{T^\sharp}/T^\sharp}$. We get the semi-logarithmic adjoint operator of $\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}}$ by setting:

$${}^\tau \underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} := \widetilde{\underline{\partial}}_{(r)}^{\langle \underline{k} \rangle_{(m)}} := \widetilde{\underline{\partial}}_\sharp^{\langle (\underline{i}, \underline{0}) \rangle_{(m)}} {}^t\underline{\partial}^{\langle (\underline{0}, \underline{j}) \rangle_{(m)}}. \tag{4.5.1.1.2}$$

Lemma 4.5.1.2. *Let $f \colon X^\sharp \to Y^\sharp$ a morphism of log smooth $T^\sharp$-log-schemes (resp. p-torsion free fine log smooth $T^\sharp$-log-formal schemes). The morphism f is log-étale if and only if the canonical morphism $f^*\Omega^1_{Y^\sharp/T^\sharp} \to \Omega^1_{X^\sharp}$ is an isomorphism.*

Proof. It follows from 3.3.2.7 that we reduce to the non-respective case. Following [Ogu18, IV.3.2.3.2 and IV.3.1.3], if the canonical morphism $f^*\Omega^1_{Y^\sharp/T^\sharp} \to \Omega^1_{X^\sharp/T^\sharp}$ is an isomorphism then f is log-étale. Conversely, if f is log-étale then f is log-smooth and therefore (by using [Ogu18, IV.3.2.3.1]) $f^*\Omega^1_{Y^\sharp/T^\sharp} \to \Omega^1_{X^\sharp/T^\sharp}$ is injective. Since f is unramified then $\Omega^1_{X^\sharp/Y^\sharp} = 0$ (see [Ogu18, IV.3.1.3]). This yields that $f^*\Omega^1_{Y^\sharp/T^\sharp} \to \Omega^1_{X^\sharp/T^\sharp}$ is surjective. □

Lemma 4.5.1.3. *With notation 4.5.1.1, let $f \colon X^\sharp \to A^{d,r}_{T^\sharp}$ be a morphism of log smooth $T^\sharp$-log-(formal) schemes given by the global sections $e_1, \dots, e_r$ of $M_{X^\sharp}$ and $a_{r+1}, \dots, a_d$ of $\mathcal{O}_X$. The morphism f is log-étale if and only if the $\mathcal{O}_X$-module $\Omega^1_{X^\sharp/T^\sharp}$ is free and if $(d\log e_1, \dots, d\log e_r, da_{r+1}, \dots, da_d)$ is a basis.*

Proof. Following 4.5.1.2, f is log-étale if and only if the canonical morphism $f^*\Omega^1_{A^{d,r}_{T^\sharp}/T^\sharp} \to \Omega^1_{X^\sharp/T^\sharp}$ is an isomorphism. This last property is equivalent to the fact that the $\mathcal{O}_X$-module $\Omega^1_{X^\sharp/T^\sharp}$ is free and that $(d\log e_1, \dots, d\log e_r, da_{r+1}, \dots, da_d)$ is a base (to compute explicitly $\Omega^1_{A^{d,r}_{T^\sharp}/T^\sharp}$, it is better to use the remark [Ogu18, IV.1.1.8]). □

Definition 4.5.1.4 (Semi-logarithmic coordinates)**.** A $T^\sharp$-morphism of the form $f\colon X^\sharp \to A^{d,r}_{T^\sharp}$ is equivalent to the data of global sections $t_1, \dots, t_r$ of $M_{X^\sharp}$ and $t_{r+1}, \dots, t_d$ of $\mathcal{O}_X$. If f is log-étale then we say that $t_1, \dots, t_d$ are semi-logarithmic coordinates (of type d, r) of $X^\sharp/T^\sharp$. In that case the canonical morphism $\mathcal{D}^{(m)}_{X^\sharp/T^\sharp} \to f^*\mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp}$ is an isomorphism. This yields a map $f^{-1}\mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp} \to \mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$, which is in fact a ring homomorphism (see 4.4.3.1.3). We still write by $\underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_{(r)}$ the global sections of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ equal to the image of $\underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_{(r)}$ via the composite morphism $f^{-1}\mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp} \to \mathcal{D}^{(m)}_{X^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. Moreover, the $\mathcal{B}_{X^\sharp}$-module (for both right or left structures) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$ is free with the base $\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}_{(r)}$, $\underline{k} \in \mathbb{N}^d$.

When $m = \infty$, we rather write $\underline{\partial}^{[\underline{i}]}_{(r)} := \underline{\partial}^{\langle \underline{i}\rangle_{(\infty)}}_{(r)}$. Finally, we have $\Omega_{X^\sharp/T^\sharp} \xrightarrow{\sim} f^*\Omega_{A^{d,r}_{T^\sharp}/T^\sharp}$.

4.5.1.5 (Semi-logarithmic adjoint operator)**.** Let $f\colon X^\sharp \to A^{d,r}_{T^\sharp}$ be a log-étale $T^\sharp$-morphism given by semi-logarithmic coordinates (of type d, r) $t_1, \dots, t_d$ of $X^\sharp/T^\sharp$. Set $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp} := \Gamma(X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$ and $B_X := \Gamma(X, \mathcal{B}_X)$. Let $\underline{k} \in \mathbb{N}^d$. We still denote by ${}^\tau\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}_{(r)}$ the image of the operator ${}^\tau\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}_{(r)}$ defined at 4.5.1.1.2 via the ring homomorphism $f^{-1}\mathcal{D}^{(m)}_{A^{d,r}_{T^\sharp}/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$.

Let $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}_{(r)} \in \widetilde{D}^{(m)}_{X^\sharp/T^\sharp}$, with $b_{\underline{k}} \in B_X$. We define the "semi-logarithmic adjoint operator" of P by setting ${}^\tau P := \sum_{\underline{k}} {}^\tau\underline{\partial}^{\langle \underline{k}\rangle_{(m)}}_{(r)} b_{\underline{k}}$, which is a mixed version of the logarithmic and non-logarithmic adjoint operator.

Let P, Q be two differential operators of $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp}$. Similarly to 4.3.5.3, we can check that the following properties hold:

(a) We have ${}^\tau({}^\tau P) = P$ and ${}^\tau(PQ) = {}^\tau Q {}^\tau P$

(b) We have the equality

$$\rho({}^\tau P) = \rho\left(\underline{t}_{(r)}{}^{\mathrm{t}}P\frac{1}{\underline{t}_{(r)}}\right), \tag{4.5.1.5.1}$$

where $\underline{t}_{(r)} = t_1 \cdots t_r$ and where ρ is the canonical map $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y/T^\sharp}$ (which is an inclusion when $\mathcal{B}_X = \mathcal{O}_X$).

Hence, ${}^\tau\colon (\widetilde{D}^{(m)}_{X^\sharp/T^\sharp})^o \to \widetilde{D}^{(m)}_{X^\sharp/T^\sharp}$ is an involution of $\mathcal{O}_T$-algebras, which is called the semi-logarithmic adjoint automorphism. Beware that this depends on the choice of the semi-logarithmic coordinates $t_1, \dots, t_d$.

Lemma 4.5.1.6. *Suppose $X^\sharp/T^\sharp$ has semi-logarithmic coordinates $t_1, \dots, t_d$ of type d, r. The sheaf $\widetilde{\omega}_{X^\sharp/T^\sharp}$ is a free $\mathcal{B}_X$-module of rank one with the basis $\widetilde{e}_0 := d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge t_d$ and $\widetilde{\omega}_{Y/T^\sharp}$ is a free $\mathcal{B}_Y$-module of rank one with the basis $dt_1 \wedge \cdots \wedge dt_d$.*

The sheaf $\widetilde{\omega}_{X^\sharp/T^\sharp}$ is a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-submodule of $j_*\widetilde{\omega}_{Y/T^\sharp}$. More precisely, the action of $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ on the section $b\,\widetilde{e}_0$, where b is section of $\mathcal{B}_X$ is given by the formula

$$(b\ \widetilde{e}_0) \cdot P = {}^\tau P(b)\ \widetilde{e}_0. \tag{4.5.1.6.1}$$

Proof. Using 4.5.1.2, we reduce to the case where $X^\sharp/T^\sharp = A^{d,r}_{T^\sharp}/T^\sharp$. By base change, we can suppose $T^\sharp = \operatorname{Spec}\mathbb{Z}$ or $T^\sharp = \operatorname{Spf}\mathbb{Z}_p$. Since $\omega_{A^{d,r}/\mathbb{Z}} = \omega_{A_{\mathbb{N}^r}/\mathbb{Z}} \boxtimes \omega_{\mathbb{A}^{d-r}_{\mathbb{Z}}/\mathbb{Z}}$ then we conclude using 3.4.5.1 and its non-logarithmic version 2.2.1.4. □

Lemma 4.5.1.7. *Consider the isomorphism*

$$\delta_{\mathcal{M}}\colon \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$$

of 4.2.5.5.1. Suppose $X^\sharp/T^\sharp$ has semi-logarithmic coordinates $t_1, \dots, t_d$ of type d, r. Then, we have the following formula

$$\delta_{\mathcal{M}}(x \otimes {}^\tau\underline{\partial}^{\langle \underline{k}\rangle}_{(r)}) = \sum_{\underline{h}\leq\underline{k}} x\,{}^\tau\underline{\partial}^{\langle \underline{k}-\underline{h}\rangle}_{(r)} \otimes \underline{\partial}^{\langle \underline{h}\rangle}_{(r)}. \tag{4.5.1.7.1}$$

Proof. Similar to 4.2.5.5. □

4.5.1.8 (Left to right $\mathcal{D}$-module)**.** Suppose $X^\sharp/T^\sharp$ has semi-logarithmic coordinates $t_1, \ldots, t_d$ of type d, r. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module. We denote by $\widetilde{e}_0 := d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge dt_d$ a basis of the free $\mathcal{B}_X$-module $\widetilde{\omega}_{X^\sharp/T^\sharp}$ has the basis, and by $\widetilde{e}_0^\vee$ its corresponding dual basis of the free $\mathcal{B}_X$-module $\widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}$.

(a) Similarly to 4.3.5.5, we compute that the right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module structure on $\widetilde{\omega}_{X^\sharp/T^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$ is given by the formula
$$(\widetilde{e}_0 \otimes x)P = \widetilde{e}_0 \otimes {}^\tau P x, \tag{4.5.1.8.1}$$
for any local section x of $\mathcal{E}$ and P of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$.

(b) The left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module structure on $\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}$ is given by the formula
$$P(y \otimes \widetilde{e}_0^\vee) = y^\tau P \otimes \widetilde{e}_0^\vee, \tag{4.5.1.8.2}$$
for any local section y of $\mathcal{M}$ and P of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$.

4.5.1.9. The formulas 4.2.3.1, 4.2.3.5 and 4.2.5.4 remain valid replacing by $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$ by $\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}}$ and $\widetilde{\underline{\partial}}_\sharp^{\langle \underline{k} \rangle}$ by $\widetilde{\underline{\partial}}_{(r)}^{\langle \underline{k} \rangle_{(m)}}$.

4.5.2 Log structures associated with relative strict normal crossing divisor

We remind first the notion of strict normal crossing divisor and then of relative strict normal crossing divisor. We give some details in the case where the base is geometrically unibranch (and locally noetherian).

Definition 4.5.2.1. Let $X \to S$ be a morphism of schemes or a morphism of formal schemes. Following [KM85, 1.1.1] (resp. [Sta22, 01WR]), an effective Cartier divisor D in X/S (resp. an effective Cartier divisor D in X) is a closed (formal) subscheme $D \subset X$ satisfying both conditions (resp. the second one):

(i) D is flat over S,

(ii) the ideal sheaf $\mathcal{I}(D) \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module, i.e., it is a locally free $\mathcal{O}_X$-module of rank one.

We denote by $\mathrm{Div}(X/S)_{\geq 0}$ (resp. $\mathrm{Div}(X)_{\geq 0}$) the integral monoid of effective Cartier divisors in X/S (resp. in X). We say that a family $\{E_i\}_{i \in I}$ of nonzero elements in $\mathrm{Div}(X/S)_{\geq 0}$ has a *locally finite intersection* if, for any point $x \in X$, there exists a Zariski open neighborhood U of x such that the set $\{i \in I \mid E_i \cap V \neq \emptyset\}$ is finite.

Lemma 4.5.2.2. *Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Let $x \in D$. The following (1) and (2) conditions are equivalent:*

1. *The local ring $\mathcal{O}_{X,x}$ is regular and there exist a regular system of parameters $t_1, \ldots, t_d \in \mathfrak{m}_{X,x}$ and an integer $1 \leq r \leq d$ such that D is cut out by $t_1 \ldots t_r$ in $\mathcal{O}_{X,x}$.*

2. *Let $D_1, \ldots, D_r$ be the irreducible components of D passing through x (viewed as reduced closed subschemes of X). There exists an open subset U of X containing x such that*

   (i) the scheme $D \cap U$ is reduced,

   (ii) each $D_i \cap U$ is an effective Cartier divisor in U,

   (iii) for any $J \subset \{1, \ldots, r\}$ the scheme theoretic intersection $D_J := \cap_{j \in J} D_j$ is a regular scheme at x and the irreducible component of $U \cap D_J$ containing x has codimension $|J|$ in X.

Proof. The proof can be found at [Sta22, 0BI9,0BIA]. For the reader convenience, let us recall the link between (1) and (2). Suppose the local ring $\mathcal{O}_{X,x}$ is regular and there exist a regular system of parameters $t_1, \ldots, t_d \in \mathfrak{m}_x$ and $1 \leq r \leq d$ such that D is cut out by $t_1 \ldots t_r$ in $\mathcal{O}_{X,x}$. Then the irreducible components $D_1, \ldots, D_r$ of D passing through x corresponds 1-to-1 to the Cartier divisors $V(t_1), \ldots, V(t_r)$. Conversely, let $t_1, \ldots, t_r$ be some local equations of $D_1, \ldots, D_r$. Set Z be the irreducible component of $D_1 \cap \cdots \cap D_r$ containing x. By choosing additional $t_{r+1}, \ldots, t_d \in \mathfrak{m}_{X,x}$ which map to a minimal system of generators of $\mathfrak{m}_{Z,x}$, we get the regular system of parameters $t_1, \ldots, t_d \in \mathfrak{m}_{X,x}$ such that D is cut out by $t_1 \ldots t_r$ in $\mathcal{O}_{X,x}$. □

Definition 4.5.2.3. Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Let $x \in D$. We say that D is a "strict normal crossing divisor on X at x" if one of the following equivalent conditions of 4.5.2.2 holds. We say that D is a "strict normal crossing divisor on X" if D is a strict normal crossing divisor on X at x for any $x \in D$.

4.5.2.4. Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Let $D_i \subset D$, $i \in I$ be its irreducible components viewed as reduced closed subschemes of X. We have the following equivalent conditions :

(a) D is a strict normal crossing divisor on X,

(b) D is a strict normal crossing divisor on X at any closed point x of D,

(c) D is reduced, each D_i is an effective Cartier divisor, and for any $J \subset I$ finite the scheme theoretic intersection $D_J := \cap_{j \in J} D_j$ is a regular scheme each of whose irreducible components has codimension $|J|$ in X (e.g. when D_J is empty this is automatic).

Indeed, a proof of the equivalence between (a) and (c) can be found at [Sta22, 0BI9,0BIA] (this is also a consequence of 4.5.2.2). Following [**?**, 0.17.3.2], to check that $\mathcal{O}_{D_J}$ is regular we reduce to check that $\mathcal{O}_{D_J,x}$ is regular for any closed point x of D_J. Hence, it follows from 4.5.2.2 that we get the equivalence between (b) and (c).

Example 4.5.2.5. Let X be a locally of finite type scheme over $S = \operatorname{Spec} k$ with k a *perfect* field. Following [Gro67, 17.15.1], a scheme locally of finite type over S is smooth at x if and if it is regular at x. Let $x \in X$ be a closed point, $d = \dim_x X$ and $t_1, \ldots, t_d$ be sections of $\mathcal{O}_X$ on X. Following [Gro67, 17.15.4], since the extension $k(x)/k$ is finite and étale, then the morphism $X \to \mathbb{A}^d$ induced by $t_1, \ldots, t_d$ is étale at x if and only if the sheaf $\mathcal{O}_{X,x}$ is regular and $(t_1)_x, \ldots, (t_d)_x$ is a regular system of parameter for $\mathcal{O}_{X,x}$. Let $D \subset X$ be an effective Cartier divisor. It follows from the above reminder that D is a strict normal crossing divisor on X if one of the following equivalent conditions holds:

(d) for every $x \in D$, there exists an open U of X containing x so that U/S is endowed with coordinates $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U,

(e) for every closed point $x \in D$, there exists an open U of X containing x so that U/S is endowed with coordinates $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U,

(f) D is reduced, and if $(D_i \subset D)_{i \in I}$ is the family of the irreducible components of D (viewed as reduced closed subschemes of X), each D_i is an effective Cartier divisor, and for any $J \subset I$ finite the scheme theoretic intersection $D_J := \cap_{j \in J} D_j$ is a smooth scheme over S each of whose irreducible components has codimension $|J|$ in X (e.g. when D_J is empty this is automatic).

Definition 4.5.2.6. Let $X \to S$ be a morphism of schemes or a morphism of formal schemes. Let $D \subset X$ be an effective Cartier divisor.

(a) We say that D admits a "decomposition by smooth components" if there exist effective Cartier divisors $(D_i)_{i \in I}$ in X/S which are smooth over S such that we have $D = \sum_{i \in I} D_i$ in $\operatorname{Div}(X/S)_{\geq 0}$.

(b) We say that D admits a decomposition by "irreducible smooth components" if there exist effective Cartier divisors $(D_i)_{i \in I}$ in X/S which are smooth over S, whose underlying topological space is irreducible and such that $D = \sum_{i \in I} D_i$ in $\operatorname{Div}(X/S)_{\geq 0}$.

We recall below the following definition given by Y. Nakkajima and A. Shiho in [NS, 2.1.7] (it is called "relative simple normal crossing divisor").

Definition 4.5.2.7. Let $X \to S$ be a morphism of schemes or a morphism of formal schemes. Let $D \subset X$ be an effective Cartier divisor.

1. Let $x \in D$. We say that D is a "relative to X/S strict normal crossing divisor at x" if there exists an open U of X containing x so that U/S is endowed with coordinates $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U.

2. We say that D is a "relative to X/S strict normal crossing divisor" if D is a relative to X/S strict normal crossing divisor at any $x \in D$ and if D admits a decomposition by smooth components.

Remark 4.5.2.8. We keep notation 4.5.2.7.

(a) When there exists an open U of X containing x so that U/S is endowed with coordinates $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U, then $D \cap U$ admits a decomposition by the smooth components given by $t_1, \ldots, t_r$ and $D \cap U$ is an effective Cartier divisor in U/S. However, since this is not clear that we can glue theses S-smooth closed subschemes, the hypothesis that D has a decomposition by smooth components is not useless in the notion of *relative to X/S strict normal crossing divisor*.

(b) A relative to X/S strict normal crossing divisor is necessarily an effective Cartier divisor in X/S.

(c) Contrary to the notion of strict normal crossing divisor, the definition 4.5.2.7 is relative. However, if S is the spectrum of a perfect field, then both notion of strict normal crossing divisor of X and of relative to X/S strict normal crossing divisor are equal (see 4.5.2.5).

Remark 4.5.2.9. Let $X \to S$ be a smooth morphism of schemes such that S is geometrically unibranch (see [Gro65, 6.15.1]). Then following [Gro67, 17.5.7], X is geometrically unibranch. In particular, for any $x \in X$, $\mathcal{O}_{X,x}$ is irreducible. Hence, when S is moreover locally noetherian, it follows from [Gro60, 6.1.10] that the irreducible components of X are equal to its connected components.

Let $\mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of $\mathcal{V}$-formal schemes.We say that $\mathfrak{S}$ is geometrically unibranch if S_i is geometrically unibranch for any $i \in \mathbb{N}$. Hence, if $\mathfrak{S}$ is geometrically unibranch and locally noetherian, then the irreducible components of $\mathfrak{X}$ are equal to its connected components.

Proposition 4.5.2.10. *Let $X \to S$ be a morphism of schemes or $\mathcal{V}$-formal schemes. Suppose S is geometrically unibranch and locally noetherian. Let $D \subset X$ be an effective Cartier divisor in X/S. Let $x \in D$. The following (a) and (b) conditions are equivalent:*

(a) There exists an open U of X containing x so that U/S is endowed with coordinates $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U.

(b) There exists an open U of X containing x, a positive integer r and effective Cartier divisors $E_1, \ldots, E_r$ in U/S such that:

   (i) $D \cap U = \sum_{i=1}^{r} E_i$ in $\mathrm{Div}(U/S)_{\geq 0}$,

   (ii) $D \cap U$ is the scheme theoretic union of $E_1, \ldots, E_r$ (see definition [Sta22, 0C4H]),

   (iii) The underlying topological spaces of $E_1, \ldots, E_r$ corresponds 1-to-1 to the irreducible components of $D \cap U$,

   (iv) For any $J \subset \{1, \ldots, r\}$, denoting by $E_J := \cap_{j \in J} E_j$ the scheme theoretic intersection, E_J is a smooth scheme over S each of whose irreducible components has codimension $|J|$ in X.

Proof. Suppose the property (a) holds. By hypothesis, there exists an open U of X containing x so that U/S is endowed with coordinate $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U. This implies that U/S and $V(t_1), \ldots, V(t_r)$ are smooth over S. With the remark 4.5.2.9, this yields that the irreducible components of $V(t_1), \ldots, V(t_r)$ are respectively equal to their connected components. Shrinking U if necessary, we can therefore suppose $V(t_1), \ldots, V(t_r)$ are irreducible. We denote by $E_i := V(t_i)$ the effective Cartier divisors in U/S for $i = 1, \ldots, r$. The fact that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U can be translated by the equality $D \cap U = \sum_{i=1}^{r} E_i$ in $\mathrm{Div}(U/S)_{\geq 0}$. Moreover, for any $J \subset \{1, \ldots, r\}$, E_J is étale over $\mathbb{A}_S^{d-|J|}$ and therefore smooth over S. This implies the codimension formula thanks to [Gro67, 17.6.4]. Hence, the underlying topological spaces of $E_1, \ldots, E_r$ correspond 1-to-1 to the irreducible components of $D \cap U$. We remark also that $E_i \cap E_J$ is an effective Cartier divisor of E_i/S. Hence, using [Sta22, 0C4R and 01WW], this yields that $D \cap U$ is the scheme theoretic union of $E_1, \ldots, E_r$.

Conversely, suppose the property (b) holds. Let U be an open of X containing x and let $E_1, \ldots, E_r$ be some effective Cartier divisors in U/S satisfying the conditions of (b). Since $E_1, \ldots, E_r$ are effective Cartier divisors, shrinking U if necessary, we can suppose there exist sections $t_1, \ldots, t_r$ of $\mathcal{O}_X$ on U which are nonzerodivisors and such that $E_i = V(t_i)$ for any $i = 1, \ldots, r$. Hence, the closed immersion $E_1 \hookrightarrow U$ is regular. Since E_1 is smooth at x, this yields that U is smooth at x (see [Gro67, 17.12.1]). Let $\mathcal{I}$ be the ideal given by the closed immersion $E_1 \cap \cdots \cap E_r \hookrightarrow U$. Then $(t_1)_x, \ldots, (t_r)_x$ is a system of generators of $\mathcal{I}_x$ whose cardinal is the smallest possible (because the irreducible component of $E_1 \cap \cdots \cap E_r$ containing x has codimension r in X and because of Krull's height theorem). Moreover, since $E_1 \cap \cdots \cap E_r$ and U are smooth at x, shrinking U if necessary, it follows from [Gro67, 17.12.2] that there exist sections $t_{r+1}, \ldots, t_d$ of $\mathcal{O}_X$ on U such that $t_1, \ldots, t_d$ are coordinates of U/S. Hence, we are done. □

Proposition 4.5.2.11. *Let $X \to S$ be a morphism of schemes or $\mathcal{V}$-formal schemes. Suppose S is geometrically unibranch and locally noetherian. Let $D \subset X$ be an effective Cartier divisor in X/S. The following (a) and (b) conditions are equivalent:*

(a) The divisor D is a relative to X/S strict normal crossing divisor.

(b) There exists a unique family of effective Cartier divisors $(D_i)_{i \in I}$ in X/S with D_i irreducible and smooth over S such that:

  (i) $D = \sum_{i \in I} D_i$ in $\mathrm{Div}(X/S)_{\geq 0}$;

  (ii) D is the scheme theoretic union of the $(D_i)_{i \in I}$ (see definition [Sta22, 0C4H]) ;

  (iii) The underlying topological spaces of $(D_i)_{i \in I}$ corresponds 1-to-1 to the irreducible components of D ;

  (iv) For any $J \subset I$ finite the scheme theoretic intersection $D_J := \cap_{j \in J} D_j$ is a smooth scheme over S each of whose irreducible components has codimension $|J|$ in X (e.g. when D_J is empty this is automatic).

Proof. Suppose (a) holds. Then, if follows from the remark 4.5.2.9 that D has a decomposition by irreducible smooth components (see definition 4.5.2.6). Since S is locally noetherian, the unicity of such a decomposition is a consequence of [NS, A.0.3]. Since the other points to be checked are local, then we get (b) from 4.5.2.10. With 4.5.2.10, the converse is obvious. □

Definition 4.5.2.12. Let $X \to S$ be a morphism of schemes or $\mathcal{V}$-formal schemes. Suppose S is geometrically unibranch and locally noetherian. Let D is a relative to X/S strict normal crossing divisor. The elements of the unique family $(D_i)_{i \in I}$ of effective Cartier divisors in X/S satisfying the conditions of (b) are called the "irreducible S-smooth components of D".

Remark 4.5.2.13. Because of the nice description of 4.5.2.11, even if it might not be always useful later, when we will consider relative strict normal crossing divisors, by convenience we will only consider the case where the base is geometrically unibranch.

4.5.2.14. Let $X \to S$ be a morphism of schemes (resp. $\mathcal{V}$-formal schemes). Suppose S is geometrically unibranch and locally noetherian. Let $D \subset X$ be a relative to X/S strict normal crossing divisor and $Y = X \setminus D$.

(a) Let $x \in D$. Choose an open U of X containing x so that U/S is endowed with coordinates $t_1, \ldots, t_d$ such that $D \cap U$ is cut out by $t_1 \ldots t_r$ in U. Shrinking U if necessary, it follows from (the proof of) 4.5.2.10 that we can suppose the underlying spaces of $V(t_1), \ldots, V(t_r)$ are equal to the irreducible components of $D \cap U$. We remark then that the submonoid $t_1^{\mathbb{N}} \cdots t_r^{\mathbb{N}} \mathcal{O}_U^{\times} \subset \mathcal{O}_U$ does not depend on the choice of $t_1, \ldots, t_d$ such that $V(t_1), \ldots, V(t_r)$ are equal to the irreducible components of $D \cap U$. As the projective limits of monoids are well defined, this yields therefore by glueing (see [Gro60, 3.3]), a sheaf of monoids on X denoted by $M(D)$ and a canonical injection $M(D) \hookrightarrow \mathcal{O}_X$. This submonoid $M(D)$ of $\mathcal{O}_X$ is characterized by the properties $M(D)|_Y = \mathcal{O}_Y^{\times}$ and $M(D)|_U = t_1^{\mathbb{N}} \cdots t_r^{\mathbb{N}} \mathcal{O}_U^{\times}$ for any $x \in D$ and U such as above.

(b) The log (formal) scheme $(X, M(D))$ is smooth over S. Indeed, let $x \in D$. Since the formal case is checked identically we can consider the non-respective case. Choose an open U of X containing x so that U/S is endowed with coordinates $t_1, \dots, t_d$ such that $D \cap U$ is cut out by $t_1 \dots t_r$ in U and the underlying spaces of $V(t_1), \dots, V(t_r)$ are equal to the irreducible components of $D \cap U$. Then $M(D)|_U = t_1^{\mathbb{N}} \cdots t_r^{\mathbb{N}} \mathcal{O}_U^{\times}$ and we have the cartesian diagram of S-morphisms

$$\begin{array}{ccccc} (U, M(D)|_U) & \xrightarrow{t} & A_{\mathbb{N}^r} \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}_S^{d-r} & \longrightarrow & A_{\mathbb{N}^r} \\ \downarrow f & \square & \downarrow & \square & \downarrow \\ U & \xrightarrow{t} & \mathbb{A}_S^d & \longrightarrow & \mathbb{A}^r \end{array} \tag{4.5.2.14.1}$$

where the right vertical arrow is induced by the canonical morphism, the bottom left arrow is induced by the coordinates $(t_\lambda)_{\lambda=1,\dots,d}$.

(c) When $r < d$, shrinking if necessary the open subset U containing x (and changing if necessary the elements $t_{r+1}, \dots, t_d$), we can suppose $U = U \setminus V(t_{r+1}, \dots, t_d)$, i.e., the $t_{r+1}, \dots, t_d$ are invertible, i.e. the top right horizontal morphism of 4.5.2.14.1 factors through the strict log étale morphism $(X, M(D)) \to A_{\mathbb{N}^r} \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{G}_m^{d-r} \times_{\operatorname{Spec} \mathbb{Z}} S$. Hence, $t_1, \dots, t_d$ induce the log étale S-morphism $(X, M(D)) \to A_{\mathbb{N}^d} \times_{\operatorname{Spec} \mathbb{Z}} S$.

Definition 4.5.2.15. Let $X \to S$ be a morphism of schemes (resp. formal schemes). Suppose S is geometrically unibranch and locally noetherian. Let $D \subset X$ be a relative to X/S strict normal crossing divisor. Let $t_1, \dots, t_d$ be some sections of $\mathcal{O}_X$ on an open subset U of X. Let r be the number of irreducible components of $D \cap U$ ($r = 0$ means $D \cap U$ is empty). We say that "$t_1, \dots, t_d$ are semi-nice (resp. nice) coordinates of $(U, M(D)|_U)/S$" if the first two (resp. the three) properties are satisfied:

(a) $t_1, \dots, t_d$ are coordinates of U/S,

(b) either $D \cap U$ is empty or $D \cap U$ is cut out by $\prod_{1 \le j \le r} t_j$ in U for some $r \ge 1$ and the underlying spaces of the family $\{V(t_j)\}_{1 \le j \le r}$ correspond 1-to-1 to the irreducible components of $D \cap U$,

(c) t_l are invertible for any integer satisfying $d \ge l > r$.

4.5.2.16. Let $X \to S$ be a smooth morphism of schemes (resp. formal schemes). Suppose S is geometrically unibranch and locally noetherian. Let $D \subset X$ be a relative to X/S strict normal crossing divisor.

(a) Let $x \in X$. It follows from 4.5.2.14 (and remark the case $x \notin D$ is obvious) that there exist an open subset U of X containing x, some sections $t_1, \dots, t_d$ such that $t_1, \dots, t_d$ are nice coordinates of $(U, M(D)|_U)/S$. Then $t_1, \dots, t_d$ are coordinates of U/S and also are logarithmic coordinates of $(U, M(D)|_U)/S$.

(b) Let U be an open subset of X such that there exists semi-nice coordinates $t_1, \dots, t_d$ of $(U, M(D)|_U)/S$. Let r be the number of irreducible components of $D \cap U$. Then $M(D)|_U = t_1^{\mathbb{N}} \cdots t_r^{\mathbb{N}} \mathcal{O}_U^{\times}$ (when $r = 0$, this means $M(D)|_U = \mathcal{O}_U^{\times}$). It follows from 4.5.2.14.1 that $t_1, \dots, t_d$ induces the strict log-étale map $(U, M(D)|_U) \to A_S^{d,r}$ (see notation 4.5.1.1), i.e. $t_1, \dots, t_d$ are semi-logarithmic coordinates of $(U, M(D)|_U)/S$ (see definition 4.5.1.4).

Example 4.5.2.17. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0, p)$ with perfect residue field k, $\mathfrak{S} = \operatorname{Spf} \mathcal{V}$ and $S = \operatorname{Spec} k$. Let X be a smooth k-scheme et $D \subset X$ be a strict normal crossing divisor on X (see 4.5.2.5). In that case, we can check $M(D) = \{g \in \mathcal{O}_X;\ g \text{ is invertible outside } D\}$.

Locally the smooth S-log scheme $(X, M(D))$ lifts to a smooth $\mathfrak{S}$-log formal scheme of the form $(\mathfrak{X}, M(\mathfrak{D}))$, with $\mathfrak{X}$ a smooth $\mathfrak{S}$-formal scheme and $\mathfrak{D} \subset \mathfrak{X}$ be a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor on $\mathfrak{X}$. Indeed, we can suppose X is affine and there exists nice coordinates $t_1, \dots, t_d$ of $(X, M(D))/S$ such that D is cut out by $\prod_{1 \le j \le r} t_j$ in D for some $r \ge 1$. Following [SGA1, III.6.10], there exists an affine smooth $\mathfrak{S}$-formal scheme which is a lifting of X. The étale morphism of the form $X \to \mathbb{A}_S^d$ given by $t_1, \dots, t_d$ lifts to an étale morphism of the form $\mathfrak{X} \to \widehat{\mathbb{A}}_{\mathfrak{S}}^d$. Let $\mathfrak{D} := V(\prod_{1 \le j \le r} t_j)$. Then $\mathfrak{D} \subset \mathfrak{X}$ is a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor on $\mathfrak{X}$ so that $(\mathfrak{X}, M(\mathfrak{D}))$ is a lifting of $(X, M(D))$.

Let $f\colon X' \to X$ be a morphism of smooth k-schemes, $D \subset X$ (resp. $D' \subset X'$) be a strict normal crossing divisor on X (resp. X'). With the description of $M(D)$ and $M(D')$, if $X' \setminus D' \subset f^{-1}(X \setminus D)$ then f canonically induces a morphism of log-schemes $(X', M(D')) \to (X, M(D))$.

Let $\mathfrak{X}$ (resp. $\mathfrak{X}'$) be a smooth $\mathfrak{S}$-formal scheme and $\mathfrak{D} \subset \mathfrak{X}$ (resp. $\mathfrak{D}' \subset \mathfrak{X}'$) be a relative to $\mathfrak{X}/\mathfrak{S}$ (resp. $\mathfrak{X}'/\mathfrak{S}$) strict normal crossing divisor. Beware we do not have the equality $M(\mathfrak{D}) = \{g \in \mathcal{O}_{\mathfrak{X}};\ g \text{ is invertible outside } \mathfrak{D}\}$. However, if $f\colon \mathfrak{X}' \to \mathfrak{X}$ is a morphism of smooth $\mathfrak{S}$-formal schemes such that the closed immersion $f^{-1}(\mathfrak{D})_{\mathrm{red}} \hookrightarrow \mathfrak{X}'$ factors through $f^{-1}(\mathfrak{D})_{\mathrm{red}} \hookrightarrow \mathfrak{D}'$, then f induces (uniquely) a morphism of log-smooth $\mathfrak{S}$-log formal schemes $(\mathfrak{X}, M(\mathfrak{D})) \to (\mathfrak{X}', M(\mathfrak{D}'))$

Suppose there exists $f_0\colon X' \to X$ such that $X' \setminus D' \subset f^{-1}(X \setminus D)$ and $\mathfrak{X}'$ is affine. Then it follows from [Kat89, 3.11], that the morphism of smooth S-log-formal schemes $f_0^\sharp\colon (X', M(D')) \to (X, M(D))$ lifts to a morphism of smooth $\mathfrak{S}$-log-formal schemes of the form $f^\sharp\colon (\mathfrak{X}', M(\mathfrak{D}')) \to (\mathfrak{X}, M(\mathfrak{D}))$.

4.5.2.18. Let $X \to S$ be a smooth morphism of schemes (resp. formal schemes). Suppose S is geometrically unibranch and locally noetherian. Let $D \subset X$ be a relative to X/S strict normal crossing divisor. Set $X^\sharp := (X, M(D))$. Let $f\colon X^\sharp \to X$ be the canonical morphism. Remark f^* is the identity in the category of $\mathcal{O}_X$-modules. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$-module structure which is compatible with its algebra structure. We set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S}$, $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp(m)} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S(m)}$ and similarly without the symbol $\sharp$.

(a) Suppose there exist nice coordinates $t_1, \ldots, t_d$ of $X^\sharp/S$. Let $\tau_i = 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{X \times X}$, the τ_i forms a regular system of generators of $\mathcal{I}$. We still denote by τ_i the image of τ_i in $\mathcal{P}^n_{X/S,(m)}$ or in $\widetilde{\mathcal{P}}^n_{X/S,(m)}$. Then following 1.4.2.3, $\mathcal{P}^n_{X/S,(m)}$ is isomorphic to an m-PD polynomial algebra with coefficients in $\mathcal{O}_X$ of order n in d variables given by $\tau_1, \ldots, \tau_d$ (see 1.4.1.5 and 1.4.1.9). Hence, $\widetilde{\mathcal{P}}^n_{X/S,(m)}$ is a free $\mathcal{B}_X$-module with basis $\{\underline{\tau}^{\{\underline{k}\}_{(m)}} : |\underline{k}| \le n\}$. Let $\{\underline{\partial}^{\langle \underline{k}\rangle_{(m)}} : |\underline{k}| \le n\}$ be the dual basis for $\widetilde{\mathcal{D}}^{(m)}_{X/S,n}$. By taking the union, we get the basis $\{\underline{\partial}^{\langle \underline{k}\rangle_{(m)}} : \underline{k} \in \mathbb{N}^d\}$ of $\widetilde{\mathcal{D}}^{(m)}_{X/S}$.

Let $\underline{\tau}_{\sharp(m)}$ and $\underline{\partial}_\sharp{}^{\langle \underline{k}\rangle_{(m)}}$ be the elements constructed from $(t_\lambda)_{\lambda=1,\ldots,r}$ as defined in 4.1.2.16. Following 3.2.3.9.3, the canonical $\mathcal{B}_X$-algebras (for left or right structures) morphism $\widetilde{\mathcal{P}}^n_{X/S,(m)} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S,(m)}$ (see 3.2.3.17) is explicitly described by

$$\underline{\tau}^{\{\underline{k}\}_{(m)}} \mapsto \underline{t}^{\underline{k}} \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}, \tag{4.5.2.18.1}$$

where the action of $\underline{t}^{\underline{k}}$ is induced by the left structure of $\mathcal{B}_X$-algebra of $\widetilde{\mathcal{P}}^n_{X^\sharp/S,(m)}$. By duality, we get $\rho\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \to \widetilde{\mathcal{D}}^{(m)}_{X/S}$ is explicitly described by

$$\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}} \mapsto \underline{t}^{\underline{k}} \underline{\partial}^{\langle \underline{k}\rangle_{(m)}}. \tag{4.5.2.18.2}$$

The morphism $\rho\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \to \widetilde{\mathcal{D}}^{(m)}_{X/S}$ is a ring morphism. Indeed, let $P \in \mathcal{D}_{X^\sharp/S^\sharp,n}$, $P' \in \mathcal{D}_{X^\sharp/S^\sharp,n'}$. Consider the diagram:

$$\begin{array}{ccccccc}
\mathcal{P}^{n+n'}_{X/S} & \xrightarrow{\delta^{n,n'}} & \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X/S} & \xrightarrow{\mathrm{id} \otimes \rho(P')} & \mathcal{P}^n_{X/S} & \xrightarrow{\rho(P)} & \mathcal{O}_X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}^{n+n'}_{X^\sharp/S} & \xrightarrow{\delta^{n,n'}} & \mathcal{P}^n_{X^\sharp/S} \otimes_{\mathcal{O}_X} \mathcal{P}^{n'}_{X^\sharp/S} & \xrightarrow{\mathrm{id} \otimes P'} & \mathcal{P}^n_{X^\sharp/S} & \xrightarrow{P} & \mathcal{O}_X
\end{array} \tag{4.5.2.18.3}$$

Using the universal property of 3.1.4.1 (or use the formula 4.5.2.18.2), we get the commutativity of the left square. The other squares are commutative by definition of the map ρ. By definition (see 3.1.4.3.1), $\rho(P)\rho(P')$ (resp. $\rho(PP')$) is the top (resp. bottom) composite morphism of 4.5.2.18.3.

Since $t_1, \ldots, t_d$ are nonzero divisors, then via the description 4.5.2.18.1 and 4.5.2.18.2 we get that $\mathcal{P}^n_{X/S,(m)} \to \mathcal{P}^n_{X^\sharp/S,(m)}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \to \widetilde{\mathcal{D}}^{(m)}_{X/S^\sharp}$ are injective. By abuse of notation, we get therefore the equality:

$$\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}} = \underline{t}^{\underline{k}} \underline{\partial}^{\langle \underline{k}\rangle_{(m)}}. \tag{4.5.2.18.4}$$

(b) Suppose there exist semi-nice coordinates $t_1, \dots, t_d$ of $X^\sharp/S$. With notation 4.5.1.4, we have the basis $\{\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} : \underline{k} \in \mathbb{N}^d\}$ of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$. The map $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \to \widetilde{\mathcal{D}}^{(m)}_{X/S}$ is explicitly described by

$$\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} \mapsto \underline{t}^{(k_1, \dots, k_r, 0, \dots, 0)} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}. \tag{4.5.2.18.5}$$

Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \to \widetilde{\mathcal{D}}^{(m)}_{X/S}$ is injective, then we can write $\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} = \underline{t}^{(k_1, \dots, k_r, 0, \dots, 0)} \underline{\partial}^{\langle \underline{k} \rangle_{(m)}}$.

(c) Let $\mathcal{E}$ and $\mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X/S}$-modules. Then the tensor product $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}$ and $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F})$ are canonically endowed with a structure of left $\widetilde{\mathcal{D}}^{(m)}_{X/S}$-module functorial in $\mathcal{E}$ and $\mathcal{F}$ (see 4.2.3.1 in the case where the log structure are trivial). Denoting by f_* the forgetful functor from the category of left $\widetilde{\mathcal{D}}^{(m)}_{X/S}$-modules to that of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-modules (via the canonical monomorphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \subset \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$ given by f), we get on $f_*(\mathcal{E}) \otimes_{\mathcal{B}_X} f_*(\mathcal{F})$ and $\mathcal{H}om_{\mathcal{B}_X}(f_*(\mathcal{E}), f_*(\mathcal{F}))$ a structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-module (see 4.2.3.1). Either by making a local computation or by identifying the m-PD-stratifications, we can easily check that we have the equalities $f_*(\mathcal{E}) \otimes_{\mathcal{B}_X} f_*(\mathcal{F}) = f_*(\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F})$ and $\mathcal{H}om_{\mathcal{B}_X}(f_*(\mathcal{E}), f_*(\mathcal{F})) = f_*(\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F}))$. We have similar results when $\mathcal{E}$ and $\mathcal{F}$ are right or left left $\widetilde{\mathcal{D}}^{(m)}_{X/S}$-modules (when this has a meaning).

4.5.3 An associativity isomorphism

Let $X \to S$ be a smooth morphism of schemes (resp. formal schemes). Let $D \subset X$ be a relative to X/S strict normal crossing divisor (see definition 4.5.2.7). Set $X^\sharp := (X, M(D))$. Suppose S is geometrically unibranch and locally noetherian (see the remark 4.5.2.13). Let $E \subset D$ be a subdivisor of D and set $X^\flat := (X, M(E))$. Moreover, let $d = \dim X$. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\flat/S}$-module structure which is compatible with its algebra structure. We set $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\flat/S}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S}$. Using a similar to 4.5.2.18.3 diagram, we easily check that the map $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S} \to \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$ is a ring morphism.

We prove in this section the associativity isomorphism 4.5.3.2.2. This corresponds to a level m variation of [CMNM05, 2.3.4]. We get similarly 4.5.3.9 (we remark that the meaning of “ associativity ” can be better understood for 4.5.3.9). Finally, we deduce from 4.5.3.2.2 the isomorphism 4.5.3.8.1 which will be used to establish 18.2.1.21.

4.5.3.1 (Local description)**.** Suppose there exist nice coordinates $t_1, \dots, t_d$ of $X^\sharp/S$ so that D is empty or D is cut out by $\prod_{1 \le j \le r} t_j$ in U for some $r \ge 1$, which is always locally possible (see 4.5.2.14). Hence, E is either empty of cut out by $\prod_{1 \le j \le s} t_j$ in U for some $r \ge s \ge 1$. Then $t_1, \dots, t_d$ of $X^\sharp/S$ are semi-nice coordinates of $X^\flat/S$ (because when $r > s$, t_r is not invertible). We get the description 4.5.2.18.(b). We get the bases $\{\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} : \underline{k} \in \mathbb{N}^d\}$ of $\mathcal{D}^{(m)}_{X^\sharp/S}$ and $\{\underline{\partial}_{(s)}^{\langle \underline{k} \rangle_{(m)}} : \underline{k} \in \mathbb{N}^d\}$ of $\mathcal{D}^{(m)}_{X^\flat/S}$. According to 4.5.1.9, in the computations of the subsection, when we refer to some logarithmic formulas, we will mean its semi-logarithmic avatar.

4.5.3.2. Let $\mathcal{E}^\sharp$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-module and $\mathcal{M}^\flat$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module. We have the canonical morphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-modules : $\mathcal{E}^\sharp \to \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp$. By functoriality of the tensor product of 4.2.3.5, this yields the morphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-modules :

$$\mathcal{M}^\flat \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp \to \mathcal{M}^\flat \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp). \tag{4.5.3.2.1}$$

The morphism 4.5.3.2.1 induces by extension the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-linear morphism:

$$(\mathcal{M}^\flat \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \to \mathcal{M}^\flat \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp). \tag{4.5.3.2.2}$$

Theorem 4.5.3.3. *Let $\mathcal{E}^\sharp$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-module and $\mathcal{M}^\flat$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module. The morphism 4.5.3.2.2 is an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-modules.*

Proof. We can compare with the proof of the complex case given in [CMNM05, A.1]. The fact that the arrow 4.5.3.2.2 is an isomorphism is local. Let us suppose then $X^\sharp/S$ endowed with nice coordinates $t_1, \ldots, t_d$ (see definition 4.5.2.15) and let us keep the notations of 4.5.2.18. It is about proving that for any right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module $\mathcal{N}$, for any $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-linear morphism $\alpha\colon \mathcal{M}^\flat \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp \to \mathcal{N}$, there exists a unique morphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-modules $\beta\colon \mathcal{M}^\flat \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \to \mathcal{N}$ making commutative the diagram

$$\mathcal{M}^\flat \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp \xrightarrow{4.5.3.2.1} \mathcal{M}^\flat \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \overset{\exists!\beta}{\dashrightarrow} \mathcal{N}, \qquad \text{(composite } \alpha\text{)}.$$

I) Let us treat first the uniqueness of β. For this purpose, we check by induction on $N \in \mathbb{N}$ that for any $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat,N}$, α uniquely determines the element $\beta(m \otimes (P \otimes e))$. When $N = 0$, we have necessarily $\beta(m \otimes (b \otimes e)) = \beta(mb \otimes (1 \otimes e)) = \alpha(mb \otimes e)$, for any $b \in \mathcal{B}_X$. By linearity, we can suppose P of the form $\underline{\partial}^{\langle \underline{k} \rangle}_{(s)}$. The following formula that β has to check (because β is $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-linear and we use the formula 4.2.3.5.3)

$$\beta(m \otimes (\underline{\partial}^{\langle \underline{k} \rangle}_{(s)} \otimes e)) = \beta(m \otimes (1 \otimes e))\widetilde{\underline{\partial}}^{\langle \underline{k} \rangle}_{(s)} - \sum_{\underline{h} \langle \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \beta(m\widetilde{\underline{\partial}}^{\langle \underline{k}-\underline{h} \rangle}_{(s)} \otimes (\underline{\partial}^{\langle \underline{h} \rangle}_{(s)} \otimes e))$$

allow us to conclude the induction.

II) Now let us establish the existence of β. By induction on $N \in \mathbb{N}$, we construct a $\mathbb{Z}$-trilinear morphism $v_N\colon \mathcal{M}^\flat \times \widetilde{\mathcal{D}}^{(m)}_{X^\flat,N} \times \mathcal{E}^\sharp \to \mathcal{N}$ inducing v_{N-1} as follows : for any $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, $b \in \mathcal{B}_X$, we set $v_0(m, b, e) := \alpha(mb \otimes e)$. Suppose v_N is defined. For any $\underline{k} \in \mathbb{N}^d$ such that $|\underline{k}| = N + 1$, for any $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, we set

$$v_{N+1}(m, \underline{\partial}^{\langle \underline{k} \rangle}_{(s)}, e) := \left(v_N(m, 1, e)\widetilde{\underline{\partial}}^{\langle \underline{k} \rangle}_{(s)} - \sum_{\underline{h} \langle \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} v_N(m\widetilde{\underline{\partial}}^{\langle \underline{k}-\underline{h} \rangle}_{(s)}, \underline{\partial}^{\langle \underline{h} \rangle}_{(s)}, e) \right). \tag{4.5.3.3.1}$$

Next, for any $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat,N}$, $Q = \sum_{|\underline{l}|=N+1} b_{\underline{l}} \underline{\partial}^{\langle \underline{l} \rangle}_{(s)} \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat,N+1}$ where the sum runs over $\underline{l} \in \mathbb{N}^d$ such that $|\underline{l}| = N + 1$ and where $b_{\underline{l}} \in \mathcal{B}_X$, we set

$$v_{N+1}(m, P + Q, e) := v_N(m, P, e) + \sum_{\underline{l}} v_{N+1}(mb_{\underline{l}}, \underline{\partial}^{\langle \underline{l} \rangle}_{(s)}, e). \tag{4.5.3.3.2}$$

In particular, for any $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat,N}$, we have $v_{N+1}(m, P, e) = v_N(m, P, e)$. Moreover, since v_N is $\mathbb{Z}$-trilinear then so is v_{N+1}. The morphisms v_N induce therefore the $\mathbb{Z}$-trilinear morphism $v\colon \mathcal{M}^\flat \times \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \times \mathcal{E}^\sharp \to \mathcal{N}$. To end the proof of the theorem, we will need the lemmas below.

Lemma 4.5.3.4. *For any $b \in \mathcal{B}_X$, $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$, $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, $v(mb, P, e) = v(m, bP, e) = v(m, P, e)b$.*

Proof. The left equality of 4.5.3.4 is checked by induction on the order of the operator P from the formula 4.5.3.3.2. By additivity, to check that the right one holds, it is sufficient to establish that $\epsilon := v(m, \underline{\partial}^{\langle \underline{k} \rangle}_{(s)}, e)b - v(mb, \underline{\partial}^{\langle \underline{k} \rangle}_{(s)}, e)$ is null. We proceed by induction on $N := |\underline{k}|$. We get by $\mathcal{B}_X$-linearity of α the equality $v(m, 1, e)b = v(mb, 1, e)$, i.e. the case $N = 0$. Now, let us suppose the formula true for $N - 1 \geq 0$. Following 4.5.3.3.1 and next by induction hypothesis, we compute :

$$v(m, 1, e)\widetilde{\underline{\partial}}^{\langle \underline{k} \rangle}_{(s)} b = \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}^{\langle \underline{k}-\underline{i} \rangle}_{(s)}, \underline{\partial}^{\langle \underline{i} \rangle}_{(s)}, e)b = \epsilon + \sum_{\underline{i} \leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}^{\langle \underline{k}-\underline{i} \rangle}_{(s)} b, \underline{\partial}^{\langle \underline{i} \rangle}_{(s)}, e). \tag{4.5.3.4.1}$$

On the other side, following 4.1.2.4.1, $\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} b = \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}(b) \widetilde{\underline{\partial}}_{(s)}^{\langle \underline{h}\rangle}$. Hence :

$$
\begin{aligned}
v(m,1,e)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} b &= \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}(b), 1, e)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{h}\rangle} \\
&\underset{4.5.3.3.1}{=} \sum_{\underline{h}\leq \underline{k}} \sum_{\underline{i}\leq \underline{h}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{h} \\ \underline{i} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}(b)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{h}-\underline{i}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{i}\rangle}, e) \\
&= \sum_{\underline{i}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} \sum_{\underline{i}\leq \underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k}-\underline{i} \\ \underline{k}-\underline{h} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}(b)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{h}-\underline{i}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{i}\rangle}, e) \\
&\underset{4.1.2.4.1}{=} \sum_{\underline{i}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{i} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{i}\rangle} b, \underline{\partial}_{(s)}^{\langle \underline{i}\rangle}, e).
\end{aligned}
\tag{4.5.3.4.2}
$$

By comparing 4.5.3.4.1 and 4.5.3.4.2, we obtain $\epsilon = 0$. □

Lemma 4.5.3.5. *For any $b \in \mathcal{B}_X$, $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$, $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, $v(m, P, be) = v(m, Pb, e)$.*

Proof. By using the relation 4.5.3.4, we reduce to the case where P is of the form $\underline{\partial}_{(s)}^{\langle \underline{k}\rangle}$, with $\underline{k} \in \mathbb{N}^d$. We have to establish that $\epsilon := v(m, \underline{\partial}_{(s)}^{\langle \underline{k}\rangle}, be) - v(m, \underline{\partial}_{(s)}^{\langle \underline{k}\rangle} b, e)$ is null. We proceed by induction on $N := |\underline{k}|$. When $N = 0$, this comes from $v(m, 1, be) := \alpha(m \otimes be) = \alpha(mb \otimes e) =: v(m, b, e)$. Now, let us suppose the formula true for $N - 1 \geq 0$ (i.e. H_{N-1} holds). We compute:

$$
\begin{aligned}
v(m,1,be)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} &\underset{4.5.3.3.1}{=} \sum_{\underline{l}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{l} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{l}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{l}\rangle}, be) \\
&\underset{H_{N-1}}{=} \epsilon + \sum_{\underline{l}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{l} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{l}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{l}\rangle} b, e) \\
&\underset{4.1.2.4.1}{=} \epsilon + \sum_{\underline{l}\leq \underline{k}} \sum_{\underline{h}\leq \underline{l}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{l} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{l} \\ \underline{h} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{l}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{l}-\underline{h}\rangle}(b)\underline{\partial}_{(s)}^{\langle \underline{h}\rangle}, e) \\
&\underset{4.5.3.4}{=} \epsilon + \sum_{\underline{l}\leq \underline{k}} \sum_{\underline{h}\leq \underline{l}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{l} \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} \underline{l} \\ \underline{h} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{l}\rangle}(\underline{\partial}_{(s)}^{\langle \underline{l}-\underline{h}\rangle}(b)), \underline{\partial}_{(s)}^{\langle \underline{h}\rangle}, e) \\
&= \epsilon + \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} v\left(m \sum_{\underline{h}\leq \underline{l}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k}-\underline{h} \\ \underline{k}-\underline{l} \end{smallmatrix} \right\} \widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{l}\rangle}(\underline{\partial}_{(s)}^{\langle \underline{l}-\underline{h}\rangle}(b)), \underline{\partial}_{(s)}^{\langle \underline{h}\rangle}, e \right) \\
&\underset{4.2.5.4.2}{=} \epsilon + \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} v(mb\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{h}\rangle}, e) \\
&\underset{4.5.3.3.1}{=} \epsilon + v(mb,1,e)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} = \epsilon + v(m,1,be)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle}.
\end{aligned}
$$

□

Lemma 4.5.3.6. *Let $\underline{a} \in \mathbb{N}^d$ such that $a_i = 0$ if $1 \leq i \leq s$ or $r + 1 \leq i \leq d$. For any $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, $\underline{k} \in \mathbb{N}^d$, the following formula is satisfied:*

$$
v(m, \underline{\partial}_{(s)}^{\langle \underline{a}\rangle}, e)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} = \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{h}\rangle}\underline{\partial}_{(s)}^{\langle \underline{a}\rangle}, e).
\tag{4.5.3.6.1}
$$

Proof. Let us check the lemma by induction on $N := |\underline{a}|$. For $N = 0$, this is a consequence of 4.5.3.3.1. Let us suppose H_N true, i.e., the equality is satisfied for N. Let us check it for $N + 1$. Set $\epsilon := v(m, \underline{\partial}_{(s)}^{\langle \underline{a}\rangle}, e)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} - \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{h}\rangle}\underline{\partial}_{(s)}^{\langle \underline{a}\rangle}, e)$. We have to check $\epsilon = 0$. We compute

$$
\begin{aligned}
v(m,1,e)\underline{\partial}_{(s)}^{\langle \underline{a}\rangle}\underline{\partial}_{(s)}^{\langle \underline{k}\rangle} &\underset{4.5.3.3.1}{=} \sum_{\underline{b}\leq \underline{a}} \left\{ \begin{smallmatrix} \underline{a} \\ \underline{b} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{a}-\underline{b}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{b}\rangle}, e)\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}\rangle} \\
&\underset{H_N}{=} \varepsilon + \sum_{\underline{b}\leq \underline{a}} \sum_{\underline{h}\leq \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \left\langle \begin{smallmatrix} \underline{h}+\underline{b} \\ \underline{b} \end{smallmatrix} \right\rangle \left\{ \begin{smallmatrix} \underline{a} \\ \underline{b} \end{smallmatrix} \right\} v(m\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{a}-\underline{b}\rangle}\widetilde{\underline{\partial}}_{(s)}^{\langle \underline{k}-\underline{h}\rangle}, \underline{\partial}_{(s)}^{\langle \underline{h}+\underline{b}\rangle}, e).
\end{aligned}
\tag{4.5.3.6.2}
$$

On the other side, by using three times the formula 2.1.3.1.2 (see the remark 4.2.3.2), we compute in $\mathcal{M}^\flat \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$ (by default we take the left structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module) :

$$\sum_{\underline{b}\leq\underline{a}}\sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} \left\langle \begin{smallmatrix}\underline{h}+\underline{b}\\ \underline{b}\end{smallmatrix}\right\rangle \left\{ \begin{smallmatrix}\underline{a}\\ \underline{b}\end{smallmatrix}\right\} m\underline{\widetilde{\partial}}^{\langle \underline{a}-\underline{b}\rangle}_{(s)}\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(s)} \otimes \underline{\partial}^{\langle \underline{h}+\underline{b}\rangle}_{(s)} = \sum_{\underline{b}\leq\underline{a}} \left\{ \begin{smallmatrix}\underline{a}\\ \underline{b}\end{smallmatrix}\right\} (m\underline{\widetilde{\partial}}^{\langle \underline{a}-\underline{b}\rangle}_{(s)} \otimes \underline{\partial}^{\langle \underline{b}\rangle}_{(s)})\underline{\partial}^{\langle \underline{k}\rangle}_{(s)}$$
$$= (m\otimes 1)\underline{\partial}^{\langle \underline{a}\rangle}_{(s)}\underline{\partial}^{\langle \underline{k}\rangle}_{(s)} = \left\langle \begin{smallmatrix}\underline{a}+\underline{k}\\ \underline{k}\end{smallmatrix}\right\rangle (m\otimes 1)\underline{\partial}^{\langle \underline{a}+\underline{k}\rangle}_{(s)} = \sum_{\underline{l}\leq\underline{a}+\underline{k}} \left\langle \begin{smallmatrix}\underline{a}+\underline{k}\\ \underline{k}\end{smallmatrix}\right\rangle \left\{ \begin{smallmatrix}\underline{a}+\underline{k}\\ \underline{l}\end{smallmatrix}\right\} m\underline{\widetilde{\partial}}^{\langle \underline{a}+\underline{k}-\underline{l}\rangle}_{(s)} \otimes \underline{\partial}^{\langle \underline{l}\rangle}_{(s)}. \tag{4.5.3.6.3}$$

Since $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$ is a free $\mathcal{B}_X$-module (for the left or right structure) with the basis consisting of the elements $\underline{\partial}^{\langle \underline{n}\rangle}_{(s)}$ with $\underline{n}$ going through $\mathbb{N}^d$, then by $\mathbb{Z}$-multilinearity of v it follows from 4.5.3.6.3 the first equality in $\mathcal{N}$:

$$\sum_{\underline{b}\leq\underline{a}}\sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} \left\langle \begin{smallmatrix}\underline{h}+\underline{b}\\ \underline{b}\end{smallmatrix}\right\rangle \left\{ \begin{smallmatrix}\underline{a}\\ \underline{b}\end{smallmatrix}\right\} v(m\underline{\widetilde{\partial}}^{\langle \underline{a}-\underline{b}\rangle}_{(s)}\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(s)}, \underline{\partial}^{\langle \underline{h}+\underline{b}\rangle}_{(s)}, e) = \sum_{\underline{l}\leq\underline{a}+\underline{k}} \left\langle \begin{smallmatrix}\underline{a}+\underline{k}\\ \underline{k}\end{smallmatrix}\right\rangle \left\{ \begin{smallmatrix}\underline{a}+\underline{k}\\ \underline{l}\end{smallmatrix}\right\} v(m\underline{\widetilde{\partial}}^{\langle \underline{a}+\underline{k}-\underline{l}\rangle}_{(s)}, \underline{\partial}^{\langle \underline{l}\rangle}_{(s)}, e)$$
$$\underset{4.5.3.3.1}{=} \left\langle \begin{smallmatrix}\underline{a}+\underline{k}\\ \underline{k}\end{smallmatrix}\right\rangle v(m,1,e)\underline{\partial}^{\langle \underline{a}+\underline{k}\rangle}_{(s)} = v(m,1,e)\underline{\partial}^{\langle \underline{a}\rangle}_{(s)}\underline{\partial}^{\langle \underline{k}\rangle}_{(s)}. \tag{4.5.3.6.4}$$

Hence, by using 4.5.3.6.2 and 4.5.3.6.4 we get $\epsilon = 0$, i.e. 4.5.3.6.1 is therefore satisfied. □

Lemma 4.5.3.7. *For any $P^\sharp \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$, $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$, $m \in \mathcal{M}^\flat$, $e \in \mathcal{E}^\sharp$, $v(m, P, P^\sharp e) = v(m, PP^\sharp, e)$.*

Proof. 1) At first, let us suppose $P = 1$, i.e., let us check the equality $v(m, 1, P^\sharp e) = v(m, P^\sharp, e)$.

i) By induction on $|\underline{k}|$, let us establish the formula : $v(m, 1, \underline{\partial}^{\langle \underline{k}\rangle}_{(r)} e) = v(m, \underline{\partial}^{\langle \underline{k}\rangle}_{(r)}, e)$. By multiplying the equality 4.5.3.6.1 by $\underline{t}^{\underline{k}} = t_1^{k_1} \dots {}_d^{k_d}$, by using the equality 4.5.2.18.4 and thanks to the formula 4.5.3.4, we obtain:

$$v(m,1,e)\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle}_{(r)} = \sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} v(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(r)}, \underline{\partial}^{\langle \underline{h}\rangle}_{(r)}, e). \tag{4.5.3.7.1}$$

Moreover, by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-linearity of α, it follows from 4.2.3.5.1 the equality:

$$(\alpha(m\otimes e))\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle}_{(r)} = \sum_{\underline{h}\leq\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} \alpha(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(r)} \otimes \underline{\partial}^{\langle \underline{h}\rangle}_{(r)} e). \tag{4.5.3.7.2}$$

By induction hypothesis, for any $\underline{h}\langle\underline{k}$, $v(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(r)}, \underline{\partial}^{\langle \underline{h}\rangle}_{(r)}, e) = v(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(r)}, 1, \underline{\partial}^{\langle \underline{h}\rangle}_{(r)} e) = \alpha(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}_{(r)} \otimes \underline{\partial}^{\langle \underline{h}\rangle}_{(r)} e)$. Since $v(m,1,e)\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle}_{(r)} = (\alpha(m\otimes e))\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle}_{(r)}$, by comparing the formulas 4.5.3.7.1 and 4.5.3.7.2, this yields $v(m, \underline{\partial}^{\langle \underline{k}\rangle}_{(r)}, e) = \alpha(m \otimes \underline{\partial}^{\langle \underline{k}\rangle}_{(r)} e) = v(m, 1, \underline{\partial}^{\langle \underline{k}\rangle}_{(r)} e)$.

ii) Finally, if $P^\sharp$ is of the form $\sum_{\underline{k}} b_{\underline{k}}\underline{\partial}^{\langle \underline{k}\rangle}_{(r)}$, where $b_{\underline{k}} \in \mathcal{B}_X$, we compute $v(m, 1, P^\sharp e) = \alpha(m \otimes P^\sharp e) = \sum_{\underline{k}} \alpha(mb_{\underline{k}} \otimes \underline{\partial}^{\langle \underline{k}\rangle}_{(r)} e) = \sum_{\underline{k}} v(mb_{\underline{k}}, 1, \underline{\partial}^{\langle \underline{k}\rangle}_{(r)} e) = \sum_{\underline{k}} v(mb_{\underline{k}}, \underline{\partial}^{\langle \underline{k}\rangle}_{(r)}, e) \underset{4.5.3.4}{=} v(m, P^\sharp, e)$.

2) Let us treat now the general case. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$ is a right $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module generated by the elements of the form $\underline{\partial}^{\langle \underline{k}\rangle}$, with $\underline{k} \in \mathbb{N}^d$ such that $k_i = 0$ if $1 \leq i \leq s$ or $r+1 \leq i \leq d$, then it is sufficient to check 4.5.3.7 when $P = \underline{\partial}^{\langle \underline{k}\rangle}$ for such $\underline{k}$. We proceed by induction on the integer $N := |\underline{k}|$. For $N = 0$, this is the step 1). Let us suppose the lemma holds for $N \geq 0$ and let us suppose $|\underline{k}| = N+1$. Following 4.5.3.3.1:

$$v(m, \underline{\partial}^{\langle \underline{k}\rangle}, P^\sharp e) = \left(v(m,1,P^\sharp e)\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle} - \sum_{\underline{h}\langle\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} v(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}, \underline{\partial}^{\langle \underline{h}\rangle}, P^\sharp e) \right). \tag{4.5.3.7.3}$$

Moreover, it comes from 4.5.3.6.1 the formula :

$$v(m, \underline{\partial}^{\langle \underline{k}\rangle} P^\sharp, e) = \left(v(m,P^\sharp,e)\underline{\widetilde{\partial}}^{\langle \underline{k}\rangle} - \sum_{\underline{h}\langle\underline{k}} \left\{ \begin{smallmatrix}\underline{k}\\ \underline{h}\end{smallmatrix}\right\} v(m\underline{\widetilde{\partial}}^{\langle \underline{k}-\underline{h}\rangle}, \underline{\partial}^{\langle \underline{h}\rangle} P^\sharp, e) \right). \tag{4.5.3.7.4}$$

Via 4.5.3.7.3, 4.5.3.7.4 and by induction hypothesis, we get $v(m, \underline{\partial}^{\langle \underline{k}\rangle}, P^\sharp e) = v(m, \underline{\partial}^{\langle \underline{k}\rangle} P^\sharp, e)$. □

Let us now conclude the proof of the theorem. It follows from the formulas 4.5.3.4 and 4.5.3.7 that the morphism v induces a morphism of $\mathcal{B}_X$-modules $\beta\colon \mathcal{M}^\flat \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \to \mathcal{N}$. Finally, using the formula of 4.5.3.6.1 and that of 4.2.3.2 (i.e. more precisely the generalization with coefficients of the formula 2.1.3.1.2) we conclude that β is $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-linear. □

Corollary 4.5.3.8. *Let $\mathcal{E}^\sharp$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-module, $\mathcal{F}^\flat$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module (resp. a $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-bimodule). We have the isomorphism:*

$$(\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{F}^\flat \xrightarrow{\sim} (\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{F}^\flat) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp. \tag{4.5.3.8.1}$$

Proof. It follows from 4.5.3.3 the canonical isomorphism

$$(\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \xrightarrow{\sim} \widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp). \tag{4.5.3.8.2}$$

By applying to it the functor $- \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}} \mathcal{F}^\flat$, this gives the first isomorphism:

$$\begin{aligned}(\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{F}^\flat &\xrightarrow{\sim} \Big(\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp)\Big) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}} \mathcal{F}^\flat \\ \xrightarrow[4.3.5.10.1]{\sim} \Big(\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{F}^\flat\Big) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) &\xrightarrow{\sim} (\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{F}^\flat) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp.\end{aligned}$$

□

Theorem 4.5.3.9. *Let $\mathcal{E}^\sharp$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-module and $\mathcal{F}^\flat$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-module. We have the morphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}$-modules $\mathcal{E}^\sharp \otimes_{\mathcal{B}_X} \mathcal{F}^\flat \to (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \otimes_{\mathcal{B}_X} \mathcal{F}^\flat$, sending $e \otimes f$ on $(1 \otimes e) \otimes f$ where $e \in \mathcal{E}^\sharp$, $f \in \mathcal{F}^\flat$. The $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-linear morphism induced by extension:*

$$\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} (\mathcal{E}^\sharp \otimes_{\mathcal{B}_X} \mathcal{F}^\flat) \to (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \otimes_{\mathcal{B}_X} \mathcal{F}^\flat \tag{4.5.3.9.1}$$

is an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S}$-modules.

Proof. Switching from left to right $\mathcal{D}$-modules, this is a consequence of 4.5.3.3. More precisely, by using 4.3.5.7, we get 4.5.3.9.1 from the canonical composite isomorphism

$$\begin{aligned}\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \Big(\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} (\mathcal{E}^\sharp \otimes_{\mathcal{B}_X} \mathcal{F}^\flat)\Big) &\xleftarrow[4.5.3.3]{\sim} (\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} (\mathcal{E}^\sharp \otimes_{\mathcal{B}_X} \mathcal{F}^\flat)) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \\ \xrightarrow[4.5.2.18.c]{\sim} ((\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{F}^\flat) \otimes_{\mathcal{B}_X} \mathcal{E}^\sharp) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} &\xrightarrow[4.5.3.3]{\sim} (\widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \mathcal{F}^\flat) \otimes_{\mathcal{B}_X} (\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \\ &\xrightarrow{\sim} \widetilde{\omega}_{X^\flat} \otimes_{\mathcal{B}_X} \Big((\widetilde{\mathcal{D}}^{(m)}_{X^\flat/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S}} \mathcal{E}^\sharp) \otimes_{\mathcal{B}_X} \mathcal{F}^\flat\Big).\end{aligned} \tag{4.5.3.9.2}$$

□

4.6 Complexes of $\mathcal{D}$-modules, first properties

4.6.1 Pseudo-coherent complexes, Theorems A and B

Let X be a scheme or a $\mathcal{V}$-formal scheme (e.g. see 3.4). We explain how to get Theorems A and B for pseudo-coherent complexes from Theorem A and B for coherent modules. First, let us recall the notion of pseudo-coherent complexes.

4.6.1.1. Let D be a ring. We denote by $D^-_{\mathrm{coh}}({}^lD)$ the full subcategory of $D^-(D)$ of pseudo-coherent complexes in the sense of [Sta22, 064Q], i.e. of complexes which are quasi-isomorphic to a bounded above complex of finite free left D-modules.

Suppose D is noetherian (resp. coherent). It follows from [Sta22, 066E] (resp. [Sta22, 0EWZ]) that a complex $E^\bullet$ of $D^-(D)$ is pseudo-coherent if and only if $H^n(E^\bullet)$ is a left D-module of finite type (resp. coherent left D-module) for any $n \in \mathbb{Z}$.

Lemma 4.6.1.2. *Let D be a coherent ring. Then the natural map*

$$D^{-}(\mathrm{Coh}(D)) \to D^{-}_{\mathrm{coh}}(D) \tag{4.6.1.2.1}$$

is an equivalence of categories.

Proof. Following [Sta22, 0EWZ] (see 4.6.1.1), the functor is well defined. The essential surjectivity is obvious by definition. It remains to check this is fully faithful. Let $E^{\bullet}, F^{\bullet} \in D^{-}(\mathrm{Coh}(D))$. Viewing $E^{\bullet}$ as a object of $D^{-}_{\mathrm{coh}}(D)$, there exists $P^{\bullet} \in K^{-}(D)$ such that P^{n} are finite free left D-modules for any $n \in \mathbb{Z}$ and endowed with a quasi-isomorphism $E^{\bullet} \xrightarrow{\sim} P^{\bullet}$ in $K^{-}(D)$. In fact, this is also a quasi-isomorphism of $K^{-}(\mathrm{Coh}(D))$ Since a finite free left D-module is a projective object in $\mathrm{Coh}(D)$ and in $M(D)$, then following [Sta22, 064B-13.19.8] we have

$$\mathrm{Hom}_{D(\mathrm{Coh}(D))}(P^{\bullet}, F^{\bullet}) = \mathrm{Hom}_{K(\mathrm{Coh}(D))}(P^{\bullet}, F^{\bullet}) = \mathrm{Hom}_{K(D)}(P^{\bullet}, F^{\bullet}) = \mathrm{Hom}_{D(D)}(P^{\bullet}, F^{\bullet}).$$

Hence we are done. □

4.6.1.3. Let $\mathcal{D}$ be a *coherent* sheaf of rings on X.

(a) With notation 1.4.3.27, it follows from [Sta22, 08FX] that if $\mathcal{E}^{\bullet} \in D^{-}_{\mathrm{coh}}(\mathcal{D})$, then $H^{n}(\mathcal{E}^{\bullet})$ is a coherent left $\mathcal{D}$-module for any $n \in \mathbb{Z}$. We will see the converse below (see 4.6.1.8) under some hypotheses.

(b) Let $\mathcal{E}^{\bullet} \in D^{\mathrm{b}}(\mathcal{D})$. By devissage, we check the converse of (a) is satisfied, i.e. that the following conditions are equivalent:

  (i) $H^{n}(\mathcal{E}^{\bullet})$ is a coherent left $\mathcal{D}$-module for any $n \in \mathbb{Z}$;

  (ii) $\mathcal{E}^{\bullet}$ is pseudo-coherent.

Definition 4.6.1.4 (Bounded cohomological dimension)**.** Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a left (resp. right) exact functor of abelian categories. Let $n \in \mathbb{N}$. We say that F has "cohomological dimension $\leq n$" or "bounded by n cohomological dimension" if

(i) every object of $\mathfrak{A}$ is a subobject (resp. quotient) of an object which is right acyclic (resp. left acyclic) for F,

(ii) $R^{n+1}F = 0$ (resp. $L^{n+1}F = 0$).

We say that F has"bounded cohomological dimension" if there exists $n \in \mathbb{N}$ such that F has cohomological dimension $\leq n$.

Notation 4.6.1.5. Let $\mathfrak{A}$ be an abelian category. Let $a \leq b$ be two integers.

(i) We denote by $K^{\leq a}(\mathfrak{A})$ (resp. $K^{\geq a}(\mathfrak{A})$, resp. $K^{[a,b]}(\mathfrak{A})$) the full subcategory of $K(\mathfrak{A})$ consisting of complexes $Y^{\bullet} \in K(\mathfrak{A})$ such that $Y^{i} = 0$ for any $i > a$ (resp. for any $i < a$, resp. $i \notin [a, b]$). By taking the union over $a \in \mathbb{Z}$ (resp. and $b \in \mathbb{Z}$), we get $K^{+}(\mathfrak{A})$ (resp. $K^{-}(\mathfrak{A})$, resp. $K^{\mathrm{b}}(\mathfrak{A})$), the strictly full subcategory of $K(\mathfrak{A})$ consisting of bounded above (resp. bounded below, resp. bounded) complexes.

(ii) We denote by $D^{\leq a}(\mathfrak{A})$ (resp. $D^{\geq a}(\mathfrak{A})$, resp. $D^{[a,b]}(\mathfrak{A})$) the full subcategory of $D(\mathfrak{A})$ consisting of complexes $Y^{\bullet} \in D(\mathfrak{A})$ such that $H^{i}(Y^{\bullet}) = 0$ for any $i > a$ (resp. for any $i < a$, resp. $i \notin [a, b]$). By taking the union over $a \in \mathbb{Z}$ (resp. and $b \in \mathbb{Z}$), we get $D^{+}(\mathfrak{A})$ (resp. $D^{-}(\mathfrak{A})$, resp. $D^{\mathrm{b}}(\mathfrak{A})$), the strictly full subcategory of $D(\mathfrak{A})$ consisting of bounded above (resp. bounded below, resp. bounded) complexes.

4.6.1.6. Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a left exact functor of abelian categories which has cohomological dimension $\leq n$ for some integer n.

(a) The functor F is exact if and only if we can choose $n = 0$. The functor F has cohomological dimension $\leq N$ for any $N \geq n$. Hence, we can suppose $n \geq 1$ in the rest of this paragraph 4.6.1.6.

(b) Let $a \leq b$ be two integers. Since F is left exact, then we have the functor $\mathbb{R}F\colon D^{\geq a}(\mathfrak{A}) \to D^{\geq a}(\mathfrak{B})$. Following [Sta22, 07K7], since F has moreover cohomological dimension $\leq n$, then 1) $\mathbb{R}F\colon D(\mathfrak{A}) \to D(\mathfrak{B})$ exists, 2) any complex consisting of right acyclic objects for F computes $\mathbb{R}F$ and 3) any complex is the source of a quasi-isomorphism into a complex consisting of right acyclic objects or F.

Let us now complete these results by giving the bounded above version. Let $Y^\bullet \in K^{\leq b}(\mathfrak{A})$ (resp. $Y^\bullet \in K^{[a,b]}(\mathfrak{A})$). Then there exists $X^\bullet \in K^{\leq b+n}(\mathfrak{A})$ (resp. $X^\bullet \in K^{[a,b+n]}(\mathfrak{A})$) with $X^i \in P$ for any i together with a quasi-isomorphism of $K(\mathfrak{A})$ of the form $Y^\bullet \to X^\bullet$. Indeed, let P be the collection of all the objects of $\mathfrak{A}$ which are right acyclic for F. Then we can check easily that the following conditions are fulfilled:

(i) Every object of $\mathfrak{A}$ is a subobject of an object which is right acyclic for F,

(ii) If $0 \to X \to Y \to Z \to 0$ is a short exact sequence of $\mathfrak{A}$, with $X \in P$, then $Y \in P \Leftrightarrow Z \in P$.

(iii) If
$$X^0 \to X^1 \to \cdots \to X^{n-1} \to Y^n \to 0$$
is an exact sequence of $\mathfrak{A}$, and $X^0, \ldots, X^{n-1} \in P$, then $Y^n \in P$.

Recall that following [Har66, Lemma I.4.6.2], we retrieve the above property 3), i.e. that for any collection P of objects of $\mathfrak{A}$ satisfying the properties (bi), (bii) and (biii), for any complex $Y^\bullet \in K(\mathfrak{A})$ there exists a complex $X^\bullet \in K(\mathfrak{A})$ with $X^i \in P$ for any i together with a quasi-isomorphism of $K(\mathfrak{A})$ of the form $Y^\bullet \to X^\bullet$.

There exists $Z^\bullet \in K(\mathfrak{A})$ (resp. $Z^\bullet \in K^{\geq a}(\mathfrak{A})$) together with a quasi-isomorphism of $K(\mathfrak{A})$ of the form $Y^\bullet \to Z^\bullet$ with $Z^i \in P$ for any $i \in \mathbb{Z}$. Let X^{b+n} be the image of $Z^{b+n-1} \to Z^{b+n}$. By applying the condition (biii), since $Z^\bullet$ is acyclic in degree $> b$, we get that $X^{b+n} \in P$. For any $i > b+n$, set $X^i := 0$ and for any $i < b+n$, set $X^i := Z^i$. In particular, $X^i \in P$ for any $i \in \mathbb{Z}$ and $X^\bullet \in K^{\leq b+n}(\mathfrak{A})$ (resp. $X^\bullet \in K^{[a,b+n]}(\mathfrak{A})$). Moreover, since $Y^i = 0$ for any $i > b$, since $b+n > b$, then the quasi-isomorphism $Y^\bullet \to Z^\bullet$ induces the quasi-isomorphism $Y^\bullet \to X^\bullet$. Hence we are done.

This yields that the functor $\mathbb{R}F\colon D(\mathfrak{A}) \to D(\mathfrak{B})$ induces $\mathbb{R}F\colon D^{\leq b}(\mathfrak{A}) \to D^{\leq b+n}(\mathfrak{B})$ and $\mathbb{R}F\colon D^{[a,b]}(\mathfrak{A}) \to D^{[a,b+n]}(\mathfrak{B})$.

(c) We clarify below the meaning of the statement: since F has bounded cohomological dimension then for any $Y^\bullet \in D(\mathfrak{A})$ we have the converging spectral sequence
$$E_2^{r,s} = R^rF(H^s(Y^\bullet)) \Rightarrow H^{r+s}\mathbb{R}F(Y^\bullet). \tag{4.6.1.6.1}$$

By shifting if necessary, we reduce to the case where $r+s=0$. Remark, we have $E_2^{r,s} = 0$ when $r \notin [0,n]$. Considering the long exact sequence induced by the exact triangle of $D(\mathfrak{B})$
$$\mathbb{R}F(\tau_{\leq 0}Y^\bullet) \to \mathbb{R}F(Y^\bullet) \to \mathbb{R}F(\tau_{\geq 1}Y^\bullet) \to \mathbb{R}F(\tau_{\leq 0}Y^\bullet)[1],$$
where $\tau_{\leq 0}$ and $\tau_{\geq 1}$ are the canonical truncations (see notation [Sta22, 0118]), we get that the morphism $H^0\mathbb{R}F(\tau_{\leq 0}Y^\bullet) \to H^0\mathbb{R}F(Y^\bullet)$ is an isomorphism. Considering the long exact sequence induced by the exact triangle of $D(\mathfrak{B})$
$$\mathbb{R}F(\tau_{\leq -n-1}Y^\bullet) \to \mathbb{R}F(\tau_{\leq 0}Y^\bullet) \to \mathbb{R}F(\tau_{\geq -n}\tau_{\leq 0}Y^\bullet) \to \mathbb{R}F(\tau_{\leq -n-1}Y^\bullet)[1],$$
since $\mathbb{R}F(\tau_{\leq -n-1}Y^\bullet) \in D^{\leq -1}(\mathfrak{B})$, then we get that the morphism $H^0\mathbb{R}F(\tau_{\leq 0}Y^\bullet) \to H^0\mathbb{R}F(\tau_{\geq -n}\tau_{\leq 0}Y^\bullet)$ is an isomorphism. Set $Z^\bullet := \tau_{\geq -n}\tau_{\leq 0}Y^\bullet$. Since $Z^\bullet \in D^+(\mathfrak{A})$ and since F is left exact, then we get the spectral sequence (see [Sta22, 015J]):
$$\widetilde{E}_2^{r,s} = R^rF(H^s(Z^\bullet)) \Rightarrow H^{r+s}\mathbb{R}F(Z^\bullet). \tag{4.6.1.6.2}$$

When $s \in [-n,0]$, we have $\widetilde{E}_2^{r,s} = E_2^{r,s}$. Hence, the spectral sequence 4.6.1.6.1 for $r+s=0$ is by definition the spectral sequence 4.6.1.6.2. In other words, we can build "locally in $(r,s) \in [0,n] \times \mathbb{Z}$" the spectral sequence 4.6.1.6.1 by using Cartan-Eilenberg resolutions.

Similarly, we get the spectral sequence
$$E_1^{r,s} = R^sF(Y^r)) \Rightarrow H^{r+s}\mathbb{R}F(Y^\bullet). \tag{4.6.1.6.3}$$

Proposition 4.6.1.7. *Suppose X is affine and noetherian. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorems A and B for coherent modules (see definition 1.4.3.14). Let $\sharp \in \{\emptyset, -, +, \mathrm{b}\}$*

a) Denote by $D^{\sharp}_{\mathrm{ccoh}}(\mathcal{D})$ (resp. $D^{\sharp}_{\mathrm{ccoh}}(D)$) the full subcategory of $D^{\sharp}(\mathcal{D})$ (resp. $D^{\sharp}(D)$) consisting of complexes $\mathcal{E}^{\bullet}$ such that $H^n(\mathcal{E}^{\bullet})$ is a coherent left $\mathcal{D}$-module (resp. coherent left D-module) for any $n \in \mathbb{Z}$. Then the functors $\mathbb{R}\Gamma(X, -)$ and $\mathcal{D} \otimes_D -$ induce canonically quasi-inverse equivalences of categories between $D^{\sharp}_{\mathrm{ccoh}}(D)$ and $D^{\sharp}_{\mathrm{ccoh}}(\mathcal{D})$.

b) Let $\mathcal{E}^{\bullet} \in D^{-}(\mathcal{D})$. The following conditions are equivalent

(i) $\mathcal{E}^{\bullet} \in D^{-}_{\mathrm{coh}}(\mathcal{D})$;

(ii) $\mathcal{E}^{\bullet} \in D^{-}_{\mathrm{ccoh}}(\mathcal{D})$;

(iii) $\mathcal{E}^{\bullet}$ is quasi-isomorphic to a bounded above complex of finite free left $\mathcal{D}$-modules.

c) Suppose moreover X is a scheme and $\mathcal{D}$ is a quasi-coherent $\mathcal{O}_X$-module. We denote by $D^{\sharp}_{\mathrm{qc}}(\mathcal{D})$ the full subcategory of $D^{\sharp}(\mathcal{D})$ consisting of complexes $\mathcal{E}^{\bullet}$ such that $H^n(\mathcal{E}^{\bullet})$ is quasi-coherent as $\mathcal{O}_X$-modules for any $n \in \mathbb{Z}$. Then the functors $\mathbb{R}\Gamma(X, -)$ and $\mathcal{D} \otimes_D -$ induce canonically quasi-inverse equivalences of categories between $D^{\sharp}(D)$ and $D^{\sharp}_{\mathrm{qc}}(\mathcal{D})$ (resp. between $D^{\sharp}_{\mathrm{tdf}}(D)$ and $D^{\sharp}_{\mathrm{qc,tdf}}(\mathcal{D})$).

Proof. a) 0) We denote by $\varpi_{X,*} := \Gamma(X, -)$ and $\varpi_X^* := \mathcal{D} \otimes_D -$. Since $D \to \mathcal{D}$ is flat, since $\mathcal{D}$ be a sheaf of rings on X satisfying theorems A and B for coherent modules, then the functor ϖ_X^* factors through $\varpi_X^* \colon D^{\sharp}_{\mathrm{ccoh}}(D) \to D^{\sharp}_{\mathrm{ccoh}}(\mathcal{D})$.

1) Let $\mathcal{E}^{\bullet} \in D^{\sharp}_{\mathrm{ccoh}}(\mathcal{D})$. We can suppose $\mathcal{E}^{\bullet} \in K^{\sharp}(\mathcal{D})$. Since X is noetherian, then $\varpi_{X,*}$ has bounded cohomological dimension. Hence, then there exist a complex $\mathcal{I}^{\bullet} \in K^{\sharp}(\mathcal{D})$ consisting of $\varpi_{X,*}$-acyclic $\mathcal{D}$-modules and a quasi-isomorphism $\mathcal{E}^{\bullet} \to \mathcal{I}^{\bullet}$ of $K^{\sharp}(\mathcal{D})$ (see the whole paragraph 4.6.1.6.b). Following [Sta22, 07K7], we have the isomorphism of $D^{\sharp}(\mathcal{D})$: $\varpi_{X,*}(\mathcal{I}^{\bullet}) \xrightarrow{\sim} \mathbb{R}\varpi_{X,*}(\mathcal{I}^{\bullet}) \xleftarrow{\sim} \mathbb{R}\varpi_{X,*}(\mathcal{E}^{\bullet})$. Since $\mathcal{I}^{\bullet} \in K^{\sharp}(\mathcal{D})$, then $I^{\bullet} := \varpi_{X,*}(\mathcal{I}^{\bullet}) \in K^{\sharp}(D)$. Since $\varpi_{X,*}$ has bounded cohomological dimension, then following 4.6.1.6.1 we have the spectral sequence $E_2^{r,s} = R^r\varpi_{X,*}(H^s(\mathcal{I}^{\bullet})) \Rightarrow H^{r+s}\mathbb{R}\varpi_{X,*}(\mathcal{I}^{\bullet})$. Since $H^s(\mathcal{I}^{\bullet})$ is $\mathcal{D}$-coherent for all $s \in \mathbb{Z}$, then $H^s(\mathcal{I}^{\bullet})$ is $\varpi_{X,*}$-acyclic (see 1.4.3.14.(iii)). Hence, the canonical map $H^n(I^{\bullet}) = H^n\varpi_{X,*}(\mathcal{I}^{\bullet}) \to \varpi_{X,*}H^n(\mathcal{I}^{\bullet})$ (coming from the above spectral sequence) is an isomorphism of D-modules. Since $H^n(\mathcal{I}^{\bullet})$ is a coherent $\mathcal{D}$-module, since $\mathcal{D}$ satisfies the condition 1.4.3.14.(iii), then this implies that $H^n(I^{\bullet})$ is a coherent D-module (and then $\mathbb{R}\varpi_{X,*}(\mathcal{E}^{\bullet}) \xrightarrow{\sim} I^{\bullet}$ is an object of $D^{\sharp}_{\mathrm{ccoh}}(D)$, i.e. we get the factorisation $\mathbb{R}\varpi_{X,*} \colon D^{\sharp}_{\mathrm{ccoh}}(\mathcal{D}) \to D^{\sharp}_{\mathrm{ccoh}}(D)$) and that the canonical morphism $\mathcal{D} \otimes_D H^n(I^{\bullet}) \to H^n(\mathcal{I}^{\bullet})$ is an isomorphism. Since $D \to \mathcal{D}$ is flat, then from the latter isomorphism we get that the canonical map $\mathcal{D} \otimes_D I^{\bullet} \to \mathcal{I}^{\bullet}$ is a quasi-isomorphism. Hence, the canonical morphism $\varpi_X^* \circ \mathbb{R}\varpi_{X,*}(\mathcal{E}^{\bullet}) \to \mathcal{E}^{\bullet}$ is an isomorphism in $D^{\sharp}_{\mathrm{ccoh}}(\mathcal{D})$.

2) Let $E^{\bullet} \in D^{\sharp}_{\mathrm{ccoh}}(D)$. Since $\mathcal{D} \otimes_D E^{\bullet}$ is a complex of $\varpi_{X,*}$-acyclic modules (see the remark 1.4.3.16), then we can check as above that the canonical map $E^{\bullet} \to \mathbb{R}\varpi_{X*}(\mathcal{D} \otimes_D E^{\bullet})$ is a quasi-isomorphism.

b) The implication $bi) \Rightarrow bii)$ is always true (see 4.6.1.3). Let us now prove the implication $bii) \Rightarrow biii)$. Let $\mathcal{E}^{\bullet} \in D^{-}_{\mathrm{ccoh}}(\mathcal{D})$. From the part a) of the proof, the canonical morphism $\varpi_X^* \circ \mathbb{R}\varpi_{X,*}(\mathcal{E}^{\bullet}) \to \mathcal{E}^{\bullet}$ is an isomorphism in $D^{-}_{\mathrm{ccoh}}(\mathcal{D})$ and $E^{\bullet} := \mathbb{R}\varpi_{X,*}(\mathcal{E}^{\bullet}) \in D^{-}_{\mathrm{ccoh}}(D)$. Since $D^{-}_{\mathrm{ccoh}}(D) = D^{-}_{\mathrm{coh}}(D)$ (see 4.6.1.1), then $E^{\bullet} \in D^{-}_{\mathrm{coh}}(D)$. Recall that by definition this means that $E^{\bullet}$ is quasi-isomorphic to a bounded above complex of finite free left D-modules (see 4.6.1.1). Since $D \to \mathcal{D}$ is flat, then $\varpi_X^* E^{\bullet}$ is quasi-isomorphic to a bounded above complex of finite free left $\mathcal{D}$-modules, and we have therefore proved the implication $bii) \Rightarrow biii)$. Since the implication $biii) \Rightarrow bi)$ is obvious, then we get the equivalence between the conditions of b.

c) To check the non-respective case, we proceed similarly to a) by using theorems of type A and B for quasi-coherent $\mathcal{O}_X$-modules (see 4.1.3.2), which is only valid when X is a scheme. Concerning the respective case, this easily follows from the following property: a quasi-coherent $\mathcal{D}$-module $\mathcal{E}$ on X is flat if and only if $\varpi_{X*}(\mathcal{E})$ is a flat D-module. Let us check this last property. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{D}$-module and set $E := \varpi_{X*}(\mathcal{E})$. Then $\mathcal{E}$ is flat if and only if $\mathcal{E}_x$ is a flat $\mathcal{D}_X$-module for any $x \in X$. Moreover, E is a flat D-module if and only if E_x is a flat D_x-module for any $x \in X$. Since $\mathcal{E}_x \xrightarrow{\sim} E_x$, then we are done. □

Corollary 4.6.1.8. *Suppose X is locally noetherian. Let $\mathcal{D}$ be a sheaf of rings on X. Suppose there exists a covering $\{U_i\}_{i \in I}$ of X by affine opens such that $\mathcal{D}|_{U_i}$ satisfies theorems A and B for coherent* *modules (see definition 1.4.3.14) for any $i \in I$. Let $\mathcal{E}^\bullet \in D^-(\mathcal{D})$. Then the following conditions are equivalent:*

(i) $H^n(\mathcal{E}^\bullet)$ is a coherent left $\mathcal{D}$-module for any $n \in \mathbb{Z}$;

(ii) $\mathcal{E}^\bullet$ is pseudo-coherent.

Corollary 4.6.1.9. *Suppose X is affine and noetherian. Let $\mathcal{D}$ be a sheaf of rings on X satisfying theorems A and B for coherent modules (see definition 1.4.3.14). The canonical functor*

$$D^-(\mathrm{Coh}(\mathcal{D})) \to D^-_{\mathrm{coh}}(\mathcal{D}) \tag{4.6.1.9.1}$$

is an equivalence of categories.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} D^-(\mathrm{Coh}(\mathcal{D})) & \longrightarrow & D^-_{\mathrm{coh}}(\mathcal{D}) \\ {\scriptstyle \mathcal{D}\otimes_D -}\uparrow & & \uparrow{\scriptstyle \mathcal{D}\otimes_D -} \\ D^-(\mathrm{Coh}(D)) & \longrightarrow & D^-_{\mathrm{coh}}(D) \end{array} \tag{4.6.1.9.2}$$

whose horizontal maps are respectively induced by the inclusions $\mathrm{Coh}(\mathcal{D}) \to M(\mathcal{D})$ and $\mathrm{Coh}(D) \to M(D)$, where the vertical arrows are induced by the exact functor $\mathcal{D} \otimes_D -$. Since D is coherent, then the bottom arrow is an equivalence of categories (see 4.6.1.2). Following 4.6.1.7.a, the right arrow of the diagram 4.6.1.9.2 is an equivalence of categories. Since the exact functor $\mathcal{D} \otimes_D -$ induces an equivalence of categories between $\mathrm{Coh}(D)$ and $\mathrm{Coh}(\mathcal{D})$ then so is the left arrow of the diagram 4.6.1.9.2. We conclude by commutativity of 4.6.1.9.2. □

4.6.2 Review on topoi, internal homomorphism

In this subsection, we fix some notation on topos, recall some facts on open subtopoi. This will be useful to understand better the (crucial in the theory of arithmetic $\mathcal{D}$-modules) topos of the form $\mathrm{Top}(X)_I$, where I is a partially ordered set and X is a topological space (7.1.2.12). We also recall the internal homomorphism functor.

4.6.2.1 (Sieves, opens of a site)**.** Let $\mathfrak{C}$ be a site. We denote by $\mathrm{PSh}(\mathfrak{C})$ (resp. $\mathrm{Sh}(\mathfrak{C})$) the category of presheaves (resp. sheaves) of set on $\mathfrak{C}$. Since the following notions are not well known, let us recall them.

(a) A "sieve" of $\mathfrak{C}$ is a full subcategory $\mathfrak{D}$ of $\mathfrak{C}$ such that for any morphism $X \to Y$ of $\mathfrak{C}$ if Y is an object of $\mathfrak{D}$ then so is X (see [SGA4.1, I.4.1]).

(b) We have a bijection between the set of sieves of $\mathfrak{C}$ and the set of subobjects of the final object e of $P\,\mathrm{Sh}(\mathfrak{C})$ described as follows. If $\mathfrak{D}$ is a sieve of $\mathfrak{C}$, then we get the subobject $R_{\mathfrak{D}}$ of e by setting $R_{\mathfrak{D}}(Y) := e(Y)$ if $Y \in \mathrm{Ob}(\mathfrak{D})$ and $R_{\mathfrak{D}}(Y)$ is empty otherwise. Conversely, if R is a subobject of e, then we get the full subcategory $\mathfrak{C}/R$ of $\mathfrak{C}$ (the inclusion is given by $(Y, u\colon Y \to R) \mapsto Y$).

(c) An "open" of $\mathfrak{C}$ is a "sieve of local nature", i.e. is a sieve $\mathfrak{D}$ such that for any covering $(X_i \to X)_{i\in I}$ of $\mathfrak{C}$, if the X_i are an object of $\mathfrak{D}$ for any i then X is an object of $\mathfrak{D}$. As above, we have a bijection between opens of $\mathfrak{C}$ and subobjects in $\mathrm{Sh}(\mathfrak{C})$ of its final object e (see [SGA4.1, IV.8.3]).

4.6.2.2. Let $\mathfrak{C}$ be a site. Let $U \in Ob(\mathfrak{C})$. We turn $\mathfrak{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathfrak{C}/U$ if and only if it is a covering in $\mathfrak{C}$. Consider the forgetful functor $j_U\colon \mathfrak{C}/U \to \mathfrak{C}$. Since j_U is cocontinuous and continuous, then we obtain a morphism of topoi $j_U\colon Sh(\mathfrak{C}/U) \to Sh(\mathfrak{C})$ given by j_U^{-1} and j_{U*}, as well as a functor $j_{U!}$.

Definition 4.6.2.3. Let $\mathfrak{C}$ be a site. Let $U \in Ob(\mathfrak{C})$.

1. The site $\mathfrak{C}/U$ is called the localization of the site $\mathfrak{C}$ at the object U.

2. The morphism of topoi $j_U\colon Sh(\mathfrak{C}/U) \to Sh(\mathfrak{C})$ is called the localization morphism.

3. The functor j_{U*} is called the direct image functor.

4. For a sheaf $\mathcal{F}$ on $\mathfrak{C}$ the sheaf $j_U^{-1}\mathcal{F}$ is called the restriction of $\mathcal{F}$ to $\mathfrak{C}/U$. For any object X/U of $\mathfrak{C}/U$ we have $j_U^{-1}\mathcal{F}(X/U) = \mathcal{F}(X)$.

5. For a sheaf $\mathcal{G}$ on $\mathfrak{C}/U$ the sheaf $j_{U!}(\mathcal{G})$ is called the extension of $\mathcal{G}$ by the empty set.

4.6.2.4. Let $\mathfrak{C}$ be a site. Let $U \in Ob(\mathfrak{C})$. Let $\star$ be the final object of $Sh(\mathfrak{C}/U)$. Since $j_{U!}(\star) = h_U^\sharp$ (use [Sta22, 03CD]), then for any $\mathcal{G} \in Sh(\mathfrak{C}/U)$ we get the functorial in $\mathcal{G}$ map $j_{U!}(\mathcal{G}) \to j_{U!}(\star)$, i.e. we get the functor $Sh(\mathfrak{C}/U) \to Sh(\mathfrak{C})/h_U^\sharp$ making commutative the diagram of categories

$$\begin{array}{ccc} Sh(\mathfrak{C}/U) & \longrightarrow & Sh(\mathfrak{C})/h_U^\sharp \\ & \searrow^{j_{U!}} & \downarrow j_{h_U^\sharp !} \\ & & Sh(\mathfrak{C}) \end{array} \tag{4.6.2.4.1}$$

where $j_{h_U^\sharp !}\colon Sh(\mathfrak{C})/h_U^\sharp \to Sh(\mathfrak{C})$ is by definition the forgetful functor. In fact, following [Sta22, 00Y1] the functor $Sh(\mathfrak{C}/U) \to Sh(\mathfrak{C})/h_U^\sharp$ is an equivalence of categories. Hence, we get a morphism of topoi $j_{h_U^\sharp}\colon Sh(\mathfrak{C})/h_U^\sharp \to Sh(\mathfrak{C})$ such that $j_{h_U^\sharp !}$, $j_{h_U^\sharp}^{-1}$ and $j_{h_U^\sharp *}$ correspond (via the equivalence of categories $Sh(\mathfrak{C}/U) \cong Sh(\mathfrak{C})/h_U^\sharp$) to $j_{U!}$, j_U^{-1} and j_{U*}, i.e., we get the commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathfrak{C}/U) & \xrightarrow{\cong} & Sh(\mathfrak{C})/h_U^\sharp \\ & \searrow^{j_U} & \downarrow j_{h_U^\sharp} \\ & & Sh(\mathfrak{C}). \end{array} \tag{4.6.2.4.2}$$

4.6.2.5. Let $\mathfrak{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathfrak{C}$. We can extend the localisation process 4.6.2.4 to $\mathcal{F}$ as follows. Then following [Sta22, 04GZ] the category $Sh(\mathfrak{C})/\mathcal{F}$ is a topos and there is a canonical morphism of topoi $j_{\mathcal{F}}\colon Sh(\mathfrak{C})/\mathcal{F} \to Sh(\mathfrak{C})$ such that

(i) the functor $j_{\mathcal{F}}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H} \times \mathcal{F}/\mathcal{F}$,

(ii) and the functor $j_{\mathcal{F}!}$ is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$.

The topos $Sh(\mathfrak{C})/\mathcal{F}$ is called the localization of the topos $Sh(\mathfrak{C})$ at $\mathcal{F}$. The morphism of topoi $j_{\mathcal{F}}\colon Sh(\mathfrak{C})/\mathcal{F}) \to Sh(\mathfrak{C})$ is called the localization morphism. We can simply write $\mathcal{F}|_{\mathcal{F}} := j_{\mathcal{F}}^{-1}$.

Let $\mathcal{A}$ be a sheaf of rings on $\mathfrak{C}$. We get the ringed topos $(\mathrm{Sh}(\mathfrak{C}), \mathcal{A})$. Then we get from the localization of the topos $Sh(\mathfrak{C})$ at $\mathcal{F}$ the morphism of ringed topoi $j_{\mathcal{F}}\colon (Sh(\mathfrak{C})/\mathcal{F}, \mathcal{A}|_{\mathcal{F}}) \to (Sh(\mathfrak{C}), \mathcal{A})$.

4.6.2.6. Let $\mathfrak{C}$ be a site. Following [Sta22, 08LW], the following assertions are equivalent for any sheaf $\mathcal{F}$ on $\mathfrak{C}$:

(a) The sheaf $\mathcal{F}$ is an open of $Sh(\mathfrak{C})$, i.e. $\mathcal{F}$ is a subobject of the final object of $Sh(\mathfrak{C})$.

(b) The morphism $j_{\mathcal{F}}$ is an embedding, i.e. $j_{\mathcal{F}*}\colon Sh(\mathfrak{C})/\mathcal{F} \to Sh(\mathfrak{C})$ is fully faithful.

A strictly full subcategory $\mathfrak{E} \subset Sh(\mathfrak{C})$ is by definition an "open subtopos of $Sh(\mathfrak{C})$" if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(C)$ such that $\mathfrak{E}$ is the essential image of $j_{\mathcal{F}}$ (see Definition [Sta22, 08LX]). Finally, a morphism of topoi $f\colon Sh(\mathfrak{D}) \to Sh(\mathfrak{C})$ is said to be an "open immersion" if f is an embedding (i.e. f_* is fully faithful) and the essential image of f_* is an open subtopos. We will use the open immersion given at 7.1.2.18.3.

4.6.2.7. Let $\mathfrak{C}$ be a site. Let $\mathcal{A}, \mathcal{B}$ be two (not necessarily commutative) sheaves of rings on $\mathfrak{C}$. Writing $\mathscr{T} := \mathrm{Sh}(\mathfrak{C})$. We denote by $\mathbb{Z}_{\mathscr{T}}$ be the sheaf of $\mathscr{T}$ associated with the constant presheaf of $\mathscr{T}$ with value $\mathbb{Z}$.

(a) Let e be a final object of $\mathrm{Sh}(\mathfrak{C})$. The global sections functor $\Gamma(\mathscr{T}, -)\colon \mathscr{T} \to \mathrm{Sets}$, also denoted by $\Gamma(\mathfrak{C}, -)\colon \mathrm{Sh}(\mathfrak{C}) \to \mathrm{Sets}$, is defined by setting for any object K of $\mathscr{T}$,

$$\Gamma(\mathscr{T}, K) := \mathrm{Hom}_{\mathscr{T}}(e, K). \tag{4.6.2.7.1}$$

(b) Following [SGA4.1, IV.10.1], for any objects E, F of $\mathscr{T}$, Grothendieck has constructed an object of $\mathscr{T}$ that we will denote by $\mathscr{H}om_{\mathscr{T}}(E, F)$ which is characterized by the property: for any object K of $\mathscr{T}$,

$$\mathrm{Hom}_{\mathscr{T}}(K, \mathscr{H}om_{\mathscr{T}}(E, F)) = \mathrm{Hom}_{\mathscr{T}|_K}(E|_K, F|_K)). \tag{4.6.2.7.2}$$

With 4.6.2.7.1, in the case where K is a final object, this yields

$$\Gamma(\mathscr{T}, \mathscr{H}om_{\mathscr{T}}(E, F)) = \mathrm{Hom}_{\mathscr{T}}(E, F). \tag{4.6.2.7.3}$$

(c) Following [SGA4.1, IV.12.1], for any left (resp. right) $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, Grothendieck has constructed an abelian sheaf on $\mathscr{T}$ that we will denote by $\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ which is characterized by the property: with notation 4.6.2.5, for any object K of $\mathscr{T}$,

$$\mathrm{Hom}_{\mathscr{T}}(K, \mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})) = \mathrm{Hom}_{\mathcal{A}|_K}(\mathcal{E}|_K, \mathcal{F}|_K)). \tag{4.6.2.7.4}$$

With 4.6.2.7.1, in the case where K is a final object, this yields

$$\Gamma(\mathscr{T}, \mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}). \tag{4.6.2.7.5}$$

When $\mathcal{F}$ is injective, then $\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is flasque (see [SGA4.2, V.4.10.2)]) and then is acyclic for the functor $\Gamma(\mathscr{T}, -)$. Hence, this yields from [Sta22, 05TA-13.16.8] that for any $\mathcal{E}^\bullet \in D^-(\mathcal{A})$, $\mathcal{F}^\bullet \in D^+(\mathcal{A})$ we have the isomorphism

$$\mathbb{R}\Gamma(\mathscr{T}, \mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)) = \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet). \tag{4.6.2.7.6}$$

Following [SGA4.1, IV.12.6], for any object K of $\mathscr{T}$ and left $\mathcal{A}$-module $\mathcal{F}$, we have the isomorphism of left $\mathcal{A}$-modules:

$$\mathscr{H}om_{\mathscr{T}}(K, \mathcal{F}) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{A}}(j_{K!}j_K^*\mathcal{A}, \mathcal{F}), \tag{4.6.2.7.7}$$

where $j_K \colon \mathscr{T}/K \to \mathscr{T}$ is the localisation morphism and the functors $j_{K!}$ and j_K^* are explicitely defined at [SGA4.1, IV.5.2]. Hence, if $\mathcal{F}$ is injective then $\mathscr{H}om_{\mathscr{T}}(K, \mathcal{F})$ is flasque. This yields from [Sta22, 05TA-13.16.8] that for any $K \in \mathscr{T}$, $\mathcal{F}^\bullet \in D^+(\mathcal{A})$ we have the isomorphism

$$\mathbb{R}\Gamma(\mathscr{T}, \mathbb{R}\mathscr{H}om_{\mathscr{T}}(K, \mathcal{F}^\bullet)) = \mathbb{R}\mathrm{Hom}_{\mathscr{T}}(K, \mathcal{F}^\bullet). \tag{4.6.2.7.8}$$

(d) Let $\mathcal{E}$ be an abelian sheaf on $\mathscr{T}$. The data of a left $(\mathcal{A}, \mathcal{B})$-bimodule on $\mathcal{E}$ (extending its structure of abelian sheaf) is equivalent to that of a left $\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{B}$-module. Let $\mathcal{E}, \mathcal{F}$ be two left $(\mathcal{A}, \mathcal{B})$-bimodules. We denote by $\mathrm{Hom}_{({}^l\mathcal{A}, {}^l\mathcal{B})}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{B}}(\mathcal{E}, \mathcal{F})$ the set of $(\mathcal{A}, \mathcal{B})$-bilinear homomorphism. We denote by $\mathscr{H}om_{({}^l\mathcal{A}, {}^l\mathcal{B})}(\mathcal{E}, \mathcal{F}) := \mathscr{H}om_{\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{B}}(\mathcal{E}, \mathcal{F})$, which is characterized by the property: with notation 4.6.2.5, for any object K of $\mathscr{T}$,

$$\mathrm{Hom}_{\mathscr{T}}(K, \mathscr{H}om_{({}^l\mathcal{A}, {}^l\mathcal{B})}(\mathcal{E}, \mathcal{F})) = \mathrm{Hom}_{({}^l\mathcal{A}|_K, {}^l\mathcal{B}|_K)}(\mathcal{E}|_K, \mathcal{F}|_K)). \tag{4.6.2.7.9}$$

We have similar notation for $(\mathcal{A}, \mathcal{B})$-bimodules or right $(\mathcal{A}, \mathcal{B})$-bimodules.

4.6.3 Derived tensor products, derived homomorphism functors, Cartan isomorphisms in general

Notation 4.6.3.1. Let $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{C}$ be four sheaves of rings (not necessarily commutative) on some topos $\mathscr{T}$.

Following [SGA4.1, IV.12.1], for any left (resp. right) $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, Grothendieck has constructed an abelian sheaf on $\mathscr{T}$ that we will denote by $\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ which is characterized by the property: with notation 4.6.2.5, for any object K of $\mathscr{T}$,

$$\mathrm{Hom}_{\mathscr{T}}(K, \mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})) = \mathrm{Hom}_{\mathcal{A}|_K}(\mathcal{E}|_K, \mathcal{F}|_K)). \tag{4.6.3.1.1}$$

When the topos $\mathscr{T}$ it the topos associated with a topological space, then we prefer to write $\mathcal{H}om$ instead of $\mathscr{H}om$. When $\mathcal{E}$ is a $(\mathcal{C}, \mathcal{A})$-bimodule and $\mathcal{F}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule, then $\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is endowed with a canonical structure of $(\mathcal{C}, \mathcal{B})$-bimodule (see [SGA4.1, IV.12.5]). For any $* \in \{l, r\}$, we define the bifunctor

$$\mathscr{H}om^\bullet_{\mathcal{A}}(-,-) \colon K({}^{\mathrm{l}}\mathcal{C}, {}^*\mathcal{A}) \times K({}^{\mathrm{l}}\mathcal{B}, {}^*\mathcal{A}) \to K({}^{\mathrm{l}}\mathcal{B}, \mathcal{C}^{\mathrm{r}}). \tag{4.6.3.1.2}$$

by setting for any $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{C}, \mathcal{A}^{\mathrm{r}})$, $\mathcal{F}^\bullet \in K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$, for any integer $n \in \mathbb{Z}$ setting:

$$\mathscr{H}om^n_{\mathcal{A}}(\mathcal{E}^\bullet,\ \mathcal{F}^\bullet) := \prod_{p\in\mathbb{Z}} \mathscr{H}om_{\mathcal{A}}(\mathcal{E}^p,\ \mathcal{F}^{p+n}) \tag{4.6.3.1.3}$$

and the transition morphisms are given by the formula $d = d_{\mathcal{E}} + (-1)^n d_{\mathcal{F}}$. We have the relation:

$$H^n(\Gamma(\mathscr{T}, \mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{E}, \mathcal{F}))) = \mathrm{Hom}_{K(\mathcal{A})}(\mathcal{E}, \mathcal{F}[n]). \tag{4.6.3.1.4}$$

For any $(\mathcal{C}, \mathcal{A})$-bimodule $\mathcal{E}$ and $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{F}$, the abelian sheaf $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ is fact a $(\mathcal{C}, \mathcal{B})$-bimodule. We define the bifunctor

$$- \otimes_{\mathcal{A}} -\colon K({}^*\mathcal{B}, \mathcal{A}^{\mathrm{r}}) \times K({}^{\mathrm{l}}\mathcal{A}, {}^{**}\mathcal{C}) \to K({}^*\mathcal{B}, {}^{**}\mathcal{C}), \tag{4.6.3.1.5}$$

by setting for any $\mathcal{E}^\bullet \in K({}^*\mathcal{B}, \mathcal{A}^{\mathrm{r}})$, $\mathcal{F}^\bullet \in K({}^{\mathrm{l}}\mathcal{A}, {}^{**}\mathcal{C})$, for any integer $n \in \mathbb{Z}$ setting:

$$(\mathcal{E}^\bullet \otimes_{\mathcal{A}} \mathcal{F}^\bullet)^n := \sum_{p\in\mathbb{Z}} \mathcal{E}^p \otimes_{\mathcal{A}} \mathcal{F}^{n-p} \tag{4.6.3.1.6}$$

and the transition morphisms are given by the formula $d = d_{\mathcal{E}} + (-1)^n d_{\mathcal{F}}$. We have similar bifunctors by changing the indices l and r.

4.6.3.2. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three sheaves of rings on some site $\mathscr{T}$. In this paragraph, e.g. in order to get some associativity of the tensor product, we would like to extend the functor

$$\mathbb{R}\mathscr{H}om_{\mathcal{A}}(-,-)\colon D({}^{\mathrm{l}}\mathcal{B}, {}^*\mathcal{A}) \times D({}^*\mathcal{A}) \to D({}^{\mathrm{r}}\mathcal{B}), \tag{4.6.3.2.1}$$

$$- \otimes^{\mathbb{L}}_{\mathcal{A}} -\colon D({}^*\mathcal{B}, \mathcal{A}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{A}) \to D({}^*\mathcal{B}). \tag{4.6.3.2.2}$$

Let $\mathcal{E} \in K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$, let $\mathcal{F} \in K({}^{\mathrm{l}}\mathcal{A}, \mathcal{C}^{\mathrm{r}})$, $\mathcal{G} \in K({}^{\mathrm{l}}\mathcal{C}, \mathcal{A}^{\mathrm{r}})$.

(a) Consider the functor $- \otimes_{\mathcal{A}} \mathcal{F}\colon K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{B}, \mathcal{C}^{\mathrm{r}})$ defined by $\mathcal{E}' \mapsto \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{F}$. In general, it is not clear that the left derived functor (localizing by quasi-isomorphisms) of $- \otimes_{\mathcal{A}} \mathcal{F}$ is defined at $\mathcal{E}$ (see Definition [Sta22, 05S9]). Indeed, this is not clear that there exists a complex of $(\mathcal{B}, \mathcal{A})$-bimodules representing $\mathcal{E}$ which is moreover a K-flat complex of $\mathcal{A}$-modules.

Similarly, since the existence of a complex of $(\mathcal{B}, \mathcal{A})$-bimodules representing $\mathcal{E}$ which is moreover a K-injective complex of $\mathcal{A}$-modules is not clear then it seems problematic to assert the functor $\mathscr{H}om_{\mathcal{A}}(\mathcal{G}, -)\colon K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{B}, \mathcal{C}^{\mathrm{r}})$ has a right derived functor at $\mathcal{E}$.

(b) However, under Berthelot's practical hypotheses below, theses derived functors can be computed as follows. Let us introduce the following definition. A commutative ring $\mathcal{R}$ is said to be a *left solving ring* (resp. *right solving ring*, resp. *solving ring*) of $(\mathcal{A}, \mathcal{B})$ if the following conditions are satisfied:

   (i) There exist ring homomorphisms $\mathcal{R} \to \mathcal{A}$ and $\mathcal{R} \to \mathcal{B}$ such that $\mathcal{R}$ is sent to the center of $\mathcal{A}$ and of $\mathcal{B}$ (in other words the ring structures of $\mathcal{A}$ and $\mathcal{B}$ are induced by a unital and associative $\mathcal{R}$-algebra structure) ;

   (ii) $\mathcal{B}$ is flat on $\mathcal{R}$ (resp. $\mathcal{A}$ is flat on $\mathcal{R}$, resp. $\mathcal{A}$ and $\mathcal{B}$ are flat on $\mathcal{R}$).

In that case, we say that the pair of rings $(\mathcal{A}, \mathcal{B})$ is left solvable (resp. right solvable, resp. solvable).

Let $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$. Let $\mathcal{R}$ be a left solving ring (resp. right solving ring, resp. solving ring) of $(\mathcal{A}, \mathcal{B})$. Then the following properties are equivalent

(1) The structures of $\mathcal{R}$-module induced on the $(\mathcal{B}, \mathcal{A})$-bimodules $\mathcal{E}^n$ by the structure of right $\mathcal{A}$-module and by that of left $\mathcal{B}$-module coincide for any $n \in \mathbb{Z}$;

(2) $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^o)$.

The complex $\mathcal{E}$ is then said to be *left solvable by* $\mathcal{R}$ (resp. *right solvable by* $\mathcal{R}$, resp. *solvable by* $\mathcal{R}$) as object of $K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$. The complex $\mathcal{E}$ is then said to be *left solvable* (resp. *right solvable*, resp. *solvable*) as object of $K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$ if there exists an left solving ring $\mathcal{R}$ (resp. right solving ring, resp. solving ring) of $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^o)$.

Let $\mathcal{R}$ be a left or right solving ring of $(\mathcal{A}, \mathcal{B})$. We denote by $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{R}, \mathcal{A}^{\mathrm{r}})$ the strictly full subcategory of $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$ consisting of complexes isomorphic in $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$ to a complex which is also an object of $K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{o})$. We denote by $D_{\mathrm{l-sol}}({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$ (resp. $D_{\mathrm{r-sol}}({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$, resp. $D_{\mathrm{sol}}({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$) the strictly full subcategory of $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$ consisting of complexes isomorphic to some left solvable (resp. right solvable, resp. solvable) complex.

Beware that the sheaf $\mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp}$ is solvable as $(\mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp}, \mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp})$-bimodule but is not (a priori) solvable as $(\mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp}, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$-bimodule.

(c) Assume $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{B}, \mathcal{A}^{\mathrm{r}})$ has a right solving ring $\mathcal{R}$ and we can view $\mathcal{E}$ as a complex of left $\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{o}$-modules (where $\mathcal{A}^{o}$ is the ring $\mathcal{A}$ with the opposite multiplication). By using (bi), $\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{o}$ can be endowed with a canonical ring structure. It follows from (bii) that the ring homomorphism $\mathcal{A}^{0} \to \mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{0}$ is flat. Hence, a K-flat complex of left $\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{0}$-modules representing $\mathcal{E}$, is also a K-flat complex of right $\mathcal{A}$-modules representing $\mathcal{E}$. This implies the functor $L(- \otimes_{\mathcal{A}} \mathcal{F})$ is defined at $\mathcal{E}$ and we write $\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{F} := L(- \otimes_{\mathcal{A}} \mathcal{F})(\mathcal{E}) \in D({}^{\mathrm{l}}\mathcal{B}, \mathcal{C}^{\mathrm{r}})$.

Moreover, a K-injective complex of left $\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{0}$-modules representing $\mathcal{E}$ (see definition [Sta22, 070H]) is also a K-injective complex of right $\mathcal{A}$-modules because $\mathcal{A}^{0} \to \mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{0}$ is flat. Hence, $R\mathscr{H}om_{\mathcal{A}}(\mathcal{G}, -)$ is defined at $\mathcal{E}$ and we set $\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{E}) := R\mathscr{H}om_{\mathcal{A}}(\mathcal{G}, -)(\mathcal{E})$.

(d) A commutative ring $\mathcal{R}$ on $\mathscr{T}$ is said to be a *solving ring of* $\mathcal{A}$ if it is a solving ring of $\mathcal{A}$ as a $(\mathcal{A}, \mathcal{A})$-bimodule, i.e. if the ring structure of $\mathcal{A}$ comes from a structure of unital and associative $\mathcal{R}$-algebra.

(e) Suppose there exists a left solving ring $\mathcal{R}$ of $(\mathcal{A}, \mathcal{B})$. For any $*, ** \in \{\mathrm{r}, \mathrm{l}\}$, we obtain the bifunctors

$$\mathbb{R}\mathscr{H}om_{\mathcal{A}}(-,-) \colon D({}^{\mathrm{l}}\mathcal{C}, {}^{*}\mathcal{A}) \times D({}^{**}\mathcal{B}, \mathcal{R}, {}^{*}\mathcal{A}) \to D({}^{**}\mathcal{B}, {}^{\mathrm{r}}\mathcal{C}), \tag{4.6.3.2.3}$$

$$- \otimes^{\mathbb{L}}_{\mathcal{A}} - \colon D({}^{*}\mathcal{B}, \mathcal{R}, \mathcal{A}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{A}, {}^{**}\mathcal{C}) \to D({}^{*}\mathcal{B}, {}^{**}\mathcal{C}). \tag{4.6.3.2.4}$$

If moreover $(\mathcal{A}, \mathcal{C})$ is right or left solvable by $\mathcal{R}$, then we get the functors

$$\mathbb{R}\mathscr{H}om_{\mathcal{A}}(-,-) \colon D({}^{\mathrm{l}}\mathcal{C} \otimes_{\mathcal{R}} \mathcal{A}) \times D({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}) \to D({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{C}^{o}), \tag{4.6.3.2.5}$$

$$- \otimes^{\mathbb{L}}_{\mathcal{A}} - \colon D({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{A}^{o}) \times D({}^{\mathrm{l}}\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C}) \to D({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}} \mathcal{C}), \tag{4.6.3.2.6}$$

$$\mathbb{R}\mathscr{H}om_{\mathcal{A}}(-,-) \colon D({}^{\mathrm{l}}\mathcal{C}, \mathcal{R}, {}^{*}\mathcal{A}) \times D({}^{**}\mathcal{B}, \mathcal{R}, {}^{*}\mathcal{A}) \to D({}^{**}\mathcal{B}, \mathcal{R}, {}^{\mathrm{r}}\mathcal{C}), \tag{4.6.3.2.7}$$

$$- \otimes^{\mathbb{L}}_{\mathcal{A}} - \colon D({}^{*}\mathcal{B}, \mathcal{R}, \mathcal{A}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{A}, \mathcal{R}, {}^{**}\mathcal{C}) \to D({}^{*}\mathcal{B}, \mathcal{R}, {}^{**}\mathcal{C}). \tag{4.6.3.2.8}$$

Example 4.6.3.3. We will use essentially in this book the following cases.

(a) Suppose $X^\sharp \to S^\sharp$ is quasi-flat (see Definition 3.1.1.5 or respectively 3.3.1.11). Then $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a solvable ring. More precisely, choose a morphism of schemes $S \to B$ such that the composition morphism $g \colon X \to S \to B$ is flat (see 3.1.1.5). Then $g^{-1}\mathcal{O}_B$ is a solving ring of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$. This yields the bifunctors:

$$\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}(-,-) \colon D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, {\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, g^{-1}\mathcal{O}_B, {\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, {\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}^{\mathrm{r}}), \tag{4.6.3.3.1}$$

$$- \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} - \colon D({}^{*}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, {\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, g^{-1}\mathcal{O}_B, {}^{*}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{*}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, {}^{*}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}). \tag{4.6.3.3.2}$$

(b) More generally, suppose that $\mathcal{B}_X$ is a quasi-flat $\mathcal{O}_S$-algebra (see Definition 3.1.1.5 or respectively 3.3.1.11). Then $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a solvable ring. More precisely, choose a morphism of schemes $S \to B$ such that the composition morphism $\widetilde{g} \colon (X, \mathcal{B}_X) \to B$ is flat. Then $\widetilde{g}^{-1}\mathcal{O}_B$ is a solving ring of $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.

For example, suppose $X^\sharp$ is a p-torsion free log smooth $S^\sharp$-formal log scheme. For any integer $i \geq 0$, set $X_i^\sharp := X^\sharp \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$. Let Z be a divisor of X_0. Then it follows from 8.7.4.2

that we can choose $\mathcal{V}/\pi^{i+1}\mathcal{V}$ as a solving ring of $\mathcal{B}_{X_i}(Z,r) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$ and $\mathcal{V}$ as a solving ring of $\varprojlim_i \mathcal{B}_{X_i}(Z,r) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$.

(c) To get nice properties of pullbacks and pushforwards in the theory of arithmetic $\mathcal{D}$-modules, we will see later in 5.1.1.3 other important cases.

4.6.3.4. Let $\mathcal{A}$, $\mathcal{B}$ be two sheaves of rings on some site $\mathscr{T}$. Suppose there exists a right solving ring $\mathcal{R}$ of $(\mathcal{A},\mathcal{B})$. For any $? \in \{\mathrm{tdf}, \mathrm{perf}, \mathrm{coh}\}$, we denote by $D_{.,?}({}^{\mathrm{l}}\mathcal{B},\mathcal{R},\mathcal{A}^{\mathrm{r}})$ (resp. $D_{.,?}({}^{\mathrm{l}}\mathcal{B},\mathcal{A}^{\mathrm{r}})$) the strictly full subcategory of $D({}^{\mathrm{l}}\mathcal{B},\mathcal{R},\mathcal{A}^{\mathrm{r}})$ (resp. $D_{\mathrm{r-sol}}({}^{\mathrm{l}}\mathcal{B},\mathcal{A}^{\mathrm{r}})$) consisting of complexes $\mathcal{E}$ whose image in $D(\mathcal{A}^{\mathrm{r}})$ belongs to $D_?(\mathcal{A}^{\mathrm{r}})$. We use similar notation when ? is on the left etc.

Concerning finite tor dimension complexes, we have the following remark: if $\mathcal{E} \in D_{.,\mathrm{tdf}}({}^{\mathrm{l}}\mathcal{B},\mathcal{R},\mathcal{A}^{\mathrm{r}})$, then $\mathcal{E}$ is isomorphism in $D({}^{\mathrm{l}}\mathcal{B},\mathcal{A}^{\mathrm{r}})$ to a complex which is also an object of $K^{\mathrm{b}}({}^{\mathrm{l}}\mathcal{B}\otimes_{\mathcal{R}}\mathcal{A}^o)$ whose term are flat as right $\mathcal{A}$-modules (moreover, except the first nonzero term, the other terms are flat left $\mathcal{B}\otimes_{\mathcal{R}}\mathcal{A}^o$-modules).

Proposition 4.6.3.5. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be four sheaves of rings on some topos $\mathscr{T}$. Let $*, ** \in \{r,l\}$. Let $\mathcal{E}^\bullet \in D_{\mathrm{r-sol}}({}^*\mathcal{B},\mathcal{A}^{\mathrm{r}})$, $\mathcal{F}^\bullet \in D({}^{\mathrm{l}}\mathcal{A},{}^{\mathrm{r}}\mathcal{C})$, $\mathcal{G}^\bullet \in D_{\mathrm{l-sol}}({}^{\mathrm{l}}\mathcal{C},{}^{**}\mathcal{D})$. The associativity of the derived tensor product holds, i.e. we have the isomorphism in $D({}^*\mathcal{B},{}^{**}\mathcal{D})$:*

$$\left(\mathcal{E}^\bullet \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{F}^\bullet\right) \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{G}^\bullet \xrightarrow{\sim} \mathcal{E}^\bullet \otimes^{\mathbb{L}}_{\mathcal{A}} \left(\mathcal{F}^\bullet \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{G}^\bullet\right). \tag{4.6.3.5.1}$$

Proof. Let $\mathcal{R}$ be a right solving ring of $\mathcal{E}$. Then we can choose a K-flat complex (of left $\mathcal{B}\otimes_{\mathcal{R}}\mathcal{A}^0$-modules if $* = l$ and of left $\mathcal{B}^0 \otimes_{\mathcal{R}} \mathcal{A}^0$-modules if $* = r$) representing $\mathcal{E}^\bullet$. Similarly, by choosing a left solving ring $\mathcal{R}'$ of $\mathcal{G}^\bullet$ and a K-flat complex representing $\mathcal{G}^\bullet$, we reduce to the case of the non-derived associativity isomorphism of the tensor product, which is well known. □

Proposition 4.6.3.6. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be sheaves of rings on some topos $\mathscr{T}$. Let $\mathcal{R}$ be either a left solving ring of $(\mathcal{A},\mathcal{B})$, $(\mathcal{B},\mathcal{C})$ and $(\mathcal{C},\mathcal{D})$ or, only in the case where $\mathcal{C} = \mathcal{D} = \mathbb{Z}_{\mathscr{T}}$, let $\mathcal{R}$ be a left solving ring of $(\mathcal{A},\mathcal{B})$. Let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A})$, $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{A},\mathcal{R},\mathcal{B}^{\mathrm{r}})$, and $\mathcal{G} \in D({}^{\mathrm{l}}\mathcal{B},\mathcal{R},\mathcal{C}^{\mathrm{r}})$ (or $\mathcal{G} \in D({}^{\mathrm{l}}\mathcal{B})$ in the case $\mathcal{C} = \mathbb{Z}_{\mathscr{T}}$).*

(i) Then there exists a canonical functorial in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ homomorphism in $D({}^*\mathcal{C})$:

$$\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{G} \to \mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{G}). \tag{4.6.3.6.1}$$

This one is an isomorphism if $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{A})$.

(ii) The isomorphism of i) is transitive, i.e., if $\mathscr{H} \in D({}^{\mathrm{l}}\mathcal{C},\mathcal{R},\mathcal{D}^{\mathrm{r}})$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{G} \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{H} & \longrightarrow & \mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{G} \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{H}) \\
 & \searrow \qquad \nearrow & \\
 & \mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{G}) \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{H}. &
\end{array}$$

Proof. For the first part, it is sufficient to choose a K-injective complex representing $\mathcal{F}$ and a K-flat complex representing $\mathcal{G}$. The transitivity of this morphism is straightforward. □

Proposition 4.6.3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be some sheaves of rings on some topos $\mathscr{T}$. Let $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{A})$, $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{l-sol}}({}^{\mathrm{l}}\mathcal{A},{}^*\mathcal{B})$. Suppose $\mathscr{T}$ is coherent. We have :*

(a) $\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F}) \in D^{\mathrm{b}}({}^*\mathcal{B})$,

(b) If $\mathcal{F} \in D_{(.,\mathrm{perf})}({}^{\mathrm{l}}\mathcal{A},{}^*\mathcal{B})$ (resp. $D_{(.,\mathrm{tdf})}({}^{\mathrm{l}}\mathcal{A},{}^*\mathcal{B})$), then $\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F}) \in D_{\mathrm{perf}}({}^*\mathcal{B})$ (resp. $\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{F}) \in D_{\mathrm{tdf}}({}^*\mathcal{B})$).

Proof. This is checked by localisation and devissage (and use 4.6.3.6.1 to check the finiteness of the tor dimension). □

Remark 4.6.3.8. In proposition 4.6.3.7, if $\mathscr{T}$ is not coherent, then we have to replace "bounded complexes" by "locally bounded complexes" and "complexes of finite tor-dimension" by "complexes locally of finite tor-dimension".

The following lemma is obvious.

Lemma 4.6.3.9 (Cartan isomorphism)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three sheaves of rings on some topos $\mathscr{T}$.*

(a) Let $\mathcal{E}$ be a $(\mathcal{A},\mathcal{B})$-bimodule, $\mathcal{F}$ be a left $(\mathcal{B},\mathcal{C})$-bimodule and $\mathcal{G}$ be a left $(\mathcal{A},\mathcal{C})$-bimodule. We have the functorial isomorphisms:

$$\mathscr{H}om_{({}^{\mathrm{l}}\mathcal{A},{}^{\mathrm{l}}\mathcal{C})}(\mathcal{E}\otimes_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathscr{H}om_{({}^{\mathrm{l}}\mathcal{B},{}^{\mathrm{l}}\mathcal{C})}(\mathcal{F},\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G}))$$
$$\mathrm{Hom}_{({}^{\mathrm{l}}\mathcal{A},{}^{\mathrm{l}}\mathcal{C})}(\mathcal{E}\otimes_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{({}^{\mathrm{l}}\mathcal{B},{}^{\mathrm{l}}\mathcal{C})}(\mathcal{F},\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G})). \tag{4.6.3.9.1}$$

(b) Let $\mathcal{R}$ be a commutative sheaf of rings on $\mathscr{T}$ endowed with ring morphisms $\mathcal{R}\to\mathcal{A}$, $\mathcal{R}\to\mathcal{B}$ and $\mathcal{R}\to\mathcal{C}$ whose respective images are included in the center of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Let $\mathcal{E}$ be a left $\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}^{\mathrm{o}}$-bimodule, $\mathcal{F}$ be a left $\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C}$-bimodule and $\mathcal{G}$ be a left $\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C}$-module. We have the functorial isomorphisms:

$$\mathscr{H}om_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{E}\otimes_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{F},\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G}))$$
$$\mathrm{Hom}_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{E}\otimes_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{F},\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G})). \tag{4.6.3.9.2}$$

Lemma 4.6.3.10. *Let $\mathcal{A}$, $\mathcal{B}$ be two sheaves of rings on some topos $\mathscr{T}$, $\mathcal{P}^\bullet\in K({}^{\mathrm{l}}\mathcal{A},{}^{\mathrm{r}}\mathcal{B})$, $\mathcal{I}^\bullet\in K({}^{\mathrm{l}}\mathcal{A})$. If $\mathcal{P}^\bullet$ is K-flat as an object of $K({}^{\mathrm{r}}\mathcal{B})$ and $\mathcal{I}^\bullet$ is K-injective as a complex of $K({}^{\mathrm{l}}\mathcal{A})$, then $\mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{P}^\bullet,\mathcal{I}^\bullet)$ is K-injective as an object of $K({}^{\mathrm{l}}\mathcal{B})$.*

Proof. For any acyclic complex $\mathcal{F}\in K({}^{\mathrm{l}}\mathcal{B})$, we get the isomorphisms

$$\mathrm{Hom}_{K(\mathcal{B})}\left(\mathcal{F}^\bullet,\mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{P}^\bullet,\mathcal{I}^\bullet)\right) \underset{4.6.3.1.4}{\xrightarrow{\sim}} H^0\Gamma\left(\mathscr{T},\mathscr{H}om^\bullet_{\mathcal{B}}\left(\mathcal{F}^\bullet,\mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{P}^\bullet,\mathcal{I}^\bullet)\right)\right)$$
$$\overset{4.6.3.9.1}{\xrightarrow{\sim}} H^0\Gamma\left(\mathscr{T},\mathscr{H}om^\bullet_{\mathcal{A}}\left(\mathcal{P}^\bullet\otimes_{\mathcal{B}}\mathcal{F}^\bullet,\mathcal{I}^\bullet\right)\right) \underset{4.6.3.1.4}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{A}}\left(\mathcal{P}^\bullet\otimes_{\mathcal{B}}\mathcal{F}^\bullet,\mathcal{I}^\bullet\right),$$

where 4.6.3.9.1 is used in the case $\mathcal{C}=\mathbb{Z}$. Since $\mathcal{P}^\bullet$ is K-flat as an object of $K({}^{\mathrm{r}}\mathcal{B})$, since $\mathcal{F}$ is acyclic, then $\mathcal{P}^\bullet\otimes_{\mathcal{B}}\mathcal{F}^\bullet$ is an acyclic object of $K({}^{\mathrm{l}}\mathcal{A})$. Since $\mathcal{I}^\bullet$ is K-injective as a complex of $K({}^{\mathrm{l}}\mathcal{A})$, then $\mathrm{Hom}_{K(\mathcal{A})}\left(\mathcal{P}^\bullet\otimes_{\mathcal{B}}\mathcal{F}^\bullet,\mathcal{I}^\bullet\right)=0$ and we are done. □

Proposition 4.6.3.11 (Cartan isomorphisms)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be two sheaves of rings on some topos $\mathscr{T}$, $\mathcal{E}\in D_{\mathrm{r-sol}}({}^{\mathrm{l}}\mathcal{A},\mathcal{B}^{\mathrm{r}})$, $\mathcal{F}\in D({}^{\mathrm{l}}\mathcal{B})$ and $\mathcal{G}\in D({}^{\mathrm{l}}\mathcal{A})$. We have the functorial isomorphisms:*

$$\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathcal{E}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathcal{B}}(\mathcal{F},\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G}))$$
$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{F},\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G}))$$
$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{E}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{B})}(\mathcal{F},\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G})). \tag{4.6.3.11.1}$$

Proof. Let us choose $\mathcal{R}$, a right solving ring of $\mathcal{E}$ (see 4.6.3.2). Let $\mathcal{P}$ be a K-flat complex of left $\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}^{\mathrm{o}}$-modules representing $\mathcal{E}$ and $\mathcal{I}$ be a K-injective complex of left $\mathcal{A}$-modules representing $\mathcal{G}$. These resolutions allow us to compute the left terms of the isomorphisms of the proposition. Moreover, since $\mathcal{P}^\bullet$ is K-flat as an object of $K({}^{\mathrm{r}}\mathcal{B})$ and $\mathcal{I}^\bullet$ is K-injective as a complex of $K(\mathcal{A})$, then it follows from 4.6.3.10 that $\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{P},\mathcal{I})$ is a K-injective complex of left $\mathcal{B}$-modules. Hence, these two resolutions compute also the right terms of the isomorphisms of the proposition. □

Corollary 4.6.3.12. *Let $\mathcal{A}$, $\mathcal{B}$ be two sheaves of rings on some topos $\mathscr{T}$. Let $\mathcal{E}\in D_{\mathrm{r-sol}}({}^{\mathrm{l}}\mathcal{A},\mathcal{B}^{\mathrm{r}})$, $\mathcal{F}\in D({}^{\mathrm{l}}\mathcal{B})$ and $\mathcal{G}\in D({}^{\mathrm{l}}\mathcal{A})$. We have the functorial in $\mathcal{E}$ and $\mathcal{G}$ morphism of $D({}^{\mathrm{l}}\mathcal{A})$:*

$$\mathcal{E}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G})\to\mathcal{G}. \tag{4.6.3.12.1}$$

Proof. The arrow 4.6.3.12.1 is the map associated with the identity via the bijection:

$$\mathrm{Hom}_{D({}^{\mathrm{l}}\mathcal{A})}(\mathcal{E}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G}),\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{D({}^{\mathrm{l}}\mathcal{B})}(\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G}),\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{A}}(\mathcal{E},\mathcal{G})).$$

□

Lemma 4.6.3.13. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three sheaves of rings on some topos $\mathscr{T}$ such that there exists a left solving ring $\mathcal{R}$ of both $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{C})$. Let $\mathcal{P}^\bullet \in K({}^1\mathcal{A}, \mathcal{R}, {}^1\mathcal{B})$, $\mathcal{I}^\bullet \in K({}^1\mathcal{A}, \mathcal{R}, {}^1\mathcal{C})$. If $\mathcal{P}^\bullet$ is K-flat as an object of $K({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})$ and $\mathcal{I}^\bullet$ is K-injective as a complex of $K({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C})$, then $\mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{P}^\bullet, \mathcal{I}^\bullet)$ is K-injective as an object of $K({}^1\mathcal{B} \otimes_{\mathcal{R}} \mathcal{C})$.*

Proof. For any acyclic complex $\mathcal{F} \in K({}^1\mathcal{B} \otimes_{\mathcal{R}} \mathcal{C})$, we get the isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{K(\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C})}\left(\mathcal{F}^\bullet, \mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{P}^\bullet, \mathcal{I}^\bullet)\right) \underset{4.6.3.1.4}{\xrightarrow{\sim}} H^0\Gamma\left(\mathscr{T}, \mathscr{H}om^\bullet_{\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C}}\left(\mathcal{F}^\bullet, \mathscr{H}om^\bullet_{\mathcal{A}}(\mathcal{P}^\bullet, \mathcal{I}^\bullet)\right)\right) \\ &\overset{4.6.3.9.2}{\xrightarrow{\sim}} H^0\Gamma\left(\mathscr{T}, \mathscr{H}om^\bullet_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C}}\left(\mathcal{P}^\bullet \otimes_{\mathcal{B}} \mathcal{F}^\bullet, \mathcal{I}^\bullet\right)\right) \underset{4.6.3.1.4}{\xrightarrow{\sim}} \mathrm{Hom}_{K(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C})}\left(\mathcal{P}^\bullet \otimes_{\mathcal{B}} \mathcal{F}^\bullet, \mathcal{I}^\bullet\right). \end{aligned}$$

Since $\mathcal{P}^\bullet$ is also K-flat as an object of $K({}^1\mathcal{B})$, since $\mathcal{F}^\bullet$ is acyclic, then $\mathcal{P}^\bullet \otimes_{\mathcal{B}} \mathcal{F}^\bullet$ is an acyclic object of $K({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C})$. Since $\mathcal{I}^\bullet$ is K-injective as a complex of $K({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C})$, then $\mathrm{Hom}_{K(\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C})}\left(\mathcal{P}^\bullet \otimes_{\mathcal{B}} \mathcal{F}^\bullet, \mathcal{I}^\bullet\right) = 0$ and we are done. □

Proposition 4.6.3.14 (Cartan isomorphisms)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three sheaves of rings on some topos $\mathscr{T}$ such that there exists a left solving ring $\mathcal{R}$ of both $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{C})$. Let $\mathcal{E} \in D({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}^{\mathrm{o}})$, $\mathcal{F} \in D({}^1\mathcal{B} \otimes_{\mathcal{R}} \mathcal{C})$ and $\mathcal{G} \in D({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C})$. We have the functorial isomorphisms:*

$$\begin{aligned} \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{F}, \mathcal{G}) &\xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{{}^1\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{F}, \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G})) \\ \mathbb{R}\mathrm{Hom}_{{}^1\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{F}, \mathcal{G}) &\xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{{}^1\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C}}(\mathcal{F}, \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G})) \\ \mathrm{Hom}_{D({}^1\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C})}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{F}, \mathcal{G}) &\xrightarrow{\sim} \mathrm{Hom}_{D({}^1\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C})}(\mathcal{F}, \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G})). \end{aligned} \tag{4.6.3.14.1}$$

Proof. Let $\mathcal{P}$ be a K-flat complex of left $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}^{\mathrm{o}}$-modules representing $\mathcal{E}$ and $\mathcal{I}$ be a K-injective complex of left $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C}$-modules representing $\mathcal{G}$. These resolutions allow us to compute the left terms of the isomorphisms of the proposition. Moreover, it follows from 4.6.3.13 that $\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{P}, \mathcal{I})$ is a K-injective complex of left $\mathcal{B} \otimes_{\mathcal{R}} \mathcal{C}$-modules. Hence, these two resolutions compute also the right terms of the isomorphisms of the proposition. Hence, we conclude by using 4.6.3.9.2. □

Corollary 4.6.3.15. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three sheaves of rings on some topos $\mathscr{T}$ such that there exists a left solving ring $\mathcal{R}$ of both $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{C})$. Let $\mathcal{E} \in D({}^1\mathcal{A}, \mathcal{R}, \mathcal{B}^{\mathrm{r}})$ (resp. $\mathcal{E} \in D(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}^{\mathrm{o}})$), $\mathcal{F} \in D({}^1\mathcal{B}, {}^1\mathcal{C})$ (resp. $\mathcal{F} \in D({}^1\mathcal{B} \otimes_{\mathcal{R}} {}^1\mathcal{C})$) and $\mathcal{G} \in D({}^1\mathcal{A}, \mathcal{R}, {}^1\mathcal{C})$ (resp. $\mathcal{G} \in D({}^1\mathcal{A} \otimes_{\mathcal{R}} {}^1\mathcal{C})$). We have the functorial in $\mathcal{E}$ and $\mathcal{G}$ morphism of $D({}^1\mathcal{A}, \mathcal{R}, {}^1\mathcal{C})$ (resp. $D({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C})$):*

$$\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G}) \to \mathcal{G}. \tag{4.6.3.15.1}$$

Proof. Since the canonical functor $D({}^1\mathcal{A} \otimes_{\mathcal{R}} \mathcal{C}) \to D({}^1\mathcal{A}, \mathcal{R}, {}^1\mathcal{C})$ is essentially surjective, it is sufficient to construct the morphism in the respective case. The arrow 4.6.3.15.1 is the map associated with the identity via the bijection:

$$\mathrm{Hom}_{D({}^1\mathcal{A}\otimes_{\mathcal{R}}\mathcal{C})}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G}), \mathcal{G}) \to \mathrm{Hom}_{D({}^1\mathcal{B}\otimes_{\mathcal{R}}\mathcal{C})}(\mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G}), \mathbb{R}\mathscr{H}om_{{}^1\mathcal{A}}(\mathcal{E}, \mathcal{G})).$$

□

4.6.4 Extension by a ring homomorphism, duality : commutation, biduality

Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be two ring homomorphisms. We suppose that there exists two ring homomorphisms $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}''$ so that $\mathcal{R}$ (resp. $\mathcal{R}'$, resp. $\mathcal{R}''$) is a ring of resolution $\mathcal{A}$ (resp. $\mathcal{B}$, resp. $\mathcal{C}$).

Lemma 4.6.4.1. *Let $\mathcal{E} \in D({}^1\mathcal{A})$, $\mathcal{G} \in D({}^1\mathcal{B})$. Let $f\colon \mathcal{E} \to \mathcal{G}$ be a morphism of $D({}^1\mathcal{A})$. Then, there exists a unique morphism $g\colon \mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E} \to \mathcal{G}$ of $D({}^1\mathcal{B})$ making commutative the diagram:*

$$\begin{array}{ccc} \mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E} & \overset{g}{\dashrightarrow} & \mathcal{G} \\ \uparrow & \overset{f}{\nearrow} & \\ \mathcal{E} & & \end{array} \tag{4.6.4.1.1}$$

The functor $\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} -\colon D({}^1\mathcal{A}) \to D({}^1\mathcal{B})$ is a left adjoint to the forgetful functor. Similarly, we have the left adjoint functor $- \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}\colon D({}^{\mathrm{r}}\mathcal{A}) \to D({}^{\mathrm{r}}\mathcal{B})$ is a left adjoint to the forgetful functor.

Proof. The morphism f is represented by a quasi-isomorphism $\mathcal{E}' \xrightarrow{\sim} \mathcal{E}$ of $K({}^{\mathrm{l}}\mathcal{A})$ and a morphism $\mathcal{E}' \to \mathcal{G}$ of $K({}^{\mathrm{l}}\mathcal{A})$. We can suppose that $\mathcal{E}'$ is a K-flat complex of $K({}^{\mathrm{l}}\mathcal{A})$. Hence, $\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}'$ and $\mathcal{E} \to \mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E}$ is represented by $\mathcal{E} \xleftarrow{\sim} \mathcal{E}' \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}'$. Finally let $g\colon \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}' \to \mathcal{G}$ be the morphism of $K({}^{\mathrm{l}}\mathcal{B})$ induced by extension from $\mathcal{E}' \to \mathcal{G}$. This is the unique morphism making commutative the diagram 4.6.4.1.1. □

4.6.4.2. Let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A})$, $\mathcal{G} \in D({}^{\mathrm{l}}\mathcal{B},, \mathcal{R}', {}^{\mathrm{r}}\mathcal{B})$. Let $\mathcal{I} \to \mathcal{J}$ be a quasi-isomorphism of K-injective complexes of $K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0)$. Then, the morphism $\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{J})$ is a quasi-isomorphism of $K({}^{\mathrm{r}}\mathcal{B})$ because it is canonically isomorphic to $\mathcal{H}om_{{}^{\mathrm{l}}\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E},\mathcal{I}) \to \mathcal{H}om_{{}^{\mathrm{l}}\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E},\mathcal{J})$. Hence, following [Sta22, 13.14.15], we get the functor $\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ -)\colon D({}^{\mathrm{l}}\mathcal{B},\mathcal{R}',\mathcal{B}^{\mathrm{r}}) \to D({}^{\mathrm{r}}\mathcal{B})$, i.e. the right derived functor (with respect to quasi-isomophisms of $K({}^{\mathrm{l}}\mathcal{B})$) of $\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ -)\colon K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0) \to D({}^{\mathrm{r}}\mathcal{B})$ exists and is computed by K-injective complexes. Moreover, the functor $K({}^{\mathrm{l}}\mathcal{A}) \to D({}^{\mathrm{r}}\mathcal{B})$, given by $\mathcal{E} \mapsto \mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{G})$ has a derived functor (with respect to quasi-isomophisms of $K({}^{\mathrm{l}}\mathcal{A})$). Indeed, following [Sta22, 13.14.15], it is sufficient to check that if $\mathcal{P} \to \mathcal{P}'$ is a quasi-isomorphism of K-flat complexes of $K({}^{\mathrm{l}}\mathcal{A})$, then $\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{P}',\ \mathcal{G}) \to \mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\ \mathcal{G})$ is an isomorphism. Let $\mathcal{I}$ be a K-injective complex of $K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0)$ representing $\mathcal{G}$. Then, the map $\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{P}',\ \mathcal{G}) \to \mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{P},\ \mathcal{G})$ corresponds to the map $\mathcal{H}om_{\mathcal{A}}(\mathcal{P}',\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{P},\ \mathcal{I})$ induced by the quasi-isomorphism $\mathcal{P} \to \mathcal{P}'$. Moreover, this latter is canonically isomorphic to $\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{P}',\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{P},\ \mathcal{I})$. Since $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{P} \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{P}'$ is a quasi-isomorphism of $K({}^{\mathrm{l}}\mathcal{B})$, since $\mathcal{I}$ is a K-injective complex of $K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0)$ then $\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{P}',\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{P},\ \mathcal{I})$ is an isomorphism. We denote by $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(-,\mathcal{G})\colon D({}^{\mathrm{l}}\mathcal{A})^{\mathrm{op}} \to D({}^{\mathrm{r}}\mathcal{B})$ the derived functor. Since this is functorial in $\mathcal{G}$, this yields the bifunctor

$$\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(-,-)\colon D({}^{\mathrm{l}}\mathcal{A})^{\mathrm{op}} \times D({}^{\mathrm{l}}\mathcal{B},\mathcal{R}',\mathcal{B}^{\mathrm{r}}) \to D({}^{\mathrm{r}}\mathcal{B}). \tag{4.6.4.2.1}$$

We remark that $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{G})$ could have been directly defined by setting $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{G}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{P},\mathcal{I})$ where $\mathcal{P}$ is a K-flat complex representing $\mathcal{E}$ and $\mathcal{I}$ is a K-injective complex representing $\mathcal{G}$ and by checking its independence relatively to the choices (it is a way of defining analogous to the one of [Sta22, 20.38.7]). By using [Sta22, 13.14.16], we check that we have the 2-commutative diagram

$$\begin{array}{ccc} D({}^{\mathrm{l}}\mathcal{A})^{\mathrm{op}} \times D({}^{\mathrm{l}}\mathcal{B},\mathcal{R}',\mathcal{B}^{\mathrm{r}}) & \xrightarrow{\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(-,-)} & D({}^{\mathrm{r}}\mathcal{B}) \\ \uparrow & & \| \\ D({}^{\mathrm{l}}\mathcal{A})^{\mathrm{op}} \times D({}^{\mathrm{l}}\mathcal{A},\mathcal{R},\mathcal{A}^{\mathrm{r}}) & \xrightarrow{\mathbb{R}\mathcal{H}om_{\mathcal{A}}(-,-)} & D({}^{\mathrm{r}}\mathcal{A}). \end{array} \tag{4.6.4.2.2}$$

Hence, this is not confusing to simply denote by $\mathbb{R}\mathcal{H}om_{\mathcal{A}}(-,-)\colon D({}^{\mathrm{l}}\mathcal{A})^{\mathrm{op}} \times D({}^{\mathrm{l}}\mathcal{B}) \to D({}^{\mathrm{r}}\mathcal{B})$, instead of $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{A}}(-,-)$.

Lemma 4.6.4.3. *Let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A})$, $\mathcal{G} \in D({}^{\mathrm{l}}\mathcal{B},\mathcal{R}',{}^{\mathrm{r}}\mathcal{B})$. We have the canonical isomorphism of $D({}^{\mathrm{r}}\mathcal{B})$:*

$$\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\ \mathcal{G}) \tag{4.6.4.3.1}$$

This isomorphism is transitive, i.e. for any $\mathcal{H} \in D({}^{\mathrm{l}}\mathcal{C},\mathcal{R}'',{}^{\mathrm{r}}\mathcal{C})$ we have the commutative diagram

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{H}) & \xrightarrow{4.6.4.3.1} & \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\ \mathcal{H}) \\ \downarrow{\scriptstyle 4.6.4.3.1} & & \downarrow{\scriptstyle 4.6.4.3.1} \\ \mathbb{R}\mathcal{H}om_{\mathcal{C}}(\mathcal{C} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\ \mathcal{H}) & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{C}}(\mathcal{C} \otimes^{\mathbb{L}}_{\mathcal{B}} (\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E}),\ \mathcal{H}) \end{array} \tag{4.6.4.3.2}$$

Proof. We construct the morphism by choosing a K-flat complex of $K({}^{\mathrm{l}}\mathcal{A})$ representing $\mathcal{E}$ and a K-injective complex of $K({}^{\mathrm{l}}\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0)$ representing $\mathcal{G}$. The transitivity is left to the reader as an easy exercice. □

Lemma 4.6.4.4. *For any $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A})$, $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{A},\mathcal{R},\mathcal{A}^{\mathrm{r}})$ such that $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}$ is "canonically" in the image of the forgetful functor $D({}^{\mathrm{l}}\mathcal{B},\mathcal{R}',\mathcal{B}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{A},\mathcal{B}^{\mathrm{r}})$ (see 4.6.4.5 for some examples). Still denoting by $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}$ the object of $D({}^{\mathrm{l}}\mathcal{B},\mathcal{R}',\mathcal{B}^{\mathrm{r}})$, we have the canonical morphism of $D({}^{\mathrm{r}}\mathcal{B})$:*

$$\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B} \to \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\ \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}). \tag{4.6.4.4.1}$$

This one is an isomorphism if $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{A})$. Moreover, this is transitive: if $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{C}$ is canonically in the image of the forgetful functor $D({}^{\mathrm{l}}\mathcal{C}, \mathcal{R}'', \mathcal{C}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{A}, \mathcal{C}^{\mathrm{r}})$ then we have the following commutative diagram :

$$\begin{array}{ccc} (\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}) \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{C} & \xrightarrow[\sim]{4.6.4.4.1} & \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}) \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{C} \\ \sim\downarrow & & \sim\downarrow 4.6.4.4.1 \\ \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{C} & \xrightarrow[\sim]{4.6.4.4.1} & \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{C}). \end{array}$$

Proof. It follows from (the right version of) 4.6.4.1.1 that the morphism $\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \to \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B})$ of $D({}^{\mathrm{r}}\mathcal{A})$ induces canonically 4.6.4.4.1. □

Examples 4.6.4.5. The condition "$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}$ is canonically in the image of the forgetful functor $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{R}', \mathcal{B}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{A}, \mathcal{B}^{\mathrm{r}})$" of lemma 4.6.4.4 holds

(i) if $\mathcal{F} = \mathcal{A}$;

(ii) if $\mathcal{A}$ and $\mathcal{B}$ are commutative;

(iii) if $\mathcal{B} = \mathcal{A} \otimes_{\mathcal{R}} \mathcal{R}'$. Indeed let $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{A}, \mathcal{A}^{\mathrm{r}})$ and $\mathcal{P}$ be a K-flat complex of $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^0$-modules representing $\mathcal{F}$. Since $\mathcal{B} \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^0) \xrightarrow{\sim} \mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0 \xrightarrow{\sim} (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^0) \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim}$, then we get the isomorphisms of K-flat complex of $\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0$-modules: $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{P} \xrightarrow{\sim} (\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0) \otimes_{(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^0)} \mathcal{P}$ and $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} \mathcal{P} \otimes_{(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}^0)} (\mathcal{B} \otimes_{\mathcal{R}'} \mathcal{B}^0)$. This yields the isomorphism $\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{F} \xrightarrow{\sim} \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}$ of $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{R}', \mathcal{B}^{\mathrm{r}})$.

For instance with notation 3.3, $\mathcal{A} = \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ or $\mathcal{D}^{(m)}_{X^\sharp_{i+1}/S^\sharp_{i+1}}$ and $\mathcal{B} = \mathcal{D}^{(m)}_{X^\sharp_i/S^\sharp_i}$.

Proposition 4.6.4.6 (biduality)**.** *For any $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A})$, we have the morphism*

$$\mathcal{E} \to \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}), \mathcal{A}) \tag{4.6.4.6.1}$$

which is an isomorphism when $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{A})$.

Proof. The construction is standard: let $\mathcal{I}$ be a K-injective complex of $K({}^{\mathrm{l}}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}))$ representing $\mathcal{A}$. Then

$$\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}), \mathcal{A}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{A}}(\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{I}), \mathcal{I}).$$

The morphism $\mathcal{E} \to \mathbb{D}(\mathbb{D}(\mathcal{E}))$ is simply the evaluation morphism

$$\mathcal{E} \to \mathcal{H}om_{\mathcal{A}}(\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{I}), \mathcal{I}).$$

When $\mathcal{E}$ is perfect, to check that this is an isomorphism we are reduced by dévissage to the case $\mathcal{E} = \mathcal{A}$, in which case it is clear. □

Proposition 4.6.4.7. *Let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A})$, $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{A}, \mathcal{R}, \mathcal{A}^{\mathrm{r}})$ such that $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}$ is canonically in the image of the forgetful functor $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{R}', \mathcal{B}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{A}, \mathcal{B}^{\mathrm{r}})$ (see the examples 4.6.4.5). Still denoting by $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}$ the object of $D({}^{\mathrm{l}}\mathcal{B}, \mathcal{R}', \mathcal{B}^{\mathrm{r}})$, we have the canonical morphism of $D({}^{\mathrm{r}}\mathcal{B})$:*

$$\alpha\colon \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B} \to \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}), \tag{4.6.4.7.1}$$

which is an isomorphism when $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{A})$. Moreover, they are transitive, i.e., if $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{C}$ is canonically in the image of the forgetful functor $D({}^{\mathrm{l}}\mathcal{C}, \mathcal{R}'', \mathcal{C}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\mathcal{A}, \mathcal{C}^{\mathrm{r}})$ then we have the following commutative diagram :

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{C} & \xrightarrow{\alpha \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{C}} & \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}) \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{C} \\ \sim\downarrow & & \downarrow\alpha \\ \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{C} & \xrightarrow{\alpha} & \mathbb{R}\mathcal{H}om_{\mathcal{C}}(\mathcal{C} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{C}). \end{array}$$

Proof. This follows by composition from 4.6.4.3.1 4.6.4.4.1:

$$\alpha\colon \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B} \overset{4.6.4.4.1}{\longrightarrow} \mathbb{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}) \underset{\sim}{\overset{4.6.4.3.1}{\longrightarrow}} \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{B} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E},\,\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{B}).$$

□

4.6.5 Morphisms of ringed topoi, adjunction, internal homomorphism

In this subsection, we give some links between internal homorphism, pushforward and pullbacks induces by a ringed topoi homomorphism. Moreover, we check the acyclicity Lemma 4.6.5.2 which extends [SGA4.2, V.4.10.2)] and which will be a key ingredient (see 7.1.3.21).

4.6.5.1 (Adjunction)**.** Let $f\colon (\mathscr{T},\mathcal{A}) \to (\mathscr{T}',\mathcal{A}')$ be a morphism of ringed topoi.

(a) It follows from [Sta22, 09T5] that the functors $\mathbb{R}f_*\colon D(\mathcal{A}) \to D(\mathcal{A}')$ and $\mathbb{L}f^*\colon D(\mathcal{A}') \to D(\mathcal{A})$ are adjoint:

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathbb{L}f^*(\mathcal{E}'^{\bullet}), \mathcal{E}^{\bullet}) = \mathrm{Hom}_{D(\mathcal{A}')}(\mathcal{E}'^{\bullet}, \mathbb{R}f_*(\mathcal{E}^{\bullet})) \quad (4.6.5.1.1)$$

for any $\mathcal{E}^{\bullet} \in D(\mathcal{A})$ and any $\mathcal{E}'^{\bullet} \in D(\mathcal{A}')$.

(b) Let $\mathcal{B}'$ be a sheaf of rings on $\mathscr{T}'$ such that there exists an left solving ring $\mathcal{R}'$ of $(\mathcal{A}',\mathcal{B}')$ (in the sense of 4.6.3.2.(b)) such that the image of the composition $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ is included in the center of $\mathcal{A}$. Then $f^{-1}\mathcal{R}'$ is a left solving ring of $(f^{-1}\mathcal{A}', f^{-1}\mathcal{B}')$ and of $(\mathcal{A}, f^{-1}\mathcal{B}')$.

We have the functor $f^* := \mathcal{A} \otimes_{f^{-1}\mathcal{A}'} f^{-1}(-)$ from the category of $(\mathcal{A}',\mathcal{B}')$-bimodules (resp. left, resp. right $(\mathcal{A}',\mathcal{B}')$-bimodules) to that of $(\mathcal{A}, f^{-1}\mathcal{B}')$-bimodules (resp. left, resp. right $(\mathcal{A}, f^{-1}\mathcal{B}')$-bimodules) which is a left adjoint of f_* (use the remark of 4.6.2.7.(d)). The derived functor $\mathbb{L}f^*\colon D(\mathcal{A}',\mathcal{R}',\mathcal{B}') \to D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}')$ is well defined: for any $\mathcal{E}' \in D(\mathcal{A}',\mathcal{R}',\mathcal{B}')$, by choosing a K-flat complex $\mathcal{P}'^{\bullet}$ of $K(\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}')$ representing $\mathcal{E}'$, we get $\mathbb{L}f^*(\mathcal{E}') = f^*(\mathcal{P}'^{\bullet})$. Since $\mathcal{A} \to \mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}'$ is flat, then the functor $\mathbb{L}f^*$ commutes with the forgetful functor, i.e., we have the commutative diagram (up to canonical isomorphism)

$$\begin{array}{ccccc}
D(\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}') & \longrightarrow & D(\mathcal{A}',\mathcal{R}',\mathcal{B}') & \longrightarrow & D(\mathcal{A}') \\
\downarrow{\scriptstyle \mathbb{L}f^*} & & \downarrow{\scriptstyle \mathbb{L}f^*} & & \downarrow{\scriptstyle \mathbb{L}f^*} \\
D(\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}') & \longrightarrow & D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}') & \longrightarrow & D(\mathcal{A}).
\end{array}$$

The derived functor $\mathbb{R}f_*\colon D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}') \to D(\mathcal{A}',\mathcal{R}',\mathcal{B}')$ is well defined: for any $\mathcal{E} \in D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}')$, by choosing a K-injective complex $\mathcal{I}^{\bullet}$ of $K(\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}')$ representing $\mathcal{E}$, we get $\mathbb{R}f_*(\mathcal{E}) = f_*(\mathcal{I}^{\bullet})$. Since $\mathcal{A} \to \mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}'$ is flat, then this functor corresponds modulo the forgetful functor to $\mathbb{R}f_*\colon D(\mathcal{A}) \to D(\mathcal{A}')$.

It follows from [Sta22, 09T5] that the above functors $\mathbb{L}f^*\colon D(\mathcal{A}',\mathcal{R}',\mathcal{B}') \to D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}')$ and $\mathbb{R}f_*\colon D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}') \to D(\mathcal{A}',\mathcal{R}',\mathcal{B}')$ are adjoint:

$$\mathrm{Hom}_{D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}')}(\mathbb{L}f^*(\mathcal{E}'^{\bullet}), \mathcal{E}^{\bullet}) = \mathrm{Hom}_{D(\mathcal{A}',\mathcal{R}',\mathcal{B}')}(\mathcal{E}'^{\bullet}, \mathbb{R}f_*(\mathcal{E}^{\bullet})) \quad (4.6.5.1.2)$$

for any $\mathcal{E}^{\bullet} \in D(\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{B}')$ and any $\mathcal{E}'^{\bullet} \in D(\mathcal{A}',\mathcal{R}',\mathcal{B}')$. Similarly, we get

$$\mathrm{Hom}_{D(\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}')}(\mathbb{L}f^*(\mathcal{E}'^{\bullet}), \mathcal{E}^{\bullet}) = \mathrm{Hom}_{D(\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}')}(\mathcal{E}'^{\bullet}, \mathbb{R}f_*(\mathcal{E}^{\bullet})) \quad (4.6.5.1.3)$$

for any $\mathcal{E}^{\bullet} \in D(\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}')$ and any $\mathcal{E}'^{\bullet} \in D(\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}')$.

Lemma 4.6.5.2. *Let $f\colon (\mathscr{T},\mathcal{A}) \to (\mathscr{T}',\mathcal{A}')$ be a morphism of ringed topoi. Let $\mathcal{P}'$ be a left $\mathcal{A}'$-module and $\mathcal{I}$ be an injective left $\mathcal{A}$-module.*

(a) The abelian sheaf $\mathscr{H}om_{\mathcal{A}'}(\mathcal{P}', f_*\mathcal{I})$ is flasque.

(b) If $\mathcal{P}'$ is a flat left $\mathcal{A}'$-module, then the left $\mathcal{A}'$-module $f_*\mathcal{I}$ is right acyclic for the functors $\mathscr{H}om_{\mathcal{A}'}(\mathcal{P}', -)$ and $\mathrm{Hom}_{\mathcal{A}'}(\mathcal{P}', -)$.

Proof. It follows from the isomorphism $\mathscr{H}om_{\mathcal{A}'}(\mathcal{P}',\mathcal{I}) = f_*\mathscr{H}om_{\mathcal{A}'}(f^*\mathcal{P}',\mathcal{I})$ that $\mathscr{H}om_{\mathcal{A}'}(\mathcal{P}',\mathcal{I})$ is flasque (use [SGA4.2, V.4.9.1 and V.4.10.2)]).

Let $\mathcal{I} \hookrightarrow \mathcal{J}$ be a monomorphism of sheaves on $\mathscr{T}$ of $\mathbb{Z}$-modules such that $\mathcal{J}$ is an injective sheaf on $\mathscr{T}$ of $\mathbb{Z}$-modules. This yields by adjointness (see [SGA4.1, IV.12.8)]) the injective morphism of $\mathcal{A}$-modules $\mathcal{I} \hookrightarrow \mathscr{H}om_{\mathbb{Z}}(\mathcal{A},\mathcal{J})$. Since $\mathscr{H}om_{\mathbb{Z}}(\mathcal{A},\mathcal{J})$ is an injective left $\mathcal{A}$-module (this follows by using the adjoint formula [SGA4.1, IV.12.8)]), since this latter injection splits (because $\mathcal{I}$ is an injective $\mathcal{A}$-module), then we reduce to check the acyclity in the case where $\mathcal{I} = \mathscr{H}om_{\mathbb{Z}}(\mathcal{A}, \mathcal{J})$ with $\mathcal{J}$ an injective sheaf on $\mathscr{T}$ of $\mathbb{Z}$-modules.

Let $\mathcal{M}^\bullet$ be a resolution of $\mathcal{A}$ by flat right $f^{-1}\mathcal{A}'$-modules. For any $n \in \mathbb{Z}$, via the adjoint formula

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}^n \otimes_{f^{-1}\mathcal{A}'} \mathcal{E}, \mathcal{J}) \xrightarrow{\sim} \mathrm{Hom}_{f^{-1}\mathcal{A}'}(\mathcal{E}, \mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^n, \mathcal{J}))$$

which holds for any left $f^{-1}\mathcal{A}'$-module $\mathcal{E}$ (see [SGA4.1, IV.12.8)]), we get that $\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^n, \mathcal{J})$ is an injective right $f^{-1}\mathcal{A}'$-modules. Since $\mathcal{J}$ is an injective sheaf on $\mathscr{T}$ of $\mathbb{Z}$-modules, this yields that $\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^\bullet, \mathcal{J})$ is resolution of $\mathcal{I}$ by injective right $f^{-1}\mathcal{A}'$-modules. Since the morphism of $f\colon (\mathscr{T}, f^{-1}\mathcal{A}') \to (\mathscr{T}', \mathcal{A}')$ of ringed topos induced by f is flat, then the functor f_* from the category of right $f^{-1}\mathcal{A}'$-modules to that of right $\mathcal{A}'$-module preserves injective objects. Hence, $f_*\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^n, \mathcal{J})$ is an injective $\mathcal{A}'$-module for any $n \in \mathbb{Z}$. Since $\mathcal{I}$ and $\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^n, \mathcal{J})$ are f_*-acyclic (because $\mathcal{I}$ is an injective $\mathcal{A}$-module and $\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^n, \mathcal{J})$ is an injective $f^{-1}\mathcal{A}'$-module), this implies that $f_*\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^\bullet, \mathcal{J})$ is a resolution of $f_*\mathcal{I}$ by injective $\mathcal{A}'$-modules. Hence, $\mathcal{E}xt^n_{\mathcal{A}'}(\mathcal{P}', f_*\mathcal{I})$ is equal to the nth cohomological space of the complex

$$\mathscr{H}om_{\mathcal{A}'}(\mathcal{P}', f_*\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^\bullet, \mathcal{J})) \underset{[\mathrm{SGA4.1}, IV.13.4.2)]}{\xrightarrow{\sim}} f_*\mathscr{H}om_{f^{-1}\mathcal{A}'}(f^{-1}\mathcal{P}', \mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^\bullet, \mathcal{J}))$$
$$\xrightarrow{\sim} f_*\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^\bullet \otimes_{f^{-1}\mathcal{A}'} f^{-1}\mathcal{P}', \mathcal{J}).$$

Since $f^{-1}\mathcal{P}'$ is a flat left $f^{-1}\mathcal{A}'$-module and $\mathcal{J}$ is an injective $\mathbb{Z}$-module, then $f_*\mathscr{H}om_{\mathbb{Z}}(\mathcal{M}^\bullet \otimes_{f^{-1}\mathcal{A}'} f^{-1}\mathcal{P}', \mathcal{J})$ is acyclic in positive degree. Hence, $\mathscr{E}xt^n_{\mathcal{A}'}(\mathcal{P}', f_*\mathcal{I}) = 0$ for any $n \geq 1$, i.e. the left $\mathcal{A}'$-module $f_*\mathcal{I}$ is right acyclic for the functor $\mathscr{H}om_{\mathcal{A}'}(\mathcal{P}', -)$. By using 4.6.2.7.6, this yields $f_*\mathcal{I}$ is right acyclic for the functor $\mathrm{Hom}_{\mathcal{A}'}(\mathcal{P}', -)$. □

4.6.5.3. Let $f\colon (\mathscr{T}, \mathcal{A}) \to (\mathscr{T}', \mathcal{A}')$ be a morphism of ringed topoi. We denote by $f_{\mathrm{top}}\colon (\mathscr{T}, \mathbb{Z}_{\mathscr{T}}) \to (\mathscr{T}', \mathbb{Z}_{\mathscr{T}'})$ be the induced morphism of ringed topoi, where $\mathbb{Z}_{\mathscr{T}}$ (resp. $\mathbb{Z}_{\mathscr{T}'}$) is the constant abelian sheaf of $\mathscr{T}$ (resp. $\mathscr{T}'$) associated with $\mathbb{Z}$.

(a) Let $\mathcal{E}'^\bullet \in D^-(\mathcal{A}')$ and $\mathcal{E}^\bullet \in D^+(\mathcal{A})$. We have the canonical isomorphism:

$$\mathbb{R}f_{\mathrm{top}*}\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathbb{L}f^*(\mathcal{E}'^\bullet), \mathcal{E}^\bullet) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}'^\bullet, \mathbb{R}f_*(\mathcal{E}^\bullet)). \tag{4.6.5.3.1}$$

Indeed, choose $\mathcal{I}^\bullet$ a bounded below complexe of injective left $\mathcal{A}$-modules together with a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ of $K^+(\mathcal{A})$, $\mathcal{P}'^\bullet$ a bounded above complexe of flat left $\mathcal{A}'$-modules together with a quasi-isomorphism $\mathcal{P}'^\bullet \to \mathcal{E}'^\bullet$ of $K^-(\mathcal{A}')$, then it follows from 4.6.5.2 that the map 4.6.5.3.1 is canonically isomorphic to the isomorphism

$$f_*\mathscr{H}om^\bullet_{\mathcal{A}}(f^*(\mathcal{P}'^\bullet), \mathcal{I}^\bullet) \underset{[\mathrm{SGA4.1}, IV.13.4.2)]}{\xrightarrow{\sim}} \mathscr{H}om^\bullet_{\mathcal{A}'}(\mathcal{P}'^\bullet, f_*(\mathcal{I}^\bullet)).$$

(b) Let $\mathcal{E}'^\bullet \in D^-(\mathcal{A}')$ and $\mathcal{F}'^\bullet \in D^+(\mathcal{A}')$. We have the canonical morphism of $D(\mathbb{Z}_{\mathscr{T}})$:

$$f^{-1}_{\mathrm{top}}\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}'^\bullet, \mathcal{F}'^\bullet) \to \mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathbb{L}f^*(\mathcal{E}'^\bullet), \mathbb{L}f^*(\mathcal{F}'^\bullet)). \tag{4.6.5.3.2}$$

Indeed, via the adjunction equality 4.6.5.1.1, the map 4.6.5.3.2 corresponds to the composition

$$\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}'^\bullet, \mathcal{F}'^\bullet) \xrightarrow{4.6.5.1.1} \mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}'^\bullet, \mathbb{R}f_*\mathbb{L}f^*(\mathcal{F}'^\bullet)) \overset{4.6.5.3.1}{\xrightarrow{\sim}} \mathbb{R}f_{\mathrm{top}*}\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathbb{L}f^*(\mathcal{E}'^\bullet), \mathbb{L}f^*(\mathcal{F}'^\bullet)).$$

We have the unbounded version of 4.6.5.3.2 (but we do not know if this is the case for 4.6.5.3.1).

Lemma 4.6.5.4. *Let $f\colon (\mathscr{T}, \mathcal{A}) \to (\mathscr{T}', \mathcal{A}')$ be a morphism of topoi. Let $\mathcal{B}'$ and $\mathcal{C}'$ be two sheaves of rings on $\mathscr{T}'$. Let $*, ** \in \{l, r\}$.*

(a) Suppose there exists a solving ring $\mathcal{R}'$ of both $(\mathcal{A}', \mathcal{B}')$ and $(\mathcal{A}', \mathcal{C}')$ such that the composition $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ is flat and its image is included in the center of $\mathcal{A}$. Let $\mathcal{F}' \in D({}^*\mathcal{A}', \mathcal{R}', {}^1\mathcal{B}')$ (resp. $\mathcal{F}' \in D({}^1\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}')$) and $\mathcal{E}' \in D({}^{**}\mathcal{C}', \mathcal{R}', {}^1\mathcal{B}')$ (resp. $\mathcal{E}' \in D({}^1\mathcal{C}' \otimes_{\mathcal{R}'} \mathcal{B}')$). We have the canonical morphism of $D({}^*\mathcal{A}, f^{-1}\mathcal{R}', {}^{**}f^{-1}\mathcal{C}'^o)$ (resp. $D({}^1\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}'^o)$):

$$\mathbb{L}f^*\left(\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}', \mathcal{F}')\right) \to \mathbb{R}\mathscr{H}om_{f^{-1}\mathcal{B}'}(f^{-1}(\mathcal{E}'), \mathbb{L}f^*(\mathcal{F}')). \tag{4.6.5.4.1}$$

(b) *Suppose there exists a left solving ring* $\mathcal{R}'$ *of both* $(\mathcal{A}',\mathcal{B}')$ *and* $(\mathcal{A}',\mathcal{C}')$ *such that the image of the composition* $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ *is included in the center of* $\mathcal{A}$. *Let* $\mathcal{E}' \in D({}^{1}\mathcal{A}',\mathcal{R}',{}^{*}\mathcal{B}')$ *(resp.* $\mathcal{E}' \in D({}^{1}\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}')$*) and* $\mathcal{F}' \in D({}^{1}\mathcal{A}',\mathcal{R}',{}^{**}\mathcal{C}')$ *(resp.* $\mathcal{F}' \in D({}^{1}\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{C}')$*). We have the canonical morphism of* $D({}^{*}f^{-1}\mathcal{B}'^{o}, f^{-1}\mathcal{R}', {}^{**}f^{-1}\mathcal{C}')$ *(resp.* $D({}^{1}f^{-1}\mathcal{B}'^{o} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$*):*

$$f^{-1}\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}',\mathcal{F}') \to \mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathbb{L}f^{*}(\mathcal{E}'),\mathbb{L}f^{*}(\mathcal{F}')). \tag{4.6.5.4.2}$$

(c) *Suppose there exists a left solving ring* $\mathcal{R}'$ *of* $(\mathcal{A}',\mathcal{B}')$ *such that the image of the composition* $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ *is included in the center of* $\mathcal{A}$. *Let* $\mathcal{E}' \in D({}^{1}\mathcal{A}',\mathcal{R}',{}^{*}\mathcal{B}')$ *and* $\mathcal{F}' \in D({}^{1}\mathcal{A}')$. *We have the canonical morphism of* $D({}^{*}f^{-1}\mathcal{B}'^{o})$*:*

$$f^{-1}\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}',\mathcal{F}') \to \mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathbb{L}f^{*}(\mathcal{E}'),\mathbb{L}f^{*}(\mathcal{F}')). \tag{4.6.5.4.3}$$

Proof. a) Let us check 4.6.5.4.1. It is sufficient to treat the respective case. Since $\mathcal{R}'$ is a left solving ring of $(\mathcal{B}',\mathcal{A}')$ then $\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}',\mathcal{F}') \in D({}^{1}\mathcal{A}'^{o} \otimes_{\mathcal{R}'} \mathcal{C}')$ (see 4.6.3.2.3). Hence, the left term of 4.6.5.4.1 is an object of $D({}^{1}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}'^{o})$. Moreover, $f^{-1}\mathcal{R}'$ is a left solving ring of $(f^{-1}\mathcal{B}',\mathcal{A})$. Hence, the right term of 4.6.5.4.1 is also an object of $D({}^{1}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}'^{o})$. Let $\mathcal{P}'$ be a K-flat complex of $K({}^{1}\mathcal{B}' \otimes_{\mathcal{R}'} \mathcal{C}')$ representing $\mathcal{E}'$ and $\mathcal{Q}'$ be a K-flat complex of $K({}^{1}\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{C}')$ representing $\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}',\mathcal{F}')$. Similarly to [Sta22, 0G7E], we check that $f^{-1}(\mathcal{P}')$ is a K-flat complex of $K({}^{1}f^{-1}\mathcal{B}' \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$. Since $\mathcal{Q}' \otimes_{\mathcal{C}'} \mathcal{P}'$ is K-flat as a complex of $K({}^{1}\mathcal{A}')$, we get therefore the canonical isomorphisms of $D({}^{1}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}')$:

$$\begin{array}{ccc} \mathbb{L}f^{*}\left(\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}',\mathcal{F}')\right) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{C}'} f^{-1}\mathcal{E}' & \xrightarrow{\sim} & \mathbb{L}f^{*}\left(\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}',\mathcal{F}') \otimes^{\mathbb{L}}_{\mathcal{C}'} \mathcal{E}'\right) \\ \downarrow\sim & & \downarrow\sim \\ f^{*}(\mathcal{Q}') \otimes_{f^{-1}\mathcal{C}'} f^{-1}\mathcal{P}' & \xrightarrow{\sim} & f^{*}(\mathcal{Q}' \otimes_{\mathcal{C}'} \mathcal{P}'). \end{array} \tag{4.6.5.4.4}$$

Following 4.6.3.15.1, we have the morphism $\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}',\mathcal{F}') \otimes^{\mathbb{L}}_{\mathcal{C}'} \mathcal{E}' \to \mathcal{F}'$ of $D({}^{1}\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{B}')$. By applying $\mathbb{L}f^{*}$ to this latter map and by composition with 4.6.5.4.4, we get the morphism of $D({}^{1}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}')$

$$\mathbb{L}f^{*}\mathbb{R}\mathscr{H}om_{\mathcal{B}'}(\mathcal{E}',\mathcal{F}') \otimes^{\mathbb{L}}_{f^{-1}\mathcal{C}'} f^{-1}(\mathcal{E}') \to \mathbb{L}f^{*}(\mathcal{F}'). \tag{4.6.5.4.5}$$

By using the Cartan isomorphism (see 4.6.3.14.1), we can conclude.

b) The check of 4.6.5.4.2 is analogous: It is sufficient to treat the respective case. Moreover, since $f^{-1}\mathcal{R}'$ is a left solving ring of $(\mathcal{A}, f^{-1}\mathcal{C}')$ then the objects of 4.6.5.4.2 are well defined. Since $\mathcal{R}'$ is a solving ring of $(\mathcal{B}',\mathcal{C}')$ and $f^{-1}\mathcal{R}'$ is a solving ring of $(f^{-1}\mathcal{B}', f^{-1}\mathcal{C}')$, then we get the the first (iso)morphism of $D({}^{1}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$:

$$f^{-1}\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}',\mathcal{F}') \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}'} \mathbb{L}f^{*}(\mathcal{E}') \xrightarrow{\sim} \mathbb{L}f^{*}\left(\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}',\mathcal{F}') \otimes^{\mathbb{L}}_{\mathcal{B}'} (\mathcal{E}')\right) \xrightarrow{4.6.3.15.1} \mathbb{L}f^{*}\mathcal{F}'. \tag{4.6.5.4.6}$$

By using the Cartan isomorphism (see 4.6.3.14.1), we can conclude.

c) The proof is the same than the one of b). □

Remark 4.6.5.5. Since $(\mathcal{A}',\mathbb{Z}_{\mathscr{T}'})$ is not necessarily right solvable, then the Lemma 4.6.5.4 does not give a morphism of the form 4.6.5.3.2. However, suppose there exists a sheaf of commutative rings $\mathcal{R}'$ on $\mathscr{T}'$ endowed with a flat morphism of rings $\mathcal{R}' \to \mathcal{A}'$ whose image is included in the center of $\mathcal{A}'$ and whose composition $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ is included in the center of $\mathcal{A}$. Then $(\mathcal{A}',\mathcal{R}')$ is solvable by $\mathcal{R}'$. For any $\mathcal{E}',\mathcal{F}' \in D({}^{1}\mathcal{A}')$, we get therefore from 4.6.5.4.3 (in the case $\mathcal{B}' = \mathcal{R}'$) the homomorphism of $D(f^{-1}\mathcal{R}')$:

$$f^{-1}\mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathcal{E}',\mathcal{F}') \to \mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathbb{L}f^{*}(\mathcal{E}'),\mathbb{L}f^{*}(\mathcal{F}')). \tag{4.6.5.5.1}$$

Lemma 4.6.5.6. *Let* $f\colon (\mathscr{T},\mathcal{A}) \to (\mathscr{T}',\mathcal{A}')$ *be a morphism of topoi. Let* $\mathcal{B}'$, $\mathcal{C}'$ *be two sheaves of rings on* $\mathscr{T}'$. *Suppose there exists a solving ring* $\mathcal{R}'$ *of* $(\mathcal{A}',\mathcal{B}')$ *which is a left solving ring of* $(\mathcal{A}',\mathcal{C}')$ *such that the image of the composition* $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ *is included in the center of* $\mathcal{A}$. *Let* $*, ** \in \{l, r\}$. *For any* $\mathcal{E} \in D({}^{*}\mathcal{A}, f^{-1}\mathcal{R}', {}^{\mathrm{r}}f^{-1}\mathcal{B}')$ *(resp.* $\mathcal{E} \in D({}^{1}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}'^{o})$*) and* $\mathcal{F} \in D({}^{1}f^{-1}\mathcal{B}' \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$ *(resp.* $\mathcal{F} \in D({}^{1}f^{-1}\mathcal{B}' \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$*), there exists a canonical morphism of* $D({}^{*}\mathcal{A}',\mathcal{R}',{}^{**}\mathcal{C}')$ *(resp.* $D({}^{1}\mathcal{A}' \otimes_{\mathcal{R}'} \mathcal{C}')$*) of the form:*

$$\mathbb{R}f_{*}(\mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{B}'} \mathbb{R}f_{*}(\mathcal{F}) \to \mathbb{R}f_{*}(\mathcal{E} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}'} \mathcal{F}). \tag{4.6.5.6.1}$$

Proof. It is sufficient to treat the respective case. By adjunction (see 4.6.5.1.3), we have the map $\mathbb{L}f^*\mathbb{R}f_*(\mathcal{E}) \to \mathcal{E}$ of $D({}^{\mathrm{l}}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}'^{o})$ and the map $f^{-1}\mathbb{R}f_*(\mathcal{F})$ of $D({}^{\mathrm{l}}f^{-1}\mathcal{B}' \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$. This yields the second morphism of $D({}^{\mathrm{l}}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$:

$$\mathbb{L}f^*\left(\mathbb{R}f_*(\mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{B}'} \mathbb{R}f_*(\mathcal{F})\right) \xrightarrow{\sim} \mathbb{L}f^*\mathbb{R}f_*(\mathcal{E}) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}'} f^{-1}\mathbb{R}f_*(\mathcal{F}) \to \mathcal{E} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}'} \mathcal{F}, \qquad (4.6.5.6.2)$$

the first isomorphism being checked by choosing a K-flat representation of $\mathbb{R}f_*(\mathcal{E})$ and of $\mathbb{R}f_*(\mathcal{E})$ (similarly to 4.6.5.4.4). Again by adjunction adjunction (see 4.6.5.1.3), we get 4.6.5.6.1 from 4.6.5.6.2. □

Lemma 4.6.5.7. *Let $f\colon (\mathscr{T}, \mathcal{A}) \to (\mathscr{T}', \mathcal{A}')$ be a morphism of topoi. Let $\mathcal{B}'$, $\mathcal{C}'$ be two sheaves of rings on $\mathscr{T}'$. Suppose there exists a solving ring $\mathcal{R}'$ of $(\mathcal{A}', \mathcal{B}')$ which is a left solving ring of $(\mathcal{A}', \mathcal{C}')$ such that the image of the composition $f^{-1}\mathcal{R}' \to f^{-1}\mathcal{A}' \to \mathcal{A}$ is included in the center of $\mathcal{A}$.*

Let $*, ** \in \{l, r\}$, $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A}, f^{-1}\mathcal{R}', {}^*f^{-1}\mathcal{B}')$ (resp. $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{B}')$) and $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{A}, f^{-1}\mathcal{R}', f^{-1}\mathcal{C}')$ (resp. $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{A} \otimes_{f^{-1}\mathcal{R}'} f^{-1}\mathcal{C}')$). We have the canonical morphism of $D({}^*(\mathcal{B}')^o, \mathcal{R}', {}^{**}\mathcal{C}')$ (resp. $D({}^{\mathrm{l}}(\mathcal{B}')^o \otimes_{\mathcal{R}'} \mathcal{C}')$):

$$\mathbb{R}f_*\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \to \mathbb{R}\mathscr{H}om_{\mathcal{A}'}(\mathbb{R}f_*(\mathcal{E}), \mathbb{R}f_*(\mathcal{F})). \qquad (4.6.5.7.1)$$

Proof. It is sufficient to treat the respective case. We have the morphisms

$$\mathbb{R}f_*\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{B}'} \mathbb{R}f_*(\mathcal{E}) \overset{4.6.5.6.1}{\longrightarrow} \mathbb{R}f_*\left(\mathbb{R}\mathscr{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}'} \mathcal{E}\right) \overset{4.6.3.15.1}{\longrightarrow} \mathbb{R}f_*(\mathcal{F}), \qquad (4.6.5.7.2)$$

where the first one is a symmetric version of 4.6.5.6.1. By using the Cartan isomorphism (see 4.6.3.14.1), we can conclude. □

We will need (see 7.3.1.22.1) the (non-commutative) projection formula stated as below:

Lemma 4.6.5.8. *Let $f\colon (\mathscr{T}, \mathcal{A}) \to (\mathscr{T}', \mathcal{A}')$ be a morphism of ringed topoi. Let $\mathcal{M}'^{\bullet} \in D({}^{\mathrm{r}}\mathcal{A}')$ and $\mathcal{E}^{\bullet} \in D({}^{\mathrm{l}}\mathcal{A})$. We have the projection map of $D(\mathbb{Z}_{\mathscr{T}'})$:*

$$\mathcal{M}'^{\bullet} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathbb{R}f_*(\mathcal{E}^{\bullet}) \to \mathbb{R}f_*(f^{-1}(\mathcal{M}'^{\bullet}) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{A}'} \mathcal{E}^{\bullet}) \xrightarrow{\sim} \mathbb{R}f_*(\mathbb{L}f^*(\mathcal{M}'^{\bullet}) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{E}^{\bullet}) \qquad (4.6.5.8.1)$$

constructed by adjointness from 4.6.5.1.1. The morphism 4.6.5.8.1 is an isomorphism if $\mathcal{M}'^{\bullet}$ is perfect. When $\mathcal{A}$ and $\mathcal{A}'$ are commutative, we construct similarly a map of the form 4.6.5.8.1 in $D(\mathcal{A}')$.

Proof. Using the ringed topoi morphism $(\mathscr{T}, \mathbb{Z}_{\mathscr{T}}) \to (\mathscr{T}', \mathbb{Z}_{\mathscr{T}'})$ induced by f, it follows by adjunction from 4.6.5.1.1 that the data of the left morphism of 4.6.5.8.1 is equivalent to the data of a morphism of the form $f^{-1}(\mathcal{M}'^{\bullet} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathbb{R}f_*(\mathcal{E}^{\bullet})) \to f^{-1}(\mathcal{M}'^{\bullet}) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{A}'} \mathcal{E}^{\bullet}$. This latter one is constructed by adjunction from 4.6.5.1.1 as follows:

$$f^{-1}(\mathcal{M}'^{\bullet} \otimes^{\mathbb{L}}_{\mathcal{A}} \mathbb{R}f_*(\mathcal{E}^{\bullet})) \xrightarrow{\sim} f^{-1}(\mathcal{M}'^{\bullet}) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{A}} f^{-1}\mathbb{R}f_*(\mathcal{E}^{\bullet}) \to f^{-1}(\mathcal{M}'^{\bullet}) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{A}'} \mathcal{E}^{\bullet}.$$

The fact that this morphism becomes an isomorphism is standard (e.g., we can copy the proof of [Sta22, 0944] which still works with non-commutative rings). □

4.6.6 Derived internal tensor products, derived internal homomorphism functors, Cartan isomorphisms, derived coefficients extensions and pullbacks for $\mathcal{D}$-modules

We keep notations and hypotheses of 4.2.

4.6.6.1. We have the canonical bifunctors

$$\mathcal{H}om_{\mathcal{B}_X}(-,-)\colon K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \times K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}), \qquad (4.6.6.1.1)$$

$$- \otimes_{\mathcal{B}_X} -\colon K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \times K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to K({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}), \qquad (4.6.6.1.2)$$

and similarly by replacing some l by r (when this has a meaning). Beware that for some authors, if $\mathcal{E}$ and $\mathcal{F}$ are complexes of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules, then $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}$ is a bicomplex and what we denote by $\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F}$ is denoted by $Tot(\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{F})$, where Tot means the total complex induced by the commutative bicomplex.

Since $\mathcal{B}_X \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is flat, then a K-injective (resp. K-flat) complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules is a K-injective (resp. K-flat) complex of $\mathcal{B}_X$-modules. Hence, we obtain by derivation

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(-,-)\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \times D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}), \tag{4.6.6.1.3}$$

$$-\otimes^{\mathbb{L}}_{\mathcal{B}_X} -\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \times D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}), \tag{4.6.6.1.4}$$

and similarly by replacing l by r.

By functoriality, we have bimodule versions of theses bifunctors, for instance:

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(-,-)\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{l}});$$

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(-,-)\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \times D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{r}}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{r}})\ etc. \tag{4.6.6.1.5}$$

Proposition 4.6.6.2 (Associativity)**.** *Let $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, $\mathcal{N} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, {}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. We have the following isomorphisms of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:*

$$\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{N} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{E}) \xrightarrow{\sim} (\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{N}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{E}, \tag{4.6.6.2.1}$$

$$\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} (\mathcal{N} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}) \xrightarrow{\sim} (\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{N}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}. \tag{4.6.6.2.2}$$

Proof. It is sufficient to choose K-flat complexes representing $\mathcal{E}$ and $\mathcal{N}$ and then to use 4.2.4.3.1 and 4.2.4.3.2. ☐

Proposition 4.6.6.3 (Switching $\mathcal{B}$ and $\mathcal{D}$)**.** *Let $\mathcal{M}$ be a complex of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{r}})$, $\mathcal{E}$ and $\mathcal{F}$ be two complexes of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. Then we have a canonical isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:*

$$(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} (\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{E}). \tag{4.6.6.3.1}$$

Proof. This is a consequence of 4.2.6.1. ☐

Proposition 4.6.6.4. *Let $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{r}})$ and $\mathcal{G} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. There exists a canonical morphism in $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}{}^{\mathrm{r}})$ of the form:*

$$\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{G} \to \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{G}). \tag{4.6.6.4.1}$$

The morphism 4.6.6.4.1 is an isomorphism when $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

Proof. By choosing a K-injective complex representing $\mathcal{F}$ and a K-flat complex representing $\mathcal{G}$, this is a consequence of 4.2.4.9. ☐

Proposition 4.6.6.5. *Let $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathcal{G} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, {}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. We have the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism*

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{B}_X} \mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F}, \mathcal{G})).$$

Proof. Let $\mathcal{P}$ be a K-flat complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules representing $\mathcal{F}$ and $\mathcal{I}$ be a K-injective complex of right (resp. left if $*=\mathrm{l}$) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodule representing $\mathcal{G}$. We compute in $D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\mathcal{B}_X} \mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{P}, \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E} \otimes_{\mathcal{B}_X} \mathcal{P}, \mathcal{I})$$

$$\underset{\sim}{\overset{4.2.4.5}{\longrightarrow}} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{P}, \mathcal{I})) \underset{\sim}{\overset{4.2.4.6}{\longrightarrow}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{P}, \mathcal{I})) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F}, \mathcal{G})).$$

☐

Corollary 4.6.6.6 (Cartan isomorphism)**.** *Let* $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*,* $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ *and* $\mathcal{G} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*. The canonical isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F}, \mathcal{G})),$$

is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$*-linear.*

Proof. We choose a K-flat complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules representing $\mathcal{F}$ and a K-injective complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules representing $\mathcal{G}$. The corollary 4.2.4.6 and the proposition **??** allow us to conclude. □

Proposition 4.6.6.7. *Let* $\mathcal{E} \in D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*,* $\mathcal{F} \in D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ *and* $\mathcal{G} \in D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$*. We have a canonical homomorphism of* $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ *of the form:*

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{G} \to \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{G}). \tag{4.6.6.7.1}$$

If $\mathcal{E}$ *is moreover in* $D_{\mathrm{perf}}(\mathcal{B}_X)$*, this morphism is an isomorphism.*

Proof. We construct the morphism 4.6.6.7.1 by choosing a K-injective complex of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules representing $\mathcal{F}$ and a K-flat complex of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules representing $\mathcal{G}$. By using 4.2.4.8, this one is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear and compatible with Frobenius.

Since a K-injective (resp. K-flat) complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules is a K-injective (resp. K-flat) complex of $\mathcal{B}_X$-modules, then if $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{B}_X)$, using 4.6.3.6.1 (with $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{B}_X$) that 4.6.6.7.1 is an isomorphism in $D(\mathcal{B}_X)$ and then in $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. □

4.6.6.8 (Coefficient extensions)**.** Let $\rho\colon \mathcal{B}_X \to \mathcal{B}'_X$ be a homomorphism of commutative $\mathcal{O}_X$-algebras. We suppose $\mathcal{B}_X$ and $\mathcal{B}'_X$ are endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module which is compatible to its underlying $\mathcal{O}_X$-algebra structure such that $\mathcal{B}_X \to \mathcal{B}'_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is a flat $\mathcal{B}_X$-module, then for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, a K-flat complex of left $\widetilde{\mathcal{D}}$-modules representing $\mathcal{E}$ is also a K-flat complex $\mathcal{B}_X$-modules representing $\mathcal{E}$. This yields a canonical isomorphism in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$

$$\mathcal{B}'_X \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{E} \xrightarrow{\sim} (\mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}. \tag{4.6.6.8.1}$$

Moreover, if $\mathcal{M} \in D({}^{\mathrm{r}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X)$, then any K-flat complex of right $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X$-modules representing $\mathcal{M}$ is also a K-flat complex of $\mathcal{B}_X$-modules representing $\mathcal{M}$. We get therefore in $D({}^{\mathrm{r}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}'_X)$ the canonical isomorphism:

$$\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{B}'_X \xrightarrow{\sim} \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}_X} (\mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{B}'_X). \tag{4.6.6.8.2}$$

4.6.6.9 (Pullbacks)**.** With notation 4.4.2, let $\star \in \{\emptyset, -, \mathrm{b}\}$. In the case where $\star = \mathrm{b}$, we suppose moreover $f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ has finite tor dimension, i.e. we suppose the functor $\widetilde{f}^* = \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}-$ from the category of $\mathcal{B}_Y$-modules to that of $\mathcal{B}_X$-modules has bounded cohomological dimension (see definition 4.6.1.4).

Since $\mathcal{B}_X \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is flat, then a K-flat complex of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules is a K-flat complex of $\mathcal{B}_X$-modules. Hence, it follows from 4.4.2.4 that the functor $\mathbb{L}\widetilde{f}^* = \mathcal{B}_X \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_Y} f^{-1}-\colon D^\star(\mathcal{B}_X) \to D^\star(\mathcal{B}_Y)$ induces the functor $\mathbb{L}\widetilde{f}^*\colon D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ making commutative the diagram:

$$\begin{array}{ccc} D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) & \xrightarrow{\mathbb{L}\widetilde{f}^*} & D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \\ \downarrow & & \downarrow \\ D^\star(\mathcal{B}_Y) & \xrightarrow{\mathbb{L}\widetilde{f}^*} & D^\star(\mathcal{B}_X) \end{array} \tag{4.6.6.9.1}$$

where the vertical functors are the forgetful ones. Following 4.4.2.9.1, for any $\mathcal{E} \in D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, we get the canonical isomorphism of $D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\mathbb{L}\widetilde{f}^{*}\mathcal{E} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{E}. \tag{4.6.6.9.2}$$

4.6.7 Comparison between $\mathcal{B}$-linear duality and $\mathcal{D}$-linear duality

We keep notations and hypotheses of 4.2.

Proposition 4.6.7.1. *Let $\mathcal{F} \in D^{\mathrm{b}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. The following assertions hold.*

(a) There exists a canonical morphism of $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$;

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{B}_X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F}, \mathcal{B}_X) \to \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \omega_X^{-1}).$$

(b) If $\mathcal{B}_X \in D_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathcal{F} \in D_{\mathrm{perf}}(\mathcal{B}_X)$, then this one is an isomorphism.

Proof. We will write $\otimes\omega^{-1}$ for $\otimes_{\mathcal{O}_X}\omega_X^{-1}$, $\widetilde{\mathcal{D}}$ for $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $\mathcal{B}$ for $\mathcal{B}_X$. It follows from the proposition 4.6.6.4 (valid by replacing "r" by "l") the arrow :

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \widetilde{\mathcal{D}} \otimes \omega^{-1}) \otimes^{\mathbb{L}}_{\mathcal{B}} \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B}) \to \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \widetilde{\mathcal{D}} \otimes \omega^{-1} \otimes_{\mathcal{B}} \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B})). \tag{4.6.7.1.1}$$

The transposition isomorphism 4.2.5.3.1 of $\mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B})$ induces the isomorphism :

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \widetilde{\mathcal{D}} \otimes \omega^{-1} \otimes_{\mathcal{B}} \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B})) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B}) \otimes_{\mathcal{B}} \widetilde{\mathcal{D}} \otimes \omega^{-1}). \tag{4.6.7.1.2}$$

Via 4.6.6.7, we have by functoriality a morphism in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}, {}^{\mathrm{l}}\widetilde{\mathcal{D}})$:

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B}) \otimes_{\mathcal{B}} \widetilde{\mathcal{D}} \otimes \omega^{-1} \to \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \widetilde{\mathcal{D}} \otimes \omega^{-1}). \tag{4.6.7.1.3}$$

By applying to 4.6.7.1.3 the functor $\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, -)$, we get in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}})$ the morphism :

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{B}) \otimes_{\mathcal{B}} \widetilde{\mathcal{D}} \otimes \omega^{-1}) \to \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \widetilde{\mathcal{D}} \otimes \omega^{-1})). \tag{4.6.7.1.4}$$

We have the isomorphism of Cartan (4.6.6.5):

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{B}, \mathbb{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \widetilde{\mathcal{D}} \otimes \omega^{-1})) \xleftarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{F}, \widetilde{\mathcal{D}} \otimes \omega^{-1}). \tag{4.6.7.1.5}$$

By composing 4.6.7.1.1, 4.6.7.1.2, 4.6.7.1.4 and 4.6.7.1.5, we get the morphism of 6.3.4.15.(i).

Finally, when $\mathcal{B}_X \in D_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ (resp. $\mathcal{F} \in D_{\mathrm{perf}}(\mathcal{B})$), the morphism 4.6.7.1.1 (resp. 4.6.7.1.3, and then 4.6.7.1.4) becomes an isomorphism. □

Example 4.6.7.2. In order to get the comparison isomorphism, we need the hypothesis $\mathcal{B}_X \in D_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. However, when $m = 0$, we have $\mathcal{B}_X \in D_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$ (see 4.7.3.7.4). After tensorising with $\mathbb{Q}$, see also 4.7.3.15 for other examples.

4.7 Level 0 case

We keep notations and hypotheses of 4.2.

4.7.1 Stratifications of order ≤ 1 and logarithmic connections

Definition 4.7.1.1. Let $\mathcal{E}$ be a $\mathcal{B}_X$-module. With notations 3.2.2.2, an *m-PD stratification relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$* is the data of a $\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)}$-linear isomorphism $\varepsilon_1 \colon \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)}$ such that the following diagram

$$\begin{array}{ccccc}
\widetilde{\psi}^{1,0*}_{X^\sharp/S^\sharp,(m)}(\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \mathcal{E}) & \xrightarrow{\sim} & \widetilde{\mathcal{P}}^0_{X^\sharp/S^\sharp,(m)} \otimes_{\mathcal{B}_X} \mathcal{E} & \xrightarrow{\sim} & \mathcal{E}\,, \\
\downarrow{\scriptstyle \widetilde{\psi}^{1,0*}_{X^\sharp/S^\sharp,(m)}(\varepsilon_1)} & & \downarrow{\scriptstyle \sim} & & \| \\
\widetilde{\psi}^{n',n*}_{X^\sharp/S^\sharp,(m)}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)}) & \xrightarrow{\sim} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^0_{X^\sharp/S^\sharp,(m)} & \xrightarrow{\sim} & \mathcal{E}
\end{array} \tag{4.7.1.1.1}$$

whose horizontal isomorphisms are the canonical ones, is commutative. When $m = \infty$, we simply say *stratification relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$*.

Remark 4.7.1.2. We have a canonical map from the set of m-PD stratifications relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$ of $\mathcal{E}$ (see 3.4.2.1) to that of m-PD stratifications relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$ given by $(\varepsilon_n)_{n\in\mathbb{N}} \mapsto \varepsilon_1$.

Proposition 4.7.1.3. *Given a $\mathcal{B}_X$-module $\mathcal{E}$. The following are equivalent.*

(a) A $\mathcal{B}_X$-linear homomorphism $\theta_1 \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)})$ (the $\mathcal{B}_X$-module structure of this latter is induced by the right structure of $\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)}$) making commutative the diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\sim} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^0_{X^\sharp/S^\sharp,(m)} \\
& {\scriptstyle \theta_1} \searrow & \uparrow{\scriptstyle \mathrm{id}_{\mathcal{E}} \otimes \widetilde{\psi}^{1,0}_{X^\sharp/S^\sharp,(m)}} \\
& & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)}
\end{array} \tag{4.7.1.3.1}$$

whose top isomorphism is the canonical one (recall $\widetilde{\mathcal{P}}^0_{X^\sharp/S^\sharp,(m)} = \mathcal{B}_X$).

(b) An m-PD stratification $\varepsilon = (\varepsilon^{\mathcal{E}}_n)$ of order ≤ 1 with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$.

The bijection between both data is given by $\epsilon_1 \mapsto \epsilon_1 \circ \widetilde{p}^1_{1,(m),\mathcal{E}} = \theta_1$ (see notation 4.1.2.7).

Proof. This comes from the proof of 4.2.1.5. □

4.7.1.4. Thanks to the local description 4.1.2.16.1 and the local computation of 3.2.2.10.1, we can check that the homomorphism $\widetilde{\psi}^1_m := \mathrm{id}_{\mathcal{B}_X} \otimes \psi^1_m \colon \widetilde{\mathcal{P}}^{\sharp 1}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{P}}^{\sharp 1}_{X^\sharp/S^\sharp,(m)}$ is an isomorphism. Hence, the data of a stratification relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$ is equivalent to that of an m-PD stratification relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$. In other words, the integer m has no importance and by default we simply consider stratifications relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$. Moreover, denoting by $\mathcal{I}^1_{X^\sharp/S^\sharp,(m)}$ the ideal of the closed immersion $\Delta^1_{X^\sharp/S^\sharp,(m)}$ and setting $\Omega^1_{X^\sharp/S^\sharp,(m)} := (\Delta^1_{X^\sharp/S^\sharp,(m)})^{-1}(\mathcal{I}^1_{X^\sharp/S^\sharp,(m)})$, the isomorphism ψ^1_m induces the isomorphism $\psi^1_m \colon \Omega^1_{X^\sharp/S^\sharp} \xrightarrow{\sim} \Omega^1_{X^\sharp/S^\sharp,(m)}$. We can therefore simply write $\Omega^1_{X^\sharp/S^\sharp}$ instead of $\Omega^1_{X^\sharp/S^\sharp,(m)}$.

4.7.1.5. We have the following proprieties making some links between the sheaf of relative differentials and the sheaf of principal parts of order ≤ 1 of $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$.

(a) We set $\widetilde{\Omega}^1_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \Omega^1_{X^\sharp/S^\sharp}$. Denoting by $\widetilde{j} := \mathrm{id}_{\mathcal{B}_X} \otimes j \colon \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \to p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$, where j is the canonical inclusion (see notation 3.1.3.2), by applying the functor $\mathcal{B}_X \otimes_{\mathcal{O}_X} -$ to the exact sequence 3.1.3.2.1, we get the exact sequence of $\mathcal{B}_X$-modules

$$0 \to \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \xrightarrow{\widetilde{j}} p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \xrightarrow{\widetilde{\psi}^{1,0}_{X^\sharp/S^\sharp}} \mathcal{B}_X \to 0. \tag{4.7.1.5.1}$$

The exact sequence 4.7.1.5.1 splits via the section $\widetilde{p}_0^1 \colon \mathcal{B}_X \to p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$ (see notation 4.1.2.7), which yields the isomorphism of $\mathcal{B}_X$-modules

$$(\widetilde{p}_0^1, \widetilde{j}) \colon \mathcal{B}_X \oplus \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \xrightarrow{\sim} p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}. \tag{4.7.1.5.2}$$

In particular, since $p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$ is a locally free $\mathcal{B}_X$-module, then so is $\widetilde{\Omega}^1_{X^\sharp/S^\sharp}$.

(b) Via the isomorphism 4.7.1.5.2, we get the $\mathcal{B}_X$-linear epimorphism

$$\widetilde{\varpi}_{X^\sharp/S^\sharp} \colon p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \twoheadrightarrow \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \tag{4.7.1.5.3}$$

which is a left inverse of the inclusion $\widetilde{j} \colon \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \subset p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$. We compute $\widetilde{\varpi}_{X^\sharp/S^\sharp} = \mathrm{id} - \widetilde{p}_0^1 \circ \widetilde{\psi}^{1,0}_{X^\sharp/S^\sharp}$.

(c) We define the map $d_{\mathcal{B}_X} : \mathcal{B}_X \to \widetilde{\Omega}^1_{X^\sharp/S^\sharp}$ by setting $d_{\mathcal{B}_X} := \widetilde{\varpi}_{X^\sharp/S^\sharp} \circ \widetilde{p}_1^n$ (see notation 4.1.2.5). Since $\widetilde{\psi}^{1,0}_{X^\sharp/S^\sharp} \circ \widetilde{p}_i^1 = \mathrm{id}$ for $i = 0, 1$ (see just after 4.1.2.12.1), then $\widetilde{p}_1^1(b) - \widetilde{p}_0^1(b) \in \widetilde{\Omega}^1_{X^\sharp/S^\sharp}$ for any section b of $\mathcal{B}_X$. Since $\widetilde{\varpi}_{X^\sharp/S^\sharp} \circ \widetilde{p}_0^n = 0$, this yields $d_{\mathcal{B}_X} = \widetilde{p}_1^1 - \widetilde{p}_0^1$.

Definition 4.7.1.6. We can extend the definition 3.1.3.3 as follows: Let $\mathcal{E}, \mathcal{F}$ be two $\mathcal{B}_X$-modules, $n \geq 0$ be an integer. We say that a $f^{-1}\mathcal{O}_S$-linear homomorphism $D \colon \mathcal{E} \to \mathcal{F}$ is a "differential operator of order $\leq n$ (relatively to $X^\sharp/S^\sharp$) with coefficient in $\mathcal{B}_X$" if there exists a homomorphism of $\mathcal{B}_X$-modules $u \colon p^n_{0*}(\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}) \to \mathcal{F}$ such that $D = u \circ \widetilde{p}^n_{1,\mathcal{E}}$ (see notation 4.1.2.7). Beware that this is not clear that such a u is unique. Hence one might prefer to call such a u (instead of D) a differential operator of order $\leq n$ relatively to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$.

4.7.1.7 (Local computation)**.** Suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. We still denote by $d\log u_\lambda$ its image in $\widetilde{\Omega}^1_{X^\sharp/S^\sharp}$ in $\widetilde{\Omega}^1_{X^\sharp/S^\sharp}$. Let $\tau_{\sharp\lambda,1} := \mu^1(u_\lambda) - 1$ in $\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$ for $\lambda = 1, \dots, d$. It follows from 3.1.3.5 that $\tau_{\sharp\lambda,1} = d\log u_\lambda$. Moreover, following 4.1.2.16.1, $\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$ is $\mathcal{B}_X$-free with the basis $1, \tau_{\sharp 1,1}, \dots, \tau_{\sharp d,1}$. Since $\widetilde{\psi}^{1,0}_{X^\sharp/S^\sharp}(\tau_{\sharp\lambda,1}) = 0$ and $\widetilde{\psi}^{1,0}_{X^\sharp/S^\sharp}(1) = 1$, then $\widetilde{\Omega}^1_{X^\sharp/S^\sharp}$ is $\mathcal{B}_X$-free with the basis $\tau_{\sharp 1,1}, \dots, \tau_{\sharp d,1}$, i.e. $d\log u_1, \dots, d\log u_d$. Moreover, $\widetilde{\varpi}_{X^\sharp/S^\sharp}(1) = 0$ (and $\ker \widetilde{\varpi}_{X^\sharp/S^\sharp} = \mathcal{B}_X$) and $\widetilde{\varpi}_{X^\sharp/S^\sharp}(\tau_{\sharp,\lambda,1}) = \tau_{\sharp\lambda,1}$.

Definition 4.7.1.8. Let $\mathcal{E}$ be a $\mathcal{B}_X$-module. A "(logarithmic) connection relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$" on $\mathcal{E}$ is an additive map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^1_{X^\sharp/S^\sharp}$ such that for any open $U \subseteq X$, the map $\nabla_U \colon \mathcal{E}(U) \to (\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^1_{X^\sharp/S^\sharp})(U)$ satisfies the condition : for $x \in \mathcal{E}(U)$, $b \in \mathcal{B}_X(U)$ we have

$$\nabla(bx) = b\nabla(x) + x \otimes d_{\mathcal{B}_X}(b).$$

Proposition 4.7.1.9. *Let $\mathcal{E}$ be a $\mathcal{B}_X$-module. The map $\epsilon_1 \mapsto (\mathrm{id} \otimes \varpi) \circ \epsilon_1 \circ \widetilde{p}^1_{1,\mathcal{E}}$, where $\widetilde{p}^1_{1,\mathcal{E}}$ is defined at 4.1.2.7, gives a bijection between the set of connections relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$ to that of stratifications relative to $X^\sharp/S^\sharp$ of order ≤ 1 with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$.*

Proof. We copy the proof of 2.3.1.6. □

4.7.2 Integrable connections and $\mathcal{D}$-modules of level 0

4.7.2.1. Let $\Omega^i_{X^\sharp/S^\sharp} = \wedge^i_{\mathcal{O}_X} \Omega_{X^\sharp/S^\sharp}$ for $i \geq 0$ be the ith exterior power. For any nonnegative integers i, j, denote by $c_{i,j} \colon \Omega^i_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \Omega^j_{X^\sharp/S^\sharp} \to \Omega^{i+j}_{X^\sharp/S^\sharp}$ the canonical projection. Recall (e.g. see [Ogu18, V.2.1.1]), for any integer $i \geq 0$, there is a unique collection of homomorphisms of $f^{-1}\mathcal{O}_S$-modules

$$d^i_{X^\sharp/S^\sharp} \colon \Omega^i_{X^\sharp/S^\sharp} \to \Omega^{i+1}_{X^\sharp/S^\sharp}$$

called the exterior derivative, such that:

(a) $d^0_{X^\sharp/S^\sharp} = d_{X^\sharp/S^\sharp}$,

(b) $d^1_{X^\sharp/S^\sharp} d\log m = 0$ if m is any section of $M_{X^\sharp}$,

(c) $d^i_{X^\sharp/S^\sharp} d^{i-1}_{X^\sharp/S^\sharp}\omega = 0$ if ω is any section of $\Omega^{i-1}_{X^\sharp/S^\sharp}$,

(d) $d^{i+j}_{X^\sharp/S^\sharp}(\omega\wedge\omega') = (d^i_{X^\sharp/S^\sharp}\omega)\wedge\omega' + (-1)^i\omega\wedge(d^j_{X^\sharp/S^\sharp}\omega')$ if $\omega\in\Omega^i_{X^\sharp/S^\sharp}$ and $\omega'\in\Omega^j_{X^\sharp/S^\sharp}$.

In particular, we get the formula: $d^i_{X^\sharp/S^\sharp}(ad\log m_1\wedge\cdots\wedge d\log m_i) = d_{X^\sharp/S^\sharp}a\wedge d\log m_1\wedge\cdots\wedge d\log m_i$ for any sections a of $\mathcal{O}_X$ and $m_1,\ldots,m_i$ of $M_{X^\sharp}$. We get the complex of $f^{-1}\mathcal{O}_S$-modules

$$0\to\Omega^0_{X^\sharp/S^\sharp}\to\Omega^1_{X^\sharp/S^\sharp}\to\Omega^2_{X^\sharp/S^\sharp}\to\ldots$$

which is called the de Rham complex of $X^\sharp/S^\sharp$.

Let $i\geq 0$ be an integer. The map $d^i_{X^\sharp/S^\sharp}$ is a differential operator of order 1 with respect to $X^\sharp/S^\sharp$ (see definition 3.1.3.3). More precisely, we construct a canonical $\mathcal{O}_X$-linear homomorphism $\varpi^i_{X^\sharp/S^\sharp}\colon\mathcal{P}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp}\to\Omega^{i+1}_{X^\sharp/S^\sharp}$ whose composition with $p^1_{1,\Omega^i_{X^\sharp/S^\sharp}}\colon\Omega^i_{X^\sharp/S^\sharp}\to\mathcal{P}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp}$ is equal to $d^i_{X^\sharp/S^\sharp}$ as follows. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\Omega^i_{X^\sharp/S^\sharp} & \xrightarrow{p^1_{1,\Omega^i_{X^\sharp/S^\sharp}}} & \mathcal{P}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp} & = & \mathcal{P}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp} \\
 & & \uparrow{\scriptstyle (p^1_1,j)\otimes\mathrm{id}}\ \wr & & \\
 & & \left(\mathcal{O}_X\oplus\Omega^1_{X^\sharp/S^\sharp}\right)\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp} & & \\
\downarrow{\scriptstyle d^i_{X^\sharp/S^\sharp}} & & \downarrow\wr & & \downarrow{\scriptstyle\varpi^i_{X^\sharp/S^\sharp}} \\
 & & \Omega^i_{X^\sharp/S^\sharp}\oplus\left(\Omega^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp}\right) & & \\
 & & \downarrow{\scriptstyle (d^i_{X^\sharp/S^\sharp},c_{1,i})} & & \\
\Omega^{i+1}_{X^\sharp/S^\sharp} & = & \Omega^{i+1}_{X^\sharp/S^\sharp} & = & \Omega^{i+1}_{X^\sharp/S^\sharp}
\end{array}\qquad(4.7.2.1.1)$$

where (p^1_1,j) is the isomorphism 3.1.3.2.5, where $\varpi^i_{X^\sharp/S^\sharp}$ is by definition the map making commutative the right rectangle of the diagram 4.7.2.1.1. When $i=0$, we remark $\varpi_{X^\sharp/S^\sharp}=\varpi^0_{X^\sharp/S^\sharp}$. Let us check the $\mathcal{O}_X$-linearity of ϖ^i. For any sections a,b of $\mathcal{O}_X$, for any section ω of $\Omega^1_{X^\sharp/S^\sharp}$, for any section ω_i of $\Omega^i_{X^\sharp/S^\sharp}$, the map $\varpi^i_{X^\sharp/S^\sharp}$ sends $(p^1_1(a)+\omega)\otimes\omega_i$ to $d^i_{X^\sharp/S^\sharp}(a\omega_i)+\omega\wedge\omega_i = d_{X^\sharp/S^\sharp}(a)\wedge\omega_i + ad^i_{X^\sharp/S^\sharp}(\omega_i)+\omega\wedge\omega_i$. We compute in the ring $\mathcal{P}^1_{X^\sharp/S^\sharp}$ the equalities $p^1_0(b)(p^1_1(a)+\omega) = (p^1_1(b)-d_{X^\sharp/S^\sharp}(b))(p^1_1(a)+\omega) = p^1_1(ab)-ad_{X^\sharp/S^\sharp}(b)+b\omega$, the first one coming from 4.7.1.5.c and the second one following from the fact that $\Delta^1_{X^\sharp/S^\sharp}$ is a closed immersion of order 1. This yields $p^1_0(b)\left((p^1_1(a)+\omega)\otimes\omega_i\right)$ is sent via $\varpi^i_{X^\sharp/S^\sharp}$ to $d_{X^\sharp/S^\sharp}(ab)\wedge\omega_i+abd^i_{X^\sharp/S^\sharp}(\omega_i)+(-ad_{X^\sharp/S^\sharp}(b)+b\omega)\wedge\omega_i = bd_{X^\sharp/S^\sharp}(a)\wedge\omega_i+abd^i_{X^\sharp/S^\sharp}(\omega_i)+b\omega\wedge\omega_i$. Hence, we are done.

This implies (via the canonical morphism $\mathcal{P}^1_{X/S}\to\mathcal{P}^1_{X^\sharp/S^\sharp}$ which is $\mathcal{O}_X$-linear for both structures) that the map $d^i_{X^\sharp/S^\sharp}$ is also a differential operator of order 1 with respect to X/S (see definition 1.1.3.2). In fact, we compute $d^i_{X^\sharp/S^\sharp}(aw)-ad^i_{X^\sharp/S^\sharp}(\omega) = d_{X^\sharp/S^\sharp}a\wedge\omega$ for any section a of $\mathcal{O}_X$ and section ω of $\Omega^i_{X^\sharp/S^\sharp}$, and we retrieve the fact that $\omega\mapsto d^i_{X^\sharp/S^\sharp}(aw)-ad^i_{X^\sharp/S^\sharp}(\omega)$ is linear (see 1.1.3.4.1).

Notation 4.7.2.2. Let $\varepsilon^{\mathcal{B}_X}=(\varepsilon^{\mathcal{B}}_n)$ be the m-PD stratification on $\mathcal{B}_X$, and let $\theta^{\mathcal{B}_X}=(\varepsilon^{\mathcal{B}_X}_n)$ be the corresponding family (see notation of 3.4.2.5). We define the map $d^i_{\mathcal{B}_X}\colon\mathcal{B}_X\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp}\to\mathcal{B}\otimes_{\mathcal{O}_X}\Omega^{i+1}_{X^\sharp/S^\sharp}$ by composition as follows

$$\begin{array}{ccccc}
\mathcal{B}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp} & \xrightarrow{\theta^{\mathcal{B}_X}_1\otimes\mathrm{id}_{\Omega^i_{X^\sharp/S^\sharp}}} & \mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{P}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp} & \xrightarrow{\mathrm{id}_{\mathcal{B}}\otimes\varpi^i_{X^\sharp/S^\sharp}} & \mathcal{B}\otimes_{\mathcal{O}_X}\Omega^{i+1}_{X^\sharp/S^\sharp}. \\
 & \searrow{\scriptstyle p^1_{1,\mathcal{B}\otimes\Omega^i}} & \uparrow{\scriptstyle\epsilon^{\mathcal{B}_X}_1\otimes\mathrm{id}} & & \\
 & & \mathcal{P}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{O}_X}\mathcal{B}\otimes_{\mathcal{O}_X}\Omega^i_{X^\sharp/S^\sharp} & &
\end{array}\qquad(4.7.2.2.1)$$

We remark $d_{\mathcal{B}_X}=d^0_{\mathcal{B}_X}$.

Notation 4.7.2.3. We set $\widetilde{\Omega}^i_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \Omega^i_{X^\sharp/S^\sharp}$ for any integer $i \geq 0$. When i is equal to the rank d of $\Omega_{X^\sharp/S^\sharp}$, we set $\omega_{X^\sharp/S^\sharp} := \Omega^d_{X^\sharp/S^\sharp}$ and $\widetilde{\omega}_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \omega_{X^\sharp/S^\sharp}$

Notation 4.7.2.4. Let $\mathcal{E}$ be a $\mathcal{B}_X$-module and $i, j \geq 0$ be two integers. Let y (resp. z) be a section of $\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp}$ (resp. $\widetilde{\Omega}^i_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$) and ω be a section of $\widetilde{\Omega}^j_{X^\sharp/S^\sharp}$. Then we denote by $y \wedge \omega$ (resp. $\omega \wedge z$) the image of $y \otimes \omega$ (resp. $\omega \otimes z$) via $\mathrm{id}_{\mathcal{E}} \otimes c_{i,j}$ (resp. $c_{i,j} \otimes \mathrm{id}_{\mathcal{E}}$). Denote by $y \mapsto {}^t y$ the canonical isomorphism $\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$. Beware ${}^t(y \wedge \omega) = (-1)^{ij} \omega \wedge {}^t y$.

4.7.2.5 (Integrable connection, de Rham complex)**.** Let $\mathcal{E}$ be a $\mathcal{B}_X$-module endowed with a connection relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$. Denote by ϵ_1 the associated relative to $X^\sharp/S^\sharp$ stratification of order ≤ 1 with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$ (see 4.7.1.9). Following 4.7.1.3, let $\theta_1 \colon \mathcal{E} \to p_{1*}(\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp})$ be the corresponding $\mathcal{B}_X$-linear homomorphism making commutative the diagram 4.7.1.3.1. We get the map $\nabla^i \colon \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp}$ from θ_1 and d^i by composition as follows

$$\begin{array}{ccccc}
\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} & \xrightarrow{\theta_1 \otimes \mathrm{id}_{\widetilde{\Omega}^i_{X^\sharp/S^\sharp}}} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} & \xrightarrow{\mathrm{id}_{\mathcal{E}} \otimes \varpi^i_{X^\sharp/S^\sharp}} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp}, \\
& \searrow^{\widetilde{p}^1_{1,\mathcal{E} \otimes \widetilde{\Omega}^i}} & \uparrow{\scriptstyle \epsilon_1 \otimes \mathrm{id}} & & \\
& & \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} & &
\end{array} \qquad (4.7.2.5.1)$$

where the triangle is commutative. Beware that the terms $\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp}$ and $\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp}$ have two structures of $\mathcal{B}_X$-modules: the left one and the right one. The morphisms $\widetilde{p}^1_{1,\mathcal{E} \otimes \widetilde{\Omega}^i}$ and $\theta_1 \otimes \mathrm{id}_{\widetilde{\Omega}^i_{X^\sharp/S^\sharp}}$ are $\mathcal{B}_X$-linear for the right one, whereas $\mathrm{id}_{\mathcal{E}} \otimes \varpi^i_{X^\sharp/S^\sharp}$ is $\mathcal{B}_X$-linear for the left one. The composition ∇^i is not $\mathcal{B}_X$-linear but since $\epsilon_1 \otimes \mathrm{id}$ is $\mathcal{B}_X$-linear for both structure (an in particular for the left structure), then $\nabla^i \colon \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp}$ is a differential operator of order ≤ 1 with coefficients in $\mathcal{B}_X$.

For any sections y_i of $\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp}$ and ω_j of $\widetilde{\Omega}^j_{X^\sharp/S^\sharp}$, with notation 4.7.2.4, as for 2.3.2.3.3, we get the formula:

$$\nabla^{i+j}(y_i \wedge \omega_j) = \nabla^i(y_i) \wedge \omega_j + (-1)^i y_i \wedge d^j_{\mathcal{B}_X}(\omega_j). \qquad (4.7.2.5.2)$$

We say that the connection ∇ of $\mathcal{E}$ is integrable if $\nabla^1 \circ \nabla = 0$. In that case $\nabla^{i+1} \circ \nabla^i = 0$ for any i and we get the complex

$$0 \to \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^2_{X^\sharp/S^\sharp} \to \cdots \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^d_{X^\sharp/S^\sharp} \to 0, \qquad (4.7.2.5.3)$$

which is called the de Rham complex.

4.7.2.6 (de Rham complex: exchanging the position)**.** Let $\mathcal{E}$ be a $\mathcal{B}_X$-module endowed with a connection. To define the map $\nabla^i \colon \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} \to \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp}$ (see 4.7.2.5.1), it was more natural to put $\mathcal{E}$ on the left. However, we can define $\nabla^i \colon \widetilde{\Omega}^i_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \to \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$ to be the $\mathcal{O}_S$-linear map making commutative the diagram

$$\begin{array}{ccc}
\widetilde{\Omega}^i_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} & \xrightarrow{\nabla^i} & \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
\mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} & \xrightarrow[\text{4.7.2.5.1}]{\nabla^i} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp},
\end{array} \qquad (4.7.2.6.1)$$

where the vertical isomorphism are the canonical ones. Via the formulas given at 4.7.2.4 and 4.7.2.5.2, for any sections z_i of $\widetilde{\Omega}^i_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$ and ω_j of $\widetilde{\Omega}^j_{X^\sharp/S^\sharp}$, with notation 4.7.2.4, we get

$$\nabla^{i+j}(\omega_j \wedge z_i) = (-1)^j \omega_j \wedge \nabla^i(z_i) + d^j_{\mathcal{B}_X}(\omega_j) \wedge z_i. \qquad (4.7.2.6.2)$$

When the connection is integrable, we denote by $\mathrm{DR}(\mathcal{E})$ the de Rham complex, by convention with $\mathcal{E}$ on the right side.

4.7.2.7. Let $\mathcal{E}$ be a $\mathcal{B}_X$-module endowed with a connection relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$. Suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d$. For any section x of $\mathcal{E}$, we denote by $\partial_{\sharp 1}(x), \ldots, \partial_{\sharp d}(x)$ the elements of $\mathcal{E}$ such that

$$\nabla(x) = \sum_{i=1}^{d} \partial_{\sharp i}(x) \otimes d\log u_i. \tag{4.7.2.7.1}$$

Using 4.7.2.5.2, we compute:

$$\nabla^1 \circ \nabla(x) = \nabla^1(\sum_{i=1}^{d} \partial_{\sharp i}(x) \otimes d\log u_i) = \sum_{j=1}^{d}\sum_{i=1}^{d} \partial_{\sharp j}(\partial_{\sharp i}(x)) \otimes d\log u_j \wedge d\log u_i.$$

Hence, the connection is integral is equivalent to saying that the maps $\partial_{\sharp i} \in \mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$ commute two by two, i.e., for any $i, j \in \{1, \ldots, d\}$, for any section x of $\mathcal{E}$ we have the equality:

$$\partial_{\sharp j}(\partial_{\sharp i}(x)) = \partial_{\sharp i}(\partial_{\sharp j}(x)).$$

For any $\underline{k} \in \mathbb{N}^d$, we get the $\mathcal{O}_S$-linear map $\underline{\partial}_\sharp^{\underline{k}} \colon \mathcal{E} \to \mathcal{E}$ by setting $\underline{\partial}_\sharp^{\underline{k}} = \partial_{\sharp 1}^{k_1} \circ \cdots \circ \partial_{\sharp d}^{k_d}$. When, the connection is integrable, for any $\underline{k}, \underline{l} \in \mathbb{N}^d$, we get the equality

$$\underline{\partial}_\sharp^{\underline{l}} \circ \underline{\partial}_\sharp^{\underline{k}} = \underline{\partial}_\sharp^{\underline{k}} \circ \underline{\partial}_\sharp^{\underline{l}} = \underline{\partial}_\sharp^{\underline{k}+\underline{l}}. \tag{4.7.2.7.2}$$

Theorem 4.7.2.8. *Let $\mathcal{E}$ be a $\mathcal{B}_X$-module. The following are equivalent.*

(a) An integrable connection relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$ on $\mathcal{E}$.

(b) A structure of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module on $\mathcal{E}$ extending its structure of $\mathcal{B}_X$-module.

Proof. We copy the proof of 2.3.2.6 □

The following proposition states that the equivalence of Theorem 4.7.2.8 is in fact an equivalence of categories.

Proposition 4.7.2.9. *Let $\mathcal{E}, \mathcal{F}$ be two left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules and $f \colon \mathcal{E} \to \mathcal{F}$ be a $\mathcal{B}_X$-linear homomorphism. The following are equivalent.*

(a) The morphism f is $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-linear.

(b) The square below is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \\ \downarrow f & & \downarrow f\otimes \mathrm{id} \\ \mathcal{F} & \xrightarrow{\nabla} & \mathcal{F} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^1_{X^\sharp/S^\sharp}, \end{array} \tag{4.7.2.9.1}$$

where the connections are the ones associated via 4.7.2.8 with the left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module structures.

Proof. Since the proposition is local, then we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d$. By using the formula 4.7.2.7.1, we compute that the square 4.7.2.9.1 is commutative if and only if $\partial_{\sharp i}(f(x)) = f(\partial_{\sharp i}(x))$ for any $i = 1, \ldots, r$. Since the ring $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is generated as $\mathcal{O}_S$-algebra by $\mathcal{B}_X$ and by $\partial_{\sharp 1}, \ldots, \partial_{\sharp d}$, then we are done. □

4.7.2.10. Let $\phi \colon \mathcal{E} \to \mathcal{F}$ be a homomorphism of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules. It follows from 4.7.2.6.2 that the commutativity of 4.7.2.9.1 implies that of

$$\begin{array}{ccc} \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} & \xrightarrow{\nabla^i} & \mathcal{E} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp} \\ \downarrow \phi\otimes \mathrm{id} & & \downarrow \phi\otimes \mathrm{id} \\ \mathcal{F} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^i_{X^\sharp/S^\sharp} & \xrightarrow{\nabla^i} & \mathcal{F} \otimes_{\mathcal{B}_X} \widetilde{\Omega}^{i+1}_{X^\sharp/S^\sharp}. \end{array} \tag{4.7.2.10.1}$$

Hence, we get the morphism of $C(f^{-1}\mathcal{O}_S)$ of the form $\mathrm{DR}(\mathcal{E}) \to \mathrm{DR}(\mathcal{F})$ (see notation 4.7.2.6).

4.7.2.11. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module. We get the integrable connection $\nabla\colon \mathcal{E} \to \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}$ (see 4.7.2.8 and 4.7.2.6). This yields (see the construction of 4.7.2.5 and 4.7.2.6) the $\mathcal{O}_S$-linear map

$$\nabla^{n-1}\colon \widetilde{\Omega}^{n-1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \to \widetilde{\Omega}^{n}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}. \tag{4.7.2.11.1}$$

Suppose now $X^\sharp/S^\sharp$ has logarithmic coordinates $u_1, \dots, u_d$. Let $\{i_1, \dots, i_{n-1}\}$ be $n-1$ elements of $\{1, \dots, d\}$. Let $\{j_1, \dots, j_{d-n+1}\}$ be the complementary. For any section x of $\mathcal{E}$, the formula 4.7.2.6.2 yields

$$\begin{aligned}\nabla^{n-1}((d\log u_{i_1} \wedge \cdots \wedge d\log u_{i_{n-1}}) \otimes x) &= (-1)^{n-1} d\log u_{i_1} \wedge \cdots \wedge d\log u_{i_{n-1}} \wedge \nabla(x)\\ &= (-1)^{n-1} \sum_{a=1}^{d-n+1} (d\log u_{i_1} \wedge \cdots \wedge d\log u_{i_{n-1}} \wedge d\log u_{j_a}) \otimes \partial_{\sharp j_a}(x).\end{aligned} \tag{4.7.2.11.2}$$

4.7.2.12. Viewing $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ as a left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module, we get from 4.7.2.11.1 the $\mathcal{O}_S$-linear map:

$$\nabla^{n-1}\colon \widetilde{\Omega}^{n-1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \to \widetilde{\Omega}^{n}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}. \tag{4.7.2.12.1}$$

In fact, since $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ as a $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-bimodule then we get by functoriality (see 4.7.2.9) that 4.7.2.12.1 is a homomorphism of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules.

Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module. Similarly to 2.3.2.10.3, we get the commutative diagram:

$$\begin{array}{ccc}(\widetilde{\Omega}^{n-1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}} \mathcal{E} & \xrightarrow{\nabla^{n-1}\otimes \mathrm{id}} & (\widetilde{\Omega}^{n-1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}} \mathcal{E} \\ \downarrow\wr & & \downarrow\wr \\ \widetilde{\Omega}^{n-1}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} & \xrightarrow{\nabla^{n-1}} & \widetilde{\Omega}^{n}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E},\end{array} \tag{4.7.2.12.2}$$

where the vertical isomorphisms are the canonical ones. Hence we have the canonical isomorphism of $C({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$ of the form $\mathrm{DR}(\mathcal{E}) \xrightarrow{\sim} \mathrm{DR}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}} \mathcal{E}$.

4.7.3 Tangent sheaf, homological dimension, Spencer resolutions

4.7.3.1 (Tangent sheaf)**.** When $\mathcal{E}$ is a $\mathcal{B}_X$-module, we write $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{B}_X}(\mathcal{E}, \mathcal{B}_X)$. We set $\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} := (\widetilde{\Omega}^1_{X^\sharp/S^\sharp})^\vee$, the tangent sheaf relative to $X^\sharp/S^\sharp$ with coefficients in $\mathcal{B}_X$. From the canonical inclusion $\widetilde{j}\colon \widetilde{\Omega}^1_{X^\sharp/S^\sharp} \hookrightarrow \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp}$, we obtain by duality the canonical epimorphism $\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp,1} \twoheadrightarrow \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$ whose kernel is $\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp,0} = \mathcal{B}_X$. From the canonical epimorphism $\widetilde{\varpi}_{X^\sharp/S^\sharp}\colon p_{0*}\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \twoheadrightarrow \widetilde{\Omega}^1_{X^\sharp/S^\sharp}$ (see 4.7.1.5.3), we get by duality the $\mathcal{B}_X$-linear monomorphism $\widetilde{\varpi}^\vee_{X^\sharp/S^\sharp}\colon \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \hookrightarrow \widetilde{\mathcal{D}}_{X^\sharp/S^\sharp,1}$ (for the left structure of $\widetilde{\mathcal{D}}_{X^\sharp/S^\sharp,1}$).

The morphisms $\widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{P}}^1_{X^\sharp/S^\sharp,(m)}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,1} \to \widetilde{\mathcal{D}}_{X^\sharp/S^\sharp,1}$ are isomorphisms for any $m \in \mathbb{N}$. This yields $\mathrm{gr}_1 \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$. Moreover, we get the $\mathcal{B}_X$-linear monomorphism

$$\iota^{(m)}\colon \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \overset{\widetilde{\varpi}^\vee_{X^\sharp/S^\sharp}}{\hookrightarrow} \widetilde{\mathcal{D}}_{X^\sharp/S^\sharp,1} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,1} \hookrightarrow \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}. \tag{4.7.3.1.1}$$

Suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. Following the local description given at 4.7.1.7, $\iota^{(m)}(\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp})$ is equal to the free $\mathcal{B}_X$-submodule (for the left or the right structure) of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ generated by elements $\partial_{\sharp 1}, \dots, \partial_{\sharp d}$.

Proposition 4.7.3.2. *The homomorphism $\iota^{(0)}\colon \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ (see 4.7.3.1.1) induces the isomorphism of graded $\mathcal{B}_X$-algebras:*

$$\mathbb{S}(\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \mathrm{gr}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}.$$

Proof. Since this is local, we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. Following 4.7.3.1, $\iota^{(0)}$ identifies $\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$ with the free $\mathcal{B}_X$-submodule of $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ generated by elements $\partial_{\sharp 1},\dots,\partial_{\sharp d}$. Denote by $\xi_{\sharp i}$ the image of $\partial_{\sharp i}$ in $\mathrm{gr}_1\,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$. Then it follows from 4.1.2.17.a that $\mathrm{gr}\,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is equal to the commutative polynomial $\mathcal{B}_X$-algebra with the variable $\xi_{\sharp 1},\dots,\xi_{\sharp d}$.

□

We suppose from now $\underline{f}\colon X\to S$ has pure relative dimension d.

Notation 4.7.3.3. We sometimes remove the canonical inclusion $\iota^{(0)}$ in the notation, i.e. we might canonically identify $\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$ with a sub-$\mathcal{B}_X$-module of $\mathcal{D}^{(0)}_{X^\sharp/S^\sharp}$. For any sections v_1,v_2 of $\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$, we denote by $[v_1,v_2]$ the section of $\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$ which corresponds to the section $v_1v_2-v_2v_1$ (we use the ring structure of $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ and we remark $v_1v_2-v_2v_1\in\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$). The $f^{-1}\mathcal{O}_S$-bilinear map $[-,-]\colon\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\times\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\to\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$ satisfies the Jacobi identity, i.e. we get a Lie bracket on the tangent space with coefficients in $\mathcal{B}_X$.

Definition 4.7.3.4. Let $\mathcal{E}=(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a filtered left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module, i.e a left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module endowed with an exhaustive filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ by $\mathcal{B}_X$-submodules so that $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp,n'}\cdot\mathcal{E}_n\subset\mathcal{E}_{n+n'}$.

For any sections v_1,v_2 of $\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}$ and b of $\mathcal{B}_X$, we compute $v_1b-bv_1=v_1(b)$ and $[bv_1,v_2]=b[v_1,v_2]-v_2(b)v_1$. Hence, similarly to [Kas95, 1.6], we can check that the morphism of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$\delta\colon\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^i\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}_{j-1}\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^{i-1}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}_j \tag{4.7.3.4.1}$$

given by

$$\begin{aligned}\delta\left(P\otimes(v_1\wedge\cdots\wedge v_i)\otimes u\right)=&\sum_{a=1}^{i}(-1)^{a-1}Pv_a\otimes(v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge v_i)\otimes u\\&-\sum_{a=1}^{i}(-1)^{a-1}P\otimes(v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge v_i)\otimes v_au\\&+\sum_{1\le a<b\le i}(-1)^{a+b}P\otimes([v_a,v_b]\wedge v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge\widehat{v_b}\wedge\cdots\wedge v_i)\otimes u\end{aligned}$$

is well defined. Moreover, we compute easily that we get the following complex of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$0\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^d\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}_{n-d}\cdots\xrightarrow[\delta]{}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}_{n-1}\xrightarrow[\delta]{}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}_n\to\mathcal{E}\to 0. \tag{4.7.3.4.2}$$

We call 4.7.3.4.2 the first Spencer sequence of degree n of $\mathcal{E}$ and denote it by $\mathrm{Sp}_{n,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{E})$ or $\mathrm{Sp}_n(\mathcal{E})$.

Remark 4.7.3.5. With the notations of 4.7.3.4, we put $\mathcal{M}:=\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}$ and $\mathcal{M}_s:=\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}_s$. We define a homomorphism of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules $\mathcal{M}_{s-1}\otimes_{\mathcal{B}_X}\wedge^r\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\to\mathcal{M}_s\otimes_{\mathcal{B}_X}\wedge^{r-1}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ by applying the functor $\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}-$ to 4.7.3.4.1 and by functoriality of the transposition isomorphism $\delta_{X^\sharp}\colon\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ (see 4.2.5.5). We get therefore a complex denoted by $\mathrm{Sp}^\bullet_{s,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{M})$ and inducing by construction the isomorphism $\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\mathrm{Sp}^\bullet_{s,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{E})\xrightarrow{\sim}\mathrm{Sp}^\bullet_{s,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{M})$.

Theorem 4.7.3.6. *With the notations of 4.7.3.4, let us suppose moreover that the filtration of $\mathcal{E}$ is good. Hence, for s large enough, $\mathrm{Sp}^\bullet_{s,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{E})$ is exact.*

Proof. This is checked similarly to [Kas95, 1.6.1]. More precisely, the proof has the following two steps.

Step 1. We check by induction on the integer $s\ge 0$ that $\mathrm{Sp}^\bullet_{s,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S})$ is an exact sequence when $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S}$ is endowed with its order filtration. When $s=0$, this is straightforward since $\mathrm{Sp}^\bullet_{0,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S})$

is equal to the exact sequence

$$0 \to 0 \xrightarrow{\delta} \cdots \xrightarrow{\delta} 0 \xrightarrow{\delta} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S,0} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S} \to 0, \tag{4.7.3.6.1}$$

Suppose now $\mathrm{Sp}^{\bullet}_{s-1,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S})$ is an exact sequence for $s \geq 1$ and consider the commutative diagram where we have removed $/S$ in the notation

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \wedge^d \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp,s-1-d} & \cdots \xrightarrow{\delta} & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp,s-2} & \xrightarrow{\delta} & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp,s-1} & \to & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \wedge^d \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp,s-d} & \cdots \xrightarrow{\delta} & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp,s-1} & \xrightarrow{\delta} & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp,s} & \to & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \wedge^d \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \mathrm{gr}_{s-d}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} & \cdots \xrightarrow{\mathrm{id}\otimes\beta} & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \mathrm{gr}_{s-1}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} & \xrightarrow{\mathrm{id}\otimes\beta} & \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} \otimes_{\mathcal{B}_X} \mathrm{gr}_s\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp} & \to & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & 0 & & &
\end{array} \tag{4.7.3.6.2}$$

where the first row (resp. second row) is $\mathrm{Sp}^{\bullet}_{s-1,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S})$ (resp. $\mathrm{Sp}^{\bullet}_{s,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S})$), where $\beta \colon \wedge^i \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \mathrm{gr}_{j-1}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \to \wedge^{i-1} \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \mathrm{gr}_j\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is given by the formula

$$\beta\left((v_1 \wedge \cdots \wedge v_i) \otimes \overline{u}\right) = \sum_{a=1}^{i} (-1)^a (v_1 \wedge \cdots \wedge \widehat{v_a} \wedge \cdots \wedge v_i) \otimes v_a \overline{u}.$$

The fact that the sequence

$$0 \longrightarrow \wedge^d \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \mathrm{gr}_{s-d}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \cdots \xrightarrow{\beta} \widetilde{\mathcal{T}}_{X^\sharp/S} \otimes_{\mathcal{B}_X} \mathrm{gr}_{s-1}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \xrightarrow{\beta} \mathrm{gr}_s\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \longrightarrow 0 \tag{4.7.3.6.3}$$

is exact is local. Hence, we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,d}$ that we fix. This yields $\mathrm{gr}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is a ring of polynomial with coefficients over $\mathcal{B}_X$ with coordinates $\xi_{\sharp 1}, \ldots, \xi_{\sharp d}$ the elements induced by $\partial_{\sharp 1}, \ldots, \partial_{\sharp d}$ (see 4.1.2.17). Moreover, we see that the exact sequence 4.7.3.6.3 is equal to the s-graded part of the Koszul complex of $\mathrm{gr}\, \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ associated with the regular elements $\xi_{\sharp 1}, \ldots, \xi_{\sharp d}$. Hence, 4.7.3.6.3 is exact. We conclude by noticing that the forth row of the diagram 4.7.3.6.2 is equal to the image of 4.7.3.6.3 by the exact functor $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} -$.

Step 2. We conclude therefore similarly to the end of the proof of [Kas95, 1.6.1]. □

Example 4.7.3.7. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module which is $\mathcal{B}_X$-coherent. Then its trivial filtration, i.e. the filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ with $\mathcal{E}_n = \mathcal{E}$, is a good filtration. Hence, we get the exact sequence of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$0 \to \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \wedge^d \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \cdots \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \wedge \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E} \to \mathcal{E} \to 0, \tag{4.7.3.7.1}$$

that we denote by $\mathrm{Sp}^{\bullet}_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{E})$. In particular, taking the trivial filtration of $\mathcal{B}_X$, we get the exact sequence of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$0 \to \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \wedge^d \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \cdots \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \wedge \widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \to \mathcal{B}_X \to 0 \tag{4.7.3.7.2}$$

where the map
$$\delta\colon \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^i\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\to \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^{i-1}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp} \tag{4.7.3.7.3}$$
is given by the formula
$$\begin{aligned}\delta\left(P\otimes(v_1\wedge\cdots\wedge v_i)\right)=&\sum_{a=1}^{i}(-1)^{a-1}Pv_a\otimes(v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge v_i)\\ &+\sum_{1\le a<b\le i}(-1)^{a+b}P\otimes([v_a,v_b]\wedge v_1\wedge\cdots\wedge\widehat{v_a}\wedge\cdots\wedge\widehat{v_b}\wedge\cdots\wedge v_i).\end{aligned}$$

In particular, we get
$$\mathcal{B}_X\in D^{\mathrm{b}}_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}). \tag{4.7.3.7.4}$$

Remark 4.7.3.8. Let us denote by $\widetilde{\mathrm{Sp}}^{(0)}_{X^\sharp/S^\sharp}:=\mathrm{Sp}^{\bullet}_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{B}_X)$ and $\mathrm{Sp}^{(0)}_{X^\sharp/S^\sharp}:=\mathrm{Sp}^{\bullet}_{\mathcal{D}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{O}_X)$. Then by construction we have the canonical isomorphism of $C({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$:
$$\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{D}^{(0)}_{X^\sharp/S^\sharp}}\mathrm{Sp}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\widetilde{\mathrm{Sp}}^{(0)}_{X^\sharp/S^\sharp}. \tag{4.7.3.8.1}$$

Since $\widetilde{\mathrm{Sp}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\mathcal{B}_X$ and $\mathrm{Sp}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\mathcal{O}_X$ (see 4.7.3.7.2), this yields the well known (e.g. use 4.3.4.6.1) isomorphism
$$\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{D}^{(0)}_{X^\sharp/S^\sharp}}\mathcal{O}_X\xrightarrow{\sim}\mathcal{B}_X. \tag{4.7.3.8.2}$$

Conversely, from the 4.7.3.8.2 and $\mathrm{Sp}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\mathcal{O}_X$ we get $\widetilde{\mathrm{Sp}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\mathcal{B}_X$ (i.e. Spencer resolution with constant coefficients implies Spencer resolution with coefficients, which was not obvious since $\mathcal{D}^{(0)}_{X^\sharp/S^\sharp}\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is not flat).

4.7.3.9. We can construct the map 4.7.2.12.1 in a second way by duality from the Spencer morphisms as follows. Since $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is a locally free $\mathcal{B}_X$-module for its left structure (in particular), since $\widetilde{\Omega}^i_{X^\sharp/S^\sharp}$ is locally free of finite type for any integer $0\le i\le d$, then we get the canonical isomorphism of $(\mathcal{B}_X,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$-bimodules:
$$\begin{aligned}&\widetilde{\Omega}^i_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\wedge^i(\widetilde{\mathcal{T}}^\vee_{X^\sharp/S^\sharp})\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\underset{2.3.3.7.1}{\xrightarrow{\sim}}\wedge^i(\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp})^\vee\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\sim}\\ &\xrightarrow{\sim}\mathcal{H}om_{\mathcal{B}_X}\left(\wedge^i\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp},\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\right)\xrightarrow{\sim}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}\left(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^i\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp},\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\right).\end{aligned} \tag{4.7.3.9.1}$$

Hence, for any integer $0\le n\le d$, we get the morphism of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules $\delta^\star_n$ making commutative the diagram
$$\begin{array}{ccc}\widetilde{\Omega}^{n-1}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} & \xrightarrow{\delta^\star_n} & \widetilde{\Omega}^{n}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\\ \sim\downarrow 4.7.3.9.1 & & \sim\downarrow 4.7.3.9.1\\ \mathcal{H}om_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}\left(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^{n-1}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp},\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\right) & \xrightarrow{\delta^\vee_n} & \mathcal{H}om_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}\left(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^{n}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp},\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\right),\end{array} \tag{4.7.3.9.2}$$
where $\delta^\vee_n$ is the image under the functor $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(-,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$ of the Spencer morphism of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules 4.7.3.7.3.

Lemma 4.7.3.10. *Both maps 4.7.2.12.1 and 4.7.3.9.2 are equal, i.e.* $\delta^\star_n=\nabla^{n-1}$.

Proof. We copy the proof of 2.3.3.9. □

4.7.3.11. By applying the functor $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(-,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$ to the complex of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^d\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\xrightarrow[\delta_d]{}\cdots\xrightarrow[\delta]{}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\xrightarrow[\delta_1]{}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \tag{4.7.3.11.1}$$

via the canonical isomorphisms 4.7.3.9.1, we get the complex of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow[\nabla^0]{}\widetilde{\Omega}^1_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow[\nabla^1]{}\cdots\xrightarrow[\nabla]{}\widetilde{\Omega}^{d-1}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow[\nabla^{d-1}]{}\widetilde{\Omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}. \tag{4.7.3.11.2}$$

It follows from 4.7.3.10 that the complex of $C({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$ defined at 4.7.3.11.2 is the de Rham complex of $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ and is denoted by $\mathrm{DR}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$ (see 4.7.2.5).

If $\mathcal{E}\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$, we define the de Rham complex of $\mathcal{E}$ to be the complex

$$\mathrm{DR}(\mathcal{E}):=\mathrm{DR}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}\mathcal{E}.$$

Remark, when $\mathcal{E}$ is a left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module, following 4.7.2.12 we retrieve the usual de Rham complex (up to canonical isomorphisms) as defined at 4.7.2.5.

4.7.3.12. By applying the functor $\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}-$ to the morphism of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules 4.7.3.7.3, we get the morphism of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules:

$$\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^n\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp})\to\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^{n-1}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}). \tag{4.7.3.12.1}$$

We have moreover the isomorphism of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$\begin{aligned}\widetilde{\Omega}^{d-n}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}&\xrightarrow[2.3.3.7.5]{\sim}(\wedge^n\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\omega}_{X^\sharp/S^\sharp})\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\\&\xrightarrow[\widetilde{\delta}_{X^\sharp/S^\sharp}\otimes\mathrm{id}_{\wedge^n\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}}]{\sim}\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^n\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}),\end{aligned} \tag{4.7.3.12.2}$$

where $\widetilde{\delta}_{X^\sharp/S^\sharp}$ is the transposition isomorphism (see 4.2.5.6.1)

Lemma 4.7.3.13. *For any integer $0\le n\le d$, the following square of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules*

$$\begin{array}{ccc}\widetilde{\Omega}^{d-n}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} & \xrightarrow[4.7.3.12.2]{\sim} & \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^n\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp})\\ {\scriptstyle (-1)^{d-n+1}\nabla^{d-n}}\big\downarrow{\scriptstyle 4.7.2.12.1} & & {\scriptstyle 4.7.3.12.1}\big\downarrow{\scriptstyle \mathrm{id}\otimes\delta_n}\\ \widetilde{\Omega}^{d-n+1}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} & \xrightarrow[4.7.3.12.2]{\sim} & \widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\wedge^{n-1}\widetilde{\mathcal{T}}_{X^\sharp/S^\sharp}),\end{array} \tag{4.7.3.13.1}$$

is commutative.

Proof. We can copy the proof of 2.3.3.12. □

Proposition 4.7.3.14. *We have the following properties.*

(i) The map $\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\xrightarrow{\beta}\widetilde{\omega}_{X^\sharp/S^\sharp}$ given by the structure of a right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module on $\widetilde{\omega}_{X^\sharp/S^\sharp}$ induces a $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-linear resolution $\mathrm{DR}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})[d]\xrightarrow{\sim}\widetilde{\omega}_{X^\sharp/S^\sharp}$ of $\widetilde{\omega}_{X^\sharp/S^\sharp}$.

(ii) $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{B}_X,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})=0$ for $i\neq d$. There is a canonical isomorphism of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules

$$\mathcal{E}xt^d_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}}(\mathcal{B}_X,\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})\xrightarrow{\sim}\widetilde{\omega}_{X^\sharp/S^\sharp}.$$

Proof. By applying the functor $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} -$ to the exact sequence 4.7.3.7.2, with 4.7.3.13, we get that $\mathrm{DR}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})[d]$ is a resolution of $\widetilde{\omega}_{X^\sharp/S^\sharp}$. Moreover, we compute that the map $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \xrightarrow{\beta} \widetilde{\omega}_{X^\sharp/S^\sharp}$ of the resolution $\mathrm{DR}(\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})[d] \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp/S^\sharp}$ is given by the structure of a right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-module on $\widetilde{\omega}_{X^\sharp/S^\sharp}$. □

Remark 4.7.3.15 (Level m case). We get the level $m \in \mathbb{N} \cup \{\infty\}$ of the formal case when we tensor by $\mathbb{Q}$. More precisely, suppose we are of the formal case and write $X^\sharp/S^\sharp$ by $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$. Suppose that $\mathcal{B}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X},\mathbb{Q}}$ is an isomorphism. The extension $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \mathcal{B}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is therefore an isomorphism. Hence denoting $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, we get $\mathcal{B}_{\mathfrak{X}} \in D^{\mathrm{b}}_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and 4.7.3.6 and 4.7.3.14 are still valid by replacing 0 by m.

Corollary 4.7.3.16 (Homological dimension)**.** *Suppose S is affine, X is affine and regular, and $f\colon X^\sharp \to S^\sharp$. Let $r := \sup_{s \in f(X)} \dim \mathcal{O}_{S,s}$. Then the ring $D^{(0)}_{X^\sharp/S^\sharp} := \Gamma(X, \mathcal{D}^{(0)}_{X^\sharp/S^\sharp})$ has homological global dimension equal to $2d + r$.*

Proof. We can copy the proof of 2.3.4.5. □

Chapter 5

Operations on differential of level m of finite order modules

5.1 Definitions of the functors and first properties

This section deals with cohomological operations at fixed level $m \in \mathbb{N}$. Over $\mathbb{C}$ we have in the theory of $\mathcal{D}$-modules four operations basing on which we can deduce the others: the direct image f_+, extraordinary inverse image $f^!$, dual, external tensor product. In the arithmetic theory, it is also necessary to consider two supplementary operations:

- the base change, the base scheme is not necessarily the spectrum of a field;

- the extension of ring of differential operators via ring homomorphisms of the form $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp} \to \mathcal{B}'_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$. with $m \leq m'$ and induced by coefficient homomorphisms $\mathcal{B}_X \to \mathcal{B}'_X$.

5.1.1 Extraordinary inverse image of complexes of $\mathcal{D}$-modules, base change

Unless otherwise stated, we use in this subsection the notation of 5.1.1.1.

5.1.1.1. Let

$$\begin{array}{ccc} X^\sharp & \xrightarrow{f} & Y^\sharp \\ \downarrow{\scriptstyle p_{X^\sharp}} & & \downarrow{\scriptstyle p_{Y^\sharp}} \\ S^\sharp & \xrightarrow{\mathscr{b}} & T^\sharp, \end{array} \tag{5.1.1.1.1}$$

be a commutative diagram where $S^\sharp$ and $T^\sharp$ are nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ as defined in 3.1.1.1 with i an integer (resp. $S^\sharp$ and $T^\sharp$ are nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10), where $X^\sharp$ is a log smooth $S^\sharp$-log-scheme (resp. log smooth $S^\sharp$-log formal scheme) and $Y^\sharp$ is a log smooth $T^\sharp$-log scheme (resp. a log smooth $T^\sharp$-log formal scheme). Let $\mathcal{B}_X$ (resp. $\mathcal{B}_Y$) be a commutative $\mathcal{O}_X$-algebra (resp. $\mathcal{O}_Y$-algebra) endowed with a compatible structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module). Let us recall that the action of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $f^*\mathcal{B}_Y$ is compatible with its structure of $\mathcal{O}_X$-algebra (see 3.4.4.6). We suppose finally that we have a morphism of algebras $f^*\mathcal{B}_Y \to \mathcal{B}_X$ which is moreover $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. We will again denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$. We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{Y}^\sharp$) the ringed logarithmic ($\mathcal{V}$-formal) scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(Y^\sharp, \mathcal{B}_Y)$), and by $\tilde{f} \colon \widetilde{X}^\sharp/S^\sharp \to \widetilde{Y}^\sharp/T^\sharp$ the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by the diagram 5.1.1.1.1 and by $f^*\mathcal{B}_Y \to \mathcal{B}_X$. When $S^\sharp \to T^\sharp$ in understood, by abuse of notation, we sometimes also denote by $\tilde{f}$ the induced morphism $\widetilde{X}^\sharp \to \widetilde{Y}^\sharp$ of ringed logarithmic ($\mathcal{V}$-formal) schemes.

Let $U := X^{\sharp *}$ be the open of X where $M_{X^\sharp}$ is trivial and $j_U \colon U \hookrightarrow X^\sharp$ be the canonical open immersion. Let $V := Y^{\sharp *}$ be the open of Y where $M_{Y^\sharp}$ is trivial and $j_V \colon V \hookrightarrow X^\sharp$ be the canonical open immersion.

Notation 5.1.1.2. We deduce by functoriality from 4.4.2.4 that we get a structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodule on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} := \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$. When $S^\sharp \to T^\sharp$ is the identity, we can simply write $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/T^\sharp}$ and when moreover there is no doubt about S we write $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp}$. When $\mathcal{B}_X = \mathcal{O}_X$ and $\mathcal{B}_Y = \mathcal{O}_Y$, we denote this bimodule by $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$, and when $\mathscr{b} = \mathrm{id}$, we simply write $\mathcal{D}^{(m)}_{X^\sharp\to Y^\sharp/T^\sharp}$ or $\mathcal{D}^{(m)}_{X^\sharp\to Y^\sharp}$.

We deduce by functoriality from 4.4.2.4 and 4.3.5.7, we get a structure of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodule on

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} := \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{f}^*_r\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right),$$

where the index "r" means that we have chosen the right (i.e. the twisted) structure of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$ on $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}$ to compute structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ via the functor $\widetilde{f}^*$.

When $S \to T$ is the identity, we can simply write $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp}$ and when moreover there is no doubt about S we write $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp}$. When $\mathcal{B}_X = \mathcal{O}_X$ and $\mathcal{B}_Y = \mathcal{O}_Y$, we denote this bimodule by $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp}$, and when $\mathscr{b} = \mathrm{id}$, we simply write $\mathcal{D}^{(m)}_{X^\sharp\to Y^\sharp/T^\sharp}$ or $\mathcal{D}^{(m)}_{Y^\sharp\leftarrow X^\sharp}$.

We have the isomorphism of left $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodules

$$\widetilde{f}^*_r\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right) \underset{\widetilde{f}^*(4.2.5.6.3)}{\overset{\sim}{\longrightarrow}} \widetilde{f}^*_l\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right),$$

where the index "l" means that we have chosen the left structure. By tensoring this latter isomorphism with $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} -$, we get the isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodules

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \overset{\sim}{\longrightarrow} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}. \tag{5.1.1.2.1}$$

Lemma 5.1.1.3. *Suppose $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3). Then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\to X^\sharp/S^\sharp}$) is solvable in the sense of 4.6.3.2.b as a complex of $C(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $C(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$).*

Proof. By definition, there exists a morphism of schemes (resp. of $\mathcal{V}$-formal schemes) of the form $T \to U$ such that both induced morphisms of ringed spaces $g\colon (X, \mathcal{B}_X) \to U$ and $h\colon (Y, \mathcal{B}_Y) \to U$ are flat. Since $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}/\mathcal{B}_Y$ is flat, since h is flat we get that $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}/h^{-1}(\mathcal{O}_U)$ is flat. Moreover, $h^{-1}(\mathcal{O}_U)$ is sent into the center of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$. Hence, $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}/g^{-1}(\mathcal{O}_U)$ is flat and $g^{-1}(\mathcal{O}_U)$ is sent in the center of $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}/\mathcal{B}_X$ is flat, since g is flat, we get that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}/g^{-1}(\mathcal{O}_U)$ is flat. Hence, $g^{-1}(\mathcal{O}_U)$ is a solving ring of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$. Similarly, we check that $g^{-1}(\mathcal{O}_U)$ is a solving ring of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp}$. □

Definition 5.1.1.4. We keep notation 5.1.1.2. Set $d_f := \delta_{X^\sharp/S^\sharp} - \delta_{Y^\sharp/T^\sharp} \circ f$, where $\delta_{X^\sharp/S^\sharp}$, $\delta_{Y^\sharp/T^\sharp}$ are respectively the rank (as a locally constant function on X' or X respectively) of the locally free modules $\Omega_{X^\sharp/S^\sharp}$ and $\Omega_{Y^\sharp/T^\sharp}$.

(a) The (left version of the) extraordinary inverse image functor of level m by $\widetilde{f}$ is the functor $\widetilde{f}^{(m)!}\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ which is defined by setting for any $\mathcal{F} \in D(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$:

$$\widetilde{f}^{(m)!}(\mathcal{F}) := \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F}\,[d_f].$$

(b) The (right version of the) extraordinary inverse image functor of level m by $\widetilde{f}$ is the functor $\widetilde{f}^{(m)!}\colon D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ which is defined by setting

$$\widetilde{f}^{(m)!}(\mathcal{M}) := f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp}\,[d_f],$$

where $\mathcal{M} \in D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.

5.1.1.5 (Left to right)**.** The left and the right versions of the extraordinary inverse image functors of level m by $\widetilde{f}$ of 5.1.1.4 are compatible with the quasi-inverse exact functors of 4.3.5.7 exchanging left and right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structures (resp. left and right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module structures). More precisely, for any $\mathcal{M} \in D({}^r\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, we have the isomorphisms:

$$\begin{aligned}\widetilde{f}^{(m)!}(\mathcal{M} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}) &\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \left(f^{-1}(\mathcal{M}) \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp})\right)\ [d_f]\\ &\xrightarrow{\sim} f^{-1}(\mathcal{M}) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp})\right)\ [d_f]\\ &\underset{5.1.1.2.1}{\xrightarrow{\sim}} \widetilde{f}^{(m)!}(\mathcal{M}) \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}. \end{aligned} \tag{5.1.1.5.1}$$

5.1.1.6. Let $\mathscr{P}_{Y^\sharp/T^\sharp}$ be the collection of complexes of $K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ which are K-flat as a complex of $K(\mathcal{B}_Y)$.

(a) Following 4.4.2.4, we have the functor $\widetilde{f}^* := \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}-\colon K({}^l\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to K({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. Moreover, for any $\mathcal{G} \in K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, it follows from 4.4.2.9.1 that the canonical morphism

$$\widetilde{f}^*(\mathcal{G}) := \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\mathcal{G} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{G} \tag{5.1.1.6.1}$$

is an isomorphism of $K(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. When the coefficients are constant, i.e. $\mathcal{B}_X = \mathcal{O}_X$ and $\mathcal{B}_Y = \mathcal{O}_Y$, we simply write f^* this functor.

(b) The collection $\mathscr{P}_{Y^\sharp/T^\sharp}$ satisfies the conditions (1) and (2) of [Sta22, 13.14.15] with respect to the functor

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}-\colon K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}),$$

which implies that for any quasi-isomorphism $\mathcal{P} \xrightarrow{\sim} \mathcal{E}$ of $K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ such that $\mathcal{P}$ is an object of $\mathscr{P}$, (in that case we say that $\mathcal{P}$ is representing $\mathcal{E}$), then

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{P} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{E}.$$

Indeed, since $\mathscr{P}_{Y^\sharp/T^\sharp}$ contains the collection of K-flat complexes of $K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ then we get the condition (1). Moreover, let $u\colon \mathcal{P} \to \mathcal{P}'$ be a quasi-isomorphism between two objects of $\mathscr{P}_{Y^\sharp/T^\sharp}$. It follows from 5.1.1.6.1 that the morphism

$$id \otimes u\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{P} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{P}'$$

is canonically isomorphic to $id\otimes u\colon \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\mathcal{P} \to \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}\mathcal{P}'$. Since $\mathcal{P}$ and $\mathcal{P}'$ are K-flat in $K(\mathcal{B}_Y)$, then it follows from [Sta22, 20.26.13] that this latter morphism is a quasi-isomorphism.

(c) Similarly, since $\mathscr{P}_{Y^\sharp/T^\sharp}$ satisfies also the condition (1) and (2) of [Sta22, 13.14.15] with respect to the functor $\widetilde{f}^*$, then $\widetilde{f}^*$ is left derivable. Let $\mathbb{L}\widetilde{f}^*\colon D({}^l\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ be the left derived functor of $\widetilde{f}^*$. For any $\mathcal{F} \in D({}^l\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, we have the canonical isomorphism of $D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\mathbb{L}\widetilde{f}^*(\mathcal{F})[d_f] \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{F}). \tag{5.1.1.6.2}$$

(d) Since 5.1.1.6.1 is an isomorphism then for any $\mathcal{P} \in \mathscr{P}_{Y^\sharp/T^\sharp}$, we have

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{P} \in \mathscr{P}_{X^\sharp/S^\sharp}. \tag{5.1.1.6.3}$$

Lemma 5.1.1.7. *For $\mathcal{E}$ and $\mathcal{F}$ two objects of $D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, with notation 5.1.1.6, we have the isomorphisms of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:*

$$\mathbb{L}\widetilde{f}^*(\mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^*(\mathcal{F}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^*(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathcal{F}), \qquad \mathbb{L}\widetilde{f}^{(m)!}(\mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^{(m)!}(\mathcal{F}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^{(m)!}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathcal{F})[d_f]. \tag{5.1.1.7.1}$$

Proof. We choose a complex $\mathcal{P}$ (resp. $\mathcal{Q}$) of $\mathscr{P}_{Y^\sharp/T^\sharp}$ (see notation of 5.1.1.6) representing $\mathcal{E}$ (resp. $\mathcal{F}$). Following 4.4.5.14.1, we get an isomorphism of $C({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ of the form

$$\widetilde{f}^*(\mathcal{P}) \otimes_{\mathcal{B}_X} \widetilde{f}^*(\mathcal{Q}) \xrightarrow{\sim} \widetilde{f}^*(\mathcal{P} \otimes_{\mathcal{B}_Y} \mathcal{Q}).$$

Hence, we are done. □

Proposition 5.1.1.8. *Suppose $\mathcal{B}_X = f^*\mathcal{B}_Y$. Suppose $f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_X$ are tor independent over $f^{-1}\mathcal{B}_Y$. Let $* \in \{\mathrm{l}, \mathrm{r}\}$, let $\mathcal{F} \in D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ and $\mathrm{forg}(\mathcal{F})$ (or simply $\mathcal{F}$) be the induced object of $D({}^*\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp})$. The canonical homomorphism*

$$f^{(m)!}(\mathrm{forg}(\mathcal{F})) \to \widetilde{f}^{(m)!}(\mathcal{F}) \tag{5.1.1.8.1}$$

is an isomorphism of $D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

Proof. When $* = \mathrm{l}$, the proposition was checked at 4.4.5.2. By using 5.1.1.5.1 (and 4.3.5.4.(c)), this yields the case $* = \mathrm{r}$. □

Remark 5.1.1.9. By functoriality, it follows from 4.4.5.2 and 5.1.1.5.1 (and exactness of the functors of 4.3.5.4.(c) that the canonical top (resp. bottom) homomorphism of $(\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules (resp. $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$-bimodules)

$$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}, \tag{5.1.1.9.1}$$

$$f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \tag{5.1.1.9.2}$$

is an isomorphism. When $f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_X$ are tor independent over $f^{-1}\mathcal{B}_Y$, then then we can replace the tensor product by the derived tensor product (e.g. this follows by functoriality from 5.1.1.8).

Proposition 5.1.1.10. *Suppose $f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ has finite tor dimension, i.e. the functor $\widetilde{f}^* = \mathcal{B}_X \otimes_{f^{-1}\mathcal{B}_Y} f^{-1}-$ from the category of $\mathcal{B}_Y$-modules to that of $\mathcal{B}_X$-modules has bounded cohomological dimension (see definition 4.6.1.4). Then the functor $\widetilde{f}^{(m)!} \colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ where $* \in \{\mathrm{r}, \mathrm{l}\}$ is way-out in both direction.*

Proof. This follows from 5.1.1.6.2. □

Lemma 5.1.1.11. *We keep notation 4.4.5.5.*

(a) We have the canonical isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, (g \circ f)^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp})$-bimodules:

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \to Z^\sharp/U^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Z^\sharp/U^\sharp}. \tag{5.1.1.11.1}$$

(b) We have the canonical isomorphism of $((g \circ f)^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodules:

$$f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp \leftarrow Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp \leftarrow X^\sharp/S^\sharp}. \tag{5.1.1.11.2}$$

Proof. a) Let us check 5.1.1.11.1. We have the isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \to Z^\sharp/U^\sharp} \xrightarrow{\sim} \widetilde{f}^*\widetilde{g}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \xrightarrow{\sim} (\widetilde{g} \circ \widetilde{f})^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}$. It follows from 4.4.5.6 that we get the canonical isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-modules

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \to Z^\sharp/U^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Z^\sharp/U^\sharp}.$$

We get by functoriality the fact that this isomorphism is an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, (g\circ f)^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp})$-bimodules.

b) Finally, we get the isomorphism 5.1.1.11.2 from 5.1.1.11.1 by twisting. □

Lemma 5.1.1.12. *Suppose $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3). We keep notation 4.4.5.5.*

(a) We have the canonical isomorphism of $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, (g\circ f)^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\to Z^\sharp/U^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Z^\sharp/U^\sharp}. \tag{5.1.1.12.1}$$

(b) We have the canonical isomorphism of $D((g\circ f)^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow X^\sharp/S^\sharp}. \tag{5.1.1.12.2}$$

Proof. a) By quasi-flatness of $\widetilde{f}$, it follows from 5.1.1.3 that the morphisms are well defined. Moreover, since $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}$ is $\mathcal{B}_Z$-flat, and $\widetilde{g}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}$ is $\mathcal{B}_Y$-flat then $\mathbb{L}\widetilde{f}^*\left(\widetilde{g}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}\right) \xrightarrow{\sim} \widetilde{f}^*\left(\widetilde{g}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp}\right)$. Hence,

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\to Z^\sharp/U^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\to Z^\sharp/U^\sharp}.$$

b) Finally, we get the isomorphism 5.1.1.12.2 from 5.1.1.12.1 by twisting (use 5.1.1.2.1 and 4.3.5.6.1). □

Proposition 5.1.1.13. *With notation 4.4.5.5, let $\mathcal{G}$ be a complex of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp})$. We have the canonical isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:*

$$\tilde{f}^{(m)!}\circ\tilde{g}^{(m)!}(\mathcal{G}) \xrightarrow{\sim} \widetilde{g\circ f}^{(m)!}(\mathcal{G}).$$

Proof. This is a consequence of 5.1.1.6.3 and 5.1.1.12. □

5.1.1.14. By functoriality, it follows from 5.1.1.13, 4.3.4.6.1 and 4.3.4.8.1 that the canonical top (resp. bottom) homomorphism of $K(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $K(f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$)

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}, \tag{5.1.1.14.1}$$

$$\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \tag{5.1.1.14.2}$$

is an isomorphism. Remark that contrary to 5.1.1.9, we do not need flatness condition with the derived tensor products version.

5.1.1.15 (Base change)**.** Suppose the diagram 5.1.1.1.1 is cartesian and the morphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism (such case is called the base change one). Then it follows from 4.4.4.1.a that we have the homomorphism of sheaves of rings

$$f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \tag{5.1.1.15.1}$$

such that the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$ is an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules. Let $\mathcal{F} \in D({}^{l}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. Since $d_{\widetilde{f}} = 0$, this yields the canonical isomorphism of $D({}^{l}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{F}). \tag{5.1.1.15.2}$$

Remark that if $p^{-1}_{X^\sharp}\mathcal{O}_S$ and $f^{-1}\mathcal{B}_Y$ are tor independent over $f^{-1}p^{-1}_{Y^\sharp}\mathcal{O}_T$, then the canonical morphism

$$p^{-1}_{X^\sharp}\mathcal{O}_S \otimes^{\mathbb{L}}_{f^{-1}p^{-1}_{Y^\sharp}\mathcal{O}_T} f^{-1}\mathcal{F} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F} \tag{5.1.1.15.3}$$

is an isomorphism. By abuse of notation, under this flatness condition we can simply denote by $\mathcal{O}_S \otimes^{\mathbb{L}}_{\mathcal{O}_T} \mathcal{F} := \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F}$.

Let $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. It follows from 4.4.4.4.4 and 5.1.1.5.1 that we have the canonical isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{M}), \qquad (5.1.1.15.4)$$

where the structure of left $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is given via 5.1.1.15.1. By functoriality, this implies that we have the canonical isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodules of the form $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$.

As for the left case, when $p^{-1}_{X^\sharp}\mathcal{O}_S$ and $f^{-1}\mathcal{B}_Y$ are tor independent over $f^{-1}p^{-1}_{Y^\sharp}\mathcal{O}_T$, the canonical morphism

$$f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}p^{-1}_{Y^\sharp}\mathcal{O}_T} p^{-1}_{X^\sharp}\mathcal{O}_T \to f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$$

is an isomorphism and we can simply set in this case $\mathcal{O}_S \otimes^{\mathbb{L}}_{\mathcal{O}_T} \mathcal{M} := f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$.

Remark 5.1.1.16. We can split the diagram 5.1.1.1.1 as follows

$$\begin{array}{ccccc} X^\sharp & \xrightarrow{g} & Z^\sharp & \xrightarrow{h} & Y^\sharp \\ \downarrow{\scriptstyle p_{X^\sharp}} & & \downarrow{\scriptstyle p_{Z^\sharp}} \ \square & & \downarrow{\scriptstyle p_{Y^\sharp}} \\ S^\sharp & = & S^\sharp & \longrightarrow & T^\sharp, \end{array} \qquad (5.1.1.16.1)$$

where the right square is cartesian. Let $\mathcal{B}_Z := h^*\mathcal{B}_Y$, $\widetilde{Z}^\sharp$ be the ringed logarithmic ($\mathcal{V}$-formal) scheme $(Z^\sharp, \mathcal{B}_Z)$, and $\widetilde{h}\colon \widetilde{Z}^\sharp/T^\sharp \to \widetilde{Y}^\sharp/T^\sharp$ be the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by the cartesian square of the diagram 7.5.6.7.1 and by $\mathcal{B}_Z = h^*\mathcal{B}_Y$. Hence, since $\tilde{g}^{(m)!} \circ \tilde{h}^{(m)!} \xrightarrow{\sim} \widetilde{f}^{(m)!}$ (see 5.1.1.13), then to study the extraordinary pullback functor (for instance), we reduce to the case of the base change or to the case where the bottom morphism of 5.1.1.1.1 is the identity.

5.1.1.17. Suppose we are in the non-respective case. To highlight the "crystalline nature" of the operations such as $f^!$ and f_+, we suppose in this paragraph we are in the context of 4.4.5.10, i.e. we generalize their construction to the case where $S^\sharp$ is equipped with a quasi-coherent m-PD-ideal $(\mathfrak{a}, \mathfrak{b}, \alpha)$, which we suppose to be m-PD-nilpotent, and where the morphisms are only defined modulo $\mathfrak{a}$. According to 4.4.5.11, we get a $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodule by setting

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} := \widetilde{f}_0^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}. \qquad (5.1.1.17.1)$$

We deduce by functoriality from 4.4.2.4 and 4.3.5.7, we get a structure of $(f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodule on

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} := \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{f}^{*}_{0r} \left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp} \right), \qquad (5.1.1.17.2)$$

where the index "r" means that we have chosen the right (i.e. the twisted) structure of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$ on $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}$ to compute structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ via the functor $\widetilde{f}^*$. For any $* \in \{\mathrm{l}, \mathrm{r}\}$, via these above bimodules we can define the pullback $\widetilde{f}_0^{(m)!}\colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ as in 5.1.1.4. The properties of the subsection extends to this context.

5.1.2 Quasi-coherence, projection formula as $\mathcal{B}$-modules

Suppose we are in the non-respective case of 5.1.1.1.

Definition 5.1.2.1. We extend the notion of quasi-coherence as follows.

(a) Let $\mathcal{E}$ be a $\mathcal{B}_X$-module. We say that $\mathcal{E}$ is $\mathcal{B}_X$-quasi-coherent if and only if locally in X we have an exact sequence of the form $\mathcal{B}_X^{(I)} \to \mathcal{B}_X^{(J)} \to \mathcal{E} \to 0$.

(b) Let $\mathcal{E}$ be a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. We say that $\mathcal{E}$ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-quasi-coherent if and only if locally in X we have an exact sequence of the form $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^{(I)} \to (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})^{(J)} \to \mathcal{E} \to 0$.

Remark that since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a locally free $\mathcal{B}_X$-module (for both structures), then the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-quasi-coherence implies the $\mathcal{B}_X$-quasi-coherence. When $\mathcal{B}_X$ is not $\mathcal{O}_X$-quasi-coherent, the converse is not clear.

Lemma 5.1.2.2. *Suppose here $\mathcal{B}_X$ is $\mathcal{O}_X$-quasi-coherent. Let $\mathcal{E}$ be a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.*

(i) The following assertion are equivalent.

(a) $\mathcal{E}$ is $\mathcal{O}_X$-quasi-coherent ;

(b) $\mathcal{E}$ is $\mathcal{B}_X$-quasi-coherent ;

(c) $\mathcal{E}$ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-quasi-coherent.

(ii) Suppose X is affine and $\mathcal{E}$ is quasi-coherent. Then the canonical morphism

$$\mathcal{C} \otimes_{\Gamma(X,\mathcal{C})} \Gamma(X,\mathcal{E}) \to \mathcal{E}$$

where $\mathcal{C}$ either equal to $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ or $\mathcal{B}_X$ or $\mathcal{O}_X$.

Proof. Since the notion of quasi-coherence is local, then we reduce to check the part (ii). Suppose X is affine. The case where $\mathcal{C} = \mathcal{O}_X$ is already known. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a locally free $\mathcal{B}_X$-module (for both structures), then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is also $\mathcal{O}_X$-quasi-coherent. Let $\mathcal{A}$ be either $\mathcal{B}_X$ or $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. Since $\mathcal{A}$ is $\mathcal{O}_X$-quasi-coherent, then the canonical morphism $\mathcal{O}_X \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma(X,\mathcal{A}) \to \mathcal{A}$ is an isomorphism. Hence, we get (ii). □

Notation 5.1.2.3. Let $\star \in \{\emptyset, \mathrm{b}, -, +\}$ and $* \in \{\mathrm{r}, \mathrm{l}\}$. We denote by $D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ (resp. $D^\star_{\mathrm{qc}}(\mathcal{B}_X)$) the full subcategory of $D^\star({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ (resp. $D^\star(\mathcal{B}_X)$) consisting of complexes $\mathcal{E}$ such that, for any $j \in \mathbb{Z}$, $H^j(\mathcal{E})$ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-quasi-coherent. We have the fully faithful forgetful functor $D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^\star_{\mathrm{qc}}(\mathcal{B}_X)$. Beware this is not clear that we have the factorization $D^\star_{\mathrm{qc}}(\mathcal{B}_X) \to D^\star_{\mathrm{qc}}(\mathcal{O}_X)$, except when $\mathcal{B}_X$ is $\mathcal{O}_X$-quasi-coherent.

5.1.2.4 (Properties of the functor $\mathbb{R}f_*$)**.** Assume that T is a noetherian scheme of finite Krull dimension; and f is quasi-compact and quasi-separated.

(i) Following [Gro57, 3.6.5], since Y is noetherian of finite Krull dimension d_Y, then for $i > d_Y$, for every sheaf $\mathcal{E}$ of abelian groups we have $H^i(Y,\mathcal{E}) = 0$. Then, following [Gro61, 12.2.1], we get that $R^i f_*(\mathcal{E}) = 0$ for $i > d_Y$ and every sheaf $\mathcal{E}$ of abelian groups. In particular, by definition (see [Gro61, 12.1.1] or 4.6.1.4), the functor f_* has bounded by d_Y cohomological dimension on $\mathrm{Mod}(f^{-1}\mathcal{O}_Y)$, the category of $f^{-1}\mathcal{O}_Y$-modules, or on $\mathrm{Mod}(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.

Let P be the subset of all the objects of $\mathrm{Mod}(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ which are f_*-acyclic. Remark that P contains injective $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-modules. Using the cohomological dimension finiteness of f_*, following 4.6.1.6.b if

$$\mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \to \mathcal{G}^{d_Y-1} \to \mathcal{E} \to 0 \tag{5.1.2.4.1}$$

is an exact sequence of $\mathrm{Mod}(f^{-1}\mathcal{O}_Y)$, and $\mathcal{G}^0, \ldots, \mathcal{G}^{d_Y-1} \in P$, then $\mathcal{E} \in P$ (see 4.6.1.6). This implies that for any complex $\mathcal{E} \in K(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\mathcal{E} \in K^+(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$) there exists a quasi-isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{I}$ where $\mathcal{I} \in K(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\mathcal{I} \in K^+(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$) is a complex whose modules belong to P. This yields by truncation (and using the above property involving the exact sequence) that for any $\mathcal{E} \in K^-(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\mathcal{E} \in K^{\mathrm{b}}(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$) there exists a quasi-isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{I}$ where $\mathcal{I} \in K^-(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\mathcal{I} \in K^{\mathrm{b}}(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$) is a complex whose modules belong to P. We get the functor $\mathbb{R}f_*\colon D(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\mathbb{R}f_*\colon D^-(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D^-(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, resp. $\mathbb{R}f_*\colon D^+(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D^+(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, resp. $\mathbb{R}f_*\colon D^{\mathrm{b}}(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D^{\mathrm{b}}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$) which is computed by taking a resolution with objects in P. Moreover, following [Har66, II.2.1] $\mathbb{R}f_*$ takes $D^?_{\mathrm{qc}}(\mathcal{O}_X)$ into $D^?_{\mathrm{qc}}(\mathcal{O}_Y)$ with $? \in \{\emptyset, +, -, \mathrm{b}\}$.

(ii) Since f is quasi-compact and quasi-separated, then it follows from [SGA4.2, VI.5.1] that the functors R^if_* for any $i \geq 0$ commute with filtered inductive limits of abelian groups (or see [FK18, 0.3.1.17] for a less general reference).

Proposition 5.1.2.5. Let $\mathcal{F} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ and $\mathcal{G} \in D({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.

(i) We have the canonical morphism in $D(\mathbb{Z}_Y)$:

$$\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathbb{R}f_*(\mathcal{G}) \to \mathbb{R}f_*\left(f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{G}\right). \tag{5.1.2.5.1}$$

Let $\mathcal{D}$ be a sheaf of rings such that $(\mathcal{D}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ is right solvable and let $\mathcal{F} \in D_{\mathrm{r-sol}}(\mathcal{D}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (see definition and notation 4.6.3.2). Then the morphism 5.1.2.5.1 can also be viewed as a morphism of $D(\mathcal{D})$.

(ii) Suppose f is quasi-compact and quasi-separated. Suppose moreover one of the following conditions:

(a) either $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, and $\mathcal{G} \in D({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$,

(b) or T is a noetherian scheme of finite Krull dimension, and $\mathcal{F} \in D^-_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, and $\mathcal{G} \in D^-({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.

Then the morphism 5.1.2.5.1 is an isomorphism.

Proof. In the bimodule case, let $\mathcal{R}$ be a right solving ring of $(\mathcal{D}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ such that $\mathcal{F} \in D(\mathcal{D}, \mathcal{R}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. Choosing a K-flat complex $\mathcal{P}$ of $K({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{R}} \mathcal{D})$) representing $\mathcal{F}$, a K-injective complex $\mathcal{I}$ of $K(f^{-1}{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ representing $\mathcal{G}$, we get

$$\begin{aligned}\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathbb{R}f_*(\mathcal{G}) &\xrightarrow{\sim} \mathcal{P} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f_*(\mathcal{I}) \to f_*\left(f^{-1}\mathcal{P} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{I}\right) \\ &\to \mathbb{R}f_*\left(f^{-1}\mathcal{P} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{I}\right) \xrightarrow{\sim} \mathbb{R}f_*\left(f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{G}\right),\end{aligned} \tag{5.1.2.5.2}$$

the last isomorphism coming from the fact that $f^{-1}\mathcal{P}$ a K-flat complex of $K({}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ representing $f^{-1}\mathcal{F}$.

To check that this is an isomorphism, using the remark 5.1.2.1 and using [Har66, I.7.1 (ii), (iii) and (iv)] and 5.1.2.4, we reduce to the case where $\mathcal{F} = \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$, which is obvious. □

Remark 5.1.2.6. Inverting r and l, similarly to 5.1.2.5, for any $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ and $\mathcal{G} \in D({}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, we get the morphism

$$\mathbb{R}f_*(\mathcal{G}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{F} \to \mathbb{R}f_*\left(\mathcal{G} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F}\right), \tag{5.1.2.6.1}$$

which is an isomorphism under the same corresponding hypotheses.

5.1.2.7. Let $*, ** \in \{\mathrm{l}, \mathrm{r}\}$ such that both are not equal to r. Let $\mathcal{F} \in D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ and $\mathcal{G} \in D^-({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. Then $f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_Y} \mathcal{G} \in D^-({}^\star f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ where $\star = \mathrm{l}$ if $* = ** = \mathrm{l}$ and otherwise $\star = \mathrm{r}$. Indeed, for instance, let us treat the case $** = \mathrm{l}$. This follows from the canonical isomorphisms

$$f^{-1}(\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{G} \xrightarrow{\sim} f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_Y} \mathcal{G} \tag{5.1.2.7.1}$$

Corollary 5.1.2.8. *Let $*, ** \in \{\mathrm{l}, \mathrm{r}\}$ such that both are not equal to* r. *Suppose f is quasi-compact and quasi-separated and $\widetilde{Y}^\sharp/T^\sharp$ is quasi-flat (see Definition 4.4.1.3). Suppose moreover one of the following conditions:*

(a) either $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, and $\mathcal{G} \in D({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$,

(b) or T is a noetherian scheme of finite Krull dimension, and $\mathcal{F} \in D^{-}_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, and $\mathcal{G} \in D^-({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.

Then we have the following isomorphism of $D^-({}^{**}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (see 5.1.2.7 for the right term):

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*(\mathcal{G}) \xrightarrow{\sim} \mathbb{R}f_*\left(f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_Y} \mathcal{G}\right). \tag{5.1.2.8.1}$$

Proof. For instance, let us treat the case $** = \mathrm{l}$. Following 4.6.3.3, since $\widetilde{Y}^\sharp/T^\sharp$ is quasi-flat then $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$ is solvable. Since $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \in D(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, we get

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*(\mathcal{G}) \xrightarrow{\sim} (\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathbb{R}f_*(\mathcal{G})$$

$$\underset{5.1.2.5.1}{\xrightarrow{\sim}} \mathbb{R}f_*\left(f^{-1}(\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{G}\right) \xrightarrow{\sim} \mathbb{R}f_*\left(f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_Y} \mathcal{G}\right).$$

□

5.1.3 Direct image

Unless otherwise stated, we keep notation and hypotheses of 5.1.2 and we suppose $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3). Assume that T is a noetherian scheme of finite Krull dimension and f is quasi-compact and quasi-separated.

Definition 5.1.3.1. We keep notation 5.1.1.2.

(a) The (left version of the) direct image functor of level m by $\widetilde{f}$ is the functor $\widetilde{f}^{(m)}_+ \colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ which is defined by setting

$$\widetilde{f}^{(m)}_+(\mathcal{E}) := \mathbb{R}f_*\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right),$$

where $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

(b) The (right version of the) direct image functor of level m by $\widetilde{f}$ is the functor $\widetilde{f}^{(m)}_+ \colon D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ which is defined by setting

$$\widetilde{f}^{(m)}_+(\mathcal{M}) := \mathbb{R}f_*\left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}\right),$$

where $\mathcal{M} \in D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

If there is no ambiguity with the level, we might simply write $\widetilde{f}^{(m)}_+$.

5.1.3.2. The left and the right versions of the direct image functor of level m by $\widetilde{f}$ of 5.1.3.1 are compatible with the quasi-inverse exact functors of 4.3.5.7 exchanging left and right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module structures (resp. left and right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-module structures). More precisely, for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ we have the canonical isomorphism

$$\widetilde{\omega}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{f}^{(m)}_+(\mathcal{E}) \xrightarrow{\sim} \widetilde{f}^{(m)}_+(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}), \tag{5.1.3.2.1}$$

which is constructed as follows:

$$\widetilde{\omega}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \mathbb{R}f_* \left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right) \underset{5.1.2.8.1}{\xrightarrow{\sim}} \mathbb{R}f_* \left(f^{-1}\widetilde{\omega}_{Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{B}_Y} (\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E})\right)$$

$$\xrightarrow{\sim} \mathbb{R}f_* \left((f^{-1}\widetilde{\omega}_{Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)$$

$$\overset{4.3.5.6.1}{\xrightarrow{\sim}} \mathbb{R}f_* \left((\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} (f^{-1}\widetilde{\omega}_{Y^\sharp/T^\sharp} \otimes_{f^{-1}\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp})\right)$$

$$\underset{5.1.1.2.1}{\xrightarrow{\sim}} \mathbb{R}f_* \left((\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}\right).$$

Proposition 5.1.3.3. *Let $* \in \{\mathrm{l}, \mathrm{r}\}$, let $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathrm{forg}(\mathcal{E})$ (or simply $\mathcal{E}$) be the induced object of $D({}^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$. We have the canonical isomorphism of $D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.*

$$f^{(m)}_+(\mathrm{forg}(\mathcal{E})) \xrightarrow{\sim} \widetilde{f}^{(m)}_+(\mathcal{E}). \tag{5.1.3.3.1}$$

Proof. Since the other case is checked similarly, we can suppose $* = \mathrm{l}$.

$$f^{(m)}_+(\mathcal{E}) = \mathbb{R}f_* \left(\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right) \xrightarrow{\sim} \mathbb{R}f_* \left(\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)$$

$$\overset{5.1.1.14.2}{\xrightarrow{\sim}} \mathbb{R}f_* \left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right) = \widetilde{f}^{(m)}_+(\mathcal{E})$$

□

Proposition 5.1.3.4. *Suppose $\mathcal{B}_Y$ is $\mathcal{O}_Y$-quasi-coherent and $\mathcal{B}_X = f^*\mathcal{B}_Y$. Suppose $f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_X$ are tor independent over $f^{-1}\mathcal{B}_Y$. Let $* \in \{\mathrm{l}, \mathrm{r}\}$, let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathrm{forg}(\mathcal{E})$ (or simply $\mathcal{E}$) be the induced object of $D({}^*\mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$. We have the canonical isomorphism of $D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$*

$$\mathcal{B}_Y \otimes^{\mathbb{L}}_{\mathcal{O}_Y} f^{(m)}_+(\mathcal{E})) \xrightarrow{\sim} \widetilde{f}^{(m)}_+(\mathcal{B}_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}). \tag{5.1.3.4.1}$$

Proof. Since the other proof is similar, we can suppose $* = \mathrm{l}$.

$$\mathcal{B}_Y \otimes^{\mathbb{L}}_{\mathcal{O}_Y} f^{(m)}_+(\mathcal{E})) = \mathcal{B}_Y \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathbb{R}f_* \left(\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)$$

$$\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}} \mathbb{R}f_* \left(\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)$$

$$\overset{5.1.2.5}{\xrightarrow{\sim}} \mathbb{R}f_* \left(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)$$

$$\overset{5.1.1.9.2}{\xrightarrow{\sim}} \mathbb{R}f_* \left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right) \xrightarrow{\sim} \mathbb{R}f_* \left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)$$

$$= \widetilde{f}^{(m)}_+(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}) \xrightarrow{\sim} \widetilde{f}^{(m)}_+(\mathcal{B}_X \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}).$$

□

Proposition 5.1.3.5. *For any $* \in \{r, l\}$, for any $\mathcal{E} \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, we have $\widetilde{f}^{(m)}_+(\mathcal{E}) \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.*

Proof. Recall that since f is quasi-compact and quasi-separated, then f_* preserves $\mathcal{O}$-quasi-coherence (see [Sta22, 26.24.1]). We conclude by using 5.1.2.2. □

5.1.3.6. Suppose the bottom square of 5.1.1.1.1 is the identity and f is log-étale. Let $(u'_\lambda)_{\lambda=1,\dots,r}$ be logarithmic coordinates of $Y^\sharp/S^\sharp$. This induces $(u_\lambda = \widetilde{f}^*(u'_\lambda))_{\lambda=1,\dots,r}$ be logarithmic coordinates of $X^\sharp/S^\sharp$. Put $\tau_{\sharp\lambda(m),\gamma,n} := \mu_{(m),\gamma}(u_\lambda) - 1$, and $\tau'_{\sharp\lambda(m),\gamma,n} := \mu_{(m),\gamma}(u'_\lambda) - 1$. The canonical morphism $\widetilde{f}^*\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m),\gamma} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m),\gamma}$ sends $\tau'_{\sharp\lambda(m),\gamma,n}$ to $\tau_{\sharp\lambda(m),\gamma,n}$. Hence, by using the local description 3.2.2.4, we check that $\widetilde{f}^*\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m),\gamma} \to \widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m),\gamma}$ is an isomorphism. By duality, this yields that the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ is an isomorphism. Moreover, we check (e.g. by computation in local coordinates) that the induced composition morphism $\widetilde{f}^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is a homomorphism of rings. Hence, for any $\mathcal{M} \in D({}^r\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ we get the canonical isomorphism

$$\widetilde{f}^{(m)}_+(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}f_*(\mathcal{M}). \tag{5.1.3.6.1}$$

Similarly, the canonical morphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp}$ is an isomorphism and for any $\mathcal{E} \in D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$

$$\widetilde{f}^{(m)}_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_*(\mathcal{E}). \tag{5.1.3.6.2}$$

In both cases the functor $\widetilde{f}^{(m)}_+$ preserves the boundedness of the cohomology, i.e. we have the induced functors $\widetilde{f}^{(m)}_+ \colon D^\star({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^\star({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ with $* \in \{r, l\}$, and $\star \in \{+, -, \mathrm{b}\}$.

5.1.3.7. It follows from 5.1.2.4.i , that we get the factorization $\widetilde{f}^{(m)}_+ \colon D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$.

Proposition 5.1.3.8. *With notation 4.4.5.5, let $\mathcal{E}$ be a complex of $D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, with $* \in \{\mathrm{r}, \mathrm{l}\}$. We have the canonical isomorphism of $D({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp})$:*

$$\tilde{g}^{(m)}_+ \circ \widetilde{f}^{(m)}_+(\mathcal{E}) \xrightarrow{\sim} \widetilde{g \circ f}^{(m)}_+(\mathcal{E}). \tag{5.1.3.8.1}$$

Proof. We can suppose $* = \mathrm{l}$. Since $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow Y^\sharp/T^\sharp} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, we get the projection isomorphism

$$\begin{aligned}
&\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \mathbb{R}f_*\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)\\
&\underset{5.1.2.5}{\xrightarrow{\sim}} \mathbb{R}f_*\left(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right).
\end{aligned}$$

By applying $\mathbb{R}f_*$, we get the first isomorphism

$$\begin{aligned}
\tilde{g}^{(m)}_+ \circ \widetilde{f}^{(m)}_+(\mathcal{E}) &\xrightarrow{\sim} \mathbb{R}g_*\mathbb{R}f_*\left(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)\\
&\underset{5.1.1.12.2}{\xrightarrow{\sim}} \mathbb{R}(g\circ f)_*\left(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/U^\sharp\leftarrow X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \mathcal{E}\right)\\
&= \widetilde{g\circ f}^{(m)}_+(\mathcal{E}).
\end{aligned}$$

□

5.1.3.9 (Base change case)**.** Suppose the diagram 5.1.1.1.1 is cartesian and the morphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism, i.e. suppose we are in the base change case. Let $* \in \{r, l\}$. It follows from 5.1.1.15 that for any $\mathcal{E} \in D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, we have the canonical isomorphism

$$\widetilde{f}^{(m)}_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_*(\mathcal{E}). \tag{5.1.3.9.1}$$

Following 5.1.2.4, in that case the functor $\widetilde{f}^{(m)}_+$ preserves the boundedness of the cohomology, i.e. we have the induced functors $\widetilde{f}^{(m)}_+ \colon D^\star({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^\star({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, with $\star \in \{+, -, \mathrm{b}\}$.

5.1.3.10. We suppose in this paragraph we are in the context of the non-respective case of 4.4.5.10. Assume that T is a noetherian scheme of finite Krull dimension and f_0 is quasi-compact and quasi-separated. For any $* \in \{\mathrm{l}, \mathrm{r}\}$, via the bimodules 5.1.1.17.1 and 5.1.1.17.2, we can define the pushforward $\widetilde{f}^{(m)}_{0+} \colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ as in 5.1.3.1. The properties of the subsection extends to this context.

5.1.4 Duality

We keep notation 5.1.2. Suppose that $\mathcal{B}_X$ is quasi-flat over $\mathcal{O}_S$ (see Definition 3.1.1.5). From 4.6.3.3.b, we get the following dual functors.

Definition 5.1.4.1. The dual functor of level m on $\widetilde{X}^\sharp/S^\sharp$ is the functor $\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp} \colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ which is defined by setting for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$

$$\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}\ [\delta_{X^\sharp/S^\sharp}].$$

Recall, the functor is computed by choosing a K-injective complex of $K({}^{\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_S} \widetilde{\mathcal{D}}^{(m)\,\mathrm{op}}_{X^\sharp/S^\sharp}))$ representing $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ (copy [Sta22, 20.39]). We have the factorization $\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp} \colon D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

Similarly, we have the right version one: the dual functor of level m on $\widetilde{X}^\sharp/S^\sharp$ is the functor $\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp} \colon D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ which is defined by setting for any $\mathcal{M} \in D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$

$$\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{M}) := \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{M}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\ [\delta_{X^\sharp/S^\sharp}].$$

5.1.4.2. We have for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ the isomorphism

$$\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{E}) \underset{4.6.6.4.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp})_{\mathrm{l}})[\delta_{X^\sharp/S^\sharp}], \tag{5.1.4.2.1}$$

where the index "l" means that we use the left structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}$ to compute $\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}$.

$$\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{M}) \underset{4.6.6.4.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{M}, (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_{\mathrm{r}})\ [\delta_{X^\sharp/S^\sharp}], \tag{5.1.4.2.2}$$

where the index r in $(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_{\mathrm{r}})$ means that we take the right structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module to compute $\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}$..

5.1.4.3 (Right and left dual functors)**.** We have the isomorphisms

$$\begin{gathered}\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\ [\delta_{X^\sharp/S^\sharp}] \\ \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}, (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_{\mathrm{l}})\ [\delta_{X^\sharp/S^\sharp}] \\ \underset{4.2.5.5}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}, (\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})_{\mathrm{r}})\ [\delta_{X^\sharp/S^\sharp}] \underset{5.1.4.2.2}{\xrightarrow{\sim}} \mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}),\end{gathered} \tag{5.1.4.3.1}$$

where the index r (resp. l) added to $\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ means that we take the right structure (resp. left structure) of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module.

Lemma 5.1.4.4 (Biduality)**.** *Let $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. We have the morphism $\mathcal{E} \to \mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp} \circ \mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{E})$, which is an isomorphism when $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.*

Proof. We copy the proof of 4.6.4.6 : let $\mathcal{I}$ be a K-injective complex of $K({}^{\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_S} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}))$ representing $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}[\delta_{X^\sharp/S^\sharp}]$. Then

$$\mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp} \circ \mathbb{D}^{(m)}_{\widetilde{X}^\sharp/S^\sharp}(\mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{I}),\mathcal{I}).$$

The morphism $\mathcal{E} \to \mathbb{D}(\mathbb{D}(\mathcal{E}))$ is simply the evaluation morphism

$$\mathcal{E} \to \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\mathcal{I}),\mathcal{I}).$$

When $\mathcal{E}$ is perfect, to check that this is an isomorphism we are reduced, by dévissage, to the case $\mathcal{E} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$, in which case it is clear. □

5.1.5 Exterior tensor product

Let $S^\sharp$ be a nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{r+1}\mathbb{Z})$ as defined in 3.1.1.1 with $r \in \mathbb{N}$. Since the base scheme $S^\sharp$ is fixed, so we can remove it in the notation concerning relative object with base $S^\sharp$. If $p\colon X^\sharp \to S^\sharp$ is a morphism, by abuse of notation, we sometimes denote $p^{-1}\mathcal{O}_S$ simply by $\mathcal{O}_S$.

For $i = 1, \ldots, n$, let $X^\sharp_i$ is a log smooth and log integral $S^\sharp$-log-scheme. Set $X^\sharp := X^\sharp_1 \times_{S^\sharp} X^\sharp_2 \times_{S^\sharp} \cdots \times_{S^\sharp} X^\sharp_n$. For $i = 1, \ldots, n$, let $\mathrm{pr}_i\colon X^\sharp \to X^\sharp_i$, be the projections. We denote by $\varpi\colon X^\sharp \to S^\sharp$ and $\varpi_i\colon X^\sharp_i \to S^\sharp$ the structural morphisms.

For $i = 1, \ldots, n$, let $\mathcal{B}_{X_i}$ be a commutative $\mathcal{O}_{X_i}$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp_i/S^\sharp}$-module. Let us recall that the action of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{B}_i := \mathrm{pr}_i^* \mathcal{B}_{X_i}$ is compatible with its structure of $\mathcal{O}_X$-algebra (see 3.4.4.6). Following 4.1.1.6, the $\mathcal{O}_X$-algebra $\mathcal{B}_X := \mathcal{B}_1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{B}_n$ is endowed with a structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module a compatible its structure of $\mathcal{O}_X$-algebra.

We will denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i} = \mathcal{B}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X^\sharp_i/S^\sharp}$. We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{X}^\sharp_i$) the ringed logarithmic scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(X^\sharp_i, \mathcal{B}_{X_i})$), and by $\widetilde{\mathrm{pr}}_i\colon \widetilde{X}^\sharp/S^\sharp \to \widetilde{X_i}^\sharp/S^\sharp$ the morphism of relative ringed logarithmic schemes induced by by pr_i and by $\mathrm{pr}_i^* \mathcal{B}_{X_i} \to \mathcal{B}_X$. We also denote by $\widetilde{\mathrm{pr}}_i$ the induced morphism $\widetilde{X}^\sharp \to \widetilde{X}^\sharp_i$ of ringed logarithmic schemes.

5.1.5.1. It follows from 5.1.1.15.1 the canonical morphism $\mathrm{pr}_i^{-1}(\mathcal{B}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X^\sharp_i/S^\sharp}) \to \mathcal{B}_i \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/X^\sharp_{\hat{i}}}$, where $X^\sharp_{\hat{i}} := \prod_{j\neq i} X^\sharp_j$ is a ring homomorphism. From 4.4.2.7.1, the natural morphism $\mathcal{B}_i \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/X^\sharp_{\hat{i}}} \to \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ is an isomorphism. By composition, this yields the ring homomorphism

$$\mathrm{pr}_i^{-1} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp} \tag{5.1.5.1.1}$$

5.1.5.2. (a) For $i = 1, \ldots, n$, let $\mathcal{E}_i$ be a sheaf of $\varpi_i^{-1}\mathcal{O}_S$-module. We get the $\varpi^{-1}\mathcal{O}_S$-module by setting

$$\boxtimes_{i}{}_{\mathrm{top}}\mathcal{E}_i := \mathrm{pr}_1^{-1} \mathcal{E}_1 \otimes_{\mathcal{O}_S} \mathrm{pr}_2^{-1} \mathcal{E}_2 \otimes_{\mathcal{O}_S} \cdots \otimes_{\mathcal{O}_S} \mathrm{pr}_n^{-1} \mathcal{E}_n.$$

(b) For $i = 1, \ldots, n$, let $\mathcal{E}_i$ be an $\mathcal{B}_{X_i}$-module. The sheaf $\boxtimes_{i\,\mathrm{top}}\mathcal{E}_i$ has a canonical structure of $\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}$-module. We put $\widetilde{\boxtimes}_i \mathcal{E}_i := \mathcal{B}_X \otimes_{\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}} \boxtimes_{i\,\mathrm{top}}\mathcal{E}_i$. When $\mathcal{B}_{X_i} = \mathcal{O}_{X_i}$ holds for any $i = 1, \ldots, n$, we simply write $\boxtimes_i \mathcal{E}_i$. Moreover, by commutativity and associativity of tensor products, we get the canonical isomorphism of $\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}$-modules

$$\boxtimes_{i\,\mathrm{top}}\mathcal{E}_i \xrightarrow{\sim} \left(\mathrm{pr}_1^{-1} \mathcal{E}_1 \otimes_{\mathrm{pr}_1^{-1} \mathcal{B}_{X_1}} \boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}\right) \otimes_{\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}} \cdots \otimes_{\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}} \left(\mathrm{pr}_n^{-1} \mathcal{E}_n \otimes_{\mathrm{pr}_n^{-1} \mathcal{B}_{X_n}} \boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}\right). \tag{5.1.5.2.1}$$

Using the isomorphism 5.1.5.2.1, we get the isomorphism of $\mathcal{B}_X$-modules

$$\widetilde{\boxtimes}_i \mathcal{E}_i \xrightarrow{\sim} \widetilde{\mathrm{pr}}_1^* \mathcal{E}_1 \otimes_{\mathcal{B}_X} \cdots \otimes_{\mathcal{B}_X} \widetilde{\mathrm{pr}}_n^* \mathcal{E}_n. \tag{5.1.5.2.2}$$

(c) Since $\mathrm{pr}_i^{-1}\,\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$ are $\mathcal{O}_S$-algebras, we get a canonical structure of $\mathcal{O}_S$-algebra on $\boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$. Using 5.1.5.1.1, we get the canonical homomorphism of $\mathcal{O}_S$-algebras $\boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$. Via a local computation, we can check that the induced morphism of $(\mathcal{B}_X, \boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$-bimodules (resp. $(\boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}, \mathcal{B}_X)$-bimodules)

$$\widetilde{\boxtimes_i}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} = \mathcal{B}_X \otimes_{\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}} \boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}, \qquad (\text{resp. } \boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} \otimes_{\boxtimes_{i\,\mathrm{top}}\mathcal{B}_{X_i}} \mathcal{B}_X \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}) \quad (5.1.5.2.3)$$

is an isomorphism.

(d) For $i = 1, \ldots, n$, let $\mathcal{F}_i$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module (resp. $\mathcal{G}_i$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module). Then $\boxtimes_{i}{}_{\mathrm{top}}\mathcal{F}_i$ (resp. $\boxtimes_{i}{}_{\mathrm{top}}\mathcal{G}_i$) has a canonical structure of left (resp. right) $\boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module. It follows from the left (resp. right) isomophism of 5.1.5.2.3 that we get the isomorphism of $\mathcal{B}_X$-modules $\widetilde{\boxtimes_i}\mathcal{F}_i \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp} \otimes_{\boxtimes_{i\,\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \boxtimes_{i}{}_{\mathrm{top}}\mathcal{F}_i$ (resp. $\widetilde{\boxtimes_i}\mathcal{G}_i \xrightarrow{\sim} \boxtimes_{i}{}_{\mathrm{top}}\mathcal{F}_i \otimes_{\boxtimes_{i\,\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$). Via this isomorphism, we endowed $\widetilde{\boxtimes_i}\mathcal{F}_i$ (resp. $\widetilde{\boxtimes_i}\mathcal{G}_i$) with a structure of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module.

(e) For $i = 1, \ldots, n$, let $\mathcal{F}_i$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module. Then $\widetilde{\mathrm{pr}}_1^*\mathcal{F}_1 \otimes_{\mathcal{B}_X} \cdots \otimes_{\mathcal{B}_X} \widetilde{\mathrm{pr}}_n^*\mathcal{F}_n$ has a canonical structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module (see 4.2.3.1). We check that the isomorphism 5.1.5.2.2 is in fact an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-modules.

Lemma 5.1.5.3. *Let $\mathcal{R}$ be a commutative sheaf on X. By convention, $\mathcal{R}$-algebras are always unital and associative. For $i = 1, \ldots, n$, let $\mathcal{D}_i$ be $\mathcal{R}$-algebras, $\mathcal{E}_i$ be a flat left $\mathcal{D}_i$-module. Then $\mathcal{E}_1 \otimes_{\mathcal{R}} \mathcal{E}_2 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathcal{E}_n$ is a flat left $\mathcal{D}_1 \otimes_{\mathcal{R}} \mathcal{D}_2 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathcal{D}_n$-module.*

Proof. By proceeding by induction we reduce to the case $n = 2$. From the structure of left $\mathcal{D}_1$-module of $\mathcal{E}_1$ and of $\mathcal{D}_2$-bimodule of $\mathcal{D}_2$, we endow the sheaf $\mathcal{E}_1 \otimes_{\mathcal{R}} \mathcal{D}_2$ with a structure of $(\mathcal{D}_1 \otimes_{\mathcal{R}} \mathcal{D}_2, \mathcal{D}_2)$-bimodule. Since the canonical isomorphisme $\mathcal{E}_1 \otimes_{\mathcal{R}} \mathcal{D}_2 \xrightarrow{\sim} (\mathcal{D}_1 \otimes_{\mathcal{R}} \mathcal{D}_2) \otimes_{\mathcal{D}_1} \mathcal{E}_1$ is $\mathcal{D}_1 \otimes_{\mathcal{R}} \mathcal{D}_2$-linear, then $\mathcal{E}_1 \otimes_{\mathcal{R}} \mathcal{D}_2$ is flat as left $\mathcal{D}_1 \otimes_{\mathcal{R}} \mathcal{D}_2$-module. Since $\mathcal{E}_2$ is flat as $\mathcal{D}_2$-module, this implies that $\mathcal{E}_1 \otimes_{\mathcal{R}} \mathcal{E}_2 \xrightarrow{\sim} (\mathcal{E}_1 \otimes_{\mathcal{R}} \mathcal{D}_2) \otimes_{\mathcal{D}_2} \mathcal{E}_2$ is a flat left $\mathcal{D}_1 \otimes_{\mathcal{R}} \mathcal{D}_2$-module. □

5.1.5.4. (a) Since the extension $\boxtimes_{i}{}_{\mathrm{top}}\mathcal{O}_{X_i} \to \mathcal{O}_X$ is flat, since $\mathcal{O}_X \otimes_{\boxtimes_{i\,\mathrm{top}}\mathcal{O}_{X_i}} \boxtimes_{i}{}_{\mathrm{top}}\mathcal{B}_{X_i} \xrightarrow[5.1.5.2.2]{\sim} \mathrm{pr}_i^*\,\mathcal{B}_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{pr}_n^*\,\mathcal{B}_{X_n} = \mathcal{B}_X$, then we get the flatness of the extension $\boxtimes_{i}{}_{\mathrm{top}}\mathcal{B}_{X_i} \to \mathcal{B}_X$. By using 5.1.5.2.3, this implies that $\boxtimes_{i}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$ are right and left flat.

(b) When S is the spectrum of a field, the multi-functor $\boxtimes_{i}{}_{\mathrm{top}}$ is exact. Hence, it follows from (a) that the multi-functor $\widetilde{\boxtimes_i}$ is exact.

(c) When S is not the spectrum of a field, the multi-functor $\boxtimes_{i}{}_{\mathrm{top}}$ is not necessarily exact. Set $\mathcal{B}_X^{\mathrm{top}} := \boxtimes_{i}{}_{\mathrm{top}}\mathcal{B}_{X_i}$. We get the multi-functor $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \colon D^-(\mathcal{B}_{X_1}) \times \cdots \times D^-(\mathcal{B}_{X_n}) \to D^-(\mathcal{B}_X^{\mathrm{top}})$ by setting for any $\mathcal{E}_i \in D^-(\mathcal{B}_{X_i})$

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}\mathcal{E}_i := \left(\mathrm{pr}_1^{-1}\,\mathcal{E}_1 \otimes^{\mathbb{L}}_{\mathrm{pr}_1^{-1}\,\mathcal{B}_{X_1}} \mathcal{B}_X^{\mathrm{top}}\right) \otimes^{\mathbb{L}}_{\mathcal{B}_X^{\mathrm{top}}} \cdots \otimes^{\mathbb{L}}_{\mathcal{B}_X^{\mathrm{top}}} \left(\mathrm{pr}_n^{-1}\,\mathcal{E}_n \otimes^{\mathbb{L}}_{\mathrm{pr}_n^{-1}\,\mathcal{B}_{X_n}} \mathcal{B}_X^{\mathrm{top}}\right). \qquad (5.1.5.4.1)$$

Remark when $\mathcal{B}_i$ is flat over $\mathcal{O}_S$ for any i, then we get the isomorphism of $D^-(\mathcal{B}_X^{\mathrm{top}})$:

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}\mathcal{E}_i \xrightarrow{\sim} \mathrm{pr}_1^{-1}\,\mathcal{E}_1 \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathrm{pr}_2^{-1}\,\mathcal{E}_2 \otimes^{\mathbb{L}}_{\mathcal{O}_S} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathrm{pr}_n^{-1}\,\mathcal{E}_n. \qquad (5.1.5.4.2)$$

(d) We have the multi-functor $\widetilde{\boxtimes}^{\mathbb{L}}_i \colon D^-(\mathcal{B}_{X_1}) \times \cdots \times D^-(\mathcal{B}_{X_n}) \to D^-(\mathcal{B}_X)$ by setting for any $\mathcal{E}_i \in D^-(\mathcal{B}_{X_i})$

$$\widetilde{\boxtimes}^{\mathbb{L}}_i\mathcal{E}_i := \mathcal{B}_X \otimes_{\mathcal{B}_X^{\mathrm{top}}} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}\mathcal{E}_i \xrightarrow{\sim} \mathbb{L}\widetilde{\mathrm{pr}}_1^*\mathcal{E}_1 \otimes^{\mathbb{L}}_{\mathcal{B}_X} \cdots \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{\mathrm{pr}}_n^*\mathcal{E}_n, \qquad (5.1.5.4.3)$$

where the last isomorphism is, after using flat resolutions, a consequence of 5.1.5.2.2. Beware that pr_i is flat but $\widetilde{\mathrm{pr}}_i$ might not be flat.

(e) Let $* \in \{\mathrm{l}, \mathrm{r}\}$. For any $i = 1, \ldots, n$, let $\mathcal{F}_i \in D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$, $\mathcal{M}_i \in D^-({}^\mathrm{l}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$. Set $\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp} := \boxtimes_{i}{}_{\mathrm{top}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$. Then the functor 5.1.5.4.1 induces the functor $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \colon D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X_1^\sharp}) \times \cdots \times D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X_n^\sharp}) \to D^-({}^*\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp})$. Indeed, since $\mathcal{B}_{X_i} \to \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$ is flat, by using $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-flat resolution $\mathcal{P}_i$ of $\mathcal{E}_i$, by commutativity and associativity of tensor products, we get therefore the canonical isomorphism

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{E}_i \xrightarrow{\sim} \mathrm{pr}_1^{-1} \mathcal{P}_1 \otimes_{\mathcal{O}_S} \mathrm{pr}_2^{-1} \mathcal{P}_2 \otimes_{\mathcal{O}_S} \cdots \otimes_{\mathcal{O}_S} \mathrm{pr}_n^{-1} \mathcal{P}_n \in C({}^\mathrm{l}\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp}), \tag{5.1.5.4.4}$$

whose terms of the right side are even flat left $\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp}$ (see 5.1.5.3).

We set $\widetilde{\boxtimes}^{\mathbb{L}}_{i} \mathcal{F}_i := \widetilde{\mathcal{D}}^{(m)}_{X^\sharp} \otimes_{\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp}} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{F}_i$ and $\widetilde{\boxtimes}^{\mathbb{L}}_{i} \mathcal{M}_i := \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{M}_i \otimes_{\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$, which defines the multi-functor

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i} \colon D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X_1^\sharp}) \times \cdots \times D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X_n^\sharp}) \to D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}). \tag{5.1.5.4.5}$$

It follows from 5.1.5.2.3 that the functors 5.1.5.4.3 and 5.1.5.4.5 are compatible.

Proposition 5.1.5.5. *Let $* \in \{\mathrm{l}, \mathrm{r}\}$. Suppose $? \in \{\mathrm{tdf}, \mathrm{perf}\}$ (resp. suppose $? = \mathrm{coh}$ and S is locally noetherian).*

(a) The functor 5.1.5.4.1 induces the functor $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \colon D^-_?({}^*\widetilde{\mathcal{D}}^{(m)}_{X_1^\sharp}) \times \cdots \times D^-_?({}^*\widetilde{\mathcal{D}}^{(m)}_{X_n^\sharp}) \to D^-_?({}^*\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp})$.

(b) The functor 5.1.5.4.5 preserves the finite tor-dimension and perfectness (resp. bounded above coherent complexes), i.e. induces

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i} \colon D^-_?({}^*\widetilde{\mathcal{D}}^{(m)}_{X_1^\sharp}) \times \cdots \times D^-_?({}^*\widetilde{\mathcal{D}}^{(m)}_{X_n^\sharp}) \to D^-_?({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}). \tag{5.1.5.5.1}$$

Proof. The first assertion is clear from the isomorphism 5.1.5.4.4. The second one is an obvious consequence. □

Lemma 5.1.5.6. *Let $\mathcal{R}$ be a commutative sheaf on X. By convention, $\mathcal{R}$-algebras are always unital and associative. For $i = 1, \ldots, n$, let $\mathcal{A}_i$, $\mathcal{B}_i$ be $\mathcal{R}$-algebras, $\mathcal{M}_i$ be a $(\mathcal{A}_i, \mathcal{B}_i)$-bimodule, $\mathcal{E}_i$ be a left $\mathcal{B}_i$-module. Set $\otimes_i \mathcal{A}_i := \mathcal{A}_1 \otimes_{\mathcal{R}} \mathcal{A}_2 \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} \mathcal{A}_n$, and similarly replacing $\mathcal{A}_i$ by $\mathcal{R}$-modules. We have the canonical isomorphism of left $\otimes_i \mathcal{A}_i$-modules of the form*

$$\otimes_i (\mathcal{M}_i \otimes_{\mathcal{B}_i} \mathcal{E}_i) \xrightarrow{\sim} \otimes_i(\mathcal{M}_i) \otimes_{\otimes_i(\mathcal{B}_i)} \otimes_i(\mathcal{E}_i). \tag{5.1.5.6.1}$$

Proof. We check that the morphism 5.1.5.6.1 is well defined by the formula $\otimes_i(y_i \otimes x_i) \mapsto (\otimes_i(y_i)) \otimes (\otimes_i(x_i))$ and its inverse by the formula $(\otimes_i(y_i)) \otimes (\otimes_i(x_i)) \mapsto \otimes_i(y_i \otimes x_i)$ for any $y_i \in \mathcal{M}_i$ and $x_i \in \mathcal{E}_i$. □

Lemma 5.1.5.7. *For $i = 1, \ldots, n$, let $\mathcal{D}_i$ be a sheaf of rings such that $(\mathcal{D}_i, \mathcal{B}_{X_i})$ has the ring of resolution $\mathcal{R}$. For $i = 1, \ldots, n$, let $\mathcal{M}_i \in D^-(\mathcal{D}_i, \mathcal{R}, \mathcal{B}_{X_i})$, $\mathcal{E}_i \in D^-(\mathcal{B}_{X_i})$, $\mathcal{N}_i \in D^-(\mathcal{D}_i, \mathcal{R}, \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$ (recall our notation 4.6.3.2), $\mathcal{F}_i \in D^-(\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$.*

(a) We have the canonical isomorphism of $D^-(\boxtimes_{i\,\mathrm{top}} \mathcal{D}_i, \mathcal{R}, \mathcal{B}^{\mathrm{top}}_X)$

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} (\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) \xrightarrow{\sim} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}^{\mathrm{top}}_X} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{E}_i. \tag{5.1.5.7.1}$$

(b) We have the canonical isomorphism of $\boxtimes_{i\,\mathrm{top}} \mathcal{D}_i$-modules

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} (\mathcal{N}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \mathcal{F}_i) \xrightarrow{\sim} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{N}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{\mathrm{top}}_{X^\sharp}} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{F}_i. \tag{5.1.5.7.2}$$

Proof. By using flat resolutions, we remove $\mathbb{L}$ (use also 5.1.5.3 for 5.1.5.7.2. Then, this is a consequence of Lemma 5.1.5.6. □

Lemma 5.1.5.8. *For $i = 1, \ldots, n$, let $\mathcal{D}_i$ be a sheaf of rings such that $(\mathcal{D}_i, \mathcal{B}_{X_i})$ has the ring of resolution $\mathcal{R}$.*

(a) For $i = 1, \ldots, n$, let $\mathcal{M}_i \in D^-(\mathcal{D}_i, \mathcal{R}, \mathcal{B}_{X_i})$, $\mathcal{E}_i \in D^-(\mathcal{B}_{X_i})$. We have the isomorphism of the form $\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) \xrightarrow{\sim} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{E}_i$ of $D^-(\boxtimes_{i\,\mathrm{top}} \mathcal{D}_i, \mathcal{R}, \mathcal{B}_X)$ fitting into the commutative up to isomorphism following diagram of $D^-(\boxtimes_{i\,\mathrm{top}} \mathcal{D}_i, \mathcal{R}, \boxtimes_{i\,\mathrm{top}} \mathcal{B}_{X_i})$:

$$\begin{array}{ccc}
\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) & \xrightarrow[5.1.5.7.1]{\sim} & \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_X^{\mathrm{top}}} \widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}} \mathcal{E}_i \\
\downarrow & & \downarrow \\
\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) & \xrightarrow{\sim} & \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{E}_i.
\end{array} \tag{5.1.5.8.1}$$

(b) For $i = 1, \ldots, n$, let $\mathcal{M}_i \in D^-({}^l\mathcal{D}_i, \mathcal{R}, {}^l\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$, $\mathcal{E}_i \in D^-(\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$. Then, the above isomorphism $\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) \xrightarrow{\sim} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{E}_i$ of (a) is in fact an isomorphism of $D^-(\boxtimes_{i\,\mathrm{top}} \mathcal{D}_i, \mathcal{R}, {}^l\mathcal{D}^{(m)}_{X^\sharp})$.

Proof. We construct the isomorphism of the form $\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) \xrightarrow{\sim} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{E}_i$ as follows:

$$\begin{aligned}
&\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_{X_i}} \mathcal{E}_i) \xrightarrow[5.1.5.4.3]{\sim} \widetilde{\mathrm{pr}}_1^*(\mathcal{M}_1 \otimes^{\mathbb{L}}_{\mathcal{B}_{X_1}} \mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \cdots \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\mathrm{pr}}_n^*(\mathcal{M}_n \otimes^{\mathbb{L}}_{\mathcal{B}_{X_n}} \mathcal{E}_n) \\
\xrightarrow{\sim}\ &\big(\widetilde{\mathrm{pr}}_1^* \mathcal{M}_1 \otimes^{\mathbb{L}}_{\mathcal{B}_X} \cdots \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\mathrm{pr}}_n^* \mathcal{M}_n\big) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \big(\widetilde{\mathrm{pr}}_1^* \mathcal{E}_1 \otimes^{\mathbb{L}}_{\mathcal{B}_X} \cdots \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\mathrm{pr}}_n^* \mathcal{E}_n\big) \xrightarrow[5.1.5.4.3]{\sim} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{M}_i \otimes^{\mathbb{L}}_{\mathcal{B}_X} \widetilde{\boxtimes}_i^{\mathbb{L}} \mathcal{E}_i.
\end{aligned} \tag{5.1.5.8.2}$$

We check by an easy computation the commutativity of the diagram 5.1.5.8.1. Finally, when $\mathcal{M}_i \in D^-(\mathcal{D}_i, \mathcal{R}, \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$, $\mathcal{E}_i \in D^-(\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$, the isomorphisms of 5.1.5.8.2 are $\mathcal{D}^{(m)}_{X^\sharp}$-linear. □

5.1.5.9. We have the splitting of $\mathcal{B}_X$-modules $\oplus_{i=1}^n \Omega^1_{X_i^\sharp} \xrightarrow{\sim} \Omega^1_{X^\sharp}$. By applying determinants, this yields the isomorphism of $\mathcal{B}_X$-modules $\widetilde{\boxtimes}_i \widetilde{\omega}_{X_i^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp}$. Using the canonical structure of right $\mathcal{D}^{(m)}_{X_i^\sharp}$-module on $\widetilde{\omega}_{X_i^\sharp}$, we get a structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module on $\widetilde{\boxtimes}_i \widetilde{\omega}_{X_i^\sharp}$. By local computations, we check the canonical isomorphism $\widetilde{\boxtimes}_i \widetilde{\omega}_{X_i^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp}$ is in fact an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-modules.

For $i = 1, \ldots, n$, $\mathcal{E}_i$ be left $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module, and $\mathcal{F}_i$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module. Then we have the canonical morphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-modules (resp. left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-modules) $\widetilde{\boxtimes}_i(\widetilde{\omega}_{X_i^\sharp} \otimes_{\mathcal{B}_{X_i}} \mathcal{E}_i) \xrightarrow{\sim} \omega_{X^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\boxtimes}_i \mathcal{E}_i$ (resp. $\widetilde{\boxtimes}_i(\mathcal{E}_i \otimes_{\mathcal{B}_{X_i}} \widetilde{\omega}^{-1}_{X_i^\sharp}) \xrightarrow{\sim} \widetilde{\boxtimes}_i \mathcal{E}_i \otimes_{\mathcal{B}_X} \omega^{-1}_{X^\sharp}$). Taking flat resolutions, we have similar isomorphisms in derived categories.

5.2 Exact closed immersion of log (formal) smooth schemes

The study of closed immersions is an important bootstrapping step in the proof of general results.

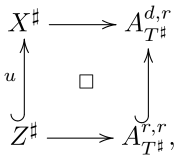
Let $T^\sharp$ be a noetherian nice (see definition 3.1.1.1) fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ with i an integer (resp. $T^\sharp$ be a p-torsion free noetherian nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10). Moreover, let $u\colon Z^\sharp \hookrightarrow X^\sharp$ be an exact closed immersion of log smooth $T^\sharp$-log schemes (resp. of p-torsion free log smooth $T^\sharp$-log formal schemes) such that the underlying closed immersion of schemes of u (resp. of u_0) is regular (for example, this is the case when $Z^\sharp$ and $X^\sharp$ are regular in the sense of log schemes: see [GR, 12.5.14], to get example of regular log schemes, recall that a log smooth fine saturated log scheme over a (log) regular log scheme is (log) regular: see [Ogu18, IV.3.5.3]). Let $\mathcal{I}$ be the ideal defining u.

The level $m \in \mathbb{N}$ is fixed. Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$-module structure which is compatible with its algebra structure. We suppose $\mathcal{O}_Z$ and $u^{-1}\mathcal{B}_X$ are tor independent over $u^{-1}\mathcal{O}_X$ and we set $\mathcal{B}_Z := u^*(\mathcal{B}_X)$. We keep similar notation than 4.1.2 (we replace S by T, and $X^\sharp$ by Z if necessary), in particular we set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} := \mathcal{B}_Z \otimes_{\mathcal{O}_Z} \mathcal{D}^{(m)}_{Z^\sharp/T^\sharp}$. We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{Z}^\sharp$) the ringed logarithmic ($\mathcal{V}$-formal) scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(Z^\sharp, \mathcal{B}_Z)$), and by $\tilde{u}\colon \widetilde{Z}^\sharp/T^\sharp \to \widetilde{X}^\sharp/T^\sharp$ the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by the diagram 5.1.1.1.1 and by $u^*\mathcal{B}_X \xrightarrow{\sim} \mathcal{B}_Z$. We suppose $\widetilde{u}$ is quasi-flat (see Definition 4.4.1.3). We set $\widetilde{\mathcal{I}} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{I}$.

5.2.1 Charts subordinate to an exact closed immersion of log smooth schemes

We will work with the following charts subordinate to u:

Proposition 5.2.1.1. *Let z be a point of $|Z|$. Then, replacing $X^\sharp$ by an open set containing z if necessary, there exist some integers $n \geq r$ and the left (resp. right) cartesian diagram of morphisms of $T^\sharp$-log-schemes (resp. $T^\sharp$-log-formal schemes) of the form:*

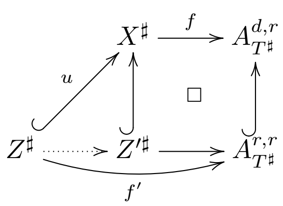


such that the horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion (see notation 4.5.1.1).

Proof. Since the respective case is a consequence of the non-respective one, let us check this latter case. The proof is analogous to that of theorem [SGA1, II.4.10]: Since u is an exact closed immersion, denoting by $\mathcal{I}$ the ideal given by u, following [Ogu18, IV.3.2.2] we get the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to u^*\Omega^1_{X^\sharp/T^\sharp} \to \Omega^1_{Z^\sharp/T^\sharp} \to 0. \tag{5.2.1.1.1}$$

Since $Z^\sharp$ and $X^\sharp$ are log-smooth over $T^\sharp$, following [Ogu18, IV.3.2.1], $\Omega^1_{X^\sharp/T^\sharp}$ and $\Omega^1_{Z^\sharp/T^\sharp}$ are locally free. Following [Ogu18, IV,1.2.11], by replacing $X^\sharp$ by an open set containing $X^\sharp$ if necessary (and $Z^\sharp$ by the trace of this open set on $Z^\sharp$), we check that $\Omega^1_{Z^\sharp/T^\sharp}$ is generated as D_Z-module by the elements of the form $d\log a$ with a a global section of $M_{Z^\sharp}$. Hence, (by replacing $X^\sharp$ by an open set containing $X^\sharp$ if necessary) we get some global sections $a_1, \ldots, a_r$ of $M_{X^\sharp}$ such that $d\log \overline{a}_1, \ldots, d\log \overline{a}_r$ is a basis of $\Omega^1_{Z^\sharp/T^\sharp}$, $\overline{a}$ means the image of a via the surjection $u^{-1}M_{X^\sharp} \to M_{Z^\sharp}$. Via the exact sequence 5.2.1.1.1, we get some global sections $a_{r+1}, \ldots, a_d$ of $\mathcal{I}$ such that $1 \otimes d\log a_1, \ldots, 1 \otimes d\log a_r, 1 \otimes da_{r+1}, \ldots, 1 \otimes da_d$ form a basis of $u^*\Omega^1_{X^\sharp/T^\sharp}$. Since $\Omega^1_{X^\sharp/T^\sharp}$ is a locally free $\mathcal{O}_X$-module, replacing $X^\sharp$ by an open set containing $X^\sharp$ if necessary, we can therefore suppose that $d\log a_1, \ldots, 1 \otimes d\log a_r, da_{r+1}, \ldots, da_d$ is a basis of $\Omega^1_{X^\sharp/T^\sharp}$. The sections $a_1, \ldots, a_d$ (resp. $\overline{a}_1, \ldots, \overline{a}_r$) induce a morphism of log smooth $T^\sharp$-log-schemes of the form $f\colon X^\sharp \to A^{d,r}_{T^\sharp}$ (resp. $f'\colon Z^\sharp \to A^{r,r}_{T^\sharp}$). Following 4.5.1.3, f and f' are log-étale. Consider the diagram



where the three closed immersions are strict and where the square is cartesian. Since f' and f are log-étale, then so is $Z^\sharp \to Z'^\sharp$. Since the closed immersions are exact, the morphism $Z^\sharp \to Z'^\sharp$ is moreover strict. The underlying morphism of schemes of $Z \to Z'$ is therefore an étale closed immersion, and therefore an open immersion. By shrinking $X^\sharp$ if necessary, we can therefore conclude. □

5.2.2 Preliminaries: some computations in local coordinates

In this subsection, with notation 4.5.1.1, we suppose $X^\sharp$ is affine and there exist some integers $r, s, d \geq 0$ such that $r + s \leq d$ and a cartesian diagram of morphisms of nice fine $T^\sharp$-log-schemes (resp. p-torsion free nice fine log smooth $T^\sharp$-log-formal schemes) of the form:

$$\begin{array}{ccc} X^\sharp & \xrightarrow{\alpha} & A^{d,r}_{T^\sharp} \\ {\scriptstyle u}\uparrow & \square & \uparrow \\ Z^\sharp & \longrightarrow & A^{r+s,r}_{T^\sharp} \end{array}$$

such that the horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion given by $t_{r+s+1}, \ldots, t_d$. Recall following 5.2.1.1, this is Zariski locally possible to suppose $s = 0$, but we will need the computation of this section when s is not necessarily 0 (see the condition $(\star)$ of 5.2.4). Moreover, without structures, we can similarly suppose $r = 0$.

Let $Y := \alpha^{-1}(B^{n,r}_{T^\sharp})$ be the open $T^\sharp$-(formal) subscheme of $X^\sharp$ with trivial log-structure (see [Ogu18, III.1.2.8]). Let $t_1, \ldots, t_r \in M_{X^\sharp}$ and $t_{r+1}, \ldots, t_d \in \Gamma(X, \mathcal{O}_X)$ be the element given by α. Then $t_{r+s+1}, \ldots, t_d$ generate $I := \Gamma(X, \mathcal{I})$, $\bar{t}_1, \ldots, \bar{t}_{r+s}$ are semi-logarithmic coordinates of $Z^\sharp$ over $T^\sharp$, and $\bar{t}_{r+s+1}, \ldots, \bar{t}_d$ is a basis of $\mathcal{I}/\mathcal{I}^2$, where $\bar{t}_1, \ldots, \bar{t}_r \in \Gamma(Z, M_{Z^\sharp})$ (resp. $\bar{t}_{r+1}, \ldots, \bar{t}_d \in \Gamma(X, \mathcal{O}_Z)$) are the images of $t_1, \ldots, t_r$ (resp. $t_{r+1}, \ldots, t_d$) via $\Gamma(X, M_{X^\sharp}) \to \Gamma(Z, M_{Z^\sharp})$ (resp. $\Gamma(X, \mathcal{I}) \to \Gamma(Z, \mathcal{O}_Z)$). Remark that since the closed immersion $\underline{u}$ is regular then it follows from [Gro67, 16.9.3] that $t_{r+s+1}, \ldots, t_d$ is a quasi-regular sequence of $I := \Gamma(X, \mathcal{I})$ and then by noetherianity (see [Gro67, 16.9.10]) is a regular sequence of I.

Since the level m is fixed, we simply write $\tau_{i\sharp} := \mu^n_{(m),\gamma}(t_i) - 1 \in \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)}$ for $i = 1, \ldots, r$ (see notation 3.2.2.4), $\tau_j := 1 \otimes t_j - t_j \otimes 1 \in \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)}$ for $j = r+1, \ldots, d$. We write $\overline{\tau}_{i\sharp} := \mu^n_{(m),\gamma}(\bar{t}_i) - 1 \in \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$ for $i = 1, \ldots, r$, $\overline{\tau}_j := 1 \otimes \bar{t}_j - \bar{t}_j \otimes 1 \in \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$ for $j = r+1, \ldots, r+s$. The sheaf of $\mathcal{B}_X$-algebras $\widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)}$ is a free $\mathcal{B}_X$-module with the basis $\{\underline{\tau}_{(r)}{}^{\{(\underline{i},\underline{j})\}_{(m)}} := \underline{\tau}_\sharp{}^{\{(\underline{i},\underline{0})\}_{(m)}} \underline{\tau}^{\{(\underline{0},\underline{j})\}_{(m)}} \mid \underline{i} \in \mathbb{N}^r,\ \underline{j} \in \mathbb{N}^{d-r}$ such that $|\underline{i}| + |\underline{j}| \leq n\}$, and $\widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$ is a free $\mathcal{B}_Z$-module with the basis $\{\underline{\overline{\tau}}_{(r)}^{\{(\underline{i},\underline{j})\}_{(m)}} := \underline{\overline{\tau}}_\sharp^{\{(\underline{i},\underline{0})\}_{(m)}} \underline{\overline{\tau}}^{\{(\underline{0},\underline{j})\}_{(m)}} \mid \underline{i} \in \mathbb{N}^r, \underline{j} \in \mathbb{N}^s$ such that $|\underline{i}| + |\underline{j}| \leq n\}$. According to 4.5.1.1, the corresponding dual basis of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp,n}$ is denoted $\{\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} \mid \underline{k} \in \mathbb{N}^d,\ |\underline{k}| \leq n\}$ and the corresponding dual basis of $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp,n}$ is denoted by $\{\underline{\partial}_{(r)}^{\langle \underline{i} \rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^{r+s},\ |\underline{i}| \leq n\}$ (we hope the similar notation is not too confusing). The sheaf $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ is a free $\mathcal{B}_X$-module with the basis $\{\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} \mid \underline{k} \in \mathbb{N}^d\}$, and $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$ is a free $\mathcal{B}_Z$-module with the basis $\{\underline{\partial}_{(r)}^{\langle \underline{i} \rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^{r+s}\}$.

We denote by straight letter, the global section of a sheaf on X, e.g. $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp} := \Gamma(X, \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$.

5.2.2.1. We compute the canonical homomorphism $\widetilde{u}^* \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$ sends $\underline{\tau}_{(r)}^{\{(\underline{i},\underline{j})\}_{(m)}}$ (where $\underline{i} \in \mathbb{N}^{r+s}$, $\underline{j} \in \mathbb{N}^{d-r-s}$) to $\underline{\overline{\tau}}_{(r)}^{\{\underline{i}\}_{(m)}}$ if $\underline{j} = (0, \ldots, 0)$ and to 0 otherwise.

We denote by $\theta \colon \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \to u_* \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp \to X^\sharp/T^\sharp}$ the canonical homomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules (which is build by duality from the canonical homomorphisms $\widetilde{u}^* \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$) and which sends 1 to $1 \otimes 1$. For any $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, we denote by $[P]_Z$ its image via the canonical morphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} / \mathcal{I} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} = u_* \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp \to X^\sharp/T^\sharp}$. We set $\underline{\xi}_{(r)}^{\langle \underline{k} \rangle_{(m)}} := [\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}}]_Z$. By duality, we compute $\theta(\underline{\partial}_{(r)}^{\langle \underline{i} \rangle_{(m)}}) = \underline{\xi}_{(r)}^{\langle (\underline{i},\underline{0}) \rangle_{(m)}}$, for any $\underline{i} \in \mathbb{N}^{r+s}$.

Notation 5.2.2.2. By abuse of notations, we write $\mathcal{O}_T$ for the sheaf deduced from $\mathcal{O}_T$ by topological inverse image (e.g. $p_X^{-1} \mathcal{O}_T$ is denoted by $\mathcal{O}_T$ where $p_X \colon X^\sharp \to T^\sharp$ is the map induced by α). We denote by $\mathcal{O}_T\{\partial_{r+s+1}, \ldots, \partial_d\}^{(m)}$ the sub-$\mathcal{O}_T$-algebra of $\mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$ generated by the elements $\{\partial_{r+s+1}^{\langle j_1 \rangle_{(m)}}, \partial_{r+s+2}^{\langle j_2 \rangle_{(m)}}, \ldots, \partial_d^{\langle j_{d-r-s} \rangle_{(m)}} \mid j_1, \ldots, j_{d-r-s} \in \mathbb{N}\}$. It is equal to the free $\mathcal{O}_T$-module whose basis is given by $\{\underline{\partial}_{(r)}^{\langle (\underline{0},\underline{j}) \rangle_{(m)}} \mid \underline{j} \in \mathbb{N}^{d-r-s}\}$, where $\underline{0} := (0, \ldots, 0) \in \mathbb{N}^{r+s}$. Beware that this commutative

$\mathcal{O}_T$-algebra has not to be confused with the partial divided powers of level m polynomial $\mathcal{O}_T$-algebra in the variables $\partial_{r+s+1},\dots,\partial_d$ of 1.3.3.6 which was denoted by $\mathcal{O}_T\langle\partial_{r+s+1},\dots,\partial_d\rangle_{(m)}$.

Notation 5.2.2.3. We denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ the free $\mathcal{B}_X$-module with the basis $\{\underline{\partial}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}} \mid \underline{i}\in\mathbb{N}^{r+s}\}$, where $\underline{0}:=(0,\dots,0)\in\mathbb{N}^{d-r-s}$. The sheaf $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ is equal to the sub-$\mathcal{O}_T$-algebra of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ which is generated by $\mathcal{B}_X$, by $\partial_{\sharp,i}^{\langle p^h\rangle_{(m)}}$ and $\partial_j^{\langle p^h\rangle_{(m)}}$ for any $1\le i\le r$, $r+1\le j\le r+s$ and $0\le h\le p^m$. Since there is no ambiguity concerning the local coordinates (resp. and $T^\sharp$), we might sometimes simply denote $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp}$).

5.2.2.4. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ is a free left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}$-module with the basis $\{\underline{\partial}^{\langle(\underline{0},\underline{h})\rangle_{(m)}}\mid \underline{h}\in\mathbb{N}^{d-r-s}\}$, where $\underline{0}:=(0,\dots,0)\in\mathbb{N}^{r+s}$. This means that the canonical homomorphism of $\mathcal{O}_T$-algebras

$$\mu\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \tag{5.2.2.4.1}$$

given by $P\otimes Q\mapsto PQ$ is also an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules.

Since $t_{r+s+1},\dots,t_d$ generate $\mathcal{I}$ and are in the center of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$, then we compute $\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}=\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\mathcal{I}$. Hence, we get a canonical $\mathcal{O}_T$-algebra structure on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ induced by that of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$. Since $\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}=\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\cap\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, we get the inclusion

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\hookrightarrow\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}.$$

The morphism 5.2.2.4.1 induces the isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules:

$$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp})\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}=u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}. \tag{5.2.2.4.2}$$

Notation 5.2.2.5. Let $\iota\colon\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\hookrightarrow\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$ be the canonical homomorphism of $\mathcal{O}_T$-algebra. Taking the structure of $\mathcal{B}_X$-module induced by the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ (resp. induced by the structure of left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$), we denote by

$$\sigma^{(m)}_{X^\sharp,\widetilde{t}}\colon\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\to u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \tag{5.2.2.5.1}$$

the $\mathcal{B}_X$-linear map given by $\sigma^{(m)}_{X^\sharp,\widetilde{t}}(\underline{\partial}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}})=\underline{\partial}_{(r)}^{\langle\underline{i}\rangle_{(m)}}\otimes\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}$, for any $\underline{i}\in\mathbb{N}^{r+s}$ and $\underline{j}\in\mathbb{N}^{d-r-s}$.

Moreover, $\sigma^{(m)}_{X^\sharp,\widetilde{t}}$ is surjective with kernel equal to $\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. We write

$$\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}\colon\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\xrightarrow{\sim}u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \tag{5.2.2.5.2}$$

the induced $\mathcal{B}_Z$-linear isomorphism characterized by the formula $\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}(\underline{\xi}_{(r)}^{\langle(\underline{i},\underline{j})\rangle_{(m)}})=\underline{\partial}_{(r)}^{\langle\underline{i}\rangle_{(m)}}\otimes\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}$, for any $\underline{i}\in\mathbb{N}^{r+s}$ and $\underline{j}\in\mathbb{N}^{d-r-s}$. In fact, we will show this isomorphism is even $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear (see 5.2.2.9).

Taking the structure of $\mathcal{B}_X$-module induced by the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ (resp. induced by the structure of left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$), we denote by

$$\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}\colon\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\to u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \tag{5.2.2.5.3}$$

the $\mathcal{B}_X$-linear morphism given by $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(\underline{\partial}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}})=\underline{\partial}_{(r)}^{\langle\underline{i}\rangle_{(m)}}$, for any $\underline{i}\in\mathbb{N}^{r+s}$. In other words, $\iota\circ\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ is the restriction of $\sigma^{(m)}_{X^\sharp,\widetilde{t}}$ on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$.

The kernel of $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ is $\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ and we get therefore the $\mathcal{B}_Z$-linear isomorphism (for the left structures):

$$\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\mathcal{I} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}. \tag{5.2.2.5.4}$$

which satisfies the formula $\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(\underline{\xi}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}}) = \underline{\partial}_{(r)}^{\langle\underline{i}\rangle_{(m)}}$, for any $\underline{i}\in\mathbb{N}^{r+s}$.

Moreover, for any $P\in\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}$, $Q\in\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$, we have the formula

$$\sigma^{(m)}_{X^\sharp,\widetilde{t}}(PQ) = \overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}([PQ]_Z) = \sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P)\otimes Q = \overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([P]_Z)\otimes Q. \tag{5.2.2.5.5}$$

By construction we have the commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} & \xrightarrow[\sim]{\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}} & u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \\
\Vert & & \uparrow{\scriptstyle \overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}\otimes\mathrm{id}}\ \wr \\
u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp} & \xleftarrow[5.2.2.4.2]{\sim} & (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp})\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}.
\end{array} \tag{5.2.2.5.6}$$

Since $\theta(\underline{\partial}_{(r)}^{\langle\underline{i}\rangle_{(m)}}) = \underline{\xi}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}}$ for any $\underline{i}\in\mathbb{N}^{r+s}$, since $\mathcal{B}_Z = u^*(\mathcal{B}_X)$, then we have the following $\mathcal{B}_Z$-linear homomorphism $\vartheta\colon u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\to\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ making commutative the diagram:

$$\begin{array}{ccccc}
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} & \xrightarrow{[-]_Z} & \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} & \hookrightarrow & \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \\
{\scriptstyle 5.2.2.5.3}\downarrow{\scriptstyle \sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}} & \nearrow{\scriptstyle \vartheta}\ \sim & & & \Vert \\
u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} & & \xrightarrow{u_*\theta} & & u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}.
\end{array} \tag{5.2.2.5.7}$$

Hence, $\vartheta = (\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}})^{-1}$ and then $\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ do not depend on the choice of the semi logarithmic coordinates (contrary to $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ or $\sigma^{(m)}_{X^\sharp,\widetilde{t}}$).

Lemma 5.2.2.6. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module, $x\in\mathcal{E}$, $P\in\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$, $[-]_Z$ the canonical surjection $\mathcal{E}\to\mathcal{E}/\mathcal{I}\mathcal{E}$. With notation 5.2.2.5.4, the structure of left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module of $\mathcal{E}/\mathcal{I}\mathcal{E}$ is characterized by the formula:*

$$\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P)\cdot[x]_Z = \overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([P]_Z)\cdot[x]_Z = [P\cdot x]_Z. \tag{5.2.2.6.1}$$

Proof. Let $Q := \sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P) = \overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([P]_Z)\in\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$. Since $[P]_Z = \vartheta(Q)$ (see 5.2.2.5.7), then this is a consequence of 4.4.2.11.2. □

Lemma 5.2.2.7. *The map $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ (see 5.2.2.5.3) is a morphism of $\mathcal{O}_T$-algebras. The maps $\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ and ϑ (see 5.2.2.5.4) are isomorphisms of $\mathcal{O}_T$-algebras.*

Proof. Since the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\to\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\mathcal{I}$ is a morphism of $\mathcal{B}_X$-rings (i.e. this is $\mathcal{B}_X$-linear and this is an homomorphism of rings), since $\vartheta = (\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}})^{-1}$, then we reduce to check ϑ is an isomorphism of $\mathcal{O}_T$-algebras.

By $\mathcal{B}_Z$-linearity of this map, we reduce to check that $\vartheta(\underline{\partial}_{(r)}^{\langle\underline{k}\rangle_{(m)}}\overline{b}\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}}) = \vartheta(\underline{\partial}_{(r)}^{\langle\underline{k}\rangle_{(m)}})\vartheta(\overline{b})\vartheta(\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}})$, for any $b\in\Gamma(X,\mathcal{B}_X)$, $\underline{k},\underline{l}\in\mathbb{N}^{r+s}$. By using the formula 3.2.3.7.2 where $\underline{\partial}_\sharp$ is replaced by $\underline{\partial}_{(r)}$, we get the formula $\vartheta(\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}}) = \underline{\xi}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$ and by $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linearity of ϑ we get the equality

$$\vartheta(\underline{\partial}_{(r)}^{\langle\underline{k}\rangle_{(m)}}[b]_Z\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}}) = \sum_{\underline{k}''+\underline{k}'=\underline{k}}\left\{\begin{matrix}\underline{k}\\ \underline{k}'\end{matrix}\right\}_{(m)}\underline{\partial}_{(r)}^{\langle\underline{k}'\rangle_{(m)}}([b]_Z)\underline{\partial}_{(r)}^{\langle\underline{k}''\rangle_{(m)}}\cdot\underline{\xi}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$$

By using 5.2.2.6.1 in the case where $\mathcal{E} = \mathcal{B}_X$, since $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(\underline{\partial}^{\langle(\underline{k}',\underline{0})\rangle_{(m)}}_{(r)}) = \underline{\partial}^{\langle\underline{k}'\rangle_{(m)}}_{(r)}$ then we get: $\underline{\partial}^{\langle\underline{k}'\rangle_{(m)}}([b]_Z) = [\underline{\partial}^{\langle(\underline{k}',\underline{0})\rangle_{(m)}}_{(r)}(b)]_Z$. Since $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ is $\mathcal{B}_X$-linear (for the left structures), then we get $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(\underline{\partial}^{\langle(\underline{k}',\underline{0})\rangle_{(m)}}_{(r)}(b)\underline{\partial}^{\langle(\underline{k}'',\underline{0})\rangle_{(m)}}_{(r)}) = \underline{\partial}^{\langle\underline{k}'\rangle_{(m)}}([b]_Z)\underline{\partial}^{\langle\underline{k}''\rangle_{(m)}}_{(r)}$.

Since $\underline{\xi}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}_{(r)} = [\underline{\partial}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}_{(r)}]_Z$, then using 5.2.2.6.1 in the case where $\mathcal{E} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, we get the first equality:

$$\sum_{\underline{k}''+\underline{k}'=\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{k}' \end{smallmatrix} \right\}_{(m)} \underline{\partial}^{\langle\underline{k}'\rangle_{(m)}}([b]_Z)\underline{\partial}^{\langle\underline{k}''\rangle_{(m)}}_{(r)} \cdot \underline{\xi}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}_{(r)} \overset{5.2.2.6.1}{=} \sum_{\underline{k}''+\underline{k}'=\underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{k}' \end{smallmatrix} \right\}_{(m)} [\underline{\partial}^{\langle(\underline{k}',\underline{0})\rangle_{(m)}}_{(r)}(b)\underline{\partial}^{\langle(\underline{k}'',\underline{0})\rangle_{(m)}}_{(r)}\underline{\partial}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}_{(r)}]_Z$$

$$= [\underline{\partial}^{\langle(\underline{k},\underline{0})\rangle_{(m)}}_{(r)} b\underline{\partial}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}_{(r)})]_Z = [\underline{\partial}^{\langle(\underline{k},\underline{0})\rangle_{(m)}}_{(r)}]_Z[b]_Z[\underline{\partial}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}_{(r)})]_Z = \vartheta(\underline{\partial}^{\langle\underline{k}\rangle_{(m)}}_{(r)})\vartheta([b]_Z)\vartheta(\underline{\partial}^{\langle\underline{l}\rangle_{(m)}}_{(r)}),$$

the third one being a consequence of the fact that $[]_Z\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ is a ring homomorphism. □

5.2.2.8. Since $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ is a $(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$-bimodule. Using the formula 5.2.2.6.1 applied to the case $\mathcal{E} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, we compute that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ is also a left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-submodule of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, and then a $(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp})$-sub-bimodule of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$.

Since $u_*\theta\colon u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \to u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}$ is a homomorphism of left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules, then via the commutativity of 5.2.2.5.7, this implies that the bijection $\vartheta\colon u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ (and its inverse $\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$) is an isomorphism of left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules.

Proposition 5.2.2.9. *Taking the structure of left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module induced by the $(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$-bimodule structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ (see 5.2.2.8), the isomorphism $\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}$ (see 5.2.2.5.2) is $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear.*

Proof. Let $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. The element P can be written (uniquely) in the form $\sum_{\underline{j}\in\mathbb{N}^{d-r-s}} P_{\underline{j}}\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}_{(r)}$, with $P_{\underline{j}} \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ zero except for a finite number of terms. Then, we compute

$$\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}([P]_Z) = \sum_{\underline{j}\in\mathbb{N}^{d-r-s}} \sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P_{\underline{j}}) \otimes \underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}. \tag{5.2.2.9.1}$$

Hence, we compute the $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linearity of $\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ follows from the fact that $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ is a morphism of $\mathcal{O}_T$-algebras (see 5.2.2.7) and from the formula 5.2.2.6.1 applied to the case $\mathcal{E} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. □

5.2.2.10. It follows from 5.2.2.9 that $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}$ is a free left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module with the basis $\{\underline{\xi}^{\langle(\underline{0},\underline{h})\rangle_{(m)}}_{(r)} \mid \underline{h} \in \mathbb{N}^{d-r-s}\}$, where $\underline{0} := (0,\ldots,0) \in \mathbb{N}^{r+s}$.

5.2.2.11. It follows from 5.2.2.9 that via the isomorphism $\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}$ we equip $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)}$ with a structure of $(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$-bimodule extending its structure of left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module. The right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module structure is given later by the explicit formula 5.2.2.20. Remark, the morphism $\sigma^{(m)}_{X^\sharp,\widetilde{t}}$ is also $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-linear.

5.2.2.12 (Semi-logarithmic adjoint operator)**.** The semi-logarithmic adjoint automorphism ${}^\tau\colon (\widetilde{D}^{(m)}_{X^\sharp/T^\sharp})^o \to \widetilde{D}^{(m)}_{X^\sharp/T^\sharp}$ (see 4.5.1.5) induces ${}^\tau\colon \widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} \to \widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ such that ${}^\tau(I\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}) = I\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$. This yields the automorphism ${}^\tau\colon \widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} \to \widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$. On the other hand, via the local logarithmic coordinates $\overline{t}_1,\ldots,\overline{t}_{r+s}$ of $Z^\sharp$ over $T^\sharp$, we get the logarithmic adjoint operator automorphism ${}^\tau\colon \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp} \to \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$ given by $Q = \sum_{\underline{i}\in\mathbb{N}^{r+s}} b_{\underline{i}}\underline{\partial}^{\langle\underline{i}\rangle_{(m)}}_{(r)} \mapsto {}^\tau(Q) := \sum_{\underline{i}\in\mathbb{N}^{r+s}} {}^\tau\underline{\partial}^{\langle\underline{i}\rangle_{(m)}}_{(r)} b_{\underline{i}}$.

Since $\vartheta(\underline{\partial}_{(r)}^{\langle \underline{i} \rangle_{(m)}}) = \underline{\partial}_{(r)}^{\langle (\underline{i},\underline{0}) \rangle_{(m)}}$, $\vartheta({}^{\tau}\underline{\partial}_{(r)}^{\langle \underline{i} \rangle_{(m)}}) = {}^{\tau}\underline{\partial}_{(r)}^{\langle (\underline{i},\underline{0}) \rangle_{(m)}}$ (use the formula 3.4.1.2.2) for any $\underline{i} \in \mathbb{N}^{r+s}$, then the following diagram

$$\begin{array}{ccccc} \widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp} & \xrightarrow{\sigma^{(m)}_{Z^\sharp, X^\sharp, \widetilde{t}}} & \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp} & \underset{\vartheta}{\xrightarrow{\sim}} & \widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp} \\ \sim\downarrow\tau & & \sim\downarrow\tau & & \sim\downarrow\tau \\ \widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp} & \xrightarrow{\sigma^{(m)}_{Z^\sharp, X^\sharp, \widetilde{t}}} & \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}, & \underset{\vartheta}{\xrightarrow{\sim}} & \widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp} \end{array} \tag{5.2.2.12.1}$$

is commutative.

5.2.2.13. For any left (resp. right) $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp}$-module E (resp. M), we denote by $[-]_Z \colon E \to E/IE$ (resp. $[-]'_Z \colon M \to M/MI$) the canonical surjection. Remark we add a prime to avoid some confusion in the case of a bimodule, e.g. see 5.2.2.14. We denote by $\widetilde{e}_0 := d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge dt_d$ a basis of the free $\mathcal{B}_X$-module $\widetilde{\omega}_{X^\sharp/T^\sharp}$ and by $\widetilde{e}_0^\vee$ its corresponding dual basis of the free $\mathcal{B}_X$-module $\widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}$. We denote by $\widetilde{f}_0 := d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge dt_{r+s}$ the basis of the free $\mathcal{B}_Z$-module $\widetilde{\omega}_{Z^\sharp/T^\sharp}$, and by $\widetilde{f}_0^\vee$ its dual basis.

Following 4.5.1.8, the functors $\widetilde{\omega}_{Z^\sharp/T^\sharp} \otimes_{\mathcal{B}_Z} -$ and $\otimes_{\mathcal{B}_Z} \widetilde{\omega}^{-1}_{Z^\sharp/T^\sharp}$ induce quasi-inverse equivalence between the category of left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules and of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules, with explicit computation involving the semi-logarithmic adjoint operator via the identification of $\widetilde{\omega}_{Z^\sharp/T^\sharp}$ (resp. $\widetilde{\omega}^{-1}_{Z^\sharp/T^\sharp}$) with $\mathcal{B}_Z$ via the choice of the basis $\widetilde{f}_0$ (resp. $\widetilde{f}_0^\vee$). Let $\mathcal{M}$ be a quasi-coherent right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module. We have the isomorphism of abelian groups:

$$\iota_{\widetilde{t}} \colon \Gamma(Z, \widetilde{u}^*(\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}) \otimes_{\mathcal{B}_Z} \widetilde{\omega}_{Z^\sharp/T^\sharp}) \xrightarrow{\sim} M/MI \tag{5.2.2.13.1}$$

given by $\widetilde{f}_0 \otimes [y \otimes \widetilde{e}_0^\vee]_Z \mapsto [y]'_Z$. Via this map $\iota_{\widetilde{t}}$, M/MI is equipped with a structure of right $\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$-module. The structure of right $\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$-module of M/MI is characterized by the formula,

$$[y]'_Z \cdot \sigma^{(m)}_{Z^\sharp, X^\sharp, \widetilde{t}}(P) = [y]'_Z \cdot \overline{\sigma}^{(m)}_{Z^\sharp, X^\sharp, \widetilde{t}}([P]_Z) = [y \cdot P]'_Z, \tag{5.2.2.13.2}$$

for any $y \in M$ and $P \in \widetilde{D}^{(m)}_{X^\sharp/T^\sharp}$. Indeed, set $Q := \sigma^{(m)}_{Z^\sharp, X^\sharp, \widetilde{t}}(P)$. Following 4.5.1.8.1, we have $\left(\widetilde{f}_0 \otimes [y \otimes \widetilde{e}_0^\vee]_Z\right) Q = \widetilde{f}_0 \otimes {}^{\tau}Q\left([y \otimes \widetilde{e}_0^\vee]_Z\right)$. Since ${}^{\tau}Q = \sigma^{(m)}_{Z^\sharp, X^\sharp, \widetilde{t}}({}^{\tau}P)$ (see 5.2.2.12.1), then following 5.2.2.6.1 we get the first equality

$${}^{\tau}Q\left([y \otimes \widetilde{e}_0^\vee]_Z\right) = [{}^{\tau}P\left(y \otimes \widetilde{e}_0^\vee\right)]_Z \overset{4.5.1.8.2}{=} [yP \otimes \widetilde{e}_0^\vee]'_Z.$$

Hence, we are done.

Since $\mathcal{M}/\mathcal{M}\mathcal{I}$ is quasi-coherent and $\Gamma(X, \mathcal{M}/\mathcal{M}\mathcal{I}) \xrightarrow{\sim} M/MI$, this yields $\mathcal{M}/\mathcal{M}\mathcal{I}$ is equipped with a structure of right $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module so that

$$\iota_{\widetilde{t}} \colon u_*\widetilde{u}^*(\mathcal{M} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}) \otimes_{\mathcal{B}_Z} \widetilde{\omega}_{Z^\sharp/T^\sharp} \xrightarrow{\sim} \mathcal{M}/\mathcal{M}\mathcal{I} \tag{5.2.2.13.3}$$

is an isomorphism of right $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules.

5.2.2.14. Following 5.2.2.13.3 and with its notation, we get the isomorphism of abelian sheaves:

$$\iota_{\widetilde{t}} \colon u_*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \leftarrow Z^\sharp/T^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I}. \tag{5.2.2.14.1}$$

Via this map, we get a structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}, u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$-bimodule on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I}$. By functoriality of the construction of the structure of $(u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$-bimodule of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \leftarrow Z^\sharp/T^\sharp}$, we check that the underlying structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I}$ is equal to its natural structure. The structure of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module is given by the formula 5.2.2.13.2 applied to the case where $\mathcal{M} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. Moreover, we compute that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp}$ is also a left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-submodule of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, and therefore a $(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp})$-sub-bimodule of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$.

5.2.2.15. We denote by

$$\mu' \colon \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \tag{5.2.2.15.1}$$

the canonical homomorphism of $\mathcal{O}_T$-algebras given by $Q\otimes P \mapsto QP$, which is also an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}$-modules. The morphism 5.2.2.15.1 induces the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules:

$$\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}\mathcal{I}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I}. \tag{5.2.2.15.2}$$

Notation 5.2.2.16. Taking the $\mathcal{B}_X$-module structure induced by the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ (resp. induced by the structure of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module on $\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$), we denote by

$$\varsigma^{(m)}_{X^\sharp,\widetilde{t}} \colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \to \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \tag{5.2.2.16.1}$$

the $\mathcal{B}_X$-linear map given by $\varsigma^{(m)}_{X^\sharp,\widetilde{t}}(\underline{\partial}^{\langle(\underline{i},\underline{j})\rangle_{(m)}}_{(r)}) = \underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}} \otimes \underline{\partial}^{\langle\underline{i}\rangle_{(m)}}_{(r)}$, for any $\underline{i}\in\mathbb{N}^{r+s}$ and $\underline{j}\in\mathbb{N}^{d-r-s}$.

Moreover, $\varsigma^{(m)}_{X^\sharp,\widetilde{t}}$ is surjective with kernel equal to $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I}$ and we denote by

$$\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}} \colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I} \xrightarrow{\sim} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \tag{5.2.2.16.2}$$

the induced $\mathcal{B}_Z$-linear isomorphism characterized by the formula $\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}(\underline{\xi}^{\langle(\underline{i},\underline{j})\rangle_{(m)}}_{(r)}) = \underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}} \otimes \underline{\partial}^{\langle\underline{i}\rangle_{(m)}}_{(r)}$, for any $\underline{i}\in\mathbb{N}^{r+s}$ and $\underline{j}\in\mathbb{N}^{d-r-s}$. For any $P\in\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}$, $Q\in\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$, we have the formula

$$\varsigma^{(m)}_{X^\sharp,\widetilde{t}}(QP) = \overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}([QP]'_Z) = Q\otimes\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P) = Q\otimes\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([P]_Z). \tag{5.2.2.16.3}$$

We have the commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I} & \xrightarrow[\sim]{\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}} & \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \\
\Vert & & \uparrow{\scriptstyle \mathrm{id}\otimes\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}}\ \wr \\
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I} & \xleftarrow[5.2.2.15.2]{\sim} & \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}).
\end{array} \tag{5.2.2.16.4}$$

where $\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ is the isomorphism 5.2.2.5.4.

5.2.2.17. By using the commutativity of 5.2.2.12.1, we can check that the semi-logarithmic adjoint operator (see 5.2.2.12) induces the bijection

$$\overline{\tau} \colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I},$$

making commutative the diagram

$$\begin{array}{ccc}
u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} & \xrightarrow[\alpha]{\sim} & \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \\
\uparrow{\scriptstyle \overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}}\ \wr & & \uparrow{\scriptstyle \overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}}\ \wr \\
\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} & \xrightarrow[\sim]{\overline{\tau}} & \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I} \\
\uparrow{\scriptstyle 5.2.2.4.2}\ \wr & & \uparrow{\scriptstyle 5.2.2.15.2}\ \wr \\
(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}) \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} & \xrightarrow[\beta]{\sim} & \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp/T^\sharp}\mathcal{I}).
\end{array} \tag{5.2.2.17.1}$$

where α is given by $P_Z\otimes Q\mapsto {}^tQ\otimes{}^\tau P_Z$ and where β is given by $[P_X]_Z\otimes Q\mapsto {}^tQ\otimes[{}^\tau P_X]'_Z$.

Proposition 5.2.2.18. *Taking the structure of right $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module induced by the $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}, u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$-bimodule structure of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ (see 5.2.2.14), the isomorphism $\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}$ (see 5.2.2.16.2) is $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear.*

Proof. Let $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. The element P can be written (uniquely) in the form $\sum_{\underline{j}\in\mathbb{N}^{d-r-s}} \underline{\partial}_{(r)}^{\langle(\underline{0},\underline{j})\rangle_{(m)}} P_{\underline{j}}$, with $P_{\underline{j}} \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ zero except for a finite number of terms. Then, following 5.2.2.16.3 we have the formula

$$\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}([P]'_Z) = \sum_{\underline{j}\in\mathbb{N}^{d-r-s}} \underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}} \otimes \sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P_{\underline{j}}). \qquad (5.2.2.18.1)$$

Hence, we compute the $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linearity of $\overline{\varsigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ follows from the fact that $\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}$ is a morphism of $\mathcal{O}_T$-algebras (see 5.2.2.7) and from the formula 5.2.2.13.2 applied to the case $\mathcal{M} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. □

5.2.2.19. It follows from 5.2.2.18 that via the isomorphism $\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}}$ we equip $\mathcal{O}_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$ with a structure of $(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$-bimodule extending its structure of left $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module. The left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module structure is given below by the explicit formula 5.2.2.20. Finally, remark the morphism $\varsigma^{(m)}_{X^\sharp,\widetilde{t}}$ is also $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-linear.

Lemma 5.2.2.20. *Let $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$, $R \in \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$, and $Q, Q' \in \mathcal{O}_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)}$. Then we get respectively in $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)}$ and $\mathcal{O}_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$ the formulas*

$$(R\otimes Q)\cdot PQ' = R\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P)\otimes QQ', \qquad Q'P\cdot(Q\otimes R) = Q'Q\otimes\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P)R. \qquad (5.2.2.20.1)$$

Proof. Choose any $R_X \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ such that $\vartheta(R) = [R_X]_Z$ (see 5.2.2.5.7). $\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}([PQ]_Z) = \overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}([PQ]_Z)$. By using the formula 5.2.2.5.5 and the $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-linearity of $\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}$, we get $(R\otimes Q)\cdot PQ' = \overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}([R_XQ]_Z\cdot PQ')$. Since $[R_XQ]_Z\cdot PQ' = [R_XPQQ']_Z$, using again formula 5.2.2.5.5, we get $\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}([R_XQ]_Z\cdot PQ') = \overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([R_XP]_Z)\otimes QQ' \overset{5.2.2.7}{=} \overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([R_X]_Z)\overline{\sigma}^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}([P]_Z)\otimes QQ' = R\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P)\otimes QQ'$.

The second equality of 5.2.2.20.1 follows similar from the formula 5.2.2.16.3 and the Lemma 5.2.2.7. □

5.2.2.21 (Base change and change of level)**.** Suppose we are in the non-respective case of log-schemes of 5.2. Take an integer $m' \geq m$. Let $T'^\sharp \to T^\sharp$ be a morphism of noetherian affine nice fine log-schemes. Put $X'^\sharp := X^\sharp \times_{T^\sharp} T'^\sharp$, $Z'^\sharp := Z^\sharp \times_{T^\sharp} T'^\sharp$, and let $u' \colon Z'^\sharp \hookrightarrow X'$ be the induced closed immersion and $f\colon X'^\sharp \to X^\sharp$, $g\colon Z'^\sharp \to Z^\sharp$ be the canonical projection. We set $\mathcal{B}_{X'} := f^*(\mathcal{B}_X)$, $\mathcal{B}_{Z'} := g^*(\mathcal{B}_Z)$. We keep similar notation than 4.1.2, in particular we set $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/T^\sharp} := \mathcal{B}_{X'} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/T^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp} := \mathcal{B}_{Z'} \otimes_{\mathcal{O}_{Z'}} \widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp}$. We denote by $\widetilde{X}'^\sharp$ (resp. $\widetilde{Z}'^\sharp$) the ringed logarithmic scheme $(X'^\sharp, \mathcal{B}_{X'})$ (resp. $(Z'^\sharp, \mathcal{B}_{Z'})$), and by $\tilde{u}'\colon \widetilde{Z}'^\sharp/T'^\sharp \to \widetilde{X}'^\sharp/T'^\sharp$ the morphism of relative ringed logarithmic schemes induced by the diagram 5.1.1.1.1 and by $u'^*\mathcal{B}_{X'} \xrightarrow{\sim} \mathcal{B}_{Z'}$

We have ring homomorphisms $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \to f^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/T'^\sharp} \to \widetilde{\mathcal{D}}^{(m')}_{X'^\sharp/T'^\sharp}$ (see 4.4.4.1). Thus we get the ring homomorphism $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp} \to \widetilde{D}^{(m')}_{X'^\sharp/T'^\sharp}$. Similarly, we get $\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp} \to \widetilde{D}^{(m')}_{Z'^\sharp/T'^\sharp}$.

The semi-logarithmic coordinates $t_1,\ldots,t_d$ of $X^\sharp/T^\sharp$ (see 4.5.1.4) induces canonically the semi-logarithmic coordinates $t'_1,\ldots,t'_d$ on $X'^\sharp/T'^\sharp$. Denote by $\partial_i'^{\langle k\rangle_{(m)}}$ for $i = r+1,\ldots,d$ and $k \in \mathbb{N}$ the associated operators of $\widetilde{D}^{(m')}_{X'^\sharp/T'^\sharp}$.

Write $\mathcal{O}_{T'^\sharp}\{\partial'_{r+s+1},\ldots,\partial'_d\}^{(m')}$ for the sub-$\mathcal{O}_{T'^\sharp}$-algebra of $D^{(m')}_{X'^\sharp/T'^\sharp}$ generated by the elements $\partial_i'^{\langle k\rangle_{(m')}}$ for $r+s+1 \leq i \leq r$ and $k \in \mathbb{N}$. Since the homomorphism $\widetilde{D}^{(m)}_{X^\sharp/T^\sharp} \to \widetilde{D}^{(m')}_{X'^\sharp/T'^\sharp}$ carries $\partial_i^{\langle k\rangle_{(m)}}$ to $\frac{q_k^{(m)}!}{q_k^{(m')}!}\partial_i'^{\langle k\rangle_{(m')}}$ for $r+s+1 \leq i \leq r$ and $k \in \mathbb{N}$ (use formula 1.4.2.5.2), then it induces the factorization $\mathcal{O}_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)} \to O_{T'^\sharp}\{\partial'_{r+s+1},\ldots,\partial'_d\}^{(m')}$. We compute that the maps $\sigma^{(m)}_{X^\sharp,\widetilde{t}}$ and $\varsigma^{(m)}_{X^\sharp,\widetilde{t}}$ of

5.2.2.5.1 commute with base change and with change of level, i.e. the canonical diagram (and similarly for $\varsigma^{(m)}_{X^\sharp,\widetilde{t}}$) is commutative:

$$\begin{array}{ccc} \widetilde{D}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp} & \xrightarrow[\sim]{\sigma^{(m)}_{X^\sharp,\underline{t}}} & \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \\ \downarrow & & \downarrow \\ \widetilde{D}^{(m)}_{Z'\to X'^\sharp/T'^\sharp} & \xrightarrow[\sim]{\sigma^{(m')}_{X'^\sharp,\underline{t}'}} & \widetilde{D}^{(m')}_{Z'^\sharp/T'^\sharp}\otimes_{\mathcal{O}_{T'^\sharp}}\mathcal{O}_{T'^\sharp}\{\partial'_{r+s+1},\dots,\partial'_d\}^{(m')}. \end{array} \tag{5.2.2.21.1}$$

5.2.2.22. Suppose we are in the non-respective case of log-schemes of 5.2. Let $\mathcal{F}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module. We get the isomorphisms

$$\widetilde{u}^{(m)}_{+}(\mathcal{F}) = u_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\hookleftarrow Z^\sharp/T^\sharp}\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} u_*\mathcal{F}) \xrightarrow[\iota_{\widetilde{t}}\otimes\mathrm{id}]{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{I})\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} u_*\mathcal{F} \xrightarrow[\varsigma^{(m)}_{X^\sharp,\widetilde{t}}\otimes\mathrm{id}]{\sim}$$
$$(\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\otimes_{\mathcal{O}_T} u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} u_*\mathcal{F} \xrightarrow{\sim} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\otimes_{\mathcal{O}_T} u_*\mathcal{F}. \tag{5.2.2.22.1}$$

Let $x\in\mathcal{M}$, $P\in\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$, and $Q,Q'\in\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$. By using the right formula of 5.2.2.20.1, we can check that the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module of $\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\otimes_{\mathcal{O}_T}\mathcal{M}$ is given by the formula:

$$Q'P\cdot(Q\otimes x) = Q'Q\otimes(\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P)\cdot x). \tag{5.2.2.22.2}$$

Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module. We have the isomorphisms

$$\widetilde{u}^{(m)}_{+}(\mathcal{M}) = u_*(\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}) \xrightarrow{\sim} u_*\mathcal{M}\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$$
$$\xrightarrow[\mathrm{id}\otimes\overline{\sigma}^{(m)}_{X^\sharp,\widetilde{t}}]{\sim} u_*\mathcal{M}\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}) \xrightarrow{\sim} u_*\mathcal{M}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}. \tag{5.2.2.22.3}$$

Let $x\in\mathcal{M}$, $P\in\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$, and $Q,Q'\in\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$. By using the left formula of 5.2.2.20.1, we can check that the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module of $\mathcal{M}\otimes_{\mathcal{O}_T}\mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$ is given by the formula:

$$(x\otimes Q)\cdot PQ' = (x\cdot\sigma^{(m)}_{Z^\sharp,X^\sharp,\widetilde{t}}(P))\otimes QQ'. \tag{5.2.2.22.4}$$

5.2.3 Exactness of the pushforward by a closed immersion

Proposition 5.2.3.1. *Suppose the hypotheses and notations of 5.1.1.1 hold. We suppose f is a (not necessarily exact) closed immersion and ℓ is the identity*

(a) The bimodule $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\hookrightarrow Y^\sharp}$ is a locally free as left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module. The bimodule $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\hookleftarrow X^\sharp}$ is a locally free as right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module.

(b) Suppose we are in the non-respective case of 5.1.1.1. For any $*\in\{r,l\}$, the functor $\widetilde{f}^{(m)}_{+}\colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp})\to D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp})$ is exact.

Proof. 0) Since f is a closed immersion, then the functor f_* is exact. Hence, we reduce to check the part a). By twisting, we reduce to check that the bimodule $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\hookrightarrow Y^\sharp}$ is a locally free as left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module. Since the morphism $\tilde{f}$ is the composition $(X^\sharp,\mathcal{B}_X)\to(X^\sharp,f^*\mathcal{B}_Y)\to(Y^\sharp,\mathcal{B}_Y)$, then by transitivity (see 5.1.1.12), we reduce to the case where $f=\mathrm{id}$ and to the case where the morphism $f^*\mathcal{B}_Y\to\mathcal{B}_X$ is an isomorphism Since the first case is clear, let us suppose the morphism $f^*\mathcal{B}_Y\to\mathcal{B}_X$ is an isomorphism.

1) When f is an exact closed immersion, then following 5.2.2.10, $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \hookrightarrow Y^\sharp}$ is a locally free right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-modules.

2) Let us prove the general case. Let $\overline{x}$ be a geometric point of $X^\sharp$. Following 3.1.1.14 (resp. 3.3.3.2), there exists a commutative diagram of the form

$$\begin{array}{ccccc} \widetilde{Y}^\sharp & \xrightarrow{u} & Y'^\sharp & \xrightarrow{v} & Y^\sharp \\ & {\scriptstyle g}\nwarrow & \uparrow{\scriptstyle f'} & \square & \uparrow{\scriptstyle f} \\ & & X'^\sharp & \xrightarrow{w} & X^\sharp \end{array}$$

such that the square is cartesian, u is log étale, $\underline{u}$ is affine, v is étale, g is an exact closed $S^\sharp$-immersion and v is an étale neighborhood of $\overline{x}$ in $X^\sharp$. By étale descent, we reduce to the case where $g = \mathrm{id}$. Since f is log-étale, then it follows from 5.1.3.6 that the canonical $\widetilde{\mathcal{D}}^{(m)}_{\widetilde{Y}^\sharp}$-linear morphism $\widetilde{\mathcal{D}}^{(m)}_{\widetilde{Y}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\widetilde{Y}^\sharp \to Y^\sharp}$ is an isomorphim. By using 5.1.1.11.1 this yields the isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-modules of the form $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \hookrightarrow \widetilde{Y}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp \hookrightarrow Y^\sharp}$. From the case 1), we get that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \hookrightarrow \widetilde{Y}^\sharp}$ is a locally free as left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$-module. Hence, we are done. □

5.2.4 On some base change of an exact closed immersion by an exact closed immersion

Suppose we are in the non-respective case of log-schemes of 5.2. Let $T'^\sharp \to T^\sharp$ be a morphism of nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$, where i is an integer (see definition 3.1.1.1). Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathcal{O}_X$, let $a\colon X'^\sharp \hookrightarrow X^\sharp$ and $u\colon Z^\sharp \hookrightarrow X^\sharp$ be the respective associated exact closed immersions. We set $Z'^\sharp := X'^\sharp \times_{X^\sharp} Z^\sharp$. We get both cartesian squares

$$\begin{array}{ccc} Z^\sharp & \xhookrightarrow{u} & X^\sharp \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle a} \\ Z'^\sharp & \xhookrightarrow{u'} & X'^\sharp, \end{array} \qquad \begin{array}{ccc} Z^\sharp_{T'^\sharp} & \xhookrightarrow{u_{T'^\sharp}} & X^\sharp_{T'^\sharp} \\ {\scriptstyle b_{T'^\sharp}}\uparrow & & \uparrow{\scriptstyle a_{T'^\sharp}} \\ Z'^\sharp_{T'^\sharp} & \xhookrightarrow{u'_{T'^\sharp}} & X'_{T'^\sharp}, \end{array} \tag{5.2.4.0.1}$$

whose morphisms of the left square are the canonical exact closed immersions, and whose right square is the induced square by base change via $T'^\sharp \to T^\sharp$. We suppose Zariski locally in X the following property holds:

($\star$) There exist three integers $d, r, s \geq 1$ such that $r + s \leq d$, a log-étale homomorphism of affine $T^\sharp$-log-schemes $\alpha\colon X^\sharp \to A^{d,r}_{T^\sharp}$ such that, denoting by $t_1, \ldots, t_r \in M_{X^\sharp}$ and $t_{r+1}, \ldots, t_d \in \mathcal{O}_X$ the elements given by α, we have $X'^\sharp = V(t_{r+s+1}, \ldots, t_d)$ and $Z^\sharp = V(t_{r+1}, \ldots, t_{r+s})$.

Let $\mathcal{B}_X$ be a commutative $\mathcal{O}_X$ algebra equipped with a left $\mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$-module structure which is compatible with its algebra structure. We set $\mathcal{B}_Z := u^*(\mathcal{B}_X)$. We keep similar notation than 4.1.2 (we replace $S^\sharp$ by $T^\sharp$, and $X^\sharp$ by $Z^\sharp$ if necessary), in particular we set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/T^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} := \mathcal{B}_Z \otimes_{\mathcal{O}_Z} \mathcal{D}^{(m)}_{Z^\sharp/T^\sharp}$. We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{Z}^\sharp$) the ringed logarithmic scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(Z^\sharp, \mathcal{B}_Z)$), and by $\tilde{u}\colon \widetilde{Z}^\sharp/T^\sharp \to \widetilde{X}^\sharp/T^\sharp$. the morphism of relative ringed logarithmic schemes induced by the diagram 5.2.4.0.1. Let $f\colon X^\sharp_{T'^\sharp} \to X^\sharp$ be the projection and $\mathcal{B}_{X^\sharp_{T'^\sharp}} := f^*(\mathcal{B}_X)$. We keep similar notation by adding the index $T'^\sharp$. We suppose $\widetilde{u}$, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{u}'$ are quasi-flat (see Definition 4.4.1.3).

5.2.4.1. To lighten the notation, we forget indicating the functors u_* and also u^{-1} for objects in the essential image of u_* ; and similarly for the other exact closed immersions. By applying the functor $\widetilde{b}^*$ to the canonical $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear morphism $\theta\colon \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{J}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ (given by $1 \mapsto [1]_Z$), we get the $\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp}$-linear morphism:

$$\widetilde{b}^*(\theta)\colon \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/(\mathcal{I}+\mathcal{J})\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$$

given by $[1]_{Z'} \mapsto [1]_{Z'}$. By composing the later morphism with the $\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp}$-linear morphism $\theta \colon \widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$ we get the canonical morphism $\theta \colon \widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/(\mathcal{I}+\mathcal{J})\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. When we are in the $(\star)$ local situation, this yields the isomorphisms ϑ are transitive (see the notation 5.2.2.6.1), i.e. the composition $\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp} \xrightarrow[\vartheta]{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp,Z'^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp,Z'^\sharp/T^\sharp} \xrightarrow[\vartheta]{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z'^\sharp/T^\sharp}/(\mathcal{I}+\mathcal{J})\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z'^\sharp/T^\sharp}$ is ϑ.

Lemma 5.2.4.2. *Suppose we are in the $(\star)$ local situation. We have the equalities*

$$(\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)})\mathcal{J} = \widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \tag{5.2.4.2.1}$$

$$\mathcal{I}(\mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}) = \mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}. \tag{5.2.4.2.2}$$

Proof. The equality 5.2.4.2.1 (resp. 5.2.4.2.2) comes from the left (resp. right) formula 5.2.2.20.1 □

Lemma 5.2.4.3. *Let $\mathcal{F}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp}$-module. We have the canonical isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}$-modules:*

$$\widetilde{a}^* \circ \widetilde{u}^{(m)}_+(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}'^{(m)}_+ \circ \widetilde{b}^*(\mathcal{F}). \tag{5.2.4.3.1}$$

Moreover, this one commutes with morphisms of change of level and of base, i.e., for any $m' \geq m$, for any $\widetilde{\mathcal{D}}^{(m')}_{Z_{T'^\sharp}/T^\sharp}$-module $\mathcal{F}'$, for any $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear morphism of the form $\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F} \to \mathcal{F}'$, the canonical diagram

$$\begin{array}{ccccc}
\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} (\widetilde{a}^* \circ \widetilde{u}^{(m)}_+(\mathcal{F})) & \xrightarrow{\sim} & \widetilde{a}^*_{T'^\sharp} \circ \widetilde{u}^{(m)}_{T'^\sharp +}(\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}) & \longrightarrow & \widetilde{a}^*_{T'^\sharp} \circ \widetilde{u}^{(m')}_{T'^\sharp +}(\mathcal{F}') \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} (\widetilde{u}'^{(m)}_+ \circ \widetilde{b}^*(\mathcal{F})) & \xrightarrow{\sim} & \widetilde{u}'^{(m)}_{T'^\sharp +} \circ \widetilde{b}^*_{T'^\sharp}(\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}) & \longrightarrow & \widetilde{u}'^{(m')}_{T'^\sharp +} \circ \widetilde{b}^*_{T'^\sharp}(\mathcal{F}'),
\end{array} \tag{5.2.4.3.2}$$

where vertical isomorphisms are that of the form 5.2.4.3.1 and where the horizontal morphisms follows from 5.2.5.11.2, is commutative.

Proof. By using 5.2.2.14.1 (still valid for arbitrary exact closed immersion), we get the isomorphisms:

$$\widetilde{a}^* \circ \widetilde{u}^{(m)}_+(\mathcal{F}) \xrightarrow{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{J}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} \mathcal{F} \tag{5.2.4.3.3}$$

$$\widetilde{u}'^{(m)}_+ \circ \widetilde{b}^*(\mathcal{F}) \xrightarrow{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} \mathcal{F}. \tag{5.2.4.3.4}$$

Hence, it is sufficient to check that we have an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$-bimodules of the form

$$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{J}) \xrightarrow{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}). \tag{5.2.4.3.5}$$

which sends $[1]_{X'} \otimes [1]'_Z$ to $[1]'_{Z'} \otimes [1]_{Z'}$. Since this is local, we can suppose we are in the $(\star)$ local situation. We denote by $\underline{t}_{X'} := (t_{r+s+1},\dots,t_d)$ and $\underline{t}_Z = (t_{r+1},\dots,t_{r+s})$. With notation 5.2.2.5.2 and 5.2.2.16.2, we get the $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}$-linear isomorphisms:

$$\begin{aligned}
&(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{J}) \\
&\xrightarrow[\widetilde{\sigma}^{(m)}_{X^\sharp,\underline{t}_{X'}} \otimes \mathrm{id}]{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{J}) \\
&\xrightarrow{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)})/(\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)})\mathcal{J} \\
&\xrightarrow{\sim} (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J}) \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)},
\end{aligned} \tag{5.2.4.3.6}$$

the last isomorphism resulting from the equality 5.2.4.2.1. We have the $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear isomorphisms:

$$\begin{aligned}
&(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{J}) \\
\xrightarrow[\mathrm{id}\otimes\overline{\varsigma}^{(m)}_{X^\sharp,\widetilde{t}_Z}]{\sim}\ & (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}) \\
\xrightarrow{\sim}\ & \mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp},
\end{aligned} \tag{5.2.4.3.7}$$

whose last isomorphism follows from the equality 5.2.4.2.2. This yields the following square:

$$\begin{array}{ccc}
(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}\mathcal{J}) & \xrightarrow[5.2.4.3.6]{\sim} & (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J}) \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \\
{\scriptstyle 5.2.4.3.7}\downarrow\wr & & {\scriptstyle \overline{\varsigma}^{(m)}_{X'^\sharp,\widetilde{t}_Z}\otimes\mathrm{id}}\downarrow\wr \\
\mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} & \xrightarrow[\mathrm{id}\otimes\overline{\sigma}^{(m)}_{Z^\sharp,\widetilde{t}_{X'}}]{\sim} & \mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}.
\end{array} \tag{5.2.4.3.8}$$

It follows by transitivity of the isomorphisms of the type ϑ (see 5.2.4.1) that the square 5.2.4.3.8 is a commutative diagram.

Let us now consider the canonical diagram:

$$\begin{array}{ccc}
(\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}) & \xrightarrow[\mathrm{id}\otimes\overline{\sigma}^{(m)}_{Z^\sharp,\widetilde{t}_{X'}}]{\sim} & (\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}\mathcal{J}) \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)} \\
{\scriptstyle \overline{\varsigma}^{(m)}_{X'^\sharp,\widetilde{t}_Z}\otimes\mathrm{id}}\downarrow\wr & & {\scriptstyle \overline{\varsigma}^{(m)}_{X'^\sharp,\widetilde{t}_Z}\otimes\mathrm{id}}\downarrow\wr \\
\mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp} & \xrightarrow[\mathrm{id}\otimes\overline{\sigma}^{(m)}_{Z^\sharp,\widetilde{t}_{X'}}]{\sim} & \mathcal{O}_T\{\partial_{r+1},\dots,\partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp} \otimes_{\mathcal{O}_T} \mathcal{O}_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}.
\end{array} \tag{5.2.4.3.9}$$

This square 5.2.4.3.9 is commutative diagram by functoriality.

Since the right arrows (resp. of the bottom) of the squares 5.2.4.3.8 and 5.2.4.3.9 are identical, since the left (resp. top) arrows of the squares 5.2.4.3.8 and 5.2.4.3.9 are $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-linear (resp. $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp}$-linear), this yields the canonical isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp},\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$-bimodules of the form 5.2.4.3.5. We easily compute that it sends $[1]_{X'}\otimes[1]'_Z$ to $[1]'_{Z'}\otimes[1]_{Z'}$. Hence the isomorphism 5.2.4.3.1 is proved.

Now let us check its commutation to the change of levels, i.e. suppose $T'^\sharp\to T^\sharp$ is the identity. By functoriality of the isomorphism 5.2.4.3.1, it is a question of checking the commutativity of the diagram

$$\begin{array}{ccc}
\widetilde{a}^*\circ\widetilde{u}^{(m)}_+(\mathcal{F}) & \longrightarrow & \widetilde{a}^*\circ\widetilde{u}^{(m')}_+(\mathcal{F}) \\
\downarrow\wr & & \downarrow\wr \\
\widetilde{u}'^{(m)}_+\circ\widetilde{b}^*(\mathcal{F}) & \longrightarrow & \widetilde{u}'^{(m')}_+\circ\widetilde{b}^*(\mathcal{F}).
\end{array} \tag{5.2.4.3.10}$$

For that, it is sufficient to notice that the isomorphisms of the squares 5.2.4.3.8 and 5.2.4.3.9 commute with the change of levels. This check comes from the commutation with level change of the isomorphisms of the form $\overline{\sigma}$ (i.e., we have the commutative square 5.2.2.21.1) and of that of the form $\overline{\vartheta}$ of 5.2.2.18. Likewise, we easily compute that it commutes with base changes, i.e. we validate the commutativity of the left square of 5.2.4.3.2. □

Lemma 5.2.4.4. *Let $\mathcal{F}$ be a flat $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp}$-module. Then we have the vanishing:*

$$\forall\, n\neq 0,\ \mathcal{H}^n\mathbb{L}\widetilde{a}^*\widetilde{u}^{(m)}_+(\mathcal{F})=0. \tag{5.2.4.4.1}$$

Proof. Since this is local, we are in the $(\star)$ local situation. We proceed by induction on $N:=d-r-s$ the cardinal of the elements $t_{r+s+1},\dots,t_d$ defining $X'^\sharp$. At first, suppose $N=1$, i.e. $X'=V(t_d)$. In this case, for $n \notin \{-1, 0\}$, $\mathcal{H}^n \mathbb{L}\widetilde{a}^* \widetilde{u}_+^{(m)}(\mathcal{F}) = 0$. Moreover, $\mathcal{H}^{-1} \mathbb{L}\widetilde{a}^* \widetilde{u}_+^{(m)}(\mathcal{F}) = \ker(\widetilde{u}_+^{(m)}(\mathcal{F}) \underset{t_d}{\longrightarrow} \widetilde{u}_+^{(m)}(\mathcal{F}))$. However, following 5.2.2.22.1, $\widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \mathcal{O}_T\{\partial_{r+1}, \dots, \partial_{r+s}\}^{(m)} \otimes_{\mathcal{O}_T} \mathcal{F}$. Since $\mathcal{F}$ is t_d torsion free, the lemma 5.2.2.20 allows us to conclude (indeed t_d commutes with the elements of $\mathcal{O}_T\{\partial_{r+1}, \dots, \partial_{r+s}\}^{(m)}$.

Suppose now $N \geq 2$ and denote by $X'' = V(t_{r+s+2}, \dots, t_d)$ and $Z'^\sharp = Z \cap X''^\sharp$, $b' \colon Z''^\sharp \hookrightarrow Z$, $\widetilde{a}' \colon X''^\sharp \hookrightarrow X^\sharp$, $\widetilde{a}'' \colon X'^\sharp \hookrightarrow X''^\sharp$, $u'' \colon Z''^\sharp \hookrightarrow X''^\sharp$ the induced closed immersions. By induction hypothesis and following 5.2.4.3.1, we get: $\mathbb{L}\widetilde{a}'^* \circ \widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \widetilde{a}'^* \circ \widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}''_+ \circ b'^*(\mathcal{F})$. Following the case $N = 1$ and following 5.2.4.3.1, we check also $\mathbb{L}\widetilde{a}''^* \circ \widetilde{u}''_+ \circ b'^*(\mathcal{F}) \xrightarrow{\sim} \widetilde{a}''^* \circ \widetilde{u}''_+ \circ b'^*(\mathcal{F}) \xrightarrow{\sim} \widetilde{a}''^* \circ \widetilde{a}'^* \circ \widetilde{u}_+^{(m)}(\mathcal{F})$. This yields the isomorphism $\mathbb{L}\widetilde{a}''^* \circ \mathbb{L}\widetilde{a}'^* \circ \widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \widetilde{a}''^* \circ \widetilde{a}'^* \circ \widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \widetilde{a}^* \circ \widetilde{u}_+^{(m)}(\mathcal{F})$. The isomorphism $\mathbb{L}\widetilde{a}^* \xrightarrow{\sim} \mathbb{L}\widetilde{a}''^* \circ \mathbb{L}\widetilde{a}'^*$ allows us to conclude. □

Proposition 5.2.4.5. *We have for any $\mathcal{F} \in D^-(\widetilde{\mathcal{D}}_{Z^\sharp}^{(m)})$ the isomorphism*

$$a^! \circ \widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}'_+ \circ b^!(\mathcal{F}). \tag{5.2.4.5.1}$$

Proof. Let $\mathcal{P}$ be a resolution of $\mathcal{F}$ by flat $\widetilde{\mathcal{D}}_{Z^\sharp}^{(m)}$-modules. Since the functors $\widetilde{u}_+^{(m)}$ and $\widetilde{u}'_+$ are exact, by 5.2.4.4, we get $\mathbb{L}\widetilde{a}^* \widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \widetilde{a}^* \widetilde{u}_+^{(m)}(\mathcal{P})$ and $\widetilde{u}'_+ \circ \mathbb{L}\widetilde{b}^*(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}'_+ \circ \widetilde{b}^*(\mathcal{P})$. The lemma 5.2.4.3 allows us to conclude. □

5.2.5 The fundamental isomorphism for schemes

In this subsection, by the local situation 5.2.1.1, we mean the local situation of 5.2.2 in the case where $s = 0$.

5.2.5.1. We can clarify 4.4.2.13 as follows in the context of an exact closed immersion. We have the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\Delta}^n_{X^\sharp/T^\sharp,(m)}(2) & \overset{p_{12}}{\underset{p_{01}}{\overset{p_{02}}{\rightrightarrows}}} & \widetilde{\Delta}^n_{X^\sharp/T^\sharp,(m)} & \overset{p_1}{\underset{p_0}{\rightrightarrows}} & \widetilde{X} \\
{\scriptstyle \widetilde{\Delta}^n(u)(2)} \big\uparrow & & {\scriptstyle \widetilde{\Delta}^n(u)} \big\uparrow & & {\scriptstyle \widetilde{u}} \big\uparrow \\
\widetilde{\Delta}^n_{X^\sharp/T^\sharp,(m)}(2) & \overset{p_{12}}{\underset{p_{01}}{\overset{p_{02}}{\rightrightarrows}}} & \widetilde{\Delta}^n_{Z^\sharp/T^\sharp,(m)} & \overset{p_1}{\underset{p_0}{\rightrightarrows}} & \widetilde{Z}
\end{array} \tag{5.2.5.1.1}$$

where p_0, p_1 correspond to the homomorphisms $\mathcal{B}_X \to \widetilde{\mathcal{P}}^n_{X(m)}$ given by respectively the left and right structures of $\mathcal{B}_X$-algebra on $\widetilde{\mathcal{P}}^n_{X(m)}$. Using the local description of the homomorphism $\widetilde{u}^* \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$ given in 5.2.2, we check that $\widetilde{u}^* \widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$.

We denote $\overline{u} \colon (Z, \mathcal{B}_Z) \to (X, u_* \mathcal{B}_Z)$ the morphism of ringed spaces induced by u. We remark that $\overline{u}$ is flat and that $\overline{u}^* = u^{-1} \colon D^+(u_* \mathcal{B}_Z) \to D^+(\mathcal{B}_Z)$. Recall that for any $\mathcal{M} \in D^+(\mathcal{B}_X)$, by definition $\widetilde{u}^\flat(\mathcal{M}) := u^{-1} \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(u_* \mathcal{B}_Z, \mathcal{M})$ (see [Har66, III.6]).

If $\mathcal{M}$ is a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module, we denote by

$$\widetilde{u}^{\flat 0}(\mathcal{M}) := u^{-1} \mathcal{H}om_{\mathcal{B}_X}(u_* \mathcal{B}_Z, \mathcal{M}).$$

To simplify notation, we will write sometimes $\widetilde{u}^{\flat 0}(\mathcal{M}) := \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z, \mathcal{M})$. By definition, the right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module structure of $\widetilde{u}^{\flat 0}(\mathcal{M})$ is given by the following m-PD-costratification $\epsilon_n^{\widetilde{u}^{\flat 0}(\mathcal{M})}$ making commutative the diagram:

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{B}_Z}(p^n_{0*}\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M})) & \xrightarrow[\epsilon_n^{\widetilde{u}^{\flat 0}(\mathcal{M})}]{\sim} & \mathcal{H}om_{\mathcal{B}_Z}(p^n_{1*}\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M})) \\
\downarrow\wr & & \downarrow\wr \\
\mathcal{H}om_{\mathcal{B}_X}(p^n_{0*}\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{M}) & & \mathcal{H}om_{\mathcal{B}_X}(p^n_{1*}\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{M}) \\
\uparrow\wr & & \uparrow\wr \\
\mathcal{H}om_{\widetilde{\mathcal{P}}^n_{X^\sharp,(m)}}(\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(p^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)},\mathcal{M})) & \xrightarrow[\epsilon_n^{\mathcal{M}}]{\sim} & \mathcal{H}om_{\widetilde{\mathcal{P}}^n_{X^\sharp,(m)}}(\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(p^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)},\mathcal{M})) \\
\big\downarrow \mathrm{ev}_1 & & \big\downarrow \mathrm{ev}_1 \\
\mathcal{H}om_{\mathcal{B}_X}(p^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)}, \mathcal{M}) & \xrightarrow[\epsilon_n^{\mathcal{M}}]{\sim} & \mathcal{H}om_{\mathcal{B}_X}(p^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)}, \mathcal{M}).
\end{array} \tag{5.2.5.1.2}$$

Both left (resp. right) top vertical isomorphisms correspond the Cartan isomorphisms $p_0^{\flat 0}\circ\widetilde{u}^{\flat 0} \xrightarrow{\sim} (u\circ p_0)^{\flat 0} = (p_0\circ\widetilde{\Delta}^n(u))^{\flat 0} \xleftarrow{\sim} (\widetilde{\Delta}^n(u))^{\flat 0}\circ p_0^{\flat 0}$ (resp. $p_1^{\flat 0}\circ\widetilde{u}^{\flat 0} \xrightarrow{\sim} (u\circ p_1)^{\flat 0} = (p_1\circ\widetilde{\Delta}^n(u))^{\flat 0} \xleftarrow{\sim} (\widetilde{\Delta}^n(u))^{\flat 0}\circ p_1^{\flat 0}$). The bottom vertical arrows are monomorphisms because $u^{-1}\widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}$ is surjective. We check the cocycle conditions via similar isomorphisms.

Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ is a flat $\mathcal{B}_X$-module, then an injective right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module is an injective $\mathcal{B}_X$-module. Hence, taking an injective resolution of a complex of $D^+({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$, we check the functor $\widetilde{u}^\flat$ sends $D^+({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$ to $D^+({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$, i.e. it induces

$$\widetilde{u}^\flat\colon D^+({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}) \to D^+({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}). \tag{5.2.5.1.3}$$

When the level m is ambiguous, we denote it more specifically by $\widetilde{u}^{\flat(m)}$.

Since $X^\sharp$ is locally noetherian, then $\widetilde{u}^\flat$ preserves the quasi-coherence and sends $D^+_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$ to $D^+_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$.

Lemma 5.2.5.2. *Suppose we are in the local situation 5.2.1.1. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module. Let $x\in\Gamma(Z,\widetilde{u}^{\flat 0}(\mathcal{M}))$ and $Q\in\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$. Choose any $Q_X\in\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ such that $\vartheta(Q)=[Q_X]_Z$ (see 5.2.2.5.7). We have the formula*

$$\mathrm{ev}_1(x\cdot Q)=\mathrm{ev}_1(x)\cdot Q_X, \tag{5.2.5.2.1}$$

where $\mathrm{ev}_1\colon\Gamma(Z,\widetilde{u}^{\flat 0}(\mathcal{M}))\hookrightarrow\Gamma(X,\mathcal{M})$ *is the evaluation at* 1 *homomorphism.*

Proof. Let $\underline{l}\in\mathbb{N}^r$ and $x\in\Gamma(Z,\widetilde{u}^{\flat 0}(\mathcal{M}))$. Let us consider the case where $Q=\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}}$ and $Q_X=\underline{\partial}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{H}om_{\mathcal{B}_Z}(p^n_{0*}\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M})) & \xrightarrow[\epsilon_n^{\widetilde{u}^{\flat 0}(\mathcal{M})}]{\sim} & \mathcal{H}om_{\mathcal{B}_Z}(p^n_{1*}\widetilde{\mathcal{P}}^n_{Z^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M})) & \xhookrightarrow{\mathrm{ev}_1} & \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M}) \\
\downarrow & & \downarrow & & \big\downarrow \mathrm{ev}_1 \\
\mathcal{H}om_{\mathcal{B}_X}(p^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)},\mathcal{M}) & \xrightarrow[\epsilon_n^{\mathcal{M}}]{\sim} & \mathcal{H}om_{\mathcal{B}_X}(p^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)},\mathcal{M}) & \xhookrightarrow{\mathrm{ev}_1} & \mathcal{M},
\end{array} \tag{5.2.5.2.2}$$

where the right square is constructed at 5.2.5.1.2. Modulo the isomorphisms 4.2.2.4.1, we identify $x\otimes\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}}$ with a global section of $\mathcal{H}om_{\mathcal{B}_Z}(p^n_{0*}\widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M}))$ and $\mathrm{ev}_1(x)\otimes\underline{\partial}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$ with a global section of $\mathcal{H}om_{\mathcal{B}_X}(p^n_{0*}\widetilde{\mathcal{P}}^n_{X^\sharp,(m)},\mathcal{M})$. We compute the left vertical arrows of 5.2.5.2.2 sends $x\otimes\underline{\partial}_{(r)}^{\langle\underline{l}\rangle_{(m)}}$ to $\mathrm{ev}_1(x)\otimes\underline{\partial}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$. Using the formula 4.2.2.6.2, the image of $\mathrm{ev}_1(x)\otimes\underline{\partial}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$ via the composition of $\epsilon_n^{\mathcal{M}}$ with the evaluation at 1 homomorphism $\mathcal{H}om_{\mathcal{B}_X}(p^n_{1*}\widetilde{\mathcal{P}}^n_{X^\sharp/T^\sharp,(m)},\mathcal{M})\to\mathcal{M}$ is equal to $\mathrm{ev}_1(x)\cdot$

$\underline{\partial}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}} \in \Gamma(X,\mathcal{M})$. Using agaof the formula 4.2.2.6.2, we compute the image of $x\otimes\underline{\partial}_{(r)}^{\langle \underline{l}\rangle_{(m)}}$ via the composition of $\epsilon_n^{\widetilde{u}^{\flat 0}\mathcal{M}}$ with the evaluation at 1 homomorphism $\mathcal{H}om_{\mathcal{B}_Z}(p_{1*}^n\widetilde{\mathcal{P}}^n_{Z^\sharp/T^\sharp,(m)}, \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M})) \to \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_Z,\mathcal{M})$ is $x\cdot\underline{\partial}_{(r)}^{\langle \underline{l}\rangle_{(m)}}$. Hence, by using the commutativity of the diagram 5.2.5.1.2 we get

$$\mathrm{ev}_1(x\cdot\underline{\partial}_{(r)}^{\langle \underline{l}\rangle_{(m)}}) = \mathrm{ev}_1(x)\cdot\underline{\partial}_{(r)}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}. \tag{5.2.5.2.3}$$

Finally, we check easily the formula 5.2.5.2.1 from 5.2.5.2.3. □

5.2.5.3 (Local description of the right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module structure of $\widetilde{u}^{\flat 0}(\mathcal{M})$)**.** Suppose we are in the local situation of 5.2.1.1. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module. Since $\vartheta\colon u_*\mathcal{B}_Z\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$, we have the isomorphism

$$\widetilde{u}^{\flat 0}(\mathcal{M}) \xrightarrow{\sim} u^{-1}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp},\mathcal{M}). \tag{5.2.5.3.1}$$

Following 5.2.2.7, we have the isomorphism of $\mathcal{O}_T$-algebras $\vartheta\colon u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ (see notation 5.2.2.5.7). Hence, we get from 5.2.5.3.1 the isomorphism

$$\widetilde{u}^{\flat 0}(\mathcal{M}) \xrightarrow{\sim} u^{-1}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp},\mathcal{M}) \tag{5.2.5.3.2}$$

By using Lemma 5.2.5.2, we compute that this isomorphism 5.2.5.3.2 is an isomorphism of right $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp}$-modules, and 5.2.5.3.1 is therefore an isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules. If there is no ambiguity, we can avoid writing u^{-1} pr u_*, e.g. we can simply write $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp},\mathcal{M}) = u^{-1}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp},\mathcal{M})$.

5.2.5.4. Suppose we are in the local situation of 5.2.2. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module.

(a) By derivating 9.3.1.6.1, we get the isomorphism of $D^{\mathrm{b}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp})$ of the form

$$\widetilde{u}^{\flat}(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp},\mathcal{M}). \tag{5.2.5.4.1}$$

Let $s := d-r$, and $f_1 = t_{r+1},\dots,f_T := t_d$. Let $K_\bullet(\underline{f})$ be the Koszul complex of $\mathcal{O}_X$-free modules given by the sequence of global section $\underline{f} = (f_1,\dots,f_s)$ of $\mathcal{O}_X$. Let $e_1,\dots,e_s$ be the canonical basis of $\mathcal{O}_X^s$. Recall $K_i(\underline{f}) = \wedge^i(\mathcal{O}_X^s)$ and $d_{i,\underline{f}}\colon K_i(\underline{f}) \to K_{i-1}(\underline{f})$ (or simply d_i) is the $\mathcal{O}_X$-linear map defined by

$$d_i(e_{n_1}\wedge\cdots\wedge e_{n_i}) = \sum_{j=1}^{i}(-1)^{j-1}f_{n_j}e_{n_1}\wedge\cdots\wedge\widehat{e}_{n_j}\wedge\cdots\wedge e_{n_i}.$$

We set $\widetilde{K}_\bullet(\underline{f}) := \mathcal{B}_X\otimes_{\mathcal{O}_X}K_\bullet(\underline{f})$.

The canonical projection $\mathcal{O}_X\to\mathcal{O}_X/\mathcal{I}$ induces the quasi-isomorphism $K_\bullet(\underline{f})\to u_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. Since $f_1,\dots,f_s$ is a regular sequence of $\mathcal{I}$, since $u_*\mathcal{O}_Z$ and $\mathcal{B}_X$ are tor independent over $\mathcal{O}_Z$, then the canonical morphism $\widetilde{K}_\bullet(\underline{f})\to\mathcal{B}_X/\mathcal{I}\mathcal{B}_X$ (given by the canonical map $\widetilde{K}_0(\underline{f}) = \mathcal{B}_X\to\mathcal{B}_X/\mathcal{I}\mathcal{B}_X$) is a quasi-isomorphism. Hence, we get the isomorphism of $D^{\mathrm{b}}(\mathcal{B}_X)$

$$\phi_{\underline{f}}\colon\widetilde{u}^{\flat}(\mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\widetilde{K}_\bullet(\underline{f}),\mathcal{M}).$$

Since $f_1,\dots f_s$ are in the center of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ and since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ is a flat $\mathcal{B}_X$-algebra, then the quasi-isomorphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ in the category of complexes of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-bimodules. We get the commutativity of diagram

$$\begin{array}{ccc}
\widetilde{u}^\flat(\mathcal{M}) & \xrightarrow[\sim]{\phi_{\underline{f}}} & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{K}_\bullet(\underline{f}),\mathcal{M}) \\
{\scriptstyle 5.2.5.4.1}\downarrow\wr & \overset{\phi_{\underline{t}}}{\searrow} & \downarrow\wr \\
\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp},\mathcal{M}) & \xleftarrow[\sim]{} & \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}),\mathcal{M}),
\end{array}$$

(5.2.5.4.2)

where $\phi_{\underline{t}}$ is the homomorphism making commutative the upper triangle. By commutativity of 5.2.5.4.2, $\phi_{\underline{t}}$ is an isomorphism of $D^{\mathrm{b}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp})$. Hence, we get the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules

$$\phi^s_{\underline{t}} := \mathcal{H}^s(\phi_{\underline{t}})\colon R^s\widetilde{u}^{\flat 0}(\mathcal{M}) \xrightarrow{\sim} \mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}),\mathcal{M}). \qquad (5.2.5.4.3)$$

We have the homomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_s(\underline{f}),\mathcal{M}) \to \mathcal{M}$ (the structure of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-module on $\mathcal{M}$ comes from its structure of $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$ via ϑ) given by $\phi\mapsto\phi(e_1\wedge\cdots\wedge e_s)$. Since $\mathcal{M}\to\mathcal{M}/\mathcal{I}\mathcal{M}$ is a morphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules, then this induces the morphism of complex of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules of the form $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}),\mathcal{M})\to\mathcal{M}/\mathcal{I}\mathcal{M}$. This yields the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules

$$\mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}),\mathcal{M}) \xrightarrow{\sim} \mathcal{M}/\mathcal{I}\mathcal{M}. \qquad (5.2.5.4.4)$$

(b) *Varying the coordinates.* Make a second choice: Suppose there exist a second cartesian diagram of morphisms of $T^\sharp$-log-(formal) schemes of the form:

$$\begin{array}{ccc}
X^\sharp & \xrightarrow{\alpha'} & A^{d,r}_{T^\sharp} \\
{\scriptstyle u}\uparrow & \square & \uparrow \\
Z^\sharp & \longrightarrow & A^{r,r}_{T^\sharp}
\end{array}$$

such that the horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion. Let $t'_1,\dots,t'_r\in M_{X^\sharp}$ and $t'_{r+1},\dots,t'_d\in\Gamma(X,\mathcal{I})$ be the element given by α'. Let $A:=\Gamma(X,\mathcal{B}_X)$ and $B:=\Gamma(Z,\mathcal{B}_Z)$. Set $f'_1=t'_{r+1},\dots,f'_s:=t'_d$. Let $\widetilde{K}_\bullet(\underline{f}')$ be the Koszul complex of $\underline{f}'=(f'_1,\dots,f'_s)$. Let $M_I=(c_{ij})\in M_T(A)$ be the matrix such that $\sum_{i=1}^s f'_ic_{ij}=f_j$. Let $\varphi\colon A^s\to A^s$ be the morphism associated with M_I. It corresponds to a morphism $\varphi\colon\widetilde{K}_1(\underline{f})\to\widetilde{K}_1(\underline{f}')$. We compute that the composition of φ with $d_{1,\underline{f}}\colon\widetilde{K}_1(\underline{f}')\to\widetilde{K}_0(\underline{f}')=A$ is equal to $d_{1,\underline{f}'}\colon\widetilde{K}_1(\underline{f})\to\widetilde{K}_0(\underline{f})=A$. Since $\widetilde{K}_\bullet(\underline{f})=\wedge\widetilde{K}_1(\underline{f})$ this yields the morphism of complexes $\wedge\varphi\colon\widetilde{K}_\bullet(\underline{f})\to\widetilde{K}_\bullet(\underline{f}')$. Hence, we get the commutative diagram

$$\begin{array}{ccc}
\widetilde{u}^\flat(\mathcal{M}) & \xrightarrow[\sim]{\phi_{\underline{f}}} & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{K}_\bullet(\underline{f}),\mathcal{M}) \\
& {\scriptstyle\phi_{\underline{f}'}}\searrow\sim & \uparrow{\scriptstyle\wedge\varphi} \\
& & \mathcal{H}om_{\mathcal{B}_X}(\widetilde{K}_\bullet(\underline{f}'),\mathcal{M}).
\end{array}$$

This yields the commutativity of the top square of the diagram

$$\begin{array}{ccc}
R^s\widetilde{u}^{\flat 0}(\mathcal{M}) & = = = = = & R^s\widetilde{u}^{\flat 0}(\mathcal{M}) \\
\downarrow{\scriptstyle H^s\phi_{\underline{f}}}\ \wr & & \wr\ \downarrow{\scriptstyle H^s\phi_{\underline{f}'}} \\
\mathcal{H}^s\mathcal{H}om_{\mathcal{B}_X}(\widetilde{K}_\bullet(\underline{f}),\mathcal{M}) & \xleftarrow{\det\varphi} & \mathcal{H}^s\mathcal{H}om_{\mathcal{B}_X}(\widetilde{K}_\bullet(\underline{f}'),\mathcal{M}) \\
\downarrow\wr & & \downarrow\wr \\
\mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}),\mathcal{M}) & & \mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}'/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}'/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}'),\mathcal{M}) \\
\wr\downarrow{\scriptstyle 5.2.5.4.4} & & \wr\downarrow{\scriptstyle 5.2.5.4.4} \\
\mathcal{M}/\mathcal{I}\mathcal{M} & \xleftarrow{\overline{\det\varphi}} & \mathcal{M}/\mathcal{I}\mathcal{M},
\end{array}$$

(5.2.5.4.5)

whose compositions of vertical morphisms are $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-linear. Hence, the diagram 5.2.5.4.5 is commutative.

Notation 5.2.5.5. Following 4.6.6.9.2, for any $\mathcal{E}\in D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$, the canonical morphism

$$\mathbb{L}\widetilde{u}^*(\mathcal{E})\to\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}\otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}}u^{-1}\mathcal{E} \tag{5.2.5.5.1}$$

is an isomorphism, which yields $\mathbb{L}\widetilde{u}^*(\mathcal{E})[d_{Z/X}]\xrightarrow{\sim}\widetilde{u}^!(\mathcal{E})$.

Suppose now we are in the local situation of 5.2.1.1. Let $Q\in\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$. Choose $Q_X\in\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ such that $[Q_X]_Z=\vartheta(Q)$. From 5.2.2.6.1, for any section x of $\mathcal{E}$, we have the formula in $\widetilde{u}^*(\mathcal{E})$:

$$Q(\widetilde{u}^*(x))=\widetilde{u}^*(Q_X\cdot x). \tag{5.2.5.5.2}$$

Via the monomorphism of rings $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}}\hookrightarrow\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$, we check the canonical homorphism

$$\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp}\otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}}}u^{-1}\mathcal{E}\to\mathbb{L}\widetilde{u}^*(\mathcal{E})$$

is an isomorphism of $D({}^l\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp})$. This yields the isomorphism of $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}})$:

$$(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}))\otimes_{u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}}}u^{-1}\mathcal{E}\xrightarrow{\sim}\mathbb{L}\widetilde{u}^*(\mathcal{E}).$$

Proposition 5.2.5.6. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module (resp. a $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-bimodule). We have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules (resp. of right $(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp},u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$-bimodules):*

$$R^{-d_{Z/X}}\widetilde{u}^{\flat 0}(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\xrightarrow{\sim}\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}\widetilde{u}^*(\mathcal{E}). \tag{5.2.5.6.1}$$

Proof. By functoriality, we reduce to check the non-respective case. Set $r:=-d_{Z/X}\in\mathbb{N}$.

0) First, let us suppose the conditions of 5.2.1.1 are satisfied, i.e. suppose $X^\sharp$ is affine and there exist some integers $d\geq r$ and a cartesian diagram of morphisms of $T^\sharp$-log-(formal) schemes of the form:

$$\begin{array}{ccc}
X^\sharp & \xrightarrow{\alpha} & A^{d,r}_{T^\sharp} \\
{\scriptstyle u}\uparrow & \square & \uparrow \\
Z^\sharp & \longrightarrow & A^{r,r}_{T^\sharp}
\end{array}$$

such that the horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion. Let $t_1,\dots,t_r\in M_{X^\sharp}$ and $t_{r+1},\dots,t_d\in\Gamma(X,\mathcal{I})$ be the element corresponding via α to the coordinates $x_1,\dots,x_d$ of $A^{d,r}_{T^\sharp}/T^\sharp$. In that case $\widetilde{\omega}_{Z^\sharp/T^\sharp}$ is a free $\mathcal{B}_Z$-module with the basis $d\log\bar{t}_1\wedge\cdots\wedge d\log\bar{t}_r$, and $\widetilde{\omega}_{X^\sharp/T^\sharp}$ is a free $\mathcal{B}_X$-module with the basis $d\log t_1\wedge\cdots\wedge d\log t_r\wedge dt_{r+1}\wedge\cdots\wedge dt_d$.

1) Since the isomorphism $\vartheta\colon \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}\xrightarrow{\sim}\widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ commutes with the adjoint operator automorphisms (see 5.2.2.12.1), by using 4.3.5.5 and 4.5.1.8, we get the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$-modules

$$\psi_{\underline{t}}\colon(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})/(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\mathcal{I}\xrightarrow{\sim}\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}(\mathcal{E}/\mathcal{I}\mathcal{E}),$$

which is given by $\widetilde{e}_0\otimes x\mod\mathcal{I}\mapsto\widetilde{f}_0\otimes(x\mod\mathcal{I})$, where $\widetilde{e}_0:=d\log t_1\wedge\cdots\wedge d\log t_r\wedge dt_{r+1}\wedge\cdots\wedge dt_d$ and $\widetilde{f}_0:=d\log\bar{t}_1\wedge\cdots\wedge d\log\bar{t}_r$.

2) Let $s:=d-r$, and $f_1=t_{r+1},\dots,f_T:=t_d$. Using the isomorphism $\phi^s_{\underline{t}}$ constructed in 5.2.5.4.3, we get by composition the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp}$-modules

$$\begin{aligned}R^s\widetilde{u}^{\flat0}(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})&\underset{\phi^s_{\underline{t}}}{\xrightarrow{\sim}}\mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{K}_\bullet(\underline{f}),\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\\&\underset{5.2.5.4.4}{\xrightarrow{\sim}}(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})/(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\mathcal{I}\underset{\psi_{\underline{t}}}{\xrightarrow{\sim}}\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}(\mathcal{E}/\mathcal{I}\mathcal{E}).\end{aligned}\tag{5.2.5.6.2}$$

3) It remains to check that the composition of the isomorphisms of 5.2.5.6.2 does not depend on the choice of the coordinates. Make some second choice: Suppose there exist a second cartesian diagram of morphisms of $T^\sharp$-log-(formal) schemes of the form:

$$\begin{array}{ccc}X^\sharp&\xrightarrow{\alpha'}&A^{d,r}_{T^\sharp}\\ {\scriptstyle u}\uparrow&\square&\uparrow\\ Z^\sharp&\longrightarrow&A^{r,r}_{T^\sharp}\end{array}$$

such that the horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion. Let $t'_1,\dots,t'_r\in M_{X^\sharp}$ and $t'_{r+1},\dots,t'_d\in\Gamma(X,\mathcal{I})$ be the element given by α'. Let $A:=\Gamma(X,\mathcal{B}_X)$, $B:=\Gamma(Z,\mathcal{B}_Z)$, $R:=\Gamma(T,\mathcal{B}_T)$. Set $f'_1=t'_{r+1},\dots,f'_s:=t'_d$. Let $(c_{ij})\in M_T(A)$ be the matrix such that $\sum_{i=1}^s f'_ic_{ij}=f_j$. We denote by $\delta_i=d\log t_i$ if $i=1,\dots,r$ and $\delta_i=d\,t_i$ if $i=r+1,\dots,d$ (resp. $\delta'_i=d\log t'_i$ if $i=1,\dots,r$ and $\delta'_i=d\,t'_i$ if $i=r+1,\dots,d$). Let $\varphi\colon A^s\to A^s$ be the morphism associated with (c_{ij}). Let $(d_{ij})\in M_d(A)$ be the matrix such that $\delta'_i=\sum_{j=1}^d d_{ij}\delta_j$. Let $D=(\bar{d}_{ij})_{1\le i,j\le d}\in M_d(B)$, $D_1=(\bar{d}_{ij})_{1\le i,j\le r}\in M_r(B)$, and $D_2=(\bar{d}_{ij})_{r+1\le i,j\le d}\in M_T(B)$. We denote by $\overline{\delta_i}$ and $\overline{\delta'_i}$ the image of δ_i and δ'_i in $\widetilde{u}^*\widetilde{\omega}_{X^\sharp/T^\sharp}$. We get $\overline{\delta'_1}\wedge\cdots\wedge\overline{\delta'_d}=(\det D)\left(\overline{\delta_1}\wedge\cdots\wedge\overline{\delta_d}\right)$.

When $i\ge r+1$, we write $d\log\overline{t_i}=0$ and $d\log\overline{t'_i}=0$. By considering the images via the canonical morphism $\widetilde{u}^*\widetilde{\omega}_{X^\sharp/T^\sharp}\to\widetilde{\omega}_{Z^\sharp/T^\sharp}$ which sends $\overline{\delta_i}$ (resp. $\overline{\delta'_i}$) to $d\log\overline{t_i}$ (resp. $d\log\overline{t'_i}$), we get the equality $d\log\overline{t'_i}=\sum_{j=1}^d\bar{d}_{ij}d\log\overline{t_j}$ for any $i=1,\dots,d$. Since $d\log\overline{t_1},\dots,d\log\overline{t_r}$ is a basis of $\widetilde{\omega}_{Z^\sharp/T^\sharp}$, then $\bar{d}_{ij}=0$ for any $i\ge r+1$ and $j\le r$ and then $d\log\overline{t'_i}=\sum_{j=1}^r\bar{d}_{ij}d\log\overline{t_j}$ for any $i=1,\dots,r$. This yields both equalities $\det D=\det D_1\det D_2$ and $d\log\overline{t'_1}\wedge\cdots\wedge d\log\overline{t'_r}=(\det D_1)d\log\overline{t_1}\wedge\cdots\wedge d\log\overline{t_r}$.

For any $i\ge r+1$, we have $\overline{dt_i}=\overline{f_{i-r}}$ and $\overline{dt'_i}=\overline{f'_{i-r}}$ in $\widetilde{\mathcal{I}}/\widetilde{\mathcal{I}}^2$. Since for any $i\ge r+1$, we have $\overline{\delta'_i}=\sum_{j=r+1}^d\overline{d_{ij}}\,\overline{\delta_j}$, this means that D_2 is the inverse of the transposition matrix of $(\bar{c}_{ij})$. Hence, $\overline{\det\varphi}=(\det D_2)^{-1}$. This implies the commutativity of the following diagram:

$$\begin{array}{ccc}(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})/(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\mathcal{I}&\underset{\sim}{\xrightarrow{\psi_{\underline{t}}}}&\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}(\mathcal{E}/\mathcal{I}\mathcal{E})\\ {\scriptstyle\overline{\det\varphi}}\uparrow&&\|\\ (\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})/(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E})\mathcal{I}&\underset{\sim}{\xrightarrow{\psi_{\underline{t}'}}}&\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}(\mathcal{E}/\mathcal{I}\mathcal{E}).\end{array}\tag{5.2.5.6.3}$$

By composing the commutative diagram 5.2.5.4.5 for $\mathcal{M}=\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\mathcal{E}$ with 5.2.5.6.3, we get the independence on the choice of the coordinates of the composition of the isomorphisms of 5.2.5.6.2. $\square$

Theorem 5.2.5.7. *Let $\mathcal{E} \in D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$ (resp. $\mathcal{E} \in D({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}, {}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$). We have the canonical isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$ (resp. $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, {}^{\mathrm{r}}u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$)*

$$\widetilde{\omega}_{Z^\sharp/T^\sharp} \otimes_{\mathcal{B}_Z} \widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^\flat(\widetilde{\omega}_{X^\sharp/T^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}). \tag{5.2.5.7.1}$$

The functors $\widetilde{u}^\flat$ and $\widetilde{u}^!$ are way-out in both direction.

Proof. We already know that there exists an isomorphism of the form 5.2.5.7.1 in the category $D(\mathcal{O}_Z)$ (see [Har66, III.7.3]), i.e. after applying the forgetful functor $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}) \to D(\mathcal{O}_Z)$ we have the canonical isomorphism

$$\omega_{Z^\sharp/T^\sharp} \otimes_{\mathcal{O}_Z} u^!(\mathcal{E}) \xrightarrow{\sim} u^\flat(\omega_{X^\sharp/T^\sharp} \otimes_{\mathcal{O}_X} \mathcal{E}) \tag{5.2.5.7.2}$$

whose left (resp. right) terms are canonically isomorphic to the left (resp. right) terms of 5.2.5.7.1.This yields that if $\mathcal{E}$ is a flat left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module, then for any $i \neq s$, $\mathcal{H}^i\widetilde{u}^\flat(\widetilde{\omega}_{X^\sharp/T^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}) = 0$. Using [Har66, I.7.4], we conclude using 5.2.5.6. □

Corollary 5.2.5.8. *Let $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$. We have the canonical isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$*

$$\widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^\flat(\mathcal{E}). \tag{5.2.5.8.1}$$

Proof. This is a consequence of the left to right isomorphism 5.1.1.5.1 and of Theorem 5.2.5.7. □

Corollary 5.2.5.9. *We have the canonical isomorphism of right $(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}, u^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$-bimodules of the form*

$$\widetilde{\omega}_{Z^\sharp/T^\sharp} \otimes_{\mathcal{B}_Z} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp}[d_{Z/X}] \xrightarrow{\sim} \widetilde{u}^\flat(\widetilde{\omega}_{X^\sharp/T^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}). \tag{5.2.5.9.1}$$

Proof. We apply Theorem 5.2.5.7 in the case $\mathcal{E} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. □

5.2.5.10. Suppose we are in the local situation of 5.2.1.1. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module. Following 5.2.5.7.1 we have the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules:

$$\widetilde{\omega}_{Z^\sharp/T^\sharp} \otimes_{\mathcal{B}_Z} H^0\widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^{\flat 0}(\widetilde{\omega}_{X^\sharp/T^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}). \tag{5.2.5.10.1}$$

The sheaf $\widetilde{\omega}_{X^\sharp/T^\sharp}$ is a free $\mathcal{B}_X$-module of rank one with the basis $d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge t_d$ and $\widetilde{\omega}_{Y/T^\sharp}$ is a free $\mathcal{B}_Y$-module of rank one with the basis $dt_1 \wedge \cdots \wedge dt_d$ (see 4.5.1.6). Using these bases, we get the isomorphism of $\mathcal{B}_Z$-modules:

$$u_*H^0\widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_X/\mathcal{I}\mathcal{B}_X, \mathcal{E}) = \cap_{s=r+1}^d \ker(\mathcal{E} \xrightarrow{t_s} \mathcal{E}). \tag{5.2.5.10.2}$$

This yields on $\mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_X/\mathcal{I}\mathcal{B}_X, \mathcal{E})$ a structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}$-module extending its structure of $\mathcal{B}_X$ (beware this structure depends a priori on the choice of the semi-logarithmic coordinates). Let us denote by $\mathrm{ev}_1 \colon \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_X/\mathcal{I}\mathcal{B}_X, \mathcal{E}) \hookrightarrow \mathcal{E}$ the canonical inclusion (this is the evaluation at 1) and by $\mathrm{ev}_1 \colon H^0\widetilde{u}^!(\mathcal{E}) \hookrightarrow \mathcal{E}$ its composition with 5.2.5.10.2. Let $x \in \Gamma(Z, H^0u^!(\mathcal{E}))$, $y \in \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_X/\mathcal{I}\mathcal{B}_X, \mathcal{E})$ and $Q \in \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$. Choose any $Q_X \in \widetilde{D}^{(m)}_{X^\sharp, Z^\sharp, \underline{t}/T^\sharp}$ such that $\vartheta(Q) = [Q_X]_Z$ (see 5.2.2.5.7). It follows from 5.2.5.2.1 that we have the formula

$$\mathrm{ev}_1(Q \cdot x) = Q_X \cdot \mathrm{ev}_1(x), \qquad \mathrm{ev}_1(Q \cdot y) = Q_X \cdot \mathrm{ev}_1(y). \tag{5.2.5.10.3}$$

Let us finish the subsection with the following result.

5.2.5.11 (Commutation of pushforwards with base and level change)**.** Suppose we are in the non-respective case of log-schemes of 5.2. Take an integer $m' \geq m$. Let $T'^\sharp \to T^\sharp$ be a morphism of noetherian affine schemes. Put $X'^\sharp := X^\sharp \times_{T^\sharp} T'^\sharp$, $Z'^\sharp := Z^\sharp \times_{T^\sharp} T'^\sharp$, and let $u' \colon Z'^\sharp \hookrightarrow X'^\sharp$ be the induced closed immersion, $\mathcal{I}'$ be the underlying ideal of $\mathcal{O}_{X'}$ and $f \colon X'^\sharp \to X^\sharp$, $g \colon Z'^\sharp \to Z^\sharp$ be the canonical projection. We set $\mathcal{B}_{X'} := f^*(\mathcal{B}_X)$, $\mathcal{B}_{Z'} := g^*(\mathcal{B}_Z)$. We keep similar notation than 4.1.2, in particular we set $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/T^\sharp} := \mathcal{B}_{X'} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/T^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp} := \mathcal{B}_{Z'} \otimes_{\mathcal{O}_{Z'}} \widetilde{\mathcal{D}}^{(m)}_{Z'^\sharp/T^\sharp}$. We denote by $\widetilde{X}'^\sharp$ (resp. $\widetilde{Z}'^\sharp$) the ringed logarithmic scheme $(X'^{\sharp}, \mathcal{B}_{X'})$ (resp. $(Z'^{\sharp}, \mathcal{B}_{Z'})$), and by $\tilde{u}' \colon \widetilde{Z}'^{\sharp}/T'^{\sharp} \to \widetilde{X}'^{\sharp}/T'^{\sharp}$ the morphism of relative ringed logarithmic schemes induced by the diagram 5.1.1.1.1 and by $u'^{*}\mathcal{B}_{X'} \xrightarrow{\sim} \mathcal{B}_{Z'}$

For any right $\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}$-module $\mathcal{M}$, we have the isomorphism:

$$\begin{aligned} f^{-1}\left(u_{*}\mathcal{M} \otimes_{u_{*}\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}} \widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}\right) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}} \widetilde{\mathcal{D}}^{(m')}_{X'^{\sharp}/T'^{\sharp}} \\ \xrightarrow{\sim} f^{-1}u_{*}\mathcal{M} \otimes_{f^{-1}u_{*}\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}} \widetilde{\mathcal{D}}^{(m')}_{X'^{\sharp}/T'^{\sharp}}/\mathcal{I}'\widetilde{\mathcal{D}}^{(m')}_{X'^{\sharp}/T'^{\sharp}} \\ \xrightarrow{\sim} u'_{*}\left(g^{-1}\mathcal{M} \otimes_{g^{-1}\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}} \widetilde{\mathcal{D}}^{(m')}_{Z'^{\sharp}/T'^{\sharp}}\right) \otimes_{u_{*}\widetilde{\mathcal{D}}^{(m')}_{Z'^{\sharp}/T'^{\sharp}}} \widetilde{\mathcal{D}}^{(m')}_{X'^{\sharp}/T'^{\sharp}}/\mathcal{I}'\widetilde{\mathcal{D}}^{(m')}_{X'^{\sharp}/T'^{\sharp}}. \end{aligned} \tag{5.2.5.11.1}$$

In the case where $m' = m$, the isomorphism 5.2.5.11.1 is the commutation of pushforward with base change $\widetilde{u}^{(m)}_{+}(\mathcal{M}) \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \xrightarrow{\sim} \widetilde{u}'^{(m)}_{+}(\mathcal{M} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'})$. When $T'^{\sharp} \to T^{\sharp}$ is the identity, this is the commutation of pushforward with level change $\widetilde{u}^{(m)}_{+}(\mathcal{M}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}} \widetilde{\mathcal{D}}^{(m')}_{X^{\sharp}/T^{\sharp}} \xrightarrow{\sim} \widetilde{u}'^{(m')}_{+}(\mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}} \widetilde{\mathcal{D}}^{(m')}_{Z^{\sharp}/T^{\sharp}})$. Similarly, for any left $\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}$-module $\mathcal{F}$, we get the isomorphisms

$$\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \widetilde{u}^{(m)}_{+}(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}'^{(m)}_{+}(\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}), \qquad \widetilde{\mathcal{D}}^{(m')}_{X^{\sharp}/T^{\sharp}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}} \widetilde{u}^{(m)}_{+}(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}'^{(m')}_{+}(\widetilde{\mathcal{D}}^{(m')}_{Z^{\sharp}/T^{\sharp}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}} \mathcal{F}). \tag{5.2.5.11.2}$$

One might compare with the construction of 5.3.3.3 (remark this latter needs some quasi-coherence hypotheses), but since this is useless here, this is left to the reader.

Suppose we are in the local situation of 5.2.1.1. Via an easy computation (e.g. use 5.2.2.21.1), for any $\mathcal{D}^{(m)}_{Z'^{\sharp}/T'^{\sharp}}$-linear morphism of the form $\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F} \to \mathcal{F}'$, we get the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \widetilde{u}^{(m)}_{+}(\mathcal{F}) & \xrightarrow[\sim]{5.2.2.22.1} & \mathcal{O}_{T'} \otimes_{\mathcal{O}_T} (O_T\{\partial_{r+s+1}, \ldots, \partial_d\}^{(m)} \otimes_{\mathcal{O}_T} \mathcal{F}) \\ \downarrow\sim & & \downarrow\sim \\ \widetilde{u}'^{(m)}_{+}(\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}) & \xrightarrow[\sim]{5.2.2.22.1} & \mathcal{O}_{T'}\{\partial'_{r+s+1}, \ldots, \partial'_d\}^{(m)} \otimes_{\mathcal{O}_{T'}} (\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}) \\ \downarrow & & \downarrow \\ \widetilde{u}'^{(m')}_{+}(\mathcal{F}') & \xrightarrow[\sim]{5.2.2.22.1} & \mathcal{O}_{T'}\{\partial'_{r+s+1}, \ldots, \partial'_d\}^{(m')} \otimes_{\mathcal{O}_{T'}} (\mathcal{F}'). \end{array} \tag{5.2.5.11.3}$$

Remark 5.2.5.12. The derived version of the base change can be found later at 5.3.3.3.

5.2.6 Adjointness, stability of the perfectness under pushforward, relative duality isomorphism for schemes

Suppose we are in the non-respective case of log-schemes of 5.2.

Proposition 5.2.6.1. *Let $\mathcal{M}$ be a right (resp. left) $\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}$-module, $\mathcal{N}$ be a right (resp. left) $\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}$-module.*

(a) There exists a canonical functorial $\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}$-linear morphism adj: $\widetilde{u}^{(m)}_{+}H^0\widetilde{u}^{!}(\mathcal{M}) \to \mathcal{M}$ *and a canonical functorial $\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}$-linear morphism* adj: $\mathcal{N} \to H^0\widetilde{u}^{!}\widetilde{u}^{(m)}_{+}(\mathcal{N})$ *so that the compositions* $H^0\widetilde{u}^{!}(\mathcal{M}) \xrightarrow{\mathrm{adj}} H^0\widetilde{u}^{!}\widetilde{u}^{(m)}_{+}H^0\widetilde{u}^{!}(\mathcal{M}) \xrightarrow{\mathrm{adj}} H^0\widetilde{u}^{!}(\mathcal{M})$ *and* $\widetilde{u}^{(m)}_{+}(\mathcal{N}) \xrightarrow{\mathrm{adj}} \widetilde{u}^{(m)}_{+}H^0\widetilde{u}^{!}\widetilde{u}^{(m)}_{+}(\mathcal{N}) \xrightarrow{\mathrm{adj}} \widetilde{u}^{(m)}_{+}(\mathcal{N})$ *are the identity.*

(b) Using the above adjoint morphisms, we construct maps

$$\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}}(\widetilde{u}^{(m)}_{+}(\mathcal{N}), \mathcal{M}) \to u_{*}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}}(\mathcal{N}, H^0\widetilde{u}^{!}(\mathcal{M})),$$
$$u_{*}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}}(\mathcal{N}, H^0\widetilde{u}^{!}(\mathcal{M})) \to \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}}(\widetilde{u}^{(m)}_{+}(\mathcal{N}), \mathcal{M}),$$

which are inverse of each other.

(c) If $\mathcal{M}$ is an injective right (resp. left) $\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/T^{\sharp}}$-module, then $H^0\widetilde{u}^{!}(\mathcal{M})$ is an injective right (resp. left) $\widetilde{\mathcal{D}}^{(m)}_{Z^{\sharp}/T^{\sharp}}$-module.

Proof. I) Let us check the non-respective case. 0) Following 5.2.5.8.1, we reduce to check the proposition with "$H^0\widetilde{u}^!$" replaced by "$\widetilde{u}^{\flat 0}$". 1) Let us check the first assertion.

a) Since the construction of the *canonical* morphism $\mathrm{adj}\colon \widetilde{u}_+^{(m)}\widetilde{u}^{\flat 0}(\mathcal{M}) \to \mathcal{M}$ is local, then we can suppose the local conditions of 5.2.1.1 are satisfied (and we use notations 5.2.2 and 5.2.2.3). We have $\widetilde{u}_+^{(m)}\widetilde{u}^{\flat 0}(\mathcal{M}) := u_*\left(u^{-1}\mathcal{H}om_{\mathcal{B}_X}(u_*\mathcal{B}_Z,\mathcal{M}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp}\right) = \mathcal{H}om_{\mathcal{B}_X}(u_*\mathcal{B}_Z,\mathcal{M})\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}$. Let $B := \Gamma(X,\mathcal{B}_X)$. We reduce to construct a canonical morphism of the form $\mathrm{Hom}_B(B/IB, M) \otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \widetilde{D}^{(m)}_{X^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp} \to M$.

i) Let us check that the canonical map $\mathrm{adj}\colon \mathrm{Hom}_B(B/IB, M) \otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \widetilde{D}^{(m)}_{X^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp} \to M$, defined by setting $x\otimes[P]_Z \mapsto \mathrm{ev}_1(x)P$ for any $x \in \mathrm{Hom}_B(B/IB, M)$, $P \in \widetilde{D}^{(m)}_{X^\sharp}$, is well defined. The independence in the choice of P lifting $[P]_Z$ is clear. Let $x \in \mathrm{Hom}_B(B/IB, M)$, $P \in \widetilde{D}^{(m)}_{X^\sharp}$, $Q \in \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$. Choose $Q_X \in \widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ such that $\vartheta(Q) = [Q_X]_Z$. Using the formula 5.2.5.2.1, we get $\mathrm{ev}_1(x\cdot Q)P = \mathrm{ev}_1(x)\cdot Q_XP$. Following 5.2.2.6.1, we have $Q\cdot[P]_Z = [Q_XP]_Z$, and then we get the equality $x\cdot Q\otimes[P]_Z = x\otimes[Q_XP]_Z$ in $\mathrm{Hom}_B(B/IB, M) \otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \widetilde{D}^{(m)}_{X^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp}$. Hence, we have checked the map is well defined.

ii) The $\widetilde{D}^{(m)}_{X^\sharp}$-linearity of the canonical map adj of the part i) is obvious.

b) The restriction of $[-]_Z$ or $[-]'_Z$ on B induces the canonical map $B \to B/IB$ which can sometimes be denoted by $b \mapsto \bar{b}$. Similarly, to construct the morphism $\mathrm{adj}\colon \mathcal{N} \to \widetilde{u}^{\flat 0}\widetilde{u}_+^{(m)}(\mathcal{N})$, we reduce to the case where the assumptions of 5.2.2 are satisfied and to check that the morphism of the form $N \to \mathrm{Hom}_B(B/IB, N\otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \widetilde{D}^{(m)}_{X^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp})$ given by $y \mapsto (\bar{b} \mapsto y\otimes\bar{b})$ is $\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$-linear. Let $y \in N$, $Q \in \widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$. Let x be the element of $\mathrm{Hom}_B(B/IB, N\otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \widetilde{D}^{(m)}_{X^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp})$ given by $(\bar{b}\mapsto y\otimes\bar{b})$. Choose $Q_X \in \widetilde{D}^{(m)}_{X^\sharp,Z^\sharp,\underline{t}/T^\sharp}$ such that $\vartheta(Q) = [Q_X]_Z$. Using 5.2.5.2.1, $\mathrm{ev}_1(x\cdot Q) = \mathrm{ev}_1(x)\cdot Q_X = y\otimes[Q_X]_Z \overset{5.2.2.6.1}{=} y\cdot Q\otimes\bar{1}$. This yields that the canonical map $N \to \mathrm{Hom}_B(B/IB, N\otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \widetilde{D}^{(m)}_{Z^\sharp\to X^\sharp})$ is $\widetilde{D}^{(m)}_{Z^\sharp/T^\sharp}$-linear.

c) Reducing to the local context 5.2.2, we compute easily that both compositions are the identity maps.

2) The fact that the first statement of the proposition implies the second one is standard. Finally, since the functor $\widetilde{u}_+^{(m)}$ is exact, we get the last statement from the second one.

II) By using the right to left isomorphisms 5.1.3.2.1 and 5.1.1.5.1, the respective case follows from 5.2.6.1. □

5.2.6.2 (Local computation of the adjunction morphisms in the left case)**.** Let $\mathcal{E}$ be a let $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$-module, $\mathcal{F}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module. Suppose the local conditions of 5.2.1.1 are satisfied (and we use notations 5.2.2 and 5.2.2.3). We denote by $\widetilde{e}_0 := d\log t_1\wedge\cdots\wedge d\log t_r\wedge dt_{r+1}\wedge\cdots\wedge dt_d$ a basis of the free $\mathcal{B}_X$-module and by $\widetilde{e}_0^\vee$ its corresponding dual basis of the free $\mathcal{B}_X$-module $\widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}$. We denote by $\widetilde{f}_0 := d\log t_1\wedge\cdots\wedge d\log t_r\wedge dt_{r+1}\wedge\cdots\wedge dt_{r+s}$ the basis of the free $\mathcal{B}_Z$-module $\widetilde{\omega}_{Z^\sharp/T^\sharp}$, and by $\widetilde{f}_0^\vee$ its dual basis. By using these bases, we can identify $\widetilde{\omega}_{X^\sharp/T^\sharp}$ and $\widetilde{\omega}^{-1}_{X^\sharp/T^\sharp}$ with $\mathcal{B}_X$ (resp. $\widetilde{\omega}_{Z^\sharp/T^\sharp}$ and $\widetilde{\omega}^{-1}_{Z^\sharp/T^\sharp}$ with $\mathcal{B}_Z$), which yields the (non-canonical) isomorphisms (see 5.2.2.14.1 and from 5.2.5.10.2):

$$\widetilde{u}_+^{(m)}H^0\widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}/\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}\mathcal{I}\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}} \mathcal{H}om_{\mathcal{B}_X}(u_*\mathcal{B}_Z,\mathcal{E}), \tag{5.2.6.2.1}$$

$$u_*H^0\widetilde{u}^!\widetilde{u}_+^{(m)}(\mathcal{F}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_X}(\mathcal{B}_X/\mathcal{I}\mathcal{B}_X, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}\otimes_{u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp}} u_*\mathcal{F}). \tag{5.2.6.2.2}$$

Let $B := \Gamma(X,\mathcal{B}_X)$. The restriction of $[-]_Z$ or $[-]'_Z$ on B induces the canonical map $B \to B/IB$ which can sometimes be denoted by $b \mapsto \bar{b}$.

a) The adjunction morphism $\mathrm{adj}\colon \widetilde{u}_+^{(m)}H^0\widetilde{u}^!(\mathcal{E}) \to \mathcal{E}$ of 5.2.6.1 corresponds via the bijection 5.2.6.2.1 to a canonical morphism of the form $\widetilde{D}^{(m)}_{X^\sharp}/\widetilde{D}^{(m)}_{X^\sharp}I\otimes_{\widetilde{D}^{(m)}_{Z^\sharp}} \mathrm{Hom}_B(B/IB, E) \to E$, which is given by $[P]'_Z\otimes x \mapsto P\,\mathrm{ev}_1(x)$ for any $x \in \mathrm{Hom}_B(B/IB, E)$, $P \in \widetilde{D}^{(m)}_{X^\sharp}$. Moreover, modulo this isomorphism 5.2.2.22.1, the map corresponds to a morphism of the form

$$O_T\{\partial_{r+s+1},\ldots,\partial_d\}^{(m)}\otimes_{O_T}\mathrm{Hom}_B(B/IB, E) \to E \tag{5.2.6.2.3}$$

given by $Q\otimes x\mapsto Q\cdot x$, for any $Q\in O_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}$ and $x\in \mathrm{Hom}_B(B/IB,E)$.

b) The adjunction morphism $\mathrm{adj}\colon \mathcal{F}\to H^0\widetilde{u}^!\widetilde{u}_+^{(m)}(\mathcal{F})$, corresponds to a canonical morphism of the form $F\to \mathrm{Hom}_B(B/IB,\widetilde{D}^{(m)}_{X^\sharp}/I\widetilde{D}^{(m)}_{X^\sharp}\otimes_{\widetilde{D}^{(m)}_{Z^\sharp}}F)$ which is given by $y\mapsto(\bar{b}\mapsto\bar{b}\otimes y)$, for any $y\in F$ and $b\in B$. Moreover, modulo this isomorphism 5.2.2.22.1, the map corresponds to a morphism of the form

$$F\to \mathrm{Hom}_B(B/IB,O_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\otimes_{O_T}F)\subset O_T\{\partial_{r+s+1},\dots,\partial_d\}^{(m)}\otimes_{O_T}F \qquad (5.2.6.2.4)$$

given by $y\mapsto 1\otimes y$ for any $y\in F$.

Corollary 5.2.6.3. *Let $*\in\{\mathrm{r},\mathrm{l}\}$, $\mathcal{M}\in D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$, $\mathcal{N}\in D({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$. We have the isomorphisms*

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}}(\widetilde{u}_+^{(m)}(\mathcal{N}),\mathcal{M})\xrightarrow{\sim}u_*\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}(\mathcal{N},\widetilde{u}^!(\mathcal{M})).$$

Proof. Taking an injective resolution of $\mathcal{M}$, this is a consequence of 5.2.6.1.(b–c). □

Before giving another corollary, let us check the stability of the perfectness as follows.

Lemma 5.2.6.4. *Let $*\in\{\mathrm{r},\mathrm{l}\}$. For any $\mathcal{N}\in D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$, we have $\widetilde{u}_+^{(m)}(\mathcal{N})\in D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$.*

Proof. By using 5.1.4.3.1 and 5.1.3.2.1, we reduce to the case $*=r$. Since the perfectness is local, then we can suppose the local conditions of 5.2.1.1 are satisfied (and we use notations 5.2.2). By devissage, we reduce to the case where $\mathcal{N}$ is a projective right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$ module of finite type. Since $\widetilde{u}_+^{(m)}$ is exact, we reduce to check that $\widetilde{u}_+^{(m)}(\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})=u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}$ is a projective right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$ of finite type. Let $s:=d-r$, and $f_1=t_{r+1},\dots,f_T:=t_d$ and $K_\bullet(\underline{f})$ be the Koszul complex of free $\mathcal{O}_X$-modules given by the sequence of global section $\underline{f}=(f_1,\dots,f_s)$ of $\mathcal{O}_X$. We set $\widetilde{K}_\bullet(\underline{f}):=\mathcal{B}_X\otimes_{\mathcal{O}_X}K_\bullet(\underline{f})$. We get $u_*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp/T^\sharp}\xrightarrow{\sim}\widetilde{K}_\bullet(\underline{f})\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}$. Hence, we are done. □

Remark 5.2.6.5. We will check later the a version of the stability of the perfecness under pushforward by a proper morphism (more precisely, see 5.3.2.13).

Corollary 5.2.6.6. *Let $*\in\{\mathrm{r},\mathrm{l}\}$. Let $\mathcal{N}\in D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp})$. We have the isomorphism of $D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})$:*

$$\widetilde{\mathbb{D}}^{(m)}_{X^\sharp/T^\sharp}\circ\widetilde{u}_+^{(m)}(\mathcal{N})\xrightarrow{\sim}\widetilde{u}_+^{(m)}\circ\widetilde{\mathbb{D}}^{(m)}_{Z^\sharp/T^\sharp}(\mathcal{N}). \qquad (5.2.6.6.1)$$

Proof. By using 5.1.4.3.1 and 5.1.3.2.1, we reduce to the case $*=r$. In this case, the isomorphism 5.2.6.6.1 is the composition of the following isomorphisms:

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp}}(\widetilde{u}_+^{(m)}(\mathcal{N}),\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/T^\sharp})[d_X]\overset{5.2.6.3}{\xrightarrow{\sim}}u_*\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}(\mathcal{N},\widetilde{u}^\flat(\widetilde{\omega}_{X^\sharp/T^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp T^\sharp}))[d_X]\overset{5.2.5.9}{\xrightarrow{\sim}}$$

$$u_*\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}(\mathcal{N},\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp})[d_Z]\underset{4.6.3.6.1}{\xrightarrow{\sim}}u_*\left(\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}(\mathcal{N},\widetilde{\omega}_{Z^\sharp/T^\sharp}\otimes_{\mathcal{B}_Z}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}[d_Z])\otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp\to X^\sharp}\right).$$

□

5.3 Commutations and relations between functors

5.3.1 Extraordinary inverse image, direct image: varying log-smoothly the basis

Let $S^\sharp$ be a nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$, where i is an integer (resp. $S^\sharp$ is a nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10) Let $\alpha\colon Z^\sharp\to S^\sharp$ be a log smooth morphism. Let $h\colon X^\sharp\to Y^\sharp$ be a morphism of log smooth $Z^\sharp$-log-scheme (resp. log smooth $Z^\sharp$-log formal scheme). We denote by $g\colon Y^\sharp\to Z^\sharp$ and $f\colon X^\sharp\to Z^\sharp$ the structural morphisms.

Let $\mathcal{B}_Z$ be an $\mathcal{O}_Z$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{Z^\sharp/S^\sharp}$-module. Set $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}:=\mathcal{B}_Z\otimes_{\mathcal{O}_Z}\mathcal{D}^{(m)}_{Z^\sharp/S^\sharp}$, and for any $n\in\mathbb{N}$, $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp,n}:=\mathcal{B}_Z\otimes_{\mathcal{O}_Z}\mathcal{D}^{(m)}_{Z^\sharp/S^\sharp,n}$, $\widetilde{\mathcal{P}}^n_{Z^\sharp/S^\sharp,(m)}:=\mathcal{B}_Z\otimes_{\mathcal{O}_Z}\mathcal{P}^n_{Z^\sharp/S^\sharp,(m)}$.

Let $\mathcal{B}_Y$ be a $g^*(\mathcal{B}_Z)$-algebra which is endowed with a compatible structure of left $\mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$-module such that the structural homomorphism $g^*\mathcal{B}_Z \to \mathcal{B}_Y$ is $\mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$-linear. Since $\mathcal{D}^{(m)}_{Y^\sharp/Z^\sharp} \to \mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$ is in fact a morphism of rings, then $\mathcal{B}_Y$ is also an $g^*(\mathcal{B}_Z)$-algebra which is endowed with a compatible structure of left $\mathcal{D}^{(m)}_{Y^\sharp/Z^\sharp}$-module. Set $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$, and for any $n \in \mathbb{N}$, $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp,n} := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/S^\sharp,n}$, $\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)} := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y^\sharp/S^\sharp,(m)}$. Set $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/Z^\sharp}$, and for any $n \in \mathbb{N}$, $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp,n} := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/Z^\sharp,n}$, $\widetilde{\mathcal{P}}^n_{Y^\sharp/Z^\sharp,(m)} := \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y^\sharp/Z^\sharp,(m)}$.

Let $\mathcal{B}_X$ be a $h^*(\mathcal{B}_Y)$-algebra which is endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module such that the structural homomorphism $h^*(\mathcal{B}_Y) \to \mathcal{B}_X$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear. Set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$, and for any $n \in \mathbb{N}$, $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp,n}$, $\widetilde{\mathcal{P}}^n_{X^\sharp/S^\sharp,(m)} := \mathcal{B}_Y \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/S^\sharp,(m)}$. Set $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/Z^\sharp}$, and for any $n \in \mathbb{N}$, $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp,n} := \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/Z^\sharp,n}$, $\widetilde{\mathcal{P}}^n_{X^\sharp/Z^\sharp,(m)} := \mathcal{B}_Y \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X^\sharp/Z^\sharp,(m)}$.

We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{Y}^\sharp$) the ringed logarithmic ($\mathcal{V}$-formal) scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(Y^\sharp, \mathcal{B}_Y)$), and by $\widetilde{h}_{/S^\sharp} \colon \widetilde{X}^\sharp/S^\sharp \to \widetilde{Y}^\sharp/S^\sharp$ and by $\widetilde{h}_{/Z^\sharp} \colon \widetilde{X}^\sharp/Z^\sharp \to \widetilde{Y}^\sharp/Z^\sharp$ the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by h and by $f^*\mathcal{B}_Y \to \mathcal{B}_X$. We denote by $\widetilde{f} \colon \widetilde{X}^\sharp/S^\sharp \to Z^\sharp/S^\sharp$ (resp. $\widetilde{g} \colon \widetilde{Y}^\sharp/S^\sharp \to Z^\sharp/S^\sharp$) the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by f (resp. g) and by $f^*\mathcal{B}_Z \to \mathcal{B}_X$ (resp. $g^*\mathcal{B}_Z \to \mathcal{B}_Y$). We suppose $\widetilde{g}$ and $\widetilde{h}_{/S^\sharp}$ (and then $\widetilde{f}$ and $\widetilde{h}_{/Z^\sharp}$) are quasi-flat (see Definition 4.4.1.3).

5.3.1.1. Following 4.4.2.7.1, the canonical homomorphism $\mathcal{D}^{(m)}_{Y^\sharp/Z^\sharp} \to \mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$ is in fact a ring homomorphism. We denote by $\mathrm{forg}_{Y^\sharp/Z^\sharp/S^\sharp}$ the forgetful functor (via the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$) from the category of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-modules to that of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-modules ; and similarly by replacing $Y^\sharp$ by $X^\sharp$.

We have the functor $\widetilde{h}^*_{/S^\sharp} := \mathcal{B}_X \otimes_{h^{-1}\mathcal{B}_Y} h^{-1}(-)$ from the category of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-modules to that of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and the functor $\widetilde{h}^*_{/Z^\sharp} := \mathcal{B}_X \otimes_{h^{-1}\mathcal{B}_Y} h^{-1}(-)$ from the category of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-modules to that of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}$-modules. From the commutative diagram 4.4.2.3.2 (still valid with some tildes), we get the commutation

$$\mathrm{forg}_{X^\sharp/Z^\sharp/S^\sharp} \circ \widetilde{h}^*_{/Z^\sharp} \xrightarrow{\sim} \widetilde{h}^*_{/S^\sharp} \circ \mathrm{forg}_{Y^\sharp/Z^\sharp/S^\sharp}. \tag{5.3.1.1.1}$$

By functoriality, we get the $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$-bimodule $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \to Y^\sharp/S^\sharp} := \widetilde{h}^*_{/S^\sharp}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ and the $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}, h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$-bimodule $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \to Y^\sharp/Z^\sharp} := \widetilde{h}^*_{/Z^\sharp}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$.

Lemma 5.3.1.2. *We have the isomorphism of* $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}, h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$*-bimodules*

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp \to Y^\sharp/Z^\sharp} \otimes_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp \to Y^\sharp/S^\sharp}. \tag{5.3.1.2.1}$$

Proof. By functoriality, this is a consequence of 5.3.1.1.1. □

Notation 5.3.1.3 (Local description of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$)**.** Suppose $Y^\sharp/Z^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,d}$. Since the level m is fixed (even if n is not fixed), we simply write $\tau_{\sharp\lambda} := \mu^n_{(m),\gamma}(u_\lambda) - 1 \in \widetilde{\mathcal{P}}^n_{Y^\sharp/Z^\sharp,(m)}$ for $\lambda = 1,\dots,d$. For any $\underline{i} = (i_1,\cdots,i_d) \in \mathbb{N}^d$, let $\underline{\tau}_\sharp^{\{\underline{i}\}_{(m)}} := \tau_{1\sharp}{}^{\{i_1\}_{(m)}} \cdots \tau_{d\sharp}{}^{\{i_d\}_{(m)}}$. Following 3.2.3.4 or 3.3.4.5, we get the basis of the free $\mathcal{B}_Y$-module $\widetilde{\mathcal{P}}^n_{Y^\sharp/Z^\sharp,(m)}$ given by $\underline{\tau}_\sharp{}^{\{\underline{i}\}_{(m)}}$, where $\underline{i} = (i_1,\cdots,i_d) \in \mathbb{N}^d$ satisfy $|\underline{i}| \le n$. By taking the dual basis and taking the inverse limits, we get a basis on the free (for the left or right structure) $\mathcal{B}_Y$-module $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$ (for its left structure this is by definition but this is also the case its right structure) which is denoted by $\{\underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^d\}$. Hence, a section of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$ can uniquely be written as a *finite* sum of the form $\sum_{\underline{i}\in\mathbb{N}^d} a_{\underline{i}}\underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}}$ (resp. $\sum_{\underline{i}\in\mathbb{N}^d} \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}} a_{\underline{i}}$) with $a_{\underline{i}} \in \mathcal{B}_Y$.

Notation 5.3.1.4 (Local description of $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$)**.** Suppose $Z^\sharp/S^\sharp$ is endowed with logarithmic coordinates $\tilde{u}_1, \cdots, \tilde{u}_r$. Since the level m is fixed (even if n is not fixed), we simply write $\widetilde{\tau}_{j\sharp} := \mu^n_{(m),\gamma}(\tilde{u}_j) - 1 \in \widetilde{\mathcal{P}}^n_{Y^\sharp/Z^\sharp,(m)}$ for $j = 1, \ldots, r$. For any $\underline{j} = (j_1, \cdots, j_r) \in \mathbb{N}^r$, let $\underline{\widetilde{\tau}}^{\{\underline{j}\}_{(m)}}_\sharp := \widetilde{\tau}^{\{j_1\}_{(m)}}_{1\sharp} \cdots \widetilde{\tau}^{\{j_r\}_{(m)}}_{r\sharp} \in \widetilde{\mathcal{P}}^n_{Z^\sharp/S^\sharp,(m)}$. We get the basis of the free $\mathcal{B}_Z$-module $\widetilde{\mathcal{P}}^n_{Z^\sharp/S^\sharp,(m)}$ given by $\underline{\widetilde{\tau}}_\sharp{}^{\{\underline{j}\}_{(m)}}$, with $|\underline{j}| \leq n$. By taking the dual basis and taking the inverse limits, we get a basis on the free (for the left or right structure) $\mathcal{B}_Z$-module $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ which is denoted by $\{\underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}} \mid \underline{j} \in \mathbb{N}^r\}$. Hence, a section of $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ can uniquely be written as a *finite* sum of the form $\sum_{\underline{j}\in\mathbb{N}^r} a_{\underline{j}} \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}}$ (resp. $\sum_{\underline{j}\in\mathbb{N}^r} \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}} a_{\underline{j}}$) with $a_{\underline{j}} \in \mathcal{B}_Z$.

Notation 5.3.1.5 (Local description of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$)**.** Suppose $Z^\sharp/S^\sharp$ is endowed with logarithmic coordinates $\tilde{u}_1, \cdots, \tilde{u}_r$ and suppose moreover that $Y^\sharp/Z^\sharp$ is endowed with logarithmic coordinates $u_1, \cdots, u_d$. We denote by $\tilde{u}_1, \cdots, \tilde{u}_r$ the element of $\Gamma(Y, M_{Y^\sharp})$ induced by $\tilde{u}_1, \cdots, \tilde{u}_r$ via g. We get the logarithmic coordinates $\tilde{u}_1, \cdots, \tilde{u}_r, u_1, \cdots, u_d$ of $Y^\sharp/S^\sharp$.

We set $\tau_{\sharp\lambda} := \mu^n_{(m),\gamma}(u_\lambda) - 1 \in \widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)}$ for $\lambda = 1, \ldots, d$, $\widetilde{\tau}_{j\sharp} := \mu^n_{(m),\gamma}(\tilde{u}_j) - 1 \in \widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)}$ for any $j = 1, \cdots, r$. For any $\underline{i} = (i_1, \cdots, i_d) \in \mathbb{N}^d$, let $\underline{\tau}_\sharp{}^{\{\underline{i}\}_{(m)}} := \tau_{1\sharp}{}^{\{i_1\}_{(m)}} \cdots \tau_{d\sharp}{}^{\{i_d\}_{(m)}}$; for any $\underline{j} = (j_1, \cdots, j_r) \in \mathbb{N}^r$, let $\underline{\widetilde{\tau}}_\sharp{}^{\{\underline{j}\}_{(m)}} := \widetilde{\tau}_{1\sharp}{}^{\{j_1\}_{(m)}} \cdots \widetilde{\tau}_{r\sharp}{}^{\{j_r\}_{(m)}} \in \widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)}$. We get the basis of the free $\mathcal{B}_Y$-module $\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)}$ given by $\underline{\tau}_\sharp{}^{\{\underline{i}\}_{(m)}} \underline{\widetilde{\tau}}_\sharp{}^{\{\underline{j}\}_{(m)}}$, with $|\underline{i}| + |\underline{j}| \leq n$. We denote by $\{\underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}}$, with $|\underline{i}| + |\underline{j}| \leq n\}$ the corresponding dual basis of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp,n}$. By taking the inductive limits, this yields the basis $\{\underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}}$, with $\underline{i} \in \mathbb{N}^d$ and $\underline{j} \in \mathbb{N}^r\}$ of the free $\mathcal{B}_Y$-module $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$. In other words, a section of the sheaf $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ can uniquely be written as a *finite* sum of the form $\sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} a_{\underline{i},\underline{j}} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}}$ (resp. $\sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}} a_{\underline{i},\underline{j}}$) with $a_{\underline{i},\underline{j}} \in \mathcal{B}_Y$.

We hope this is not too confusing that $\underline{\partial}_\sharp{}^{\langle \underline{i}\rangle_{(m)}}$ (resp. $\underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}}$) is either a global section of $\mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$ or of $\mathcal{D}^{(m)}_{Y^\sharp/Z^\sharp}$ (resp. of $\mathcal{D}^{(m)}_{Z^\sharp/S^\sharp}$).

5.3.1.6. We keep notations and hypotheses of 5.3.1.5.

(a) We compute the natural ring homomorphism $\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Y^\sharp/Z^\sharp,(m)}$ sends $\underline{\tau}_\sharp{}^{\{\underline{i}\}_{(m)}}$ to $\underline{\tau}_\sharp{}^{\{\underline{i}\}_{(m)}}$, which justifies why we took the same notation. Hence, the morphism $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ corresponds to the inclusion given by

$$\sum_{\underline{i}\in\mathbb{N}^d} a_{\underline{i}} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp \mapsto \sum_{\underline{i}\in\mathbb{N}^d} a_{\underline{i}} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp,$$

where $a_{\underline{i}}$ are global sections of $\mathcal{B}_Y$. Since $\widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Y^\sharp/Z^\sharp,(m)}$ is a homomorphism of $\mathcal{B}_Y$-algebras for the right structure (and also for the left one, but this is useless here), then the action of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$ on $\mathcal{B}_Y$ and of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ on $\mathcal{B}_Y$ are compatible with the canonical inclusion $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$. This implies the homomorphism $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ is also given by the formula

$$\sum_{\underline{i}\in\mathbb{N}^d} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp a_{\underline{i}} \mapsto \sum_{\underline{i}\in\mathbb{N}^d} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp a_{\underline{i}},$$

where $a_{\underline{i}}$ are global sections of $\mathcal{B}_Y$.

(b) Using the universal property of m-PD-envelopes, we get the homomorphisms of rings $\widetilde{g}^* \widetilde{\mathcal{P}}^n_{Z^\sharp/S^\sharp,(m)} \to \widetilde{\mathcal{P}}^n_{Y^\sharp/S^\sharp,(m)}$. We compute this map sends $1 \otimes \underline{\widetilde{\tau}}_\sharp{}^{\{\underline{j}\}_{(m)}}$ to $\underline{\widetilde{\tau}}_\sharp{}^{\{\underline{j}\}_{(m)}}$, which justifies a bit why we took the same notation. This yields that the homomorphism $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \to \widetilde{g}^* \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ is given by

$$\sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} \alpha_{\underline{i},\underline{j}} \underline{\partial}^{\langle \underline{i}\rangle_{(m)}}_\sharp \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}} \mapsto \sum_{\underline{j}\in\mathbb{N}^r} \alpha_{\underline{0},\underline{j}} \otimes \underline{\widetilde{\partial}}_\sharp{}^{\langle \underline{j}\rangle_{(m)}},$$

where $\alpha_{\underline{i},\underline{j}} \in \mathcal{B}_Y$. In particular, $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \to \widetilde{g}^* \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ is an epimorphism.

(c) The left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-module (resp. right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-module) $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ canonically splits as follows:

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} = \oplus_{\underline{j}\in\mathbb{N}^r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}},\ \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} = \oplus_{\underline{j}\in\mathbb{N}^r}\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}, \tag{5.3.1.6.1}$$

where $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}$ (resp. $\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$) is the left (resp. right) free $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-submodule of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ generated by $\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}$. We get the exhausted filtration of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ by left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-submodules (resp. right $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-submodules) $F^{\mathrm{l}}_n\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} := \oplus_{|\underline{j}|\leq n}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}$ (resp. $F^{\mathrm{r}}_n\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} := \oplus_{|\underline{j}|\leq n}\underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$).

5.3.1.7. From 5.3.1.6.c, we have that $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ is a left (resp. right) flat $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}$-module. This yields from 5.3.1.2.1 the isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}\otimes^{\mathbb{L}}_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}. \tag{5.3.1.7.1}$$

5.3.1.8. For any $\mathcal{E}\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$, we have the extraordinary inverse images $\widetilde{h}^{(m)!}_{/Z^\sharp}(\mathcal{E}) := \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}\otimes^{\mathbb{L}}_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} h^{-1}\mathcal{E}[d_h]$, and for any $\mathcal{E}\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$, we will write $\widetilde{h}^{(m)!}_{/S^\sharp}(\mathcal{E}) := \widetilde{\mathcal{D}}^{(m)}_{X\to Y}\otimes^{\mathbb{L}}_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} h^{-1}\mathcal{E}[d_h]$. We denote by $\mathrm{forg}_{Y^\sharp/Z^\sharp/S^\sharp}\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})\to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$ the canonical forgetful functor (and similarly by replacing Y with X).

Proposition 5.3.1.9. For any $\mathcal{E}\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$, we get the isomorphism

$$\mathrm{forg}_{X^\sharp/Z^\sharp/S^\sharp}\circ\widetilde{h}^{(m)!}_{/S^\sharp}(\mathcal{E})\xrightarrow{\sim}\widetilde{h}^{(m)!}_{/Z^\sharp}\circ\mathrm{forg}_{Y^\sharp/Z^\sharp/S^\sharp}(\mathcal{E}). \tag{5.3.1.9.1}$$

Proof. By associativity of the tensor product, this is a consequence of 5.3.1.7.1. □

5.3.1.10. Consider the diagram of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules

$$\begin{array}{ccc}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} & \longrightarrow & \widetilde{h}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\\ \downarrow & & \downarrow\\ \widetilde{f}^{*}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp} & \underset{\sim}{\xleftarrow{4.4.5.6}} & \widetilde{h}^{*}\widetilde{g}^{*}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp},\end{array} \tag{5.3.1.10.1}$$

where except the natural bottom isomorphism, the morphisms are given by 4.4.2.6.3. We compute that both paths $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\to\widetilde{f}^{*}\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ send 1 to $1\otimes 1$. Hence, the diagram 5.3.1.10.1 is commutative.

Suppose $Z^\sharp/S^\sharp$ is endowed with logarithmic coordinates $\tilde{u}_1,\cdots,\tilde{u}_r$. Suppose moreover that $Y^\sharp/Z^\sharp$ is endowed with logarithmic coordinates $u_1,\cdots,u_d$. By abuse of notation, we denote by $\tilde{u}_1,\cdots,\tilde{u}_r$ the element of $\Gamma(Y,M_{Y^\sharp})$ induced by $\tilde{u}_1,\cdots,\tilde{u}_r$ via g. We get the logarithmic coordinates $\tilde{u}_1,\cdots,\tilde{u}_r,u_1,\cdots,u_d$ of $Y^\sharp/S^\sharp$. We keep notation 5.3.1.5: we get the basis $\{\underline{\partial}_\sharp^{\langle\underline{i}\rangle_{(m)}}\underline{\widetilde{\partial}}_\sharp^{\langle\underline{j}\rangle_{(m)}}$, with $\underline{i}\in\mathbb{N}^d$ and $\underline{j}\in\mathbb{N}^r\}$ of the free $\mathcal{B}_Y$-module $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$.

Suppose moreover that $X^\sharp/Z^\sharp$ is endowed with logarithmic coordinates $u'_1,\cdots,u'_{d'}$. We denote by $\tilde{u}'_1,\cdots,\tilde{u}'_r$ the elements of $\Gamma(X,M_{X^\sharp})$ induced by $\tilde{u}_1,\cdots,\tilde{u}_r$ via f (we add some prime to avoid any confusion). We get the logarithmic coordinates $\tilde{u}'_1,\cdots,\tilde{u}'_r,u'_1,\cdots,u'_{d'}$ of $Y^\sharp/S^\sharp$. Similarly to notation 5.3.1.5, we get the basis $\{\underline{\partial}'^{\langle\underline{i}'\rangle_{(m)}}_\sharp\underline{\widetilde{\partial}}'^{\langle\underline{j}\rangle_{(m)}}_\sharp$, with $\underline{i}'\in\mathbb{N}^{d'}$ and $\underline{j}\in\mathbb{N}^r\}$ of the free $\mathcal{B}_X$-module $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$.

Let $n\in\mathbb{N}$. Fix $\underline{l}\in\mathbb{N}^r$ such that $|\underline{l}|=n$.

i) The morphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\to\widetilde{h}^{*}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$ factors through $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,n}\to\widetilde{h}^{*}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp,n})$. This yields $\underline{\widetilde{\partial}}'^{\langle\underline{l}\rangle_{(m)}}_\sharp\cdot(1\otimes 1)\in\widetilde{h}^{*}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp,n})$. Hence, we can write in $\widetilde{h}^{*}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$ the equality:

$$\underline{\widetilde{\partial}}'^{\langle\underline{l}\rangle_{(m)}}_\sharp\cdot(1\otimes 1) = \sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r,|\underline{i}|+|\underline{j}|\leq n} a_{\underline{i},\underline{j}}\otimes\underline{\partial}_\sharp^{\langle\underline{i}\rangle_{(m)}}\underline{\widetilde{\partial}}_\sharp^{\langle\underline{j}\rangle_{(m)}}, \tag{5.3.1.10.2}$$

where the sum is finite and where $a_{\underline{i},\underline{j}}\in\mathcal{B}_X$ are uniquely determined.

ii) Let us denote by ψ the composition $\psi\colon \widetilde{h}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \to \widetilde{h}^*\widetilde{g}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$. By using 5.3.1.6.b we compute in $\widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ the equality:

$$\psi\left(\sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} a_{\underline{i},\underline{j}} \otimes \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}} \underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}\right) = \sum_{\underline{j}\in\mathbb{N}^r} a_{\underline{0},\underline{j}} \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}. \tag{5.3.1.10.3}$$

iii) The maps $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{h}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ send $\underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}}$ to $\underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1)$. Since ψ is $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear, by commutativity of 5.3.1.10.1, we have $\psi(\underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1)) = \underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot \psi(1\otimes 1) = \underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1)$. By using the computation of 5.3.1.6.b (applied to $\widetilde{X}^\sharp/S^\sharp$ instead of $\widetilde{X}^\sharp/S^\sharp$), we get $\underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1) = 1 \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{l}\rangle_{(m)}}$. Hence, we get in $\widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ the equality:

$$\psi(\underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1)) = 1 \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{l}\rangle_{(m)}}. \tag{5.3.1.10.4}$$

iv) It follows from 5.3.1.10.2, 5.3.1.10.3 and 5.3.1.10.4 that we have in $\widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/S^\sharp}$ the formula:

$$\sum_{\underline{j}\in\mathbb{N}^r} a_{\underline{0},\underline{j}} \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}} = 1 \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{l}\rangle_{(m)}}.$$

This yields $a_{\underline{0},\underline{l}} = 1$ and $a_{\underline{0},\underline{j}} = 0$ if $\underline{j} \neq \underline{l}$. Hence, we have in $\widetilde{h}^*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp,n})$ the equality:

$$\underline{\widetilde{\partial}}_\sharp'^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1) = 1 \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{l}\rangle_{(m)}} + \sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r,|\underline{i}|+|\underline{j}|\leq n,|\underline{i}|\neq 0} a_{\underline{i},\underline{j}} \otimes \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}} \underline{\widetilde{\partial}}_\sharp^{\langle \underline{j}\rangle_{(m)}}. \tag{5.3.1.10.5}$$

Hence, we have in $\widetilde{h}^*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp,n})$ the congruence:

$$\underline{\widetilde{\partial}}_\sharp^{\langle \underline{l}\rangle_{(m)}} \cdot (1\otimes 1) \equiv 1 \otimes \underline{\widetilde{\partial}}_\sharp^{\langle \underline{l}\rangle_{(m)}} \mod \widetilde{h}^*(F^{\mathrm{l}}_{n-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}), \tag{5.3.1.10.6}$$

where $(F^{\mathrm{l}}_n\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})_n$ is the filtration defined at 5.3.1.6.c.

Lemma 5.3.1.11. *The canonical morphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$-bimodules*

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp} \tag{5.3.1.11.1}$$

is an isomorphism.

Proof. The canonical homomorphism 5.3.1.11.1 is constructed as follows. By applying the functor $\widetilde{h}^*$ to the homomorphism $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$, we get the homomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}, h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$-bimodules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp} = \widetilde{h}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp} \to \widetilde{h}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}$. This yields the homomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$-bimodules

$$\phi\colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}.$$

We have to check that this is an isomorphism. Since this is local, we can suppose Suppose $Z^\sharp/S^\sharp$ is endowed with logarithmic coordinates $\tilde{u}_1, \cdots, \tilde{u}_r$, $Y^\sharp/Z^\sharp$ is endowed with logarithmic coordinates $u_1, \cdots, u_d$, $X^\sharp/Z^\sharp$ is endowed with logarithmic coordinates $u'_1, \cdots, u'_{d'}$. We follow notation 5.3.1.10.

Let $P \in \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}$. By using 5.3.1.3 and by using 5.3.1.6.c for $X^\sharp/S^\sharp$ instead of $Y^\sharp/S^\sharp$, we can uniquely write P of the form

$$P = \sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} \underline{\widetilde{\partial}}_\sharp'^{\langle \underline{j}\rangle_{(m)}} \otimes (a_{\underline{i},\underline{j}} \otimes \underline{\partial}_\sharp^{\langle \underline{i}\rangle_{(m)}})$$

where the sum is finite and $a_{\underline{i},\underline{j}} \in \mathcal{B}_X$. We get

$$\phi(P) = \sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} \widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{j}\rangle_{(m)}} a_{\underline{i},\underline{j}} \cdot (1 \otimes \underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}}) = \sum_{\underline{i}\in\mathbb{N}^d,\underline{j}\in\mathbb{N}^r} \widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{j}\rangle_{(m)}} a_{\underline{i},\underline{j}} \cdot (1 \otimes 1) \cdot \underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}}.$$

Let $n := \max\{k \in \mathbb{N} \mid \exists \underline{j} \in \mathbb{N}^r, \exists \underline{i} \in \mathbb{N}^d, \text{such that } |\underline{j}| = k \text{ and } a_{\underline{i},\underline{j}} \neq 0\}$. Let $\underline{l} \in \mathbb{N}^r$ be such that $|\underline{l}| = n$. For any integer s, we denote by $\mathcal{D}^{(m)}_{X,Z^\sharp/S^\sharp,s}$ the free $\mathcal{B}_X$-submodule (for both structure) of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ whose basis is given by $\widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{j}\rangle_{(m)}}$ for any $\underline{j} \in \mathbb{N}^r$ such that $|\underline{j}| \leq s$. We remark that $\widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} a_{\underline{i},\underline{l}} - a_{\underline{i},\underline{l}}\widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} \in \mathcal{D}^{(m)}_{X,Z^\sharp/S^\sharp,n-1}$. Hence, by using 5.3.1.10.6, we compute

$$\widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} a_{\underline{i},\underline{l}} \cdot (1 \otimes 1) \equiv a_{\underline{i},\underline{l}} \widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} \cdot (1 \otimes 1) \equiv a_{\underline{i},\underline{l}} \otimes \widetilde{\underline{\partial}}_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} \mod \widetilde{h}^*(F^{\mathrm{l}}_{n-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}).$$

Since the action of $\underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}}$ via the right $h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$-module structure of $\widetilde{h}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ preserves $\widetilde{h}^*(F^{\mathrm{l}}_{n-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$ (because $\underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}}$ and $\widetilde{\underline{\partial}}_{\sharp}{}^{\langle \underline{j}\rangle_{(m)}}$ commute), we get

$$\widetilde{\underline{\partial}}'_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} a_{\underline{i},\underline{l}} \cdot (1 \otimes 1) \cdot \underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}} \equiv a_{\underline{i},\underline{l}} \otimes \widetilde{\underline{\partial}}_{\sharp}{}^{\langle \underline{l}\rangle_{(m)}} \underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}} \mod \widetilde{h}^*(F^{\mathrm{l}}_{n-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}).$$

Since $\widetilde{h}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ is a free $\mathcal{B}_X$-module with the basis $\{\underline{\partial}_{\sharp}^{\langle \underline{i}\rangle_{(m)}} \widetilde{\underline{\partial}}_{\sharp}{}^{\langle \underline{j}\rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^d,\ \underline{j} \in \mathbb{N}^r\}$ then from this latter congruence, we check easily by induction on n the injectivity and the surjectivity of ϕ. □

Proposition 5.3.1.12. For any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$, we get the isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{h}^{(m)!}_{/Z^\sharp}(\mathcal{E}) \xrightarrow{\sim} \widetilde{h}^{(m)!}_{/S^\sharp}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} \mathcal{E}). \tag{5.3.1.12.1}$$

Proof. By associativity of the tensor product, we get

$$\begin{aligned}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{h}^{(m)!}_{/Z^\sharp}(\mathcal{E}) &= \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp} \otimes^{\mathbb{L}}_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} h^{-1}\mathcal{E}\right)[d_h] \\ &\underset{5.3.1.11.1}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} h^{-1}\mathcal{E}[d_h] \\ \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} h^{-1}&\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} \mathcal{E}\right)[d_h] = \widetilde{h}^{(m)!}_{/S^\sharp}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}} \mathcal{E}).\end{aligned}$$

□

Lemma 5.3.1.13. *Suppose we are in the non-respective case. Let $\mathcal{M} \in D^-_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. Then the canonical morphism of $D^-({}^{\mathrm{r}}h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$*

$$\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp} \to \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp} \tag{5.3.1.13.1}$$

is an isomorphism.

Proof. Since this is local, we can suppose X affine. Using the way-out left version of [Har66, I.7.1.(iv)], since the functors $\mathcal{M} \mapsto \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}$ and $\mathcal{M} \mapsto \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}$ are way-out left, we reduce to check the isomorphism when $\mathcal{M}$ is a free right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Hence, we come down to the case where $\mathcal{M} = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. In that case, $\mathcal{M}$ is a flat right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module and a flat right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}$-module (see 5.3.1.6.c). Hence, we conclude using 5.3.1.11. □

Proposition 5.3.1.14. *Suppose we are in the non-respective case. Assume that S and Z are noetherian of finite Krull dimension and that (the underlying schemes morphim) h is quasi-compact and quasi-separated. We have for any $\mathcal{M} \in D^-_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ the isomorphism*

$$\mathrm{forg}_{Y^\sharp/Z^\sharp/S^\sharp} \circ \widetilde{h}^{(m)}_{/S^\sharp\,+}(\mathcal{M}) \xrightarrow{\sim} \widetilde{h}^{(m)}_{/Z^\sharp\,+} \circ \mathrm{forg}_{X^\sharp/Z^\sharp/S^\sharp}(\mathcal{M}). \tag{5.3.1.14.1}$$

Proof. Recall $\widetilde{h}^{(m)}_{/S^\sharp\,+}(\mathcal{M}) := \mathbb{R}h_*\left(\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}\right)$, $\widetilde{h}^{(m)}_{/Z^\sharp\,+}(\mathcal{M}) := \mathbb{R}h_*\left(\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}\right)$. Hence, this is straightforward from 5.3.1.13. □

Proposition 5.3.1.15. With hypotheses of 5.3.1.14, for $\mathcal{M}\in D^-_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp})$, we have the canonical isomorphism

$$\widetilde{h}^{(m)}_{/Z^\sharp+}(\mathcal{M})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\xrightarrow{\sim}\widetilde{h}^{(m)}_{/S^\sharp+}(\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}). \tag{5.3.1.15.1}$$

Proof. Using the projection isomorphism, we get

$$\widetilde{h}^{(m)}_{/Z^\sharp+}(\mathcal{M})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}=\mathbb{R}h_*\left(\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}\right) \tag{5.3.1.15.2}$$

$$\underset{5.1.2.6.1}{\xrightarrow{\sim}}\mathbb{R}h_*\left(\left(\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Z^\sharp}\right)\otimes_{h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/Z^\sharp}}h^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\right) \tag{5.3.1.15.3}$$

$$\underset{5.3.1.2.1}{\xrightarrow{\sim}}\mathbb{R}h_*\left(\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}\right)\xrightarrow{\sim}\widetilde{h}^{(m)}_{/S^\sharp+}(\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Z^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}). \tag{5.3.1.15.4}$$

□

5.3.2 Way-out properties of pushforwards and extraordinary pullbacks, stability of the coherence, tor dimension finiteness, perfectness

We keep notation and hypotheses of 5.1.1.1 and we suppose $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3).

5.3.2.1. We suppose f is log-smooth and the bottom arrow of 5.1.1.1.1 is the identity. Following 5.3.1.11.1 (applied in the case where $g=\mathrm{id}$ and therefore $f=h$), since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/Y^\sharp}=\mathcal{B}_X$, we have therefore the canonical morphism

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}}\mathcal{B}_X\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp/S^\sharp}, \tag{5.3.2.1.1}$$

given by $P\otimes 1\mapsto P\cdot(1\otimes 1)$, is an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp},\widetilde{f}^{-1}\mathcal{B}_Y)$-bimodules Since f is log smooth, following 4.7.3.7.2 we have the exact sequence of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}$-modules

$$0\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}\otimes_{\mathcal{B}_X}\wedge^d\widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}\cdots\underset{\delta}{\longrightarrow}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}\otimes_{\mathcal{B}_X}\wedge\widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}\underset{\delta}{\longrightarrow}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}\to\mathcal{B}_X\to 0. \tag{5.3.2.1.2}$$

By applying the exact functor $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}}-$ to the Spencer exact sequence 5.3.2.1.2, by using the isomorphism 5.3.2.1.1, we get the exact sequence of left $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules:

$$0\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{B}}^{(0)}_X}\wedge^d\widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}\cdots\underset{\delta}{\longrightarrow}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{B}}^{(0)}_X}\widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}\underset{\delta}{\longrightarrow}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp\to Y^\sharp/S^\sharp}\to 0. \tag{5.3.2.1.3}$$

In other words, the morphism $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{B}}^{(0)}_X}\wedge^\bullet\widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}\to\widetilde{\mathcal{D}}^{(0)}_{X^\sharp\to Y^\sharp/S^\sharp}$ is a quasi-isomorphism (where $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{B}}^{(0)}_X}\wedge^0\widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}=\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ is at the 0th place.

5.3.2.2. We suppose f is log-smooth and the bottom arrow of 5.1.1.1.1 is the identity. By applying the functor $\mathscr{F}:=\widetilde{\omega}_{X^\sharp/S^\sharp}\otimes\widetilde{f}^*(-\otimes\widetilde{\omega}^{-1}_{Y^\sharp/S^\sharp})$ to the canonical ring homomorphism $\mathcal{B}_Y\to\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$, we get the canonical map of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}$-modules $\widetilde{\omega}_{X^\sharp/Y^\sharp}\to\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp}$ (indeed the induced structure via $\mathscr{F}$ of the right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}$-module of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp}$ is given by the right structure of $\mathcal{B}_Y$-module of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$). This yields the morphism

$$\widetilde{\omega}_{X^\sharp/Y^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\to\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp}. \tag{5.3.2.2.1}$$

Via a local computation (recall 4.3.5.4), it follows from 5.3.2.1.1 that the map 5.3.2.2.1 is an isomorphism. Following 4.7.3.14, we have the exact sequence

$$0 \to \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp} \underset{d}{\longrightarrow} \widetilde{\Omega}^1_{X^\sharp/Y^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp} \underset{d}{\longrightarrow} \cdots \underset{d}{\longrightarrow} \widetilde{\omega}_{X^\sharp/Y^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp} \to \widetilde{\omega}_{X^\sharp/Y^\sharp} \to 0 \qquad (5.3.2.2.2)$$

given by the de Rham complex of $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}$ and for the last arrow by the right structure of $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}$ on $\widetilde{\omega}_{X^\sharp/Y^\sharp}$. By applying the exact functor $-\otimes_{\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/Y^\sharp}} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$ to the exact sequence 5.3.2.2.2, by using the isomorphism 5.3.2.2.1, we get the exact sequence of right $\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}$-modules:

$$0 \to \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \underset{d}{\longrightarrow} \widetilde{\Omega}^1_{X^\sharp/Y^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \underset{d}{\longrightarrow} \cdots \underset{d}{\longrightarrow} \widetilde{\omega}_{X^\sharp/Y^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp} \to 0 \quad (5.3.2.2.3)$$

In other words, the map $\widetilde{\Omega}^\bullet_{X^\sharp/Y^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp}[d_f] \to \widetilde{\mathcal{D}}^{(0)}_{Y^\sharp\leftarrow X^\sharp/S^\sharp}$ is a quasi-isomorphism.

Proposition 5.3.2.3. *Assume we are in the non-respective case of 5.1.1.1, T is a noetherian scheme of finite Krull dimension and f is a log-smooth, quasi-compact and quasi-separated morphism and the bottom arrow of 5.1.1.1.1 is the identity. Then we have the following isomorphisms:*

1. *For any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$, we have the isomorphism*

$$f^{(0)}_{+}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_*\left(\widetilde{\Omega}^\bullet_{X^\sharp/Y^\sharp} \otimes_{\mathcal{B}_X} \mathcal{E}\right)[d_f]; \qquad (5.3.2.3.1)$$

2. *For any $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(0)}_{X^\sharp/S^\sharp})$, we have the isomorphism*

$$f^{(0)}_{+}(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}f_*\left(\mathcal{M} \otimes_{\mathcal{B}_X} \wedge^\bullet \widetilde{\mathcal{T}}_{X^\sharp/Y^\sharp}\right). \qquad (5.3.2.3.2)$$

Proof. The isomorphism 5.3.2.3.1 (resp. 5.3.2.3.2) is a consequence of 5.3.2.2.3 (resp. 5.3.2.1.3). □

Proposition 5.3.2.4. *Suppose one of the following conditions holds:*

(i) either $m = 0$,

(ii) or we are the non-respective case of 5.1.1.1 and log-structures are trivial.

Then the left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$ has finite tor-dimension and the right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp}$ has finite tor-dimension.

Proof. I) Let us prove the case $m = 0$. a) Suppose the diagram 5.1.1.1.1 is cartesian and the morphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism, i.e. suppose we are in the base change case. Following 5.1.1.15, the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$ is an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules and moreover there exists a canonical isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$-bimodules of the form $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp\leftarrow X^\sharp/S^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$. Hence we get done.

b) By using the splitting of 7.5.6.7.1 and 5.1.1.12.1, by using the part a) of the proof, we reduce to the case where the bottom morphism of 5.1.1.1.1 is the identity. When f is log-smooth, we can check the first (resp. second) statement is a consequence of 5.3.2.1.3 (resp. 5.3.2.2.3). By decomposing f by its graph $\gamma_f\colon X^\sharp \hookrightarrow X^\sharp \times_{S^\sharp} Y^\sharp$, which is a closed immersion, followed by the log-smooth projection $X^\sharp \times_{S^\sharp} Y^\sharp \to Y^\sharp$ this yields the first statement thanks to 5.2.3.1.

II) Let us prove the case (ii). Since p is nilpotent and log structures are trivial, we are under conditions 6.1. Hence, by Frobenius descent (see 6.1.3.6.b), this is a consequence of the case where $m = 0$. □

The following corollary completes 5.1.3.5.

Corollary 5.3.2.5. *Suppose we are the non-respective case of 5.1.1.1. Suppose one of the following conditions holds:*

(i) either $m = 0$,

(ii) or log-structures are trivial.

Then, the functor $\widetilde{f}_+^{(m)}$ is way out in both direction and for any $* \in \{\mathrm{r}, \mathrm{l}\}$, we have the factorization

$$\widetilde{f}_+^{(m)} \colon D^{\mathrm{b}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D^{\mathrm{b}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}).$$

Proof. The factorization is a consequence of Proposition 5.3.2.4. □

Corollary 5.3.2.6. *Let $* \in \{\mathrm{r}, \mathrm{l}\}$. Suppose one of the following conditions holds:*

(i) either $m = 0$,

(ii) or we are the non-respective case of 5.1.1.1 and log-structures are trivial.

Then, we have the factorization $\widetilde{f}^{(m)!} \colon D_{\mathrm{tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \to D_{\mathrm{tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

Proof. By twisting if necessary, we can suppose $* = \mathrm{l}$. Let $\mathcal{F} \in D_{\mathrm{tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$. In both cases, the left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}$ has finite tor-dimension (see 5.3.2.4). Hence, there exists a bounded resolution $\mathcal{P}$ of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp}$ by $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$-bimodules which are moreover flat as left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules (see the remark 4.6.3.4). Let $\mathcal{Q}$ be a bounded complex of flat left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules representing $\mathcal{F}$. Hence,

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{P} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}\mathcal{Q},$$

which is a bounded complex of flat left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules. □

Proposition 5.3.2.7. *Suppose f is log-smooth, the bottom arrow of 5.1.1.1.1 is the identity and both conditions of 4.1.2.17.(e) are satisfied for $\mathcal{B}_Y$ and $\mathcal{B}_X$. Let $* \in \{\mathrm{r}, \mathrm{l}\}$.*

(a) For any $\mathcal{E} \in D^-_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$, we have $\widetilde{f}^{(m)!}(\mathcal{E}) \in D^-_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$.

(b) Suppose either $m = 0$, or we are the non-respective case of 5.1.1.1 and log-structures are trivial. We have the factorization

$$\widetilde{f}^{(m)!} \colon D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}) \to D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}).$$

Proof. Let us check the first statement. Since this is local on $X^\sharp$, using locally free resolution, we reduce to the case $\mathcal{E} = \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$. Since the map $\alpha \colon \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp} \to \mathcal{B}_X$ (given by the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}$-module of $\mathcal{B}_X$) is surjective, since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ is flat, then it follows form 5.3.2.1.1 the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ is surjective and its kernel is equal to $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}} \ker \alpha$. Remark, when $X^\sharp/Y^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$, then $\ker \alpha$ is the left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/Y^\sharp}$-submodule generated $\partial_{\sharp 1}, \dots, \partial_{\sharp r}$. In particular, the kernel of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \to \widetilde{f}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}$ is a left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module of finite type. Following [Sta22, 08G8], a complex is perfect if and only if it is pseudo-coherent and locally has finite tor dimension. Hence, the second statement is a consequence of the first one and of the propositions 4.1.2.17.(e) and 5.3.2.6. □

Proposition 5.3.2.8. *Suppose we are the non-respective case of 5.1.1.1. Let $\mathcal{C}_Y$ be a $\mathcal{B}_Y$-algebra endowed with a compatible structure of $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-module such that $\mathcal{B}_Y \to \mathcal{C}_Y$ is $\mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$-linear. Let $\mathcal{F} \in D(\mathcal{B}_Y)$ (resp. $\mathcal{F} \in D(\mathcal{C}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}, \mathcal{B}_Y)$), and $\mathcal{G} \in D(\mathcal{B}_X)$.*

(i) We have the canonical morphism of $D(\mathcal{B}_Y)$ (resp. of $D(\mathcal{C}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{B}_Y)$):

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*(\mathcal{G}) \to \mathbb{R}f_*\left(\mathbb{L}\widetilde{f}^*(\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{G}\right). \tag{5.3.2.8.1}$$

(ii) Suppose f is quasi-compact and quasi-separated. Suppose one of the following conditions:

(a) *either* $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{B}_Y)$*, and* $\mathcal{G} \in D(\mathcal{B}_X)$*,*

(b) *or* S *and* T *are noetherian schemes of finite Krull dimension and* $\mathcal{F} \in D^{-}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{B}_Y)$*, and* $\mathcal{G} \in D^{-}(\mathcal{B}_X)$*.*

Then the morphism 5.3.2.8.1 is an isomorphism.

Proof. Choosing a K-flat complex $\mathcal{P}$ of $K(\mathcal{B}_Y)$ (resp. of $K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{B}_Y)$ representing $\mathcal{F}$, a K-injective complex $\mathcal{I}$ of $K(\mathcal{B}_X)$ representing $\mathcal{G}$, we get

$$\begin{aligned}\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*(\mathcal{G}) &\xrightarrow{\sim} \mathcal{P} \otimes_{\mathcal{B}_Y} f_*(\mathcal{I}) \to f_*\left(\widetilde{f}^*\mathcal{P} \otimes_{\mathcal{B}_X} \mathcal{I}\right)\\ &\to \mathbb{R}f_*\left(\widetilde{f}^*\mathcal{P} \otimes_{\mathcal{B}_X} \mathcal{I}\right) \xrightarrow{\sim} \mathbb{R}f_*\left(\mathbb{L}f^*\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{G}\right), \qquad (5.3.2.8.2)\end{aligned}$$

the second morphism is built by adjointness from the morphism of ringed spaces $\widetilde{f} \colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$, the last isomorphism coming from the fact that $f^*\mathcal{P}$ a K-flat complex of $K(\mathcal{B}_X)$ representing $\mathbb{L}\widetilde{f}^*\mathcal{F}$. In the respective case, since we have by functoriality $f^*\mathcal{P} \otimes_{\mathcal{B}_X} \mathcal{I} \in K(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{B}_X)$ which yields $f_*\left(\widetilde{f}^*\mathcal{P} \otimes_{\mathcal{B}_X} \mathcal{I}\right) \in K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{B}_Y)$, then the second arrow do is a morphism of $K(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{B}_Y)$.

To check that this is an isomorphism, using the remark 5.1.2.1 and using [Har66, I.7.1 (ii), (iii) and (iv)] and 5.1.2.4, we reduce to the case where $\mathcal{F} = \mathcal{B}_Y$, which is obvious. □

5.3.2.9. Suppose we are the non-respective case of 5.1.1.1, $\mathcal{B}_Y$ is quasi-coherent and the morphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism. Then $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}$ is endowed with a canonical structure of $(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}, \mathcal{B}_Y)$-bimodule (induced by its structure of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-bimodule) such that both underlying $\mathcal{O}_Y$-module structure are quasi-coherent. Since $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism, then $\widetilde{f}^*_{\mathrm{r}}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right) \xrightarrow{\sim} f^*_{\mathrm{r}}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right)$. Let $\widetilde{\mathcal{G}}$ be a $\mathcal{B}_X$-module. Then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{G}}$ is endowed with a canonical structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \mathcal{B}_X)$-bimodule. We get the isomorphisms of $D(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}, \mathcal{B}_Y)$:

$$\begin{aligned}\widetilde{f}^{(m)}_{+}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{G}}) &\xrightarrow{\sim} \mathbb{R}f_*\left(\left(\widetilde{f}^*_{\mathrm{r}}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right) \otimes_{\mathcal{B}_X} \widetilde{\omega}_{X^\sharp/S^\sharp}\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{G}})\right)\\ &\xrightarrow{\sim} \mathbb{R}f_*\left(\widetilde{f}^*_{\mathrm{r}}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}\right) \otimes_{\mathcal{B}_X} \widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{G}}\right)\\ &\underset{5.3.2.8.1}{\xleftarrow{\sim}} (\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*\left(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{G}}\right)\\ &\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \left(\widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*\left(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\mathcal{G}}\right)\right). \qquad (5.3.2.9.1)\end{aligned}$$

Lemma 5.3.2.10. *Suppose we are the non-respective case of 5.1.1.1. Suppose* f *is proper,* S *and* T *are noetherian schemes of finite Krull dimension and* $f^*\mathcal{B}_Y \to \mathcal{B}_X$ *is an isomorphism. Suppose* $\mathcal{B}_Y$ *is a quasi-coherent* $\mathcal{O}_Y$*-module, is an* $\mathcal{O}_Y$*-algebra of finite type and the conditions 4.1.2.17.(e) are satisfied for* $\mathcal{B}_Y$*. For any* $\star \in \{\mathrm{b}, +, -, \emptyset\}$*, the functor* $\mathbb{R}f_*$ *sends* $D^{\star}_{\mathrm{coh}}(\mathcal{B}_X)$ *to* $D^{\star}_{\mathrm{coh}}(\mathcal{B}_Y)$*.*

Proof. Recall that following 1.4.5.2, $\mathcal{B}_Y$ is a coherent sheaf of rings. We can suppose $\mathcal{B}_X = f^*\mathcal{B}_Y$. Let $\mathcal{E}$ be a coherent $\mathcal{B}_X$-module. Since $\mathbb{R}f_*$ is way-out, then by using [Har66, I.7.3.(iii)], we reduce to check $\mathbb{R}f_*(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{B}_Y)$. Since this is Zariski local in Y, we can suppose Y is affine. Let $\widetilde{Y} := \mathrm{Spec}(\Gamma(Y, \mathcal{B}_Y))$, $\widetilde{X} := \widetilde{Y} \times_Y X$, $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ and $\varpi \colon \widetilde{X} \to X$ be the projections. In other words, $\widetilde{X} = \mathrm{Spec}_X(\mathcal{B}_X)$ is the relative spectrum of $\mathcal{B}_X$ over X and ϖ is the natural affine projection. Since f is proper, so is $\widetilde{f}$. Since $\widetilde{Y}$ and Y are noetherian, then so are $\widetilde{X}$ and X. Hence, a $\mathcal{B}_X$-module (resp. $\mathcal{B}_{\widetilde{X}}$-module) is coherent if and only if it is both quasi-coherent and of finite type. Following [Gro61, 1.4.5], there exists therefore a coherent $\mathcal{O}_{\widetilde{X}}$-module $\widetilde{\mathcal{E}}$ such that $\varpi_*(\widetilde{\mathcal{E}}) = \mathcal{E}$. Since $\widetilde{f}$ is proper, then $\mathbb{R}\widetilde{f}_*(\widetilde{\mathcal{E}}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{B}_{\widetilde{Y}})$. This yields $\Gamma(\widetilde{Y}, \mathbb{R}\widetilde{f}_*(\widetilde{\mathcal{E}}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\Gamma(Y, \mathcal{B}_Y))$.

Since we already know $\mathbb{R}f_*(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{B}_Y)$, since Y is affine and the conditions 4.1.2.17.(e) are satisfied for $\mathcal{B}_Y$, then $\mathbb{R}f_*(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{B}_Y)$ if and only if $\Gamma(Y, \mathbb{R}f_*(\mathcal{E})) \in D^{\mathrm{b}}_{\mathrm{coh}}(\Gamma(Y, \mathcal{B}_Y))$. Since $\mathbb{R}f_*(\mathcal{E}) \xrightarrow{\sim} \Gamma(\widetilde{Y}, \mathbb{R}\widetilde{f}_*(\widetilde{\mathcal{E}})$, then we are done. □

Proposition 5.3.2.11. *We keep notation and hypotheses of 5.3.2.10. Suppose moreover one of the following conditions hold:*

(a) either $\star = -$,

(b) or $m = 0$,

(c) or log structures are trivial.

The functor $\widetilde{f}_{+}^{(m)}$ sends $D^{\star}_{\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ to $D^{\star}_{\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, where $* \in \{\mathrm{r}, \mathrm{l}\}$.

Proof. Following 5.1.3.5 and 5.3.2.5, $\widetilde{f}_{+}^{(m)}$ is way out left (or in both direction in the second and third case). Following [Har66, I.7.3] in order to check the coherence of $\widetilde{f}_{+}^{(m)}(\mathcal{F})$ for $\mathcal{F} \in D^{\star}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ we can suppose that $\mathcal{F}$ is a coherent left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. Since $\mathcal{F}$ is the inductive limit of its coherent $\mathcal{O}_X$-submodules (see [Gro60, 9.4.9]), there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{G}$ of $\mathcal{F}$ such that the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear map $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{G} \twoheadrightarrow \mathcal{F}$ is surjective. Hence, by using [Har66, I.7.3.(iv)] (in fact, a left way-out version), we reduce to the case where $\mathcal{F}$ is of the form $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{G}$ where $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module.

Since f is proper and S is locally noetherian, then the functor $\mathbb{R}f_*$ preserves the $\mathcal{O}$-coherence, hence If follows from 5.3.2.10, that $\widetilde{\omega}^{-1}_{Y^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_*(\widetilde{\omega}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} (\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{G})) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{B}_Y)$. By using the isomorphism 5.3.2.9.1 (applied to the case $\widetilde{\mathcal{G}} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{G}$, we can conclude. □

Proposition 5.3.2.12. *Suppose we are the non-respective case of 5.1.1.1, $\mathcal{B}_Y$ is quasi-coherent, $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism and the bottom arrow of 5.1.1.1.1 is the identity. Suppose $f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ has finite tor dimension, S is a noetherian scheme of finite Krull dimension, f is quasi-compact and quasi-separated. The functor $\widetilde{f}_{+}^{(m)}$ sends $D_{\mathrm{tdf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ (resp. $D_{\mathrm{qc,tdf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$) to $D_{\mathrm{tdf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$ (resp. $D_{\mathrm{qc,tdf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$), where $* \in \{\mathrm{r}, \mathrm{l}\}$.*

Proof. Let $\mathcal{M} \in D_{\mathrm{tdf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathcal{F}$ be a quasi-coherent left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module. It follows from 5.1.2.6.1 that we have the canonical isomorphism

$$\mathbb{R}f_*\left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/S^\sharp}\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \mathcal{F} \xrightarrow{\sim} \mathbb{R}f_*\left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{F}\right).$$

It follows from 5.1.1.10 that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp \to Y^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} f^{-1}\mathcal{F} = \widetilde{f}^{(m)!}(\mathcal{F})[d_f]$ is bounded. Since $\mathcal{M}$ has finite tor-dimension, since $\widetilde{f}_{+}^{(m)}(\mathcal{M}) = \mathbb{R}f_*\left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp \to Y^\sharp/S^\sharp}\right)$, then this yields that $\widetilde{f}_{+}^{(m)}(\mathcal{M}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}} \mathcal{F}$ is bounded. We conclude by using 1.4.3.3. □

Proposition 5.3.2.13. *Suppose we are the non-respective case of 5.1.1.1, $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism and the bottom arrow of 5.1.1.1.1 is the identity. Suppose $\mathcal{B}_Y$ is a quasi-coherent $\mathcal{O}_Y$-module, is an $\mathcal{O}_Y$-algebra of finite type and the conditions 4.1.2.17.(e) are satisfied for $\mathcal{B}_Y$. Suppose $f^{-1}\mathcal{B}_Y \to \mathcal{B}_X$ has finite tor dimension, S is a noetherian scheme of finite Krull dimension, f is proper. The functor $\widetilde{f}_{+}^{(m)}$ sends $D_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ to $D_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp})$, where $* \in \{\mathrm{r}, \mathrm{l}\}$.*

Proof. Following [Sta22, 08G8], a complex is perfect if and only if it is pseudo-coherent and locally has finite tor dimension. Hence, this is a consequence of 5.3.2.11 and 5.3.2.12. □

Notation 5.3.2.14 (Varying m notation)**.** We keep notation and hypotheses 5.1.1.1. Fix $m \geq m' \geq 0$ a second integer. Let $\mathcal{B}'_X$ (resp. $\mathcal{B}'_Y$) be a commutative $\mathcal{O}_X$-algebra (resp. $\mathcal{O}_Y$-algebra) endowed with a compatible structure of left $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$-module (resp. left $\mathcal{D}^{(m')}_{Y^\sharp/T^\sharp}$-module) and satisfying the hypotheses of 7.3.2. We suppose that we have algebras morphisms $\mathcal{B}'_X \to \mathcal{B}_X$,$\mathcal{B}'_Y \to \mathcal{B}_Y$, $f^*\mathcal{B}'_Y \to \mathcal{B}'_X$ which are respectively $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}$-linear (resp. $\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp}$-linear, resp. $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear) and inducing the commutative diagram

$$\begin{array}{ccc} f^*\mathcal{B}_Y & \longrightarrow & \mathcal{B}_X \\ \uparrow & & \uparrow \\ f^*\mathcal{B}'_Y & \longrightarrow & \mathcal{B}'_X. \end{array}$$

We denote by $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp} := \mathcal{B}'_X \widehat{\otimes}_{\mathcal{O}_X} \mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp} := \mathcal{B}'_Y \widehat{\otimes}_{\mathcal{O}_Y} \mathcal{D}^{(m')}_{Y^\sharp/T^\sharp}$. We keep similar to 5.1.1.2 notation by replacing m by m' and $\mathcal{B}$ by $\mathcal{B}'$.

Proposition 5.3.2.15. *With notation 5.3.2.14, suppose we are the non-respective case of 5.1.1.1, S and T are noetherian schemes of finite Krull dimension, f is quasi-compact and quasi-separated, $\mathcal{B}'_Y$ is quasi-coherent and $f^*\mathcal{B}'_Y \to \mathcal{B}'_X$ is an isomorphism, $\mathcal{B}_Y$ is quasi-coherent and $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism. Let $\star \in \{+, -, \emptyset\}$, $* \in \{\mathrm{r}, \mathrm{l}\}$ and $\mathcal{F} \in D^\star_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp})$. Suppose moreover one of the following conditions holds:*

(a) either $\star = -$,

(b) or log structures are trivial.

Suppose moreover either $\mathcal{B}'_X \to \mathcal{B}_X$ is an isomorphism or $f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_X$ are tor independent over $f^{-1}\mathcal{B}_Y$.

Then we have in $D^\star_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp})$ the canonical isomorphism

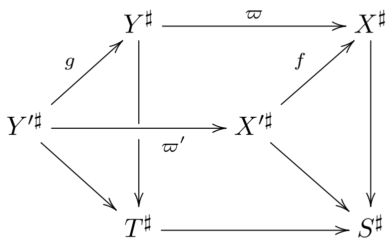
$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}} \mathcal{F}\right). \tag{5.3.2.15.1}$$

Proof. We can suppose $* = \mathrm{l}$. It follows from 5.1.3.4 that we can suppose $\mathcal{B}'_X \to \mathcal{B}_X$ is an isomorphism. By using the canonical morphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp}, \widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp})$-bimodules $\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp \leftarrow X^\sharp/S^\sharp}$ and the $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}$-linear morphism $\mathcal{F} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}} \mathcal{F}$, we get the $\widetilde{\mathcal{D}}^{(m')}_{Y^\sharp/T^\sharp}$-linear morphism $\widetilde{f}^{(m')}_{+}(\mathcal{F}) \to \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}} \mathcal{F}\right)$, which yields 5.3.2.15.1 by extension. To check that this is an isomorphism, since $\widetilde{f}^{(m')}_{+}$ is way out left (resp. in both direction by using 3.2.4.3), following [Har66, I.7.1.(iii)], we can suppose that $\mathcal{F}$ is a quasi-coherent left $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp}$-module. Since the canonical morphism $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ is surjective, by using [Har66, I.7.3.(iv)] (in fact, a left way-out version), we reduce to the case where $\mathcal{F}$ is of the form $\widetilde{\mathcal{D}}^{(m')}_{X^\sharp/S^\sharp} \otimes_{\mathcal{O}_X} \mathcal{G}$ where $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module. We conclude via the isomorphism 5.3.2.9.1. □

Remark 5.3.2.16. When $m' = m$, we get 5.1.3.4.

5.3.3 Commutation with base change

Consider the commutative diagram



(5.3.3.0.1)

where the squares are cartesian, where $S^\sharp$ and $T^\sharp$ are Noetherian nice fine log scheme over $\mathrm{Spec}(\mathbb{Z}/p^{i+1}\mathbb{Z})$ (see definition 3.1.1.1), where $X^\sharp$ and $Y^\sharp$ are log smooth $S^\sharp$-log-scheme, where morphisms are morphisms of log schemes whose underlying morphism of schemes are quasi-compact and quasi-separated. Let $\mathcal{B}_X$ (resp. $\mathcal{B}_{X'}$) be a commutative quasi-coherent $\mathcal{O}_X$-algebra (resp. $\mathcal{O}_{X'}$-algebra) endowed with of a compatible structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module (resp. $\mathcal{D}^{(m)}_{X'^\sharp/S^\sharp}$-module). We suppose finally that we have a morphism of algebras $f^*\mathcal{B}_X \to \mathcal{B}_{X'}$ which is moreover $\mathcal{D}^{(m)}_{X'^\sharp/S^\sharp}$-linear. We will again denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/S^\sharp}$. We denote by $\widetilde{X}^\sharp$ (resp. $\widetilde{X'}^\sharp$) the ringed logarithmic ($\mathcal{V}$-formal) scheme $(X^\sharp, \mathcal{B}_X)$ (resp. $(X'^\sharp, \mathcal{B}_{X'})$), and by $\tilde{f} \colon \widetilde{X'}^\sharp/S^\sharp \to \widetilde{X}^\sharp/S^\sharp$ the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by the diagram 5.3.3.0.1 and by $f^*\mathcal{B}_X \to \mathcal{B}_{X'}$. We suppose $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3).

Set $\mathcal{B}_Y := \varpi^*\mathcal{B}_X$ and $\mathcal{B}_{Y'} := \varpi'^*\mathcal{B}_{X'}$, $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp} = \mathcal{B}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y^\sharp/T^\sharp}$. We denote by $\widetilde{Y}^\sharp$ (resp. $\widetilde{Y}'^\sharp$) the ringed logarithmic ($\mathcal{V}$-formal) scheme $(Y^\sharp, \mathcal{B}_Y)$ (resp. $(Y'^\sharp, \mathcal{B}_{Y'})$), and by $\tilde{g}\colon \widetilde{Y}'^\sharp/T^\sharp \to \widetilde{Y}^\sharp/T^\sharp$ the morphism of relative ringed logarithmic ($\mathcal{V}$-formal) schemes induced by the diagram 5.3.3.0.1 and by $g^*\mathcal{B}_Y \to \mathcal{B}_{Y'}$. We get moreover the morphisms $\widetilde{\varpi}\colon \widetilde{Y}^\sharp/T^\sharp \to \widetilde{X}^\sharp/S^\sharp$ and $\widetilde{\varpi}'\colon \widetilde{Y}'^\sharp/T^\sharp \to \widetilde{X}'^\sharp/S^\sharp$. Since the squares of the diagram 5.3.3.0.1 are cartesian, then the functors $\widetilde{\varpi}^{(m)!}$ and $\widetilde{\varpi}'^{(m)!}$ are the base change inverse image (see 5.1.1.15).

Proposition 5.3.3.1. Let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X^\sharp/S^\sharp})$. There exists a canonical isomorphism in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T'^\sharp})$ of the form:

$$\widetilde{\varpi}'^{(m)!} \circ \widetilde{f}^{(m)!}(\mathcal{E}) \xrightarrow{\sim} \widetilde{g}^{(m)!} \circ \widetilde{\varpi}^{(m)!}(\mathcal{E}). \tag{5.3.3.1.1}$$

Proof. Following 5.1.1.13, they both are canonically isomorphic to $\widetilde{h}^{(m)!}(\mathcal{E})$ where $\widetilde{h}\colon (Y', \mathcal{B}_{Y'})/T^\sharp \to (X, \mathcal{B}_X)/S^\sharp$ is the morphism induced by composition. □

Proposition 5.3.3.2. Tensor products commutes with base change, i.e. we have the canonical isomorphism in $D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ for any $\mathcal{M} \in D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\widetilde{\varpi}^{(m)!}(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathcal{E}) \xrightarrow{\sim} \widetilde{\varpi}^{(m)!}(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \widetilde{\varpi}^{(m)!}(\mathcal{E}). \tag{5.3.3.2.1}$$

Proof. Obvious. □

Theorem 5.3.3.3. *Let $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp})$. There exists a canonical homorphism in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ of the form:*

$$\widetilde{\varpi}^{(m)!} \circ \widetilde{f}^{(m)}_+(\mathcal{E}') \to \widetilde{g}^{(m)}_+ \circ \widetilde{\varpi}'^{(m)!}(\mathcal{E}'). \tag{5.3.3.3.1}$$

This morphism is an isomorphism in the following cases:

(a) either $\mathcal{E}' \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp})$;

(b) or $m = 0$, $f^{-1}\mathcal{B}_X \to \mathcal{B}_Y$ and $f^{-1}\mathcal{B}_{X'} \to \mathcal{B}_{Y'}$ have finite tor dimension and $\mathcal{E}' \in D_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp})$;

(c) or the log structures are trivial, p is nilpotent, $f^{-1}\mathcal{B}_X \to \mathcal{B}_Y$ and $f^{-1}\mathcal{B}_{X'} \to \mathcal{B}_{Y'}$ have finite tor dimension and $\mathcal{E}' \in D_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp})$.

Proof. a) By computing in local coordinates, we can check the canonical isomorphisms of $\mathcal{B}_Y$-modules (resp. of $\mathcal{B}_{Y'}$-modules) $\widetilde{\varpi}^!(\widetilde{\omega}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\omega}_{Y^\sharp/T^\sharp}$ (resp. $\widetilde{\varpi}'^!(\widetilde{\omega}_{X'^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\omega}_{Y'^\sharp/T^\sharp}$) is $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-linear (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}$-linear). Hence, we get the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}$-modules:

$$\begin{aligned} \widetilde{\varpi}'^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}) &\xrightarrow{\sim} \widetilde{\varpi}'^*(\widetilde{\omega}_{X'^\sharp/S^\sharp} \otimes_{\mathcal{B}_{X'}} \widetilde{f}^*_{\mathrm{r}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp})) \\ \xrightarrow{\sim} \widetilde{\omega}_{Y'^\sharp/T^\sharp} \otimes_{\mathcal{B}_{Y'}} \widetilde{\varpi}'^* \widetilde{f}^*_{\mathrm{r}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}) &\xrightarrow{\sim} \widetilde{\omega}_{Y'^\sharp/T^\sharp} \otimes_{\mathcal{B}_{Y'}} \widetilde{g}^*_{\mathrm{r}} \widetilde{\varpi}^*_{\mathrm{r}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}) \\ &\xrightarrow{\sim} \widetilde{\omega}_{Y'^\sharp/T^\sharp} \otimes_{\mathcal{B}_{Y'}} g^*_{\mathrm{r}}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp})) = \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}. \end{aligned} \tag{5.3.3.3.2}$$

In fact, since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}$ is a $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S})$-bimodule, we check by functoriality that the homomorphisms of 5.3.3.3.2 are homomorphisms of $((\varpi \circ g)^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/S})$-bimodules. Let $\mathcal{P}'$ be a K-flat complex representing $\mathcal{E}'$. Hence, we get the morphism of $D((\varpi \circ g)^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\varpi'^{-1}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}} \mathcal{P}') \to \widetilde{\varpi}'^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}} \widetilde{\varpi}'^!(\mathcal{P}') \underset{5.3.3.3.2}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}} \widetilde{\varpi}'^!(\mathcal{P}'). \tag{5.3.3.3.3}$$

b) We have the adjoint morphism (or called base change morphism which is constructed by adjointness: see [Sta22, 20.28.3]) adj: $\varpi^{-1}\mathbb{R}f_* \to \mathbb{R}g_*\varpi'^{-1}$ of functors $D(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \to D(\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$. Hence, we get the morphism of $D(\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$:

$$\varpi^{-1}\circ \mathbb{R}f_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}}\mathcal{P}')\xrightarrow{\text{adj}}\mathbb{R}g_*\circ\varpi'^{-1}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}}\mathcal{P}')$$
$$\underset{5.3.3.3.3}{\longrightarrow}\mathbb{R}g_*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}}\widetilde{\varpi}'^!(\mathcal{P}')). \tag{5.3.3.3.4}$$

Via the extension $\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\to\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$, this yields the morphism of $D(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$

$$\widetilde{\varpi}^!\circ\widetilde{f}^{(m)}_+(\mathcal{E}')=\widetilde{\varpi}^!\circ\mathbb{R}f_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}}\mathcal{E}')\to\mathbb{R}g_*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}}\widetilde{\varpi}'^!(\mathcal{E}'))=\widetilde{g}^{(m)}_+\circ\widetilde{\varpi}'^!(\mathcal{E}'). \tag{5.3.3.3.5}$$

It remains to check that this morphism is an isomorphism. In the first case (resp. in the other two cases), since the functors $\widetilde{\varpi}^!\circ\widetilde{f}^{(m)}_+$ and $\widetilde{g}^{(m)}_+\circ\varpi'^!$ are way out left (resp. way out in both directions: see 5.1.1.10 and 5.3.2.5), by using (the way out left version of) Proposition [Har66, I.7.1], we reduce to the case where $\mathcal{E}'$ is a quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}$-module. By using [Har66, I.7.1.(iv)], we reduce to the case where $\mathcal{E}'$ is of the form $\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}'$, where $\mathcal{F}'$ is a quasi-coherent $\mathcal{B}_{X'}$-module. The morphism 5.3.3.3.4 is canonically isomorphism to the composite of the top arrow of the following diagram:

$$\begin{array}{ccccc}
\varpi^{-1}\mathbb{R}f_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}}\mathcal{E}') & \xrightarrow{\text{adj}} & \mathbb{R}g_*\varpi'^{-1}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'^\sharp/S^\sharp}}\mathcal{E}') & \to & \mathbb{R}g_*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y'^\sharp/T^\sharp}}\widetilde{\varpi}'^!(\mathcal{E}')) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\varpi^{-1}\circ\mathbb{R}f_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}') & \xrightarrow{\text{adj}} & \mathbb{R}g_*\circ\varpi'^{-1}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}') & \longrightarrow & \mathbb{R}g_*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}\otimes_{\mathcal{B}_{Y'}}\widetilde{\varpi}'^!(\mathcal{F}')) \\
\downarrow & & \downarrow & & \| \\
\widetilde{\varpi}^!\circ\mathbb{R}f_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}') & \underset{\sim}{\xrightarrow{\text{adj}}} & \mathbb{R}g_*\circ\widetilde{\varpi}'^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}') & \xrightarrow{\sim} & \mathbb{R}g_*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}\otimes_{\mathcal{B}_{Y'}}\widetilde{\varpi}'^!(\mathcal{F}')),
\end{array} \tag{5.3.3.3.6}$$

where, by identifying $\widetilde{\varpi}^!$ with the functor ϖ^*, the adjoint isomorphism of the bottom row can be viewed at the one constructed in the categories of $\mathcal{O}$-modules (this latter is indeed an isomorphism because Y and X' are tor-independent over X: see [Sta22, 36.22.5]). Remark that the left bottom square is indeed commutative by transitivity of the base change morphism (see [Sta22, 20.28.4]). This yields the commutative diagram:

$$\begin{array}{ccc}
\widetilde{\varpi}^!\circ\widetilde{f}^{(m)}_+(\mathcal{E}') & \longrightarrow & \widetilde{g}^{(m)}_+\circ\widetilde{\varpi}'^!(\mathcal{E}') \\
\downarrow\wr & & \downarrow\wr \\
\widetilde{\varpi}^!\circ\mathbb{R}f_*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}')\underset{\sim}{\xrightarrow{\text{adj}}}\mathbb{R}g_*\circ\widetilde{\varpi}'^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp\leftarrow X'^\sharp/S^\sharp}\otimes_{\mathcal{B}_{X'}}\mathcal{F}') & \xrightarrow{\sim} & \mathbb{R}g_*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow Y'^\sharp/T^\sharp}\otimes_{\mathcal{B}_{U'}}\widetilde{\varpi}'^!(\mathcal{F}')).
\end{array} \tag{5.3.3.3.7}$$

□

Proposition 5.3.3.4. Let $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp,\mathbb{Q}})$. We have the canonical isomorphism

$$\widetilde{\varpi}^!(\mathbb{D}_{X^\sharp/S^\sharp}(\mathcal{E}))\xrightarrow{\sim}\mathbb{D}_{Y^\sharp/T^\sharp}(\widetilde{\varpi}^!(\mathcal{E})). \tag{5.3.3.4.1}$$

Proof. Since $\mathcal{E}$ is a perfect complex, then we have the last canonical isomorphism

$$\widetilde{\varpi}^!(\mathbb{D}_{X^\sharp/S^\sharp}(\mathcal{E}))=\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\otimes^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\varpi^{-1}\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\mathcal{E},\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\omega}^{-1}_{X^\sharp/S^\sharp})[\delta^{S^\sharp}_{X^\sharp}] \tag{5.3.3.4.2}$$

$$\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/S^\sharp}\otimes^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}\mathbb{R}\mathcal{H}om_{\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\varpi^{-1}\mathcal{E},\varpi^{-1}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}))[\delta^{S^\sharp}_{X^\sharp}] \tag{5.3.3.4.3}$$

$$\underset{4.6.3.6.1}{\xrightarrow{\sim}}\mathbb{R}\mathcal{H}om_{\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\varpi^{-1}\mathcal{E},\widetilde{\varpi}^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}\otimes_{\mathcal{B}_X}\widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}))[\delta^{S^\sharp}_{X^\sharp}]. \tag{5.3.3.4.4}$$

Using **??**, since $\widetilde{\varpi}^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) = \varpi^*(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$ and $\widetilde{\varpi}^!(\widetilde{\omega}^{(m)}_{X^\sharp/S^\sharp}) = \varpi^*(\widetilde{\omega}^{(m)}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\omega}^{(m)}_{Y^\sharp/T^\sharp}$, then we get the isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}$-bimodules

$$\widetilde{\varpi}^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}.$$

Using **??**, this yields the first isomorphism

$$\mathbb{R}\mathcal{H}om_{\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\varpi^{-1}\mathcal{E}, \widetilde{\varpi}^!(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}))[\delta^{S^\sharp}_{X^\sharp}] \quad (5.3.3.4.5)$$

$$\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\varpi^{-1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}}(\varpi^{-1}\mathcal{E}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}))[\delta^{T^\sharp}_{Y^\sharp}] \quad (5.3.3.4.6)$$

$$\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}}(\widetilde{\varpi}^!\mathcal{E}, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \widetilde{\omega}^{-1}_{Y^\sharp/T^\sharp}))[\delta^{T^\sharp}_{Y^\sharp}] = \mathbb{D}_{Y^\sharp/T^\sharp}(\widetilde{\varpi}^!(\mathcal{E})). \quad (5.3.3.4.7)$$

□

5.3.4 Projection formula as $\mathcal{D}$-module

We keep notation 5.1.2 and we suppose $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3). Moreover, assume that T is a noetherian scheme of finite Krull dimension; and f is quasi-compact and quasi-separated. We check here a projection formula involving the pushforward as $\mathcal{D}$-module and the extraordinary inverse image as $\mathcal{D}$-module (compare 5.3.4.1 with 5.1.2.8).

Proposition 5.3.4.1 (Projection formula)**.** For $\mathcal{M} \in D^-({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathcal{E} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$ and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$, with notation 5.1.1.4 and 5.1.3.1 we have the canonical isomorphism

$$\widetilde{f}^{(m)}_+\left(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^*(\mathcal{E})\right) \xrightarrow{\sim} \widetilde{f}^{(m)}_+(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathcal{E}. \quad (5.3.4.1.1)$$

Proof. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$ are flat $\mathcal{B}_X$-modules, we have the isomorphisms of $D({}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$:

$$\left(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^*(\mathcal{E})\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \xrightarrow{\sim} \left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp} \otimes_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^*(\mathcal{E}))\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}$$
$$\underset{4.2.5.1}{\xrightarrow{\sim}} \left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} (\mathbb{L}\widetilde{f}^*(\mathcal{E}) \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \xrightarrow{\sim} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \left(\mathbb{L}\widetilde{f}^*(\mathcal{E}) \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}\right). \quad (5.3.4.1.2)$$

We have the isomorphism of $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$:

$$\mathbb{L}\widetilde{f}^*(\mathcal{E}) \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \underset{5.1.1.7.1}{\xrightarrow{\sim}} \mathbb{L}\widetilde{f}^*(\mathcal{E} \otimes_{\mathcal{B}_Y} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}) \underset{4.2.5.1}{\xleftarrow{\sim}} \mathbb{L}\widetilde{f}^*(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E})$$
$$\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E}). \quad (5.3.4.1.3)$$

By using 5.3.4.1.3 and 5.3.4.1.3, we get the isomorphism

$$\left(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^*(\mathcal{E})\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \xrightarrow{\sim} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathcal{E}) \quad (5.3.4.1.4)$$

Since $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$ (resp. $\in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp})$), we get

$$\widetilde{f}^{(m)}_+\left(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^*(\mathcal{E})\right) \underset{\mathbb{R}f_*(5.3.4.1.4)}{\xrightarrow{\sim}} \mathbb{R}f_*\left(\left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}\right) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E})\right)$$
$$\underset{5.1.2.6.1}{\xleftarrow{\sim}} \mathbb{R}f_*\left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp\to Y^\sharp/T^\sharp}\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp/T^\sharp} \otimes_{\mathcal{B}_Y} \mathcal{E}) \xrightarrow{\sim} \widetilde{f}^{(m)}_+(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathcal{E}.$$

□

5.3.5 Commutations with exterior tensor products

Let $S^\sharp$ be a nice fine log schemes over $\mathrm{Spec}(\mathbb{Z}/p^{r+1}\mathbb{Z})$ as defined in 3.1.1.1 with $r \in \mathbb{N}$. We suppose the underlying scheme S is noetherian of finite Krull dimension. Since the base scheme $S^\sharp$ is fixed, so we can remove it in the notation concerning relative object with base $S^\sharp$. If $p\colon X^\sharp \to S^\sharp$ is a morphism, by abuse of notation, we sometimes denote $p^{-1}\mathcal{O}_S$ simply by $\mathcal{O}_S$.

For $i = 1, \dots, n$, let $f_i\colon X_i^\sharp \to Y_i^\sharp$ be a quasi-separated and quasi-compact morphism of log smooth $S^\sharp$-log schemes. Set $X^\sharp := X_1^\sharp \times_{S^\sharp} X_2^\sharp \times_{S^\sharp} \cdots \times_{S^\sharp} X_n^\sharp$, $Y^\sharp := Y_1^\sharp \times_{S^\sharp} Y_2^\sharp \times_{S^\sharp} \cdots \times_{S^\sharp} Y_n^\sharp$, and $f := f_1 \times \cdots \times f_n\colon X^\sharp \to Y^\sharp$. For $i = 1, \dots, n$, let $\mathrm{pr}_i\colon X \to X_i$, $\mathrm{pr}'_i\colon Y^\sharp \to Y_i^\sharp$ be the projections. We denote by $\varpi\colon X^\sharp \to S^\sharp$ and $\varpi_i\colon X_i^\sharp \to S^\sharp$, $\varpi'\colon Y^\sharp \to S^\sharp$ and $\varpi'_i\colon Y_i^\sharp \to S^\sharp$ the structural morphisms.

For $i = 1, \dots, n$, let $\mathcal{B}_{X_i}$ (resp. $\mathcal{B}_{Y_i}$) be a commutative $\mathcal{O}_{X_i}$-algebra (resp. $\mathcal{O}_{Y_i}$-algebras) endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp}$-module (resp. left $\mathcal{D}^{(m)}_{Y_i^\sharp/S^\sharp}$-module). For $i = 1, \dots, n$, we suppose there exists a homomorphism of $\mathcal{O}_{Y_i}$-algebras $f_i^*\mathcal{B}_{X_i} \to \mathcal{B}_{Y_i}$ which is also $\mathcal{D}^{(m)}_{Y_i^\sharp/S^\sharp}$-linear. We set $\mathcal{B}_i := \mathrm{pr}_i^* \mathcal{B}_{X_i}$ and $\mathcal{B}'_i := \mathrm{pr}_i'^* \mathcal{B}_{Y_i}$, $\mathcal{B}_X := \mathcal{B}_1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{B}_n$ and $\mathcal{B}_Y := \mathcal{B}'_1 \otimes_{\mathcal{O}_Y} \cdots \otimes_{\mathcal{O}_Y} \mathcal{B}'_n$.

We will denote by $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} = \mathcal{B}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} = \mathcal{B}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S^\sharp}$. We get the following ringed logarithmic schemes $\widetilde{X}^\sharp := (X^\sharp, \mathcal{B}_X)$, $\widetilde{X}_i^\sharp := (X_i^\sharp, \mathcal{B}_{X_i})$, $\widetilde{Y}^\sharp := (Y^\sharp, \mathcal{B}_Y)$, $\widetilde{Y}_i^\sharp := (Y_i^\sharp, \mathcal{B}_{Y_i})$. We denote by $\widetilde{\mathrm{pr}}_i\colon \widetilde{X}^\sharp/S^\sharp \to \widetilde{X_i}^\sharp/S^\sharp$, $\widetilde{\mathrm{pr}}'_i\colon \widetilde{X}'^\sharp/S^\sharp \to \widetilde{X_i'}^\sharp/S^\sharp$, $\widetilde{f}_i\colon \widetilde{X_i}^\sharp/S^\sharp \to \widetilde{Y_i}^\sharp/S^\sharp$ and $\widetilde{f}\colon \widetilde{X}^\sharp/S^\sharp \to \widetilde{Y}^\sharp/S^\sharp$ the induced morphisms of relative to $S^\sharp$ ringed logarithmic schemes and similarly the induced morphisms of ringed logarithmic schemes.

Remark 5.3.5.1. Suppose $n = 2$ and f_2 is the identity. In that case, denoting by $T^\sharp := X_2^\sharp = Y_2^\sharp$, we get the cartesian square

$$\begin{array}{ccc} X^\sharp = X_1^\sharp \times_{S^\sharp} T^\sharp & \xrightarrow{f = f_1 \times \mathrm{id}} & Y^\sharp = Y_1^\sharp \times_{S^\sharp} T^\sharp \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}'_1} \\ X_1^\sharp & \xrightarrow{f_1} & Y_1^\sharp. \end{array} \tag{5.3.5.1.1}$$

Proposition 5.3.5.2. *For $i = 1, \dots, n$, let $\mathcal{D}_i$ be a sheaf of rings such that $(\mathcal{D}_i, \widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp})$ has the ring of resolution $\mathcal{R}$. Let $\mathcal{N}_i \in D^-(\mathcal{D}_i, \mathcal{R}, \widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp})$. With notation 5.1.5, we have the canonical isomorphism of $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i, \mathcal{R}, \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$:*

$$\mathbb{L}\widetilde{f}^*(\widetilde{\boxtimes}^{\mathbb{L}}_i \mathcal{N}_i) \xrightarrow{\sim} \widetilde{\boxtimes}^{\mathbb{L}}_i \mathbb{L}\widetilde{f}_i^*(\mathcal{N}_i).$$

Proof. This is a consequence of 5.1.5.4.3 and of the commutation of inverse images with tensor products. □

Remark 5.3.5.3. The proposition means in particular that tensor products commute with extensions of coefficients.

5.3.5.4. We recall the following fact (see [Sta22, 09T5]). Let $F\colon \mathfrak{A} \to \mathfrak{B}$ and $G\colon \mathfrak{B} \to \mathfrak{A}$ be functors of abelian categories such that F is a right adjoint to G. Let $\mathcal{M}$ be a complex of $\mathfrak{A}$ and let $\mathcal{N}$ be a complex of $\mathfrak{B}$. If $\mathbb{R}F$ is defined at $\mathcal{M}$ and $\mathbb{L}G$ is defined at $\mathcal{N}$, then there is a canonical isomorphism

$$\mathrm{Hom}_{D(\mathfrak{B})}(\mathcal{N}, \mathbb{R}F(\mathcal{M})) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathfrak{A})}(\mathbb{L}G(\mathcal{N}), \mathcal{M}).$$

This isomorphism is functorial in both variables on the triangulated subcategories of $D(\mathfrak{A})$ and $D(\mathfrak{B})$ where $\mathbb{R}F$ and $\mathbb{L}G$ are defined.

5.3.5.5. Let $u\colon (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ be a morphism of ringed spaces (or more generally ringed topoi). Let $\mathcal{M} \in \mathrm{Ob}(D(\mathcal{O}_U))$ and $\mathcal{N} \in \mathrm{Ob}(D(\mathcal{O}_V))$.

(a) Since the functors $\mathbb{R}u_*$ and $\mathbb{L}u^*$ are well defined, we get from 5.3.5.4

$$\mathrm{Hom}_{D(\mathcal{O}_V)}(\mathcal{N}, \mathbb{R}u_*(\mathcal{M})) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathbb{L}u^*(\mathcal{N}), \mathcal{M}) \tag{5.3.5.5.1}$$

bifunctorially in $\mathcal{M}$ and $\mathcal{N}$ (see [Sta22, 079W]). We get the adjunction functorial morphisms $\mathrm{adj}_{RL}(\mathcal{N})\colon \mathcal{N} \to \mathbb{R}u_*\mathbb{L}u^*(\mathcal{N})$ and $\mathrm{adj}_{LR}\colon \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) \to \mathcal{M}$. In fact, we can directly prove 5.3.5.5.1 and describe (which will be useful) theses adjunction morphisms as follows.

i) Let $\mathcal{P}$ be a K-flat complex of $\mathrm{Ob}(K(\mathcal{O}_V))$ endowed with a quasi-isomorphism $\mathcal{P} \to \mathcal{N}$ (we call such $\mathcal{P}$ a K-flat representation or K-flat resolution of $\mathcal{N}$). Modulo the isomorphism $Q(\mathcal{P}) \xrightarrow{\sim} \mathcal{N}$, to construct $\mathrm{adj}_{RL}(\mathcal{N})$ we reduce to build $\mathrm{adj}_{RL}(Q(\mathcal{P}))$ which is simply $Q(\mathcal{P}) \to Q(u_*u^*(\mathcal{P}) \to \mathbb{R}u_* \circ Q(u^*(\mathcal{P})) = \mathbb{R}u_* \circ \mathbb{L}u^*(Q(\mathcal{P}))$, where Q means the localisation functors $K(\mathcal{O}_U) \to D(\mathcal{O}_U)$ or $K(\mathcal{O}_V) \to D(\mathcal{O}_V)$ and where the first morphism is given by adjunction via the adjoint paire (u^*, u_*) (in the categories of complexes).

ii) Let $\mathcal{I}$ be a K-injective complex of $\mathrm{Ob}(K(\mathcal{O}_U))$ endowed with a quasi-isomorphism $\mathcal{M} \to \mathcal{I}$ (we call such $\mathcal{I}$ a K-injective representation or K-injective resolution of $\mathcal{M}$). Modulo the isomorphism $\mathcal{M} \xrightarrow{\sim} Q(\mathcal{I})$, to construct $\mathrm{adj}_{LR}(\mathcal{M})$ we reduce to build $\mathrm{adj}_{LR}(Q(\mathcal{I}))$ which is simply $\mathbb{L}u^*\mathbb{R}u_*(Q(\mathcal{I})) = \mathbb{L}u^* \circ Q(u_*(\mathcal{I})) \to Q \circ u^*(u_*(\mathcal{I})) \to Q(\mathcal{I})$ where the last morphism is given by adjunction via the adjoint paire (u^*, u_*) (in the categories of complexes).

iii) This is an easy exercice to check that check that $\mathrm{adj}_{LR}(\mathbb{L}u^*\mathcal{N}) \circ \mathbb{L}u^*(\mathrm{adj}_{RL}(\mathcal{N}))$ is the identity of $\mathbb{L}u^*\mathcal{N}$ and $\mathbb{L}u_* \,\mathrm{adj}_{LR}(\mathcal{M}) \circ \mathrm{adj}_{RL}(\mathbb{L}u_*\mathcal{M})$ is the identity of $\mathbb{L}u_*\mathcal{M}$. This yields a check of 5.3.5.5.1.

(b) For any $\mathcal{M}, \mathcal{M}' \in D(\mathcal{O}_U)$, we construct the canonical morphism of $D(\mathcal{O}_V)$:

$$\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathbb{R}u_*(\mathcal{M}') \to \mathbb{R}u_*(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathcal{M}') \tag{5.3.5.5.2}$$

bifunctorially in $\mathcal{M}, \mathcal{M}' \in \mathrm{Ob}(D(\mathcal{O}_U))$ as follows. Using 5.3.5.5.1, since tensor products commute with inverses images, we reduce to construct a canonical morphism $\mathrm{adj}_{LR} \otimes \mathrm{adj}_{LR}\colon \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}') \to \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathcal{M}'$, which again a consequence of 5.3.5.5.1.

(c) For any $\mathcal{M} \in D(\mathcal{O}_U)$, $\mathcal{N} \in D(\mathcal{O}_V)$, we construct the canonical morphism

$$\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{N} \to \mathbb{R}u_*(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N}). \tag{5.3.5.5.3}$$

as follows. Using 5.3.5.5.1, since tensor products commute with inverses images, we reduce to construct a canonical morphism $\mathrm{adj}_{LR} \otimes \mathrm{id}\colon \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N} \to \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N}$, which again a consequence of 5.3.5.5.1.

(d) Let $\mathcal{M} \in D(\mathcal{O}_U)$, $\mathcal{N} \in D(\mathcal{O}_V)$, $\mathcal{I}$ be a K-injective resolution of $\mathcal{M}$, $\mathcal{Q}$ be a K-flat resolution of $\mathcal{N}$ (see [Sta22, 06YF]). Then the map 5.3.5.5.3 is equal to the composition $\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{N} \xrightarrow{\sim} u_*(\mathcal{I}) \otimes_{\mathcal{O}_V} \mathcal{Q} \to u_*(\mathcal{I} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) \to \mathbb{R}u_*(\mathcal{I} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) \xrightarrow{\sim} \mathbb{R}u_*(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N})$. Indeed, taking $\mathcal{P}$ a K-flat object representing $u_*\mathcal{I}$, this follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{N} & \xrightarrow{\sim} & \mathcal{P} \otimes_{\mathcal{O}_V} \mathcal{Q} & \xrightarrow{\sim} & u_*\mathcal{I} \otimes_{\mathcal{O}_V} \mathcal{Q} & \longrightarrow & u_*(\mathcal{I} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) \\
\downarrow{\scriptstyle \mathrm{adj}_{RL}} & & \downarrow{\scriptstyle \mathrm{adj}} & & \downarrow{\scriptstyle \mathrm{adj}} & & \| \\
\mathbb{R}u_*\mathbb{L}u^*(\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{N}) & \longleftarrow & u_*u^*(\mathcal{P} \otimes_{\mathcal{O}_V} \mathcal{Q}) & \longrightarrow & u_*u^*(u_*\mathcal{I} \otimes_{\mathcal{O}_V} \mathcal{Q}) & & \| \\
\downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \| \\
\downarrow{\scriptstyle \sim} & & u_*(u^*\mathcal{P} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) & \longrightarrow & u_*(u^*u_*(\mathcal{I}) \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) & \xrightarrow{\mathrm{adj}} & u_*(\mathcal{I} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \mathbb{R}u_*(u^*\mathcal{P} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) & \longrightarrow & \mathbb{R}u_*(u^*u_*(\mathcal{I}) \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) & \xrightarrow{\mathrm{adj}} & \mathbb{R}u_*(\mathcal{I} \otimes_{\mathcal{O}_U} u^*\mathcal{Q}) \\
\downarrow & \overset{\sim}{\nearrow} & & & & & \downarrow{\scriptstyle \sim} \\
\mathbb{R}u_*(\mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N}) & & & \xrightarrow{\mathrm{adj}_{LR}} & & & \mathbb{R}u_*(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N})
\end{array} \tag{5.3.5.5.4}$$

where we have avoid writing the localisation functor of the form $Q\colon K(\mathcal{D}) \to D(\mathcal{D})$, where 5.3.5.5.3 is by definition the composition of the left and bottom arrows (to check the commutativity of the top left square or of the bottom trapeze see the explicit construction of adj_{RL} and adj_{LR} given at (a)). Hence, we get the commutativity of the canonical diagram

$$\begin{array}{ccc} \mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{O}_V & \xrightarrow{5.3.5.5.3} & \mathbb{R}u_*(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{O}_V) \\ \uparrow\sim & \nearrow\sim & \\ \mathbb{R}u_*(\mathcal{M}). & & \end{array} \tag{5.3.5.5.5}$$

This yields (using standard methods) that when u is a quasi-separated and quasi-compact morphism of noetherian schemes of finite Krull dimension, the projection morphism 5.3.5.5.3 is an isomorphism for complexes with bounded above and quasi-coherent cohomology. The commutativity of 5.3.5.5.5 will be used to check the commutativity of 5.3.6.1.3.

Lemma 5.3.5.6. *Let $u\colon (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ be a morphism of ringed spaces. For any $\mathcal{M} \in D(\mathcal{O}_U)$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathbb{R}u_*\mathbb{L}u^*(\mathcal{N}) & \xrightarrow{5.3.5.5.2} & \mathbb{R}u_*(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N}) \\ \uparrow{\scriptstyle \mathrm{id}\otimes \mathrm{adj}_{RL}} & \nearrow{\scriptstyle 5.3.5.5.3} & \\ \mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{N}. & & \end{array} \tag{5.3.5.6.1}$$

Proof. Since the composition $\mathbb{L}u^*(\mathcal{N}) \stackrel{\mathrm{adj}_{RL}}{\longrightarrow} \mathbb{L}u^*\mathbb{R}u_*\mathbb{L}u^*(\mathcal{N}) \stackrel{\mathrm{adj}_{LR}}{\longrightarrow} \mathbb{L}u^*(\mathcal{N})$ is the identity, we get the commutativity of the triangle of the diagram:

$$\begin{array}{ccccc} \mathbb{L}u^*\left(\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathbb{R}u_*\mathbb{L}u^*(\mathcal{N})\right) & \xrightarrow{\sim} & \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathbb{R}u_*\mathbb{L}u^*(\mathcal{N}) & \xrightarrow{\mathrm{adj}_{LR}\otimes \mathrm{adj}_{LR}} & \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N} \\ \uparrow{\scriptstyle \mathrm{id}\otimes \mathrm{adj}_{RL}} & & \uparrow{\scriptstyle \mathrm{id}\otimes \mathrm{adj}_{RL}} & \nearrow{\scriptstyle \mathrm{adj}_{LR}\otimes \mathrm{id}} & \\ \mathbb{L}u^*\left(\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_V} \mathcal{N}\right) & \xrightarrow{\sim} & \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{L}u^*\mathcal{N}. & & \end{array} \tag{5.3.5.6.2}$$

By construction of both morphisms 5.3.5.5.2 and 5.3.5.5.3, from the commutative diagram 5.3.5.6.2 we get by adjointness (via 5.3.5.5.1) the commutative diagram 5.3.5.6.1. □

Lemma 5.3.5.7. *Let $u\colon (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ be a morphism of ringed spaces. Let $\mathcal{M} \in \mathrm{Ob}(D(\mathcal{O}_U))$ and $\mathcal{N} \in \mathrm{Ob}(D(\mathcal{O}_V))$. For any $\mathcal{M} \in D(\mathcal{O}_U)$, the following commutative diagram of $D(\mathcal{O}_V)$:*

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\mathrm{adj}_{RL}(\mathcal{N})} & \mathbb{R}u_*u^{-1}(\mathcal{N}) \\ \| & & \downarrow \\ \mathcal{N} & \xrightarrow{\mathrm{adj}_{RL}(\mathcal{N})} & \mathbb{R}u_*\mathbb{L}u^*(\mathcal{N}) \end{array}, \qquad \begin{array}{ccc} u^{-1}\mathbb{R}u_*(\mathcal{M}) & \xrightarrow{\mathrm{adj}_{LR}(\mathcal{M})} & \mathcal{M} \\ \downarrow & & \| \\ \mathbb{L}u^*\mathbb{R}u_*(\mathcal{M}) & \xrightarrow{\mathrm{adj}_{LR}(\mathcal{M})} & \mathcal{M} \end{array} \tag{5.3.5.7.1}$$

Proof. By using the explicit description of the adjunction maps adj_{RL} and adj_{LR} of 5.3.5.5.(a), this is easy. □

Lemma 5.3.5.8. *Let $u\colon (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$, $v\colon (V, \mathcal{O}_V) \to (V', \mathcal{O}_{V'})$, $v'\colon (U, \mathcal{O}_U) \to (U', \mathcal{O}_{U'})$ and $u'\colon (U', \mathcal{O}_{U'}) \to (V', \mathcal{O}_{V'})$ be some morphisms of ringed spaces such that $v \circ u = u' \circ v'$. For any $\mathcal{E}' \in D(\mathcal{O}_{U'})$, the canonical diagram*

$$\begin{array}{ccc} v^{-1}\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\mathrm{adj}} & \mathbb{R}u_*v'^{-1}(\mathcal{E}') \\ \downarrow & & \downarrow \\ \mathbb{L}v^*\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\mathrm{adj}} & \mathbb{R}u_*\mathbb{L}v'^*(\mathcal{E}') \end{array} \tag{5.3.5.8.1}$$

is commutative.

Proof. Consider the diagram

$$\begin{array}{ccccccc}
v^{-1}\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\mathrm{adj}_{RL}} & \mathbb{R}u_*u^{-1}v^{-1}\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\sim} & \mathbb{R}u_*v'^{-1}u'^{-1}\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\mathrm{adj}_{LR}} & \mathbb{R}u_*v'^{-1}(\mathcal{E}') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{L}v^*\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\mathrm{adj}_{RL}} & \mathbb{R}u_*\mathbb{L}u^*\mathbb{L}v^*\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\sim} & \mathbb{R}u_*\mathbb{L}v'^*\mathbb{L}u'^*\mathbb{R}u'_*(\mathcal{E}') & \xrightarrow{\mathrm{adj}_{LR}} & \mathbb{R}u_*\mathbb{L}v'^*(\mathcal{E}')
\end{array} \tag{5.3.5.8.2}$$

whose the composition of the top (resp. below) maps is the top (resp. below) map of the diagram 5.3.5.8.1. It follows from 5.3.5.8 that the left and right squares of 5.3.5.8.2 is commutative. The middle one is obvious. □

Lemma 5.3.5.9. *For $i = 1, \ldots, n$, we suppose $\mathcal{B}_{X_i}$ and $\mathcal{B}_{Y_i}$ are flat $\mathcal{O}_S$-algebras. For $i = 1, \ldots, n$, let $\mathcal{D}_i$ be a sheaf of $\varpi_i'^{-1}\mathcal{O}_S$-algebras.*

(a) For $i = 1, \ldots, n$, let $\mathcal{E}_i \in D^-(f_i^{-1}\mathcal{D}_i)$. Then we have the canonical morphism $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathbb{R}f_{i*}\mathcal{E}_i) \to \mathbb{R}f_*(\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}\mathcal{E}_i)$ of $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i)$.

(b) For $i = 1, \ldots, n$, let $\mathcal{E}_i \in D^-(f_i^{-1}\mathcal{D}_i, \mathcal{B}_{X_i})$. Then the canonical morphism $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathbb{R}f_{i*}\mathcal{E}_i) \to \mathbb{R}f_*(\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}\mathcal{E}_i)$ is also a morphism of $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i, \boxtimes_{i\,\mathrm{top}}\mathcal{B}_{Y_i})$. Moreover we have the canonical morphism $\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i) \to \mathbb{R}f_*(\widetilde{\boxtimes}_i^{\mathbb{L}}\mathcal{E}_i)$ of $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i, \mathcal{B}_Y)$ making commutative the diagram

$$\begin{array}{ccc}
\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathbb{R}f_{i*}\mathcal{E}_i) & \longrightarrow & \widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i) \\
\downarrow & & \downarrow \\
\mathbb{R}f_*(\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}\mathcal{E}_i) & \longrightarrow & \mathbb{R}f_*(\widetilde{\boxtimes}_i^{\mathbb{L}}\mathcal{E}_i).
\end{array} \tag{5.3.5.9.1}$$

Proof. 0) If $\mathcal{G}_i \in D^-(\varpi_i^{-1}\mathcal{O}_S)$ (resp. $\mathcal{G}_i \in D^-(\varpi_i'^{-1}\mathcal{O}_S)$), then we set $\overset{\mathbb{L}}{\underset{i}{\otimes}}\mathcal{G}_i := \mathcal{G}_1 \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathcal{G}_2 \otimes^{\mathbb{L}}_{\mathcal{O}_S} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_S} \mathcal{G}_n$.

a) By applying 5.3.5.5.2 to the morphism of ringed spaces $(X, \varpi^{-1}\mathcal{O}_S) \to (Y, \varpi'^{-1}\mathcal{O}_S)$, we get the morphism $\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{R}f_*\,\mathrm{pr}_i^{-1}\mathcal{E}_i) \to \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathrm{pr}_i^{-1}\mathcal{E}_i)\right)$. By functoriality, since $\mathrm{pr}_i^{-1}\mathcal{E}_i \in D^-(f^{-1}\,\mathrm{pr}_i'^{-1}\mathcal{D}_i, \varpi^{-1}\mathcal{O}_S)$ this latter morphism belongs to $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i)$.

b) Similarly, by applying 5.3.5.5.2 we get the morphisms $\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{R}f_*\,\mathrm{pr}_i^{-1}\mathcal{E}_i) \to \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathrm{pr}_i^{-1}\mathcal{E}_i)\right)$ and $\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{R}f_*\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i) \to \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i)\right)$ of $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i, \boxtimes_{i\,\mathrm{top}}\mathcal{B}_{Y_i})$. Finally, by applying 5.3.5.5.2 to the morphism $X \to Y$, we get the morphisms $\underset{\mathcal{B}_Y}{\otimes}^{\mathbb{L}}(\mathbb{R}f_*\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i) \to \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i)\right)$ of $D^-(\boxtimes_{i\,\mathrm{top}}\mathcal{D}_i, \mathcal{B}_Y)$.

1) The canonical morphism $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathbb{R}f_{i*}\mathcal{E}_i) \to \mathbb{R}f_*\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathcal{E}_i)$ (resp. $\widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i) \to \mathbb{R}f_*(\widetilde{\boxtimes}_i^{\mathbb{L}}\mathcal{E}_i)$ is by definition is the one making commutative the right (resp. left) rectangle of the diagram 5.3.5.9.2 below:

$$\begin{array}{ccccccccc}
\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathbb{R}f_{i*}\mathcal{E}_i) & \xrightarrow[5.1.5.4.2]{\sim} & \overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathrm{pr}_i'^{-1}\mathbb{R}f_{i*}\mathcal{E}_i) & \longrightarrow & \overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{L}\widetilde{\mathrm{pr}}_i'^*\mathbb{R}f_{i*}\mathcal{E}_i) & \longrightarrow & \overset{\mathbb{L}}{\underset{\mathcal{B}_Y}{\otimes}}(\mathbb{L}\widetilde{\mathrm{pr}}_i'^*\mathbb{R}f_{i*}\mathcal{E}_i) & \xrightarrow{\sim} & \widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i) \\
\vdots & & \downarrow{\scriptstyle \mathrm{adj}} & & \downarrow{\scriptstyle \mathrm{adj}} & & \downarrow{\scriptstyle \mathrm{adj}} & & \vdots \\
 & & \overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{R}f_*\,\mathrm{pr}_i^{-1}\mathcal{E}_i) & \longrightarrow & \overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{R}f_*\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i) & \longrightarrow & \overset{\mathbb{L}}{\underset{\mathcal{B}_Y}{\otimes}}(\mathbb{R}f_*\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i) & & \\
\vdots & & \downarrow{\scriptstyle 5.3.5.5.2} & & \downarrow{\scriptstyle 5.3.5.5.2} & & \downarrow{\scriptstyle 5.3.5.5.2} & & \vdots \\
\mathbb{R}f_*\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathcal{E}_i) & \xrightarrow[5.1.5.4.2]{\sim} & \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathrm{pr}_i^{-1}\mathcal{E}_i)\right) & \longrightarrow & \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{i}{\otimes}}(\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i)\right) & \longrightarrow & \mathbb{R}f_*\left(\overset{\mathbb{L}}{\underset{\mathcal{B}_X}{\otimes}}(\mathbb{L}\widetilde{\mathrm{pr}}_i^*\mathcal{E}_i)\right) & \xrightarrow{\sim} & \widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i)
\end{array} \tag{5.3.5.9.2}$$

Using 5.3.5.8, we get the commutativity of the top left square. We check the commutativity of the bottom right square by construction of both vertical arrows. The other squares are commutative by functoriality. Since the right and left rectangles are commutative by definition, then the diagram 5.3.5.9.2 is commutative. Finally, the composition of the top (resp. bottom) morphisms of 5.3.5.9.2 is the canonical morphism $\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathbb{R}f_{i*}\mathcal{E}_i) \to \widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i)$ (resp. $\mathbb{R}f_*\widetilde{\boxtimes}^{\mathbb{L}}_{i\,\mathrm{top}}(\mathcal{E}_i) \to \widetilde{\boxtimes}_i^{\mathbb{L}}(\mathbb{R}f_{i*}\mathcal{E}_i)$). □

5.3.5.10. By flatness, the morphism $\widetilde{\boxtimes}_i^{\mathbb{L}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp} \to \widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}$ is an isomorphism. We have by functoriality the canonical isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}, f^{-1}\boxtimes_{i\,\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp})$-bimodules

$$\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp} = \widetilde{\boxtimes}_i\widetilde{f}_i^*(\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp} \underset{5.3.5.2}{\xrightarrow{\sim}} \widetilde{f}^*(\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp}, \tag{5.3.5.10.1}$$

where the last isomorphism is a consequence of the canonical isomorphism $\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}$ (see 5.1.5.2.3). This yields a canonical structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp})$-bimodule on $\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}$ making $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp})$-bilinear the composite isomorphism 5.3.5.10.1.

Similarly, by flatness, the morphism $\widetilde{\boxtimes}_i^{\mathbb{L}}\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp\leftarrow X_i^\sharp} \to \widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp\leftarrow X_i^\sharp}$ is an isomorphism. Moreover, using 5.1.5.9, we get by functoriality the canonical isomorphism of $(f^{-1}\boxtimes_{i\,\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp})$-bimodules

$$\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp\leftarrow X_i^\sharp} = \widetilde{\boxtimes}_i\left(\widetilde{\omega}_{X_i^\sharp}\otimes_{\mathcal{B}_{X_i}}\widetilde{f}_i^*(\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp}\otimes_{\mathcal{B}_{Y_i}}\omega_{Y_i}^{-1})\right) \underset{5.3.5.2}{\xrightarrow{\sim}} \omega_X\otimes_{\mathcal{B}_X}\widetilde{f}^*(\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp}\otimes_{\mathcal{B}_Y}\omega_{Y^\sharp}^{-1}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp\leftarrow X^\sharp}, \tag{5.3.5.10.2}$$

where the last isomorphism is a consequence of the canonical isomorphism $\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}$ (see 5.1.5.2.3). This yields a canonical structure of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp})$-bimodule on $\widetilde{\boxtimes}_i\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp\leftarrow X_i^\sharp}$ making $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp})$-bilinear the composite isomorphism 5.3.5.10.2.

Theorem 5.3.5.11. *Suppose $\mathcal{B}_{X_i} = \mathcal{O}_{X_i}$ and $\mathcal{B}_{Y_i} = \mathcal{O}_{Y_i}$ for any $i = 1, \ldots, n$ and suppose $\mathcal{E}_i \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$. The canonical morphism of 5.3.5.9*

$$\boxtimes_i^{\mathbb{L}}\,\mathbb{R}f_{i*}(\mathcal{E}_i) \to \mathbb{R}f_*(\boxtimes_i^{\mathbb{L}}\mathcal{E}_i). \tag{5.3.5.11.1}$$

is an isomorphism.

Proof. i) By construction of the morphism 5.3.5.11.1 (i.e. the one making commutative the right rectangle of 5.3.5.9.2), we reduce to the case where $n = 2$. We have to check that the composition

$$\mathrm{pr}_1'^*\,\mathbb{R}f_{1*}(\mathcal{E}_1)\otimes^{\mathbb{L}}_{\mathcal{B}_Y}\mathrm{pr}_2'^*\,\mathbb{R}f_{2*}\mathcal{E}_2 \to \mathbb{R}f_*\,\mathrm{pr}_1^*(\mathcal{E}_1)\otimes^{\mathbb{L}}_{\mathcal{B}_Y}\mathbb{R}f_*\,\mathrm{pr}_2^*\,\mathcal{E}_2 \to \mathbb{R}f_*(\mathrm{pr}_1^*(\mathcal{E}_1)\otimes^{\mathbb{L}}_{\mathcal{B}_X}\mathrm{pr}_2^*\,\mathcal{E}_2) \tag{5.3.5.11.2}$$

is an isomorphism.

ii) We reduce to the case where $f_2 = \mathrm{id}$ as follows. Consider the commutative diagram

$$\begin{array}{ccccc}
X_1 & \xleftarrow{\mathrm{pr}_1} & X = X_1\times X_2 & \xrightarrow{\mathrm{pr}_2} & X_2 \\
\| & & \downarrow{\scriptstyle g_2:=\mathrm{id}\times f_2} & & \downarrow{\scriptstyle f_2} \\
X_1 & \xleftarrow{\mathrm{pr}_1''} & Y' := X_1\times Y_2 & \xrightarrow{\mathrm{pr}_2''} & X_2 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1:=f_1\times\mathrm{id}} & & \| \\
Y_1 & \xleftarrow{\mathrm{pr}_1'} & Y := Y_1\times Y_2 & \xrightarrow{\mathrm{pr}_2'} & X_2.
\end{array} \tag{5.3.5.11.3}$$

By adjointness with respect to the left bottom square of 5.3.5.11.3 (resp. the left top square of 5.3.5.11.3) we get the morphism $\mathrm{pr}_1'^*\,\mathbb{R}f_{1*}(\mathcal{E}_1) \xrightarrow{\mathrm{adj}} \mathbb{R}g_{1*}\,\mathrm{pr}_1''^*(\mathcal{E}_1)$ and $\mathrm{pr}_1''^*(\mathcal{E}_1) = \mathrm{pr}_1''^*\,\mathrm{id}_*(\mathcal{E}_1) \xrightarrow{\mathrm{adj}} \mathbb{R}g_{2*}\,\mathrm{pr}_1^*(\mathcal{E}_1)$. By transitivity, we get that the composition $\mathrm{pr}_1'^*\,\mathbb{R}f_{1*}(\mathcal{E}_1) \xrightarrow{\mathrm{adj}} \mathbb{R}g_{1*}\,\mathrm{pr}_1''^*(\mathcal{E}_1) \xrightarrow{\mathrm{adj}} \mathbb{R}g_{1*}\mathbb{R}g_{2*}\,\mathrm{pr}_1^*(\mathcal{E}_1) \xrightarrow{\sim} \mathbb{R}f_*\,\mathrm{pr}_1^*(\mathcal{E}_1)$ is the adjoint morphism with respect to the left rectangle of 5.3.5.11.3 (i.e. the composite of both left squares). Similarly, we get by transitivity that the composition $\mathrm{pr}_2'^*\,\mathbb{R}f_{2*}\mathcal{E}_2 \xrightarrow{\mathrm{adj}}$

$\mathbb{R}g_{1*}\,\mathrm{pr}_2''^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 \xrightarrow{\mathrm{adj}} \mathbb{R}g_{1*}\mathbb{R}g_{2*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2 \xrightarrow{\sim} \mathbb{R}f_{*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2$ is the adjoint morphism. This yields the commutativity of the top left square of the following diagram:

$$\begin{array}{ccccc}
\mathrm{pr}_1'^{*}\,\mathbb{R}f_{1*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_Y}{\otimes}} \mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 & \xrightarrow{\mathrm{adj}} & \mathbb{R}g_{1*}\,\mathrm{pr}_1''^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_Y}{\otimes}} \mathbb{R}g_{1*}\,\mathrm{pr}_2''^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 & \xrightarrow{5.3.5.5.2} & \mathbb{R}g_{1*}\left(\mathrm{pr}_1''^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_{Y'}}{\otimes}} \mathrm{pr}_2''^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2\right) \\
\downarrow{\scriptstyle \mathrm{adj}} & & \downarrow{\scriptstyle \mathrm{adj}} & & \downarrow{\scriptstyle \mathrm{adj}} \\
\mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_Y}{\otimes}} \mathbb{R}f_{*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2 & \xrightarrow{\sim} & \mathbb{R}g_{1*}\mathbb{R}g_{2*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_Y}{\otimes}} \mathbb{R}g_{1*}\mathbb{R}g_{2*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2 & \xrightarrow{5.3.5.5.2} & \mathbb{R}g_{1*}\left(\mathbb{R}g_{2*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_{Y'}}{\otimes}} \mathbb{R}g_{2*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2\right) \\
\downarrow{\scriptstyle 5.3.5.5.2} & & & & \downarrow{\scriptstyle 5.3.5.5.2} \\
\mathbb{R}f_{*}\left(\mathrm{pr}_1^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_X}{\otimes}} \mathrm{pr}_2^{*}\,\mathcal{E}_2\right) & & \xrightarrow{\qquad\sim\qquad} & & \mathbb{R}g_{1*}\mathbb{R}g_{2*}\left(\mathrm{pr}_1^{*}(\mathcal{E}_1) \overset{\mathbb{L}}{\underset{\mathcal{B}_X}{\otimes}} \mathrm{pr}_2^{*}\,\mathcal{E}_2\right)
\end{array} \tag{5.3.5.11.4}$$

By transitivity of the morphism of the form 5.3.5.5.2, we get the commutativity of the bottom rectangle. The top right square is commutative by functoriality. Hence, 5.3.5.11.4 is commutative. By stability of the quasi-coherence under (topological) pushforwards, $\mathbb{R}f_{2*}\mathcal{E}_2$ is quasi-coherent. Hence, the case where f_1 or f_2 is the identity implies the general case. By symmetry, we can suppose $f_2 = \mathrm{id}$.

iii) Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{pr}_1'^{*}\,\mathbb{R}f_{1*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 & \longrightarrow & \mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_{*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2 & \longrightarrow & \mathbb{R}f_{*}(\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathrm{pr}_2^{*}\,\mathcal{E}_2) \\
\downarrow{\scriptstyle \sim} & \nearrow & \uparrow & & \uparrow \\
\mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 & & \mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_{*}\,\mathrm{pr}_2^{*}\,\mathbb{L}f_2^{*}\mathbb{R}f_{2*}\mathcal{E}_2 & \longrightarrow & \mathbb{R}f_{*}(\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathrm{pr}_2^{*}\,\mathbb{L}f_2^{*}\mathbb{R}f_{2*}\mathcal{E}_2) \\
\| & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\
\mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 & \xrightarrow{\mathrm{id}\otimes\mathrm{adj}} & \mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathbb{R}f_{*}\mathbb{L}f^{*}\,\mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 & \xrightarrow{5.3.5.5.2} & \mathbb{R}f_{*}(\mathrm{pr}_1^{*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}f^{*}\,\mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2)
\end{array} \tag{5.3.5.11.5}$$

where the top horizontal morphisms are 5.3.5.11.2, where the left vertical isomorphism is induced by the base change morphism $\mathrm{pr}_1'^{*}\,\mathbb{R}f_{1*}(\mathcal{E}_1) \to \mathbb{R}f_{*}\,\mathrm{pr}_1^{*}(\mathcal{E}_1)$ which is in our case an isomorphism (see [Har66, II.5.12]), where the trapeze is commutative by construction of the base change isomorphism $\mathrm{pr}_2'^{*}\,\mathbb{R}f_{2*}\mathcal{E}_2 \xrightarrow{\sim} \mathbb{R}f_{*}\,\mathrm{pr}_2^{*}\,\mathcal{E}_2$, where the composition of the bottom horizontal morphisms is the projection morphism and hence is an isomorphism (see 5.3.5.6). Since $f_2 = \mathrm{id}$, then the top middle and right vertical arrows are identity morphisms. Hence, the composition of the top arrows of the diagram 5.3.5.11.5 is an isomorphism. □

Notation 5.3.5.12. Let $\mathcal{M}_i \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$. We denote by $\mathrm{T}_{f_i} : \mathbb{R}f_{i,*}(\mathcal{M}_i) \to \mathbb{R}f_{i,*}(\mathcal{M}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp \to Y_i^\sharp}) = \widetilde{f}^{(m)}_{i,+}(\mathcal{M}_i)$, the morphism induced by the canonical homomorphism $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp \to Y_i^\sharp}$ (see 4.4.2.6.3).

Theorem 5.3.5.13. *For $i = 1, \ldots, n$, let $\mathcal{E}_i \in D^{-}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$, with $* = r$ or $* = l$.*

(a) We have the canonical homomorphism of $D^{-}({}^{*}\mathcal{D}^{(m)}_{Y^\sharp})$:

$$\widetilde{\boxtimes}^{\mathbb{L}}_{i}\, \widetilde{f}^{(m)}_{i+}(\mathcal{E}_i) \to \widetilde{f}^{(m)}_{+}(\widetilde{\boxtimes}^{\mathbb{L}}_{i}\, \mathcal{E}_i) \tag{5.3.5.13.1}$$

(b) Suppose $\mathcal{B}_{X_i} = \mathcal{O}_{X_i}$ and $\mathcal{B}_{Y_i} = \mathcal{O}_{Y_i}$ for any $i = 1, \ldots, n$ and suppose $\mathcal{E}_i \in D^{-}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp})$. The homomorphism 5.3.5.13.1 is therefore an isomorphism making commutative the canonical diagram

$$\begin{array}{ccccc}
\boxtimes^{\mathbb{L}}_{\mathrm{top}\, i}(\mathbb{R}f_{i*}\mathcal{E}_i) & \longrightarrow & \boxtimes^{\mathbb{L}}_{i}\, \mathbb{R}f_{i*}(\mathcal{E}_i) & \xrightarrow[5.3.5.12]{\boxtimes^{\mathbb{L}}_{i}\mathrm{T}_{f_i}} & \boxtimes^{\mathbb{L}}_{i}\, \widetilde{f}^{(m)}_{i+}(\mathcal{E}_i) \\
\downarrow & & \downarrow{\scriptstyle 5.3.5.11.1} & & \downarrow{\scriptstyle \sim} \\
\mathbb{R}f_{*}(\boxtimes^{\mathbb{L}}_{\mathrm{top}\, i}\mathcal{E}_i) & \longrightarrow & \mathbb{R}f_{*}(\boxtimes^{\mathbb{L}}_{i}\mathcal{E}_i) & \xrightarrow[5.3.5.12]{\mathrm{T}_f} & \widetilde{f}^{(m)}_{+}(\boxtimes^{\mathbb{L}}_{i}\mathcal{E}_i).
\end{array} \tag{5.3.5.13.2}$$

Proof. 0) To get 5.3.5.13.1, since the case where $* = l$ is a consequence of the case suppose $* = r$ we can prove the whole theorem in the case where $*=$ r.

I) 1) We have the morphisms of $D^-({}^{\mathrm{r}} \underset{i}{\boxtimes}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp})$:

$$\overset{\mathbb{L}}{\underset{i}{\boxtimes}}{}_{\mathrm{top}}\mathbb{R}f_{i*}(\mathcal{E}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}) \underset{5.3.5.9.a}{\longrightarrow} \mathbb{R}f_*\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}_{\mathrm{top}}(\mathcal{E}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}) \underset{5.1.5.7.2}{\xrightarrow{\sim}} \mathbb{R}f_*\left(\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}_{\mathrm{top}}\mathcal{E}_i \otimes^{\mathbb{L}}_{\underset{i}{\boxtimes}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \underset{i}{\boxtimes}{}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}\right)$$

$$\to \mathbb{R}f_*\left(\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}\mathcal{E}_i \otimes^{\mathbb{L}}_{\underset{i}{\widetilde{\boxtimes}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \underset{i}{\widetilde{\boxtimes}}\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}\right) \underset{5.3.5.10.1}{\xrightarrow{\sim}} \mathbb{R}f_*\left(\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}\mathcal{E}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp\to Y^\sharp}\right) = f^{(m)}_+(\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}\mathcal{E}_i)$$

Since $f^{(m)}_+(\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}\mathcal{E}_i) \in D^-({}^{\mathrm{r}}\mathcal{D}^{(m)}_{Y^\sharp})$, this yields the morphism of $D^-({}^{\mathrm{r}}\mathcal{D}^{(m)}_{Y^\sharp})$:

$$\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}\widetilde{f}^{(m)}_{i+}(\mathcal{E}_i) = \widetilde{\mathcal{D}}^{(m)}_{Y} \otimes_{\underset{i}{\boxtimes}_{\mathrm{top}}\widetilde{\mathcal{D}}^{(m)}_{Y_i^\sharp}} \overset{\mathbb{L}}{\underset{i}{\boxtimes}}{}_{\mathrm{top}}\mathbb{R}f_{i*}(\mathcal{E}_i \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp\to Y_i^\sharp}) \to \widetilde{f}^{(m)}_+(\underset{i}{\widetilde{\boxtimes}}{}^{\mathbb{L}}\mathcal{E}_i). \tag{5.3.5.13.3}$$

Since the left square is commutative (see 5.3.5.9.1), to check the commutativity of the diagram 5.3.5.13.2, we notice it is enough to check the commutativity of the outer. This latter fact is easy.

II) It remains to check the morphism const ructed at the step I.1) is an isomorphism. Using [Har66, I.7.1.(i)], we reduce to the case where $\mathcal{E}_i$ is a quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module. We remark that such a $\mathcal{E}_i$ is a quotient of a $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}$-module of the form $\mathcal{L}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp}$, where $\mathcal{L}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module (e.g. take $\mathcal{L}_i = \mathcal{E}_i$). Since both functors of 5.3.5.13.2 are way-out right, using [Har66, I.7.1.(iv)] we reduce to the case where $\mathcal{E}_i = \mathcal{L}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp}$.

3) To simplify notation, put $\overset{\mathbb{L}}{\underset{\mathrm{top}}{\boxtimes}} := \overset{\mathbb{L}}{\underset{i}{\boxtimes}}{}^{\mathrm{top}}$, $\mathcal{M}_i := \mathcal{D}^{(m)}_{X_i^\sharp\to Y_i^\sharp}$, $\mathcal{D}_i := \mathcal{D}^{(m)}_{X_i^\sharp}$, $\mathcal{O}_{X_i} := \mathcal{O}_i$, $\mathcal{D}'_i := \mathcal{D}^{(m)}_{Y_i^\sharp}$, $\mathcal{O}'_i := \mathcal{O}_{Y_i}$. Since $\underset{i}{\boxtimes}{}^{\mathrm{top}} f_i^{-1}\mathcal{D}^{(m)}_{Y_i^\sharp} = f^{-1}\underset{i}{\boxtimes}{}^{\mathrm{top}}\mathcal{D}^{(m)}_{Y_i^\sharp}$, we get the the following diagram in the category $D^-(f^{-1}\underset{i}{\boxtimes}{}^{\mathrm{top}}\mathcal{D}^{(m)}_{Y_i^\sharp}, \mathcal{O}_X)$:

$$\begin{array}{ccccc}
\overset{\mathbb{L}}{\underset{\mathrm{top}}{\boxtimes}}\big((\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{M}_i\big) & \underset{5.1.5.7.2}{\xrightarrow{\sim}} \overset{\mathbb{L}}{\underset{\mathrm{top}}{\boxtimes}}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\underset{\mathrm{top}}{\boxtimes}\mathcal{D}_i} \underset{\mathrm{top}}{\boxtimes}\mathcal{M}_i & \underset{5.1.5.7.1}{\xrightarrow{\sim}} & (\overset{\mathbb{L}}{\underset{\mathrm{top}}{\boxtimes}}\mathcal{L}_i \otimes_{\underset{\mathrm{top}}{\boxtimes}\mathcal{O}_i} \underset{\mathrm{top}}{\boxtimes}\mathcal{D}_i) \otimes^{\mathbb{L}}_{\underset{\mathrm{top}}{\boxtimes}\mathcal{D}_i} \underset{\mathrm{top}}{\boxtimes}\mathcal{M}_i \\
\downarrow\sim & & & \downarrow\sim \\
\overset{\mathbb{L}}{\underset{\mathrm{top}}{\boxtimes}}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{M}_i) & \xrightarrow[5.1.5.7.1]{\sim} & & \underset{\mathrm{top}}{\boxtimes}\mathcal{L}_i \otimes_{\underset{\mathrm{top}}{\boxtimes}\mathcal{O}_i} \overset{\mathbb{L}}{\underset{\mathrm{top}}{\boxtimes}}\mathcal{M}_i \\
\downarrow & & & \downarrow \\
\overset{\mathbb{L}}{\underset{i}{\boxtimes}}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{M}_i) & \xrightarrow[5.1.5.8.2]{\sim} & & \underset{i}{\boxtimes}\mathcal{L}_i \otimes_{\underset{i}{\boxtimes}\mathcal{O}_i} \overset{\mathbb{L}}{\underset{i}{\boxtimes}}\mathcal{M}_i.
\end{array} \tag{5.3.5.13.4}$$

Using 5.1.5.8.1, we get the commutativity of the bottom rectangle. The commutativity of the top rectangle is straightforward. Consider now the following diagram:

$$\begin{array}{ccccc}
\overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_{\mathrm{top}} \mathcal{D}_i} \boxtimes_{\mathrm{top}} \mathcal{M}_i & \longrightarrow & \overset{\mathbb{L}}{\boxtimes}_{i}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_{\mathrm{top}} \mathcal{D}_i} \boxtimes_{\mathrm{top}} \mathcal{M}_i & \longrightarrow & \overset{\mathbb{L}}{\boxtimes}_{i}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_i \mathcal{D}_i} \boxtimes_i \mathcal{M}_i \\
{\scriptstyle 5.1.5.7.1}\downarrow\wr & & {\scriptstyle 5.1.5.8.2}\downarrow\wr & & \downarrow\wr \\
(\boxtimes_{\mathrm{top}} \mathcal{L}_i \otimes_{\boxtimes_{\mathrm{top}} \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_{\mathrm{top}} \mathcal{D}_i} \boxtimes_{\mathrm{top}} \mathcal{M}_i & \longrightarrow & (\boxtimes_i \mathcal{L}_i \otimes_{\boxtimes_i \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_{\mathrm{top}} \mathcal{D}_i} \boxtimes_{\mathrm{top}} \mathcal{M}_i & \longrightarrow & (\boxtimes_i \mathcal{L}_i \otimes_{\boxtimes_i \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_i \mathcal{D}_i} \boxtimes_i \mathcal{M}_i \\
\downarrow\wr & & & & \downarrow\wr \\
\boxtimes_{\mathrm{top}} \mathcal{L}_i \otimes_{\boxtimes_{\mathrm{top}} \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} \mathcal{M}_i & & \longrightarrow & & \boxtimes_i \mathcal{L}_i \otimes_{\boxtimes_i \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{M}_i .
\end{array} \tag{5.3.5.13.5}$$

The left top square is commutative because of that of 5.1.5.8.1. The right top square is commutative by functoriality. Taking $\mathcal{O}_{X_i}$-flat resolutions of $\mathcal{L}_i$, we check the commutativity of the bottom rectangle.

Compositing both diagrams 5.3.5.13.4 and 5.3.5.13.5, we get the commutative diagram

$$\begin{array}{ccc}
\overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} \big((\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{M}_i\big) & \longrightarrow & \overset{\mathbb{L}}{\boxtimes}_i(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_i \mathcal{D}_i} \boxtimes_i \mathcal{M}_i \\
\downarrow\wr & & \downarrow\wr \\
\overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}}(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{M}_i) & & (\boxtimes_i \mathcal{L}_i \otimes_{\boxtimes_i \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_i \mathcal{D}_i} \boxtimes_i \mathcal{M}_i \\
\downarrow & & \downarrow\wr \\
\overset{\mathbb{L}}{\boxtimes}_i(\mathcal{L}_i \otimes_{\mathcal{O}_i} \mathcal{M}_i) & \underset{5.1.5.8.2}{\overset{\sim}{\longrightarrow}} & \boxtimes_i \mathcal{L}_i \otimes_{\boxtimes_i \mathcal{O}_i} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{M}_i ,
\end{array} \tag{5.3.5.13.6}$$

where the top arrow is the composite of the left top horizontal arrow of 5.3.5.13.4 with top horizontal arrows of 5.3.5.13.5. We get from the commutativity of 5.3.5.13.6 that of the right rectangle of the diagram 5.3.5.13.7 below:

$$\begin{array}{ccccc}
\overset{\mathbb{L}}{\boxtimes}_i f^{(m)}_{i+}(\mathcal{E}_i) & & \longrightarrow & & f^{(m)}_{+}(\overset{\mathbb{L}}{\boxtimes}_i \mathcal{E}_i) \\
\| & & & & \uparrow\wr\, {\scriptstyle 5.3.5.10.1} \\
\overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} \mathbb{R} f_{i*}((\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{D}_i) \overset{\mathbb{L}}{\underset{\mathcal{D}_i}{\otimes}} \mathcal{M}_i) \underset{\boxtimes_{\mathrm{top}} \mathcal{D}'_i}{\otimes} \mathcal{D}' & \xrightarrow{5.3.5.9.a} & \mathbb{R} f_* \overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} \left((\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{D}_i) \overset{\mathbb{L}}{\underset{\mathcal{D}_i}{\otimes}} \mathcal{M}_i\right) \underset{\boxtimes_{\mathrm{top}} \mathcal{D}'_i}{\otimes} \mathcal{D}' & \longrightarrow & \mathbb{R} f_* \left(\overset{\mathbb{L}}{\boxtimes}_i(\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_i \mathcal{D}_i} \boxtimes_i \mathcal{M}_i\right) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} \mathbb{R} f_{i*}(\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{M}_i) \underset{\boxtimes_{\mathrm{top}} \mathcal{D}'_i}{\otimes} \mathcal{D}' & \xrightarrow{5.3.5.9.a} & \mathbb{R} f_* \overset{\mathbb{L}}{\boxtimes}_{\mathrm{top}} (\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{M}_i) \underset{\boxtimes_{\mathrm{top}} \mathcal{D}'_i}{\otimes} \mathcal{D}' & & \mathbb{R} f_* \left((\boxtimes_i \mathcal{L}_i \underset{\boxtimes_i \mathcal{O}_i}{\otimes} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{D}_i) \otimes^{\mathbb{L}}_{\boxtimes_i \mathcal{D}_i} \boxtimes_i \mathcal{M}_i\right) \\
\uparrow\wr & & \downarrow\wr & & \downarrow\wr \\
\overset{\mathbb{L}}{\boxtimes}_i \mathbb{R} f_{i*}(\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{M}_i) & \xrightarrow{5.3.5.9.b} & \mathbb{R} f_* \overset{\mathbb{L}}{\boxtimes}_i (\mathcal{L}_i \underset{\mathcal{O}_i}{\otimes} \mathcal{M}_i) & \xrightarrow{\sim} & \mathbb{R} f_* (\boxtimes_i \mathcal{L}_i \underset{\boxtimes_i \mathcal{O}_i}{\otimes} \overset{\mathbb{L}}{\boxtimes}_i \mathcal{M}_i).
\end{array} \tag{5.3.5.13.7}$$

The commutativity of the top rectangle is by construction of the top arrow (see 5.3.5.13.3). The commutativity of the top square is checked by functoriality. Using the commutativity of the diagram 5.3.5.9.1, we obtain the commutativity of the bottom square of 5.3.5.13.7. Hence, the diagram 5.3.5.13.7 is commutative. Following Theorem 5.3.5.11, the left bottom morphism is an isomorphism. Hence, using the commutativity of the diagram 5.3.5.13.7, this yields that the top morphism is an isomorphism. □

5.3.6 Base change in the projection case

We keep notation 5.3.5 and we suppose $n = 2$ and f_2 is the identity.

Proposition 5.3.6.1. *For any $\mathcal{E}_1 \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_{X_1})$, we have the canonical isomorphism $pr_1'^{(m)!} \circ f^{(m)}_{1,+}(\mathcal{E}_1) \xrightarrow{\sim} f^{(m)}_{+} \circ pr_1^{(m)!}(\mathcal{E}_1)$ of $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_{Y})$ making commutative the diagram*

$$\begin{array}{ccc} pr_1'^* \circ \mathbb{R}f_{1,*}(\mathcal{E}_1) & \xrightarrow{\sim} & \mathbb{R}f_* \circ pr_1^*(\mathcal{E}_1) \\ \downarrow & & \downarrow \\ pr_1'^* \circ f^{(m)}_{1,+}(\mathcal{E}_1) & \xrightarrow{\sim} & f^{(m)}_{+} \circ pr_1^*(\mathcal{E}_1), \end{array} \tag{5.3.6.1.1}$$

where the top isomorphism is the usual base change isomorphism.

Proof. This is an easy consequence of Theorem 5.3.5.13. Indeed, with notation 5.3.5.1, recall $pr_1^{(m)!} = pr_1^*[\dim T]$, $pr_1'^{(m)!} = pr_1'^*[\dim T]$. Next, consider the following diagram

$$\begin{array}{ccccccc} pr_1'^* \circ \mathbb{R}f_{1,*}(\mathcal{E}_1) & \xrightarrow{\sim} & pr_1'^*\mathbb{R}f_{1*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^*\mathcal{O}_T & \longrightarrow & \mathbb{R}f_*(pr_1^*(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_X} pr_2^*\mathcal{O}_T) & \xrightarrow{\sim} & \mathbb{R}f_* \circ pr_1^*(\mathcal{E}_1) \\ \| & & \| & & \| & & \| \\ pr_1'^* \circ \mathbb{R}f_{1,*}(\mathcal{E}_1) & \xrightarrow{\sim} & \mathbb{R}f_{1,*}(\mathcal{E}_1) \overset{\mathbb{L}}{\boxtimes} \mathcal{O}_T & \underset{\sim}{\xrightarrow{5.3.5.11.1}} & \mathbb{R}f_*(\mathcal{E}_1 \overset{\mathbb{L}}{\boxtimes} \mathcal{O}_T) & \xrightarrow{\sim} & \mathbb{R}f_* \circ pr_1^*(\mathcal{E}_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ pr_1'^* \circ f^{(m)}_{1,+}(\mathcal{E}_1) & \xrightarrow{\sim} & f^{(m)}_{1,+}(\mathcal{E}_1) \overset{\mathbb{L}}{\boxtimes} \mathcal{O}_T & \underset{5.3.5.13}{\xrightarrow{\sim}} & f^{(m)}_{+}(\mathcal{E}_1 \overset{\mathbb{L}}{\boxtimes} \mathcal{O}_T) & \xrightarrow{\sim} & f^{(m)}_{+}(pr_1^*(\mathcal{E}_1)) \end{array} \tag{5.3.6.1.2}$$

Following 5.3.5.13, the middle square (of the bottom) is commutative. The left and right squares are commutative by functoriality. By compositing the bottom isomorphisms, we get $pr_1^* \circ f^{(m)}_{1,+}(\mathcal{E}_1) \xrightarrow{\sim} f^{(m)}_{+} \circ pr_1^*(\mathcal{E}_1)$. It remains to check that the composition of the top isomorphisms is the base change isomorphism.

Consider the commutative diagram

$$\begin{array}{ccccccc} pr_1'^* \circ \mathbb{R}f_{1,*}(\mathcal{E}_1) & \xrightarrow{\sim} & pr_1'^*\mathbb{R}f_{1*}(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^*\mathcal{O}_T & \xrightarrow{\sim} & \mathbb{R}f_*(pr_1^*(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_X} pr_2^*\mathcal{O}_T) & \xrightarrow{\sim} & \mathbb{R}f_* pr_1^*(\mathcal{E}_1) \\ \downarrow\sim & & \downarrow\sim & & \uparrow\sim & & \| \\ \mathbb{R}f_* pr_1^*(\mathcal{E}_1) & \xrightarrow{\sim} & \mathbb{R}f_* pr_1^*(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^*\mathcal{O}_T & \xrightarrow{5.3.5.5.3} & \mathbb{R}f_*(pr_1^*(\mathcal{E}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathbb{L}f^* pr_2'^*\mathcal{O}_T) & \xrightarrow{\sim} & \mathbb{R}f_* pr_1^*(\mathcal{E}_1), \end{array} \tag{5.3.6.1.3}$$

where the middle square is commutative (this is the outer of the diagram 5.3.5.11.5), the top arrows are the same as that of 5.3.6.1.2, where the right and left squares are commutative by functoriality. Using 5.3.5.5.5, we get that the composition of the bottom morphisms is the identity. Hence we are done. □

Notation 5.3.6.2. Let $g \colon Z \to T$ be a smooth morphism of S-schemes. As for [Har66, III.2], we define a functor $g^\sharp \colon D(\mathcal{O}_T) \to D(\mathcal{O}_Z)$ by $g^\sharp(\mathcal{M}) := g^*(\mathcal{M}) \otimes_{\mathcal{O}_T} \omega_{Z/T}[d_{Z/T}]$. We remark that if $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_{T})$ then $g^\sharp(\mathcal{M}) \xrightarrow{\sim} g^{(m)!}(\mathcal{M})$.

Proposition 5.3.6.3. *We keep notation 5.3.6.2.*

(a) For any $\mathcal{M}_1 \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{X_1})$, we have the isomorphism

$$pr_1'^\sharp \circ \mathbb{R}f_{1,*}(\mathcal{M}_1) \xrightarrow{\sim} \mathbb{R}f_* \circ pr_1^\sharp(\mathcal{M}_1) \tag{5.3.6.3.1}$$

of $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_Y)$ canonically induced by the usual base change isomorphism.

(b) For any $\mathcal{M}_1 \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{X_1})$, we have the isomorphism the canonical $pr_1'^{(m)!} \circ f^{(m)}_{1,+}(\mathcal{M}_1) \xrightarrow{\sim} f^{(m)}_+ \circ pr_1^{(m)!}(\mathcal{M}_1)$ of $D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_Y)$ making commutative the diagram

$$\begin{array}{ccc} pr_1'^{\sharp} \circ \mathbb{R}f_{1,*}(\mathcal{M}_1) & \xrightarrow[5.3.6.3.1]{\sim} & \mathbb{R}f_* \circ pr_1^{\sharp}(\mathcal{M}_1) \\ \downarrow & & \downarrow \\ pr_1'^{(m)!} \circ f^{(m)}_{1,+}(\mathcal{M}_1) & \xrightarrow{\sim} & f^{(m)}_+ \circ pr_1^{(m)!}(\mathcal{M}_1). \end{array} \tag{5.3.6.3.2}$$

Proof. Again, this is an easy consequence of Theorem 5.3.5.13. Indeed, with notation 5.3.5.1, since $pr_2'^*\omega_T \xrightarrow{\sim} \omega_{Y/T}$ and $pr_2^*\omega_T \xrightarrow{\sim} \omega_{X/T}$, we get the commutative diagram

$$\begin{array}{ccccccc} pr_1'^{\sharp} \circ \mathbb{R}f_{1,*}(\mathcal{M}_1) & \xrightarrow{\sim} & pr_1'^{*}\mathbb{R}f_{1*}(\mathcal{M}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^*\omega_T[d_T] & \xrightarrow[5.3.5.11.2]{\sim} & \mathbb{R}f_*(pr_1^*(\mathcal{M}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_X} pr_2^*\omega_T)[d_T] & \xrightarrow{\sim} & \mathbb{R}f_* \circ pr_1^{\sharp}(\mathcal{M}_1) \\ \| & & \| & & \| & & \| \\ pr_1'^{\sharp} \circ \mathbb{R}f_{1,*}(\mathcal{M}_1) & \xrightarrow{\sim} & \mathbb{R}f_{1,*}(\mathcal{M}_1) \overset{\mathbb{L}}{\boxtimes} \omega_T[d_T] & \xrightarrow[\sim]{5.3.5.11.1} & \mathbb{R}f_*(\mathcal{M}_1 \overset{\mathbb{L}}{\boxtimes} \omega_T)[d_T] & \xrightarrow{\sim} & \mathbb{R}f_* \circ pr_1^{\sharp}(\mathcal{M}_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ pr_1'^{(m)!} \circ f^{(m)}_{1,+}(\mathcal{M}_1) & \xrightarrow{\sim} & f^{(m)}_{1,+}(\mathcal{M}_1) \overset{\mathbb{L}}{\boxtimes} \omega_T[d_T] & \xrightarrow[5.3.5.13]{\sim} & f^{(m)}_+(\mathcal{M}_1 \overset{\mathbb{L}}{\boxtimes} \omega_T)[d_T] & \xrightarrow{\sim} & f^{(m)}_+(pr_1^{(m)!}(\mathcal{M}_1)). \end{array} \tag{5.3.6.3.3}$$

The commutative diagram 5.3.6.3.2 corresponds to the outer of 5.3.6.3.3.

□

5.3.7 Trace map, relative duality isomorphism, adjunction for proper morphisms

We just review in this subsection the results of [Vir00], [Vir04] concerning Virrion's relative duality isomorphism and its related properties.

Let S be a smooth scheme on $\mathrm{Spec}(\mathcal{V}/\mathfrak{m}^{i+1})$ for some integer $i \in \mathbb{N}$ or a regular scheme over $\mathrm{Spec}\, \mathbb{Z}_{(p)}$. We assume that S is Noetherian and of finite Krull dimension. Let $f\colon X \to Y$ be a proper morphism of smooth S schemes.

Notation 5.3.7.1. We get a locally free $\mathcal{O}_X$-module by setting

$$\mathcal{H}^k_{X/S} := \varinjlim_{n \geq 0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S,(m)}(k), \mathcal{O}_X).$$

where $\mathcal{P}^n_{X/S,(m)}(k)$ is the partial divided power envelope of level m and order n defined by the closed immersion $X \hookrightarrow X^{k+1}/S$.

The Čech-Alexander complex was mentioned in [Grot68, 6.2]; [Ber74, V, 1.2.3] ; [BO78, 5.29] ; [Vir04, p.1048]. Virrion proved this arithmetic analogue:

Theorem 5.3.7.2 (Čech-Alexander resolution)**.** *The following properties hold.*

(a) The complex $\mathcal{D}^{(m)}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{H}^{\bullet}_{X/S}$ is a left resolution of $\mathcal{O}_X$ as a left $\mathcal{D}^{(m)}_{X/S}$-module.

(b) Let $\mathcal{F}$ be a right $\mathcal{D}^{(m)}_{X/S}$-module. Then the Čech-Alexander complex $\check{C}A^*(\mathcal{F})$ defined by

$$\check{C}A^*(\mathcal{F})^k := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}^k_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S}$$

is a canonical left resolution of $\mathcal{F}$ by induced right $\mathcal{D}^{(m)}_{X/S}$-modules.

(c) Let $\mathcal{F}^{\bullet}$ be a bounded above complex of right $\mathcal{D}^{(m)}_{X/S}$-modules. Then $\check{C}A^*(\mathcal{F}^{\bullet})$ is a resolution of $\mathcal{F}^{\bullet}$ by induced right $\mathcal{D}^{(m)}_{X/S}$-modules.

Sketch of the proof. The part (a) is the result of a study of the $\mathcal{O}_X$-linear dual of the Čech-Alexander complex of the linearization of $\mathcal{O}_X$ - see [Vir04] pp. 1047-1061. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are much easier and can be checked as an exercice. The reader can found a complete proof at [Vir04, II.2.4, II.2.5, II.3.2, II.3.4.1]. □

5.3.7.3 (Trace map)**.** The Grothendieck-Hartshorne duality theory of [Har66] then provides an $\mathcal{O}_Y$-linear trace morphism:

$$\mathrm{Tr}_f : \mathbb{R}f_*(\omega_{X/S}[d_{X/S}]) \to \omega_{Y/S}[d_{Y/S}]. \tag{5.3.7.3.1}$$

The $\mathcal{D}^{(m)}_{X/S}$-linear canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{D}^{(m)}_{X\to Y/S}$ induces

$$\mathbb{R}f_*(\omega_{X/S}[d_{X/S}]) \to \mathbb{R}f_*(\omega_{X/S}[d_{X/S}] \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X/S}} \mathcal{D}^{(m)}_{X\to Y/S}) = f_+(\omega_{X/S}[d_{X/S}]). \tag{5.3.7.3.2}$$

Let $\mathcal{K}^\bullet_{X/S}$ be the Cousin complex of $\omega_{X/S}[d_{X/S}]$ (see [Har66, IV.§2,3]), which is a resolution of $\omega_{X/S}[d_{X/S}]$ by injective right $\mathcal{D}^{(m)}_{X/S}$-modules. Hence, using 5.3.7.2 , we get the isomorphisms:

$$\mathbb{R}_*(\omega_{X/S}[d_{X/S}]) \xrightarrow{\sim} \mathbb{R}f_*(\check{C}A^*(\mathcal{K}^\bullet_{X/S})) \xrightarrow{\sim} f_*(\mathcal{K}^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y/S})$$

$$f_+(\omega_{X/S}[d_{X/S}]) \xrightarrow{\sim} \mathbb{R}f_*(\check{C}A^*(\mathcal{K}^\bullet_{X/S}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X/S}} \mathcal{D}^{(m)}_{X\to Y/S}) \xrightarrow{\sim} f_*(\mathcal{K}^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{H}^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y/S})$$

$$\omega_{Y/S}[d_{Y/S}] \xrightarrow{\sim} \check{C}A^*(\mathcal{K}^\bullet_{Y/S}) = \mathcal{K}^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y/S}.$$

Virrion constructs in ([Vir04, pp. 1074-1090]) the following horizontal morphism

$$\begin{array}{ccc} f_*(\mathcal{K}^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{H}^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y/S}) & \longrightarrow & \mathcal{K}^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{H}^\bullet_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_{Y/S} \\ \uparrow & \nearrow & \\ f_*(\mathcal{K}^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{H}^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S}) & & \end{array}$$

where the oblique arrow is induced by the map $f_*(\mathcal{K}^\bullet_{X/S}) \to \mathcal{K}^\bullet_{Y/S}$ which appears in the construction of 5.3.7.3.1, where the vertical arrow is induced by functoriality from $\mathcal{D}^{(m)}_{X/S} \to \mathcal{D}^{(m)}_{X\to Y/S}$, making commutative the diagram. This yields the construction of

$$\mathrm{Tr}_{+,f} : f_+(\omega_{X/S}[d_{X/S}]) \to \omega_{Y/S}[d_{Y/S}] \tag{5.3.7.3.3}$$

making commutative the diagram

$$\begin{array}{ccc} f_+(\omega_{X/S}[d_{X/S}] & \xrightarrow{\mathrm{Tr}_{+,f}} & \omega_{Y/S}[d_{Y/S}] \\ {\scriptstyle 5.3.7.3.2}\uparrow & \nearrow^{\mathrm{Tr}_f}_{5.3.7.3.1} & \\ \mathbb{R}f_*(\omega_{X/S})[d_{X/S}]. & & \end{array} \tag{5.3.7.3.4}$$

Theorem 5.3.7.4 (Virrion)**.** *Let* $* \in \{\mathrm{l}, \mathrm{r}\}$*. Let* $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^*\mathcal{D}^{(m)}_{X/S})$*. We have in* $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y/S})$ *the canonical isomorphism*

$$\chi\colon f^{(m)}_+ \circ \mathbb{D}^{(m)}_X(\mathcal{E}) \xrightarrow{\sim} \mathbb{D}^{(m)}_Y \circ f^{(m)}_+(\mathcal{E}) \tag{5.3.7.4.1}$$

satisfying the transitivity condition for the composition of two proper morphisms.

Proof. 1) Using the trace map, we construct the morphism 5.3.7.4.1 as follows: We can suppose $* = \mathrm{r}$. We construct the composite morphism:

$$\begin{aligned} f^{(m)}_+ \circ \mathbb{D}_X(\mathcal{E}) &\xrightarrow{\sim} \mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X/S}} \mathcal{D}^{(m)}_{X\to Y}\right)[d_X] \\ &\xrightarrow[4.6.3.6.1]{\sim} \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y})[d_X] \\ &\to \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{f^{-1}\mathcal{D}^{(m)}_Y}(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}, (\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y})[d_X] \\ &\xrightarrow[4.6.5.7.1]{} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_Y}\left(\mathbb{R}f_*(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}), \mathbb{R}f_*\left((\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}\right)\right)[d_X]. \end{aligned} \tag{5.3.7.4.2}$$

By using flat resolutions, it follows by functoriality from 4.2.6.1.1 that we have the isomorphism of complexes of right $\mathcal{D}_X^{(m)}$-bimodules:

$$\mathbb{R}f_*\left((\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}\right) \xrightarrow{\sim} \mathbb{R}f_*\left(\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} (\mathcal{D}^{(m)}_{X\to Y} \otimes_{\mathcal{O}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y})\right). \tag{5.3.7.4.3}$$

We have the isomorphism of left $\mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_S} f^{-1}(\mathcal{D}_X^{(m)})^{\mathrm{op}} \otimes_{\mathcal{O}_S} f^{-1}(\mathcal{D}_X^{(m)})^{\mathrm{op}}$-modules

$$\begin{aligned}\mathcal{D}^{(m)}_{X\to Y} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X\to Y} &\xrightarrow{\sim} f^*_{\mathrm{l}}(\mathcal{D}^{(m)}_Y \otimes^{\mathrm{l,l}}_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y) \underset{\gamma}{\xleftarrow{\sim}} f^*_{\mathrm{l}}(\mathcal{D}^{(m)}_Y \otimes^{\mathrm{r,l}}_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y)\\ &\xrightarrow{\sim} \mathcal{D}^{(m)}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}^{(m)}_Y,\end{aligned} \tag{5.3.7.4.4}$$

where $\mathcal{D}^{(m)}_Y \otimes^{\mathrm{l,l}}_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y$ is a left, right, right $\mathcal{D}^{(m)}_Y$-trimodule whose left $\mathcal{D}^{(m)}_Y$-module structure comes from the tensor product structure given by the left $\mathcal{D}^{(m)}_Y$-module structure of both $\mathcal{D}^{(m)}_Y$ and whose the two structures of right $\mathcal{D}^{(m)}_Y$-modules are induced by functoriality (the 'second" one is given by $\mathcal{D}^{(m)}_Y$ at the right by convention), where $\mathcal{D}^{(m)}_Y \otimes^{\mathrm{r,l}}_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y$ is a left, right, right $\mathcal{D}^{(m)}_Y$-trimodules whose second right $\mathcal{D}^{(m)}_Y$-module structure comes from the right (resp. left) $\mathcal{D}^{(m)}_Y$-module structure of the left (resp. right) term $\mathcal{D}^{(m)}_Y$ (and the two other $\mathcal{D}^{(m)}_Y$-modules structures are obtained by functoriality) and where γ is the transposition isomorphism of $\mathcal{D}^{(m)}_Y$ (see 4.2.5.1.1). This yields

$$\mathbb{R}f_*\left(\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} (\mathcal{D}^{(m)}_{X\to Y} \otimes_{\mathcal{O}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y})\right) \xrightarrow{\sim} \mathbb{R}f_*\left((\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}^{(m)}_Y\right). \tag{5.3.7.4.5}$$

By applying the projection isomorphism 7.5.7.3, we get

$$\mathbb{R}f_*\left((\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}^{(m)}_Y)\right) \xrightarrow{\sim} \mathbb{R}f_*\left(\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}\right) \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y. \tag{5.3.7.4.6}$$

By applying the functor $(-\otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y)$ to the trace map 5.3.7.3.3, we get

$$\mathbb{R}f_*\left(\omega_X \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}\right) \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y \to \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y[-d_{X/Y}] \tag{5.3.7.4.7}$$

By composing 5.3.7.4.3, 5.3.7.4.5, 5.3.7.4.6, 5.3.7.4.7, we get

$$\mathbb{R}f_*\left((\omega_X \otimes_{\mathcal{O}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}\right) \to \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m)}_Y[-d_{X/Y}]. \tag{5.3.7.4.8}$$

By applying the functor $\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_Y}(\mathbb{R}f_*(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}), -)$ to 5.3.7.4.8 and composing it to 5.3.7.4.2, we get the first morphism:

$$f^{(m)}_+ \circ \mathbb{D}_X(\mathcal{E}) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_Y}(\mathbb{R}f_*(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathcal{D}^{(m)}_{X\to Y}), \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y)[d_X] \xrightarrow{\sim} \mathbb{D}_Y \circ f^{(m)}_+(\mathcal{E}). \tag{5.3.7.4.9}$$

2) Let us now sketch the fact that is an isomorphism. By applying base change we can reduce to the case when S is a regular scheme. By using [Har66, I.7.1.(i)], we reduce to the case where $\mathcal{E}$ is a coherent $\mathcal{D}^{(m)}_{X/S}$-module. And then because both functors $f^{(m)}_+ \circ \mathbb{D}^{(m)}_X$ and $\mathbb{D}^{(m)}_Y \circ f^{(m)}_+$ are way-out right, since any coherent $\mathcal{D}^{(m)}_{X/S}$-module is a quotient of a $\mathcal{D}^{(m)}_{X/S}$-module of the form $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X/S}$, where $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of finite type, then following [Har66, I.7.1.(iv)] we reduce to check χ is an isomorphism for such a module. For such an induced module, using the duality for coherent $\mathcal{O}_X$-modules as given in [Har66, VII, (3.4,)c], Virrion constructed an isomorphism χ' (see [Vir04, IV.2.2.4]). The equality $\chi = \chi'$ and is a consequence of the commutativity of 5.3.7.3.4: see [Vir04, IV.2.2.5]. □

Corollary 5.3.7.5 (Virrion)**.** *Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{X/S})$, and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y/S})$. We have the isomorphisms*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{Y/S}}(f^{(m)}_+(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, f^{(m)!}(\mathcal{F})). \tag{5.3.7.5.1}$$

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}^{(m)}_{Y/S}}(f^{(m)}_+(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, f^{(m)!}(\mathcal{F})). \tag{5.3.7.5.2}$$

Proof. Since we have $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y/S}) = D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{(m)}_{Y/S})$, then using 4.6.3.6.1, we get the canonical isomorphism:

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{Y/S}}(f^{(m)}_{+}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} (\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{Y/S}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{Y/S}}\left(f^{(m)}_{+}(\mathcal{E}), \mathcal{D}^{(m)}_{Y/S} \otimes_{\mathcal{O}_Y} \omega^{-1}_Y\right).$$

Via 5.3.7.4, this yields by composition the isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{Y/S}}(f^{(m)}_{+}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} (\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{Y/S}} f^{(m)}_{+}\left(\mathbb{D}_X(\mathcal{E})\right)[-d_Y]. \tag{5.3.7.5.3}$$

Via the projection formula of 5.1.2.5.1, the right term of 5.3.7.5.3 is isomorphic to

$$\mathbb{R}f_*\left(f^{-1}\left(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{F}\right) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{(m)}_{Y/S}} \mathcal{D}^{(m)}_{Y\leftarrow X} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X/S}} \mathbb{D}_X(\mathcal{E})\right)[-d_Y]. \tag{5.3.7.5.4}$$

Using the isomorphisms $\left(f^{-1}\left(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{F}\right) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{(m)}_{Y/S}} \mathcal{D}^{(m)}_{Y\leftarrow X}\right) \otimes_{\mathcal{O}_X} \omega^{-1}_X[d_f] \xrightarrow[5.1.1.5.1]{\sim} f^{(m)!}(\mathcal{F})$ and $\omega_X \otimes_{\mathcal{O}_X} \mathbb{D}_X(\mathcal{E})[-d_X] \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, \mathcal{D}^{(m)}_{X/S})$, the term of 5.3.7.5.4 is isomorphic to

$$\mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, \mathcal{D}^{(m)}_{X/S}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X/S}} f^{(m)!}(\mathcal{F})\right) \xrightarrow[4.6.3.6.1]{\sim} \mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X/S}}(\mathcal{E}, f^{(m)!}(\mathcal{F})\right).$$

□

5.3.8 Trace map, relative duality isomorphism and adjunction for projective morphisms

We have seen at 5.3.7 that following Virrion proved the relative duality isomorphism and the adjoint pair $(f_+, f^!)$ for a proper morphism f. The key point is to construct a trace map which is compatible with that of Grothendieck for coherent $\mathcal{O}$-modules (i.e. we have a commutative diagram of the form 5.3.8.3.2). In general, this key point is highly technical and corresponds to the hard part of the proof of a relative duality isomorphism (see [Vir04]). In this subsection, we show how this is much easier to construct such a trace map in the case of projective morphisms by using base change in the projection case and by using the case of a closed immersion (see 5.2.6.6).

We keep notation 5.3.5 and we suppose $n = 2$, log structures are trivial (so we remove $\sharp$ in the notation), f_2 is the identity, $X_1 = \mathbb{P}^d_{Y_1}$, $f_1 \colon \mathbb{P}^d_{Y_1} \to Y_1$ is the canonical projection. We set $T := X_2 = Y_2$.

Lemma 5.3.8.1. *For any $\mathcal{N}_1 \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{Y_1})$, we have the commutative diagram (see notation 5.3.6.2)*

$$\begin{array}{ccccc} pr'^{\sharp}_1 \circ \mathbb{R}f_{1,*} \circ f^{\sharp}_1(\mathcal{N}_1) & \xrightarrow[5.3.6.3.1]{\sim} & \mathbb{R}f_* \circ pr^{\sharp}_1 \circ f^{\sharp}_1(\mathcal{N}_1) & \xrightarrow{\sim} & \mathbb{R}f_* \circ f^{\sharp} \circ pr'^{\sharp}_1(\mathcal{N}_1) \\ \downarrow{\scriptstyle \mathrm{Tr}_{f_1}} & & & & \downarrow{\scriptstyle \mathrm{Tr}_f} \\ pr'^{\sharp}_1(\mathcal{N}_1) & = & = & = & pr'^{\sharp}_1(\mathcal{N}_1). \end{array} \tag{5.3.8.1.1}$$

Proof. Following the commutativity of the diagram 5.3.5.11.5, since $\omega_{Y/Y_1} \xrightarrow{\sim} pr'^{*}_2\omega_T$ and $\omega_{X/X_1} \xrightarrow{\sim} pr^{*}_2\omega_T$ the isomorphism $pr'^{\sharp}_1 \circ \mathbb{R}f_{1,*} \circ f^{\sharp}_1(\mathcal{N}_1) \xrightarrow[5.3.6.3.1]{\sim} \mathbb{R}f_* \circ pr^{\sharp}_1 \circ f^{\sharp}_1(\mathcal{N}_1)$ is given by the composition of the left vertical arrows of the diagram below:

$$\begin{array}{ccccc}
pr_1'^{\sharp} \circ \mathbb{R}f_{1,*} \circ f_1^{\sharp}(\mathcal{N}_1) & \xrightarrow{\mathrm{Tr}_{f_1}} & pr_1'^{\sharp}(\mathcal{N}_1) & & \\
\downarrow \wr & & \downarrow \wr & & \\
pr_1'^{*}\mathbb{R}f_{1*}f_1^{\sharp}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^{*}\omega_T[d_T] & \xrightarrow{\mathrm{Tr}_{f_1}\otimes \mathrm{id}} & pr_1'^{*}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^{*}\omega_T[d_T] & = & \\
\downarrow \wr & & \uparrow \mathrm{Tr}_f \otimes \mathrm{id} & & \\
\mathbb{R}f_* pr_1^{*} f_1^{\sharp}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^{*}\omega_T[d_T] & \xrightarrow{\sim} & \mathbb{R}f_* f^{\sharp} pr_1'^{*}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^{*}\omega_T[d_T] & \xrightarrow{\mathrm{Tr}_f\otimes \mathrm{id}} & pr_1'^{*}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^{*}\omega_T[d_T] \\
\downarrow \wr & & \downarrow \wr & & \uparrow \mathrm{Tr}_f \\
\mathbb{R}f_*(pr_1^{*} f_1^{\sharp}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} f^* pr_2'^{*}\omega_T)[d_T] & \xrightarrow{\sim} & \mathbb{R}f_*(f^{\sharp} pr_1'^{*}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} f^* pr_2'^{*}\omega_T)[d_T] & \xrightarrow{\sim} & \mathbb{R}f_* f^{\sharp}(pr_1'^{*}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2'^{*}\omega_T)[d_T] \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathbb{R}f_*(pr_1^{*} f_1^{\sharp}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2^{*}\omega_T)[d_T] & \xrightarrow{\sim} & \mathbb{R}f_*(f^{\sharp} pr_1'^{*}(\mathcal{N}_1) \otimes^{\mathbb{L}}_{\mathcal{O}_Y} pr_2^{*}\omega_T)[d_T] & & \\
\downarrow \wr & & & & \\
\mathbb{R}f_* \circ pr_1^{\sharp} \circ f_1^{\sharp}(\mathcal{N}_1) & & \xrightarrow{\sim} & & \mathbb{R}f_* \circ f^{\sharp} \circ pr_1'^{\sharp}(\mathcal{N}_1)
\end{array}$$

Following [Har66, III.10.5.Tra 4)], the square of the second row is commutative. Following [Har66, III.4.4], the right square of the third line is commutative. The commutativity of the other squares, of the triangle or trapeze are obvious. Hence, we are done. □

Proposition 5.3.8.2. *Let $\mathcal{N}_1 \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{Y_1})$. Suppose we have the canonical morphism $\mathrm{Tr}_{+,f_1} \colon f^{(m)}_{1,+} \circ f_1^{(m)!}(\mathcal{N}_1) \to \mathcal{N}_1$ of $D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{Y_1})$ making commutative the diagram*

$$\begin{array}{ccc}
\mathbb{R}f_{1,*} \circ f_1^{\sharp}(\mathcal{N}_1) & \xrightarrow{\mathrm{Tr}_{f_1}} & \mathcal{N}_1 \\
\downarrow & \nearrow_{\mathrm{Tr}_{+,f_1}} & \\
f^{(m)}_{1,+} \circ f_1^{(m)!}(\mathcal{N}_1). & &
\end{array} \tag{5.3.8.2.1}$$

Then, there exists a canonical morphism $\mathrm{Tr}_{+,f} \colon f^{(m)}_{+} \circ f^{(m)!} \circ pr_1'^{(m)!}(\mathcal{N}_1) \to pr_1'^{(m)!}(\mathcal{N}_1)$ of $D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{Y})$ making commutative the diagram

$$\begin{array}{ccc}
\mathbb{R}f_* \circ f^{\sharp} \circ pr_1'^{\sharp}(\mathcal{N}_1) & \xrightarrow{\mathrm{Tr}_f} & pr_1'^{\sharp}(\mathcal{N}_1) \\
\downarrow & & \downarrow \wr \\
f^{(m)}_{+} \circ f^{(m)!} \circ pr_1'^{(m)!}(\mathcal{N}_1) & \xrightarrow{\mathrm{Tr}_{+,f}} & pr_1'^{(m)!}(\mathcal{N}_1).
\end{array} \tag{5.3.8.2.2}$$

Proof. By definition, we define the morphism $\mathrm{Tr}_{+,f} \colon f^{(m)}_{+} \circ f^{(m)!} \circ pr_1'^{(m)!}(\mathcal{N}_1) \to pr_1'^{(m)!}(\mathcal{N}_1)$ to be the one making commutative the bottom of the diagram:

$$\begin{array}{ccccccc}
pr_1'^{\sharp} \circ \mathbb{R}f_{1,*} \circ f_1^{\sharp}(\mathcal{N}_1) & \underset{5.3.6.3.1}{\xrightarrow{\sim}} & \mathbb{R}f_* \circ pr_1^{\sharp} \circ f_1^{\sharp}(\mathcal{N}_1) & \xrightarrow{\sim} & \mathbb{R}f_* \circ f^{\sharp} \circ pr_1'^{\sharp}(\mathcal{N}_1) & \xrightarrow{\mathrm{Tr}_f} & pr_1'^{\sharp}(\mathcal{N}_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \wr \\
pr_1'^{(m)!} \circ f^{(m)}_{1,+} \circ f_1^{(m)!}(\mathcal{N}_1) & \xrightarrow{\sim} & f^{(m)}_{+} \circ pr_1^{(m)!} \circ f_1^{(m)!}(\mathcal{N}_1) & \xrightarrow{\sim} & f^{(m)}_{+} \circ f^{(m)!} \circ pr_1'^{(m)!}(\mathcal{N}_1) & \xrightarrow{\mathrm{Tr}_{+,f}} & pr_1'^{(m)!}(\mathcal{N}_1).
\end{array}$$

(5.3.8.2.3)

Following 5.3.8.1.1, the top of the diagram 5.3.8.2.3 is commutative. From the commutativity of 5.3.8.2.1, we get the commutativity of the outer of 5.3.8.2.3. From the commutative diagram 5.3.6.3.2, we get the commutativity of the left square of 5.3.8.2.3. The commutativity of the middle square of 5.3.8.2.3 is easy. This yields that the right square of 5.3.8.2.3 is indeed commutative. □

5.3.8.3. Suppose $Y_1 = S$, $X_1 = \mathbb{P}^d_S$, $f_1 \colon \mathbb{P}^d_S \to S$ is the canonical projection and $\mathcal{N}_1 = \mathcal{O}_S \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{Y_1/S}) = D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_S)$. We have $f_1^{(m)!}(\mathcal{O}_S) = f_1^{\sharp}(\mathcal{O}_S) = \omega_{\mathbb{P}^d_S/S}[d]$ and the trace map $\mathrm{Tr}_{f_1} : \mathbb{R}f_*(\omega_{\mathbb{P}^d_S/S})[d] \to \mathcal{O}_S$ is an isomorphism of $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_S)$. The canonical morphism $\mathbb{R}f_*(\omega_{\mathbb{P}^d_S/S})[d] \to f^{(m)}_{1,+}(\omega_{\mathbb{P}^d_S/S})[d]$ is an isomorphism after applying the truncation functor $\tau_{\geq 0}$. Indeed, $f^{(m)}_{1,+}(\omega_{\mathbb{P}^d_S/S})[d] \underset{5.1.3.2.1}{\xrightarrow{\sim}} f^{(m)}_{1,+}(\mathcal{O}_{\mathbb{P}^d_S})[d] \underset{6.2.6.2.2}{\xrightarrow{\sim}} F^{m*}_{\mathbb{P}^d_{S_0}/S_0} f^{(0)}_{1,+}(\mathcal{O}_{\mathbb{P}^d_S})[d] \underset{5.3.2.3.1}{\xrightarrow{\sim}} F^{m*}_{\mathbb{P}^d_{S_0}/S_0} \mathbb{R}f_{1*}\left(\Omega^{\bullet}_{\mathbb{P}^d_S/S}\right)[2d]$. Hence, we get the morphism $\mathrm{Tr}_{+,f_1} : f^{(m)}_{1,+}(\omega_{\mathbb{P}^d_S/S})[d] \to \mathcal{O}_S$ making commutative the diagram

$$\begin{array}{ccc} \mathbb{R}f_{1,*}(\omega_{\mathbb{P}^d_S/S})[d] & \xrightarrow{\mathrm{Tr}_{f_1}} & \mathcal{O}_S \\ \downarrow & \nearrow_{\mathrm{Tr}_{+,f_1}} & \\ f^{(m)}_{1,+}(\omega_{\mathbb{P}^d_S/S})[d]. & & \end{array} \qquad (5.3.8.3.1)$$

Hence, following Proposition 5.3.8.2 and using $pr_1'^{(m)!}(\mathcal{O}_S) = \omega_{T/S}[d_{T/S}]$, there exists a canonical morphism

$$\mathrm{Tr}_{+,f} \colon f^{(m)}_{+}(\omega_{\mathbb{P}^d_T/S})[d + d_{T/S}] \to \omega_{T/S}[d_{T/S}]$$

of $D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{T/S})$ making commutative the diagram

$$\begin{array}{ccc} \mathbb{R}f_* \circ (\omega_{\mathbb{P}^d_T/S})[d] & \xrightarrow{\mathrm{Tr}_{f}} & \omega_{T/S}. \\ \downarrow & \nearrow^{\mathrm{Tr}_{+,f}} & \\ f^{(m)}_{+}(\omega_{\mathbb{P}^d_T/S})[d] & & \end{array} \qquad (5.3.8.3.2)$$

Corollary 5.3.8.4. *Let $f \colon X \to Y$ be a morphism of smooth S-schemes which is the composition of a closed immersion of the form $X \hookrightarrow \mathbb{P}^d_Y$ and of the projection $\mathbb{P}^d_Y \to Y$. Let $* \in \{\mathrm{r}, \mathrm{l}\}$, $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^*\mathcal{D}^{(m)}_X)$. We have the isomorphism of $D^{\mathrm{b}}_{\mathrm{coh}}({}^l\mathcal{D}^{(m)}_Y)$:*

$$\mathbb{D}^{(m)} \circ f^{(m)}_{+}(\mathcal{E}) \xrightarrow{\sim} f^{(m)}_{+} \circ \mathbb{D}^{(m)}(\mathcal{E}). \qquad (5.3.8.4.1)$$

Proof. Following 5.2.6.6, the case of a closed immersion is easily checked. Hence, we reduce to the case where f is the projection $\mathbb{P}^d_Y \to Y$. Using 5.3.8.3.2, to check such an isomorphism, we can copy Virrion's proof of 5.3.7.4. □

Chapter 6

Frobenius

6.1 Frobenius descent

Let $m \geq 0$ be an integer. Let S be a $\mathbb{Z}_{(p)}$-scheme endowed with quasi-coherent m-PD-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S))$, X be a smooth S-scheme. We denote by $S_0 = V(\mathfrak{a}_S)$ and by $X_0 := X \times_S S_0$. We suppose also that the following conditions are satisfied :

(a) p is nilpotent on S (then, thanks to 1.2.4.5.1, $|X| = |X_0|$) ;

(b) $p \in \mathfrak{a}_S$ (then we have a Frobenius on X_0).

Let $s \geq 0$ be a fixed integer, $X_0^{(s)}$ be the S_0-scheme induced from X_0 by base change via the sth power of the absolute Frobenius of T_0. We suppose there exists a smooth S-scheme X' which is a lifting of $X_0^{(s)}$ (i.e. $X' \times_S S_0 \xrightarrow{\sim} X_0^{(s)}$). We suppose moreover there exists $F\colon X \to X'$ an S-morphism lifting the relative Frobenius S_0-morphism $F^s_{X_0/S_0}\colon X_0 \to X_0^{(s)}$. Remark that such lifting exists always Zariski locally on X. Since F is an homeomorphism, then by abuse of notation we might avoid writing F^{-1} or F_* when no confusion is possible.

Beware that, even locally, this is not clear that there exists a lifting $F_S\colon S \to S$ of the absolute Frobenius $F^s_{S_0}\colon S_0 \to S_0$.

We suppose given an $\mathcal{O}_{X'}$-algebra $\mathcal{B}_{X'}$ endowed with a left action of $\mathcal{D}^{(m)}_{X'}$ compatible with its algebra structure. We set $\mathcal{B}_X := F^*\mathcal{B}_{X'}$. We will denote by $\widetilde{\mathcal{D}}^{(m+s)}_{X/S} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m+s)}_X$, $\widetilde{\mathcal{D}}^{(m)}_{X'/S} = \mathcal{B}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'/S}$ (we remove $/S$ in the notation if there is no doubt on S). and $\widetilde{F}$ the morphism of ringed spaces $(X, \mathcal{B}_X) \to (X', \mathcal{B}_{X'})$ induced by F.

We denote by p_0^n and $p_1^n\colon \Delta^n_{X/S(m)} \to X$ the left and right projections. Let $\widetilde{\Delta}^n_{X/S(m)}$ be the ringed space $(\Delta^n_{X/S(m)}, \mathcal{O}_{\widetilde{\Delta}^n_{X/S(m)}})$, with $\mathcal{O}_{\widetilde{\Delta}^n_{X/S(m)}} := p_0^{n*}(\mathcal{B}_X)$. We denote by $\tilde{p}_0^n\colon \widetilde{\Delta}^n_{X/S(m)} \to \widetilde{X}^\sharp$ the morphism induced by the continuous map p_0^n and by the morphism of rings canonical $(p_0^n)^{-1}(\mathcal{B}_X) \to \mathcal{O}_{\widetilde{\Delta}^n_{X/S(m)}}$ and $\tilde{p}_1^n\colon \widetilde{\Delta}^n_{X/S(m)} \to \widetilde{X}^\sharp$ whose morphism of topological spaces is p_1^n and whose ring homomorphism is $(p_1^n)^{-1}(\mathcal{B}_X) \to p_1^{n*}(\mathcal{B}_X) \xrightarrow{\sim} p_0^{n*}(\mathcal{B}_X) = \mathcal{O}_{\widetilde{\Delta}^n_{X/S(m)}}$. For any $\nu \geq 1$, we have similar notation by replacing respectively S and X by T and Y.

6.1.1 Frobenius descent for left $\mathcal{D}$-modules

Notation 6.1.1.1. Let $\nu \geq 1$ be an integer. If Y is a S-scheme, we will denote by $Y^{\nu+1} = Y^{\nu+1}/S$, $\mathcal{I}_\nu$ the ideal of the diagonal $\Delta_{Y/S}(\nu)\colon Y \hookrightarrow Y^{\nu+1}$ and $(\mathcal{P}_{Y/S,(m)}(\nu), \overline{\mathcal{I}}_\nu, \widetilde{\mathcal{I}}_\nu)$ the m-PD-envelope of $\mathcal{I}_\nu$, i.e. with the notation of 1.3.3.5, $(\mathcal{P}_{Y/S,(m)}(\nu), \overline{\mathcal{I}}_\nu, \widetilde{\mathcal{I}}_\nu) = (\mathcal{P}^n_{(m),\gamma}(\Delta_{Y/S}(\nu)), (\mathscr{I}_{(m),\gamma}(\Delta_{Y/S}(\nu)), \mathscr{I}_{(m),\gamma}(\Delta_{Y/S}(\nu)),^{[\,]}))$. When $\nu = 1$, we simply write $(\mathcal{P}_{Y/S,(m)}, \overline{\mathcal{I}}, \widetilde{\mathcal{I}})$.

We denote by $F_\nu\colon X^{\nu+1} \to X'^{\nu+1}$ the morphism $F \times \cdots \times F$ induced by F.

Notation 6.1.1.2 (Local coordinates)**.** Suppose we have an étale morphism of the form $g_0 \colon X_0 \to \mathbb{A}^d_{S_0}$. Since $(\mathbb{A}^d_{S_0})^{(s)} = \mathbb{A}^d_{S_0}$, then we get by definition the right cartesian squares of the diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{F^s_{X_0/S_0}} & X_0^{(s)} & \longrightarrow & X_0 \\ \downarrow{\scriptstyle g_0} & \square & \downarrow{\scriptstyle g_0^{(s)}} & \square & \downarrow{\scriptstyle g_0} \\ \mathbb{A}^d_{S_0} & \xrightarrow[F^s_{\mathbb{A}^d_{S_0}/S_0}]{} & \mathbb{A}^d_{S_0} & \longrightarrow & \mathbb{A}^d_{S_0} \\ & & \downarrow & \square & \downarrow \\ & & S_0 & \xrightarrow{F_{S_0}} & S_0. \end{array} \tag{6.1.1.2.1}$$

Since $g_0 \colon X_0 \to \mathbb{A}^d_{S_0}$ is étale, then the left square is also cartesian.

Lemma 6.1.1.3. *The morphism $F \colon X \to X'$ is finite and is locally free of finite type, i.e. $F_*\mathcal{O}_X$ is a locally free $\mathcal{O}_{X'}$-module of finite type.*

Proof. Since this is local, we come down to the local context of 6.1.1.2. Since $F^s_{\mathbb{A}^d_{S_0}/S_0}$ is locally free of rank ds, using the cartesian left square of 6.1.1.2.1, then so is $F^s_{X_0/S_0}$. Since X/S is flat, then using [Gro66, 11.3.10], we get that F is flat. By reducing to the case where S is noetherian and therefore $\mathfrak{a}_S$ is nilpotent, this yields that F is also finite and then F is free of rank ds. □

Proposition 6.1.1.4. *We have the following properties.*

(a) There exists a unique PD-morphism

$$\Phi^*_\nu \colon F_\nu^{-1}\mathcal{P}_{X',(m)}(\nu) \to \mathcal{P}_{X,(m+s)}(\nu) \tag{6.1.1.4.1}$$

sending $F_\nu^{-1}\widetilde{\mathcal{I}}'_\nu$ to $\widetilde{\mathcal{I}}_\nu + \mathfrak{b}\overline{\mathcal{I}}_\nu$. This yields the morphism $\Phi_\nu \colon \Delta_{X,(m+s)}(\nu) \to \Delta_{X',(m)}(\nu)$. When $\nu = 1$, we remove ν in the notation.

(b) For any $n \in \mathbb{N}$, we have the inclusion

$$\Phi^*_\nu(F_\nu^{-1}\overline{\mathcal{I}}'^{\{n\}_{(m)}}_\nu) \subset \overline{\mathcal{I}}^{\{n\}_{(m+s)}}_\nu.$$

Proof. See a proof in [Ber00, 2.2.2]). □

Corollary 6.1.1.5. *There exists a canonical factorization $\Phi^n_\nu \colon \widetilde{\Delta}^n_{X/S(m+s)}(\nu) \to \widetilde{\Delta}^n_{X'/S(m)}(\nu)$ making commutative the following diagram of ringed spaces*

$$\begin{array}{ccc} \widetilde{\Delta}^n_{X/S(m)}(\nu) & \longrightarrow & \widetilde{\Delta}^n_{X/S(m+s)}(\nu) \\ \downarrow{\scriptstyle \widetilde{F}^n_{\nu,(m)}} & \swarrow{\scriptstyle \Phi^n_\nu} & \downarrow{\scriptstyle \widetilde{F}^n_{\nu,(m+s)}} \\ \widetilde{\Delta}^n_{X'/S(m)}(\nu) & \longrightarrow & \widetilde{\Delta}^n_{X'/S(m+s)}(\nu). \end{array} \tag{6.1.1.5.1}$$

Proof. We have to establish that the canonical morphism $\widetilde{F}^*_{n,(m)} \colon \widetilde{\mathcal{P}}^n_{X'/T(m)} \to \widetilde{\mathcal{P}}^n_{X/T(m)}$ factors through a morphism $\widetilde{\mathcal{P}}^n_{X'/T(m)} \to \widetilde{\mathcal{P}}^n_{X/T(m+s)}$. Following the proposition 6.1.1.4, we have such a factorization without the tildes. This yields the composition morphism $\mathcal{P}^n_{X'/T(m)} \to \mathcal{P}^n_{X/T(m+s)} \to \widetilde{\mathcal{P}}^n_{X/T(m+s)}$. We get a unique factorization $\widetilde{\mathcal{P}}^n_{X'/T(m)} \to \widetilde{\mathcal{P}}^n_{X/T(m+s)}$ which is semi-linear with respect to homomorphism $\mathcal{B}_{X'} \to \mathcal{B}_X = F_*F^*\mathcal{B}_{X'}$. □

Proposition 6.1.1.6. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module (resp. $(\widetilde{\mathcal{D}}^{(m)}_{X'}, \mathcal{D})$-bimodule). The sheaf $\widetilde{F}^*\mathcal{E}'$ is endowed with a structure of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module (resp. $(\widetilde{\mathcal{D}}^{(m+s)}_X, \mathcal{D})$-bimodule). Moreover, the canonical isomorphism $\widetilde{F}^*\mathcal{E}' \xrightarrow{\sim} \widetilde{F}^*\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}'$ is an isomorphism of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules (resp. $(\widetilde{\mathcal{D}}^{(m+s)}_X, \mathcal{D})$-bimodules).*

Proof. By taking the inverse image by the morphisms $\widetilde{\Delta}^n_{X/S(m+s)} \to \widetilde{\Delta}^n_{X'/S(m)}$ (see 6.1.1.5) of the m-PD-stratification with coefficients in $\mathcal{B}_{X'}$ of $\mathcal{E}'$, we obtain an $(m+s)$-PD-stratification with coefficients in $\mathcal{B}_X$ of $\widetilde{F}^*\mathcal{E}'$. We easily check the cocycle condition holds. The respective case is proved in a similar way. The $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-linearity is easy. □

Remark 6.1.1.7. Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module. Following [Ber00, 2.2.7], $F^*(\mathcal{E}')$ is endowed with a structure of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module extending its structure of left $\mathcal{D}^{(m+s)}_{X/S}$-module. We check, as for 4.4.5.2, that the isomorphism $F^*(\mathcal{E}') \xrightarrow{\sim} \widetilde{F}^*(\mathcal{E}')$ is $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-linear. In the rest of the chapter, we denote simply by F^* instead of $\widetilde{F}^*$.

Proposition 6.1.1.8. *Suppose $\mathfrak{a}_S = 0$, $F = F^s_{X/S}$, $X' = X^{(s)}$ and there exists coordinates $t_1, \ldots, t_d$ of X/S. Denotes by $t'_1, \ldots, t'_d$ the coordinates of X'/S induced by base change and by $\underline{\partial}'^{\langle \underline{k}\rangle_{(m)}}$ and $\underline{\tau}'^{\{\underline{k}\}_{(m)}}$ the associated elements.*

(a) The homomorphism $\Phi^* \colon F_\nu^{-1}\mathcal{P}_{X',(m)}(\nu) \to \mathcal{P}_{X,(m+s)}$ of 6.1.1.4.1 satisfies for all $\underline{k} \in \mathbb{N}^d$:

$$(\underline{\tau}'^{\{\underline{k}\}_{(m)}}) = \underline{\tau}^{\{p^s\underline{k}\}_{(m+s)}}. \tag{6.1.1.8.1}$$

(b) For any $n \in \mathbb{N}$, we have the inclusion

$$\Phi^*_\nu(F_\nu^{-1}\overline{\mathcal{I}}'^{\{n\}_{(m)}}_\nu) \subset \overline{\mathcal{I}}^{\{p^s n\}_{(m+s)}}_\nu. \tag{6.1.1.8.2}$$

(c) Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module. The structure of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module of $F^*\mathcal{E}'$ is characterized by the relations:

$$\underline{\partial}^{\langle \underline{k}\rangle_{(m+s)}}(1 \otimes e') = \begin{cases} \underline{\partial}'^{\langle \underline{k}/p^s\rangle_{(m)}}(e') & \textit{if } p^s \textit{ divides } \underline{k} \\ 0 & \textit{otherwise} \end{cases} \tag{6.1.1.8.3}$$

Proof. See [Ber00, 2.2.4]. □

Theorem 6.1.1.9 (Berthelot)**.** *The functor F^* is an equivalence between the category of left (resp. quasi-coherent) $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules and that of left (resp. quasi-coherent) $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules.*

Proof. A proof can be found in [Ber00, 2.3.6]. □

6.1.2 Frobenius descent for right $\mathcal{D}$-modules

6.1.2.1. Following 6.1.1.3, $F_*\mathcal{O}_X$ is an $\mathcal{O}_{X'}$-module of finite type. Since $\mathcal{B}_X = F^*\mathcal{B}_{X'}$, this yields that $F_*\mathcal{B}_X$ is a $\mathcal{B}_{X'}$-module of finite type. Since F is an homeomorphism, we get a structure of $\mathcal{B}_{X'}$-module on $F_*\mathcal{B}_X$ via F. Since F is supposed to be fixed, we simply write $\mathcal{B}_X$ instead of $F_*\mathcal{B}_X$. For any $\mathcal{B}_{X'}$-module $\mathcal{M}'$, this yields the isomorphism

$$\widetilde{F}^\flat\mathcal{M}' = F^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(F_*\mathcal{B}_X, \mathcal{M}') \xrightarrow{\sim} F^{-1}\mathcal{H}om_{\mathcal{B}_{X'}}(F_*\mathcal{B}_X, \mathcal{M}').$$

For simplicity, we might remove F^{-1} and F_* in the notation.

Proposition 6.1.2.2. *Let $\mathcal{M}'$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module (resp. a $(\mathcal{D}, \widetilde{\mathcal{D}}^{(m)}_{X'})$-bimodule). The sheaf $\widetilde{F}^\flat\mathcal{M}'$ is endowed with a structure of right $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module (resp. a $(\mathcal{D}, \widetilde{\mathcal{D}}^{(m+s)}_{X})$-bimodule). Moreover, the canonical isomorphism $\widetilde{F}^\flat\mathcal{M}' \xrightarrow{\sim} \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \widetilde{F}^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}$ is an isomorphism of $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules (resp. $(\mathcal{D}, \widetilde{\mathcal{D}}^{(m+s)}_{X})$-bimodules).*

Proof. Let us check the non-respective case. Following 3.4.3.4, $\mathcal{M}'$ has a structural m-PD-costratifcation. By applying $\widetilde{F}^\flat$ and using 6.1.1.5, we get a canonical structure of $(m+s)$-PD-costratifcation on $\widetilde{F}^\flat\mathcal{M}'$, i.e. $\widetilde{F}^\flat\mathcal{M}'$ is endowed with a structure of right $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module. By functoriality, we get the respective case from the non-respective case. □

Remark 6.1.2.3. Let $\mathcal{M}'$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module. The module $F^\flat(\mathcal{E}')$ is endowed with a structure of right $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module extending its structure of right $\mathcal{D}^{(m+s)}_{X/S}$-module. We can check similarly that $F^\flat(\mathcal{M}') \xrightarrow{\sim} \widetilde{F}^\flat(\mathcal{M}')$ is $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-linear. So we can simply write $F^\flat$ instead of $\widetilde{F}^\flat$.

Lemma 6.1.2.4. *There exists a canonical isomorphism of right $\mathcal{D}_{X/S}$-modules*

$$\mu_X \colon F^\flat(\omega_{X'/S}) \xrightarrow{\sim} \omega_{X/S}. \tag{6.1.2.4.1}$$

Proof. See [Ber00, 2.4.2]. □

Proposition 6.1.2.5. *For any left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module $\mathcal{E}'$, we have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules of the form*

$$\omega_{X/S} \otimes_{\mathcal{O}_X} F^*(\mathcal{E}') \xrightarrow{\sim} F^\flat(\omega_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{E}'). \tag{6.1.2.5.1}$$

Proof. See [Ber00, 2.4.3]. □

This yields the following corollaries (see [Ber00, 2.4.4–5]):

Corollary 6.1.2.6. *For any right $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module $\mathcal{M}'$, we have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules of the form*

$$F^*(\mathcal{M}' \otimes_{\mathcal{O}_X} \omega^{-1}_{X/S}) \xrightarrow{\sim} F^\flat(\mathcal{M}') \otimes_{\mathcal{O}_{X'}} \omega^{-1}_{X'/S}. \tag{6.1.2.6.1}$$

Theorem 6.1.2.7. *The functor $F^\flat$ is an equivalence between the category of right (resp. quasi-coherent) $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules and that of right (resp. quasi-coherent) $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules.*

Proof. This is a consequence of 6.1.1.9 and of 6.1.2.5 and 6.1.2.6 which allows to switch from left to right $\mathcal{D}$-modules. □

6.1.3 Quasi-inverse functor

6.1.3.1. By functoriality, we can check that the functors F^* and $F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} -$ induce exact canonically quasi-inverse equivalences between the category of complexes of left $\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\mathcal{O}_S} (\widetilde{\mathcal{D}}^{(m)}_{X'})^{\mathrm{op}}$-modules (i.e. $(\widetilde{\mathcal{D}}^{(m)}_{X'}, \widetilde{\mathcal{D}}^{(m)}_{X'})$-bimodules) and that of complexes of left $\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\mathcal{O}_S} (\widetilde{\mathcal{D}}^{(m+s)}_{X/S})^{\mathrm{op}}$-modules (i.e. $(\widetilde{\mathcal{D}}^{(m)}_{X'}, \widetilde{\mathcal{D}}^{(m+s)}_{X/S})$-bimodules).

Similarly, by functoriality, we can check that the functors $F^\flat$ and $- \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}$ induce exact canonically quasi-inverse equivalences between the category of complexes of left $\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\mathcal{O}_S} (\widetilde{\mathcal{D}}^{(m)}_{X'})^{\mathrm{op}}$-modules (i.e. $(\widetilde{\mathcal{D}}^{(m)}_{X'}, \widetilde{\mathcal{D}}^{(m)}_{X'})$-bimodules) and that of complexes of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S} \otimes_{\mathcal{O}_S} (\widetilde{\mathcal{D}}^{(m)}_{X'})^{\mathrm{op}}$-modules (i.e. $(\widetilde{\mathcal{D}}^{(m+s)}_{X/S}, \widetilde{\mathcal{D}}^{(m)}_{X'})$-bimodules).

For the convenience of the reader we collect below a number of results whose proofs are given in [Ber00] section 2.5.

Proposition 6.1.3.2. *There exists an isomorphism of $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-bimodules of the form*

$$\vartheta \colon \widetilde{\mathcal{D}}^{(m+s)}_{X/S} \xrightarrow{\sim} F^* F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S}. \tag{6.1.3.2.1}$$

Corollary 6.1.3.3. *(a) The $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules $F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S}$ and $F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'/S}$ are locally projective of finite type*

(b) A left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module $\mathcal{E}'$ (resp. $\mathcal{M}'$) is coherent if and only if $F^*(\mathcal{E}')$ (resp. $F^\flat\mathcal{M}'$) is $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-coherent.

Corollary 6.1.3.4. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module (resp. $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule). Let $\mathcal{M}'$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module (resp. $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule).*

(a) *We have the functorial isomorphisms of (complexes of) left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules (resp. $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule):*

$$F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \xrightarrow{\sim} F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'. \tag{6.1.3.4.1}$$

(b) *We have the functorial isomorphisms of (complexes of) right $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules (resp. $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodules):*

$$F^\flat \mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S} \xrightarrow{\sim} F^\flat \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S} \xrightarrow{\sim} \mathcal{M}'. \tag{6.1.3.4.2}$$

(c) *The functors $F^* = F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} -$ and $F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} -$ (resp. $F^\flat = - \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S}$ and $- \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S}$) induce exact quasi-inverse equivalences of categories between the category of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules and of left (resp. right) $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules.*

Remark 6.1.3.5. Let $\mathcal{D}$ be a sheaf of rings on the topological space $|X'| = |X|$.

(a) It follows by functoriality from 6.1.3.4.(c) that the functors $F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} -$ and $F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} -$ (resp. $- \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S}$ and $- \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S}$) induce exact quasi-inverse equivalences of categories between the category of left (resp. right) $(\widetilde{\mathcal{D}}^{(m)}_{X'/S}, \mathcal{D})$-bimodules and of left (resp. right) $(\widetilde{\mathcal{D}}^{(m+s)}_{X/S}, \mathcal{D})$-bimodules.

(b) Suppose $(\widetilde{\mathcal{D}}^{(m)}_{X'/S}, \mathcal{D})$ and $(\widetilde{\mathcal{D}}^{(m+s)}_{X/S}, \mathcal{D})$ are solved by $\mathcal{O}_S$ (see definition 4.6.3.2). In that case, the functors $F^* = F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} -$ and $F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} -$ (resp. $F^\flat = - \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S}$ and $- \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S}$) induce exact quasi-inverse equivalences of categories between the category of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{O}_S} \mathcal{D}$-modules and of left (resp. right) $\widetilde{\mathcal{D}}^{(m+s)}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{D}$-modules. Let $\mathcal{I}' \in K^-({}^l\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{O}_S} \mathcal{D})$. This yields that if $\mathcal{I}'$ is a K-injective complex of $K^-({}^l\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{O}_S} \mathcal{D})$ if and only if $F^*\mathcal{I}'$ is a K-injective complex of $K^-({}^l\widetilde{\mathcal{D}}^{(m+s)}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{D})$.

Corollary 6.1.3.6. *Let $\mathcal{E}' \in D^-({}^l\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $\mathcal{M}' \in D^-({}^r\widetilde{\mathcal{D}}^{(m)}_{X'/S})$.*

(a) *A left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module $\mathcal{E}'$ (resp. $\mathcal{M}'$) is flat if and only if $F^*\mathcal{E}'$ (resp. $F^\flat\mathcal{M}'$) is a flat left (resp. right) $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-module.*

(b) *Let $a, b \in \mathbb{Z}$ with $a \leq b$. The complex $\mathcal{E}'$ (resp. $\mathcal{M}'$) has tor-amplitude in $[a, b]$ if and only if so is $F^*\mathcal{E}' \in D^-({}^l\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$ (resp. $F^\flat\mathcal{M}' \in D^-({}^r\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$).*

(c) *Let $f\colon X \to S$ be the structural morphism. We have the functorial isomorphism in $D(f^{-1}\mathcal{O}_S)$*

$$F^\flat \mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}'. \tag{6.1.3.6.1}$$

6.1.4 Homological dimension of the sheaf of differential operators of level m

Theorem 6.1.4.1 (Homological dimension)**.** *Suppose S is affine and regular, $f\colon X \to S$ is affine. Suppose the fibers of $f\colon X \to S$ are of dimension d. Let $r := \sup_{t \in f(X)} \mathcal{O}_{T,t}$. Then for any integer $m \in \mathbb{N}$, the ring $D^{(m)}_{X/S} := \Gamma(X, \mathcal{D}^{(m)}_{X/S})$ has homological global dimension equal to $2d + r$.*

Proof. Using the Frobenius descent Theorem 6.1.1.9, we reduce to the case $m = 0$. Then, this is 4.7.3.16. □

Theorem 6.1.4.2 (Montagnon)**.** *Let R be a regular noetherian ring of characteristic p, $S = \operatorname{Spec} R$ endowed with the trivial log structure. Let X be an affine smooth S-scheme endowed with the log structure given by a strict normal crossing divisor D relative to X/S. Then $D^{(m)}_{X/S} := \Gamma(X, \mathcal{D}^{(m)}_{X/S})$ has finite homological global dimension.*

Proof. A proof can be found at [Mon02, Proposition 5.3.1]). This is based on a sort of Frobenius descent. □

6.1.5 Glueing isomorphisms and Frobenius

Proposition 6.1.5.1. *Suppose $\mathfrak{a}_S$ is m-PD-nilpotent. Suppose there exists a second morphism $F'\colon X \to X'$ which is a lifting of $F^s_{X_0/S_0}$.*

(a) Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module. Then the glueing isomorphism $\tau_{F,F'}\colon F^*(\mathcal{E}') \xrightarrow{\sim} F'^*(\mathcal{E}')$ defined in 4.4.5.3.1 is $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-linear.

(b) Let $\mathcal{M}'$ be a right $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module. Then the glueing isomorphism $\sigma_{F,F'}\colon F^\flat(\mathcal{M}') \xrightarrow{\sim} F'^\flat(\mathcal{M}')$ defined in 4.4.5.3.2 is $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-linear.

Proof. We can copy the proof of [Ber00, 2.2.5] (replace $\mathcal{D}$ by $\widetilde{\mathcal{D}}$ for the first statement, the second one replace moreover the symbole $*$ by $\flat$). □

Corollary 6.1.5.2. *Let $S \to T$ be an m-PD-morphism of schemes endowed with m-PD-nilpotent and quasi-coherent m-PD-ideals satisfying the hypotheses (a) and (b) of 6.1, X_0 (resp. Y_0) a smooth S_0-scheme (resp. smooth T_0-scheme), $f_0\colon X_0 \to Y_0$ a T_0-morphism, X (resp. Y') a smooth S-scheme (resp. smooth T-scheme) lifting X_0 (resp. $Y_0^{(s)}$).*

(a) The functor $(F^s_{Y_0/T_0} \circ f_0)^* = (f_0^{(s)} \circ F^s_{X_0/S_0})^*$ defined in 4.4.5.10 is canonically equal (up to canonical isomorphism) to the composition of a functor from the category of left $\mathcal{D}^{(m)}_{Y'/T}$-modules to that of left $\mathcal{D}^{(m+s)}_{X/S}$-modules with the restriction functor from the category of the left $\mathcal{D}^{(m+s)}_{X/S}$-modules to that of left $\mathcal{D}^{(m)}_{X/S}$-modules. We still denote by $(F^s_{Y_0/T_0} \circ f_0)^* = (f_0^{(s)} \circ F^s_{X_0/S_0})^*$ this factorization.

(b) This isomorphism is compatible with the composition: suppose given a second m-PD-morphism $T \to U$ with m-PD-nilpotent m-PD-ideals, a morphism $g_0\colon Y_0 \to Z_0$, where Z_0 is smooth over U_0, and a smooth lifting Z'' of $Z_0^{(s+s')}$. Then there exists a canonical isomorphism of functors of the category of left $\mathcal{D}^{(m)}_{Z''/S}$-modules to that of left $\mathcal{D}^{(m+s+s')}_{X/S}$-modules for the above structures of the form

$$(F^s_{Y_0/T_0} \circ f_0)^* \circ (F^{s'}_{Z_0^{(s)}/T_0} \circ g_0^{(s)})^* \xrightarrow{\sim} (F^{s+s'}_{Z_0^{(s)}/T_0} \circ g_0 \circ f_0)^*. \tag{6.1.5.2.1}$$

6.1.5.3. With notation 6.1.5.2, when f_0 is the identity we get the functor $(F^s_{X_0/S_0})^*$ from the category of left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules to that of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules.

The functor $(F^s_{X_0/S_0})^\flat$ defined in 4.4.5.10 is canonically equal (up to canonical isomorphism) to the composition of a functor from the category of right $\mathcal{D}^{(m)}_{X'/S}$-modules to that of left $\mathcal{D}^{(m+s)}_{X/S}$-modules with the restriction functor from the category of the right $\mathcal{D}^{(m+s)}_{X/S}$-modules to that of left $\mathcal{D}^{(m)}_{X/S}$-modules. We still denote by $(F^s_{X_0/S_0})^\flat$ this factorization.

6.1.5.4. With notation 6.1.5.2, let Z_0 be a divisor of X_0. The canonical $\mathcal{D}^{(m)}_{X/S}$-linear isomorphism

$$(F^s_{X_0/S_0})^*\mathcal{B}_{X'}(Z_0^{(s)}, r) \xrightarrow{\sim} \mathcal{B}_{X'}(p^s Z_0, r) = \mathcal{B}_{X'}(Z_0, p^s r)$$

is in fact $\mathcal{D}^{(m+s)}_{X/S}$-linear (see [Ber00, 2.2.9]).

Theorem 6.1.5.5. *Suppose $\mathfrak{a}_S$ is m-PD-nilpotent. Let $\mathcal{B}_{X'}$ be an $\mathcal{O}_{X'}$-quasi-coherent algebra endowed with a left action of $\mathcal{D}^{(m)}_{X'/S}$ compatible with is $\mathcal{O}_{X'}$-algebra structure. Let $\mathcal{B}_X := (F^s_{X_0/S_0})^*(\mathcal{B}_{X'})$.*

(a) The functor $(F^s_{X_0/S_0})^*$ is an equivalence from the category of left $\mathcal{B}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'/S}$-modules to that of left $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m+s)}_{X/S}$-modules.

(b) The functor $(F^s_{X_0/S_0})^\flat$ is an equivalence from the category of right $\mathcal{B}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'/S}$-modules to that of right $\mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m+s)}_{X/S}$-modules.

6.2 Commutation with Frobenius, first examples

First let us make precise the hypotheses and notations which we shall use systematically in the following sections and which we shall not mention explicitly in the statements.

(i) We denote by S the base scheme (resp. by $S \to T$ a morphism of base change). For each of the operations considered, we will first recall the general definition. No assumption on S is needed when $m = 0$ or $m = \infty$. If $m \in \mathbb{N}^*$, we shall assume that S is a $\mathbb{Z}_{(p)}$-scheme.

When we consider a smooth S-scheme X, we suppose given on X a sheaf $\mathcal{B}_X$ of $\mathcal{O}_X$-algebras equipped with a compatible $\mathcal{D}_X^{(m)}$ action, and we put $\widetilde{\mathcal{D}}_X^{(m)} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)}$. When we consider a morphism of schemes $f : X \to Y$, we suppose that $\mathcal{B}_X = f^*\mathcal{B}_Y$, equipped with the action of $\mathcal{D}_X^{(m)}$ defined by inverse image coming from that of $\mathcal{D}_Y^{(m)}$ on $\mathcal{B}_Y$ (even, if there is the action of $\mathcal{D}_X^{(m+s)}$ obtained by applying 6.1.1.6.

(ii) To highlight the "crystalline nature" of the operations such as $f^!$ and f_+, we generalize their construction to the case where S is equipped with a quasi-coherent m-PD-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha)$, which we suppose to be m-PD-nilpotent, and where the morphisms are only defined modulo $\mathfrak{a}_S$. We write $S_0 = V(\mathfrak{a}_S)$, and, in general, the index 0 denotes the reduction of a S-scheme modulo $\mathfrak{a}_S$, or the data defined over S_0. Under these hypotheses, the inverse image functors will be the functors f_0^* defined by applying 4.4.5.11. In particular, we generalize what we have said above by supposing that, when a morphism $f_0 : X_0 \to Y_0$ is given, we have $\mathcal{B}_X = f_0^*\mathcal{B}_Y$.

(iii) To state the properties of commutation with F^*, we suppose that p is nilpotent on S, and that S is equipped with a quasi-coherent m-PD-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha)$ such that $p \in \mathfrak{a}_S$. We suppose fixed an integer s, and, if X is a smooth S-scheme, we denote by X' a smooth S-scheme lifting $X_0^{(s)}$.

These properties can be stated first in a "lifted" situation: we shall not make the hypotheses of m-PD-nilpotence on $\mathfrak{a}_S$, the morphisms of schemes considered will be smooth morphisms of S-schemes, and the diagrams will be commutative on S. We denote $F : X \to X'$ a S- morphism lifting the morphism of relative Frobenius $F^s_{X_0/S_0}$.

(iv) Finally, we will give a "crystalline" variant of the preceding statements: we suppose then that S and $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha)$ satisfy the hypotheses (ii) and (iii), therefore, in particular that $\mathfrak{a}_S$ is m-PD-nilpotent. In this case, again it suffices that the morphisms are given between the reductions on S_0. The inverse image functors will still be defined by applying 4.4.5.11, reinforced by 6.1.5.2 when, as for the functor $F^*_{X_0/S_0}$, it will be necessary to take into account the raising of the level by Frobenius.

We keep notation 6.1.

6.2.1 Definition

Definition 6.2.1.1. Let M be a family of morphisms of ringed spaces $\widetilde{f}\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ where the underlying map $X \to Y$, denoted by f, is a morphism of smooth S-schemes, $\mathcal{B}_X$ (resp. $\mathcal{B}_Y$) be a commutative $\mathcal{O}_X$-algebra (resp. $\mathcal{O}_Y$-algebra) endowed with a compatible structure of left $\mathcal{D}_{X/S}^{(m)}$-module (resp. $\mathcal{D}_{Y/S}^{(m)}$-module), where the underlying morphism of algebras $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is moreover $\mathcal{D}_{X/S}^{(m)}$-linear (recall that the action of left $\mathcal{D}_{X/S}^{(m)}$-module on $f^*\mathcal{B}_Y$ is compatible with its structure of $\mathcal{O}_X$-algebra (see 3.4.4.6). For such a $\widetilde{f} \in M$, we set $\widetilde{\mathcal{D}}_{X/S}^{(m)} = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}^{(m)}$ and $\widetilde{\mathcal{D}}_{Y/S}^{(m)} = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y/S}^{(m)}$. For any integer m and every morphism $\widetilde{f}$ of M, suppose given some functors $\phi_{\widetilde{f}}^{(m)} \colon D({}^{1}\widetilde{\mathcal{D}}_{X/S}^{(m)}) \to D({}^{1}\widetilde{\mathcal{D}}_{Y/S}^{(m)})$ (resp. $\phi_{\widetilde{f}}^{(m)} \colon D(\widetilde{\mathcal{D}}_{X/S}^{(m)}{}^{d}) \to D(\widetilde{\mathcal{D}}_{Y/S}^{(m)}{}^{r})$).

We say that the family $(\phi_{\widetilde{f}}^{(m)})_{\widetilde{f} \in M}$ *commutes with Frobenius* if for any morphism $\widetilde{f}\colon (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ of M, for any integer s, for any smooth lifting X'/S (resp. Y'/S) of $X_0^{(s)}/S_0$ (resp. $Y_0^{(s)}/S_0$) and $F_X \colon X \to X'$, $F_Y \colon Y \to Y'$ liftings of $F^s_{X_0/S_0}$, $F^s_{Y_0/S_0}$ for any morphism $\widetilde{f}'\colon (X', \mathcal{B}_{X'}) \to (Y', \mathcal{B}_{Y'})$ of M such that f' is a lifting of $f_0^{(s)}$, $\mathcal{B}_X = F_X^*\mathcal{B}_{X'}$, $\mathcal{B}_Y = F_Y^*\mathcal{B}_{Y'}$ and satisfying $F_Y \circ f = f' \circ F_X$, there exists for any $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ (resp. $\mathcal{M}' \in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$) a functorial in $\mathcal{E}'$ (resp. $\mathcal{M}$) canonical isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S})$ (resp. $D(\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}{}^{\mathrm{r}})$) of the form:

$$F_Y^* \circ \phi^{(m)}_{f'}(\mathcal{E}') \xrightarrow{\sim} \phi^{(m+s)}_{\widetilde{f}} \circ F_X^*(\mathcal{E}') \text{ (resp. } F_Y^\flat \circ \phi^{(m)}_{f'}(\mathcal{M}')) \to \phi^{(m+s)}_{\widetilde{f}} \circ F_X^\flat(\mathcal{M}'). \qquad (6.2.1.1.1)$$

This definition extends to bifunctors, to complexes of bimodules, and also to functors of the form $\phi^{(m)}_{\widetilde{f}} : D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y/S}) \to D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X/S})$ or of the form $\phi^{(m)}_{\widetilde{f}} : D^-(\widetilde{\mathcal{D}}^{(m)}_{Y/S}{}^{\mathrm{r}}) \to D^-(\widetilde{\mathcal{D}}^{(m)}_{X/S}{}^{\mathrm{l}})$ etc.

6.2.2 Internal tensor products, homomorphisms

Lemma 6.2.2.1. *We assume the hypotheses 6.2 (iii).*

(a) Suppose $F : X \to X'$ is a lift of $F^s_{X_0/S_0}$ and let $\mathcal{E}', \mathcal{F}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'})$, $\mathcal{M}' \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X'})$. There exists in $D^-(\widetilde{\mathcal{D}}^{(m+s)}_X)$ a canonical isomorphism:

$$F^*\mathcal{E}' \otimes^{\mathbb{L}}_{\mathcal{B}_X} F^*\mathcal{F}' \xrightarrow{\sim} F^*(\mathcal{E}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{F}'), \qquad (6.2.2.1.1)$$

i.e. the bifunctor $- \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} -$ commutes with Frobenius (in Definition 6.2.1.1, the morphisms $\widetilde{f} \in M$ consist in the family of identities). Moreover, we have the canonical isomorphism

$$F^\flat\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{B}_X} F^*\mathcal{E}' \xrightarrow{\sim} F^*(\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{E}'). \qquad (6.2.2.1.2)$$

(b) If $\mathfrak{a}_S$ is m-PD-nilpotent and if $\mathcal{E}', \mathcal{F}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'})$, there exists in $D(\widetilde{\mathcal{D}}^{(m+s)}_X)$ a canonical isomorphism:

$$F^{s*}_{X_0/S_0}\mathcal{E}' \otimes^{\mathbb{L}}_{\mathcal{B}_X} F^{s*}_{X_0/S_0}\mathcal{F}' \xrightarrow{\sim} F^{s*}_{X_0/S_0}(\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{F}'). \qquad (6.2.2.1.3)$$

Proof. (a) i) Let $\mathcal{P}'$ be a K-flat resolution on $\widetilde{\mathcal{D}}^{(m)}_{X'}$ of $\mathcal{F}'$. As $\mathcal{P}'$ is also a K-flat complex of $\mathcal{B}_{X'}$-modules, the domain and range of (6.2.2.1.1) are complexes of $\widetilde{\mathcal{D}}^{(m+s)}_X$-modules given respectively by $F^*\mathcal{E}' \otimes_{\mathcal{B}_X} F^*\mathcal{P}'$ and $F^*(\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{P}')$. It suffices thus to verify that the canonical $\mathcal{B}_X$-linear isomorphism between these complexes are $\widetilde{\mathcal{D}}^{(m+s)}_X$-linear. We can suppose that $\mathcal{E}'$ and $\mathcal{P}'$ are reduced to a single module, and we are reduced to check that this isomorphism is horizontal for the $(m+s)$-PD-stratifications of $F^*\mathcal{E}' \otimes F^*\mathcal{F}'$ and $F^*(\mathcal{E}' \otimes \mathcal{F}')$. If (ε'_n), (η'_n) are the (m)-PD-stratifications of $\mathcal{E}'$, $\mathcal{F}'$, those of $\mathcal{E}' \otimes \mathcal{F}'$ are then $(\varepsilon'_n \otimes \eta'_n)$, and with notation of 6.1.1.5 the assertion results from that the inverse image by the morphisms $\Phi^n : \widetilde{\Delta}^n_{X/S(m+s)} \to \widetilde{\Delta}^n_{X'/S(m)}$ commutes with tensor product.

ii) By using 4.3.5.8 and 6.1.2.5, we deduce 6.2.2.1.2 from 6.2.2.1.1.

The case (b) follows from the case (a) once we observe that the isomorphisms of the glueing $\tau_{F,F'}$ themselves commute with tensor product, since they are deduced from the m-PD-stratifications by applying an inverse image functor. □

Proposition 6.2.2.2. *Let $\mathcal{E}'$ and $\mathcal{F}'$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-modules. Let $\mathcal{M}'$ and $\mathcal{N}'$ be two right $\widetilde{\mathcal{D}}^{(m)}_{X'}$-modules. The canonical morphism*

$$F^*(\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}', \mathcal{F}')) \to \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}', F^*\mathcal{F}') \qquad (6.2.2.2.1)$$

is an isomorphism of left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules, i.e. the bifunctor $\mathcal{H}om_{\mathcal{B}_{X'}}(-,-)$ therefore commutes with Frobenius. Moreover, we have respectively the canonical isomorphisms of right and left $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules

$$F^\flat(\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}', \mathcal{M}')) \to \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}', F^\flat\mathcal{M}'),$$
$$F^*(\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{M}', \mathcal{N}')) \to \mathcal{H}om_{\mathcal{B}_X}(F^\flat\mathcal{M}', F^\flat\mathcal{N}'). \qquad (6.2.2.2.2)$$

Proof. Since $\mathcal{B}_X$ is a locally free $\mathcal{B}_{X'}$-algebra, then the canonical morphism 6.2.2.2.1 is an isomorphism of $\mathcal{B}_X$-modules. We check the $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-linearity of 6.2.2.2.1 similarly to the proof of 4.4.5.14 by replacing the morphism $\tilde{f}_n$ by the morphism $\widetilde{\Delta}^n_{X/S(m+s)} \to \widetilde{\Delta}^n_{X'/S(m)}$. By using 4.3.5.8 and 6.1.2.5, this implies 6.2.2.2.2. □

Proposition 6.2.2.3. *Let $\mathcal{E}'$ be a $\widetilde{\mathcal{D}}^{(m)}_{X'}$-bimodule and $\mathcal{F}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module. So we have the canonical isomorphism:*

$$(F^*F^\flat\mathcal{E}')\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(F^*\mathcal{F}') \xrightarrow{\sim} F^*(\mathcal{E}'\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{F}'). \tag{6.2.2.3.1}$$

The bifunctor $-\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}-$ therefore commutes with Frobenius.

Proof. By functoriality, it follows from 6.1.3.6.1 that we have the canonical isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-modules :

$$F^\flat\mathcal{E}'\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}F^*\mathcal{F}' \xrightarrow{\sim} \mathcal{E}'\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{F}'.$$

By applying to it the functor F^*, this yields the proposition. □

Proposition 6.2.2.4. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module and $\mathcal{F}'$ be a $\widetilde{\mathcal{D}}^{(m)}_{X'}$-bimodule. Hence we have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-modules :*

$$F^\flat\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',\mathcal{F}') \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',F^\flat\mathcal{F}') \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(F^*\mathcal{E}',F^*F^\flat\mathcal{F}'). \tag{6.2.2.4.1}$$

The bifunctor $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(-,-)$ therefore commutes with Frobenius.

Proof. Since $\widetilde{F^\flat}\widetilde{\mathcal{D}}^{(m)}_{X'}$ is a locally free left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module, then the canonical isomorphism

$$\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',\mathcal{F}')\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\widetilde{F^\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \to \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',\mathcal{F}'\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\widetilde{F^\flat}\widetilde{\mathcal{D}}^{(m)}_{X'})$$

is an isomorphism. Since $F^\flat$ is canonically isomorphic to the functor $-\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\widetilde{F^\flat}\widetilde{\mathcal{D}}^{(m)}_{X'}$, then we get the first isomorphism of 6.2.2.4.1. Moreover, the theorem 6.1.1.9 gives us the isomorphism $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',F^\flat\mathcal{F}') \xrightarrow{\sim}$ $\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(F^*\mathcal{E}',F^*F^\flat\mathcal{F}')$. Hence, we are done. □

6.2.2.5. Suppose there exists a morphism of schemes $S\to B$ such that the composition morphism $\widetilde{g}'\colon(X',\mathcal{B}_{X'})\to B$ and $\widetilde{g}\colon(X,\mathcal{B}_X)\to B$ are flat. In particular, $\mathcal{B}_X$ and $\mathcal{B}_{X'}$ are quasi-flat $\mathcal{O}_S$-algebras (see Definition 3.1.1.5). Then $\widetilde{g}^{-1}\mathcal{O}_B$ is a solving ring of $(\widetilde{\mathcal{D}}^{(m)}_{X'/S},\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $(\widetilde{\mathcal{D}}^{(m+s)}_{X/S},\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ and $(\widetilde{\mathcal{D}}^{(m+s)}_{X/S},\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$ (see definition 4.6.3.2).

Proposition 6.2.2.6. *Suppose the hypotheses of 6.2.2.5 are satisfied. Let $\mathcal{E}'\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'})$ and $\mathcal{F}'\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'}\otimes_{\mathcal{O}_S}\widetilde{\mathcal{D}}^{(m)\,\mathrm{op}}_{X'})$. Hence we have the canonical isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$:*

$$F^\flat\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',\mathcal{F}') \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(\mathcal{E}',F^\flat\mathcal{F}') \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(F^*\mathcal{E}',F^*F^\flat\mathcal{F}'). \tag{6.2.2.6.1}$$

The bifunctor $\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(-,-)$ therefore commutes with Frobenius.

Proof. Let $\mathcal{I}'$ be K-injective complex of $K^-({}^{l}\widetilde{\mathcal{D}}^{(m)}_{X'/S}\otimes_{\mathcal{O}_S}\widetilde{\mathcal{D}}^{(m)\,\mathrm{op}}_{X'/S})$ which represents $\mathcal{F}'$. Then following 6.1.3.5.(b), $F^\flat\mathcal{I}'$ is a K-injective complex of $K^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}\otimes_{\mathcal{O}_S}\widetilde{\mathcal{D}}^{(m+s)\,\mathrm{op}}_{X/S})$ which represents $F^\flat\mathcal{F}'$ and $F^*F^\flat\mathcal{I}'$ is a K-injective complex of $K^-({}^{l}\widetilde{\mathcal{D}}^{(m+s)}_{X/S}\otimes_{\mathcal{O}_S}\widetilde{\mathcal{D}}^{(m+s)\,\mathrm{op}}_{X/S})$ which represents $F^*F^\flat\mathcal{F}'$ Hence, we get the isomorphisms 6.2.2.6.1 from 6.2.2.4.1.

□

6.2.3 Extension of the coefficients, level rising

We assume the hypotheses 6.2 (iii) holds. Let $m'\geq m$, $\mathcal{B}_{X'}$ (resp. $\mathcal{C}_{X'}$) an $\mathcal{O}_{X'}$-algebra equipped with a compatible action of $\mathcal{D}^{(m)}_{X'}$ (resp. $\mathcal{D}^{(m')}_{X'}$), $\mathcal{B}_{X'}\to\mathcal{C}_{X'}$ a $\mathcal{D}^{(m)}_{X'}$-linear momorphism of $\mathcal{O}_{X'}$-algebras. We put $\mathcal{B}_X=F^*\mathcal{B}_{X'}$, $\mathcal{C}_X=F^*\mathcal{C}_{X'}$, and we equip these algebras with actions of $\mathcal{D}^{(m+s)}_X$, $\mathcal{D}^{(m'+s)}_X$ respectively defined thanks to 6.1.1.6. We can then consider the sheaves of differential operators with coefficients in these algebras, defined by

$$\widetilde{\mathcal{D}}^{(m)}_{X'}=\mathcal{B}_{X'}\otimes_{\mathcal{O}_{X'}}\mathcal{D}^{(m)}_{X'},\qquad \widetilde{\mathcal{D}}^{(m+s)}_{X}=\mathcal{B}_{X}\otimes_{\mathcal{O}_{X}}\mathcal{D}^{(m+s)}_{X},$$
$$\widetilde{\mathcal{D}}^{(m')}_{X'}=\mathcal{C}_{X'}\otimes_{\mathcal{O}_{X'}}\mathcal{D}^{(m')}_{X'},\qquad \widetilde{\mathcal{D}}^{(m'+s)}_{X}=\mathcal{C}_{X}\otimes_{\mathcal{O}_{X}}\mathcal{D}^{(m'+s)}_{X}.$$

6.2.3.1. The isomorphism of $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-bimodules $\vartheta\colon \widetilde{\mathcal{D}}^{(m+s)}_{X/S} \xrightarrow{\sim} F^*F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}$ (see 6.1.3.2.1) is by construction (see the proof of [Ber00, 2.5.2]) built by extension from the case where $\mathcal{B}=\mathcal{O}$, i.e. the composition

$$F^*F^\flat(\mathcal{B}_{X'}\otimes_{\mathcal{O}_{X'}}\widetilde{\mathcal{D}}^{(m)}_{X'}) \xrightarrow{\sim} F^*(\mathcal{B}_{X'}\otimes_{\mathcal{O}_{X'}} F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}) \overset{6.2.2.1.1}{\xrightarrow{\sim}} \mathcal{B}_X\otimes_{\mathcal{O}_X} F^*F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'} \xrightarrow{\vartheta} \mathcal{B}_X\otimes_{\mathcal{O}_X}\widetilde{\mathcal{D}}^{(m)}_{X/S}$$

is ϑ. This yields that ϑ is compatible with extension of the coefficients. We moreover easily check its compatibility with changes of level. To sump up the following diagram is commutative :

$$\begin{array}{ccc} F^*F^\flat(\mathcal{D}^{(m)}_{X'}) & \xrightarrow{F^*F^\flat(\rho)} & F^*F^\flat(\mathcal{D}^{(m')}_{X'}) \\ {\scriptstyle\vartheta}\uparrow{\scriptstyle\sim} & & {\scriptstyle\mathrm{id}\otimes\vartheta}\uparrow{\scriptstyle\sim} \\ \mathcal{D}^{(m+s)}_{X/S} & \xrightarrow{\rho} & \mathcal{D}^{(m'+s)}_{X/S}. \end{array} \tag{6.2.3.1.1}$$

6.2.3.2. For any left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module $\mathcal{E}'$, we get the morphism of left $\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}$-modules

$$\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \xrightarrow{\sim} F^*(\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{E}') \tag{6.2.3.2.1}$$

as being the unique $\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}$-linear morphism making commutative the following diagram :

$$F^*\mathcal{E} \longrightarrow \widetilde{\mathcal{D}}^{(m'+s)}_{X/S}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \dashrightarrow F^*(\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{E}'). \tag{6.2.3.2.2}$$

On the other hand, inspired by 6.2.2.3.1, we can construct an isomorphism of the form 6.2.3.2.1 in a different way as follows: by considering $\widetilde{\mathcal{D}}^{(m')}_{X'}$ as a $(\widetilde{\mathcal{D}}^{(m')}_{X'},\widetilde{\mathcal{D}}^{(m)}_{X'})$-bimodule, we get the isomorphism of $(\widetilde{\mathcal{D}}^{(m'+s)}_{X},\widetilde{\mathcal{D}}^{(m+s)}_{X})$-bimodules:

$$F^*F^\flat\widetilde{\mathcal{D}}^{(m')}_{X'} \xrightarrow{\sim} F^*\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m')}_{X'}}\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}. \tag{6.2.3.2.3}$$

This yields the first functorial in $\mathcal{E}'$ isomorphism of left $\widetilde{\mathcal{D}}^{(m')}_{X'}$-modules:

$$\begin{aligned} &\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \underset{\vartheta\otimes\mathrm{id}}{\overset{6.1.3.2.1}{\xrightarrow{\sim}}} F^*F^\flat\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \overset{6.2.3.2.3}{\xrightarrow{\sim}} \\ &F^*\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m')}_{X'}}\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}(F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}') \overset{6.1.3.6.1}{\xrightarrow{\sim}} F^*(\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{E}'). \end{aligned} \tag{6.2.3.2.4}$$

Proposition 6.2.3.3. *Both constructions of 6.2.3.2 of the isomorphism of left $\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}$-modules of the form*

$$\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \xrightarrow{\sim} F^*(\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{E}')$$

coincide. The functor $\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} -$ therefore commutes with Frobenius.

Proof. Let us denote by ϑ the isomorphisms of the form $\widetilde{\mathcal{D}}^{(m+s)}_{X/S} \xrightarrow{\sim} F^*F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}$ and ρ the ones of the form $\widetilde{\mathcal{D}}^{(m)}_{X'}\to\widetilde{\mathcal{D}}^{(m')}_{X'}$ or $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}\to\widetilde{\mathcal{D}}^{(m'+s)}_{X/S}$ etc. By uniqueness of the factorization of the diagram 6.2.3.2.2, it is sufficient to establish the commutativity of the diagram below:

$$\begin{array}{ccccc} F^*\mathcal{E}' & = & = & = & F^*\mathcal{E}' \\ \uparrow{\scriptstyle\sim} & & & & \uparrow{\scriptstyle\sim} \\ \widetilde{\mathcal{D}}^{(m+s)}_{X/S}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' & \underset{\sim}{\xrightarrow{\vartheta\otimes\mathrm{id}}} & F^*F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' & \underset{\sim}{\xrightarrow{6.1.3.6.1}} & F^*(\widetilde{\mathcal{D}}^{(m)}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{E}') \\ \downarrow{\scriptstyle\rho\otimes\mathrm{id}} & & \downarrow{\scriptstyle F^*F^\flat(\rho)\otimes\mathrm{id}} & & \downarrow{\scriptstyle F^*(\rho)\otimes\mathrm{id}} \\ \widetilde{\mathcal{D}}^{(m'+s)}_{X/S}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' & \underset{\sim}{\xrightarrow{\vartheta\otimes\mathrm{id}}} & F^*F^\flat\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' & \underset{\sim}{\xrightarrow{6.2.3.2.4}} & F^*(\widetilde{\mathcal{D}}^{(m')}_{X'}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}}\mathcal{E}'). \end{array} \tag{6.2.3.3.1}$$

By construction of 6.2.3.2.4, the right diagram is commutative. Moreover, the canonical isomorphism $F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ was constructed such that the rectangle of the top of 6.2.3.3.1 is commutative (see the proof of [Ber00, 2.5.5.(i)]). By applying to the right diagram of 6.2.3.1.1 the functor $-\otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*\mathcal{E}'$, we get the commutativity of left bottom square of 6.2.3.3.1. □

Corollary 6.2.3.4. *Let $\mathcal{E}' \in D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'})$. There exists in $D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m'+s)}_{X})$ a canonical isomorphism*

$$\widetilde{\mathcal{D}}^{(m'+s)}_{X} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X}} F^*\mathcal{E}' \xrightarrow{\sim} F^*(\widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}') \tag{6.2.3.4.1}$$

Proof. Thanks to 6.1.3.6, we obtain the isomorphism (6.2.3.4.1) by applying (6.2.3.4) to a flat resolution of $\mathcal{E}'$. □

Corollary 6.2.3.5. *Under the hypotheses 6.2(iv), we put $\mathcal{B}_X = F^*_{X_0/S_0}\mathcal{B}_{X'}$, $\mathcal{C}_X = F^*_{X_0/S_0}\mathcal{C}_{X'}$. For any $\mathcal{E}' \in D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'})$, there exists in $D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m'+s)}_{X})$ a canonical isomorphism*

$$\widetilde{\mathcal{D}}^{(m'+s)}_{X} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}^{(m+s)}_{X}}{\otimes}} F^{s*}_{X_0/S_0}\mathcal{E}' \simeq F^{s*}_{X_0/S_0}(\widetilde{\mathcal{D}}^{(m')}_{X'} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}^{(m)}_{X'}}{\otimes}} \mathcal{E}').$$

Proof. Let $F, F' \colon U \to X'$be two liftings of F_{X^0/S_0} on an open subset U of X. For any $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module $\mathcal{E}'$, we have isomorphisms $\tau^{(m+s)}_{F,F'} \colon F'^*\mathcal{E}' \xrightarrow{\sim} F^*\mathcal{E}'$, and

$$\tau^{(m'+s)}_{F,F'} \colon F'^*(\widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}') \xrightarrow{\sim} F^*(\widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}').$$

The square

$$\begin{array}{ccc}
\widetilde{\mathcal{D}}^{(m'+s)}_{X} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X}} F'^*\mathcal{E}' & \xrightarrow{\sim} & F'^*(\widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}') \\
{\scriptstyle \mathrm{id}\otimes\tau^{(m+s)}_{F,F'}}\downarrow\wr & & {\scriptstyle \tau^{(m'+s)}_{F,F'}}\downarrow\wr \\
\widetilde{\mathcal{D}}^{(m'+s)}_{X} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X}} F^*\mathcal{E}' & \xrightarrow{\sim} & F^*(\widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}')
\end{array}$$

formed from the isomorphisms (6.2.3.4) relative to F, F' is commutative, because it suffices to prove it before taking tensor product with $\widetilde{\mathcal{D}}^{(m'+s)}_{X}$ on the left along the columns. Then it follows from the functoriality of $\tau^{(m+s)}_{F,F'}$ applied to the homomorphism $\mathcal{E}' \to \widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}'$, and from this, if $\mathcal{F}'$ is a $\widetilde{\mathcal{D}}^{(m')}_{X'}$-module, the isomorphisms $\tau^{(m+s)}_{F,F'}$, $\tau^{(m'+s)}_{F,F'}$ relative to $\mathcal{F}'$ are equal.

It follows that the isomorphisms (6.2.3.4) defined by the local liftings of Frobenius can be glued, and we obtain thus for any $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module a global isomorphism

$$\widetilde{\mathcal{D}}^{(m'+s)}_{X} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X}} F^{s*}_{X_0/S_0}\mathcal{E}' \simeq F^{s*}_{X_0/S_0}(\widetilde{\mathcal{D}}^{(m')}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'}} \mathcal{E}').$$

By taking flat resolutions on $\widetilde{\mathcal{D}}^{(m)}_{X'}$, this isomorphism extends to derived categories. □

6.2.4 Base change and extraordinary inverse image

Proposition 6.2.4.1 (Base change)**.** *Suppose that S and T are quipped with quasi-coherent m-PD ideals $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S)$, $(\mathfrak{a}_T, \mathfrak{b}_T, \alpha_T)$ satisfying the hypotheses 6.2(iii), and such that $S \to T$ is a m-PD-morphism. Let $X' = S \times_T Y'$, $f' \colon X' \to Y'$ the projection, $\mathcal{B}_{Y'}$ an $\mathcal{O}_{Y'}$-algebra equipped with compatible action of $\mathcal{D}^{(m)}_{Y'}$, $\mathcal{B}_{X'} = f^*\mathcal{B}_{Y'}$.*

(a) If $F_Y \colon Y \to Y'$ is a lifting of $F^s_{Y_0/S_0}$, let $F_X : X \to X'$ be the morphism deduced from F_Y by base change, $\mathcal{B}_Y = F^*_Y\mathcal{B}_{Y'}$, $\mathcal{B}_X = f^*\mathcal{B}_Y \simeq F^*_X\mathcal{B}_{X'}$. For any $\mathcal{F}' \in D^-(\widetilde{\mathcal{D}}^{(m)}_{Y'})$ there exists a canonical isomorphism of $D^-(\widetilde{\mathcal{D}}^{(m+s)}_{X})$

$$\widetilde{\mathcal{D}}^{(m+s)}_{X} \overset{\mathbb{L}}{\underset{f^{-1}(\widetilde{\mathcal{D}}^{(m+s)}_{Y})}{\otimes}} f^{-1}F^*_Y\mathcal{F}' \xrightarrow{\sim} F^*_X(\widetilde{\mathcal{D}}^{(m)}_{X'} \overset{\mathbb{L}}{\underset{f^{-1}(\widetilde{\mathcal{D}}^{(m+s)}_{Y'})}{\otimes}} f^{-1}\mathcal{F}').$$

(b) *If $\mathfrak{a}_S$ and $\mathfrak{a}_T$ are m-PD nilpotent, let $\mathcal{B}_Y = F^{s*}_{Y_0/S_0}\mathcal{B}_{Y'}$, $\mathcal{B}_X = f^*\mathcal{B}_Y \simeq F^{s*}_{X_0/S_0}\mathcal{B}_{X'}$. For any $\mathcal{F}' \in D^-(\widetilde{\mathcal{D}}^{(m)}_{Y'})$, there exists a canonical isomorphism of $D^-(\widetilde{\mathcal{D}}^{(m+s)}_X)$*

$$\widetilde{\mathcal{D}}^{(m+s)}_X \overset{\mathbb{L}}{\underset{f^{-1}(\widetilde{\mathcal{D}}^{(m+s)}_Y)}{\otimes}} f^{-1}F^{s*}_{Y_0/S_0}\mathcal{F}' \xrightarrow{\sim} F^{s*}_{X_0/S_0}(\widetilde{\mathcal{D}}^{(m)}_{X'} \overset{\mathbb{L}}{\underset{f^{-1}(\widetilde{\mathcal{D}}^{(m+s)}_{Y'})}{\otimes}} f^{-1}\mathcal{F}').$$

Proof. It suffices to prove the non-derived version of these isomorphisms when $\mathcal{F}'$ reduces to a $\widetilde{\mathcal{D}}^{(m)}_{Y'}$-module, because one can calculate the derived functor by replacing $\mathcal{F}'$ with a flat resolution on $\widetilde{\mathcal{D}}^{(m)}_{Y'}$. Using the isomorphism (5.1.1.15.2), we see that we just have to define a $\widetilde{\mathcal{D}}^{(m+s)}_X$-linear isomorphism $f^*F^*_Y\mathcal{F}' \xrightarrow{\sim} F^*_X f'^*\mathcal{F}'$ (resp. $f^*F^{s*}_{Y_0/S_0}\mathcal{F}' \xrightarrow{\sim} F^{s*}_{X_0/S_0}f'^*\mathcal{F}'$). In the first case we have $f' \circ F_X = F_Y \circ f$, i suffices to apply the transitivity isomorphism. In the second case we use

$$(F^{s*}_{Y_0/S_0} \circ f_0)^* \simeq f_0^* \circ F^{s\,*}_{Y_0/S_0} \simeq F^{s\,*}_{X_0/S_0} \circ f_0^{(s)^*}.$$

□

Proposition 6.2.4.2. *Under the hypotheses 6.2 (iii), let $f_0\colon X_0 \to Y_0$ be a morphism of smooth S_0-schemes, X, Y, X', Y' are smooth S-schemes lifting X_0, Y_0, $X_0^{(s)}, Y_0^{(s)}$.*

(a) *Suppose given S-morphisms $f\colon X \to Y$, $f'\colon X' \to Y'$, $F_X : X \to X'$, $F_Y : Y \to Y'$ lifting f_0, $f_0^{(s)}$, $F^s_{X_0/S_0}$, $F^s_{Y_0/S_0}$. If $F_Y \circ f = f' \circ F_X$, there exists for any $\mathcal{F}'$ in $D^-(\widetilde{\mathcal{D}}^{(m)}_{Y'})$ a canonical isomorphism of $D^-(\widetilde{\mathcal{D}}^{(m+s)}_X)$*

$$f^{!(m+s)}F^*_Y\mathcal{F}' \xrightarrow{\sim} F^*_X(f^{!(m)}\mathcal{F}'). \tag{6.2.4.2.1}$$

(b) *Suppose $\mathfrak{a}_S$ is m-PD-nilpotent. For any $\mathcal{F}'$ in $D^-(\widetilde{\mathcal{D}}^{(m)}_{Y'})$ a canonical isomorphism of $D^-(\widetilde{\mathcal{D}}^{(m+s)}_X)$*

$$f_0^{!(m+s)}F^{s*}_{Y_0/S_0}\mathcal{F}' \xrightarrow{\sim} F^{s*}_{X_0/S_0}(f_0^{(s)!(m)}\mathcal{F}'). \tag{6.2.4.2.2}$$

Proof. Note that $d_{X'/Y'} = d_{X/Y}$. Let $\mathcal{P}'$ be a resolution of $\mathcal{F}'$ flat on $\widetilde{\mathcal{D}}^{(m)}_{Y'}$. Thanks to 6.1.3.6 we are reduced in the first case to construct a canonical $\widetilde{\mathcal{D}}^{(m+s)}_X$-linear isomorphism

$$f^*F^*_Y\mathcal{P}' \xrightarrow{\sim} F^*_X f'^*\mathcal{P}' \tag{6.2.4.2.3}$$

This isomorphism holds without hypotheses by applying 6.1.1.6 (and 6.1.1.7).

In case (b) we need the same construction of a canonical $\widetilde{\mathcal{D}}^{(m+s)}_X$-linear isomorphism

$$f_0^*F^{s*}_{Y_0/S_0}\mathcal{P}' \xrightarrow{\sim} F^{s*}_{X_0/S_0}f_0{}^{(s)*}\mathcal{P}', \tag{6.2.4.2.4}$$

and this isomorphism is furnished by 6.1.5.2.1

□

Remark 6.2.4.3. Applying the assertion (ii) to the particular case where X and Y are two lifting of X_0 and $f_0 = \mathrm{Id}_{X_0}$, the independence equivalences of 4.4.5.12 are compatible with the functors F^*.

6.2.5 External tensor products

The compatibility of the functor F^* with external tensor product (see 5.1.5) is analogous to what we have just seen for the inverse images.

Proposition 6.2.5.1. *Under the hypotheses 6.2(iii), let X, Y be smooth S schemes, with reductions X_0, Y_0 and X', Y' the smooth S-schemes lifting $X_0^{(s)}$, $Y_0^{(s)}$. Write $Z = X \times_S Y$, $Z_0 = X_0 \times_{S_0} Y_0$, $Z' = X' \times_{S'} Y'$. Let $\mathcal{B}_{X'}$ (resp. $\mathcal{B}_{Y'}$) is an $\mathcal{O}_{X'}$-algebra (resp. $\mathcal{O}_{Y'}$-algebra) equipped with a compatible action of $\mathcal{D}^{(m)}_{X'}$ (resp. $\mathcal{D}^{(m)}_{Y'}$, $\mathcal{B}_{Z'} = \mathcal{B}_{X'} \boxtimes_{\mathcal{O}_S} \mathcal{B}_{Y'}$ be the $\mathcal{O}_{Z'}$-algebra equipped with a compatible action of $\mathcal{D}^{(m)}_{Z'}$. We note as before $\widetilde{\mathcal{D}}^{(m)}_{X'} = \mathcal{B}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$, $\widetilde{\mathcal{D}}^{(m)}_{Y'} = \mathcal{B}_{Y'} \otimes_{\mathcal{O}_{Y'}} \mathcal{D}^{(m)}_{Y'}$, $\widetilde{\mathcal{D}}^{(m)}_{Z'} = \mathcal{B}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{D}^{(m)}_{Z'}$. Set $\mathcal{B}_X := F^{s*}_{X_0/S_0}$, $\mathcal{B}_Y := F^{s*}_{Y_0/S_0}$ and $\mathcal{B}_Z := F^{s*}_{Z_0/S_0}$. We note as before $\widetilde{\mathcal{D}}^{(m+s)}_X = \mathcal{B}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m+s)}_X$, $\widetilde{\mathcal{D}}^{(m+s)}_Y = \mathcal{B}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}^{(m+s)}_Y$, $\widetilde{\mathcal{D}}^{(m+s)}_Z = \mathcal{B}_Z \otimes_{\mathcal{O}_Z} \mathcal{D}^{(m+s)}_Z$.*

(a) *Suppose given liftings* $F_X\colon X\to X'$, $F_Y\colon Y\to Y'$ *of* $F^s_{X_0/S_0}$, $F^s_{Y_0/S_0}$, *and put* $F_Z=F_X\times F_Y: Z\to Z'$. *If* $\mathcal{E}'\in D^-(\widetilde{\mathcal{D}}^{(m)}_{X'})$, $\mathcal{F}'\in D^-(\widetilde{\mathcal{D}}^{(m)}_{Y'})$, *there exists in* $D^-(\widetilde{\mathcal{D}}^{(m+s)}_{Z})$ *a canonical isomorphism*

$$F_X^*\mathcal{E}'\overset{\mathbb{L}}{\underset{\mathcal{O}_S}{\boxtimes}}F_Y^*\mathcal{F}'\xrightarrow{\sim}F_Z^*(\mathcal{E}'\overset{\mathbb{L}}{\underset{\mathcal{O}_S}{\boxtimes}}\mathcal{F}').$$

(b) *If* $\mathfrak{a}_S$ *is* m*-PD-nilpotent and if* $\mathcal{E}'\in D^-(\widetilde{\mathcal{D}}^{(m)}_{X'})$, $\mathcal{F}'\in D^-(\widetilde{\mathcal{D}}^{(m)}_{Y'})$, *there exists in* $D^-(\widetilde{\mathcal{D}}^{(m+s)}_{Z})$ *a canonical isomorphism:*

$$F^{s*}_{X_0/S_0}\mathcal{E}'\overset{\mathbb{L}}{\underset{\mathcal{O}_S}{\boxtimes}}F^{s*}_{Y_0/S_0}\mathcal{F}'\xrightarrow{\sim}F^{s*}_{Z_0/S_0}(\mathcal{E}'\overset{\mathbb{L}}{\underset{\mathcal{O}_S}{\boxtimes}}\mathcal{F}').$$

Proof. Let $f: Z\to X$, $g: Z\to Y$, $f': Z'\to X'$, $g': Z'\to Y'$ be projections and let $\mathcal{P}'$, $\mathcal{Q}'$ be flat resolutions of $\mathcal{E},\mathcal{F}$. For case (a) it suffices to define a $\widetilde{\mathcal{D}}^{(m+s)}_Z$-linear isomorphism

$$f^*F_X^*\mathcal{P}"\otimes_{\mathcal{O}_Z}g^*F_Y^*\mathcal{Q}'\xrightarrow{\sim}F_Z^*(f'^*\mathcal{P}'\otimes_{\mathcal{O}_{Z'}}g'^*\mathcal{Q}'),$$

and we take the composition of the isomorphisms in 6.1.1.6 (and 6.1.1.7) and (6.2.2.1.1):

$$f^*F_X^*\mathcal{P}"\otimes_{\mathcal{O}_Z}g^*F_Y^*\mathcal{Q}'\xrightarrow{\sim}F_Z^*f'^*\mathcal{P}'\otimes_{\mathcal{O}_Z}F_Z^*g'^*\mathcal{Q}'\xrightarrow{\sim}F_Z^*(f'^*\mathcal{P}'\otimes_{\mathcal{O}_{Z'}}g'^*\mathcal{Q}').$$

We treat the case (b) similarly. □

6.2.6 Direct image

We establish the commutation of f_+ with F^*.

Lemma 6.2.6.1. *Under the hypotheses 6.2(iii), let* $f\colon X\to Y$ *be a morphism of smooth* S*-schemes.*

(a) *Suppose given liftings* $F_X\colon X\to X'$, $F_Y\colon Y\to Y'$, $f'\colon X'\to Y'$ *of* $F^s_{X_0/S_0}$, $F^s_{Y_0/S_0}$, *and* $f_0^{(s)}$ *such that* $F_Y\circ f=f'\circ F_X$. *There exists canonical isomorphisms of* $(\widetilde{\mathcal{D}}^{(m+s)}_X,f^{-1}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S})$*-bimodules and of* $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y/S},\widetilde{\mathcal{D}}^{(m)}_X)$*-bimodules*

$$\widetilde{\mathcal{D}}^{(m+s)}_{X\to Y/S}\xrightarrow{\sim}F_X^*F_Y^\flat\widetilde{\mathcal{D}}^{(m)}_{Y'\to X'/S}:=F_X^*(f'^*_{\mathrm{l}}(F_Y^\flat\widetilde{\mathcal{D}}^{(m)}_{Y'/S})),\tag{6.2.6.1.1}$$

$$\widetilde{\mathcal{D}}^{(m+s)}_{Y\leftarrow X/S}\xrightarrow{\sim}F_X^\flat F_Y^*\widetilde{\mathcal{D}}^{(m)}_{Y'\to X'/S}:=F^\flat_{X\mathrm{r}}\left(f'^*_{\mathrm{r}}(F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S}\otimes_{\mathcal{O}_{Y'}}\omega^{-1}_{Y'/S}))\otimes_{\mathcal{O}_{X'}}\omega_{X'/S}\right).\tag{6.2.6.1.2}$$

(b) *If* $\mathfrak{a}_S$ *is* m*-PD-nilpotent, there exists canonical isomorphisms of* $(\widetilde{\mathcal{D}}^{(m+s)}_X,f^{-1}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S})$*-bimodules and of* $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y/S},\widetilde{\mathcal{D}}^{(m)}_X)$*-bimodules*

$$\widetilde{\mathcal{D}}^{(m+s)}_{X\to Y/S}\xrightarrow{\sim}F^{s*}_{X_0/S_0}F^{s\flat}_{Y_0/S_0}\widetilde{\mathcal{D}}^{(m)}_{Y'\to X'/S}:=F^{s*}_{X_0/S_0}(f^{(s)*}_{0\mathrm{l}}(F^{s\flat}_{Y_0/S_0}\widetilde{\mathcal{D}}^{(m)}_{Y'/S})),\tag{6.2.6.1.3}$$

$$\widetilde{\mathcal{D}}^{(m+s)}_{Y\leftarrow X/S}\xrightarrow{\sim}F^{s\flat}_{X_0/S_0}F^{s*}_{Y_0/S_0}\widetilde{\mathcal{D}}^{(m)}_{Y'\to X'/S}\tag{6.2.6.1.4}$$

$$:=F^{s\flat}_{X_0/S_0,\mathrm{r}}\left(f^{(s)*}_{0\mathrm{r}}(F^{s*}_{Y_0/S_0,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S}\otimes_{\mathcal{O}_{Y'}}\omega^{-1}_{Y'/S}))\otimes_{\mathcal{O}_{X'}}\omega_{X'/S}\right)\tag{6.2.6.1.5}$$

Proof. According to 6.1.3.2 there exits a canonical isomorphism of $\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}$-bimodules

$$\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}\xrightarrow{\sim}F_Y^*F_Y^\flat\widetilde{\mathcal{D}}^{(m)}_{Y'/S}.\tag{6.2.6.1.6}$$

Apply to this f^* for the left structure, and use the $\widetilde{\mathcal{D}}^{(m+s)}_X$-linear isomorphism $f^*F_Y^*F_Y^\flat\widetilde{\mathcal{D}}^{(m)}_{Y'/S}\xrightarrow{\sim}F_X^*f'^*F_Y^\flat\widetilde{\mathcal{D}}^{(m)}_{Y'/S}$ furnished by (6.2.4.2.3) we get (6.2.6.1.1).

Tensor (6.2.6.1.6) on the right by $\omega_{Y/S}^{-1}$ and use 6.1.2.6.1 we get a left $\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}$-bimodule isomorphism

$$\widetilde{\mathcal{D}}_{Y/S}^{(m+s)} \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \xrightarrow{\sim} F_{Y,\mathrm{l}}^*(F_{Y,\mathrm{r}}^*(\widetilde{\mathcal{D}}_{Y'/S}^{(m)} \otimes_{\mathcal{O}_{Y'}} \omega_{Y'/S}^{-1})) \xrightarrow{\sim} F_{Y,\mathrm{r}}^*(F_{Y,\mathrm{l}}^*(\widetilde{\mathcal{D}}_{Y'/S}^{(m)} \otimes_{\mathcal{O}_{Y'}} \omega_{Y'/S}^{-1})).$$

Now apply f^* for the right structure and use the isomorphism $f^*F_Y^* \simeq F_X^* f'^*$, we get an isomorphism of left $(f^{-1}\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}, \widetilde{\mathcal{D}}_X^{(m+s)})$-bimodules

$$f^*(\widetilde{\mathcal{D}}_{Y/S}^{(m+s)} \otimes_{\mathcal{O}_Y} \omega_{Y/S}^{-1}) \xrightarrow{\sim} F_{X,\mathrm{r}}^* f'^*_{\mathrm{r}}(F_{Y,\mathrm{l}}^*(\widetilde{\mathcal{D}}_{Y'/S}^{(m)} \otimes_{\mathcal{O}_{Y'}} \omega_{Y'/S}^{-1})).$$

To get (6.2.6.1.2) we tensor by $\omega_{X/S}$ on the right, and compose with the isomorphism 6.1.2.5.1 relative to the left $\widetilde{\mathcal{D}}_{X'/S}^{(m)}$-module $f'^*_{\mathrm{r}}(F_{Y,\mathrm{l}}^*(\widetilde{\mathcal{D}}_{Y'/S}^{(m)} \otimes_{\mathcal{O}_{Y'}} \omega_{Y'/S}^{-1}))$.

When $\mathfrak{a}_S$ is m-PD-nilpotent, the functors and the isomorphisms appear in the preceding construction for the local lifting of Frobenius glues to yield globally on X the isomorphisms (6.2.6.1.3) and (6.2.6.1.4). □

Theorem 6.2.6.2. *Suppose the hypotheses 6.2(iii) and the relative dimension of X on S is constant. Let X, Y be smooth S-schemes, $f_0; X_0 \to Y_0$ a S_0-morphism between their reduction modulo $\mathfrak{a}_S$, X', Y' smooth lifting on S of $X_0^{(s)}, Y_0^{(s)}$.*

(a) Let $f\colon X \to Y$, $f'\colon X' \to Y'$, $F_X : X \to X'$, $F_Y : Y \to Y'$ be lifting f_0, $f_0^{(s)}$, $F^s_{X_0/S_0}$, $F^s_{Y_0/S_0}$. If $F_Y \circ f = f' \circ F_X$, there exists for any $\mathcal{E}'$ in $D^-(\widetilde{\mathcal{D}}_{X'/S}^{(m)})$ a canonical isomorphism of $D^-(\widetilde{\mathcal{D}}_{Y/S}^{(m+s)})$

$$f_{+^{(m+s)}}(F_X^*\mathcal{E}') \xrightarrow{\sim} F_Y^*(f'^{(m)}_+\mathcal{E}'). \tag{6.2.6.2.1}$$

(b) Suppose $\mathfrak{a}_S$ is m-PD-nilpotent. For any $\mathcal{E}' \in D^-(\widetilde{\mathcal{D}}_{X'/S}^{(m)})$, there exists a canonical isomorphism of $D^-(\widetilde{\mathcal{D}}_{Y/S}^{(m+s)})$

$$f_{0+}^{(m+s)}(F^{s*}_{X_0/S_0}\mathcal{E}') \xrightarrow{\sim} F^{s*}_{Y_0/S_0}(f^{(s)})_+^{(m)}(\mathcal{E}'). \tag{6.2.6.2.2}$$

Proof. Under the hypotheses of (a), the exact functor F_Y^* induces an equivalence of categories $D^-(\widetilde{\mathcal{D}}_{Y'/S}^{(m)}) \xrightarrow{\sim} D^-(\widetilde{\mathcal{D}}_{Y/S}^{(m+s)})$, and according to 6.1.3.4.1 a quasi-inverse functor is given by $\mathcal{E} \mapsto F_Y^\flat \widetilde{\mathcal{D}}_{Y'/S}^{(m)} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}} \mathcal{E}$. To define the isomorphism (6.2.6.2.1) it suffices to define an isomorphism

$$F_Y^\flat \widetilde{\mathcal{D}}_{Y'/S}^{(m)} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}}{\otimes}} f_{+^{(m+s)}}(F_X^*\mathcal{E}') \xrightarrow{\sim} f'^{(m)}_+(\mathcal{E}')$$

in $D^-(\widetilde{\mathcal{D}}_{Y'/S}^{(m)})$. By definition, $f_{+^{(m+s)}}(F_X^*\mathcal{E}') = \mathbb{R}f_*(\mathcal{F})$, where

$$\mathcal{F} = \widetilde{\mathcal{D}}_{Y\leftarrow X}^{(m+s)} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}_{X/S}^{(m+s)}}{\otimes}} F_X^*\mathcal{E}' \in D^-(f^{-1}\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}).$$

By 6.1.3.3, $F_Y^\flat \widetilde{\mathcal{D}}_{Y'/S}^{(m)}$ is a left $\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}$-module which is locally projective of finite type, the canonical morphism

$$F_Y^\flat \widetilde{\mathcal{D}}_{Y'/S}^{(m)} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}}{\otimes}} \mathbb{R}f_*(\mathcal{F}) \to \mathbb{R}f_*\left(f^{-1}(F_Y^\flat \widetilde{\mathcal{D}}_{Y'/S}^{(m)}) \overset{\mathbb{L}}{\underset{f^{-1}\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}}{\otimes}} \mathcal{F} \right)$$

is an isomorphism in $D^-(f^{-1}\widetilde{\mathcal{D}}_{Y/S}^{(m)})$. As $f = f'$ in as much as continuous maps between topological spaces, we are reduced to defining an isomorphism

$$f^{-1}(F_Y^\flat \widetilde{\mathcal{D}}_{Y'/S}^{(m)}) \overset{\mathbb{L}}{\underset{f^{-1}\widetilde{\mathcal{D}}_{Y/S}^{(m+s)}}{\otimes}} (\widetilde{\mathcal{D}}_{Y\leftarrow X/S}^{(m+s)} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}_{X/S}^{(m+s)}}{\otimes}} F_X^*\mathcal{E}') \xrightarrow{\sim} \widetilde{\mathcal{D}}_{Y'\to X'/S}^{(m)} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}_{X'/S}^{(m)}}{\otimes}} \mathcal{E}' \tag{6.2.6.2.3}$$

in $D^-(f^{-1}\widetilde{\mathcal{D}}_{Y'/S}^{(m)})$.

Thanks to (6.2.6.1.2) we have a canonical isomorphism of bimodule

$$\widetilde{\mathcal{D}}^{(m+s)}_{Y\leftarrow X/S} \xrightarrow{\sim} F^*_{X,\mathrm{r}}\left(f'^*_{\mathrm{r}}(F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'}} \omega^{-1}_{Y'/S})) \otimes_{\mathcal{O}_{X'}} \omega_{X'/S}\right).$$

Apply the isomorphism of invariance of tensor product (6.1.3.6.1) we get in $D^-(f^{-1}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S})$ the isomorphism

$$\widetilde{\mathcal{D}}^{(m+s)}_{Y\leftarrow X/S} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^*_X \xrightarrow{\sim} \left(f'^*_{\mathrm{r}}(F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'}} \omega^{-1}_{Y'/S})) \otimes_{\mathcal{O}_{X'}} \omega_{X'/S}\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}'.$$

The left hand side of (6.2.6.2.3) can be identified with

$$f^{-1}(F^\flat_Y \widetilde{\mathcal{D}}^{(m)}_{Y'/S}) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}} \left(f'^*_{\mathrm{r}}(F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'}} \omega^{-1}_{Y'/S})) \otimes_{\mathcal{O}_{X'}} \omega_{X'/S}\right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}' \qquad (6.2.6.2.4)$$

$$\simeq \left(f^{-1}(F^\flat_Y \widetilde{\mathcal{D}}^{(m)}_{Y'/S}) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}} f^{-1}(F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'}} \omega^{-1}_{Y'/S})) \otimes_{\mathcal{O}_{X'}} \omega_{X'/S} \right) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}'. \qquad (6.2.6.2.5)$$

We have isomorphism of $\widetilde{\mathcal{D}}^{(m)}_{Y'/S}$-bimodules (6.1.3.4.1)

$$F^\flat_Y \widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}} F^*_{Y,\mathrm{l}} \widetilde{\mathcal{D}}^{(m)}_{Y'/S} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y'/S}$$

which we right tensor by $\omega^{-1}_{Y'/S}$ to deduce an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{Y'/S}$-bimodules

$$F^\flat_Y \widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}} F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'}} \omega^{-1}_{Y'/S}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'/S}} \omega^{-1}_{Y'/S}.$$

Apply f^{-1} and right tensor by $\omega_{X'/S}$ we obtain an $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y'/S}, \widetilde{\mathcal{D}}^{(m)}_{X'/S})$-bimodule isomorphism

$$f^{-1}(F^\flat_Y \widetilde{\mathcal{D}}^{(m)}_{Y'/S}) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m+s)}_{Y/S}} f^{-1}(F^*_{Y,\mathrm{l}}(\widetilde{\mathcal{D}}^{(m)}_{Y'/S} \otimes_{\mathcal{O}_{Y'}} \omega^{-1}_{Y'/S})) \otimes_{f^{-1}\mathcal{O}_{Y'}} \omega_{X'/S} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y'\to X'/S}.$$

It suffices to tensor by $\mathcal{E}'$ to get the isomorphism (6.2.6.2.3).

If we suppose that $\mathfrak{a}_S$ is m-PD-nilpotent, we can construct the isomorphism 6.2.6.2.2 following the same method : each of the isomorphism used in the construction remains well defined without assuming the morphisms are liftable (taking into account that $f = f' = f_0$ as morphisms of topological spaces). □

6.2.7 Dual functor

Suppose there exists a morphism of schemes $S \to B$ such that the composition morphism $\widetilde{g}'\colon (X', \mathcal{B}_{X'}) \to B$ and $\widetilde{g}\colon (X, \mathcal{B}_X) \to B$ are flat.

Notation 6.2.7.1. We work under the hypotheses 6.2 (i). Set $d := \delta_{X/S}$. The $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear and $\widetilde{\mathcal{D}}^{(m+s)}_{X/S}$-lineal duals (see 5.1.4.1) will be denoted by setting for any $\mathcal{E}' \in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ and $\mathcal{E} \in D(\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$

$$\mathbb{D}^{(m)}_{\widetilde{X}'/S}(\mathcal{E}') := \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X}}(\mathcal{E}', \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \otimes_{\mathcal{O}_{X'}} \omega^{-1}_{X'/S}[d],$$
$$\mathbb{D}^{(m+s)}_{\widetilde{X}/S}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(\mathcal{E}, \widetilde{\mathcal{D}}^{(m+s)}_{X/S}) \otimes_{\mathcal{O}_X} \omega^{-1}_{X/S}[d].$$

Theorem 6.2.7.2. *We keep notation 6.2.7.1. We assume that the hypotheses 6.2(iii) holds.*

(a) Let $F : X \to X'$ be a lifting of $F^s_{X_0/S_0}$. For any $\mathcal{E}' \in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, there exists a canonicial isomorphism of $D(\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$

$$\mathbb{D}^{(m+s)}_{\widetilde{X}/S} F^* \mathcal{E}' \xrightarrow{\sim} F^* \mathbb{D}^{(m)}_{\widetilde{X}'/S}(\mathcal{E}').$$

(b) *If $\mathfrak{a}_S$ is m-PD-nilpotent, for any $\mathcal{E}' \in D^-(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, there exists a canonicial isomorphism of $D^+(\widetilde{\mathcal{D}}^{(m+s)}_{X/S})$*

$$\mathbb{D}^{(m+s)}_{\widetilde{X}/S} F^{s*}_{X_0/S_0} \mathcal{E}' \xrightarrow{\sim} F^{s*}_{X_0/S_0} \mathbb{D}^{(m)}_{\widetilde{X}/S}(\mathcal{E}').$$

Proof. For any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, we have the isomorphisms

$$F^\flat \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X}}(\mathcal{E}', \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \underset{6.2.2.6}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*F^\flat \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \underset{6.1.3.2.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(F^*\mathcal{E}', \widetilde{\mathcal{D}}^{(m+s)}_{X/S}). \tag{6.2.7.2.1}$$

This yields:

$$F^*\mathbb{D}^{(m)}_{\widetilde{X}'/S}(\mathcal{E}') \overset{6.1.2.6.1}{\xrightarrow{\sim}} F^\flat \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X}}(\mathcal{E}', \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \otimes_{\mathcal{O}_X} \omega^{-1}_{X/S}[d]$$
$$\overset{6.2.7.2.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}}(F^*\mathcal{E}', \widetilde{\mathcal{D}}^{(m+s)}_{X/S}) \otimes_{\mathcal{O}_X} \omega^{-1}_{X/S}[d] = \mathbb{D}^{(m+s)}_{\widetilde{X}/S} F^*\mathcal{E}'.$$

The proof of (b) is analogous, the isomorphisms used remain defined without lifting hypotheses on $F^s_{X_0/S_0}$ when $\mathfrak{a}_S$ is m-PD-nilpotent. □

6.3 Compatibility with Frobenius, first examples

We keep notation 6.1.

6.3.1 Definition

Definition 6.3.1.1. We keep the hypotheses and notations of the definition 6.2.1.1. For any integer m, let $(\phi^{(m)}_{\widetilde{f}})_{\widetilde{f}\in M}$ and $(\psi^{(m)}_{\widetilde{f}}))_{\widetilde{f}\in M}$ be two families of functors commuting to Frobenius and, for any $\widetilde{f} \in M$, let $\theta^{(m)}_{\widetilde{f}} : \phi^{(m)}_{\widetilde{f}} \to \psi^{(m)}_{\widetilde{f}}$ be a morphism of functors. The family $(\theta^{(m)}_{\widetilde{f}})_{\widetilde{f}\in M}$ is said to be *compatible with Frobenius* if it induces a commutative diagram

$$\begin{array}{ccc} F^*_Y \circ \phi^{(m)}_{f'} & \longrightarrow & \phi^{(m+s)}_{\widetilde{f}} \circ F^*_X \\ \downarrow{\scriptstyle F^*_Y\circ\theta^{(m)}_{\widetilde{f}}} & & \downarrow{\scriptstyle \theta^{(m+s)}_{\widetilde{f}}\circ F^*_X} \\ F^*_Y \circ \psi^{(m)}_{f'} & \longrightarrow & \psi^{(m+s)}_{\widetilde{f}} \circ F^*_X, \end{array} \tag{6.3.1.1.1}$$

(resp. by replacing F^* by $F^\flat$).

6.3.2 Associativity, commutativity, switching left to right, compatibility with coefficients extensions of tensor products and homomorphisms

Proposition 6.3.2.1. *Let $\mathcal{E}'$ and $\mathcal{G}'$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules and $\mathcal{F}'$ be a right or left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module. The following canonical $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear isomorphisms (see 4.2.3.9.1)*

$$\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{G}' \xrightarrow{\sim} \mathcal{G}' \otimes_{\mathcal{B}_{X'}} \mathcal{E}', \ \mathcal{E}' \otimes_{\mathcal{B}_{X'}} (\mathcal{F}' \otimes_{\mathcal{B}_{X'}} \mathcal{G}') \xrightarrow{\sim} (\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{F}') \otimes_{\mathcal{B}_{X'}} \mathcal{G}'. \tag{6.3.2.1.1}$$

are compatible with Frobenius.

Proof. This is an exercice. □

Proposition 6.3.2.2. *The $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear switching left to right $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module structures isomorphisms for tensor products and internal homomorphisms of 4.3.5.8 are compatible with Frobenius.*

Proof. The compatibility with Frobenius is an exercice. ☐

Let $\mathcal{C}_{X'}$ be a $\mathcal{B}_{X'}$-algebra commutative endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X'}$-module such that $\mathcal{B}_{X'} \to \mathcal{C}_{X'}$ is $\mathcal{D}^{(m)}_{X'}$-linear (and then $\widetilde{\mathcal{D}}^{(m)}_{X'}$-linear). We set $\mathcal{C}_{X'} := F^*\mathcal{C}_{X'}$. Finally, we will suppose that S is a noetherian (hypothesis useful for 4.6.3.7 and then for 6.3.4.1 etc.) scheme.

Proposition 6.3.2.3. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module. The functorial in $\mathcal{E}'$ canonical isomorphism*

$$\mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}' \underset{4.3.4.6.1}{\xrightarrow{\sim}} \left(\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'/S}\right) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}' \tag{6.3.2.3.1}$$

is compatible with Frobenius.

Proof. It is about proving that the right square of the diagram

$$\begin{array}{ccccc} F^*\mathcal{E} & \longrightarrow & F^*(\mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}') & \underset{\sim}{\longrightarrow} & F^*((\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}) \otimes_{\mathcal{B}_{X'}\otimes_{\mathcal{O}_{X'}}\mathcal{D}^{(m)}_{X'}} \mathcal{E}') \\ & \searrow & \uparrow\wr & & \uparrow\wr \\ & & \mathcal{C}_X \otimes_{\mathcal{B}_X} F^*\mathcal{E}' & \underset{\sim}{\longrightarrow} & (\mathcal{C}_X \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_X) \otimes_{\mathcal{B}_X\otimes_{\mathcal{O}_X}\mathcal{D}^{(m)}_X} F^*\mathcal{E}' \end{array}$$

is commutative. By $\mathcal{C}_X$-linearity, it is sufficient to check that the outer of the diagram is commutative, which is tautological (6.2.3.2.2). ☐

Proposition 6.3.2.4. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module, $\mathcal{F}'$ a left $\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$-module and $\mathcal{E}' \to \mathcal{F}'$ be a $\widetilde{\mathcal{D}}^{(m)}_{X'}$-linear morphism. The canonical $\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$-linear morphism (see 4.3.4.9)*

$$\rho\colon \mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}' \to \mathcal{F}'$$

is compatible with Frobenius.

Proof. We have to prove that the following canonical diagram

$$\begin{array}{ccc} F^*(\mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}') & \underset{\sim}{\longrightarrow} & \mathcal{C}_X \otimes_{\mathcal{B}_X} F^*\mathcal{E}' \\ & {\scriptstyle F^*\rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & F^*\mathcal{F}' & \end{array}$$

is commutative, which is elementary. ☐

Proposition 6.3.2.5. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module, $\mathcal{F}'$ be a left $\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$-module. The canonical $\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$-linear isomorphism (see 4.3.4.12.1)*

$$\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}', \mathcal{F}') \xrightarrow{\sim} \mathcal{H}om_{\mathcal{C}_{X'}}(\mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}', \mathcal{F}')$$

is compatible with Frobenius. We have similar compatibility replacing left by right modules.

Proof. The compatibility with Frobenius means that the canonical diagram below

$$\begin{array}{ccccc} F^*\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}', \mathcal{F}') & & \underset{\sim}{\longrightarrow} & & \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}', F^*\mathcal{F}') \\ \wr\downarrow & & & & \wr\downarrow \\ F^*\mathcal{H}om_{\mathcal{C}_{X'}}(\mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}', \mathcal{F}') & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{C}_X}(F^*(\mathcal{C}_{X'} \otimes_{\mathcal{B}_{X'}} \mathcal{E}'), F^*\mathcal{F}') & \underset{\sim}{\longrightarrow} & \mathcal{H}om_{\mathcal{C}_X}(\mathcal{C}_X \otimes_{\mathcal{B}_X} F^*\mathcal{E}', F^*\mathcal{F}') \end{array}$$

is commutative, which is easy. ☐

Proposition 6.3.2.6. *Let $\mathcal{E}'$ be a left $\widetilde{\mathcal{D}}^{(m)}_{X'}$-module, $\mathcal{F}'$ be a left $\mathcal{C}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$-module. The canonical $\mathcal{B}_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{D}^{(m)}_{X'}$-linear isomorphism (see 4.3.4.12.1)*

$$(\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{C}_{X'}) \otimes_{\mathcal{C}_{X'}} \mathcal{F}' \xrightarrow{\sim} \mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{F}'. \tag{6.3.2.6.1}$$

is compatible with Frobenius. We have a similar compatibility replacing left by right modules.

Proof. This is an exercice. ☐

6.3.3 Transposition isomorphisms

We keep notation 6.1.

Proposition 6.3.3.1. *Let $\mathcal{M}' \in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$. The transposition isomorphism $\delta_{\mathcal{M}'} \colon \mathcal{M}' \otimes_{\mathcal{B}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S} \xrightarrow{\sim} \mathcal{M}' \otimes_{\mathcal{B}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}$ which exchanges the two right structures of $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules (see 4.2.5.5) is compatible with Frobenius, i.e., the following diagram is commutative :*

$$\begin{array}{ccc} F^{\flat}\mathcal{M}' \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m+s)}_{X/S} & \xrightarrow[\sim]{\delta_{F^{\flat}\mathcal{M}'}} & F^{\flat}\mathcal{M}' \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m+s)}_{X/S} \\ \downarrow\wr & & \downarrow\wr \\ F^{\flat}\mathcal{M}' \otimes_{\mathcal{B}_X} F^{*}F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'/S} & \xrightarrow[\sim]{F^{\flat}_{\mathrm{l}}F^{\flat}_{\mathrm{r}}\delta_{\mathcal{M}'}} & F^{\flat}\mathcal{M}' \otimes_{\mathcal{B}_X} F^{\flat}F^{*}\widetilde{\mathcal{D}}^{(m)}_{X'/S}. \end{array} \tag{6.3.3.1.1}$$

Proof. It is sufficient to copy the proof of Virrion of [Vir00, II.1.12.1] by adding some tildes and replacing "$\omega_{X'}$" by "$\mathcal{M}'$" and "ω_X" by "$F^{\flat}\mathcal{M}'$". ☐

Proposition 6.3.3.2. *Let $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$. The transposition isomorphism of $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodules associated to $\mathcal{E}'$, $\gamma_{\mathcal{E}'} \colon \widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{B}_{X'}} \mathcal{E}' \xrightarrow{\sim} \mathcal{E}' \otimes_{\mathcal{B}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}$ (see 4.2.5.1), is compatible with Frobenius, i.e., the canonical diagram*

$$\begin{array}{ccc} F^{*}F^{\flat}(\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{B}_{X'}} \mathcal{E}') & \xrightarrow[\sim]{F^{*}F^{\flat}(\gamma_{E'})} & F^{*}F^{\flat}(\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \\ \uparrow\wr & & \uparrow\wr \\ \widetilde{\mathcal{D}}^{(m+s)}_{X/S} \otimes_{\mathcal{B}_X} F^{*}\mathcal{E}' & \xrightarrow[\sim]{\gamma_{F^{*}E'}} & F^{*}\mathcal{E}' \otimes_{\mathcal{B}_X} \widetilde{\mathcal{D}}^{(m+s)}_{X/S} \end{array} \tag{6.3.3.2.1}$$

is commutative.

In the same way, we have the compatible with Frobenius isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodules

$$(\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{O}_{X'}} \omega^{-1}_{X'/S}) \otimes_{\mathcal{B}_{X'}} \mathcal{E}' \xrightarrow{\sim} \mathcal{E}' \otimes_{\mathcal{B}_{X'}} (\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\mathcal{O}_{X'}} \omega^{-1}_{X'/S}). \tag{6.3.3.2.2}$$

Proof. First of all, it is equivalent to prove the compatibility with Frobenius of

$$\omega_{X'/S} \otimes_{\mathcal{O}_{X'}} (\gamma_{\mathcal{E}'}) \colon (\omega_{X'} \otimes_{\mathcal{O}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \otimes_{\mathcal{B}_{X'}} \mathcal{E}' \xrightarrow{\sim} (\omega_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{E}') \otimes_{\mathcal{B}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}.$$

Moreover, the isomorphism

$$\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \delta_{\omega_{X'}} \colon \mathcal{E}' \otimes_{\mathcal{B}_{X'}} (\omega_{X'} \otimes_{\mathcal{O}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \xrightarrow{\sim} (\omega_{X'} \otimes_{\mathcal{O}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S}) \otimes_{\mathcal{B}_{X'}} \mathcal{E}'$$

is compatible with Frobenius (6.3.3.1.1). Moreover, thanks to 6.3.2.2 and 6.3.2.1.1, the canonical isomorphism

$$(\omega_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{E}') \otimes_{\mathcal{B}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S} \xrightarrow{\sim} \mathcal{E}' \otimes_{\mathcal{B}_{X'}} (\omega_{X'} \otimes_{\mathcal{O}_{X'}} \widetilde{\mathcal{D}}^{(m)}_{X'/S})$$

is compatible with Frobenius. Since by composing these three isomorphisms we obtain the transposition isomorphism $\delta_{\omega_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{E}'}$ (indeed, by $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linearity it is sufficient to check that for any section x of $\omega_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{E}'$ we have $\delta_{\omega_{X'} \otimes_{\mathcal{O}_{X'}} \mathcal{E}'}(x \otimes 1) = x \otimes 1$) which is compatible with Frobenius following 6.3.3.1.1. This yields the commutativity of 6.3.3.2.1.

By construction of the isomorphism 6.3.3.2.2 (see 4.2.5.3.2), its compatibility with Frobenius follows from that of $\gamma_{\mathcal{E}'}$. ☐

6.3.4 Cartan isomorphisms, relations between internal tensor products and homomorphisms, comparison between $\mathcal{B}$-linear and $\mathcal{D}$-linear dual

We keep notation 6.3.2. In order to be able to use the example 4.6.3.3 or 6.2.2.5, we suppose moreover that $\mathcal{B}_X$ is a quasi-flat $\mathcal{O}_S$-algebra (see Definition 3.1.1.5).

Proposition 6.3.4.1. *Let* $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$*,* $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}, \widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$*, and* $\mathcal{G} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}, \widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$*. The canonical morphism of* $D(\widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$ *(see 4.6.3.6.1)*

$$\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{G} \xrightarrow{4.6.3.6.1} \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{G}) \tag{6.3.4.1.1}$$

is compatible with Frobenius.

Proof. By transitivity of the isomorphism 4.6.3.6.1 (see 4.6.3.6.(ii)), we have the following commutative diagram.

$$\begin{array}{ccc} F^{\flat}(\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{G}) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\mathcal{G} \\ \downarrow & & \downarrow \\ F^{\flat}\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{G}) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\mathcal{G}). \end{array} \tag{6.3.4.1.2}$$

Moreover, via the isomorphism $F^{\flat}\mathcal{G} \xrightarrow{\sim} (F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}\widetilde{\mathcal{D}}^{(m)}_{X'}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\mathcal{G} \xrightarrow{\sim} F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G}$, we have by functoriality of the commutative diagram :

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\mathcal{G} & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G} \\ \downarrow & & \downarrow \\ \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\mathcal{G}) & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G}). \end{array}$$

By invoking the transitivity of 4.6.3.6.ii), we obtain the commutativity of the diagram below

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G} & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},F^{\flat}\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G} \\ \downarrow & & \downarrow \\ \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^{\flat}\widetilde{\mathcal{D}}^{(m)}_{X'} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G}) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},F^{\flat}\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G}). \end{array} \tag{6.3.4.1.3}$$

Finally, choosing a K-flat complex representing $F^{*}F^{\flat}\mathcal{G}$ and a K-injective complex representing $F^{\flat}\mathcal{F}$, we compute that the diagram

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},F^{\flat}\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G} & \xrightarrow[\sim]{F^{*}\otimes\mathrm{id}} & \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X}}(F^{*}\mathcal{E},F^{*}F^{\flat}\mathcal{F}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G} \\ \downarrow & & \downarrow \\ \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E},F^{\flat}\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G}) & \xrightarrow[\sim]{F^{*}} & \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X}}(F^{*}\mathcal{E},F^{*}F^{\flat}\mathcal{F} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{X/S}} F^{*}F^{\flat}\mathcal{G}). \end{array} \tag{6.3.4.1.4}$$

is commutative. By putting end to end these four commutative diagrams, we obtain the commutativity with Frobenius of the morphism of 4.6.3.6.1. □

Proposition 6.3.4.2. *Let* $\mathcal{M}'$ *be a right* $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$*-module,* $\mathcal{E}'$ *be a left* $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$*-module and* $\mathcal{N}'$ *be a* $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$*-bimodule. The canonical* $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$*-linear isomorphisms (see 4.2.4.3)*

$$\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (\mathcal{N}' \otimes_{\mathcal{B}_{X'}} \mathcal{E}') \xrightarrow{\sim} (\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{N}') \otimes_{\mathcal{B}_{X'}} \mathcal{E}', \tag{6.3.4.2.1}$$

$$\mathcal{M}' \otimes_{\mathcal{B}_{X'}} (\mathcal{N}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}') \xrightarrow{\sim} (\mathcal{M}' \otimes_{\mathcal{B}_{X'}} \mathcal{N}') \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}', \tag{6.3.4.2.2}$$

are compatible with Frobenius.

Proof. To prove the compatibility with Frobenius of 6.3.4.2.1, first let us check that the following canonical diagram

$$\begin{array}{ccccc} F^\flat[\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (\mathcal{N}' \otimes_{\mathcal{B}_{X'}} \mathcal{E}')] & \xrightarrow{\sim} & \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^\flat(\mathcal{N}' \otimes_{\mathcal{B}_{X'}} \mathcal{E}') & \underset{\sim}{\xrightarrow{6.2.2.1.2}} & \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (F^\flat\mathcal{N}' \otimes_{\mathcal{B}_X} F^*\mathcal{E}') \\ \sim\downarrow 6.3.4.2.1 & & & & \sim\downarrow \\ F^\flat[(\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{N}') \otimes_{\mathcal{B}_{X'}} \mathcal{E}'] & \underset{\sim}{\xrightarrow{6.2.2.1.2}} & F^\flat(\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{N}') \otimes_{\mathcal{B}_X} F^*\mathcal{E}' & \xrightarrow{\sim} & (\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^\flat\mathcal{N}') \otimes_{\mathcal{B}_X} F^*\mathcal{E}', \end{array} \tag{6.3.4.2.3}$$

where the right vertical morphism is constructed similarly to 6.3.4.2.1, is commutative. For this purpose, let m' be a local section of $\mathcal{M}'$, n' of $\mathcal{N}'$, e' of $\mathcal{E}'$ and θ of $\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_X, \mathcal{O}_{X'})$. The composite morphism of the top of the diagram 6.3.4.2.3 sends the section $[m' \otimes (n' \otimes e')] \otimes \theta$ on $m' \otimes [(n' \otimes \theta) \otimes (1 \otimes e')]$; the right one sends $m' \otimes [(n' \otimes \theta) \otimes (1 \otimes e')]$ on $[m' \otimes (n' \otimes \theta)] \otimes (1 \otimes e')$; the left one sends $[m' \otimes (n' \otimes e')] \otimes \theta$ on $[(m' \otimes n') \otimes e'] \otimes \theta$ and finally that bottom sends $[(m' \otimes n') \otimes e'] \otimes \theta$ on $[m' \otimes (n' \otimes \theta)] \otimes (1 \otimes e')$. The diagram 6.3.4.2.3 is then commutative.

Next, via (the inverse of) the canonical isomorphism $\phi\colon F^\flat\widetilde{\mathcal{D}}^{(m)}_{X'/S} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_X} F^*\widetilde{\mathcal{D}}^{(m)}_{X'/S} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X'/S}$, we check by a computation (we write the image of 1 by ϕ^{-1} etc.) the commutativity of the diagram below

$$\begin{array}{ccc} \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (F^\flat\mathcal{N}' \otimes_{\mathcal{B}_X} F^*\mathcal{E}') & \xrightarrow{\sim} & F^\flat\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_X} (F^*F^\flat\mathcal{N}' \otimes_{\mathcal{B}_X} F^*\mathcal{E}') \\ \sim\downarrow & & \sim\downarrow 6.3.4.2.1 \\ (\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} F^\flat\mathcal{N}') \otimes_{\mathcal{B}_X} F^*\mathcal{E}' & \xrightarrow{\sim} & (F^\flat\mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_X} F^*F^\flat\mathcal{N}') \otimes_{\mathcal{B}_X} F^*\mathcal{E}'. \end{array} \tag{6.3.4.2.4}$$

By composing 6.3.4.2.3 and 6.3.4.2.4, we obtain the diagram meaning that the isomorphism 6.3.4.2.1 is compatible with Frobenius.

Concerning the second isomorphism of the proposition, we proceed in a similar way □

Remark 6.3.4.3. The propositions 6.3.2.2 and 4.3.5.9 allow in the propositions of this section, to replace "left $\widetilde{\mathcal{D}}^{(m)}_X$-module(s)" by "right $\widetilde{\mathcal{D}}^{(m)}_X$-module(s)", and conversely.

Proposition 6.3.4.4. *Let $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $\mathcal{N}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}, \widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$, $\mathcal{M}' \in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S}{}^{\mathrm{r}})$. The following isomorphisms of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$*

$$\mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (\mathcal{N}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{E}') \xrightarrow{\sim} (\mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{N}') \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{E}', \tag{6.3.4.4.1}$$

$$\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} (\mathcal{N}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}') \xrightarrow{\sim} (\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{N}') \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}', \tag{6.3.4.4.2}$$

are compatible with Frobenius.

Proof. By choosing K-flat complexes representing $\mathcal{E}'$ and $\mathcal{N}'$, this is a consequence of 6.3.4.2.2 and 6.3.4.2.1. □

Proposition 6.3.4.5 (Switching $\mathcal{B}$ and $\widetilde{\mathcal{D}}$)**.** *Let $\mathcal{M}'$ be a $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule, $\mathcal{E}', \mathcal{F}'$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules. The functorial in $\mathcal{E}'$, $\mathcal{F}'$, $\mathcal{M}'$ canonical isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules (see 4.2.6.1):*

$$(\mathcal{M}' \otimes_{\mathcal{B}_{X'}} \mathcal{E}') \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{F}' \xrightarrow{\sim} \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (\mathcal{E}' \otimes_{\mathcal{B}_{X'}} \mathcal{F}'), \tag{6.3.4.5.1}$$

is compatible with Frobenius.

Proof. This is obvious by functoriality of 4.2.6.1. □

Corollary 6.3.4.6. *Let $\mathcal{M}'$ be a complex of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S}, {}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $\mathcal{E}'$ and $\mathcal{F}'$ be two complexes of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$. The canonical isomorphism in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ is compatible with Frobenius :*

$$(\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{F}') \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} \mathcal{E}' \xrightarrow{\sim} \mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}} (\mathcal{F}' \otimes^{\mathbb{L}}_{\mathcal{B}_{X'}} \mathcal{E}'). \tag{6.3.4.6.1}$$

Proof. By using K-flat resolutions, this is a consequence of 6.3.4.5. □

Proposition 6.3.4.7. *Let $\mathcal{E}'$ and $\mathcal{G}'$ be two left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules and $\mathcal{F}'$ be a $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule. The canonical $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear morphism (see 4.2.4.9)*

$$\mathcal{H}om_{{}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{F}')\otimes_{\mathcal{B}_{X'}}\mathcal{G}'\to\mathcal{H}om_{{}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{F}'\otimes_{\mathcal{B}_{X'}}\mathcal{G}'),\tag{6.3.4.7.1}$$

is compatible with Frobenius.

Proof. The morphism was constructed at 4.2.4.9. By construction, this is the composite of three morphisms. Proceeding similarly to the proof of 6.3.4.1(iii) (we remove $\mathbb{R}$ and $\mathbb{L}$), we prove that the second morphism is compatible with Frobenius. For the two other ones, this is a consequence of 6.3.4.2.1. □

Corollary 6.3.4.8. *Let $\mathcal{E}'\in D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ $\mathcal{F}'\in D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S},\widetilde{\mathcal{D}}^{(m)\,\mathrm{r}}_{X'/S})$ and $\mathcal{G}'\in D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$. There exists a compatible with Frobenius canonical morphism in $D(\widetilde{\mathcal{D}}^{(m)\,\mathrm{r}}_{X'/S})$ of the form*

$$\mathbb{R}\mathcal{H}om_{{}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{F}')\otimes^{\mathbb{L}}_{\mathcal{B}_{X'}}\mathcal{G}'\to\mathbb{R}\mathcal{H}om_{{}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{F}'\otimes^{\mathbb{L}}_{\mathcal{B}_{X'}}\mathcal{G}').\tag{6.3.4.8.1}$$

The morphism 6.3.4.8.1 is an isomorphism when $\mathcal{E}'\in D_{\mathrm{perf}}({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$.

Proof. Choosing a K-injective complex representing $\mathcal{F}'$ and a K-flat complex representing $\mathcal{G}'$, this is a consequence of 6.3.4.7. □

Proposition 6.3.4.9. *Let $\mathcal{E}'$, $\mathcal{F}'$, $\mathcal{G}'$ be three left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules. The canonical $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear isomorphism (see* **??***).*

$$\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}')\xrightarrow{\sim}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}')),\tag{6.3.4.9.1}$$

is compatible with Frobenius.

Proof. Let us denote by θ the isomorphisms of the form 6.3.4.9.1. The compatibility with Frobenius of θ means that the canonical diagram

$$\begin{array}{ccc}
F^*[\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}')] & \xrightarrow[\sim]{F^*\theta} & F^*[\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}'))] \\
\sim\downarrow 6.2.2.2.1 & & \sim\downarrow 6.2.2.1.1 \\
\mathcal{H}om_{\mathcal{B}_X}(F^*(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}'),F^*\mathcal{G}') & & \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}',F^*\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}')) \\
\sim\downarrow 6.2.2.1.1 & & \sim\downarrow 6.2.2.2.1 \\
\mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}'\otimes_{\mathcal{B}_X}F^*\mathcal{F}',F^*\mathcal{G}') & \xrightarrow[\sim]{\theta} & \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}',\mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{F}',F^*\mathcal{G}'))
\end{array}$$

is commutative. This is checked by a computation : for any $b_0\in\mathcal{B}_X$, for any $\phi\in\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}')$, the image of $b_0\otimes\phi$ into the right bottom term via by both possible paths is $b_1\otimes e'\mapsto(b_2\otimes f'\mapsto b_0b_1b_2\otimes\phi(e'\otimes f'))$, where $b_1,b_2\in\mathcal{B}_X$, $e'\in\mathcal{E}'$ and $f'\in\mathcal{F}'$. □

Corollary 6.3.4.10 (Cartan isomorphism)**.** *Let $\mathcal{E}'\in D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $\mathcal{F}'\in D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ and $\mathcal{G}'\in D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$. The canonical isomorphism of $D({}^1\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ (see 4.6.6.6)*

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}'\otimes^{\mathbb{L}}_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}')\xrightarrow{\sim}\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}')),$$

is compatible with Frobenius.

Proof. By construction (see the proof of 4.6.6.6), this is a consequence of 6.3.4.9. □

Proposition 6.3.4.11. *Let $\mathcal{E}'$, $\mathcal{F}'$, $\mathcal{G}'$ be three left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules. The canonical $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear morphism (see 4.2.4.8) :*

$$\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}')\otimes_{\mathcal{B}_{X'}}\mathcal{G}'\to\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}'\otimes_{\mathcal{B}_{X'}}\mathcal{G}')\tag{6.3.4.11.1}$$

is compatible with Frobenius.

Proof. Let us denote by θ the morphisms of the form 6.3.4.11.1. We have to establish the commutativity of the diagram below

$$\begin{array}{ccccc}
F^*[\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}')\otimes_{\mathcal{B}_{X'}}\mathcal{G}'] & \xrightarrow[\sim]{6.2.2.1.1} & F^*\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}')\otimes_{\mathcal{B}_X}F^*\mathcal{G}' & \xrightarrow[\sim]{6.2.2.2.1} & \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}',F^*\mathcal{F}')\otimes_{\mathcal{B}_X}F^*\mathcal{G}' \\
\downarrow{\scriptstyle F^*(\theta)} & & & & \downarrow{\scriptstyle \theta} \\
F^*[\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}'\otimes_{\mathcal{B}_{X'}}\mathcal{G}')] & \xrightarrow[\sim]{6.2.2.2.1} & \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}',F^*(\mathcal{F}'\otimes_{\mathcal{B}_{X'}}\mathcal{G}')) & \xrightarrow[\sim]{6.2.2.1.1} & \mathcal{H}om_{\mathcal{B}_X}(F^*\mathcal{E}',F^*\mathcal{F}'\otimes_{\mathcal{B}_X}F^*\mathcal{G}').
\end{array}$$

For this purpose, we compute that for any $b_0\in\mathcal{B}_X$, $\phi\in\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}')$, $g'\in\mathcal{G}'$, the section $b_0\otimes\phi\otimes g'$ is sent for both paths on $(b_1\otimes e'\mapsto b_0b_1\otimes\phi(e')\otimes(1\otimes g'))$, with $b_1\in\mathcal{B}_X$, $e'\in\mathcal{E}'$ and $g'\in\mathcal{G}'$. □

Proposition 6.3.4.12. *Let $\mathcal{E}'\in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $\mathcal{F}'\in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ and $\mathcal{G}'\in D(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$. The canonical homomorphism of $D(\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ (see 4.6.6.7) :*

$$\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}')\otimes^{\mathbb{L}}_{\mathcal{B}_{X'}}\mathcal{G}'\to\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{E}',\mathcal{F}'\otimes^{\mathbb{L}}_{\mathcal{B}_{X'}}\mathcal{G}'). \tag{6.3.4.12.1}$$

is compatible with Frobenius. If $\mathcal{E}'$ is moreover in $D_{\mathrm{perf}}(\mathcal{B}_{X'})$, this morphism is an isomorphism.

Proof. By construction of 6.3.4.12.1, this is a consequence of 6.3.4.11.

□

Proposition 6.3.4.13. *Let $\mathcal{E}'$, $\mathcal{F}'$ and $\mathcal{G}'$ three left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules. Suppose the structure of $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-module of $\mathcal{E}'$, $\mathcal{F}'$ or $\mathcal{G}'$ extends to a structure of $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule or left bimodule. Then the canonical the $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-linear isomorphism (see 4.2.4.5)*

$$\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}')\xrightarrow{\sim}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}')). \tag{6.3.4.13.1}$$

is compatible with Frobenius.

Proof. Suppose $\mathcal{G}'$ extends to a structure of $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodule. Since $\mathcal{O}_X$ is locally free on $\mathcal{O}_{X'}$, the morphism $F^\flat\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}')\to\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',F^\flat\mathcal{G}')$ is an isomorphism. We easily compute that the following diagram

$$\begin{array}{ccc}
F^\flat\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}') & \xrightarrow{F^\flat\rho^{(m)}_{X'}} & F^\flat\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}')) \\
\sim\downarrow & & \sim\downarrow \\
\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}',F^\flat\mathcal{G}') & \xrightarrow{\rho^{(m)}_{X'}} & \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',F^\flat\mathcal{G}')) \\
\sim\downarrow{\scriptstyle F^*} & & \sim\downarrow{\scriptstyle F^*} \\
\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X}}(F^*(\mathcal{E}'\otimes_{\mathcal{B}_{X'}}\mathcal{F}'),F^*F^\flat\mathcal{G}') & \xrightarrow{\sim} & \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X}}(F^*\mathcal{E}',F^*\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',F^\flat\mathcal{G}')) \\
{\scriptstyle 6.2.2.1.1}\downarrow\sim & & \downarrow\sim \\
\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X}}(F^*\mathcal{E}'\otimes_{\mathcal{B}_X}F^*\mathcal{F}',F^*F^\flat\mathcal{G}') & \xrightarrow{\rho^{(m+s)}_{X/S}} & \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m+s)}_{X}}(F^*\mathcal{E}',\mathcal{H}om_{\mathcal{B}_{X'}}(F^*\mathcal{F}',F^*F^\flat\mathcal{G}')).
\end{array} \tag{6.3.4.13.2}$$

is commutative, which yields the desired compatibility with Frobenius. The other cases are checked using similar computations. □

Corollary 6.3.4.14. *Let $*\in\{\mathrm{r},\mathrm{l}\}$, $\mathcal{E}'\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$, $\mathcal{F}'\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ and $\mathcal{G}'\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X'/S},{}^{*}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$. The isomorphism of $D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X'/S})$ (see 4.6.6.5)*

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}'\overset{\mathbb{L}}{\otimes}_{\mathcal{B}_{X'}}\mathcal{F}',\mathcal{G}')\xrightarrow{\sim}\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{X'/S}}(\mathcal{E}',\mathbb{R}\mathcal{H}om_{\mathcal{B}_{X'}}(\mathcal{F}',\mathcal{G}'))$$

is compatible with Frobenius

Proof. Similarly to 4.6.6.5, by using a K-flat complex of left $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-modules representing $\mathcal{F}'$ and a K-injective complex of right (resp. left if $* = \mathrm{l}$) $\widetilde{\mathcal{D}}^{(m)}_{X'/S}$-bimodules representing $\mathcal{G}'$, this follows from 6.3.4.13. □

Proposition 6.3.4.15. *Let $\mathcal{F} \in D^{\mathrm{b}}(\widetilde{\mathcal{D}}^{(m)}_X)$. The following assertions hold. The canonical morphism of $\widetilde{\mathcal{D}}^{(m)}_X$-modules (see 4.6.7.1) is compatible with Frobenius:*

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_X}(\mathcal{B}_X, \widetilde{\mathcal{D}}^{(m)}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{R}\mathcal{H}om_{\mathcal{B}_X}(\mathcal{F}, \mathcal{B}_X) \to \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_X}(\mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}).$$

Proof. By construction (see the proof of 4.6.7.1), its compatibility with Frobenius follows from 6.3.4.8, 6.3.3.2.2, 6.3.4.12, 6.3.4.14 □

Chapter 7

Completed sheaves of differential operators of level m

This chapter is mostly based on the unpublished notes of Berthelot.

7.1 Derived category of projective systems

7.1.1 Projective and inductive systems, projective and inductive limits

Let us recall some terminology and notation.

Definition 7.1.1.1. Let $\mathfrak{I}$ be a small category, $\mathfrak{C}$ be a category.

1. An inductive system of objects of $\mathfrak{C}$ indexed by $\mathfrak{I}$ is a covariant functor of the form $\mathfrak{I} \to \mathfrak{C}$. We denote by $\mathfrak{C}^{\mathfrak{I}}$ the category of inductive system indexed by $\mathfrak{I}$.

2. A projective system of objects of $\mathfrak{C}$ indexed by $\mathfrak{I}$ is a contravariant functor of the form $\mathfrak{I} \to \mathfrak{C}$. In other words, a projective system indexed by $\mathfrak{I}$ is an inductive system indexed by $\mathfrak{I}^{\mathrm{op}}$, the opposite category of $\mathfrak{I}$. We denote by $\mathfrak{C}_{\mathfrak{I}}$ be the category of projective system indexed by $\mathfrak{I}$. By definition, we have the equality $\mathfrak{C}_{\mathfrak{I}} = \mathfrak{C}^{\mathfrak{I}^{\mathrm{op}}}$.

7.1.1.2. Let $\mathfrak{I}$ be a small category, $\mathfrak{C}$ be a category. We denote by $c^{\mathfrak{I}} \colon \mathfrak{C} \to \mathfrak{C}^{\mathfrak{I}}$ the functor which sends an object X of $\mathfrak{C}$ to the constant object of $\mathfrak{C}^{\mathfrak{I}}$ with value X and identity of X as transition maps. Similarly, we have the functor $c_{\mathfrak{I}} \colon \mathfrak{C} \to \mathfrak{C}_{\mathfrak{I}}$. Let $F \in \mathfrak{C}^{\mathfrak{I}}$ (resp. $G \in \mathfrak{C}_{\mathfrak{I}}$).

When the left (resp. right) covariant functors $\mathfrak{C} \to \mathit{Sets}$

$$Y \mapsto \mathrm{Hom}_{\mathfrak{C}^{\mathfrak{I}}}(F, c^{\mathfrak{I}}(Y)), \qquad Y \mapsto \mathrm{Hom}_{\mathfrak{C}_{\mathfrak{I}}}(G, c_{\mathfrak{I}}(Y)), \tag{7.1.1.2.1}$$

is representable, we say that the inductive limits of F (resp. G) exists and the object of $\mathfrak{C}$ representing this functor is denoted by $\varinjlim F$ (resp. $\varinjlim G$) or $\varinjlim_{i \in \mathfrak{I}} F(i)$ (resp. $\varinjlim_{i \in \mathfrak{I}^{\mathrm{op}}} G(i)$). Let us explain a bit why "op" appears in the respective notation. Via the equality $\mathfrak{C}^{\mathfrak{I}} = \mathfrak{C}_{\mathfrak{I}^{\mathrm{op}}}$, we can view F as on object of $\mathfrak{C}_{\mathfrak{I}^{\mathrm{op}}}$ that we will denote by $F^{\star}$ to avoid confusion. We get $\varinjlim_{i \in \mathfrak{I}} F(i) = \varinjlim_{i \in \mathfrak{I}} F^{\star}(i)$, which justifies our notations. In order to avoid confusion, we will consider inductive limits for inductive systems by default.

When the left (resp. right) contravariant functors $\mathfrak{C} \to \mathit{Sets}$

$$X \mapsto \mathrm{Hom}_{\mathfrak{C}^{\mathfrak{I}}}(c^{\mathfrak{I}}(X), F), \qquad X \mapsto \mathrm{Hom}_{\mathfrak{C}_{\mathfrak{I}}}(c_{\mathfrak{I}}(X), G), \tag{7.1.1.2.2}$$

is representable, we say that the projective limits of F (resp. G) exists and the object of $\mathfrak{C}$ representing this functor is denoted by $\varprojlim F$ (resp. $\varprojlim G$) or $\varprojlim_{i \in \mathfrak{I}^{\mathrm{op}}} F(i)$ (resp. $\varprojlim_{i \in \mathfrak{I}} G(i)$).

Via the equality $\mathfrak{C}_{\mathfrak{I}} = (\mathfrak{C}^{\mathrm{op}})_{\mathfrak{I}^{\mathrm{op}}}$, we can view G as on object of $(\mathfrak{C}^{\mathrm{op}})_{\mathfrak{I}^{\mathrm{op}}}$ that we will denote by G^{op}. Remark we get $\varprojlim G = \varinjlim G^{\mathrm{op}}$.

7.1.2 Topos of projective systems of sheaves on a topological spaces

The topoi that we will use in the book are essentially that considered in 7.1.2.1.

Notation 7.1.2.1. Let I be a partially ordered set and let X be a topological space.

1. We denote by $\mathrm{Cat}(I)$ the category defined as follows: the objects of $\mathrm{Cat}(I)$ are the elements $i \in I$ and for any $i, j \in I$, the set of homomorphisms from i to j has only one element if $i \leq j$ and is empty otherwise. We denote by $I^\natural$ the site (see definition [Sta22, 00VH-7.6.2]), whose underlying category is $\mathrm{Cat}(I)$ and whose topology is the chaotic topology (i.e. the collection of coverings of an object $i \in I$ is $\{\mathrm{id}_i : i \to i\}$). By abuse of notation, if there is no ambiguity, we might write I instead of $\mathrm{Cat}(I)$ or $I^\natural$. We denote by I° the partially ordered set equal to I as a set but equipped with the ordering opposite to that of I. We get $\mathrm{Cat}(I^\circ) = \mathrm{Cat}(I)^{\mathrm{op}}$.

2. As in [Sta22, 00VJ-7.6.4]), we denote by X_{Zar} the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. We still denote by X_{Zar} the site whose underlying category is X_{Zar} and whose covering are families of morphisms $\{U_l \to U\}_{l \in L}$ of X_{Zar} such that $\cup_{l \in L} U_l = U$ (i.e. coverings are by definition open coverings).

3. We denote by $I^\natural \times X_{Zar}$ the site whose underlying category is $\mathrm{Cat}(I) \times X_{Zar}$ and whose covering are families of morphisms $\{(i, U_\lambda) \to (i, U)\}_{\lambda \in \Lambda}$ of $\mathrm{Cat}(I) \times X_{Zar}$ such that $\cup_{\lambda \in \Lambda} U_\lambda = U$.

4. We have the equality $\mathrm{PSh}(I^\natural) = \mathrm{Sh}(I^\natural)$ and $\mathrm{PSh}(I^\natural)$ is equal to the category of projective systems in Set indexed by I, i.e. to the category of functors $\mathrm{Cat}(I)^{\mathrm{op}} \to \mathrm{Set}$. Moreover, $\mathrm{PSh}(I^\natural \times X_{Zar})$ (resp. $\mathrm{Sh}(I^\natural \times X_{Zar})$) is equal to the category of projective systems in $\mathrm{PSh}(X_{Zar})$ (resp. $\mathrm{Sh}(X_{Zar})$) indexed by I i.e. to the category of functors $\mathrm{Cat}(I)^{\mathrm{op}} \to \mathrm{PSh}(X_{Zar})$ (resp. $\mathrm{Cat}(I)^{\mathrm{op}} \to \mathrm{Sh}(X_{Zar})$). An object of $\mathrm{Sh}(I^\natural \times X_{Zar})$ is written $\mathcal{F}_\bullet$ or $(\mathcal{F}_i)_{i \in I}$. We set

$$\mathrm{Top}(X) := \mathrm{Sh}(X_{Zar}), \qquad \mathrm{Top}(X)_I := \mathrm{Sh}(I^\natural \times X_{Zar}).$$

Even if most of the results are still valid for any topos, we will restrict in this book our attention only topos of the form $\mathrm{Top}(X)_I$. Such topos can be called I-topos or I^o-topos.

Definition 7.1.2.2. Let $u \colon I \to I'$ be an increasing map of partially ordered sets. For $i' \in I'$ denote by $I_{i'}$ the ordered subset of I consisting of i such that $u(i) \leq i'$ and by $I^{i'}$ the subset of I consisting of i such that $u(i) \geq i'$.

1. We say that u is "filtered" (resp. "cofiltered") when $I_{i'}$ is directed (resp. $(I^{i'})^\circ$ is directed), i.e. $\mathrm{Cat}(I_{i'})$ is filtered (resp. $\mathrm{Cat}(I^{i'})$ is cofiltered (see definition [KS06, 3.1.1]).

2. We say u is "cofinal" if $I^{i'}$ is non-empty for all $i' \in I'$.

Remark 7.1.2.3. Let $u \colon I \to I'$ be an increasing map of partially ordered sets. By convention, if u is cofiltered then u is cofinal. Moreover, let $u^\circ \colon I^\circ \to I'^\circ$ be the increasing map induced by u. From u°, we define such as 7.1.2.2 the partially ordered sets $(I^\circ)_{i'}$ and $(I^\circ)^{i'}$ for any $i' \in I'$. For $i' \in I'$, we have the equalities

$$(I^\circ)_{i'} = (I^{i'})^\circ \qquad (I^\circ)^{i'} = (I_{i'})^\circ. \tag{7.1.2.3.1}$$

Hence, u is filtered if and only if $u^\circ \colon I^\circ \to I'^\circ$ is cofiltered. Finally, when I' has only one element, then u is filtered (resp. cofiltered) if and only if I is filtered (resp. cofiltered).

7.1.2.4 (Varying I). Let $u \colon I \to I'$ be an increasing map of partially ordered sets. The canonical functor $u_X \colon I^\natural \times X_{Zar} \to I'^\natural \times X_{Zar}$ given by $(i, U) \mapsto (u(i), U)$ is cocontinuous (see definition [Sta22, 00XJ-7.20.1]). Hence we get the morphism of topos

$$u_X = (u_X^{-1} \dashv u_{X*}) \colon \mathrm{Top}(X)_I \to \mathrm{Top}(X)_{I'} \tag{7.1.2.4.1}$$

induced by u_X (see [Sta22, 00XO-7.21.1]). For any $\mathcal{F}'_\bullet \in \mathrm{Top}(X)_{I'}$, for any $i \in I$, we have $u_X^{-1}(\mathcal{F}'_\bullet)_i = \mathcal{F}'_{u(i)}$. For any $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$, for any $i' \in I'$ we have

$$u_{X*}(\mathcal{F}_\bullet)_{i'} = \varprojlim_{u(i) \leq i'} \mathcal{F}_i,$$

where the projective limit is that of the functor $\mathrm{Cat}(I_{i'})^{\mathrm{op}} \to \mathrm{Sh}(X_{Zar})$ induced by $\mathcal{F}_\bullet$.

Since u_X is continuous we have a left adjoint $u_{X!}$ of u_X^{-1}. We compute for any $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$ that

$$u_{X!}(\mathcal{F}_\bullet)_{i'} = \varinjlim_{u(i)\geq i'} \mathcal{F}_i, \tag{7.1.2.4.2}$$

where the inductive limit is that of the inductive system $\mathrm{Cat}((I^{i'})^{\mathrm{o}}) \to \mathrm{Sh}(X_{Zar})$ induced by $\mathcal{F}_\bullet$.

Beware that $u_{X!}$ is not necessarily exact. Indeed, suppose there exists an $i' \in I'$ such that $I_{i'}$ is empty. Denoting by e and e' the final object (i.e. the projective limit indexed by the empty set) of respectively $\mathrm{Top}(X)_I$ and $\mathrm{Top}(X)_{I'}$, then we do not have $u_{X!}(e) = e'$. This means that the adjoint pair $(u_{X!} \dashv u_X^{-1})$ do not induce a morphism of topos $\mathrm{Top}(X)_{I'} \to \mathrm{Top}(X)_I$.

When u is cofiltered, it follows from [KS06, 3.1.6] that the functor $u_{X!}$ commutes with finite projective limits. Hence, $u_{X!}$ is exact and we get the morphism of topos

$$\breve{u}_X = (u_{X!} \dashv u_X^{-1})\colon \mathrm{Top}(X)_{I'} \to \mathrm{Top}(X)_I, \tag{7.1.2.4.3}$$

i.e. $\breve{u}_X^{-1} = u_{X!}, \qquad \breve{u}_{X*} = u_X^{-1}$.

Notation 7.1.2.5. Let $I' := \{*\}$ be some one element set. Let I be a partially ordered set, $u\colon I \to \{*\}$ be the map. Since $I'^{\natural} \times X_{Zar}$ is equivalent to the site X_{Zar}, by identifying $\mathrm{Top}(X)$ with $\mathrm{Top}(X)_{\{*\}}$, then the morphism of 7.1.2.4.1 is denoted in this case

$$\underleftarrow{l}_{X,I} = (\underleftarrow{l}_{X,I}^{-1} \dashv \underleftarrow{l}_{X,I*})\colon \mathrm{Top}(X)_I \to \mathrm{Top}(X). \tag{7.1.2.5.1}$$

We have $\underleftarrow{l}_{X,I}^{-1}(\mathcal{F})_i = \mathcal{F}$ for any $\mathcal{F} \in \mathrm{Top}(X)$ and any $i \in I$; transition morphisms are the identities. Moreover, for any $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$ we have

$$\underleftarrow{l}_{X,I*}(\mathcal{F}_\bullet) = \varprojlim_i \mathcal{F}_i,$$

where the projective limit is that of the functor $\mathrm{Cat}(I)^{\mathrm{op}} \to \mathrm{Sh}(X_{Zar})$ induced by the object $\mathcal{F}_\bullet$.

Remark 7.1.2.6. Suppose I is a filtered set and $J \subset I$ is a cofinal subset. Let $u\colon J \to I$ the corresponding map. Then, with notation 7.1.2.4.1 and 7.1.2.5, we get the isomorphism $\underleftarrow{l}_{X,J*} \circ u_X^{-1} \xrightarrow{\sim} \underleftarrow{l}_{X,I*}$.

Notation 7.1.2.7. Let $I' := \{*\}$ be some one element set. Let I be a partially ordered set. Fix $i \in I$ and let $i\colon I' \to I$ be the map sending $*$ to i. Since $I'^{\natural} \times X_{Zar}$ is equivalent to the site X_{Zar}, by identifying $\mathrm{Top}(X)$ with $\mathrm{Top}(X)_{\{*\}}$, then the morphism of 7.1.2.4.1 is denoted in this case

$$i_X = (i_X^{-1} \dashv i_{X*})\colon \mathrm{Top}(X) \to \mathrm{Top}(X)_I. \tag{7.1.2.7.1}$$

We have $i_X^{-1}(\mathcal{F}_\bullet) = \mathcal{F}_i$ for any $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$ and we compute

$$(i_{X*}(\mathcal{F}))_j = \begin{cases} \mathcal{F} & \text{if } j \geq i \\ e & \text{otherwise} \end{cases}, \qquad (i_{X!}(\mathcal{F}))_j = \begin{cases} \mathcal{F} & \text{otherwise} \\ \emptyset & \text{if } j > i \end{cases} \tag{7.1.2.7.2}$$

where e (resp. $\emptyset$) is the final (resp. initial) object of $\mathrm{Top}(X)$ for any $\mathcal{F} \in \mathrm{Top}(X)$.

Notation 7.1.2.8. Let I be a partially ordered set and $i \in I$. With notation 7.1.2.18, the morphism of 7.1.2.4.1 is in this case where u is the morphism $u_i\colon I_{\leq i} \to I$ induces the morphism of topos

$$u_{i,X} = (u_{i,X}^{-1} \dashv u_{i,X*})\colon \mathrm{Top}(X)_{I_{\leq i}} \to \mathrm{Top}(X)_I, \tag{7.1.2.8.1}$$

which is a localization morphism. For any $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$, $u_{i,X}^{-1}(\mathcal{F}_\bullet)$ is the projective system $(\mathcal{F}_j)_{j\leq i}$ induced by restriction. For any $\mathcal{F}'_\bullet \in \mathrm{Top}(X)_{I_{\leq i}}$, we have also

$$(u_{i,X*}(\mathcal{F}'_\bullet))_j = \begin{cases} \mathcal{F}'_j & \text{if } j \geq i \\ e & \text{otherwise} \end{cases}, \qquad (u_{i,X!}(\mathcal{F}'_\bullet))_j = \begin{cases} \mathcal{F}'_j & \text{if } j \leq i \\ \emptyset & \text{otherwise} \end{cases} \tag{7.1.2.8.2}$$

where e (resp. $\emptyset$) is the final (resp. initial) object of $\mathrm{Top}(X)$.

7.1.2.9 (Varying X)**.** Let I be partially ordered set and $f\colon X \to X'$ be a continuous map of topological spaces. We have the topos morphism

$$f_I = (f_I^{-1} \dashv f_{I*})\colon \mathrm{Top}(X)_I \to \mathrm{Top}(X')_I \tag{7.1.2.9.1}$$

defined by setting $f_I^{-1}(\mathcal{G}_\bullet)\colon I^{\mathrm{o}} \to \mathrm{Top}(X)$ is the functor $i \mapsto f^{-1}\mathcal{G}_i$ and $f_{I*}(\mathcal{F}_\bullet)\colon I^{\mathrm{o}} \to \mathrm{Top}(X)$ is the functor $i \mapsto f_*\mathcal{F}_i$.

7.1.2.10. Let $u\colon I' \to I$ be an increasing map of partially ordered sets. Let $f\colon X \to X'$ be a continuous map of topological spaces. Then we get the equality $f_I \circ u_X = u_{X'} \circ f_{I'}$ as morphism of topos $\mathrm{Top}(X)_I \to \mathrm{Top}(X')_{I'}$. In particular, for any $i \in I$, wet get $f_I \circ i_X = i_{X'} \circ f$, where $f\colon \mathrm{Top}(X) \to \mathrm{Top}(X')$ is the morphism of topos induced by f.

7.1.2.11. This is not clear that the topos $\mathrm{Sh}(I^\natural)$ is algebraic (in the sense of [SGA4.2, VI.2.3]). We will describe when this is the case in 7.1.2.15 thanks to the equivalence of topoi 7.1.2.13.1. However, in the special case below, we can check it without using the equivalence of topoi 7.1.2.13.1.

(a) Any objects of the site $I^\natural$ is quasi-compact (in the site sense of [SGA4.2, VI.1.1]). Let us denote by $\varepsilon\colon \mathrm{Cat}(I) \to \mathrm{Sh}(I^\natural)$ the canonical functor (see [SGA4.1, II.4.4.0]), which is fully faithful (because the topology of $\mathrm{Cat}(I)$ is subcanonical) and commutes with projective limits. Let $\varepsilon(I)$ be the essential image of ε. Then following [SGA4.2, VI.1.2] the objects of $\varepsilon(I)$ are quasi-compact.

(b) Suppose I satisfies one of the following equivalent properties:

  (i) For any $i, j \in I$, the set $\{k \in I \text{ such that } k \le i, k \le j\}$ has a greatest element that we denote by $\inf\{i, j\}$.

  (ii) The fibered products in $\mathrm{Cat}(I)$ exists.

  (iii) The subsets $I_{\le i}$ with $i \in I$ are closed under intersection.

  Then $\varepsilon(I)$ is stable under fiber products. Since the kernels in $\mathrm{Sh}(I^\natural)$ of two morphisms $u, v\colon A \to B$ such that A and B are objects of $\varepsilon(I)$ is an object of $\varepsilon(I)$ (because necessarily $u = v$ by full faithfulness of ε) and is therefore quasi-compact, then following the remark [SGA4.2, VI.2.2.1], we get that $\mathrm{Sh}(I^\natural)$ is an algebraic topos.

7.1.2.12 (Open subtopos of $\mathrm{Top}(X)_I$)**.** Let I be a partially ordered set and let X be a topological space. We describe here the opens of the site $I^\natural \times X_{Zar}$ defined at 7.1.2.1.

(a) Since the topology is chaotic, a sieve of $I^\natural$ is the same as a sieve of $I^\natural$ of local nature. Let J be a subset of I. Then $\mathrm{Cat}(J)$ (simply denoted by abuse of notation by J) is a sieve of $I^\natural$ if and only if the inclusion $J \to \cup_{j\in J} I_{\le j}$ is an equality. For example, the subsets $I_{\le j}$ with $j \in I$ of I are sieves.

(b) For any $(i, U) \in I^\natural \times X_{Zar}$, we set $\mathcal{U}_{\le(i,U)} := \{(j, V) \in I^\natural \times X_{Zar}\ ;\ j \le i \text{ and } V \subset U\}$. This is straightforward that $\mathcal{U}_{\le(i,U)}$ is an open of the site $I^\natural \times X_{Zar}$. It corresponds to $\mathcal{U}_{\le(i,U)}$ a subobject of the final object of $\mathrm{Sh}(I^\natural \times X_{Zar})$ (the construction is recalled in 4.6.2.1) which is equal to $h_{(i,U)} = h^\sharp_{(i,U)}$ (we remark the topology on $\mathrm{Sh}(I^\natural \times X_{Zar})$ is subcanonical).

(c) To get all the opens of the site $I^\natural \times X_{Zar}$, we need the following construction. Let $\phi\colon \mathrm{Cat}(J)^{\mathrm{op}} \to X_{Zar}$ be a functor, where J is a sieve of $I^\natural$. We set $\mathcal{U}_\phi := \cup_{j\in J} \mathcal{U}_{\le(j,\phi(j))}$. Since a union of opens is an open then $\mathcal{U}_\phi$ is an open of $I^\natural \times X_{Zar}$. We remark that for any $(i, U) \in \mathrm{Ob}(I^\natural \times X_{Zar})$, we have the property $(i, U) \in \mathcal{U}_\phi$ if and only if $i \in J$ and $U \subset \phi(i)$. Hence, for any functor $\psi\colon \mathrm{Cat}(J')^{\mathrm{op}} \to X_{Zar}$, where J' is a sieve of $I^\natural$, we have the inclusion $\mathcal{U}_\phi \subset \mathcal{U}_\psi$ if and only if $J \subset J'$ and for any $j \in J$, $\phi(j) \subset \psi(j)$. This yields that we have the equality $\mathcal{U}_\phi = \mathcal{U}_\psi$ if and only if $J = J'$ and $\phi = \psi$.

  For any $(i, U) \in I^\natural \times X_{Zar}$, we have $\mathcal{U}_{\le(i,U)} = \mathcal{U}_{c_{I_{\le i}}(U)}$, where $c_{I_{\le i}}(U)\colon \mathrm{Cat}(I_{\le i})^{\mathrm{op}} \to X_{Zar}$ is the constant object of $\mathrm{Cat}(I_{\le i})^{\mathrm{op}} \to X_{Zar}$ with value U (see 7.1.1.2).

(d) The opens of $I^\natural \times X_{Zar}$ correspond to the subsets of the form $\mathcal{U}_\phi$. Indeed, we already know that subsets of the form $\mathcal{U}_\phi$ are opens. Conversely, let $\mathcal{U}$ be an open of $I^\natural \times X_{Zar}$. For any $i \in I$, let $E_i := \{V \in X_{Zar}\ ;\ (i, V) \in \mathcal{U}\}$. Let J be the set of the elements $i \in I$ such that E_i is not empty. Since $\mathcal{U}$ is a sieve, then so is J. Since $\mathcal{U}$ is of local nature, by setting $U_i := \cup_{V\in E_i} V$ we have $U_i \in E_i$

(in other words, U_i is the greatest element of E_i). For any $i \leq j$, since $(i, U_j) \leq (j, U_j)$ and $\mathcal{U}$ is a sieve, then we get $U_j \subset U_i$. This yields the functor $\phi\colon \mathrm{Cat}(J)^{\mathrm{op}} \to X_{Zar}$ defined by setting $\phi(i) := U_i$. We have $\mathcal{U} = \mathcal{U}_\phi$.

7.1.2.13 (An equivalence of topoi)**.** Let I be a partially ordered set and let X be a topological space. We can endow I with a canonical topology as follows. We say that a subset J of I is open if $J = \cup_{j\in J} I_{\leq j}$, i.e. $\mathrm{Cat}(J)$ is a sieve of $I^\natural$. We easily check that this gives a topology on I. Moreover, we remark that the subsets $I_{\leq i}$ where $i \in I$ form a basis of open subsets on I. We endow the set $I \times X$ with the product topology. The open subsets $(I_{\leq i}, U)$ where $i \in I$ and U is a non-empty open subset of X form a basis of open subsets of $I \times X$ that we will be denoted by $\mathfrak{B}$.

(a) The opens of $I \times X$ are described as follows. Let $X'_{Zar} := X_{Zar} \setminus \{\emptyset\}$ and let $\phi\colon \mathrm{Cat}(J)^{\mathrm{op}} \to X'_{Zar}$ be a functor, where J is an open of I. We set $\mathcal{V}_\phi := \cup_{j\in J}(I_{\leq j}, \phi(j)) \subset I \times X$. For any non-empty open subset V of $I \times X$, there exists a unique functor $\phi\colon \mathrm{Cat}(J)^{\mathrm{op}} \to X'_{Zar}$ such that $V = \mathcal{V}_\phi$. Hence, we get therefore an order preserving injection from the set of opens of the topogical space $I \times X$ to the set of opens of the site $I^\natural \times X_{Zar}$ given by $\mathcal{V}_\phi \mapsto \mathcal{U}_\phi$.

    Unicity: We remark that for any $(i, x) \in I \times X$, we have the property $(i, x) \in \mathcal{V}_\phi$ if and only if $i \in J$ and $x \in \phi(i)$. Hence, for any functor $\psi\colon \mathrm{Cat}(J')^{\mathrm{op}} \to X'_{Zar}$, where J' is an open of I, we have the inclusion $\mathcal{V}_\phi \subset \mathcal{V}_\psi$ if and only if $J \subset J'$ and for any $j \in J$, $\phi(j) \subset \psi(j)$. This yields the equality $\mathcal{V}_\phi = \mathcal{V}_\psi$ if and only if $J = J'$ and $\phi = \psi$.

    Existence: Let V be a non-empty open of $I \times X$. For any $i \in I$, let $E_i := \{U \in X'_{Zar}\ ;\ (I_{\leq i}, U) \subset V\}$. Let J be the set of the elements $i \in I$ such that E_i is not empty. Then J is an open of I. By setting $U_i := \cup_{V\in E_i} V$ we have $U_i \in E_i$ (in other words, U_i is the greatest element of E_i). For any $i \leq j$, since $I_{\leq i} \leq I_{\leq j}$, then we get $U_j \subset U_i$. This yields the functor $\phi\colon \mathrm{Cat}(J)^{\mathrm{op}} \to X'_{Zar}$ defined by setting $\phi(i) := U_i$. We have $V = \mathcal{V}_\phi$.

(b) We denote by $(I \times X)_{\mathrm{Zar}}$ the site induced by the topological space $I \times X$. Let $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$. We get a presheaf $\alpha_*(\mathcal{F}_\bullet)$ on the basis of opens subsets $\mathfrak{B}$ of $I \times X$ by setting $\alpha_*(\mathcal{F}_\bullet)(I_{\leq i}, U) := \mathcal{F}_i(U)$. In fact $\alpha_*(\mathcal{F}_\bullet)$ is a sheaf on the basis $\mathfrak{B}$, i.e. satisfies the property (F_0) of [Gro60, 3.2.2]. Indeed, let $(I_{\leq i_\alpha}, U_\alpha)_{\alpha\in A}$ be an open covering of $(I_{\leq i}, U)$. We remark that $(I_{\leq i}, U_\alpha)_{\alpha\in A'}$ is also an open covering of $(I_{\leq i}, U)$, where A' is the subset of A consisting of the elements α such that $i_\alpha = i$. Hence, to check the property (F_0) of [Gro60, 3.2.2] it is sufficient to reduce to the case where the open covering of $(I_{\leq i}, U)$ is of the form $(I_{\leq i}, U_\alpha)_{\alpha\in A'}$. Then, we can check the property (F_0) by using the fact that $\mathcal{F}_i$ is a sheaf on X. We still denote by $\alpha_*(\mathcal{F}_\bullet)$ the induced sheaf on $(I \times X)_{\mathrm{Zar}}$. We get the morphism of topoi

$$\alpha\colon \mathrm{Top}(X)_I \to \mathrm{Sh}((I \times X)_{\mathrm{Zar}}) = \mathrm{Top}(I \times X), \tag{7.1.2.13.1}$$

    where α_* is the functor constructed above and for any $\mathcal{F} \in \mathrm{Sh}((I \times X)_{\mathrm{Zar}})$, we define $\alpha^{-1}(\mathcal{F}) := \mathcal{F}_\bullet$ so that $\mathcal{F}_i$ is the sheaf on X defined by setting $\mathcal{F}_i(U) := \mathcal{F}(I_{\leq i}, U)$ for any $i \in I$ and open set U of X, the transition maps $\mathcal{F}_j(U) \to \mathcal{F}_i(U)$ are given by the restriction map $\mathcal{F}(I_{\leq j}, U) \to \mathcal{F}(I_{\leq i}, U)$ for any $i \leq j$. Since $\alpha^{-1} \circ \alpha_* = \mathrm{id}$ and $\alpha_* \circ \alpha^{-1} = \mathrm{id}$, this is in fact an equivalence of topoi.

7.1.2.14 (Finiteness properties)**.** Let X be a topological space.

(a) Following [SGA4.2, VI.1.2 and 1.6.1], for any open U of X, U is quasi-compact if and only if h_U is a quasi-compact object of the topos $\mathrm{Top}(X)$, where $U \mapsto h_U$ is the canonical functor $X_{Zar} \to \mathrm{Top}(X)$. Moreover, a sheaf F of $\mathrm{Top}(X)$ is quasi-compact if and only if the étale space over X associated with F is quasi-compact (loc. cit.). Let $f\colon X \to Y$ be a continuous morphism of topological spaces. With [SGA4.1, IV.4.1.1], this yields that the topos morphism $\mathrm{Top}(f)\colon \mathrm{Top}(X) \to \mathrm{Top}(Y)$ is quasi-compact (see definition [SGA4.2, VI.3.1]) if and only if f is quasi-compact.

(b) The topological space X is said to be coherent (in the sense of [FK18, 0.2.2.1]) if X has an open basis consisting of quasi-compact open subsets and X is quasi-compact and quasi-separated. The topological space X is said to be locally coherent (in the sense of [FK18, 0.2.2.21]) if X admits an open covering by coherent subspaces. This is equivalent to saying that has an open basis $\mathfrak{B}$ consisting of quasi-compact open subsets such that fibered product of objects of $\mathfrak{B}$ are quasi-compact (the objects of $\mathfrak{B}$ are therefore coherent). Hence, following [SGA4.2, VI.2.4.7]), $\mathrm{Top}(X)$ is algebraic (in the sense of [SGA4.2, VI.2.3]) if and only if X is locally coherent.

(c) Suppose X is locally coherent. Let $\mathcal{C}(X)$ be the family of open coherent subsets of X. The family $\mathcal{C}(X)$ is a family of topological generators (in the sense of [SGA4.1, II.3.0.1]). Then, it follows from [SGA4.1, II.4.10] that the family of objects $(h_U)_{U\in\mathcal{C}(X)}$ is a family of generators of $\mathrm{Top}(X)$. Hence, we deduce from [SGA4.2, VI.1.11], that to check that a morphism from a quasi-compact object F of $\mathrm{Top}(X)$ to h_X is quasi-compact, we reduce to the case where $F = h_U$, with $U \in \mathcal{C}(X)$. Hence, h_X is quasi-separated if and only if X is quasi-separated.

(d) This yields that X is locally coherent and quasi-separated (resp. coherent) if and only if $\mathrm{Top}(X)$ is quasi-separated (resp. coherent) in the sense of [SGA4.2, VI.2.3].

(e) Suppose X is locally coherent. Let F be a sheaf of $\mathrm{Top}(X)$. Following [SGA4.2, VI.2.4.6], the sheaf F of $\mathrm{Top}(X)$ is quasi-separated (resp. coherent) if and only if the topos $\mathrm{Top}(X)_{/F}$ is quasi-separated (resp. coherent). Since $\mathrm{Top}(X)_{/F} \xrightarrow{\sim} \mathrm{Top}(F')$ where F' is the étale space associated with F (see [SGA4.1, IV.5.7]), since F' is also locally coherent, then the object F of the topos $\mathrm{Top}(X)$ is quasi-separated if and only if to the topological space F' is quasi-separated (coherent).

(f) Let $f\colon X \to Y$ be a morphism of locally coherent topological spaces. It follows from (e) (resp. and (a)) that f is quasi-separated (resp. coherent) if and only if $\mathrm{Top}(f)\colon \mathrm{Top}(X) \to \mathrm{Top}(Y)$ is quasi-separated (resp. coherent) as morphism of topoi (see definition [SGA4.2, VI.3.1]).

7.1.2.15 (Finiteness properties)**.** Let X be a topological space. Let I be a partially ordered set. Recall I can be naturally seen as a topological space so that the subsets $I_{\leq i}$ where $i \in I$ form a basis of open subsets of I (see 7.1.2.13).

(a) The subsets $I_{\leq i}$ are quasi-compact for any $i \in I$. Indeed, if $I_{\leq i} = \cup_{\alpha\in A} J_\alpha$ where J_α are opens of I, then there exists $\alpha \in A$ such that $i \in J_\alpha$. Since J_α is a open then $I_{\leq i} \subset J_\alpha$, and therefore $I_{\leq i} = J_\alpha$. This yields the quasi-compact open subsets of I are the finite union of subsets of the form $I_{\leq i}$ with $i \in I$.

(b) Suppose the topological space I is quasi-separated, i.e. suppose the intersection of two subsets of the form $I_{\leq i}$ with $i \in I$ is quasi-compact. Then the open quasi-compact subsets of I (in particular $I_{\leq i}$) are coherent (in the sense of [FK18, 0.2.2.1]) and I is therefore locally coherent as topological space (in the sense of [FK18, 0.2.2.21]). This yields that $\mathrm{Sh}(I_{\mathrm{Zar}})$ is an algebraic topos. Hence, it follows from the equivalence of topoi 7.1.2.13.1 that so is $\mathrm{Sh}(I^{\natural})$.

(c) If I is coherent, i.e., I is quasi-separated and is a finite union of subsets of the forms $I_{\leq i}$, then $\mathrm{Sh}(I_{\mathrm{Zar}})$ is a coherent topos.

(d) If I is quasi-separated and X is locally coherent, then $I \times X$ is locally coherent and $\mathrm{Top}(X)_I$ is therefore an algebraic topos (use 7.1.2.14 and the equivalence of topoi 7.1.2.13.1).

(e) If I and X are coherent, then $I \times X$ is coherent and $\mathrm{Top}(X)_I$ is therefore a coherent topos.

7.1.2.16. Let X be a locally coherent topological space. Let I be a partially ordered set which is quasi-separated for its canonical topology. Then following [SGA4.2, VI.5.3], for any integer q, for any coherent object F of $\mathrm{Top}(X)_I$ (i.e. following 7.1.2.14.(e), the étale space associated with F' is coherent), the functor $H^q(F, -)$ commutes with filtered inductive limits of abelian sheaves. In particular, when $I \times X$ is coherent, the functor $H^q(X_\bullet, -)$ commutes with filtered inductive limits of abelian sheaves.

7.1.2.17. Let $u\colon I \to I'$ be an increasing map of partially ordered sets. Let $f\colon X \to X'$ be a continuous map of topological spaces. We get the continuous morphism of topological spaces $u \times f\colon I \times X \to I' \times X'$, where the topologies on I and I' are defined at 7.1.2.13. This yields the morphism of topoi $\mathrm{Top}(u \times f)\colon \mathrm{Top}(I \times X) \to \mathrm{Top}(I' \times X')$.

(a) We have the commutative square

$$\begin{array}{ccc}
\mathrm{Top}(X)_I & \xrightarrow[f_I \circ u_X]{7.1.2.10} & \mathrm{Top}(X')_{I'} \\
{\scriptstyle \alpha}\downarrow{\scriptstyle 7.1.2.13.1} & & {\scriptstyle \alpha}\downarrow{\scriptstyle 7.1.2.13.1} \\
\mathrm{Top}(I \times X) & \xrightarrow{\mathrm{Top}(u\times f)} & \mathrm{Top}(I' \times X').
\end{array} \tag{7.1.2.17.1}$$

Indeed, we reduce to the case where either $u = \mathrm{id}$ or $f = \mathrm{id}$. When $f = \mathrm{id}$, by using the description of α of 7.1.2.13.(b) and of u_X of 7.1.2.4, we easily compute that $\mathrm{Top}(u \times \mathrm{id})_* \circ \alpha_* = \alpha_* \circ u_{X*}$ and $u_X^{-1} \circ \alpha^{-1} = \alpha^{-1} \circ \mathrm{Top}(u \times \mathrm{id})^{-1}$. Similarly using 7.1.2.9, we compute $\mathrm{Top}(\mathrm{id} \times f)_* \circ \alpha_* = \alpha_* \circ f_{I*}$ and $f_I^{-1} \circ \alpha^{-1} = \alpha^{-1} \circ \mathrm{Top}(\mathrm{id} \times f)^{-1}$.

(b) Suppose I, I' are quasi-separated and X, X' are locally coherent. Then the morphism of topoi $f_I \circ u_X \colon \mathrm{Top}(X)_I \to \mathrm{Top}(X')_{I'}$ is quasi-separated (resp. quasi-compact, resp. coherent) if and only if u and f are quasi-separated (resp. quasi-compact, resp. coherent). Indeed, this is a consequence of 7.1.2.14.(f).

7.1.2.18. Let I be a partially ordered set and let X be a topological space. Let $(i, U) \in \mathrm{Ob}(I^\natural \times X_{Zar})$. We denote by $I_{\leq i} = \{j \in I \mid j \leq i\}$ and by $u_i \colon I_{\leq i} \to I$ the canonical inclusion. Then we have the commutative diagram of sites

$$\begin{array}{ccc} I^\natural_{\leq i} \times U_{Zar} & \xrightarrow{\sim} & I^\natural \times X_{Zar}/(i,U) \\ & \searrow^{u_i \times j_U} & \downarrow{\scriptstyle j_{(i,U)}} \\ & & I^\natural \times X_{Zar} \end{array} \qquad (7.1.2.18.1)$$

where the top right term is the localization of the site $I^\natural \times X_{Zar}$ at the object (i, U), $j_{(i,U)}$ is the forgetful functor (see 4.6.2.3) and the horizontal morphism is an isomorphism of sites. Following 4.6.2.4.2 in the case where $\mathfrak{C} = I^\natural \times X_{Zar}$ and using localisation at (i, U), since $h^\sharp_{(i,U)} = \mathcal{U}_{\leq (i,U)}$, then we have the commutative diagram

$$\begin{array}{ccccc} Sh(I^\natural_{\leq i} \times U_{Zar}) & \xrightarrow{\cong} & Sh(I^\natural \times X_{Zar}/(i,U)) & \xrightarrow{\cong} & Sh(I^\natural \times X_{Zar})/\mathcal{U}_{\leq (i,U)} \\ & \searrow^{j_{(i,U)}} & \downarrow{\scriptstyle j_{(i,U)}} & \swarrow^{j_{\mathcal{U}_{\leq (i,U)}}} & \\ & & Sh(I^\natural \times X_{Zar}) & & \end{array} \qquad (7.1.2.18.2)$$

where $j_{(i,U)} \colon Sh(I^\natural_{\leq i} \times U_{Zar}) \to Sh(I^\natural \times X_{Zar})$ is by definition the composition of the functor $u_{i,X}$ of 7.1.2.8.1 with the functor $j_{U,I}$ of 7.1.2.9.1 , where $j_U \colon U \subset X$ is the inclusion. Since $j_{\mathcal{U}_{\leq (i,U)}}$ is an open of the site $I^\natural \times X_{Zar}$, then $Sh(I^\natural \times X_{Zar})/\mathcal{U}_{\leq (i,U)}$ is an open subtopos of $Sh(I^\natural \times X_{Zar})$ and we get the open immersion (see definition 4.6.2.6):

$$j_{(i,U)} \colon \mathrm{Top}(U)_{I_{\leq i}} \to \mathrm{Top}(X)_I. \qquad (7.1.2.18.3)$$

We can simply write $|_{(i,U)}$ the functor $j^{-1}_{(i,U)}$.

7.1.2.19 (Flasque sheaves)**.** We keep notation 7.1.2.12 and 7.1.2.13.

(a) Let $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$. Then the sheaf $\mathcal{F}_\bullet$ is flasque (see Definition [SGA4.2, V.4.10]) if and only if $\alpha_*(\mathcal{F}_\bullet)$ is flasque if and only if the restriction map $\alpha_*(\mathcal{F}_\bullet)(\mathcal{U}_{\leq (j,X)}) \to \alpha_*(\mathcal{F}_\bullet)(\mathcal{U}_{\leq (i,U)})$ is surjective for any $i \leq j$ and U open subset of X if and only if the map $\mathcal{F}_j(X) \to \mathcal{F}_i(U)$ is surjective for any $i \leq j$, for any open set U of X if and only if $\mathcal{F}_\bullet$ is a projective system of flasque sheaves whose transition maps are surjective in the category of presheaves. The notion of flasque sheaves is also called totally acyclic sheaves (see [Sta22, 072Y-21.13.4]) but we keep Grothendieck's terminology.

(b) Let $\mathcal{F}_\bullet \in \mathrm{Top}(X)_I$. We define the sheaf $\mathcal{G}_i$ on X by setting for any open set U of X:

$$\mathcal{G}_i(U) := \prod_{j \leq i} \prod_{x \in U} \mathcal{F}_{j,x}. \qquad (7.1.2.19.1)$$

This is clear that $\mathcal{G}_\bullet$ is flasque and we have the canonical embedding $\mathcal{F}_\bullet \hookrightarrow \mathcal{G}_\bullet$. We say that this is the canonical embedding of $\mathcal{F}_\bullet$ into a flasque sheaf of $\mathrm{Top}(X)_I$.

(c) Suppose I and X are coherent. In that case, filtered inductive limits of flasque abelian sheaves of $\mathrm{Top}(X)_I$ are flasque. Indeed, since $I \times X$ is a coherent topological space, then $\mathrm{Top}(I \times X)$ is a coherent topos and then filtered inductive limits of abelian sheaves on $I \times X$ commute with the functors $\Gamma(V, -)$ for any open coherent subset V of $I \times X$ (see 7.1.2.16). Since the equivalence of topoi 7.1.2.13.1 preserves the flasqueness and commutes with filtered inductive limits, then we are done.

7.1.2.20 (Points of the topos $\mathrm{Top}(X)_I$)**.** Let I be a partially ordered set, X be a topological space, $i \in I$ and $x \in X$. We get the continuous map of topological spaces $f_x : \{x\} \to X$. We denote by

$$p_{x,i} : \mathrm{Top}(\{x\})_{\{i\}} \to \mathrm{Top}(X)_I \tag{7.1.2.20.1}$$

the morphism of topoi defined by setting $p_{x,i} = (f_x)_I \circ i_{\{x\}} = i_{\{x\}} \circ f_{x\{i\}}$ (see notation 7.1.2.7.1 and 7.1.2.9.1). Let $\{*\}$ be some one element set. Then $\mathrm{Top}(\{x\})_{\{i\}} = \mathrm{Top}(\{*\})$ and then the morphism of topoi $p_{x,i}$ of 7.1.2.20.1 can be identified with a point of the topos $\mathrm{Top}(X)_I$ (recall definition [Sta22, 00Y4–7.32.1]). We compute that for any $\mathcal{G}_\bullet \in \mathrm{Top}(X)_I$, we have $p_{x,i}^{-1}(\mathcal{G}_\bullet) = \mathcal{G}_{i,x}$ and for any $G \in \mathrm{Top}(\{*\})$, we have

$$(p_{x,i*}(G))_j = \begin{cases} f_{x*}(G) & \text{if } j \geq i \\ e & \text{otherwise} \end{cases}. \tag{7.1.2.20.2}$$

The family of points $\{p_{x,i}\}_{x \in X, i \in I}$ is conservative (see definition [Sta22, 00YK–7.38.1]). In particular, $\mathrm{Top}(X)_I$ has enough points.

7.1.2.21. Let I be a partially ordered set, let X be a topological space and $\mathscr{T} = \mathrm{Top}(X)$. Let $(i, U) \in I^\natural \times X_{Zar}$. Let $E_\bullet, F_\bullet$ be two objects of $\mathscr{T}_I$. By applying the formula 4.6.2.7.2 (in the case where $K = h^\sharp_{(i,U)}$ and by using Yoneda lemma), with the notation of 7.1.2.18.2 we get

$$\mathrm{Hom}_{\mathscr{T}_I}((i,U), \mathscr{H}om_{\mathscr{T}_I}(E_\bullet, F_\bullet)) = \mathrm{Hom}_{\mathscr{T}_{I_{\leq i}}}(E_\bullet|_{(i,U)}, F_\bullet|_{(i,U)}). \tag{7.1.2.21.1}$$

With the notation 7.1.2.18, this yields the isomorphism of $\mathscr{T}_{I_{\leq i}}$

$$\mathscr{H}om_{\mathscr{T}_I}(E_\bullet, F_\bullet)|_{(i,U)} \xrightarrow{\sim} \mathscr{H}om_{\mathscr{T}_{I_{\leq i}}}(E_\bullet|_{(i,U)}, F_\bullet|_{(i,U)}). \tag{7.1.2.21.2}$$

7.1.3 Modules

Let I be a partially ordered set and let X be a topological space. Let $\mathscr{T} = \mathrm{Top}(X)$. Recall an abelian sheaf (resp. a ring) of $\mathscr{T}_I$ is a projective system indexed by I of abelian sheaves (resp. of rings) on X. Let $\mathcal{D}_\bullet$ be a ring of $\mathscr{T}_I$ i.e. a functor $I^\circ \to \mathrm{Top}(X)$ denoted by $i \mapsto \mathcal{D}_i$ such that the transition maps $\mathcal{D}_j \to \mathcal{D}_i$ (if $i \leq j$) are ring morphisms. We get the ringed topos $(\mathscr{T}_I, \mathcal{D}_\bullet)$.

Notation 7.1.3.1. A left (resp. right) $\mathcal{D}_\bullet$-module $\mathcal{M}_\bullet$ is a projective system $i \mapsto \mathcal{M}_i$ indexed by I of left (resp. right) $\mathcal{D}_i$-module $\mathcal{M}_i$ such that the transition maps $\mathcal{M}_j \to \mathcal{M}_i$ are $\mathcal{D}_j$-linear. We denote by $\mathrm{Mod}({}^l\mathcal{D}_\bullet)$ (resp. $\mathrm{Mod}({}^r\mathcal{D}_\bullet)$) the category of left (resp. right) $\mathcal{D}_\bullet$-modules.

Let $f^\bullet_\bullet : \mathcal{E}^\bullet_\bullet \to \mathcal{F}^\bullet_\bullet$ be a morphism in $C({}^*\mathcal{D}_\bullet)$. Then $\ker(f^n_\bullet)$ (resp. $\mathrm{Coker}(f^n_\bullet)$) is the $\mathcal{D}_\bullet$-module $i \mapsto \ker(f^n_i)$ (resp. $i \mapsto \mathrm{Coker}(f^n_i)$) with the induced by $f^\bullet_\bullet$ transition maps. Hence, $f^\bullet_\bullet$ is a quasi-isomorphism of $C({}^*\mathcal{D}_\bullet)$ if and only if for any $i \in I$ the map $f^\bullet_i$ is a quasi-isomorphism of $C({}^*\mathcal{D}_i)$.

7.1.3.2. For any $i \in I$, with the notation 7.1.2.7,since $i_X^{-1}(\mathcal{D}_\bullet) = \mathcal{D}_i$ then we get (from the morphism of topos of 7.1.2.7.1) the morphism of ringed topoi

$$i_X = (i_X^{-1} \dashv i_{X*}) : (\mathscr{T}, \mathcal{D}_i) \to (\mathscr{T}_I, \mathcal{D}_\bullet),$$

which is such that the (exact) functor $i_X^* : \mathrm{Mod}(\mathcal{D}^*_\bullet) \to \mathrm{Mod}({}^*\mathcal{D}_i)$, with $* \in \{l, r\}$ is given for any $\mathcal{E}_\bullet \in \mathrm{Mod}({}^*\mathcal{D}_\bullet)$ by $i_X^*(\mathcal{E}_\bullet) = i_X^{-1}(\mathcal{E}_\bullet) = \mathcal{E}_i$. The functor i_X^* has a left adjoint $i_{X!} : \mathrm{Mod}(\mathcal{D}^*_i) \to \mathrm{Mod}(\mathcal{D}^*_\bullet)$ which is such that for any $\mathcal{F} \in \mathrm{Mod}(\mathcal{D}^*_i)$ we have

$$(i_{X!}(\mathcal{F}))_j = \begin{cases} \mathcal{D}_j \otimes_{\mathcal{D}_i} \mathcal{F} & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.3.2.1}$$

7.1.3.3. Fix $i \in I$ and U an open subset of X. Using the open immersion $j_{(i,U)}$ of 7.1.2.18.3, we get the morphism of ringed topoi

$$j_{(i,U)} \colon (\mathrm{Top}(U)_{I_{\leq i}}, \mathcal{D}_\bullet|_{(i,U)}) \to (\mathrm{Top}(X)_I, \mathcal{D}_\bullet) \tag{7.1.3.3.1}$$

which is such that the (exact) functor $j^*_{(i,U)} \colon \mathrm{Mod}({}^*\mathcal{D}_\bullet) \to \mathrm{Mod}({}^*\mathcal{D}_\bullet|_{(i,U)})$, with $* \in \{l, r\}$ is given for any $\mathcal{E}_\bullet \in \mathrm{Mod}({}^*\mathcal{D}_\bullet)$ by $j^*_{(i,U)}(\mathcal{E}_\bullet) = j^{-1}_{(i,U)}(\mathcal{E}_\bullet) = (\mathcal{E}_j|_U)_{j\leq i}$. We can also simply write $\mathcal{E}_\bullet|_{(i,U)}) = j^*_{(i,U)}(\mathcal{E}_\bullet)$. By definition, we have $j_{(i,U)} = u_{i,X} \circ j_{U,I}$, were $u_{i,X}$ is the functor defined at 7.1.2.8.1 and $j_{U,I}$ is the functor defined at 7.1.2.9.1 , where $j_U \colon U \subset X$ is the inclusion. Hence, we compute that the functor $j^*_{(i,U)}$ has a left adjoint $j_{(i,U)!} \colon \mathrm{Mod}({}^*\mathcal{D}_\bullet|_{(i,U)}) \to \mathrm{Mod}({}^*\mathcal{D}_\bullet)$ which is such that for any $\mathcal{F}_\bullet \in \mathrm{Mod}({}^*\mathcal{D}_\bullet|_{(i,U)})$ we have

$$(j_{(i,U)!}(\mathcal{F}_\bullet))_j = \begin{cases} j_{U!}(\mathcal{F}_j) & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.3.3.2}$$

Modulo the equivalence of topoi 7.1.2.13.1, the functor $j_{(i,U)!}$ corresponds to the extension by zero via the open immersion $I_{\geq i} \times U \subset I \times X$ (for the canonical topology).

Definition 7.1.3.4. Let $\mathcal{E}^\bullet_\bullet \in K({}^l\mathcal{D}_\bullet)$. We recall the following definition in our context (see the general one at [SGA6, I.5.2] or in the commutative case at [Sta22, 08FZ-21.44.1]).

1. Let $a, b \in \mathbb{Z}$ with $a \leq b$. We say $\mathcal{E}^\bullet_\bullet$ has tor-amplitude in $[a, b]$ if $H^i(\mathcal{M}_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathcal{E}^\bullet_\bullet) = 0$ for all right $\mathcal{D}_\bullet$-modules $\mathcal{M}_\bullet$ and all $i \notin [a, b]$.
2. We say $\mathcal{E}^\bullet_\bullet$ has finite tor dimension if it has tor-amplitude in $[a, b]$ for some integers $a \leq b$. We denote by $D_{\mathrm{tdf}}(\mathcal{D}_\bullet)$ the full subcategory of $D^{\mathrm{b}}({}^l\mathcal{D}_\bullet)$ consisting of complexes having finite tor dimension on $\mathcal{D}_\bullet$.
3. We say $\mathcal{E}^\bullet_\bullet$ *locally has finite tor dimension* if for every object (i, U) of $I^\natural \times X_{Zar}$ there exists a covering $\{(i, U_\lambda) \to (i, U)\}_{\lambda\in\Lambda}$ such that $\mathcal{E}^\bullet_\bullet|_{(i,U_\lambda)}$ has finite tor dimension and for each $\lambda \in \Lambda$. Remark that when I has a greatest element and the topological space X is quasi-compact, the complex $\mathcal{E}^\bullet_\bullet$ locally has finite tor dimension if and only if $\mathcal{E}^\bullet_\bullet$ has finite tor dimension.

Lemma 7.1.3.5. *Let $\mathcal{E}^\bullet_\bullet \in K({}^l\mathcal{D}_\bullet)$. Let $a, b \in \mathbb{Z}$ with $a \leq b$. The following properties are equivalent.*

1. *The complex $\mathcal{E}^\bullet_\bullet$ has tor-amplitude in $[a, b]$.*
2. *For any $i \in I$ and $x \in X$, the complex $p^{-1}_{x,i}(\mathcal{E}^\bullet_\bullet)$ (see notation 7.1.2.20) has tor-amplitude in $[a, b]$.*

Proof. This is a consequence of the fact that $\mathrm{Top}(X)_I$ has enough points (see 7.1.2.20) and [Sta22, 0DJJ-21.44.10] □

7.1.3.6. We denote by $\mathbb{Z}_\bullet$ the sheaf of ring of $\mathscr{T}_I$ equal to the constant projective system indexed by I equal to the sheaf on X associated to $\mathbb{Z}$. Let $\mathcal{E}^\bullet_\bullet \in K({}^l\mathcal{D}_\bullet)$. By definition, the complex $\mathcal{E}^\bullet_\bullet$ is K-flat is for any acyclic complex $\mathcal{M}^\bullet_\bullet \in K({}^r\mathcal{D}_\bullet)$, the complex $\mathcal{M}^\bullet_\bullet \otimes_{\mathcal{D}_\bullet} \mathcal{E}^\bullet_\bullet$ of $K(\mathbb{Z}_\bullet)$ is acyclic.

Since a complex $\mathcal{G}^\bullet_\bullet$ of $K(\mathbb{Z}_\bullet)$ (resp. $K(\mathcal{D}_\bullet)$) is acyclic if and only if the complex $\mathcal{G}^\bullet_i$ of $K(\mathbb{Z})$ (resp. $K(\mathcal{D}_i)$) is acyclic for any $i \in I$, then a complex $\mathcal{E}^\bullet_\bullet \in K({}^l\mathcal{D}_\bullet)$ is K-flat if and only if the complexes $\mathcal{E}^\bullet_i \in K({}^l\mathcal{D}_i)$ are K-flat for any $i \in I$. Similarly, a left $\mathcal{D}_\bullet$-module $\mathcal{E}_\bullet$ is flat if and only if the left $\mathcal{D}_i$-module $\mathcal{E}_i$ are flat for any $i \in I$. Hence, for any $\mathcal{E}^\bullet_\bullet \in D({}^l\mathcal{D}_\bullet)$, $\mathcal{M}^\bullet_\bullet \in D({}^r\mathcal{D}_\bullet)$, we have

$$\left(\mathcal{M}^\bullet_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathcal{E}^\bullet_\bullet\right)_i \xrightarrow{\sim} \mathcal{M}^\bullet_i \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i. \tag{7.1.3.6.1}$$

Let $\mathcal{E}^\bullet_\bullet \in D({}^l\mathcal{D}_\bullet)$. It follows from 7.1.3.6.1 that $\mathcal{E}^\bullet_\bullet$ has tor-amplitude in $[a, b]$ over $\mathcal{D}_\bullet$ if and only if for any $i \in I$ the complex $\mathcal{E}^\bullet_i$ has tor-amplitude in $[a, b]$ over $\mathcal{D}_i$.

7.1.3.7. Let $\mathcal{M}_\bullet$ be a left (resp. right) $\mathcal{D}_\bullet$-module. We remind the following definition (see [Sta22, 03DE-18.17.1]).

(a) $\mathcal{M}_\bullet$ is "generated by finitely many global sections" means that there exists a surjective morphism in $\mathrm{Mod}({}^*\mathcal{D}_\bullet)$ of the form

$$\mathcal{D}^N_\bullet \to \mathcal{M}_\bullet,$$

for some positive integer N.

(b) Moreover, $\mathcal{M}_\bullet$ has "a global presentation" (resp. "a global finite presentation", resp. is "free", resp. is "finite free") means that there exists an exact sequence in $\mathrm{Mod}(^*\mathcal{D}_\bullet)$ of the form

$$\oplus_{i\in I}\mathcal{D}_\bullet \to \oplus_{j\in J}\mathcal{D}_\bullet \to \mathcal{M}_\bullet \to 0,$$

for some sets I and J (resp. some finite sets I and J, resp. I is the empty set and J is a set, resp. I is the empty set and J is a finite set).

When I has a greatest element i_0, $\mathcal{M}_\bullet$ is generated by finitely many global sections (resp. has global presentation, resp. has global finite presentation, resp. is free, resp. is finite free) if and only if the $\mathcal{D}_{i_0}$-module $\mathcal{M}_{i_0}$ is generated by finitely many global sections (resp. has global presentation, resp. has global finite presentation, resp. is free, resp. is finite free) and if for any $i \in I$ the canonical homomorphism $\mathcal{D}_i \otimes_{\mathcal{D}_{i_0}} \mathcal{M}_{i_0} \to \mathcal{M}_i$ is surjective (for the other respective cases, is an isomorphism).
When I has only one element, we retrieve usual finiteness notion on sheaves on a topological space.

Except the notion of quasi-coherent which is called here "having local presentation", with 7.1.2.18.2, we follow the definitions of [Sta22, 03DL-18.23.1]) (see also [Sta22, 04IX-18.19.1] for the notation): $\mathcal{M}_\bullet$ is a $\mathcal{D}_\bullet$-module of finite type (resp. has local presentation, resp. is of finite presentation, resp. is locally free, resp. is locally finite free) if for any object $(i, U) \in I^\natural \times X_{Zar}$, there exists a covering $\{(i, U_\lambda) \to (i, U)\}_{\lambda\in\Lambda}$ such that for any $\lambda \in \Lambda$ the $\mathcal{D}_\bullet|_{(i,U_\lambda)}$-module $\mathcal{M}_\bullet|_{(i,U_\lambda)}$ on the site $I^\natural_{\leq i} \times U_{\lambda,Zar}$ is generated by finitely many global sections (resp. has global presentation, resp. has global finite presentation, resp. is free, resp. is finite free).

Proposition 7.1.3.8. *Let $\mathcal{M}_\bullet$ be a left (resp. right) $\mathcal{D}_\bullet$-module. Then $\mathcal{M}_\bullet$ is a $\mathcal{D}_\bullet$-module of finite type (resp. has local presentation, resp. is of finite presentation, resp. is locally free, resp. is locally finite free) if and only if the following two properties are satisfied:*

1. *For any $i \in I$, $\mathcal{M}_i$ is a $\mathcal{D}_i$-module of finite type (resp. has local presentation, resp. is of finite presentation, resp. is locally free , resp. is locally free of finite type)*

2. *For any $i, j \in I$ such that $j \leq i$, the canonical homomorphism $\mathcal{D}_j \otimes_{\mathcal{D}_i} \mathcal{M}_i \to \mathcal{M}_j$ is surjective (resp. is an isomorphism).*

Proof. Since the other cases are similar, let us prove the non-respective one. Let $(i, U) \in I^\natural \times X_{Zar}$. There exists a covering $\{(i, U_\lambda) \to (i, U)\}_{\lambda\in\Lambda}$ such that for any $\lambda \in \Lambda$ the $\mathcal{D}_\bullet|_{(i,U_\lambda)}$-module $\mathcal{M}_\bullet|_{(i,U_\lambda)}$ on the site $I^\natural_{\leq i} \times U_{\lambda,Zar}$ is generated by finitely many global sections. Since i is the greatest element of $I_{\leq i}$, then it follows from 7.1.3.7 that the $\mathcal{D}_\bullet$-module $\mathcal{M}_\bullet$ is of finite type if and only if for any $j \leq i$, for any open U of X, there exists an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of U such that the canonical homomorphism $\mathcal{D}_j \otimes_{\mathcal{D}_i} \mathcal{M}_i|_{U_\lambda} \to \mathcal{M}_j|_{U_\lambda}$ is surjective. Hence, we are done. □

Notation 7.1.3.9. Let us denote by $\mathrm{Mod}_{\mathrm{lp}}(^l\mathcal{D}_\bullet)$ (resp. $\mathrm{Mod}_{\mathrm{gp}}(^l\mathcal{D}_\bullet)$, resp. $\mathrm{Mod}_{\mathrm{fp}}(^l\mathcal{D}_\bullet)$, resp. $\mathrm{Mod}_{\mathrm{gfp}}(^l\mathcal{D}_\bullet)$) the category of left $\mathcal{D}_\bullet$-modules having local presentation (resp. having global presentation, resp. of finite presentation, resp. having global finite presentation).

We recall the definition (see [Sta22, 08FL-21.42.1], [Sta22, 08FT-21.43.1] and [Sta22, 08G5-21.45.1]):

Definition 7.1.3.10. Let $\mathcal{E}^\bullet_\bullet \in C(\mathcal{D}_\bullet)$.

1. We say $\mathcal{E}^\bullet_\bullet$ is "strictly perfect" if $\mathcal{E}^i_\bullet$ is zero for all but finitely many i and $\mathcal{E}^i_\bullet$ is a direct summand of a finite free $\mathcal{D}_\bullet$-module for all i.

2. Let $n \in \mathbb{Z}$. We say $\mathcal{E}^\bullet_\bullet$ is n-pseudo-coherent if for every object (i, U) of $I^\natural \times X_{Zar}$ there exists a covering $\{(i, U_\lambda) \to (i, U)\}_{\lambda\in\Lambda}$, and for each $\lambda \in \Lambda$ there exist a strictly perfect complex of $\mathcal{D}_\bullet|_{(i,U_\lambda)}$-modules $\mathcal{E}^\bullet_{\lambda\bullet}$ and a morphism $\alpha_\lambda \colon \mathcal{E}^\bullet_{\lambda\bullet} \to \mathcal{E}^\bullet_\bullet|_{(i,U_\lambda)}$ of $C(\mathcal{D}_\bullet|_{(i,U_\lambda)})$ such that $H^j(\alpha_\lambda)$ is an isomorphism for $j > n$ and $H^n(\alpha_\lambda)$ is surjective.

3. We say $\mathcal{E}^\bullet_\bullet$ is "pseudo-coherent" if it is n-pseudo-coherent for all $n \in \mathbb{Z}$.

4. We say $\mathcal{E}^\bullet_\bullet$ is "perfect" if for every object (i, U) of $I^\natural \times X_{Zar}$ there exists a covering $\{(i, U_\lambda) \to (i, U)\}_{\lambda\in\Lambda}$ and for each $\lambda \in \Lambda$ a morphism of complexes $\alpha_\lambda \colon \mathcal{E}^\bullet_{\lambda\bullet} \to \mathcal{E}^\bullet_\bullet|_{(i,U_\lambda)}$ which is a quasi-isomorphism with $\mathcal{E}^\bullet_{\lambda\bullet}$ strictly perfect.

5. Let $n \in \mathbb{Z}$. We say an object of $D(\mathcal{D}_\bullet)$ is n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) if and only if it can be represented by a n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) complex of $\mathcal{D}_\bullet$-modules. We denote by $D_{n-\mathrm{coh}}(\mathcal{D}_\bullet)$ (resp. $D_{\mathrm{coh}}(\mathcal{D}_\bullet)$, resp. $D_{\mathrm{perf}}(\mathcal{D}_\bullet)$) the full subcategory of $D(\mathcal{D}_\bullet)$ consisting of n-pseudo-coherent (resp. pseudo-coherent, resp. perfect) complexes.

Remark 7.1.3.11. Let $\mathcal{E}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$. Then $\mathcal{E}^\bullet_\bullet \in D_{n-\mathrm{coh}}(\mathcal{D}_\bullet)$ (resp. $\mathcal{E}^\bullet_\bullet \in D_{\mathrm{perf}}(\mathcal{D}_\bullet)$) if and only if for every object (i,U) of $I^\natural \times X_{Zar}$ there exists a covering $\{(i,U_\lambda) \to (i,U)\}_{\lambda\in\Lambda}$, and for each $\lambda \in \Lambda$ there exist a strictly perfect complex of $\mathcal{D}_\bullet|_{(i,U_\lambda)}$-modules $\mathcal{E}^\bullet_{\lambda\bullet}$ and a morphism $\alpha_\lambda \colon \mathcal{E}^\bullet_{\lambda\bullet} \to \mathcal{E}^\bullet_\bullet|_{(i,U_\lambda)}$ of $D(\mathcal{D}_\bullet|_{(i,U_\lambda)})$ which is an n-isomorphism (resp. is an isomorphism). Recall α_λ is an n-isomorphism of $D(\mathcal{D}_\bullet|_{(i,U_\lambda)})$ means that for any $j \geq n$, we have $H^j C(\alpha_\lambda) = 0$, where $C(\alpha_\lambda)$ is the cone of α_λ in $D(\mathcal{D}_\bullet)$.

7.1.3.12. Let $\mathcal{E}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$. Following [SGA6, I.5.8.1] (or see [Sta22, 08G8] in the case where $\mathcal{D}_\bullet$ is commutative), $\mathcal{E}^\bullet_\bullet$ is perfect if and only if $\mathcal{E}^\bullet_\bullet$ is pseudo-coherent and locally has finite tor dimension.

Proposition 7.1.3.13. *Let $\mathcal{E}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$. Then $\mathcal{E}^\bullet_\bullet \in D_{n-\mathrm{coh}}(\mathcal{D}_\bullet)$ (resp. $\mathcal{E}^\bullet_\bullet \in D_{\mathrm{coh}}(\mathcal{D}_\bullet)$, resp. $\mathcal{E}^\bullet_\bullet \in D_{\mathrm{perf}}(\mathcal{D}_\bullet)$) if and only if the following properties hold*

1. *For any $i \in I$, $\mathcal{E}^\bullet_i \in D_{n-\mathrm{coh}}(\mathcal{D}_i)$ (resp. $\mathcal{E}^\bullet_i \in D_{\mathrm{coh}}(\mathcal{D}_i)$, resp. $\mathcal{E}^\bullet_i \in D_{\mathrm{perf}}(\mathcal{D}_i)$) ;*

2. *For any $i, j \in I$ such that $j \leq i$, the canonical homomorphism $\mathcal{D}_j \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_j$ is an isomorphism.*

Proof. The fact that these condition are necessary is obvious. Conversely, suppose $\mathcal{E}^\bullet_\bullet$ satisfies both conditions. Since the other cases are either a straightforward consequence or are checked similarly, let us consider the non-respective one. Let $(i,U) \in I^\natural \times X_{Zar}$. From the first condition, there exists a covering $\{U_\lambda \to U\}_{\lambda\in\Lambda}$, and for each $\lambda \in \Lambda$ there exist a strictly perfect complex of $\mathcal{D}_i|_{U_\lambda}$-modules $\mathcal{E}^\bullet_\lambda$ and a morphism $\alpha_\lambda \colon \mathcal{E}^\bullet_\lambda \to \mathcal{E}^\bullet_i|_{U_\lambda}$ of $D(\mathcal{D}_i|_{U_\lambda})$ which is an n-isomorphism, i.e. for any $j \geq n$, we have $H^j C(\alpha_\lambda) = 0$, where $C(\alpha_\lambda)$ is the cone of α_λ (see remark 7.1.3.11). We get $\beta_\lambda := \mathcal{D}_\bullet|_{(i,U_\lambda)} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i|_{U_\lambda}} \alpha_\lambda$, which is also an n-isomorphism.

From the second condition, the canonical morphism $\gamma_\lambda \colon \mathcal{D}_\bullet|_{(i,U_\lambda)} \otimes^{\mathbb{L}}_{\mathcal{D}_i|_{U_\lambda}} \mathcal{E}^\bullet_i|_{U_\lambda} \to \mathcal{E}^\bullet_\bullet|_{(i,U_\lambda)}$ is an isomorphism. This yields the n-isomorphism $\gamma_\lambda \circ \beta_\lambda \colon \mathcal{D}_\bullet|_{(i,U_\lambda)} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i|_{U_\lambda}} \mathcal{E}^\bullet_\lambda \to \mathcal{E}^\bullet_\bullet|_{(i,U_\lambda)}$, where $\mathcal{D}_\bullet|_{(i,U_\lambda)} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i|_{U_\lambda}} \mathcal{E}^\bullet_\lambda$ is a strictly perfect complex of $\mathcal{D}_\bullet|_{(i,U_\lambda)}$-modules. Hence, we are done.

□

7.1.3.14 (Flasque resolutions)**.** Let $\mathcal{F}_\bullet$ be a left $\mathcal{D}_\bullet$-module. Let $\mathcal{F}_\bullet \hookrightarrow \mathcal{G}_\bullet$ be the canonical embedding of $\mathcal{F}_\bullet$ into a flasque sheaf of $\mathrm{Top}(X)_I$, where the flasque object $\mathcal{G}_\bullet$ of $\mathrm{Top}(X)_I$ is defined at 7.1.2.19.1. In fact, we see that $\mathcal{G}_\bullet$ is endowed with a canonical structure of left $\mathcal{D}_\bullet$-module and that the canonical embedding $\mathcal{F}_\bullet \hookrightarrow \mathcal{G}_\bullet$ is $\mathcal{D}_\bullet$-linear. We say that a $\mathcal{D}_\bullet$-module is flasque if it is flasque as sheaf of sets. Following [Sta22, 05T6], for any $\mathcal{F}^\bullet_\bullet \in K^+(\mathcal{D}_\bullet)$, there exists a quasi-isomorphism in $K^+(\mathcal{D}_\bullet)$ of the form $\mathcal{F}^\bullet_\bullet \to \mathcal{H}^\bullet_\bullet$ where $\mathcal{H}^n_\bullet$ is a flasque $\mathcal{D}_\bullet$-module for any integer n.

7.1.3.15. Let $f \colon X \to X'$ be a continuous map of topological spaces. Set $\mathscr{T}' := \mathrm{Top}(X')$. Let $\mathcal{D}_\bullet$ (resp. $\mathcal{D}'_\bullet$) be a ring of $\mathscr{T}_I$ (resp. $\mathscr{T}'_I$) and $f_I^{-1}\mathcal{D}'_\bullet \to \mathcal{D}_\bullet$ be a morphism of rings of $\mathscr{T}_I$ (see notation 7.1.2.9). This yields the ringed topos morphism

$$f_I = (f_I^{-1} \dashv f_{I*}) \colon (\mathscr{T}_I, \mathcal{D}_\bullet) \to \mathscr{T}'_I, \mathcal{D}'_\bullet).$$

We get the functor $f_I^* \colon \mathrm{Mod}(\mathcal{D}'_\bullet) \to \mathrm{Mod}(\mathcal{D}_\bullet)$ which is defined for any $\mathcal{E}'_\bullet \in \mathrm{Mod}(\mathcal{D}'_\bullet)$ by setting

$$f_I^*(\mathcal{E}'_\bullet) := \mathcal{D}_\bullet \otimes_{f_I^{-1}\mathcal{D}'_\bullet} f_I^{-1}\mathcal{E}'_\bullet.$$

We denote by

$$f_i = (f_i^{-1} \dashv f_{i*}) \colon (\mathscr{T}, \mathcal{D}_i) \to \mathscr{T}', \mathcal{D}'_i)$$

the morphism of ringed topoi (which is more precisely a morphism of ringed topological spaces here), where $f_i^{-1} = f^{-1}$ and $f_{i*} = f_*$.

For any $\mathcal{E}'^\bullet_\bullet \in D(\mathcal{D}'_\bullet)$, since the functors i_X^* and $i_{X'}^*$ are exact, we get the canonical isomorphism

$$(\mathbb{L}f_I^*(\mathcal{E}'^\bullet_\bullet))_i \xrightarrow{\sim} \mathbb{L}f_i^*(\mathcal{E}'^\bullet_i). \tag{7.1.3.15.1}$$

We have the functor $\mathbb{R}f_{I*}\colon D(\mathcal{D}_\bullet) \to D(\mathcal{D}'_\bullet)$, which is computed by using K-injective resolutions (see [Sta22, 07A5-21.19]). We have also the functor $\mathbb{R}f_{I*}\colon D^+(\mathcal{D}_\bullet) \to D^+(\mathcal{D}'_\bullet)$, which can be computed by using flasque resolutions (see 7.1.3.14). This yields by using flasque resolution [Sta22, 05TA-13.16.8] and 7.1.2.19.(a) that for any $\mathcal{E}^\bullet_\bullet \in D^+(\mathcal{D}_\bullet)$, we check that the canonical base change morphism (see [Sta22, 07A7-21.19.3])

$$(\mathbb{R}f_{I*}(\mathcal{E}^\bullet_\bullet))_i \to \mathbb{R}f_{i*}(\mathcal{E}^\bullet_i) \tag{7.1.3.15.2}$$

is an isomorphism. In particular, this implies that a left $\mathcal{D}_\bullet$-module $\mathcal{E}_\bullet$ is f_{I*}-acyclic if and only if each $\mathcal{E}_i$ is f_*-acyclic for any $i \in I$.

Recall that if $f_I^*\colon \mathrm{Mod}(\mathcal{D}'_\bullet) \to \mathrm{Mod}(\mathcal{D}_\bullet)$ is not exact then the functor $f_{I*}\colon K(\mathcal{D}_\bullet) \to K(\mathcal{D}'_\bullet)$ do not preserve K-injective complexes. But, we can still check the transitivity of the right derived of the direct image of morphism of topos (see [Sta22, 0D6E-21.19.2]). Hence, for any increasing map $u\colon I \to I'$ of partially ordered sets, with notation 7.1.2.4, we get the canonical isomorphisms

$$\mathbb{R}u_{X'*} \circ \mathbb{R}f_{I*} \xrightarrow{\sim} \mathbb{R}\left(u_{X'*} \circ f_{I*}\right) \xrightarrow{\sim} \mathbb{R}\left(f_{I'*} \circ u_{X*}\right) \xrightarrow{\sim} \mathbb{R}f_{I'*} \circ \mathbb{R}u_{X*}. \tag{7.1.3.15.3}$$

7.1.3.16 (Bounded cohomological dimension)**.** With notation 7.1.3.15, suppose moreover f_* has bounded cohomological dimension. It follows from the isomorphism 7.1.3.15.2 that f_{I*} has also bounded cohomological dimension and then the functor $\mathbb{R}f_{I*}$ is way-out in both directions. This yields that the morphism 7.1.3.15.2 is still an isomorphism for $\mathcal{E}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$ (use [Har66, I.7.1.(iii)]). Following 4.6.1.6, for any $\mathcal{E}^\bullet_\bullet \in K(\mathcal{D}_\bullet)$ (resp. for any $\mathcal{E}^\bullet_\bullet \in K^-(\mathcal{D}_\bullet)$), there exist a complex $\mathcal{I}^\bullet_\bullet \in K(\mathcal{D}_\bullet)$ (resp. $\mathcal{I}^\bullet_\bullet \in K^-(\mathcal{D}_\bullet)$) of f_{I*}-acyclic $\mathcal{D}_\bullet$-modules and a quasi-isomorphism $\mathcal{E}^\bullet_\bullet \xrightarrow{\sim} \mathcal{I}^\bullet_\bullet$. It follows from [Sta22, 07K7-13.31.2] that we have the isomorphism $\mathbb{R}f_{I*}\mathcal{E}^\bullet_\bullet \xrightarrow{\sim} f_{I*}\mathcal{I}^\bullet_\bullet$.

7.1.3.17. Let $\mathcal{E}_\bullet, \mathcal{F}_\bullet$ be two left $\mathcal{D}_\bullet$-modules.

(a) Following 4.6.2.7, we have the abelian sheaf on $\mathscr{T}_I$ that we will denote by $\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)$ which is characterized by the property: for any object $K_\bullet$ of $\mathscr{T}_I$,

$$\mathrm{Hom}_{\mathscr{T}_I}(K_\bullet, \mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)) = \mathrm{Hom}_{\mathcal{D}_\bullet|_{K_\bullet}}(\mathcal{E}_\bullet|_{K_\bullet}, \mathcal{F}_\bullet|_{K_\bullet})).$$

In particular, for any $(i, U) \in I^\natural \times X_{Zar}$, by using 7.1.2.18.2 and Yoneda lemma we get

$$\mathrm{Hom}_{\mathscr{T}_I}((i, U), \mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)) = \mathrm{Hom}_{\mathcal{D}_\bullet|_{(i,U)}}(\mathcal{E}_\bullet|_{(i,U)}, \mathcal{F}_\bullet|_{(i,U)})). \tag{7.1.3.17.1}$$

A morphism of this latter abelian group is a compatible family of $\mathcal{D}_j|_U$-linear homomorphisms $\mathcal{E}_j|_U \to \mathcal{F}_j|_U$ for any $j \leq i$.

(b) We have the abelian sheaf $\mathcal{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)$ on X by setting, for any open set U of X,

$$\Gamma(U, \mathcal{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)) = \mathrm{Hom}_{\mathcal{D}_\bullet|_U}(\mathcal{E}_\bullet|_U, \mathcal{F}_\bullet|_U).$$

With the above description of 7.1.3.17.1 and with notation 7.1.2.7.1, we get the isomorphism of abelian sheaves on X:

$$\underleftarrow{l}_{X,I*}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet). \tag{7.1.3.17.2}$$

7.1.3.18. Let $\mathcal{E}_\bullet, \mathcal{F}_\bullet$ be two left $\mathcal{D}_\bullet$-modules. Fix $i \in I$. Suppose that for any $j \leq i$ the homomorphism $\mathcal{D}_j \otimes_{\mathcal{D}_i} \mathcal{E}_i \to \mathcal{E}_j$ is an isomorphism. Then for any $j \leq i$, for any open set U of X we get the homomorphisms

$$\mathrm{Hom}_{\mathcal{D}_i|_U}(\mathcal{E}_i|_U, \mathcal{F}_i|_U) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{D}_j|_U}(\mathcal{E}_j|_U, \mathcal{F}_i|_U) \to \mathrm{Hom}_{\mathcal{D}_j|_U}(\mathcal{E}_j|_U, \mathcal{F}_j|_U), \tag{7.1.3.18.1}$$

which induce the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}_i|_U}(\mathcal{E}_i|_U, \mathcal{F}_i|_U) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_\bullet|_{(i,U)}}(\mathcal{E}_\bullet|_{(i,U)}, \mathcal{F}_\bullet|_{(i,U)})). \tag{7.1.3.18.2}$$

Hence the abelian sheaf $\mathscr{H}om_{\mathcal{D}_\bullet|_{(i,X)}}(\mathcal{E}_\bullet|_{(i,X)}, \mathcal{F}_\bullet|_{(i,X)}))$ of $\mathscr{T}_{I_{\leq 1}}$ is the projective system $\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}_j, \mathcal{F}_j) \to \mathcal{H}om_{\mathcal{D}_{j'}}(\mathcal{E}_{j'}, \mathcal{F}_{j'})$ for any $j' \leq j \leq i$ (induced by 7.1.3.18.1). In particular, we get the commutative diagram

$$\begin{array}{ccc}
(\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet))_i & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{D}_i}(\mathcal{E}_i, \mathcal{F}_i) \\
\downarrow & & \downarrow {\scriptstyle 7.1.3.18.1} \\
(\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet))_j & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}_j, \mathcal{F}_j)
\end{array} \tag{7.1.3.18.3}$$

where the horizontal isomorphisms are the natural forgetful projection.

7.1.3.19. Let $\mathcal{E}^\bullet_\bullet \in D^-(\mathcal{D}_\bullet)$, $\mathcal{F}^\bullet_\bullet \in D^+(\mathcal{D}_\bullet)$. Let $j \leq i$ in I such that the canonical morphism

$$\mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_j \tag{7.1.3.19.1}$$

is an isomorphism. Then we get the morphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_i}(\mathcal{E}^\bullet_i, \mathcal{F}^\bullet_i) \xrightarrow[\sim]{4.6.5.3.1} \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}^\bullet_j, \mathcal{F}^\bullet_i) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}^\bullet_j, \mathcal{F}^\bullet_j). \tag{7.1.3.19.2}$$

Let $\mathcal{E}^\bullet_\bullet \in C(\mathcal{D}_\bullet)$ be a stricly perfect complex (see definition 7.1.3.10) and $\mathcal{F}^\bullet_\bullet \in C(\mathcal{D}_\bullet)$. Then we construct similarly the first isomorphisms of 7.1.3.19.2 and we get therefore the composite map 7.1.3.19.2.

7.1.3.20. Let $(i, U) \in I^\natural \times X_{Zar}$. Let $\mathcal{E}_\bullet, \mathcal{F}_\bullet$ be two left $\mathcal{D}_\bullet$-modules. It follows from 7.1.3.17.1 that we have the isomorphism of abelian sheaves of $\mathscr{T}_{I_{\leq i}}$

$$\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)|_{(i,U)} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{D}_\bullet|_{(i,U)}}(\mathcal{E}_\bullet|_{(i,U)}, \mathcal{F}_\bullet|_{(i,U)}). \tag{7.1.3.20.1}$$

Since $j^{-1}_{(i,U)} = j^*_{(i,U)} \colon \mathrm{Mod}({}^*\mathcal{D}_\bullet) \to \mathrm{Mod}({}^*\mathcal{D}_\bullet|_{(i,U)})$ has an exact left adjoint functor (see 7.1.3.3.2), then it follows from [Sta22, 08BJ] that $j^{-1}_{(i,U)}$ preserves the K-injectivity. Hence, for any $\mathcal{E}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$, $\mathcal{F}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$, we get the isomorphism

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet, \mathcal{F}_\bullet)|_{(i,U)} \cong \mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet|_{(i,U)}}(\mathcal{E}_\bullet|_{(i,U)}, \mathcal{F}_\bullet|_{(i,U)}). \tag{7.1.3.20.2}$$

Lemma 7.1.3.21. *Let $\mathcal{I}_\bullet$ be an injective left $\mathcal{D}_\bullet$-module. Let $\mathcal{I}^\bullet_\bullet$ be a K-injective complex of $K({}^1\mathcal{D}_\bullet)$. Let $i \in I$ and let $\mathcal{P}$ be a left $\mathcal{D}_i$-module.*

(a) The abelian sheaf $\mathcal{H}om_{\mathcal{D}_i}(\mathcal{P}, \mathcal{I}_i)$ is flasque.

(b) If $\mathcal{P}$ is flat, then $\mathcal{I}_i$ is right acyclic for the functors $\mathcal{H}om_{\mathcal{D}_i}(\mathcal{P}, -)$ and $\mathrm{Hom}_{\mathcal{D}_i}(\mathcal{P}, -)$.

(c) If moreover for any $j \leq i$ the homomorphisms $\mathcal{D}_i \to \mathcal{D}_j$ are flat, then $\mathcal{I}_i$ is an injective $\mathcal{D}_i$-module.

(d) If moreover for any $j \leq i$ the homomorphisms $\mathcal{D}_i \to \mathcal{D}_j$ are flat, then $\mathcal{I}^\bullet_i$ is a K-injective complex of $K({}^1\mathcal{D}_i)$.

Proof. With notation 7.1.2.18.3, let us consider the canonical morphism of ringed topoi

$$f\colon (\mathscr{T}, \mathcal{D}_i) \to (\mathscr{T}_{I_{\leq i}}, \mathcal{D}_\bullet|_{(i,X)})$$

induced by the morphism of topos $i_X = (i_X^{-1} \dashv i_{X*})\colon \mathrm{Top}(X) \to \mathrm{Top}(X)_{I_{\leq i}}$, where $i\colon \{*\} \to I_{\leq i}$ is the map sending $*$ to i. Since $f_*(\mathcal{I}) = \mathcal{I}_i$, then the first two statements follows from 4.6.5.2. In the third and forth one, with the flatness hypotheses, the left adjoint $i_{X!}$ of the functor $i_X^*\colon \mathrm{Mod}(\mathcal{D}_\bullet|_{(i,X)}) \to \mathrm{Mod}(\mathcal{D}_i)$ is therefore exact (use the computation 7.1.3.2.1 in the case where $I = I_{\leq i}$). Hence, i_X^* preserves the injectivity (resp. the K-injectivity following [Sta22, 08BJ]) and we are done. □

Proposition 7.1.3.22. *Let $\mathcal{E}^\bullet_\bullet \in D^-(\mathcal{D}_\bullet)$, $\mathcal{F}^\bullet_\bullet \in D^+(\mathcal{D}_\bullet)$. Let $i \in I$ such that for any $j \leq i$ the canonical morphism*

$$\mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_j \tag{7.1.3.22.1}$$

is an isomorphism. Then for any $j' \leq j \leq i$ we have the canonical commutative diagram of $D(\mathbb{Z}_X)$:

$$\begin{array}{ccc} (\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}^\bullet_\bullet, \mathcal{F}^\bullet_\bullet))_j & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}^\bullet_j, \mathcal{F}^\bullet_j) \\ \downarrow & & \downarrow {\scriptstyle 7.1.3.19.2} \\ (\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}^\bullet_\bullet, \mathcal{F}^\bullet_\bullet))_{j'} & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{j'}}(\mathcal{E}^\bullet_{j'}, \mathcal{F}^\bullet_{j'}) \end{array} \tag{7.1.3.22.2}$$

whose horizontal arrows are isomorphisms.

Proof. By using 7.1.3.20.2, since $i_X^{-1} \circ u_{iX}^{-1} \xrightarrow{\sim} i_X^{-1}$, then we can suppose $I_{\leq i} = I$. Let $\mathcal{P}_i^\bullet \in C(\mathcal{D}_i)$ be a complex of flat $\mathcal{D}_i$-modules and $\mathcal{P}_i^\bullet \to \mathcal{E}_i^\bullet$ be a morphism of $C(\mathcal{D}_i)$ which is a quasi-isomorphism. Set $\mathcal{P}_\bullet^\bullet := \mathcal{D}_\bullet \otimes_{\mathcal{D}_i} \mathcal{P}_i^\bullet \in C(\mathcal{D}_\bullet)$. Then $\mathcal{P}_\bullet^\bullet$ is a flat resolution of $\mathcal{E}_\bullet^\bullet$. Moreover, it follows from 7.1.3.22.1 that for any $j' \leq j$ the canonical morphism $\mathcal{D}_{j'} \otimes_{\mathcal{D}_j} \mathcal{P}_j^\bullet \to \mathcal{P}_{j'}^\bullet$ is an isomorphism of $C(\mathcal{D}_{j'})$. Let $\mathcal{I}_\bullet^\bullet$ be a resolution of $\mathcal{F}_\bullet^\bullet$ by injective $\mathcal{D}_\bullet$-modules. Then, we get the isomorphisms of $D(\mathbb{Z}_X)$:

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet^\bullet, \mathcal{F}_\bullet^\bullet)_j \xrightarrow{\sim} \mathscr{H}om^\bullet_{\mathcal{D}_\bullet}(\mathcal{P}_\bullet^\bullet, \mathcal{I}_\bullet^\bullet)_j \underset{7.1.3.18.3}{\xrightarrow{\sim}} \mathcal{H}om_{\mathcal{D}_j}(\mathcal{P}_j^\bullet, \mathcal{I}_j^\bullet)$$
$$\underset{7.1.3.21b}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{P}_j^\bullet, \mathcal{I}_j^\bullet) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}_j^\bullet, \mathcal{F}_j^\bullet).$$

This yields the construction of the horizontal isomorphisms of the diagram 7.1.3.22.2. Its commutativity is easy. □

7.1.3.23. Let $\mathcal{D}_\bullet$ and $\mathcal{D}''_\bullet$ be two rings of $\mathscr{T}_I$ such that $(\mathcal{D}_\bullet, \mathcal{D}'_\bullet)$ and $(\mathcal{D}_\bullet, \mathcal{D}''_\bullet)$ are solved by some commutative ring $\mathcal{R}_\bullet$ (see definition 4.6.3.2). Following 4.6.3.2.1, we have the bifunctors:

$$\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{D}_\bullet}(-,-) \colon D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}'^{\mathrm{r}}_\bullet) \times D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}''^{\mathrm{r}}_\bullet) \to D({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}''^{\mathrm{r}}_\bullet). \qquad (7.1.3.23.1)$$

Let $\mathcal{E}_\bullet^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}'^{\mathrm{r}}_\bullet)$, $\mathcal{F}_\bullet^\bullet \in D^+({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}''^{\mathrm{r}}_\bullet)$. Let $i \in I$ such that for any $j \leq i$ the canonical morphism

$$\mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}_i^\bullet \to \mathcal{E}_j^\bullet \qquad (7.1.3.23.2)$$

is an isomorphism. Then for any $j' \leq j \leq i$, similarly to 7.1.3.22, we get the canonical commutative diagram of $D({}^{\mathrm{l}}\mathcal{D}'_j, \mathcal{R}_j, \mathcal{D}''^{\mathrm{r}}_j)$:

$$\begin{array}{ccc} (\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet^\bullet, \mathcal{F}_\bullet^\bullet))_j & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}_j^\bullet, \mathcal{F}_j^\bullet) \\ \downarrow & & \downarrow \\ (\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet^\bullet, \mathcal{F}_\bullet^\bullet))_{j'} & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{j'}}(\mathcal{E}_{j'}^\bullet, \mathcal{F}_{j'}^\bullet) \end{array} \qquad (7.1.3.23.3)$$

whose horizontal arrows are isomorphisms and whose left morphism is built similarly than 7.1.3.19.2. We get similar diagram in the case where $\mathcal{E}_\bullet^\bullet \in C({}^*\mathcal{D}_\bullet)$ is a stricly perfect complex (see definition 7.1.3.10) and $\mathcal{F}_\bullet^\bullet \in D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}''^{\mathrm{r}}_\bullet)$.

7.2 Completion on formal schemes

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. For any open $\mathfrak{U} \subseteq \mathfrak{X}$, we write

$$U_i = (|\mathfrak{U}|, \mathcal{O}_{\mathfrak{U}}/\mathcal{I}^{i+1}),\ U := U_0.$$

Denoting by $|\mathfrak{X}| = |X_i|$ the underlying topological space of $\mathcal{X}$ and X_i, we simply write by $\mathfrak{X}$ or X_i the topos $\mathrm{Top}(|\mathfrak{X}|)$. Moreover, since $\mathbb{N}$ is fixed in the subsection, we denote by $X_\bullet$ the topos $\mathrm{Top}(X)_{\mathbb{N}}$. We consider $X_\bullet$ as a ringed topos by equipping it with the sheaf of rings $\mathcal{O}_{X_\bullet} = (\mathcal{O}_{X_i})_{i\in\mathbb{N}}$. We get the morphism of ringed topoi $\underleftarrow{l}_{X,\mathbb{N}} \colon (X_\bullet, \mathcal{O}_{X_\bullet}) \to (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ (see notation 7.1.2.5) that we shall simply write by $\underleftarrow{l}_X$. If $\mathcal{E}$ is a $\mathcal{O}_{\mathfrak{X}}$-module, $\underleftarrow{l}_X^*\mathcal{E}$ is then the projective system

$$\cdots \to \mathcal{E}/\mathcal{I}^{i+1}\mathcal{E} \to \cdots \to \mathcal{E}/\mathcal{I}^2\mathcal{E} \to \mathcal{E}/\mathcal{I}\mathcal{E}.$$

7.2.1 I-adic completion of a non-commutative ring

Unless otherwise specified, modules are left modules and the results are still valid for right modules. Moreover, "completed" will mean "separated completed". Let D be a not necessarily commutative ring, $I \subset D$ be a central ideal of finite type, i.e. I is two-sided ideal which is generated as a left (or right) ideal by a finite set of elements belonging to the center of D. We set $D_i := D/I^{i+1}$ for any $i \in \mathbb{N}$.

7.2.1.1 (Completion)**.** Let M be a left D-module. The I-adic topology on D (resp. M) is given by the two-sided ideals $(I^i)_{i\in\mathbb{N}}$ (resp. the submodules $(I^iM)_{i\in\mathbb{N}}$). The completion $\widehat{D} := \varprojlim_i D_i$ is endowed with a ring structure, and the completion $\widehat{M} := \varprojlim_i M/I^{i+1}M$ is endowed with a left $\widehat{D}$-module structure.

Lemma 7.2.1.2. *Let $(M_i)_{i\in\mathbb{N}}$ be a projective system of left D_i-modules such that the homomorphisms $M_{i+1}/I^{i+1}M_{i+1} \to M_i$ are D_i-linear isomorphisms. Put $M = \varprojlim_i M_i$.*

(a) For any $n \geq 0$, the canonical homomorphisms

$$I^n M \to \varprojlim_i I^n M_i, \qquad M/I^{n+1}M \to M_n$$

are isomorphisms.

(b) If M_0 is of finite type on D_0, then M is of finite type on $\widehat{D}$. Moreover, any family of D whose image in D_0 generates D_0 spans D.

(c) If D_0 is left (resp. right) noetherian, then so is $\widehat{D}$.

Proof. See [Ber96c, 3.2.2]. □

7.2.1.3. Suppose D is left noetherian. According to [Ber96c, 3.2.3], we have the following properties.

(a) The Artin-Rees property holds: If M is a left D-module of finite type, and $N \subset M$ is a sub D-module, there exists a positive integer n_0 such that, for any $n \geq n_0$, we have

$$I^n N \subset I^n M \cap N \subset I^{n-n_0} N. \tag{7.2.1.3.1}$$

(b) The functor $M \mapsto \widehat{M}$ is exact on the category of left D-modules of finite type.

(c) For any left D-module of finite type M, the canonical homomorphism

$$\widehat{D} \otimes_D M \to \widehat{M} \tag{7.2.1.3.2}$$

is an isomorphism.

(d) $\widehat{D}$ a right flat on D.

(e) Any left $\widehat{D}$-module of finite type is separated and complete.

(f) The ring $\widehat{D}$ is left noetherian.

(g) Let $D \to E$ be a ring homomorphism such that E is left noetherian an IE is a central ideal of E. The following properties are equivalent.

  (i) $\widehat{E}$ is right flat on $\widehat{D}$

  (ii) for any $i \in \mathbb{N}$, E/I^{i+1} is right flat on D_i.

Proposition 7.2.1.4. *Let D be a left noetherian ring, I be a central ideal, $\widehat{M}$ be its I-adic completion. Let M be a right D-module such that, M/MI^n is a flat D/DI^n for any $n \in \mathbb{N}$. Then the I-adic completion $\widehat{M}$ of M is a flat right $\widehat{D}$-module. Moreover, for any left $\widehat{D}$-module of finite type E, the abelian group $\widehat{M} \otimes_{\widehat{D}} E$ is separated and complete for the topology given by the subgroups $\widehat{M} \otimes_{\widehat{D}} I^n E$.*

Proof. See [Ber96c, 3.2.4]. □

We recall the following well known Lemma.

Lemma 7.2.1.5. *Let $R\colon \mathfrak{C} \to \mathfrak{C}'$ be a functor, (L, R) be an adjoint pair of functors such that the adjoint morphism $L \circ R \to \mathrm{id}$ is an isomorphism.*

(a) The functor R is fully faithful. Denote by $\mathfrak{D}$ its essential image.

(b) An object $\mathcal{E}' \in \mathfrak{C}'$ is an object of $\mathfrak{D}$ if and only if the canonical morphism $\mathcal{E}' \to R \circ L(\mathcal{E}')$ is an isomorphism.

(c) The functors R and L induce quasi-inverse equivalences of categories between $\mathfrak{C}$ and $\mathfrak{D}$.

Proof. For any $\mathcal{E}, \mathcal{F} \in \mathfrak{C}$, since the adjoint morphism $L \circ R \to \mathrm{id}$ is an isomorphism, then the composition $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}(R(\mathcal{E}), R(\mathcal{F})) \xrightarrow{\sim} \mathrm{Hom}(L \circ R(\mathcal{E}), \mathcal{F})$ is a bijection. Hence R is fully faithful. Now, since the composition of both adjoint morphisms $R \to R \circ L \circ R \to R$ is the identity since the $R \circ L \circ R \to R$ is an isomorphism, then so is $R \to R \circ L \circ R$. The parts (b) and (c) come from the fact that the adjoint morphisms $L \circ R \to \mathrm{id}$ and $R \to R \circ L \circ R$ are isomorphisms. □

7.2.1.6. We have the following notation and properties.

(a) The ringed topos $\underleftarrow{l}_{\{*\},\mathbb{N}} \colon (\{*\}_\bullet, D_\bullet) \to (\{*\}, \widehat{D})$ will simply be denoted by $\underleftarrow{l}$. Since any left D_i-module has a (global) presentation, then by using 7.1.3.8 in the case where the topological space is $\{*\}$, we get that a left $D_\bullet$-module $E_\bullet$ has local presentation if and only if the canonical homomorphisms $E_{i+1}/I^{i+1}M_{i+1} \to E_i$ are D_i-linear isomorphisms. Hence, we can interpret the property 7.2.1.2.(a) as follows: for any left $D_\bullet$-module $E_\bullet$ having local presentation, the canonical morphism $\underleftarrow{l}^* \circ \underleftarrow{l}_*(E_\bullet) \to E_\bullet$ is an isomorphism.

A left $\widehat{D}$-module E is said to be "separated complete" if the canonical morphism

$$E \to \underleftarrow{l}_* \circ \underleftarrow{l}^* E = \varprojlim_i D_i \otimes_{\widehat{D}} E \tag{7.2.1.6.1}$$

is an isomorphism. We denote by $\mathrm{Mod}_{\mathrm{c}}({}^{1}\widehat{D})$ the category of separated complete left $\widehat{D}$-modules. With notation 7.1.3.9, it follows from 7.2.1.5 that the functors $\underleftarrow{l}_*$ and $\underleftarrow{l}^*$ induce canonically quasi-inverse equivalences of categories between $\mathrm{Mod}_{\mathrm{lp}}({}^{1}D_\bullet)$ and $\mathrm{Mod}_{\mathrm{c}}({}^{1}\widehat{D})$.

(b) We remark that any left $\widehat{D}$-module E has a (global) presentation, i.e. there exists an exact sequence in $\mathrm{Mod}({}^{1}D)$ of the form

$$\oplus_{i \in I} \widehat{D} \to \oplus_{j \in J} \widehat{D} \to E \to 0,$$

for some sets I and J. Since $\underleftarrow{l}^*$ is left exact and commutes with direct limits, it preserve a global presentation, i.e. we get the functor $\underleftarrow{l}^* \colon \mathrm{Mod}({}^{1}\widehat{D}) \to \mathrm{Mod}_{\mathrm{gp}}({}^{1}D_\bullet)$ (see definition 7.1.3.7 and notation 7.1.3.9). Hence, it follows from (a) that the inclusion $\mathrm{Mod}_{\mathrm{gp}}({}^{1}D_\bullet) \subset \mathrm{Mod}_{\mathrm{lp}}({}^{1}D_\bullet)$ is in fact an equality.

(c) Suppose D_0 is noetherian. We denote by $\mathrm{Coh}({}^{1}\widehat{D})$ the category of coherent left $\widehat{D}$-modules which is therefore that of left $\widehat{D}$-modules of finite type. It follows from 7.2.1.2 and 7.2.1.3 that the functors $\underleftarrow{l}_*$ and $\underleftarrow{l}^*$ induce canonically quasi-inverse equivalences of categories between $\mathrm{Mod}_{\mathrm{fp}}({}^{1}D_\bullet)$ and $\mathrm{Coh}({}^{1}\widehat{D})$ (see notation 7.1.3.9). Moreover, since a coherent $\widehat{D}$-module has a (global) finite presentation, we get $\mathrm{Mod}_{\mathrm{fp}}({}^{1}D_\bullet) = \mathrm{Mod}_{\mathrm{gfp}}({}^{1}D_\bullet)$.

7.2.2 Completion of sheaves of rings on formal schemes, flatness

Lemma 7.2.2.1. *Let $(\mathcal{M}_i)_{i \in \mathbb{N}}$ be a projective system of $\mathcal{O}_{X_i}$-modules satisfying both following properties*

(a) the $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ are quasi-coherent for any $i \in \mathbb{N}$,

(b) the homomorphisms $\mathcal{M}_{i+1}/\mathcal{I}^{i+1}\mathcal{M}_{i+1} \to \mathcal{M}_i$ are isomorphisms for any $i \in \mathbb{N}$.

Then, setting $\mathcal{M} = \varprojlim_i \mathcal{M}_i$, for any $n \geq 0$, the canonical homomorphisms

$$\mathcal{I}^n \mathcal{M} \to \varprojlim_i \mathcal{I}^n \mathcal{M}_i, \qquad \mathcal{M}/\mathcal{I}^{n+1}\mathcal{M} \to \mathcal{M}_n \tag{7.2.2.1.1}$$

are isomorphisms.

Proof. This follows from Lemma 7.2.1.2. □

Lemma 7.2.2.2. *Suppose $\mathfrak{X}$ affine and put $I = \Gamma(\mathfrak{X}, \mathcal{I})$.*

(a) Let $\mathcal{M}$ be an $\mathcal{O}_{\mathfrak{X}}$-module such that $\mathcal{M} \xrightarrow{\sim} \varprojlim_i \mathcal{M}/\mathcal{I}^i\mathcal{M}$, and such that $\mathcal{M}/\mathcal{I}^i\mathcal{M}$ are $\mathcal{O}_{X_i}$-quasi-coherent. Then $\forall\, n \in \mathbb{N}$, $\forall\, q \geq 1$ we have

$$H^q(\mathfrak{X}, \mathcal{I}^n\mathcal{M}) = 0, \quad \Gamma(\mathfrak{X}, \mathcal{I}^n\mathcal{M}) = I^n\Gamma(\mathfrak{X}, \mathcal{M}). \tag{7.2.2.2.1}$$

(b) *Suppose the $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is a filtered inductive limit of a family $(\mathcal{M}_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{O}_{\mathfrak{X}}$-modules satisfying both conditions of (a). Then $\mathcal{M}$ also satisfies the preceding properties. Moreover, if $\widehat{\mathcal{M}} = \varprojlim_i \mathcal{M}/\mathcal{I}^i\mathcal{M}$, and if $\Gamma(\mathfrak{X},\mathcal{M})^\wedge$ is the I-adic completion of $\Gamma(\mathfrak{X},\mathcal{M})$, then*

$$\Gamma(\mathfrak{X},\mathcal{M})^\wedge \xrightarrow{\sim} \Gamma(\mathfrak{X},\widehat{\mathcal{M}}). \tag{7.2.2.2.2}$$

Proof. The first equality of 7.2.2.2.1 follows from [Gro61, 0.13.3.1]. The second one follows from Lemma 7.2.1.2. As the functors $\mathcal{M} \mapsto H^q(\mathfrak{X},\mathcal{I}^n\mathcal{M})$ commutes with filtered inductive limits (see 7.1.2.16), both equalities of 7.2.2.2.1 remain true under the hypothesis of (b). From these we get isomorphisms

$$\Gamma(\mathfrak{X},\mathcal{M})/I^i\Gamma(\mathfrak{X},\mathcal{M}) \xrightarrow{\sim} \Gamma(\mathfrak{X},\mathcal{M}/\mathcal{I}^i\mathcal{M}). \tag{7.2.2.2.3}$$

This yields (b). ☐

7.2.2.3. Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ satisfying the following conditions.

(a) On any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ideal $\mathcal{I}$ has a family of generators with image in the center of $\mathcal{D}$.

(b) As $\mathcal{O}_{\mathfrak{X}}$-module for the left structure, $\mathcal{D}$ is a filtered inductive limit of $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{D}_\lambda$ such that $\mathcal{D}_\lambda/\mathcal{I}^{i+1}\mathcal{D}_\lambda$ is a $\mathcal{O}_{X_i}$-quasi-coherent for its left structure for any $i \in \mathbb{N}$ and such that the canonical homomorphism $\mathcal{D}_\lambda \to \varprojlim_i \mathcal{D}_\lambda/\mathcal{I}^{i+1}\mathcal{D}_\lambda$ is an isomorphism.

(c) For any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ring $\Gamma(\mathfrak{U},\mathcal{D})$ is left noetherian.

For such a ring $\mathcal{D}$, we write $\mathcal{D}_i := \mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}$ and $\widehat{\mathcal{D}} := \varprojlim_i \mathcal{D}_i$ the $\mathcal{I}$-adic completion of $\mathcal{D}$. We remark that the condition (b) implies that $\mathcal{D}_i$ are $\mathcal{O}_{X_i}$-quasi-coherent for any $i \in \mathbb{N}$. We have the ringed topos morphism: $\underset{\leftarrow}{l}_X \colon (X_\bullet,\mathcal{D}_\bullet) \to (\mathcal{X},\widehat{\mathcal{D}})$.

Example 7.2.2.4. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10. Suppose $\mathfrak{X}^\sharp$ is a log smooth $\mathfrak{S}^\sharp$-log formal scheme. Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure. We suppose the following properties hold:

(a) For any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ring $\Gamma(\mathfrak{U},\mathcal{B})$ is noetherian.

(b) $\mathcal{B}/\mathcal{I}^{i+1}\mathcal{B}$ is $\mathcal{O}_{X_i}$-quasi-coherent and the canonical morphism $\mathcal{B} \to \varprojlim_i \mathcal{B}/\mathcal{I}^{i+1}\mathcal{B}$ is an isomorphism.

Using the order filtration and the proposition 4.1.2.17.(d), we can check that the properties 7.2.2.3 are satisfied for $\mathcal{D} = \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

Definition 7.2.2.5. We keep the notation and hypotheses of 7.2.2.3. Let $\mathcal{E}$ be a left $\mathcal{D}$-module.

(a) The separated completion (for the $\mathcal{I}$-adic topology) of a $\mathcal{E}$ is defined to be the $\widehat{\mathcal{D}}$-module $\widehat{\mathcal{E}} = \underset{\leftarrow}{l}_{X,*} \circ \underset{\leftarrow}{l}^*_X(\mathcal{E}) = \varprojlim_i \mathcal{D}_i \otimes_{\mathcal{D}} \mathcal{E} \xrightarrow{\sim} \varprojlim_i \mathcal{E}/\mathcal{I}^{i+1}\mathcal{E}$.

(b) $\mathcal{E}$ is said to be separated complete if the canonical morphism

$$\mathcal{E} \to \underset{\leftarrow}{l}_{X,*} \circ \underset{\leftarrow}{l}^*_X(\mathcal{E}) = \varprojlim_i \mathcal{D}_i \otimes_{\mathcal{D}} \mathcal{E} \tag{7.2.2.5.1}$$

is an isomorphism. Remark that if $\mathcal{E}$ is a separated complete $\mathcal{D}$-module the both equalities of 7.2.2.2.1 are valid for $\mathcal{M} = \mathcal{E}$ and also 7.2.2.2.2. and then we have such results.

(c) We say that $\mathcal{E}$ is p-torsion free if for all open $\mathfrak{U}$ of $\mathfrak{X}$, $\Gamma(\mathfrak{U},\mathcal{E})$ is p-torsion free.

Proposition 7.2.2.6. *We keep the notation and hypotheses of 7.2.2.3.*

(a) *For any affine open $\mathcal{U} \subset \mathfrak{X}$, the ring $\Gamma(\mathcal{U},\widehat{\mathcal{D}})$ is left noetherian.*

(b) *If $\mathcal{D}$ is flat over $\mathcal{O}_{\mathfrak{X}}$ fo the right structure, then so is $\widehat{\mathcal{D}}$.*

Proof. The first statement is a consequence of 7.2.1.2.(c) and of 7.2.2.2.(b). The second one follows from 7.2.1.3.(g) and 7.2.2.2.2. ☐

7.2.3 Coherent and pseudo-quasi-coherent modules over a complete sheaf of rings on formal schemes, theorems A and B

Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ satisfying the following conditions.

(a) On any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ideal $\mathcal{I}$ has a family of generators with image in the center of $\mathcal{D}$.

(b) The sheaf $\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}$ is $\mathcal{O}_{X_i}$-quasi-coherent for its left structure for any $i \in \mathbb{N}$;

(c) The canonical homomorphism $\mathcal{D} \to \varprojlim_i \mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}$ is an isomorphism.

(d) For any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ring $\Gamma(\mathfrak{U}, \mathcal{D})$ is left noetherian.

We set $\mathcal{D}_i := \mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}$. Following 7.1.3.8, the property (b) means that $\mathcal{D}_\bullet$ has local presentation as an $\mathcal{O}_{X_\bullet}$-module for its left structure. It follows from 4.1.3.2, $\mathcal{D}_i$ be a sheaf of rings on X_i satisfying theorems A and B for coherent modules (see definition 1.4.3.14). We have the ringed topos morphism: $\underleftarrow{l}_X : (X_\bullet, \mathcal{D}_\bullet) \to (\mathcal{X}, \mathcal{D})$. For any left $\mathcal{D}$-module $\mathcal{E}$, we write $\widehat{\mathcal{E}} = \underleftarrow{l}_{X,*} \circ \underleftarrow{l}_X^*(\mathcal{E}) = \varprojlim_i \mathcal{E}/\mathcal{I}^{i+1}\mathcal{E}$.

Remark 7.2.3.1. The sheaf $\mathcal{D}$ of the subsection 7.2.3 satisfies the conditions of 7.2.2.3. Conversely, let $\mathscr{D}$ be a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathscr{D}$ satisfying the conditions of 7.2.2.3. It follows from 7.2.2.2 and 7.2.2.6.(a) that the sheaf $\varprojlim_i \mathscr{D}/\mathcal{I}^{i+1}\mathscr{D}$ satisfies the same properties than the sheaf $\mathcal{D}$.

Example 7.2.3.2. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10. Suppose $\mathfrak{X}^\sharp$ is a log smooth $\mathfrak{S}^\sharp$-formal log scheme. Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure and satisfying both extra conditions of 7.2.2.4. Then with the remark 7.2.3.1, the conditions of the section 7.2.3 are satisfied for $\mathcal{D} = \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. We call $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ the sheaf of differential operators of (infinite order and) level m on $(\mathfrak{X}^\sharp, \mathcal{B})/\mathfrak{S}^\sharp$.

Suppose in this paragraph, $\mathcal{D} = \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Let $\mathfrak{U} \subset \mathfrak{X}$ be an affine open on which we have logarithmic coordinates $(u_\lambda)_{\lambda=1,\ldots,n}$. Let $\underline{\tau}_{\sharp(m)}$ (resp. $\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$) be the element constructed from $(u_\lambda)_{\lambda=1,\ldots,r}$ as defined in 3.2.2.4 (resp. 3.2.3.4). An element $P \in \Gamma(\mathfrak{U}, \mathcal{D})$ can be written as an infinite serie

$$P = \sum_{\underline{k} \in \mathbb{N}^d} b_{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$$

with $b_{\underline{k}} \in \Gamma(\mathfrak{U}, \mathcal{B})$ and $b_{\underline{k}} \to 0$ for the p-adic topology. Note that we no longer have filtration by order on the sheaf $\mathcal{D}$.

Proposition 7.2.3.3. *We have the following properties.*

(a) For any pair of affine opens $\mathfrak{U}' \subset \mathfrak{U}$, the homomorphism

$$\Gamma(\mathfrak{U}, \mathcal{D}) \to \Gamma(\mathfrak{U}', \mathcal{D})$$

is right flat.

(b) The sheaf $\mathcal{D}$ is left coherent.

Proof. Let us check (a). Since $\mathcal{D}_i$ is quasi-coherent for the left structure, then $\Gamma(U_i, \mathcal{D}_i) \to \Gamma(U'_i, \mathcal{D}_i)$ is right flat. It follows from 7.2.1.3.g that the morphism $\varprojlim_i \Gamma(U_i, \mathcal{D}_i) \to \varprojlim_i \Gamma(U'_i, \mathcal{D}_i)$ is right flat. We get the desired result by using the condition (c) satisfied by $\mathcal{D}$ and the commutation of the local sections functors with projective limits. We get the second statement by using 1.4.5.2. $\square$

Notation 7.2.3.4. Assume $\mathfrak{X}$ is affine and set $O_{\mathfrak{X}} = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, $D = \Gamma(\mathfrak{X}, \mathcal{D})$, $I = \Gamma(\mathfrak{X}, \mathcal{I})$, $D_i = \Gamma(\mathfrak{X}, \mathcal{D}_i) \xrightarrow{\sim} D/I^{i+1}D$ (see 7.2.2.2.3), $D = \Gamma(\mathfrak{X}, \mathcal{D})^\wedge \xrightarrow{\sim} \Gamma(\mathfrak{X}, \mathcal{D}^\wedge)$ (see 7.2.2.2.2) and $O_{X_i} := \Gamma(X_i, \mathcal{O}_{X_i}) \xrightarrow{\sim} O_{\mathfrak{X}}/I^{i+1}O_{\mathfrak{X}}$. We denote by $\mathrm{Mod}_{\mathrm{gp}}({}^{\mathrm{l}}\mathcal{D})$ (resp. $\mathrm{Mod}_{\mathrm{gfp}}({}^{\mathrm{l}}\mathcal{D})$) is the category of left $\mathcal{D}$-module having a global presentation (resp. a global finite presentation).

Definition 7.2.3.5. A left $\mathcal{D}$-module $\mathcal{E}$ is "pseudo-quasi-coherent" if it is separated complete (see definition 7.2.2.5) and if, for any $i \in \mathbb{N}$, $\mathcal{E}_i := \mathcal{D}_i \otimes_{\mathcal{D}} \mathcal{E}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module. We denote by $\mathrm{Mod}_{\mathrm{pqc}}({}^{\mathrm{l}}\mathcal{D})$ the category of pseudo quasi-coherent left $\mathcal{D}$-modules.

7.2.3.6. We point out the fact that a left $\mathcal{D}$-module is pseudo-quasi-coherent (resp. is p-torsion free) is local on $\mathfrak{X}$.

7.2.3.7. Let $\mathcal{E}_\bullet \in \mathrm{Mod}(\mathcal{D}_\bullet)$. Since $\mathcal{D}_i$ is $\mathcal{O}_{X_i}$-quasi-coherent, then a $\mathcal{D}_i$-module is $\mathcal{O}_{X_i}$-quasi-coherent if and only if it has local presentation. Hence, following 7.1.3.8, this yields that $\mathcal{E}_\bullet \in \mathrm{Mod}_{\mathrm{lp}}(\mathcal{D}_\bullet)$ (see notation 7.1.3.9) if and only if it satisfies both conditions (a) and (b) of 7.2.2.1.

Hence, for any $\mathcal{E}_\bullet \in \mathrm{Mod}_{\mathrm{lp}}({}^{l}\mathcal{D}_\bullet)$, by using the second isomorphism of 7.2.2.1.1, we check that the adjoint morphism $\underleftarrow{l}_X^* \circ \underleftarrow{l}_{X,*}(\mathcal{E}_\bullet) \to \mathcal{E}_\bullet$ is an isomorphism. By using 7.2.1.5 and 7.2.2.1, this implies that the functor $\underleftarrow{l}_{X,*} \colon \mathrm{Mod}_{\mathrm{lp}}({}^{l}\mathcal{D}_\bullet) \to \mathrm{Mod}({}^{l}\mathcal{D})$ is fully faithful and its essential image is $\mathrm{Mod}_{\mathrm{pqc}}({}^{l}\mathcal{D})$. Moreover, the functors $\underleftarrow{l}_{X*}$ and $\underleftarrow{l}_X^*$ induce canonically quasi-inverse equivalences of categories between $\mathrm{Mod}_{\mathrm{lp}}({}^{l}\mathcal{D}_\bullet)$ and $\mathrm{Mod}_{\mathrm{pqc}}({}^{l}\mathcal{D})$.

7.2.3.8. We suppose $\mathfrak{X}$ affine and we keep notation 7.2.3.4. Let $\varpi_X \colon |\mathfrak{X}| \to \{*\}$ be the continuous morphism from the underlying topological space of $\mathfrak{X}$ to a punctual set. Taking global section of the inverse system $\mathcal{D}_\bullet$, we get the inverse system $D_\bullet$. With notation 7.1.2.9, this yields the morphism of ringed topoi $\varpi_{X,\mathbb{N}} \colon (X_\bullet, \mathcal{D}_\bullet) \to (\{*\}_\bullet, D_\bullet)$ that will we denote by $\varpi_{X_\bullet}$.

Similarly, we denote by $\varpi_{\mathfrak{X}} \to (|\mathfrak{X}|, \mathcal{D}) \to (\{*\}, D)$ the morphism of ringed topoi. We have $\varpi_{\mathfrak{X}}^* = \mathcal{D} \otimes_D -$ and $\varpi_{\mathfrak{X}*} = \Gamma(\mathfrak{X}, -)$. With notation 7.2.1.6, we get the commutative diagram of ringed topos:

$$\begin{array}{ccc} (X_\bullet, \mathcal{D}_\bullet) & \xrightarrow{\varpi_{X_\bullet}} & (\{*\}_\bullet, D_\bullet) \\ \downarrow{\scriptstyle \underleftarrow{l}_X} & & \downarrow{\scriptstyle \underleftarrow{l}} \\ (|\mathfrak{X}|, \mathcal{D}) & \xrightarrow{\varpi_{\mathfrak{X}}} & (\{*\}, D). \end{array} \tag{7.2.3.8.1}$$

Proposition 7.2.3.9. *We suppose $\mathfrak{X}$ affine and we keep notation 7.2.3.8.*

(a) The functors $\varpi_{X_\bullet *}$ and $\varpi_{X_\bullet}^*$ induce canonically quasi-inverse equivalences of categories between $\mathrm{Mod}_{\mathrm{lp}}(\mathcal{D}_\bullet)$ and $\mathrm{Mod}_{\mathrm{lp}}(D_\bullet)$ (resp. $\mathrm{Mod}_{\mathrm{fp}}(\mathcal{D}_\bullet)$ and $\mathrm{Mod}_{\mathrm{fp}}(D_\bullet)$).

(b) We have the equalities $\mathrm{Mod}_{\mathrm{lp}}(D_\bullet) = \mathrm{Mod}_{\mathrm{gp}}(D_\bullet)$, $\mathrm{Mod}_{\mathrm{fp}}(D_\bullet) = \mathrm{Mod}_{\mathrm{gfp}}(D_\bullet)$, $\mathrm{Mod}_{\mathrm{lp}}(\mathcal{D}_\bullet) = \mathrm{Mod}_{\mathrm{gp}}(\mathcal{D}_\bullet)$, $\mathrm{Mod}_{\mathrm{fp}}(\mathcal{D}_\bullet) = \mathrm{Mod}_{\mathrm{gfp}}(\mathcal{D}_\bullet)$.

Proof. a) i) Let $\mathcal{E}_\bullet \in \mathrm{Mod}_{\mathrm{lp}}(\mathcal{D}_\bullet)$. Following 7.2.3.7, this means that $\mathcal{E}_\bullet$ is a left $\mathcal{D}_\bullet$-module satisfying both conditions (a) and (b) of 7.2.2.1. Let $E_\bullet \in \mathrm{Mod}_{\mathrm{lp}}(D_\bullet)$. Following 7.2.1.6.a, $E_\bullet$ has local presentation means that the canonical homomorphisms $E_{i+1}/I^{i+1}M_{i+1} \to E_i$ are D_i-linear isomorphisms for any $i \in \mathbb{N}$. By definition, $\varpi_{X_\bullet *}(\mathcal{E}_\bullet) = (\Gamma(X_i, \mathcal{E}_i))_{i\in\mathbb{N}}$ and $\varpi_{X_\bullet}^*(E_\bullet) = (\mathcal{D}_i \otimes_{D_i} E_i)_{i\in\mathbb{N}}$ (see 7.1.3.8). Since $\mathcal{D}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module, then $D_i \to \mathcal{D}_i$ is flat. With both above descriptions of local presentation, this implies that $\varpi_{X_\bullet}^*$ preserve modules of local presentation, i.e. induces $\varpi_{X_\bullet}^* \colon \mathrm{Mod}_{\mathrm{lp}}(D_\bullet) \to \mathrm{Mod}_{\mathrm{lp}}(\mathcal{D}_\bullet)$. Similarly, using theorems of type A and B for quasi-coherent $\mathcal{D}_{X_i}$-modules (4.1.3.2), we get the factorization $\varpi_{X_\bullet *} \colon \mathrm{Mod}_{\mathrm{lp}}(\mathcal{D}_\bullet) \to \mathrm{Mod}_{\mathrm{lp}}(D_\bullet)$ and we check that the canonical morphisms $\varpi_{X_\bullet}^* \circ \varpi_{X_\bullet *}(\mathcal{E}_\bullet) \to \mathcal{E}_\bullet$ and $E_\bullet \to \varpi_{X_\bullet *} \circ \varpi_{X_\bullet}^*(E_\bullet)$ are isomorphisms.

ii) By using the theorem A and B for coherent $\mathcal{D}_i$-modules for any $i \in \mathbb{N}$ (see 4.1.3.2), we can proceed similarly to i) to check the functors $\varpi_{X_\bullet *}$ and $\varpi_{X_\bullet}^*$ induce canonically quasi-inverse equivalences of categories between $\mathrm{Mod}_{\mathrm{fp}}(\mathcal{D}_\bullet)$ and $\mathrm{Mod}_{\mathrm{fp}}(D_\bullet)$.

b) i) We already know that $\mathrm{Mod}_{\mathrm{lp}}(D_\bullet) = \mathrm{Mod}_{\mathrm{gp}}(D_\bullet)$, $\mathrm{Mod}_{\mathrm{fp}}(D_\bullet) = \mathrm{Mod}_{\mathrm{gfp}}(D_\bullet)$, (see 7.2.1.6.(b and (c)).

ii) The functor $\varpi_{X_\bullet}^*$ preserves the (finite) global presentability. Hence, to get the equality $\mathrm{Mod}_{\mathrm{fp}}(\mathcal{D}_\bullet) = \mathrm{Mod}_{\mathrm{gfp}}(\mathcal{D}_\bullet)$, thanks to (a) it remains to check that the functor $\varpi_{X_\bullet *}$ preserves the finite global presentability. Let $\mathcal{E}_\bullet \in \mathrm{Mod}_{\mathrm{gfp}}({}^{l}\mathcal{D}_\bullet)$. By definition, $\mathcal{E}_\bullet$ is the cokernel of a morphism of the form $f \colon \oplus_{i=1}^r \mathcal{D}_\bullet \to \oplus_{j=1}^s \mathcal{D}_\bullet$. By applying the functor $\varpi_{X_\bullet *}$, we get the morphism of left $D_\bullet$-modules $g \colon \oplus_{i=1}^r D_\bullet \to \oplus_{j=1}^s D_\bullet$. Let $G_\bullet$ be its cokernel. By (right) flatness of the functor $\varpi_{X_\bullet}^* = \mathcal{D}_\bullet \otimes_{D_\bullet} -$, this yields that $\mathcal{E}_\bullet \xrightarrow{\sim} \mathcal{D}_\bullet \otimes_{D_\bullet} G_\bullet$. From the part (a), this implies $\varpi_{X_\bullet *}(\mathcal{E}_\bullet) \xrightarrow{\sim} G_\bullet$. Hence, $\varpi_{X_\bullet *}(\mathcal{E}_\bullet)$ has global finite presentation.

iii) Since $\mathbb{N}$ is quasi-separated (because $\mathbb{N}$ is totally ordered), since the topological spaces X and $\{*\}$ are coherent (and in particular locally coherent), then following 7.1.2.15.d the topoi $X_\bullet$ and $\{*\}_\bullet$ are algebraic. (Beware that $\mathbb{N}$ is not quasi-compact and the topoi $X_\bullet$ and $\{*\}_\bullet$ are not coherent.) However, since $\varpi_X \colon |\mathfrak{X}| \to \{*\}$ is a coherent morphism of locally coherent topological spaces, then it follows from

7.1.2.17.(b) that the topoi morphism $\varpi_{X,\mathbb{N}} \colon X_\bullet \to \{*\}_\bullet$ is coherent. Hence, the functor $\varpi_{X_\bullet *}$ commutes with filtered inductive limits of abelian sheaves (see [SGA4.2, VI.5.1]) and in particular with direct sums of abelian sheaves. This implies that we can proceed similarly to b.ii) replacing a global finite presentation by a global presentation. □

Proposition 7.2.3.10. *We suppose $\mathfrak{X}$ affine and we keep notation 7.2.3.8. The functors $\varpi^*_{\mathfrak{X}} = \mathcal{D} \otimes_D -$ and $\varpi_{\mathfrak{X}*} = \Gamma(\mathfrak{X}, -)$ induce canonically quasi-inverse equivalences of categories between $\mathrm{Mod}({}^l D)$ and $\mathrm{Mod}_{\mathrm{gp}}({}^l \mathcal{D})$ (resp. $\mathrm{Coh}({}^l D)$ and $\mathrm{Mod}_{\mathrm{gfp}}({}^l \mathcal{D})$). Moreover, the functor $\mathcal{D} \otimes_D -$ is exact.*

Proof. a) Since the functor $\mathcal{D} \otimes_D - \colon \mathrm{Mod}({}^l D) \to \mathrm{Mod}({}^l \mathcal{D})$ is right exact (and even exact thanks to 7.2.3.3.a), we have the factorization

$$\mathcal{D} \otimes_D - \colon \mathrm{Mod}({}^l D) \to \mathrm{Mod}_{\mathrm{gp}}({}^l \mathcal{D}), \quad \mathcal{D} \otimes_D - \colon \mathrm{Coh}({}^l D) \to \mathrm{Mod}_{\mathrm{gfp}}({}^l \mathcal{D}). \tag{7.2.3.10.1}$$

b) Since $\Gamma(\mathfrak{X}, \mathcal{D}) = D$, since $\Gamma(\mathfrak{X}, -)$ commutes with filtered inductive limits of abelian sheaves (see [SGA4.2, VI.5.2]), then the canonical homomorphism $L \to \Gamma(\mathfrak{X}, \mathcal{D} \otimes_D L)$ is an isomorphism when L is a free left D-module. By using the five lemma, this yields that the canonical homomorphism $E \to \Gamma(\mathfrak{X}, \mathcal{D} \otimes_D E)$ is an isomorphism for any left D-module E.

c) Let $\mathcal{E} \in \mathrm{Mod}_{\mathrm{gp}}({}^l \mathcal{D})$. By definition, $\mathcal{M}$ is the cokernel of a morphism of the form $f \colon \oplus_{i \in I} \mathcal{D} \to \oplus_{j \in J} \mathcal{D}$. Taking the global section we get the morphism of left D-modules $g \colon \oplus_{i \in I} D \to \oplus_{j \in J} D$. Let G be its cokernel. Since the functor $\mathcal{D} \otimes_D -$ is right exact (and is the adjoint of $\Gamma(\mathfrak{X}, -)$), this yields that $\mathcal{E} \xrightarrow{\sim} \mathcal{D} \otimes_D G$. Hence, the functor $\mathcal{D} \otimes_D - \colon \mathrm{Mod}({}^l D) \to \mathrm{Mod}_{\mathrm{gp}}({}^l \mathcal{D})$ is essentially surjective. From the part b) of the proof, this yields $\Gamma(\mathfrak{X}, \mathcal{E}) \xrightarrow{\sim} G$. When $\mathcal{E}$ has global finite presentation, this implies that $\Gamma(\mathfrak{X}, \mathcal{E})$ is a left D-module of finite type. Hence, we are done. □

7.2.3.11. We suppose $\mathfrak{X}$ affine and we keep notation and hypotheses of 7.2.3.4.

(a) We have the functor ${}^\Delta \colon \mathrm{Mod}({}^l D) \to \mathrm{Mod}_{\mathrm{pqc}}({}^l \mathcal{D})$ which associates to a left D-module E the left $\mathcal{D}$-module

$$E^\Delta := \mathcal{D} \widehat{\otimes}_D E = \underset{\leftarrow X,*}{l} \circ \underset{\leftarrow X}{l^*} \circ \varpi^*_{\mathfrak{X}}(E) \xrightarrow{\sim} \underset{\leftarrow X,*}{l} \circ \varpi^*_{X_\bullet} \circ \underset{\leftarrow}{l^*}(E) \xrightarrow{\sim} \varprojlim_i (\mathcal{D}_i \otimes_{D_i} (E/I^{i+1}E)). \tag{7.2.3.11.1}$$

where we identify a left D-module with its associated constant sheaf on $\mathfrak{X}$.

(b) If E is a left D-module of finite type, then the canonical morphism

$$\mathcal{D} \otimes_D E \to \mathcal{D} \widehat{\otimes}_D E = E^\Delta \tag{7.2.3.11.2}$$

is an isomorphism. Indeed, for any $f \in O_{\mathfrak{X}}$, it follows from 7.2.1.3 and 7.2.3.3 that the morphism $\mathcal{D}(\mathfrak{D}(f)) \otimes_D E \to \mathcal{D}(\mathfrak{D}(f)) \widehat{\otimes}_D E$ is an isomorphism.

(c) It follows from 7.2.3.10 and 7.2.3.11.2 that the functor ${}^\Delta$ factors through

$${}^\Delta \colon \mathrm{Coh}({}^l D) \to \mathrm{Mod}_{\mathrm{gfp}}({}^l \mathcal{D}). \tag{7.2.3.11.3}$$

Remark 7.2.3.12. With notation 7.2.3.11, since $\mathcal{D}_i$ is an $\mathcal{O}_{X_i}$-quasi-coherent module, then the canonical morphism

$$\mathcal{O}_{\mathfrak{X}} \widehat{\otimes}_{O_{\mathfrak{X}}} M \to M^\Delta$$

is an isomorphism.

Proposition 7.2.3.13 (Characterization of the pseudo quasi-coherence, Theorem A, Theorem B)**.** *We suppose $\mathfrak{X}$ affine and we keep notation and hypotheses of 7.2.3.4.*

(i) Let $\mathcal{E}$ is a left $\mathcal{D}$-module. The following properties are equivalent.

(a) The D-module $\Gamma(\mathfrak{X}, \mathcal{E})$ belongs to $\mathrm{Mod}_{\mathrm{c}}({}^l D)$ (see notation 7.2.1.6.a) and the canonical homomorphism

$$\Gamma(\mathfrak{X}, \mathcal{E})^\Delta = \mathcal{D} \widehat{\otimes}_D \Gamma(\mathfrak{X}, \mathcal{E}) \to \mathcal{E}$$

is an isomorphism.

(b) $\mathcal{E}$ *is a pseudo quasi-coherent* $\mathcal{D}$*-module.*

When these equivalent conditions hold, the canonical morphism

$$D_i \otimes_D \Gamma(\mathfrak{X}, \mathcal{E}) \to \Gamma(\mathfrak{X}, \mathcal{D}_i \otimes_{\mathcal{D}} \mathcal{E}) \qquad (7.2.3.13.1)$$

is an isomorphism.

(ii) *The functors* ${}^{\Delta}$ *and* $\Gamma(\mathfrak{X}, -)$ *induce canonically exact quasi-inverse equivalences of categories between* $\mathrm{Mod}_{\mathrm{c}}({}^{1}D)$ *and* $\mathrm{Mod}_{\mathrm{pqc}}({}^{1}\mathcal{D})$.

(iii) *Let* $\mathcal{E}$ *be a pseudo quasi-coherent* $\mathcal{D}$*-module. We have* $H^{q}(\mathfrak{X}, \mathcal{E}) = 0$ *for all* $q \geq 1$.

Proof. Let us check the equivalence of (i). The implication $(a) \Rightarrow (b)$ is straightforward. Conversely, let $\mathcal{E}$ be a pseudo-quasi-coherent $\mathcal{D}$-module. Following 7.2.2.2.2, we have the canonical morphism $\widehat{\Gamma(\mathfrak{X}, \mathcal{E})} \xrightarrow{\sim} \Gamma(\mathfrak{X}, \mathcal{E})$. Hence, $\Gamma(\mathfrak{X}, \mathcal{E}) \in \mathrm{Mod}_{\mathrm{c}}({}^{1}D)$. It follows from (7.2.2.2.1) that for any integers $n \geq 1$ and $i \geq 0$ we have $H^{n}(\mathfrak{X}, \mathcal{I}^{i+1}\mathcal{E}) = 0$ and $\Gamma(\mathfrak{X}, \mathcal{I}^{i}\mathcal{E}) = I^{i}\Gamma(\mathfrak{X}, \mathcal{E})$. By applying the functor $\Gamma(\mathfrak{X}, -)$ to the exact sequence $0 \to \mathcal{I}^{i+1}\mathcal{E} \to \mathcal{E} \to \mathcal{E}_i \to 0$, this yields that the canonical morphism 7.2.3.13.1 is an isomorphism. The canonical morphism $\mathcal{D}_i \otimes_D \Gamma(\mathfrak{X}, \mathcal{E}) \to \mathcal{E}_i$ of quasi-coherent $\mathcal{D}_i$-modules is therefore an isomorphism. Passing to projective limit, we get the canonical morphism $\mathcal{D}\widehat{\otimes}_D \Gamma(\mathfrak{X}, \mathcal{E}) \to \mathcal{E}$ is an isomorphism and we have therefore checked the implication $(b) \Rightarrow (a)$.

It follows from (i) that both functors ${}^{\Delta}\colon \mathrm{Mod}_{\mathrm{c}}({}^{1}D) \to \mathrm{Mod}_{\mathrm{pqc}}({}^{1}\mathcal{D})$ and $\Gamma(\mathfrak{X}, -)\colon \mathrm{Mod}_{\mathrm{pqc}}({}^{1}\mathcal{D}) \to \mathrm{Mod}_{\mathrm{c}}({}^{1}D)$ are well defined. Let $E \in \mathrm{Mod}_{\mathrm{c}}({}^{1}D)$. Since $E^{\Delta} \xrightarrow{\sim} \varprojlim_i (\mathcal{D}_i \otimes_D E)$ (see 7.2.3.11.1), since $\Gamma(X, -)$ commutes with projective limits, since E is separated complete, then we get $\Gamma(\mathfrak{X}, E^{\Delta}) \xrightarrow{\sim} \varprojlim_i \Gamma(X, \mathcal{D}_i \otimes_D E) \xrightarrow{\sim} \varprojlim_i D_i \otimes_D E \xrightarrow{\sim} E$. Hence, using (i), we get the second statement. The third one is a consequence of 7.2.2.2.1.

□

Lemma 7.2.3.14. *We assume that* $\mathfrak{X}$ *is affine. Let* $\mathcal{E}$ *be a pseudo quasi-coherent* $\mathcal{D}$*-module. The following condition are equivalent.*

1. $\mathcal{E}$ *is* $\mathcal{O}_{\mathfrak{X}}$*-flat*
2. $\Gamma(\mathfrak{X}, \mathcal{E})$ *is* $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$*-flat.*

Proof. 1) Suppose $\Gamma(\mathfrak{X}, \mathcal{E})$ is $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$-flat. For all i, for any affine open $\mathfrak{Y}$ of $\mathfrak{X}$, $\Gamma(Y_i, \mathcal{O}_{X_i}) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})} \Gamma(\mathfrak{X}, \mathcal{E})$ is therefore a $\Gamma(Y_i, \mathcal{O}_{X_i})$-flat module. Now, according to 7.2.3.13.(i), the canonical homomorphism

$$\Gamma(\mathfrak{X}, \mathcal{E})^{\Delta} = \mathcal{D}\widehat{\otimes}_D \Gamma(\mathfrak{X}, \mathcal{E}) \to \mathcal{E}$$

is an isomorphism. By applying to the isomorphism 7.2.3.11.1 the functor $\Gamma(\mathfrak{Y}, -)$, which commutes to projective limits, we deduce

$$\Gamma(\mathfrak{Y}, \mathcal{E}) \xrightarrow{\sim} \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{X}}) \widehat{\otimes}_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})} \Gamma(\mathfrak{X}, \mathcal{E}).$$

As $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{X}})$ is Noetherian, it then follows from 7.2.1.4 that $\Gamma(\mathfrak{Y}, \mathcal{E})$ is $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{X}})$-flat. Hence we have checked the flatness of $\mathcal{E}$ as $\mathcal{O}_{\mathfrak{X}}$-module.

Conversely, suppose $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{X}}$-flat. For all i, $\mathcal{E}_i := \mathcal{O}_{X_i} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ is therefore $\mathcal{O}_{X_i}$-flat. As $\mathcal{E}_i$ is moreover quasi-coherent, then the sheaf $\Gamma(X_i, \mathcal{E}_i)$ is a flat $\Gamma(X_i, \mathcal{O}_{X_i})$-module. Following 7.2.3.13.1, $\Gamma(X, \mathcal{E})/\pi^{i+1}\Gamma(X, \mathcal{E}) \xrightarrow{\sim} \Gamma(X_i, \mathcal{E}_i)$. Hence, $\Gamma(X, \mathcal{E})/\pi^{i+1}\Gamma(X, \mathcal{E})$ is flat. However, by applying the functor $\Gamma(\mathfrak{X}, -)$ to the isomorphism 7.2.3.11.1, we obtain

$$\Gamma(\mathfrak{X}, \mathcal{E}) \xrightarrow{\sim} \varprojlim_i \Gamma(\mathfrak{X}, \mathcal{E})/\pi^{i+1}\Gamma(\mathfrak{X}, \mathcal{E}). \qquad (7.2.3.14.1)$$

According to 7.2.1.4, as $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is Noetherian and $\Gamma(\mathfrak{X}, \mathcal{E})/\pi^{i+1}\Gamma(\mathfrak{X}, \mathcal{E})$ is $\Gamma(X_i, \mathcal{O}_{X_i})$-flat, it follows from the isomorphism 7.2.3.14.1 that $\Gamma(\mathfrak{X}, \mathcal{E})$ is $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$-flat. □

Proposition 7.2.3.15. *We have the inclusion* $\mathrm{Coh}({}^{1}\mathcal{D}) \subset \mathrm{Mod}_{\mathrm{pqc}}({}^{1}\mathcal{D})$ *and the functors* $\underleftarrow{l}^{*}_{X}$ *and* $\underleftarrow{l}_{X*}$ *induce canonically quasi-inverse equivalences of categories between* $\mathrm{Coh}({}^{1}\mathcal{D})$ *and* $\mathrm{Mod}_{\mathrm{fp}}({}^{1}\mathcal{D}_{\bullet})$*. When* $\mathfrak{X}$ *is affine, we have the equality* $\mathrm{Coh}({}^{1}\mathcal{D}) = \mathrm{Mod}_{\mathrm{gfp}}({}^{1}\mathcal{D})$.

Proof. 1) Let $\mathcal{E} \in \mathrm{Coh}({}^1\mathcal{D})$. It follows from 7.1.3.8 that $\underleftarrow{l}^*_X(\mathcal{E}) \in \mathrm{Mod}_{\mathrm{fp}}({}^1\mathcal{D}_\bullet)$. Let us check that the canonical morphism

$$\mathcal{E} \to \underleftarrow{l}_{X,*} \circ \underleftarrow{l}^*_X(\mathcal{E}) \tag{7.2.3.15.1}$$

is an isomorphism. Since this is local, we can suppose that $\mathfrak{X}$ is affine and $\mathcal{E}$ has a global finite presentation. Hence, following 7.2.3.10, $E := \Gamma(\mathfrak{X}, \mathcal{E})$ is a left D-module of finite type and the canonical morphism $\mathcal{D} \otimes_D E \to \mathcal{E}$ is an isomorphism. This implies the morphism 7.2.3.15.1 is an isomorphism if and only if so is the canonical $\mathcal{D} \otimes_D E \to \mathcal{D}\widehat{\otimes}_D E$. We conclude thanks to 7.2.3.11.2.

2) Let $\mathcal{E}_\bullet \in \mathrm{Mod}_{\mathrm{fp}}({}^1\mathcal{D}_\bullet)$. Let us check that $\underleftarrow{l}_{X,*}(\mathcal{E}_\bullet) \in \mathrm{Coh}({}^1\mathcal{D})$ and that the canonical morphism $\underleftarrow{l}^*_X \circ \underleftarrow{l}_{X,*}(\mathcal{E}_\bullet) \to \mathcal{E}_\bullet$ is an isomorphism. Since this is local, we can suppose $\mathfrak{X}$ is affine. By using the equivalence of categories of 7.2.1.6 and 7.2.3.9, there exist $E \in \mathrm{Coh}(D)$ and an isomorphism $\varpi^*_{X_\bullet} \circ \underleftarrow{l}^*(E) \xrightarrow{\sim} \mathcal{E}_\bullet$ of $\mathrm{Mod}_{\mathrm{fp}}({}^1\mathcal{D}_\bullet)$. This yields

$$E^\Delta = \underleftarrow{l}_{X,*} \circ \underleftarrow{l}^*_X \circ \varpi^*_{\mathfrak{X}}(E) \xrightarrow{\sim} \underleftarrow{l}_{X,*} \varpi^*_{X_\bullet} \circ \underleftarrow{l}^*(E) \xrightarrow{\sim} \underleftarrow{l}_{X,*}(\mathcal{E}_\bullet).$$

Since E is a left D-module of finite type, then following 7.2.3.11.3 we have $E^\Delta \in \mathrm{Mod}_{\mathrm{gfp}}({}^1\mathcal{D})$. Hence, $\underleftarrow{l}_{X,*}(\mathcal{E}_\bullet) \in \mathrm{Coh}({}^1\mathcal{D})$. Since the adjoint morphism $\underleftarrow{l}^*_X \circ \underleftarrow{l}_{X,*}(\underleftarrow{l}^*_X \circ \varpi^*_{\mathfrak{X}}(E)) \to \underleftarrow{l}^*_X \circ \varpi^*_{\mathfrak{X}}(E)$ is an isomorphism (use 1) and the fact that $\varpi^*_{\mathfrak{X}}(E) \in \mathrm{Coh}({}^1\mathcal{D})$), then so is $\underleftarrow{l}^*_X \circ \underleftarrow{l}_{X,*}(\mathcal{E}_\bullet) \to \mathcal{E}_\bullet$.

3) When $\mathfrak{X}$ is affine, it remains to check the equality $\mathrm{Coh}({}^1\mathcal{D}) = \mathrm{Mod}_{\mathrm{gfp}}({}^1\mathcal{D})$. Let $\mathcal{E} \in \mathrm{Coh}({}^1\mathcal{D})$. From 1), $\mathcal{E}_\bullet := \underleftarrow{l}^*_X(\mathcal{E}) \in \mathrm{Mod}_{\mathrm{fp}}({}^1\mathcal{D}_\bullet)$ and $\mathcal{E} \xrightarrow{\sim} \underleftarrow{l}_{X,*}(\mathcal{E}_\bullet)$. From 2), there exists $E \in \mathrm{Coh}(D)$ such that $\underleftarrow{l}_{X,*}(\mathcal{E}_\bullet) \xrightarrow{\sim} E^\Delta \in \mathrm{Mod}_{\mathrm{gfp}}({}^1\mathcal{D})$. Hence, we are done. □

Corollary 7.2.3.16 (Characterization of the $\mathcal{D}$-coherence, Theorem A, Theorem B)**.** *We suppose $\mathfrak{X}$ affine and we keep notation and hypotheses of 7.2.3.4.*

(i) Let $\mathcal{E}$ is a left $\mathcal{D}$-module. The following properties are equivalent.

(a) For any $i \in \mathbb{N}$, the $\mathcal{D}_i$-module $\mathcal{E}/\mathcal{I}^{i+1}\mathcal{E}$ is coherent, and the canonical homomorphism $\mathcal{E} \to \varprojlim_i \mathcal{E}/\mathcal{I}^{i+1}\mathcal{E}$ is an isomorphism.

(b) The D-module $\Gamma(\mathfrak{X}, \mathcal{E})$ is of finite type and the canonical homomorphism

$$\mathcal{D} \otimes_D \Gamma(\mathfrak{X}, \mathcal{E}) \to \mathcal{E}$$

is an isomorphism.

(c) $\mathcal{E}$ is a coherent $\mathcal{D}$-module.

(ii) The functors $\varpi^*_{\mathfrak{X}} = \mathcal{D} \otimes_D -$ and $\varpi_{\mathfrak{X}*} = \Gamma(\mathfrak{X}, -)$ induce canonically exact quasi-inverse equivalences of categories between $\mathrm{Coh}({}^1D)$ and $\mathrm{Coh}({}^1\mathcal{D})$.

(iii) Let $\mathcal{E}$ be a coherent left $\mathcal{D}$-module. We have $H^q(\mathfrak{X}, \mathcal{E}) = 0$ for all $q \geq 1$.

Proof. The first two statements follows from 7.2.3.10 and 7.2.3.15. Using 7.2.2.2.a, this yields the last one. □

7.2.3.17 (Support)**.** Let $\mathcal{E}$ be a coherent $\mathcal{D}$-module. By using the theorem of type A of 7.2.3.16, we easily check that the set $\mathfrak{U}$ consisting of elements $x \in \mathfrak{X}$ such that $\mathcal{E}_x = 0$ is an open subset of $\mathfrak{X}$. The *support* of $\mathcal{E}$ is by definition the complementary of $\mathcal{U}$ in $\mathcal{X}$. We have $\mathcal{E}|\mathfrak{U} = 0$.

Let Z be a closed subset of $\mathfrak{X}$ and V be the complementary. We say that $\mathcal{E}$ has his support in Z if $\mathfrak{Z}$ if $\mathcal{E}|V = 0$.

7.3 Quasi-coherent complexes on formal schemes or on inductive systems of schemes

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. We keep notation 7.2.

7.3.1 Definitions and first properties

Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ satisfying the following conditions of 7.2.3.

7.3.1.1. With notation 7.2, we have the morphism of ringed topoi $\underleftarrow{l}_X : (X_\bullet, \mathcal{O}_{X_\bullet}) \to (|\mathcal{X}|, \mathcal{O}_{\mathfrak{X}})$. From [Sta22, 07A6] (or see 5.3.5.4), this yields the functors $\mathbb{R}\underleftarrow{l}_{X*} : D(\mathcal{O}_{X_\bullet}) \to D(\mathcal{O}_{\mathfrak{X}})$ and $\mathbb{L}\underleftarrow{l}_X^* : D(\mathcal{O}_{\mathfrak{X}}) \to D(\mathcal{O}_{X_\bullet})$ which are adjoint:

$$\mathrm{Hom}_{D(\mathcal{O}_{X_\bullet})}(\mathbb{L}\underleftarrow{l}_X^*(\mathcal{E}^\bullet), \mathcal{F}_\bullet^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_{\mathfrak{X}})}(\mathcal{E}^\bullet, \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{F}_\bullet^\bullet)) \tag{7.3.1.1.1}$$

for any $\mathcal{E}^\bullet \in D(\mathcal{O}_{\mathfrak{X}})$ and any $\mathcal{F}_\bullet^\bullet \in D(\mathcal{O}_{X_\bullet})$.

7.3.1.2 (Bounded cohomological dimension)**.** Let $\mathfrak{B}$ be a basis of open sets of $|X|$. For any abelian sheaf $\mathcal{E}_\bullet$ of $X_\bullet$ and for any integer q the sheaf $R^q\underleftarrow{l}_{X*}\mathcal{E}_\bullet$ is the sheaf associated with the presheaf $\mathcal{U} \in \mathfrak{B} \mapsto H^q(U_\bullet, \mathcal{E}_\bullet)$ (see [SGA4.2, V.5.1.1]). According to notation 7.2.3.8, let $\varpi_U : |\mathfrak{U}| \to \{*\}$ be the continuous morphism for the topological space of $\mathfrak{U}$ to a punctual set. Since $\mathbb{R}\underleftarrow{l}_{\{*\}} \circ \mathbb{R}\varpi_{U,\mathbb{N}*} \xrightarrow{\sim} \mathbb{R}(\underleftarrow{l}_{\{*\}} \circ \varpi_{U,\mathbb{N}*})$ (see 7.1.3.15.3), since of $H^q(U_\bullet, \mathcal{E}_\bullet) = R^q(\underleftarrow{l}_{\{*\}} \circ \varpi_{U,\mathbb{N}*})(\mathcal{E}_\bullet|_{U_\bullet})$, since $R^q\varpi_{U,\mathbb{N}*}(\mathcal{E}_\bullet|_{U_\bullet}) = (H^q(U, \mathcal{E}_i))_{i\in\mathbb{N}}$ (see 7.1.3.15.2), then we get the spectral sequence

$$E_2^{p,q} = R^p \varprojlim_i (H^q(U, \mathcal{E}_i)) \Rightarrow H^{p+q}(U_\bullet, \mathcal{E}_\bullet). \tag{7.3.1.2.1}$$

For any $p > 1$ or $q > d$, $E_2^{p,q} = 0$. Hence, $H^n(U_\bullet, \mathcal{E}_\bullet) = 0$ for $n > d + 1$. This yields that the functor $\underleftarrow{l}_{X*}$ has cohomological dimension bounded by $d + 1$ on the category of abelian sheaves. In some special cases, we have better estimates (see 7.3.1.3.d).

Following 4.6.1.6.b, for any $\mathcal{E}_\bullet^\bullet \in K(\mathcal{D}_\bullet)$ (resp. for any $\mathcal{E}_\bullet^\bullet \in K^-(\mathcal{D}_\bullet)$), there exist a complex $\mathcal{I}_\bullet^\bullet \in K(\mathcal{D}_\bullet)$ (resp. $\mathcal{I}_\bullet^\bullet \in K^-(\mathcal{D}_\bullet)$) of $\underleftarrow{l}_{X,*}$-acyclic left $\mathcal{D}_\bullet$-modules and a quasi-isomorphism $\mathcal{E}_\bullet^\bullet \xrightarrow{\sim} \mathcal{I}_\bullet^\bullet$. Following [Sta22, 07K7–Lemma 13.31.2], Moreover, we have the isomorphism $\mathbb{R}\underleftarrow{l}_{X,*}\mathcal{E}_\bullet^\bullet \xrightarrow{\sim} \underleftarrow{l}_{X,*}\mathcal{I}_\bullet^\bullet$.

7.3.1.3. Let $\mathcal{E}_\bullet$ be a left $\mathcal{D}_\bullet$-module.

(a) Put $[\mathcal{E}]_i = \prod_{j\leq i} \mathcal{E}_j$. By composing the transition map $\mathcal{E}_{i+1} \to \mathcal{E}_i$ with the canonical inclusion $\mathcal{E}_i \to [\mathcal{E}]_i$, we get the morphism $d_i : \mathcal{E}_{i+1} \to [\mathcal{E}]_i$. This yields the transition maps $(d_i, id) : [\mathcal{E}]_{i+1} = \mathcal{E}_{i+1} \oplus [\mathcal{E}]_i \to [\mathcal{E}]_i$. We get the $\mathcal{D}_\bullet$-module $[\mathcal{E}]_\bullet$ endowed with the canonical monomorphism of $\mathcal{D}_\bullet$-modules $\mathcal{E}_\bullet \to [\mathcal{E}]_\bullet$. Remark that this is similar to the canonical embedding of $\mathcal{E}_\bullet$ into a flasque sheaf of $X_\bullet$ (see the construction 7.1.2.19.1).

Let $\mathcal{G}_\bullet := [\mathcal{E}]_\bullet/\mathcal{E}_\bullet$. Then the transition maps $\mathcal{G}_{i+1} = \prod_{j\leq i} \mathcal{E}_j \to \prod_{j\leq i-1} \mathcal{E}_j = \mathcal{G}_i$ are the canonical projections. Hence, we get the exact sequence

$$0 \to \mathcal{E}_\bullet \to [\mathcal{E}]_\bullet \to [\mathcal{E}]_\bullet/\mathcal{E}_\bullet \to 0, \tag{7.3.1.3.1}$$

where the transition maps of $[\mathcal{E}]_\bullet$ and $[\mathcal{E}]_\bullet/\mathcal{E}_\bullet$ are surjective in the category of presheaves.

(b) We say that $\mathcal{E}_\bullet$ is "locally Γ-acyclic" if there exists a basis of open sets $\mathfrak{B}$ of $|X|$ such that for any $\mathfrak{U} \in \mathfrak{B}$, $\mathcal{E}_i|_{U_i}$ is $\Gamma(U_i, -)$-acyclic for any $i \in \mathbb{N}$. For instance, if $\mathcal{E}_i$ are quasi-coherent $\mathcal{O}_{X_i}$-modules for any $i \in \mathbb{N}$, then we can choose $\mathfrak{B}$ equal to the affine open sets of $|\mathfrak{X}|$.

(c) We say that $\mathcal{E}_\bullet$ is "ML-flasque" if there exists a basis of open subsets $\mathfrak{B}$ of $|X|$ such that for any $\mathfrak{U} \in \mathfrak{B}$ and for any $i \in \mathbb{N}$ $\mathcal{E}_i|_{U_i}$ is $\Gamma(U_i, -)$-acyclic and the maps $\mathcal{E}_{i+1}(U_{i+1}) \to \mathcal{E}_i(U_i)$ induced by the transition maps are surjective. It follows from Mittag-Leffler criterion and from 7.3.1.2.1 that if $\mathcal{E}_\bullet$ is ML-flasque then it is acyclic for $\underleftarrow{l}_{X*}$. For instance, when $\mathcal{E}_i$ are quasi-coherent $\mathcal{O}_{X_i}$-modules and $\mathcal{E}_{i+1} \to \mathcal{E}_i$ are surjective as sheaves for any $i \in \mathbb{N}$ (e.g. when $\mathcal{E}_\bullet$ is of finite type : see 7.1.3.8), then we can choose $\mathfrak{B}$ equal to the affine opens of $|\mathfrak{X}|$.

(d) Suppose $\mathcal{E}_\bullet$ is locally Γ-acyclic. Hence so are $[\mathcal{E}]_\bullet$ and $[\mathcal{E}]_\bullet/\mathcal{E}_\bullet$, and they are therefore ML-flasque. Hence the exact sequence 7.3.1.3.1 gives a resolution of length 1 of $\mathcal{E}_\bullet$ with ML-flasque $\mathcal{D}_\bullet$-modules. This yields $R^n\underleftarrow{l}_{X*}\mathcal{E}_\bullet = 0$ for any $n \geq 2$.

Lemma 7.3.1.4. *Let $n \in \mathbb{N}$ and $\mathcal{E}^\bullet \in D^-(\mathcal{O}_{X_n})$. The following conditions are equivalent* *(a) The complex $\mathcal{E}^\bullet$ has $\mathcal{O}_{X_n}$-quasi-coherent cohomology.*

(b) The complex $\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}^\bullet$ has $\mathcal{O}_{X_0}$-quasi-coherent cohomology.

Proof. Since the converse is obvious, let us check $(b) \to (a)$. We proceed by induction on n. The case $n = 0$ is tautological. Suppose $n \geq 1$ and consider now the distinguished triangle

$$\mathcal{I}^n/\mathcal{I}^{n+1} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \mathcal{O}_{X_{n-1}} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}^\bullet \to +1. \qquad (7.3.1.4.1)$$

By induction hypothesis, the complex $\mathcal{O}_{X_{n-1}} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}^\bullet$ has $\mathcal{O}_{X_{n-1}}$-quasi-coherent cohomology, which implies it has $\mathcal{O}_{X_n}$-quasi-coherent cohomology. Since $\mathcal{I}^n/\mathcal{I}^{n+1} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{I}^n/\mathcal{I}^{n+1} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_0}} (\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}^\bullet)$, then similarly we get that the left term of the exact triangle 7.3.1.4.1 has $\mathcal{O}_{X_n}$-quasi-coherent cohomology. □

Definition 7.3.1.5. Let $\mathcal{E}^\bullet \in D^-(\mathcal{O}_{\mathfrak{X}})$ be a complex of $\mathcal{O}_{\mathfrak{X}}$-modules. We say $\mathcal{E}^\bullet$ is $\mathcal{O}_{\mathfrak{X}}$-*quasi-coherent* if the following conditions are satisfied:

(a) The complex $\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\bullet$ has $\mathcal{O}_{X_0}$-quasi-coherent cohomology.

(b) The canonical morphism

$$\mathcal{E}^\bullet \to \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{L}\underleftarrow{l}^*_X \mathcal{E}^\bullet) \qquad (7.3.1.5.1)$$

is an isomorphism.

We denote by $D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ the full subcategory of $D^-(\mathcal{O}_{\mathfrak{X}})$ consisting of quasi-coherent complexes. It follows from the Lemma 7.3.1.4 that if $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ then $\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_n})$ for any integer $n \in \mathbb{N}$. This yields this notion of quasi-coherence does not depend on the choice of the ideal of definition of $\mathfrak{X}$. Moreover, this is straightforward by definition that $D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ is a triangulated subcategory of $D^-(\mathcal{O}_{\mathfrak{X}})$. Finally, the quasi-coherence is a local notion: 1) a complex $\mathcal{E}^\bullet$ of $D^-(\mathcal{O}_{\mathfrak{X}})$ is quasi-coherent if and only if there exists an covering $(\mathfrak{U}_i)_i$ by open subset of $\mathfrak{X}$ such that $\mathcal{E}^\bullet|_{\mathfrak{U}_i}$ is quasi-coherent for any i ; 2) if $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ then $\mathcal{E}^\bullet|_{\mathfrak{U}} \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{U}})$ for any open set $\mathfrak{U}$ of $\mathfrak{X}$.

Definition 7.3.1.6. Let $\mathcal{E}$ be an $\mathcal{O}_{\mathfrak{X}}$-module. We say that $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{X}}$-quasi-coherent if $\mathcal{E} \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$.

If $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ then this is false in general that for any $n \in \mathbb{Z}$ the $\mathcal{O}_{\mathfrak{X}}$-modules $H^n(\mathcal{E}^\bullet)$ are quasi-coherent.

The following proposition gives examples of quasi-coherent modules and will be useful to check 8.7.4.2.

Proposition 7.3.1.7. *Let $\mathcal{E}$ be an $\mathcal{O}_{\mathfrak{X}}$-module which is tor-independent with $\mathcal{O}_{X_n}$ for any $n \in \mathbb{N}$ (e.g. when $\mathcal{I} = (p)$ and $\mathcal{E}$ is p-torsion free). Then the following properties are equivalent:*

(a) The module $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{X}}$-quasi-coherent ;

(b) The module $\mathcal{E}$ is pseudo-quasi-coherent (see Definition 7.2.3.5).

Proof. By hypothesis, we have the isomorphism $\mathcal{E}_\bullet := \underleftarrow{l}^*_X \mathcal{E} \xleftarrow{\sim} \mathbb{L}\underleftarrow{l}^*_X \mathcal{E}$. Suppose $\mathcal{E}$ is pseudo-quasi-coherent. Since $\mathcal{E}_\bullet$ is ML-flasque (see definition 7.3.1.3.(c)), then it is acyclic for $\underleftarrow{l}_{X*}$ and we get the canonical isomorphism $\underleftarrow{l}_{X*}(\mathcal{E}_\bullet) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet)$. Hence, since the canonical morphism $\mathcal{E} \to \underleftarrow{l}_{X*}(\underleftarrow{l}^*_X \mathcal{E})$ is an isomorphism, then so is

$$\mathcal{E} \to \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{L}\underleftarrow{l}^*_X \mathcal{E}),$$

i.e. $\mathcal{E}$ is quasi-coherent. Conversely, if the morphism $\mathcal{E} \to \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{L}\underleftarrow{l}^*_X \mathcal{E})$ is an isomorphism, then so is $\mathcal{E} \to \underleftarrow{l}_{X*}(\underleftarrow{l}^*_X \mathcal{E})$, and $\mathcal{E}$ is pseudo quasi-coherent.

□

Lemma 7.3.1.8. *Let A be a noetherian commutative ring, I an ideal of A, $A_i = A/I^{i+1}$, M be an A-module of finite type. Then, for any $q \geq 1$, there exists a large enough integer k such that the homomorphisms*

$$\mathrm{tor}^A_q(A_{i+k}, M) \to \mathrm{tor}^A_q(A_i, M)$$

are null for any $i \in \mathbb{N}$.

Proof. Fix $q \geq 1$. Let $L_\bullet$ a resolution of M by free A-modules of finite type, d be its differential. It follows from the Artin-Rees lemma applied to the inclusion $d(L_q) \subset L_{q-1}$ that there exists $k \in \mathbb{N}$ such that for any $i \in \mathbb{N}$ we have the inclusion

$$d(L_q) \cap I^{i+1+k} L_{q-1} \subset I^{i+1} d(L_q) = d(I^{i+1} L_q). \tag{7.3.1.8.1}$$

For such fixed k, it remains to check that the canonical homomorphisms $A_{i+k} \otimes_A L_q \to A_i \otimes_A L_q$ send $Z_q(A_{i+k} \otimes_A L_\bullet)$ to $B_q(A_i \otimes_A L_\bullet)$ for any $i \in \mathbb{N}$. Let $x \in Z_q(A_{i+k} \otimes_A L_\bullet)$. Choose $y \in L_q$ which gives x via the surjection $L_q \to A_{i+k} \otimes_A L_q$. Since $d(y) \in d(L_q) \cap I^{i+1+k} L_{q-1}$, then from 7.3.1.8.1, there exists $z \in I^{i+1} L_q$ such that $d(y) = d(z)$. Since $H^q(L_\bullet) = 0$ (recall $q \geq 1$), then there exists $t \in L_{q+1}$ such that $y = z + d(t)$. We compute $\overline{x} = d(\overline{t})$, where $\overline{x}$ (resp. $\overline{t}$) is the image of x (resp. t) in $A_i \otimes_A L_q$ (resp. $A_i \otimes_A L_{q+1}$). □

Proposition 7.3.1.9. *Let $n \in \mathbb{N}$, and $\mathcal{E}^\bullet \in D^-(\mathcal{O}_{X_n})$. Then $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ if and only if $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_n})$.*

Proof. a) Suppose $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$. Since $\mathcal{E}^\bullet \in D^-(\mathcal{O}_{X_n})$, the canonical morphism $\mathcal{E} \to \mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ has the canonical retraction $\mathcal{O}_{X_n} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \to \mathcal{E}$. Hence, $H^i(\mathcal{E})$ is a direct summand of $H^i(\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E})$.

b) Conversely, suppose $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_n})$. Since $\mathcal{O}_{X_n} \in D^-_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{X}})$, then $\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n} \in D^-_{\mathrm{coh}}(\mathcal{O}_{X_n})$. Since $\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\sim} (\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} \mathcal{E}$, this yields that $\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_n})$. Hence, $\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\sim} \mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_n}} (\mathcal{O}_{X_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_0})$. It remains to check that the morphism 7.3.1.5.1 is an isomorphism. Since the fact that the homomorphism 7.3.1.5.1 is an isomorphism is local, we can suppose $\mathfrak{X}$ affine and noetherian and that $\mathcal{E}$ is a quotient of a free $\mathcal{O}_{X_n}$-module Since the functors $\mathbb{R}\underset{\leftarrow}{l}_{X*}$ and $\mathbb{L}\underset{\leftarrow}{l}^*_X$ are way-out left (see 7.3.1.2 for the first one), then we reduce to the case where the complex $\mathcal{E}^\bullet$ is in fact a quasi-coherent $\mathcal{O}_{X_n}$-module (use the left version of [Har66, I.7.1.(ii)]). and then that $\mathcal{E}^\bullet$ is a free $\mathcal{O}_{X_n}$-module (use the left version of [Har66, I.7.1.(iv)]). For any $i \geq n$, the morphism $\mathcal{E} \to \mathcal{O}_{X_i} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ is an isomorphism. Hence the morphism $\mathcal{E} \to \varprojlim_i \mathcal{O}_{X_i} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ is an isomorphism. Consider the spectral sequence

$$E_2^{p,q} = R^p \varprojlim_i (\mathcal{T}or_q^{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{X_i}, \mathcal{E})) = R^p \underset{\leftarrow}{l}_{X*}(L^q \underset{\leftarrow}{l}^*_X(\mathcal{E}) \Rightarrow H^n(\mathbb{R}\underset{\leftarrow}{l}_{X*}(\mathbb{L}\underset{\leftarrow}{l}^*_X \mathcal{E})).$$

Since $\underset{\leftarrow}{l}^*_X \mathcal{E}$ is ML-flasque (see definition 7.3.1.3), then $E_2^{p,0} = 0$ for any $p > 0$. Hence, since this implies that $E_2^{p,q} = 0$ for any $q \geq 1$, this is sufficient to check that the projective system $(\mathcal{T}or_q^{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{X_i}, \mathcal{E}))_{i \in \mathbb{N}}$ is essentially null for any $q \geq 1$. By freeness, we reduce to check it to the case where $\mathcal{E} = \mathcal{O}_{X_n}$. Since $\mathcal{T}or_q^{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{X_i}, \mathcal{O}_{X_n})$ are coherent $\mathcal{O}_{X_i}$-modules such that $\Gamma(X_i, \mathcal{T}or_q^{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{X_i}, \mathcal{O}_{X_n})) = \mathrm{tor}_q^A(A_i, A_n)$, where $\mathfrak{X} = \mathrm{Spf}\, A$ and $X_i = \mathrm{Spec}\, A_i$, this follows from Lemma 7.3.1.8. □

Definition 7.3.1.10. Let $\mathcal{E}^\bullet_\bullet \in D^-(\mathcal{O}_{X_\bullet})$ be a complex of $\mathcal{O}_{X_\bullet}$-modules. We say $\mathcal{E}^\bullet_\bullet$ is $\mathcal{O}_{X_\bullet}$-*quasi-coherent* if the following conditions hold:

(a) The complex $\mathcal{E}^\bullet_0$ is in $D^-_{\mathrm{qc}}(\mathcal{O}_{X_0})$.

(b) The canonical morphisms

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_{i+1}}} \mathcal{E}^\bullet_{i+1} \to \mathcal{E}^\bullet_i \tag{7.3.1.10.1}$$

are isomorphisms for all i.

We denote by $D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ the full subcategory of $D^-(\mathcal{O}_{X_\bullet})$ consisting of quasi-coherent complexes. It follows from 7.3.1.4 that for any $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ and any $i \in \mathbb{N}$, we have $\mathcal{E}^\bullet_i \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_i})$. Moreover, this is straightforward by definition that $D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ is a triangulated subcategory of $D^-(\mathcal{O}_{X_\bullet})$. Finally, the quasi-coherence is a local notion: a complex $\mathcal{E}^\bullet$ of $D^-(\mathcal{O}_{X_\bullet})$ is quasi-coherent if and only if there exists an covering $(\mathfrak{U}_i)_i$ by open subset of $\mathfrak{X}$ such that $\mathcal{E}^\bullet|_{\mathfrak{U}_i}$ is quasi-coherent for any i ; if $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ then $\mathcal{E}^\bullet|_{\mathfrak{U}} \in D^-_{\mathrm{qc}}(\mathcal{O}_{U_\bullet})$ for any open set $\mathfrak{U}$ of $\mathfrak{X}$.

Remark 7.3.1.11. Let $\mathcal{E}^\bullet_\bullet \in D^-(\mathcal{O}_{X_\bullet})$. Choose a morphism $\mathcal{P}^\bullet_\bullet \xrightarrow{\sim} \mathcal{E}^\bullet_\bullet$ in $C^-(\mathcal{O}_{X_\bullet})$ which is a quasi-isomorphism and such that $\mathcal{P}^n_\bullet$ are $\mathcal{O}_{X_\bullet}$-flat for any $n \in \mathbb{Z}$. Since $\mathcal{P}^n_i$ are $\mathcal{O}_{X_i}$-flat for any $i \in \mathbb{N}$ and $n \in \mathbb{Z}$, then the fact that the morphisms 7.3.1.10.1 are isomorphisms is equivalent to the fact that the canonical morphisms

$$\mathcal{O}_{X_i} \otimes_{\mathcal{O}_{X_{i+1}}} \mathcal{P}^\bullet_{i+1} \to \mathcal{P}^\bullet_i \tag{7.3.1.11.1}$$

are isomorphisms for all i.

7.3.1.12. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. We denote by $D^-_{\mathrm{qc}}({}^*\mathcal{D})$ (resp. $D^-_{\mathrm{qc}}({}^*\mathcal{D}_\bullet)$) the full subcategory of $D^-({}^*\mathcal{D})$ (resp. $D^-({}^*\mathcal{D}_\bullet)$) consisting of complexes of $\mathcal{O}_{\mathfrak{X}}$-quasi-coherent modules (resp. $\mathcal{O}_\bullet$-quasi-coherent modules) in the sense of 7.3.1.5 (resp. 7.3.1.10). In other words, $D^-_{\mathrm{qc}}({}^*\mathcal{D}) = D^-(\mathcal{D}) \cap D^-_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ (resp. $D^-_{\mathrm{qc}}({}^*\mathcal{D}_\bullet) = D^-(\mathcal{D}_\bullet) \cap D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$). The objects of $D^-_{\mathrm{qc}}({}^*\mathcal{D})$ (resp. $D^-_{\mathrm{qc}}({}^*\mathcal{D}_\bullet)$) are called quasi-coherent complexes of $D^-({}^*\mathcal{D})$ (resp. $D^-({}^*\mathcal{D}_\bullet)$). Moreover, $D^-_{\mathrm{qc}}({}^*\mathcal{D})$ is a triangulated subcategory of $D^-(\mathcal{D})$ and $D^-_{\mathrm{qc}}({}^*\mathcal{D}_\bullet)$ is a triangulated subcategory of $D^-(\mathcal{D}_\bullet)$. The full subcategory of $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}_\bullet)$ consisting of quasi-coherent complexes of finite tor dimension on $\mathcal{D}_\bullet$ is denoted by $D_{\mathrm{qc,tdf}}({}^*\mathcal{D}_\bullet)$.

Proposition 7.3.1.13. *Let $\mathcal{E}^\bullet_\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}_\bullet)$ be a complex such that the canonical morphisms*

$$\mathcal{D}_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1} \to \mathcal{E}^\bullet_i \tag{7.3.1.13.1}$$

are isomorphisms for any i. Then $\mathcal{E}^\bullet_\bullet$ has tor-amplitude in $[a,b]$ over $\mathcal{D}_\bullet$ if and only if $\mathcal{E}^\bullet_0$ has tor-amplitude in $[a,b]$ over $\mathcal{D}_0$.

Proof. From 7.1.3.6, we reduce to check that if $\mathcal{E}^\bullet_0$ has tor-amplitude in $[a,b]$ over $\mathcal{D}_0$ then $\mathcal{E}^\bullet_i$ has tor-amplitude in $[a,b]$ over $\mathcal{D}_i$ for any $i \in \mathbb{N}$. We proceed by induction on $i \in \mathbb{N}$. The case $i = 0$ is obvious. Let us suppose $i \geq 1$ and the property valid for $j < i$. Let $\mathcal{M}$ be a right $\mathcal{D}_i$-module. We have to prove that $H^n(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i) = 0$ for any $n \notin [a,b]$. For any $0 \leq j \leq i-1$, if $\mathcal{N}_j$ is a right $\mathcal{D}_j$-module, using the induction hypothesis we get

$$\mathcal{N}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \xrightarrow{\sim} \mathcal{N}_j \otimes^{\mathbb{L}}_{\mathcal{D}_j} \left(\mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i\right) \xrightarrow{\sim} \mathcal{N}_j \otimes^{\mathbb{L}}_{\mathcal{D}_j} \mathcal{E}^\bullet_j \in D^{[a,b]}(\mathcal{D}_i).$$

We have the exact sequence $0 \to \mathcal{M}\mathcal{I}^i \to \mathcal{M} \to \mathcal{M}/\mathcal{M}\mathcal{I}^i \to 0$, which yields the exact triangle

$$\mathcal{M}\mathcal{I}^i \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{M}/\mathcal{M}\mathcal{I}^i \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to +1.$$

Since $\mathcal{M}/\mathcal{M}\mathcal{I}^i$ is a right $\mathcal{D}_{i-1}$-module and $\mathcal{M}\mathcal{I}^i$ is a right $\mathcal{D}_0$-module, we are done. □

7.3.1.14. Suppose that $\mathcal{D}_\bullet$ satisfied the condition 7.3.1.10.b for the left structure. For any $\mathcal{E}^\bullet_\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}_\bullet)$, the canonical homomorphisms

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_{i+1}}} \mathcal{E}^\bullet_{i+1} \to \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1} \tag{7.3.1.14.1}$$

are then isomorphisms for all $i \in \mathbb{N}$. Hence, $\mathcal{E}^\bullet_\bullet$ satisfies the isomorphisms 7.3.1.13.1 if and only if it satisfies the condition 7.3.1.10.b. Finally, choose a morphism $\mathcal{P}^\bullet_\bullet \xrightarrow{\sim} \mathcal{E}^\bullet_\bullet$ in $C^-({}^{\mathrm{l}}\mathcal{D}_\bullet)$ which is a quasi-isomorphism and such that $\mathcal{P}^n_\bullet$ are $\mathcal{D}_{X_\bullet}$-flat for any $n \in \mathbb{Z}$. Since $\mathcal{P}^n_i$ are $\mathcal{D}_i$-flat for any $i \in \mathbb{N}$ and $n \in \mathbb{Z}$, then the fact that the morphisms 7.3.1.13.1 are isomorphisms is equivalent to the fact that the canonical morphisms

$$\mathcal{D}_i \otimes_{\mathcal{D}_{i+1}} \mathcal{P}^\bullet_{i+1} \to \mathcal{P}^\bullet_i \tag{7.3.1.14.2}$$

are isomorphisms for all i. We have similar results for right $\mathcal{D}_\bullet$-modules.

Proposition 7.3.1.15. *Suppose $\mathcal{D}$ is left coherent. Let $\mathcal{E}^\bullet_\bullet$ be a complex of $D^-({}^{\mathrm{l}}\mathcal{D}_\bullet)$. Then $\mathcal{E}^\bullet_\bullet$ is an object of $D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$ (resp. $D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$) if and only if the following conditions hold.*

(a) The complex $\mathcal{E}^\bullet_0$ is in $D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_0)$ (resp. $D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}_0)$).

(b) $\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1} \to \mathcal{E}^\bullet_i$ are isomorphisms for all i.

Proof. By using 7.1.3.12 and 7.3.1.13, we reduce to check the non-respective case. Suppose $\mathcal{E}^\bullet_\bullet$ satisfies both conditions (a) and (b). From 7.1.3.13, it remains to check that for any $i \in \mathbb{N}$, $\mathcal{E}^\bullet_i \in D_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_i)$. The case $i = 0$ is obvious. Let us suppose $i \geq 1$ and the property valid for $j < i$. We have the exact sequence of $\mathcal{D}_i$-bimodules $0 \to \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \to \mathcal{D}_i \to \mathcal{D}_{i-1} \to 0$, which yields the exact triangle of left $\mathcal{D}_i$-modules

$$\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_{i-1} \to +1. \tag{7.3.1.15.1}$$

By induction hypothesis, $\mathcal{E}^\bullet_{i-1} \in D_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_{i-1})$ and then $\mathcal{E}^\bullet_{i-1} \subset D_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_i)$ (use 4.1.3.2 and 4.6.1.7). Since

$$\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \xrightarrow{\sim} \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \left(\mathcal{D}_0 \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i\right) \xrightarrow{\sim} \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \mathcal{E}^\bullet_0, \tag{7.3.1.15.2}$$

then we reduce to check that $\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \mathcal{E}_0^\bullet \in D_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_0)$. By devissage, we can suppose that $\mathcal{E}_0^\bullet$ is a coherent $\mathcal{D}_0$-module denoted by $\mathcal{E}_0$. We have to check $tor_n^{\mathcal{D}_0}(\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}, \mathcal{E}_0)$ is a coherent $\mathcal{D}_0$-module for any integer n. Since this is local we can suppose there exist a resolution $\mathcal{P}_0^\bullet$ of $\mathcal{E}_0$ such that for any $l \geq -n-1$, $\mathcal{P}_0^{\mathrm{l}}$ is a finite free $\mathcal{D}_0$-module. Hence $\mathrm{tor}_n^{\mathcal{D}_0}(\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}, \mathcal{E}_0) \xrightarrow{\sim} \operatorname{Ker}\beta/\operatorname{Im}\alpha$, where α and β are $\mathcal{D}_0$-linear (for the left structure) morphisms of the form:

$$(\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D})^r \xrightarrow{\alpha} (\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D})^s \xrightarrow{\beta} (\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D})^t.$$

Hence, we are done. □

Proposition 7.3.1.16. *If $\mathcal{D}$ (resp. $\mathcal{D}_\bullet$) is left quasi-coherent i.e. satisfies the conditions of 7.3.1.5 (resp. 7.3.1.10 for the left structure of $\mathcal{O}_{\mathfrak{X}}$-module (resp. $\mathcal{O}_{X_\bullet}$-module), then the category $D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D})$ (resp. $D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$) is a triangulated subcategory $D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$ (resp. $D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$).*

Proof. First suppose $\mathcal{D}$ is left quasi-coherent. Let $\mathcal{E}^\bullet \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D})$. Let $n \in \mathbb{Z}$. Since the fact that $H^n(\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\bullet)$ is $\mathcal{O}_{X_0}$-quasi-coherent is local, we can suppose there exists a bounded complex of free $\mathcal{D}$-modules of finite type $\mathcal{L}^\bullet$ and a morphism $\mathcal{L}^\bullet \to \mathcal{E}^\bullet$ of $C^-({}^{\mathrm{l}}\mathcal{D})$ which is an $n-1$-isomorphism in $D^-({}^{\mathrm{l}}\mathcal{D})$, i.e. whose cone is acyclic in degree $\geq n-1$. This implies that the canonical morphism $H^n(\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^\bullet) \to H^n(\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\bullet)$ is an isomorphism. Using the first property of quasi-coherence (see 7.3.1.5.a), we get $\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^m \in D^-_{\mathrm{qc}}(\mathcal{O}_{X_0})$. By using some spectral sequence this yields the $\mathcal{O}_{X_0}$-quasi-coherence of $H^n(\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^\bullet)$. It remains to check that the canonical morphism

$$\mathcal{E}^\bullet \to \mathbb{R}\underset{\leftarrow}{l}_{X*}(\mathbb{L}\underset{\leftarrow}{l}^*_X \mathcal{E}^\bullet) \tag{7.3.1.16.1}$$

is an isomorphism. Since this is local, since these functors are way-out left (see 7.3.1.2 for the functor $\mathbb{R}\underset{\leftarrow}{l}_{X*}$), we reduce to the case where $\mathcal{E}^\bullet$ is a free left $\mathcal{D}$-module of finite type (use a "local left" version of [Har66, I.7.1.(iv)] i.e. replacing way-out right by way-out left and replacing the existence of an injection by the local existence of a surjection), which follows from the quasi-coherence of $\mathcal{D}$. Hence, we are done.

Let us now treat the respective case, i.e. suppose $\mathcal{D}_\bullet$ is left quasi-coherent. This is a consequence of 7.3.1.15 and 7.3.1.14. □

The non-respective cases of the following lemma 7.3.1.17 and proposition 7.3.1.18 is an erratum of [BO78, B.2.1] given to the author by Berthelot.

Lemma 7.3.1.17. *Suppose $\mathfrak{X}$ affine and set $D_i := \Gamma(\mathfrak{X}, \mathcal{D}_i)$ for any $i \in \mathbb{N}$. Let $E_\bullet^\bullet \in C^-({}^{\mathrm{l}}D_\bullet)$ such that the transition map $E_{i+1}^\bullet \to E_i^\bullet$ is an epimorphism of $C^-({}^{\mathrm{l}}D_i)$ for all $i \in \mathbb{N}$ (resp. such that $E_i^\bullet \in C^-_{\mathrm{coh}}({}^{\mathrm{l}}D_i)$ and the transition map $E_{i+1}^\bullet \to E_i^\bullet$ is an epimorphism of $C^-({}^{\mathrm{l}}D_i)$, for all $i \in \mathbb{N}$). Then there exists $F_\bullet^\bullet \in C^-(D_\bullet)$ such that each F_i^n is a free (resp. free of finite type) left D_i-module, the transition map $F_i^\bullet \to F_{i-1}^\bullet$ is an epimorphism of $C^-({}^{\mathrm{l}}D_i)$ for all $i \geq 1$ and there exists an epimorphism (resp. a morphism) $F_\bullet^\bullet \to E_\bullet^\bullet$ of $C^-(D_\bullet)$ which is a quasi-isomorphism.*

Proof. 0) It is standard (e.g. see the dual version of the proof of [Har66, I.4.6.1.(i)], or [Sta22, 0EWZ-15.62.19] or [Sta22, 066E-15.62.18] in the respective case) to find a $F_0^\bullet \in C^-(D_0)$ consisting of free left D_0-modules (resp. free left D_0-modules of finite type) and an epimorphism (resp. morphism) $F_0^\bullet \to E_0^\bullet$ of $C^-(D_0)$ which is moreover a quasi-isomorphism.

We prove the Lemma by constructing inductively on $i \geq 1$ a complex a) $F_i^\bullet \in C^-(D_i)$ consisting of free left D_i-modules (resp. free left D_i-modules of finite type), b) an epimorphism (resp. a morphism) $F_i^\bullet \to E_i^\bullet$ of $C^-(D_i)$ which is also a quasi-isomorphism and c) an epimorphism $F_i^\bullet \to F_{i-1}^\bullet$ of $C^-(D_i)$ making commutative the diagram of $C^-(D_{i-1})$

$$\begin{array}{ccc} F_i^\bullet & \longrightarrow & E_i^\bullet \\ \downarrow & & \downarrow \\ F_{i-1}^\bullet & \longrightarrow & E_{i-1}^\bullet. \end{array} \tag{7.3.1.17.1}$$

Let $i \geq 1$ and suppose constructed such a $F_j^\bullet$, $F_j^\bullet \to E_j^\bullet$, $F_j^\bullet \to F_{j-1}^\bullet$ for any $j \leq i-1$ (when $j = 0$, we forget the second map).

1) i) Let us focus in the non-respective case. As above, let $P_i^\bullet \in C^-(D_i)$ be a complex consisting of free left D_i-modules and an epimorphism $a\colon P_i^\bullet \to E_i^\bullet$ of $C^-(D_i)$ which is moreover a quasi-isomorphism.

Since P_i^n are projective for any $n \in \mathbb{Z}$, since our complexes are bounded above, since $F_{i-1}^\bullet \to E_{i-1}^\bullet$ is an epimorphism of $C^-(D_i)$, then following [Sta22, 0649-Lemma 13.19.6] there exists a morphism $b\colon P_i^\bullet \to F_{i-1}^\bullet$ of $C^-(D_i)$ making commutative in $C^-(D_i)$ the diagram

$$\begin{array}{ccc} P_i^\bullet & \xrightarrow{a} & E_i^\bullet \\ {\scriptstyle b}\downarrow & & \downarrow \\ F_{i-1}^\bullet & \longrightarrow & E_{i-1}^\bullet . \end{array} \tag{7.3.1.17.2}$$

ii) In the respective case, since $C^-(D_i)$ is an abelian category, we can consider the complex $R_i^\bullet := F_{i-1}^\bullet \times_{E_{i-1}^\bullet} E_i^\bullet$ of $C^-(D_i)$ (we do not write the forgetful functor $C^-(D_{i-1}) \to C^-(D_i)$). Let $\alpha\colon R_i^\bullet \to E_i^\bullet$ an $\beta\colon R_i^\bullet \to F_{i-1}^\bullet$ be the structural morphisms of $C^-(D_i)$. Since $E_i^\bullet \to E_{i-1}^\bullet$ is an epimorphism of $C^-(D_i)$, then this is well known (e.g., using the five lemma, consider the morphism of long exact sequences induced by the morphism of exact sequences of $C^-(D_i)$ constructed from the morphism of epimorphisms given by $(R_i^\bullet \to F_{i-1}^\bullet) \to (E_i^\bullet \to E_{i-1}^\bullet))$ that α is a quasi-isomorphism. Since $R_i^\bullet \in D^-_{\mathrm{coh}}(D_i)$, then we can choose $P_i^\bullet \in C^-(D_i)$ a complex consisting of free left D_i-modules of finite type endowed with a quasi-isomorphism $\gamma\colon P_i^\bullet \to R_i^\bullet$ of $C^-(D_i)$. Then by setting $a := \alpha \circ \gamma$ and $b := \beta \circ \gamma$, we get a commutative diagram such as 7.3.1.17.2 whose horizontal morphisms are quasi-isomorphisms.

2) We add an acyclic complex to $P_i^\bullet$ to construct $F_i^\bullet$ so that $F_i^\bullet \to F_{i-1}^\bullet$ is an epimorphism as follows.

First, let $K_{i-1}^\bullet$ be the mapping cone of the identity of $F_{i-1}^\bullet$ and $\psi\colon K_{i-1}^\bullet \to F_{i-1}^\bullet[1]$ be the canonical epimorphism of $C(D_{i-1})$. Since $K_{i-1}^\bullet$ is an acyclic complex of projective left D_{i-1}-modules and is bounded above, then by descending induction on the degree n we can check that each K_{i-1}^n is a direct sum (in the category of D_{i-1}-modules) of the form $Q_{i-1}^{n-1} \oplus Q_{i-1}^n$ and the boundary map of $K_{i-1}^n \to K_{i-1}^{n+1}$ are the maps given by the formula $d^n(q^{n-1}, q^n) = (q^n, 0)$. Now, choose for any integer n a free left D_i-module (resp. a free left D_i-module of finite type) Q_i^n such that Q_{i-1}^n is a quotient of Q_i^n (in the category of left D_i-modules) and take $K_i^n := Q_i^{n-1} \oplus Q_i^n$ with boundary maps given as above by the formula $d^n(q^{n-1}, q^n) = (q^n, 0)$. We observe that $K_i^\bullet$ is an acyclic complex of free D_i-modules (resp. free D_i-modules of finite type), still bounded above, and endowed with an epimorphism $\psi'\colon K_i^\bullet \to K_{i-1}^\bullet$ of $C(D_i)$. Now set $F_i^\bullet := P_i^\bullet \oplus K_i^\bullet[-1]$ and $\delta := (b, \psi \circ \psi'[-1])\colon F_i^\bullet \to F_{i-1}^\bullet$. Since ψ and ψ' are surjective, then so is δ.

3) We construct now the quasi-isomorphism $F_i^\bullet \to E_i^\bullet$ making commutative the diagram 7.3.1.17.1 in the next two steps.

i) We remark that, for any $G \in C(D_i)$, the map

$$\prod_n \mathrm{Hom}_{D_i}(Q_i^n, G^n) \to \mathrm{Hom}_{C(D_i)}(K_i^\bullet, G^\bullet)$$

given by $h^\bullet \mapsto \{(d_G^{n-1} \circ h^{n-1}, h^n)\colon Q_i^{n-1} \oplus Q_i^n \to G^n \ ; \ n \in \mathbb{Z}\}$ is a bijection.

ii) Since K_i^n are projective for any $n \in \mathbb{Z}$, since our complexes are bounded above, since $E_i^\bullet \to E_{i-1}^\bullet$ is surjective, then following [Sta22, 0649-Lemma 13.19.6] we can find a morphism of complexes $\phi\colon K_i^\bullet[-1] \to E_i^\bullet$ of $C^-(D_i)$ making commutative in $C^-(D_i)$ the diagram

$$\begin{array}{ccc} K_i^\bullet[-1] & \xrightarrow{\phi} & E_i^\bullet \\ {\scriptstyle \psi \circ \psi'[-1]}\downarrow & & \downarrow \\ F_{i-1}^\bullet & \longrightarrow & E_{i-1}^\bullet . \end{array}$$

Hence, we get the morphism $c := (a, \phi)\colon F_i^\bullet \to E_i^\bullet$ making commutative the diagram 7.3.1.17.1 . Finally, since a is a quasi-isomorphism and $K_i^\bullet[-1]$ is acyclic, then c is also a quasi-isomorphism.

□

Proposition 7.3.1.18. *Suppose $\mathfrak{X}$ affine and set $D_i := \Gamma(\mathfrak{X}, D_i)$ for any $i \in \mathbb{N}$. Let $E_\bullet^\bullet \in D^-({}^lD_\bullet)$ (resp. let $E_\bullet^\bullet \in D^-_{\mathrm{coh}}({}^lD_\bullet)$). Then there exists $F_\bullet^\bullet \in D^-(D_\bullet)$, such that each F_i^n is a free (resp. free of finite type) left D_i-module and each map $F_i^n \to F_{i-1}^n$ is surjective, and an isomorphism of $D^-(D_\bullet)$ of the form $E_\bullet^\bullet \xrightarrow{\sim} F_\bullet^\bullet$.*

Proof. By using ML-flasque resolution (of length 1) of 7.3.1.3.1, for each $E_n^\bullet$ and then taking the total complex, we get a morphism in $C^-({}^lD_\bullet)$ of the form $E_\bullet^\bullet \to G_\bullet^\bullet$ which is a quasi-isomorphism and $G_\bullet^\bullet$ is a complex satisfying the condition of Lemma 7.3.1.17. Hence, we are done.

□

Remark 7.3.1.19. Suppose $\mathfrak{X}$ affine. Then the functors $\Gamma(\mathfrak{X}, -)$ and $\mathcal{D}_\bullet \otimes_{D_\bullet} -$ are exact canonically quasi-inverse equivalences of categories between $C^-({}^lD_\bullet)$ and the full subcategory of $C^-({}^l\mathcal{D}_\bullet)$ consisting of complexes $\mathcal{E}_\bullet^\bullet$ such that, for all $n \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\mathcal{E}_i^n$ is $\mathcal{O}_{X_i}$-quasi-coherent. Via these equivalences, we can get a sheafified version of 7.3.1.17 and 7.3.1.18.

Lemma 7.3.1.20. *Let $\mathcal{E}_\bullet^\bullet \in C^-({}^l\mathcal{D}_\bullet)$ be a complex such that $\mathcal{E}_\bullet^n$ is $\mathcal{D}_\bullet$-flat, $\mathcal{E}_i^n$ is a quasi-coherent $\mathcal{O}_{X_i}$-module and the transition maps $\mathcal{E}_i^n \to \mathcal{E}_{i-1}^n$ are surjective for any $n \in \mathbb{Z}$ and $i \in \mathbb{N}$.*

(a) The complex $\underleftarrow{l}_{X*}(\mathcal{E}_\bullet^\bullet)$ consists of flat left $\mathcal{D}$-modules.

(b) For any coherent right $\mathcal{D}$-module $\mathcal{M}$, the natural map of $C^-({}^l\mathcal{D})$:

$$\mathcal{M} \otimes_{\mathcal{D}} \underleftarrow{l}_{X*}(\mathcal{E}_\bullet^\bullet) \to \underleftarrow{l}_{X*}(\underleftarrow{l}_X^{-1}(\mathcal{M}) \otimes_{\underleftarrow{l}_X^{-1}(\mathcal{D})} \mathcal{E}_\bullet^\bullet)$$

is an isomorphism.

Proof. 0) We can suppose $\mathcal{E}_\bullet$ is a flat left $\mathcal{D}_\bullet$-module such that $\mathcal{E}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module and the transition maps $\mathcal{E}_i \to \mathcal{E}_{i-1}$ are surjective for any $i \in \mathbb{N}$. Since the lemma is local, we can suppose $\mathfrak{X}$ is affine. Set $D_\bullet := \Gamma(\mathfrak{X}, \mathcal{D}_\bullet)$ and $D := \Gamma(\mathfrak{X}, \mathcal{D})$, $E_\bullet := \Gamma(\mathfrak{X}, \mathcal{E}_\bullet)$ and $M := \Gamma(\mathfrak{X}, \mathcal{M})$. Since the functor $\Gamma(\mathfrak{X}, -)$ commutes with inverse limits, we reduce to check that the left D-module $\underleftarrow{l}_*(E_\bullet)$ is flat and that the natural map

$$M \otimes_D \underleftarrow{l}_*(E_\bullet) \to \underleftarrow{l}_*(M \otimes_D E_\bullet), \qquad (7.3.1.20.1)$$

where by abuse of notation we removed to indicate the functor $\underleftarrow{l}^{-1}$, is an isomorphism. Since the map 7.3.1.20.1 is an isomorphism when M is free of finite type, then by using the five lemma and the right exactness of the functor $M \mapsto M \otimes_D \underleftarrow{l}_*(E_\bullet)$, it is sufficient to prove that $F\colon M \mapsto \underleftarrow{l}_*(M \otimes_D E_\bullet)$ is exact on the category of right D-modules.

1) We prove that the inverse system $\mathrm{tor}_1^D(M, E_\bullet)$ is essentially zero. Fix $i \in \mathbb{N}$.

i) Since E_i is D_i-flat, then $\mathrm{tor}_1^D(M, E_i) \xrightarrow{\sim} \mathrm{tor}_1^D(M, D_i) \otimes_{D_i} E_i$. Hence, it suffices to prove that the right $D_\bullet$-module $T_\bullet(M) := \mathrm{tor}_1^D(M, D_\bullet)$ is a an essentially zero system. By using the exact sequence of D-bimodules $0 \to I^{i+1}D \to D \to D_i \to 0$, we get the inclusion of right D-modules $T_i(M) \subset M \otimes_D I^{i+1}D$. By applying Artin-Rees with respect to the I-adic topology for this inclusion we can find an integer ν such that

$$T_i(M) \cap ((M \otimes_D I^{i+1}D)I^{m+\nu}) \subset T_i(M)I^m, \text{ for all } m \in \mathbb{N}.$$

Since $T_i(M)$ is a right D_i-module, then $T_i(M)I^{i+1} = 0$. This yields $T_i(M) \cap ((M \otimes_D I^{i+1}D)I^{m+\nu}) = 0$ for $m \rangle i + \nu$.

ii) By using the commutative diagram of D-bimodules,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{i+j+1}D & \longrightarrow & D & \longrightarrow & D_{i+j} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & I^{i+1}D & \longrightarrow & D & \longrightarrow & D_i & \longrightarrow & 0 \end{array}$$

whose horizontal sequences are exact, we get the commutative square

$$\begin{array}{ccc} T_{i+j}(M) & \longrightarrow & M \otimes_D I^{i+j+1}D \\ \downarrow & & \downarrow \\ T_i(M) & \longrightarrow & M \otimes_D I^{i+1}D. \end{array}$$

Hence, from the part 1.i) of the proof, the image $T_{i+j}(M) \to T_i(M)$ is zero for $j > i + \nu$.

2) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of right D-modules. Let $S_\bullet$ be the image of $\mathrm{tor}_1^D(M'', E_\bullet) \to M' \otimes_D E_\bullet$ and $R_\bullet$ be the image of $M' \otimes_D E_\bullet \to M \otimes_D E_\bullet$. We get the exact sequence

$0 \to S_\bullet \to M' \otimes_D E_\bullet \to R_\bullet \to 0$ where $S_\bullet$ is essentially zero. Hence, $\underset{\leftarrow *}{l}(M' \otimes_D E_\bullet) \to \underset{\leftarrow *}{l}(R_\bullet)$ is an isomorphism. From the exact sequence, $0 \to R_\bullet \to M \otimes_D E_\bullet \to M'' \otimes_D E_\bullet \to 0$, since $R_\bullet$ is flasque, then using again Mittag-Leffler condition we get the exact sequence $0 \to \underset{\leftarrow *}{l}(R_\bullet) \to \underset{\leftarrow *}{l}(M \otimes_D E_\bullet) \to \underset{\leftarrow *}{l}(M'' \otimes_D E_\bullet) \to 0$. Hence, we are done. □

Notation 7.3.1.21. Let $i \in \mathbb{N}$. Let $v_i \colon \mathbb{N}_{\geq i} \to \mathbb{N}$ be the inclusion. Then, with notation 7.1.2.4.1, we get the topoi morphism $v_{i,X} \colon \mathrm{Top}(X)_{\mathbb{N}_{\geq i}} \to \mathrm{Top}(X)_{\mathbb{N}}$. This yields the ringed topoi morphisms $v_{i,X} \colon (\mathrm{Top}(X)_{\mathbb{N}_{\geq i}}, v_{i,X}^{-1}\mathcal{D}_\bullet) \to (\mathrm{Top}(X)_{\mathbb{N}}, \mathcal{D}_\bullet)$ and $\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l} \colon (\mathrm{Top}(X)_{\mathbb{N}_{\geq i}}, v_{i,X}^{-1}\mathcal{D}_\bullet) \to (\mathrm{Top}(X), \mathcal{D})$ (see notation 7.1.2.5). We have moreover the isomorphism $\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l} \circ v_{i,X}^{-1} \xrightarrow{\sim} \underset{\leftarrow X,\mathbb{N}*}{l}$. For any $\mathcal{E}_\bullet^\bullet \in D({}^{\mathrm{l}}\mathcal{D}_\bullet)$, this yields the isomorphism

$$\mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l} \circ v_{i,X}^{-1}(\mathcal{E}_\bullet^\bullet) \xrightarrow{\sim} \mathbb{R}\underset{\leftarrow X,\mathbb{N}*}{l}(\mathcal{E}_\bullet^\bullet). \tag{7.3.1.21.1}$$

7.3.1.22. Let $i \in \mathbb{N}$ and keep notation of 7.3.1.21. Recall that $\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i)$ is the inverse system of sheaves of rings whose transition maps are the identity of $\mathcal{D}_i$. By using the projection formula of 4.6.5.8 in the case of the ringed topoi morphism $\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l} \colon (\mathrm{Top}(X)_{\mathbb{N}_{\geq i}}, \underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i)) \to (\mathrm{Top}(X), \mathcal{D})$, for any $\mathcal{M}_i^\bullet \in D^-({}^{\mathrm{r}}\mathcal{D}_i)$ and $\mathcal{E}_\bullet^\bullet \in D^-({}^{\mathrm{l}}\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i))$, we get the projection morphism

$$\mathcal{M}_i^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}(\mathcal{E}_\bullet^\bullet) \to \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}\left(\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{M}_i^\bullet) \otimes^{\mathbb{L}}_{\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i)} \mathcal{E}_\bullet^\bullet\right). \tag{7.3.1.22.1}$$

The morphism 7.3.1.22.1 is an isomorphism when $\mathcal{M}_i^\bullet \in D^-_{\mathrm{coh}}({}^{\mathrm{r}}\mathcal{D}_i)$. Indeed, even if this is not contained in 4.6.5.8, this can be checked as follow. Since the functor $\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}$ has bounded cohomological dimension (this is checked similarly to 7.3.1.2), then both functors $\mathcal{M}^\bullet \mapsto \mathcal{M}_i^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}(\mathcal{E}_\bullet^\bullet)$ and $\mathcal{M}^\bullet \mapsto \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}\left(\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{M}_i^\bullet) \otimes^{\mathbb{L}}_{\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i)} \mathcal{E}_\bullet^\bullet\right)$ are way-out left. Then it follows from lemma [Har66, I.7.1.(ii) and (iv)] that we can reduce to the case where $\mathcal{M}^\bullet$ is a free $\mathcal{D}$-module of finite type. Since this latter case is obvious, we are done.

The following proposition 7.3.1.23 is a sheafified version of [BO78, B.2.3].

Proposition 7.3.1.23. *Let $i \in \mathbb{N}$, $\mathcal{M}_i^\bullet \in D^-({}^{\mathrm{r}}\mathcal{D}_i)$ and $\mathcal{E}_\bullet^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}_\bullet)$ such that, for all $n \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\mathcal{E}_j^n$ is $\mathcal{O}_{X_j}$-quasi-coherent. With notation 7.3.1.21, there exists a natural map*

$$\mathcal{M}_i^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathbb{R}\underset{\leftarrow X*}{l}(\mathcal{E}_\bullet^\bullet) \to \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}\left(\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{M}_i^\bullet) \otimes^{\mathbb{L}}_{v_{i,X}^{-1}\mathcal{D}_\bullet} v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet\right), \tag{7.3.1.23.1}$$

which is an isomorphism when $\mathcal{M}_i^\bullet$ has coherent cohomology.

Proof. I) First suppose $\mathcal{M}_i^\bullet = \mathcal{D}_i$.

1) Let us construct the morphism 7.3.1.23.1. By choosing a resolution of $\mathcal{E}_\bullet^\bullet$ by flat left $\mathcal{D}_\bullet$-modules, we get the morphism $v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet \to \underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i) \otimes_{v_{i,X}^{-1}\mathcal{D}_\bullet} v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet$ of $D^-({}^{\mathrm{l}}v_{i,X}^{-1}\mathcal{D}_\bullet)$. This yields the morphism

$$\mathbb{R}\underset{\leftarrow X*}{l}(\mathcal{E}_\bullet^\bullet) \underset{7.3.1.21.1}{\xrightarrow{\sim}} \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l} v_{i,X}^{-1}(\mathcal{E}_\bullet^\bullet) \to \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}(\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i) \otimes^{\mathbb{L}}_{v_{i,X}^{-1}\mathcal{D}_\bullet} v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet) \tag{7.3.1.23.2}$$

of $D^-({}^{\mathrm{l}}\mathcal{D})$. In fact, since $\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i) \otimes^{\mathbb{L}}_{v_{i,X}^{-1}\mathcal{D}_\bullet} v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet$ is an object of $D^-({}^{\mathrm{l}}\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i))$ then the right term of 7.3.1.23.2 is an object of $D^-({}^{\mathrm{l}}\mathcal{D}_i)$. Hence, we get by extension via the ring homomorphism $\mathcal{D} \to \mathcal{D}_i$ the desired morphism

$$\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathbb{R}\underset{\leftarrow X*}{l}(\mathcal{E}_\bullet^\bullet) \to \mathbb{R}\underset{\leftarrow X,\mathbb{N}_{\geq i}*}{l}\left(\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i) \otimes^{\mathbb{L}}_{v_{i,X}^{-1}\mathcal{D}_\bullet} v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet\right). \tag{7.3.1.23.3}$$

Recall, the map 7.3.1.23.3 is constructed from 7.3.1.23.2 by solving $\mathbb{R}\underset{\leftarrow X*}{l}(\mathcal{E}_\bullet^\bullet)$ with flat left $\mathcal{D}$-modules.

2) Let us check 7.3.1.23.3 is an isomorphism. Since this is local, we can suppose $\mathfrak{X}$ is affine. Following 7.3.1.18 (and the remark 7.3.1.19), we can therefore suppose $\mathcal{E}_\bullet^\bullet$ is such that each $\mathcal{E}_i^n$ is a free left $\mathcal{D}_i$-module and each map $\mathcal{E}_i^n \to \mathcal{E}_{i-1}^n$ is surjective. By flatness of $\mathcal{E}_\bullet^\bullet$, we get $\underset{\leftarrow X,\mathbb{N}_{\geq i}}{l}^{-1}(\mathcal{D}_i) \otimes^{\mathbb{L}}_{v_{i,X}^{-1}\mathcal{D}_\bullet} v_{i,X}^{-1}\mathcal{E}_\bullet^\bullet =$

$\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet$. Moreover, since for any $n \in \mathbb{Z}$ the complexes $\mathcal{E}^n_\bullet$ and $\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet$ are complexes of quasi-coherent modules with surjective transitive maps, then the morphism 7.3.1.23.2 is the canonical morphism $\underset{\leftarrow}{l}_{X*}(\mathcal{E}^\bullet_\bullet) \to \underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*}(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet)$. It follows from 7.3.1.20.a that the complex $\underset{\leftarrow}{l}_{X*}(\mathcal{E}^\bullet_\bullet)$ consist of flat $\mathcal{D}$-modules and then the morphism 7.3.1.23.3 is the natural morphism $\mathcal{D}_i\otimes_{\mathcal{D}}\underset{\leftarrow}{l}_{X*}(\mathcal{E}^\bullet_\bullet) \to \underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*}(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet)$. Since $\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes_{\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D})} v^{-1}_{i,X}\mathcal{D}_\bullet \xrightarrow{\sim} \underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)$, then the morphism 7.3.1.23.3 corresponds to the natural map

$$\begin{aligned}
&\mathcal{D}_i \otimes_{\mathcal{D}} \underset{\leftarrow}{l}_{X*}(\mathcal{E}^\bullet_\bullet) \to \underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*}(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes_{\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D})} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet)\\
&\xrightarrow{\sim} \underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*} v^{-1}_{i,X}(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}}(\mathcal{D}_i)\otimes_{\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}}(\mathcal{D})} \mathcal{E}^\bullet_\bullet) \xrightarrow{\sim} \underset{\leftarrow}{l}_{X,\mathbb{N}*}(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}}(\mathcal{D}_i)\otimes_{\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}}(\mathcal{D})} \mathcal{E}^\bullet_\bullet).
\end{aligned}$$

Since $\mathcal{D}_i$ is a coherent $\mathcal{D}$-module, we conclude by using 7.3.1.20.b.

II) Let us check the general case. By applying the functor $\mathcal{M}^\bullet_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i} -$ to the isomorphism 7.3.1.23.3, we get the first isomorphism

$$\begin{aligned}
\mathcal{M}^\bullet_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i} \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathbb{R}\underset{\leftarrow}{l}_{X*}(\mathcal{E}^\bullet_\bullet) &\xrightarrow{\sim} \mathcal{M}^\bullet_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_i} \mathbb{R}\underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*}\left(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes^{\mathbb{L}}_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet\right) \qquad (7.3.1.23.4)\\
&\xrightarrow{7.3.1.22.1} \mathbb{R}\underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*}\left(\mathcal{M}^\bullet_i \overset{\mathbb{L}}{\otimes}_{\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}\mathcal{D}_i} \underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i)\otimes^{\mathbb{L}}_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet\right)\\
&\xrightarrow{\sim} \mathbb{R}\underset{\leftarrow}{l}_{X,\mathbb{N}_{\geq i}*}\left(\underset{\leftarrow}{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{M}^\bullet_i)\otimes^{\mathbb{L}}_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet\right).
\end{aligned}$$

When $\mathcal{M}^\bullet_i$ has coherent cohomology, the morphism 7.3.1.22.1 is an isomorphism and we are done.

□

7.3.2 Equivalence of categories between both notions of quasi-coherence

Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ satisfying the following conditions.

(a) On any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ideal $\mathcal{I}$ has a family of generators with image in the center of $\mathcal{D}$.

(b) The canonical homomorphism $\mathcal{D} \to \varprojlim_i \mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}$ is an isomorphism.

(c) For any affine open $\mathfrak{U} \subseteq \mathfrak{X}$, the ring $\Gamma(\mathfrak{U}, \mathcal{D})$ is left and right noetherian.

(d) The projective system $\mathcal{D}_\bullet = (\mathcal{D}_i)_{i\in\mathbb{N}}$ with $\mathcal{D}_i := \mathcal{D}/\mathcal{I}^{i+1}\mathcal{D}$ is left and right quasi-coherent in the sense of 7.3.1.10.

It follows from 7.2.3.3 that $\mathcal{D}$ is a coherent sheaf of rings.

Examples 7.3.2.1. We will use essentially in this book the following cases. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension.

(a) Let $m \in \mathbb{N}$. Let $\mathcal{B}_{\mathfrak{X}}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible structure of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module and satisfying the hypotheses of 7.3.2. Since $\mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is $\mathcal{B}_{\mathfrak{X}}$-flat (for both structures), then $\mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies also 7.3.2.

(b) Suppose $\mathfrak{X}$ is moreover p-torsion free (see 3.3.1.12 for some example). Let Z be a divisor of $\mathfrak{X}^\sharp \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi\mathcal{V})$. Let $m, r \in \mathbb{N}$ be two integers such that p^{m+1} divides r. Then following 8.7.4.2 the sheaves $\mathcal{B}_{\mathfrak{X}}(Z, r)$ and $\mathcal{B}_{\mathfrak{X}}(Z, r)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies 7.3.2.

7.3.2.2. We denote by $\underset{\leftarrow}{l}_X : (X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$ the morphism of ringed topoi. From [Sta22, 07A6], this yields the functors $\mathbb{R}\underset{\leftarrow}{l}_{X*} : D(\mathcal{D}_\bullet) \to D(\mathcal{D})$ and $\mathbb{L}\underset{\leftarrow}{l}^*_X : D(\mathcal{D}) \to D(\mathcal{D}_\bullet)$ which are adjoint:

$$\mathrm{Hom}_{D(\mathcal{D}_\bullet)}(\mathbb{L}\underset{\leftarrow}{l}^*_X(\mathcal{E}^\bullet), \mathcal{F}^\bullet_\bullet) = \mathrm{Hom}_{D(\mathcal{D})}(\mathcal{E}^\bullet, \mathbb{R}\underset{\leftarrow}{l}_{X*}(\mathcal{F}^\bullet_\bullet)) \qquad (7.3.2.2.1)$$

for any $\mathcal{E}^\bullet \in D(\mathcal{D})$ and any $\mathcal{F}^\bullet_\bullet \in D(\mathcal{D}_\bullet)$. Remark that following 7.3.2.3.d just below, the functors $\mathbb{L}\underset{\leftarrow}{l}^*_X$ when $\underset{\leftarrow}{l}_X$ is either the morphism of ringed topoi $(X_\bullet, \mathcal{O}_{X_\bullet}) \to (|\mathcal{X}|, \mathcal{O}_{\mathfrak{X}})$ or $(X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$ are canonically isomorphic. So this is harmless to switch from one to the other.

7.3.2.3. We have the following properties.

(a) Since $\mathcal{D}_\bullet$ is ML-flasque (see definition 7.3.1.3), then the canonical map $\mathcal{D} \to \mathbb{R}\underset{\leftarrow}{l}_{X*}(\mathcal{D}_\bullet)$ is an isomorphism.

(b) By using 7.3.1.23 in the case where $\mathcal{D} = \mathcal{O}_{\mathfrak{X}}$, for any $i \in \mathbb{N}$ we get the canonical isomorphism

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbb{R}\varprojlim_{j \geq i} \mathcal{D}_j \xrightarrow{\sim} \mathbb{R}\varprojlim_{j \geq i} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_j}} \mathcal{D}_j). \tag{7.3.2.3.1}$$

Since $\mathcal{D}_\bullet$ is left quasi-coherent and $\mathcal{D} \xrightarrow{\sim} \mathbb{R}\varprojlim_{j\geq i} \mathcal{D}_j$, this means that the canonical morphism $\mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D} \to \mathcal{D}_\bullet$ is an isomorphism. This yields the canonical isomorphism of functors $D(\mathcal{O}_{\mathcal{D}}) \to D(\mathcal{D}_\bullet)$ and $\mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathfrak{X}} - \xrightarrow{\sim} \mathcal{D}_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} -$, the isomorphism of functors $D(\mathcal{O}_{\mathcal{D}}) \to D(\mathcal{D}_i)$ of the form $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} - \xrightarrow{\sim} \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} -$, and both isomorphisms of functors $D(\mathcal{O}_{\mathfrak{X}}) \to D(\mathcal{D})$ of the form $- \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}_\bullet \xrightarrow{\sim} - \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}$ and $- \otimes_{\mathcal{O}_{X_i}} \mathcal{D}_i \xrightarrow{\sim} - \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}$.

Similarly, we check the canonical morphism $\mathcal{D} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_\bullet} \to \mathcal{D}_\bullet$ is an isomorphism.

(c) It follows from (a) and (b) that the sheaf $\mathcal{D}$ is left and right quasi-coherent in the sense of 7.3.1.5.

(d) For any $\mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D})$, following (b), the canonical morphism $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\bullet \to \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet$ is an isomorphism and similarly for complexes of right $\mathcal{D}$-modules. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. With notation 7.3.1.12, this yields the canonical morphism $\mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\underset{\leftarrow}{l}^{-1}_X \mathcal{O}_{\mathfrak{X}}} \underset{\leftarrow}{l}^{-1}_X(-) \to \mathcal{D}_\bullet \otimes^{\mathbb{L}}_{\underset{\leftarrow}{l}^{-1}_X \mathcal{D}} \underset{\leftarrow}{l}^{-1}_X(-)$ is an isomorphism of functors. Hence, we get $\mathbb{L}\underset{\leftarrow}{l}^*_X : D^-_{\mathrm{qc}}({}^*\mathcal{D}) \to D^-_{\mathrm{qc}}({}^*\mathcal{D}_\bullet)$.

(e) By applying the functor $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}$ to the exact sequence $0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{X_n} \to \mathcal{O}_{X_{n-1}} \to 0$, since we have also the exact sequence $0 \to \mathcal{I}^n\mathcal{D}/\mathcal{I}^{n+1}\mathcal{D} \to \mathcal{D}_n \to \mathcal{D}_{n-1} \to 0$, then we get the first isomorphism $\mathcal{I}^n\mathcal{D}/\mathcal{I}^{n+1}\mathcal{D} \xrightarrow{\sim} \mathcal{I}^n/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D} \xrightarrow{\sim} \mathcal{I}^n/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_{X_0}} \mathcal{D}_0$, the second one is checked at (b).

Similarly, we get $\mathcal{D}_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{I}^n/\mathcal{I}^{n+1} \xrightarrow{\sim} \mathcal{I}^n\mathcal{D}/\mathcal{I}^{n+1}\mathcal{D}$.

7.3.2.4. Suppose $\mathfrak{X}$ is affine and keep notation 7.2.3.8, e.g. $\varpi_X : |\mathfrak{X}| \to \{*\}$ is the continuous morphism from the topological space of $\mathfrak{X}$ to a punctual set. Let $E^\bullet_\bullet \in D^-(D_\bullet)$. Following [BO78, B.4], we say $E^\bullet_\bullet$ is *$D_\bullet$-quasi-consistent* if and only if the canonical morphisms

$$D_i \otimes^{\mathbb{L}}_{D_{i+1}} E^\bullet_{i+1} \to E^\bullet_i \tag{7.3.2.4.1}$$

are isomorphisms for all $i \in \mathbb{N}$. We denote by $D^-_{\mathrm{qc}}(D_\bullet)$ the full subcategory of $D^-(D_\bullet)$ consisting of quasi-consistent complexes. Moreover, this is straightforward by definition that $D^-_{\mathrm{qc}}(D_\bullet)$ is a triangulated subcategory of $D^-(D_\bullet)$.

7.3.2.5. We keep notation 7.3.2.4. Let $E^\bullet_\bullet$ be a complex of $D^-({}^{\mathrm{l}}D_\bullet)$. Since D is noetherian, then similarly to 7.3.1.15 $E^\bullet_\bullet$ is an object of $D^-_{\mathrm{coh}}({}^{\mathrm{l}}D_\bullet)$ (resp. $D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}D_\bullet)$) if and only if the following conditions hold.

(a) The complex $E^\bullet_0$ is in $D^-_{\mathrm{coh}}({}^{\mathrm{l}}D_0)$ (resp. $D_{\mathrm{perf}}({}^{\mathrm{l}}D_0)$).

(b) $E^\bullet_\bullet$ is quasi-consistent.

Proposition 7.3.2.6. *With notations and hypotheses of 7.3.2.4, the functors $\mathbb{R}\varpi_{X_\bullet *}$ and $\varpi^*_{X_\bullet}$ induce canonically quasi-inverse equivalences of categories between $D^-_{\mathrm{qc}}(D_\bullet)$ and $D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$ (resp. $D^-_{\mathrm{coh}}(D_\bullet)$ and $D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$).*

Proof. I) Let us check the quasi-coherent case. 1) Let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$. We prove in this step that the canonical morphism $\varpi^*_{X,\bullet}\mathbb{R}\varpi_{X,\bullet*}(\mathcal{E}^\bullet_\bullet) \to \mathcal{E}^\bullet_\bullet$ is an isomorphism.

It follows from 7.1.3.16 there exist a complex $\mathcal{I}^\bullet_\bullet \in K^-(\mathcal{D}_\bullet)$ consisting of $\varpi_{X,\bullet*}$-acyclic $\mathcal{D}_\bullet$-modules and a quasi-isomorphism $\mathcal{E}^\bullet_\bullet \xrightarrow{\sim} \mathcal{I}^\bullet_\bullet$. We get the isomorphism $\varpi_{X,\bullet*}(\mathcal{I}^\bullet_\bullet) \xrightarrow{\sim} \mathbb{R}\varpi_{X,\bullet*}(\mathcal{I}^\bullet_\bullet)$ of $D^-(\mathcal{D}_\bullet)$.

Since $\mathcal{I}^\bullet_\bullet \in D^-(\mathcal{D}_\bullet)$ then $I^\bullet_\bullet := \varpi_{X,\bullet *}(\mathcal{I}^\bullet_\bullet) \in D^-(D_\bullet)$. Since $\varpi_{X,\bullet *}$ has bounded cohomological dimension, then following 4.6.1.6.1 we have the spectral sequence $E_2^{r,s} = R^r\varpi_{X,\bullet *}(H^s(\mathcal{I}^\bullet_\bullet)) \Rightarrow H^{r+s}\mathbb{R}\varpi_{X,\bullet *}(\mathcal{I}^\bullet_\bullet)$. Since $H^s(\mathcal{I}^\bullet_i)$ is $\mathcal{O}_{X_i}$-quasi-coherent for all $s \in \mathbb{Z}$ and $i \in \mathbb{N}$, then $H^s(\mathcal{I}^\bullet_\bullet)$ is $\varpi_{X,\bullet *}$-acyclic (see 7.1.3.15.2). Hence, the canonical map $H^n(I^\bullet_\bullet) = H^n\varpi_{X,\bullet *}(\mathcal{I}^\bullet_\bullet) \to \varpi_{X,\bullet *}H^n(\mathcal{I}^\bullet_\bullet)$ (coming from the above spectral sequence) is an isomorphism. This yields that the canonical map $\mathcal{D}_\bullet \otimes_{D_\bullet} I^\bullet_\bullet \to \mathcal{I}^\bullet_\bullet$ is a quasi-isomorphism and we are done.

2) Let $E^\bullet_\bullet \in D^-(D_\bullet)$. We check in this step that $E^\bullet_\bullet \in D^-_{\mathrm{qc}}(D_\bullet)$ if and only if $\mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$.

Choose a complex $P^\bullet_\bullet \in K^-(D_\bullet)$ consisting of flat $D_\bullet$-modules together with a quasi-isomorphism $P^\bullet_\bullet \to E^\bullet_\bullet$ of $K^-(D_\bullet)$. Then $E^\bullet_\bullet$ is quasi-consistent if and only if the canonical morphism

$$D_i \otimes_{D_{i+1}} P^\bullet_{i+1} \to P^\bullet_i \tag{7.3.2.6.1}$$

is a quasi-isomorphism of $K^-(D_i)$ for all $i \in \mathbb{N}$. Set $\mathcal{P}^\bullet_\bullet := \mathcal{D}_\bullet \otimes_{D_\bullet} P^\bullet_\bullet$. Then $\mathcal{P}^\bullet_\bullet$ is a complex of $K^-(\mathcal{D}_\bullet)$ consisting of flat $\mathcal{D}_\bullet$-modules together with a quasi-isomorphism $\mathcal{P}^\bullet_\bullet \to \mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet$ of $K^-(\mathcal{D}_\bullet)$. Moreover, $\mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet$ is quasi-coherent if and only if the canonical morphism

$$\mathcal{D}_i \otimes_{\mathcal{D}_{i+1}} \mathcal{P}^\bullet_{i+1} \to \mathcal{P}^\bullet_i \tag{7.3.2.6.2}$$

is a quasi-isomorphism of $K^-(\mathcal{D}_i)$ for all i. By applying the functor $\mathcal{D}_i \otimes_{D_i} -$ to 7.3.2.6.1 we get the second morphism:

$$\mathcal{D}_i \otimes_{\mathcal{D}_{i+1}} \mathcal{P}^\bullet_{i+1} \xrightarrow{\sim} \mathcal{D}_i \otimes_{D_i} (D_i \otimes_{D_{i+1}} P^\bullet_{i+1}) \to \mathcal{D}_i \otimes_{D_i} P^\bullet_i = \mathcal{P}^\bullet_i, \tag{7.3.2.6.3}$$

whose composition is 7.3.2.6.2. If 7.3.2.6.1 is a quasi-isomorphism, then so is 7.3.2.6.2. Conversely, following 4.6.1.7.c, the functor $\mathcal{D}_i \otimes_{D_i} -\colon D^-(D_i) \to D^-_{\mathrm{qc}}(\mathcal{D}_i)$ is fully faithful which implies the following property: if 7.3.2.6.2 is an isomorphism of $D^-(\mathcal{D}_i)$, then 7.3.2.6.1 is an isomorphism of $D^-(D_i)$. Hence, we are done.

3) Since $\varpi^*_{X,\bullet} = \mathcal{D}_\bullet \otimes_{D_\bullet} -$, then it follows from 1) and 2) that the functor $\mathbb{R}\varpi_{X_\bullet *}$ factors through $D^-_{\mathrm{qc}}(\mathcal{D}_\bullet) \to D^-_{\mathrm{qc}}(D_\bullet)$.

4) Let $E^\bullet_\bullet \in D^-(D_\bullet)$. Similarly to the step 1), since $\mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet$ is a complex of $\varpi_{X,\bullet *}$-acyclic modules, we can check that the canonical map $E^\bullet_\bullet \to \mathbb{R}\varpi_{X_\bullet *}(\mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet)$ is a quasi-isomorphism.

II) Let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$. It follows from 7.1.3.15.2, that we have the isomorphism $(\mathbb{R}\varpi_{X,\bullet *}(\mathcal{E}^\bullet_\bullet))_0 \xrightarrow{\sim} \mathbb{R}\varpi_{X*}(\mathcal{E}^\bullet_0)$. It follows from 4.6.1.7 that $\mathbb{R}\varpi_{X*}(\mathcal{E}^\bullet_0) = \mathbb{R}\Gamma(X, \mathcal{E}_0) \in D^-_{\mathrm{coh}}(D_0)$. By using 7.3.2.5, this yields $\mathbb{R}\varpi_{X,\bullet *}(\mathcal{E}^\bullet_\bullet) \in D^-_{\mathrm{coh}}(D_\bullet)$. Moreover, since the pseudo-coherence is stable under derived pullbacks (see [Sta22, 08H4-21.42.3]), then for any $E^\bullet_\bullet \in D^-_{\mathrm{coh}}(D_\bullet)$, we have $\varpi^*_{X_\bullet}(E^\bullet_\bullet) \in D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$. Hence, we get the quasi-inverse equivalence for coherent complexes from the part I) of the proof.

□

Proposition 7.3.2.7. *Let $\mathcal{E}^\bullet_\bullet \in D^-(\mathcal{D}_\bullet)$. The following conditions are equivalent:*

(a) $\mathcal{E}^\bullet_\bullet$ is quasi-coherent ;

(b) For any affine open set $\mathfrak{U} = \mathrm{Spf}\, A \subset \mathfrak{X}$, there exists an inverse system of complexes $P^\bullet_\bullet \in C^-(\Gamma(\mathfrak{U}, \mathcal{D}_\bullet))$, endowed with an isomorphism $\mathcal{D}_\bullet|_{\mathfrak{U}} \otimes_{\Gamma(\mathfrak{U},\mathcal{D}_\bullet)} P^\bullet_\bullet \xrightarrow{\sim} \mathcal{E}^\bullet_\bullet|_{\mathfrak{U}}$ in $D^-(\mathcal{D}_\bullet|_{\mathfrak{U}})$, such that P^n_i is a free left $\Gamma(\mathfrak{U}, \mathcal{D}_i)$-module, the transition maps $P^n_{i+1} \to P^n_i$ are surjective, and the canonical morphisms

$$\Gamma(\mathfrak{U}, \mathcal{D}_i) \otimes_{\Gamma(\mathfrak{U},\mathcal{D}_{i+1})} P^\bullet_{i+1} \to P^\bullet_i \tag{7.3.2.7.1}$$

are quasi-isomorphisms.

(c) The complex $\mathcal{E}^\bullet_0$ belongs to $D^-_{\mathrm{qc}}(\mathcal{D}_0)$, and the canonical morphism

$$\mathbb{L}\underleftarrow{l}^*_X \circ \mathbb{R}\underleftarrow{l}_{X*}\mathcal{E}^\bullet_\bullet \to \mathcal{E}^\bullet_\bullet \tag{7.3.2.7.2}$$

is an isomorphism.

Proof. Let us check $(c) \Rightarrow (a)$. Suppose $\mathcal{E}^\bullet_\bullet$ satisfies the conditions of (c). First remark that for any $\mathcal{F}^\bullet \in D^-(\mathcal{D})$ and $\mathcal{F}^\bullet_\bullet := \mathbb{L}\underleftarrow{l}^*_X(\mathcal{F}^\bullet)$, the canonical morphism

$$\mathcal{D}_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{i+1}} \mathcal{F}^\bullet_{i+1} \to \mathcal{F}^\bullet_i \tag{7.3.2.7.3}$$

is an isomorphism. Hence, the condition 7.3.1.10.b holds for $\mathcal{E}^\bullet_\bullet$ (use 7.3.1.14).

Now let us check $(a) \Rightarrow (b)$. Suppose $\mathcal{E}^\bullet_\bullet$ is quasi-coherent. We can suppose $\mathfrak{X}$ is affine. It follows from 7.3.2.6 that we reduce to the case where there exists $E^\bullet_\bullet \in D^-_{\mathrm{qc}}(D_\bullet)$ such that $\mathcal{E}^\bullet_\bullet = \mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet$. Following 7.3.1.18, there exists $P^\bullet_\bullet \in C^-(D_\bullet)$, such that each P^n_i is a free left D_i-module and each map $P^n_i \to P^n_{i-1}$ is surjective, and an isomorphism of $D^-(D_\bullet)$ of the form $P^\bullet_\bullet \xrightarrow{\sim} E^\bullet_\bullet$. Hence, $\mathcal{D}_\bullet \otimes_{D_\bullet} P^\bullet_\bullet \xrightarrow{\sim} \mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet$. is quasi-coherent. Since P^n_i are flat left D_i-modules, then it follows the part 2) of the proof of 7.3.2.6, that the canonical morphisms 7.3.2.7.1 are isomorphisms.

It remains to check $(b) \Rightarrow (c)$. Since this is local, we can suppose $\mathfrak{X}$ affine and $\mathcal{E}^\bullet_\bullet = \mathcal{D}_\bullet \otimes_{D_\bullet} P^\bullet_\bullet$, with $P^\bullet_\bullet \in C^-(D_\bullet)$, such that each P^n_i is a free left D_i-module and each map $P^n_i \to P^n_{i-1}$ is surjective and such that the homomorphism 7.3.2.7.1 are isomorphisms. Following 7.3.1.23.1, we have the canonical isomorphism $\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}^\bullet_\bullet) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X,\mathbb{N}_{\geq i}*}(\underleftarrow{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i) \otimes^{\mathbb{L}}_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet)$. Since the morphisms 7.3.2.7.1 are isomorphism, then so are the canonical morphisms $\mathcal{D}_j \otimes_{\mathcal{D}_{j+1}} \mathcal{E}^\bullet_{j+1} \to \mathcal{E}^\bullet_j$ for any $j \geq i$. Since $\mathcal{E}^\bullet_\bullet$ is a flat $\mathcal{D}_\bullet$-module, this implies that the canonical morphism $\underleftarrow{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{D}_i) \otimes^{\mathbb{L}}_{v^{-1}_{i,X}\mathcal{D}_\bullet} v^{-1}_{i,X}\mathcal{E}^\bullet_\bullet \to \underleftarrow{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{E}^\bullet_i)$ is an isomorphism. Since $\mathbb{R}\underleftarrow{l}_{X,\mathbb{N}_{\geq i}*}(\underleftarrow{l}^{-1}_{X,\mathbb{N}_{\geq i}}(\mathcal{E}^\bullet_i)) \xrightarrow{\sim} \mathcal{E}^\bullet_i$, this yields the canonical isomorphism $\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}^\bullet_\bullet) \xrightarrow{\sim} \mathcal{E}^\bullet_i$. We conclude by using 7.3.1.14.

□

Lemma 7.3.2.8. *Suppose $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has right tor dimension $\leq d$ on $\mathcal{D}$ (resp. $\mathcal{D}_0$) for some integer d.*

(a) Then $\mathcal{D}_\bullet$ has right tor dimension $\leq 2d$ on $\underleftarrow{l}^{-1}_X\mathcal{D}$.

(b) Suppose either $\mathcal{D}$ is commutative or $\mathcal{O}_{X_0}$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{O}_{\mathfrak{X}}$) has right tor dimension $\leq d$ on $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{X_0}$). Then $\mathcal{D}_j$ has right tor dimension $\leq 2d$ on $\mathcal{D}_i$, for any integers $0 \leq j \leq i$.

Proof. a) Let $\mathcal{E}$ be a left $\mathcal{D}$-module. We have the exact sequence of $\mathcal{D}_i$-bimodules $0 \to \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \to \mathcal{D}_i \to \mathcal{D}_{i-1} \to 0$ which yields the exact triangle of left $\mathcal{D}_i$-modules

$$\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \left(\mathcal{D}_0 \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}\right) \to \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E} \to \mathcal{D}_{i-1} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E} \to +1. \tag{7.3.2.8.1}$$

For any $i \in \mathbb{N}$, the term $\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \left(\mathcal{D}_0 \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}\right)$ has tor-amplitude in $[-2d, 0]$. Hence, we prove by induction on $i \in \mathbb{N}$ that $\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}$ has tor-amplitude in $[-2d, 0]$.

b) By using 7.3.1.14.1, we reduce to the case where $\mathcal{D}$ is commutative (and then we do not have to bother with solving rings to be able to get derived tensor products in the derived categories of bimodules). Let $0 \leq j \leq i$ be two integers and $\mathcal{E}_i$ be a left $\mathcal{D}_i$-module. The canonical morphism $\mathcal{D}_i \to \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{D}_i$ has the retractation $\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{D}_i \to \mathcal{D}_i$. Hence, $\mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}_i$ is a direct summand of $\mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}_i} (\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{D}_i) \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}_i \xrightarrow{\sim} \mathcal{D}_j \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}_i$, which has tor-amplitude in $[-2d, 0]$. □

Corollary 7.3.2.9. *Suppose $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$). Then $\mathbb{L}\underleftarrow{l}^*_X$ preserves the boundedness, i.e. we get the functor $\mathbb{L}\underleftarrow{l}^*_X \colon D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}) \to D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_\bullet)$.*

Proof. This is straightforward from 7.3.2.8. □

Corollary 7.3.2.10. *The functors $\mathbb{R}\underleftarrow{l}_{X*}$ and $\mathbb{L}\underleftarrow{l}^*_X$ induce canonically quasi-inverse equivalences of categories between $D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$ and $D^-_{\mathrm{qc}}(\mathcal{D})$.*

Moreover if $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$) then the functors $\mathbb{R}\underleftarrow{l}_{X*}$ and $\mathbb{L}\underleftarrow{l}^*_X$ induce canonically quasi-inverse equivalences of categories between $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_\bullet)$ and $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D})$.

Proof. This is straightforward that we have the functor $\mathbb{L}\underleftarrow{l}^*_X \colon D^-_{\mathrm{qc}}(\mathcal{D}) \to D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$. Let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$. Then following 7.3.2.7, we have the canonical morphism $\mathbb{L}\underleftarrow{l}^*_X \circ \mathbb{R}\underleftarrow{l}_{X*}\mathcal{E}^\bullet_\bullet \to \mathcal{E}^\bullet_\bullet$ is an isomorphism. Hence, the canonical morphism $\mathbb{R}\underleftarrow{l}_{X*}\mathcal{E}^\bullet_\bullet \to \mathbb{R}\underleftarrow{l}_{X*} \circ \mathbb{L}\underleftarrow{l}^*_X \circ \mathbb{R}\underleftarrow{l}_{X*}\mathcal{E}^\bullet_\bullet$ is an isomorphism (similarly to the proof of 7.2.1.5). Both isomorphisms imply that $\mathbb{R}\underleftarrow{l}_{X*}\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$, and we get the first assertion.

We get the second one from the boundedness of the cohomological functor $\mathbb{R}\underleftarrow{l}_{X*}$ (see 7.3.1.2) and of $\mathbb{L}\underleftarrow{l}^*_X$ (see 7.3.2.9). □

Examples 7.3.2.11. Let us give an example when we can apply 7.3.2.10. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension.

(a) Let $\mathcal{B}_{\mathfrak{X}}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible structure of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module and satisfying the hypotheses of 7.3.2. We suppose $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}})/\mathfrak{S}^\sharp$ is strongly quasi-flat, i.e., by definition (see 4.4.1.3.b), there exists a morphism $\mathfrak{S} \to \mathfrak{T}$ of $\mathcal{V}$-formal schemes such that the induced morphism of ringed spaces $(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}) \to \mathfrak{T}$ is flat and such that, denoting by $\mathcal{I}_{\mathfrak{T}} = \pi\mathcal{O}_{\mathfrak{T}}$ an ideal of definition of $\mathfrak{T}$, the sheaf $\mathcal{O}_{T_0}$ (resp. $gr^\bullet_{\mathcal{I}_{\mathfrak{T}}}\mathcal{O}_{\mathfrak{T}}$) has finite tor dimension on $\mathcal{O}_{\mathfrak{T}}$ (resp. $\mathcal{O}_{T_0}$).

Following 4.4.1.5.b, this yields $\mathcal{B}_{X_0}$ (resp. $gr^\bullet_{\mathfrak{m}}\mathcal{B}_{\mathfrak{X}}$) has finite tor dimension on $\mathcal{B}_{\mathfrak{X}}$ (resp. $\mathcal{B}_{X_0}$). Moreover, since $\mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is $\mathcal{B}_{\mathfrak{X}}$-flat (for the left and the right structures), then $\mathcal{D} := \mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies 7.3.2 and also $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has left and right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$).

(b) When $\mathfrak{X}$ is moreover p-torsion free (see 3.3.1.12 for some example) and Z be a divisor of $\mathfrak{X}^\sharp \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi\mathcal{V})$, then following 8.7.4.2 the sheaf $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$ satisfies 7.3.2. Moreover, $(\mathfrak{X}^\sharp, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z))/\mathfrak{S}^\sharp$ is strongly quasi-flat.

Corollary 7.3.2.12. *Suppose $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$). Let $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D})$. Then $\mathcal{E}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D})$ if and only if $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_\bullet)$.*

Proof. This is a consequence of 7.3.2.10. □

Corollary 7.3.2.13. *With notations and hypotheses of 7.3.2.4, for any $E^\bullet_\bullet \in D^-_{\mathrm{qc}}(D_\bullet)$, $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D})$, the canonical base change morphisms (relative to the diagram 7.2.3.8.1)*

$$\mathbb{L}\underleftarrow{l}^* \circ \mathbb{R}\varpi_{X*}(\mathcal{E}^\bullet) \to \mathbb{R}\varpi_{X_\bullet *} \circ \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet), \quad \varpi^*_X \circ \mathbb{R}\underleftarrow{l}_*(E^\bullet_\bullet) \to \mathbb{R}\underleftarrow{l}_{X,*} \circ \varpi^*_{X_\bullet}(E^\bullet_\bullet) \tag{7.3.2.13.1}$$

are isomorphism.

Proof. This is a consequence of 7.3.2.6 and 7.3.2.10. □

Corollary 7.3.2.14. *We have the following properties.*

(a) Let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$. Then $\mathcal{E}^\bullet_\bullet = 0$ if and only if $\mathcal{E}^\bullet_0 = 0$.

(b) Let $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$. Then $\mathcal{E}^\bullet = 0$ if and only if $\mathcal{D}_0 \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet = 0$.

(c) A morphism $f_\bullet$ of $D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$ is an isomorphism if and only if so is f_0.

(d) A morphism f of $D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$ is an isomorphism if and only if so is $\mathcal{D}_0 \otimes^{\mathbb{L}}_{\mathcal{D}} f$.

Proof. Let us check (a). Suppose $\mathcal{E}^\bullet_0 = 0$. By induction on $i \in \mathbb{N}$, we prove that $\mathcal{E}^\bullet_i = 0$. Let us suppose $i \geq 1$ and $\mathcal{E}^\bullet_j = 0$ for $j < i$. We have the exact sequence of $\mathcal{D}_i$-bimodules $0 \to \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \to \mathcal{D}_i \to \mathcal{D}_{i-1} \to 0$, which yields the exact triangle of left $\mathcal{D}_i$-modules

$$\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_i \to \mathcal{E}^\bullet_{i-1} \to +1. \tag{7.3.2.14.1}$$

By induction hypothesis, $\mathcal{E}^\bullet_{i-1} = 0$. We conclude with the vanishing

$$\mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i \xrightarrow{\sim} \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \left(\mathcal{D}_0 \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i\right) \xrightarrow{\sim} \mathcal{I}^i\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}_0} \mathcal{E}^\bullet_0 = 0.$$

Let us check (b). Following 7.3.2.3.d, $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) = \mathcal{D}_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$. Hence, it follows from theorem 7.3.2.10 that (a) implies (b). The statement (c) and (d) are obvious consequences of respectively (a) and (b). □

Theorem 7.3.2.15. *The functors $\mathbb{R}\underleftarrow{l}_{X*}$ and $\mathbb{L}\underleftarrow{l}^*_X$ induce canonically quasi-inverse equivalences of categories between $D_{\mathrm{qc,tdf}}(\mathcal{D}_\bullet)$ and $D_{\mathrm{qc,tdf}}(\mathcal{D})$.*

More precisely let $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D})$ (resp. $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$). Then $\mathcal{E}^\bullet$ (resp. $\mathcal{E}^\bullet_\bullet$) has tor amplitude in $[a, b]$ if and only if $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet)$ (resp. $\mathbb{R}\underleftarrow{l}_{X*}\mathcal{E}^\bullet_\bullet$) has tor amplitude in $[a, b]$.

Proof. If $\mathcal{E}^\bullet \in D_{\mathrm{qc,tdf}}(\mathcal{D})$ has tor amplitude in $[a, b]$, then it is well known that $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \in D_{\mathrm{qc,tdf}}(\mathcal{D}_\bullet)$ has tor amplitude in $[a, b]$. Conversely, let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet)$ and suppose $\mathcal{E}^\bullet_\bullet$ has tor-amplitude in $[a, b]$. We can replace $\mathcal{E}^\bullet_\bullet$ by $\tau_{\leq b}\mathcal{E}^\bullet_\bullet$. The fact that $\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}^\bullet_\bullet)$ has tor-amplitude in $[a, b]$ is local. Hence, we can suppose $\mathfrak{X}$ affine and, following 7.3.2.7, there exists a complex $P^\bullet_\bullet \in C^-(D_\bullet)$ such that $P^n_\bullet = 0$ if $n \geq b+1$, endowed with an isomorphism $\mathcal{P}^\bullet_\bullet := \mathcal{D}_\bullet \otimes_{\mathcal{D}_\bullet} P^\bullet_\bullet \xrightarrow{\sim} \mathcal{E}^\bullet_\bullet$ in $D^-(\mathcal{D}_\bullet)$, such that P^n_i is a free left D_i-module, the transition maps $P^n_{i+1} \to P^n_i$ are surjective, and the canonical morphisms $D_i \otimes_{D_{i+1}} P^\bullet_{i+1} \to P^\bullet_i$ are quasi-isomorphisms. Let $F^\bullet_\bullet := \tau_{\geq a} P^\bullet_\bullet$ and $\mathcal{F}^\bullet_\bullet := \mathcal{D}_\bullet \otimes_{\mathcal{D}_\bullet} F^\bullet_\bullet$. Then $\mathcal{P}^\bullet_\bullet \to \mathcal{F}^\bullet_\bullet$ is a quasi-isomorphism. By construction, we remark that $F^n_{i+1} \to F^n_i$ are surjective for any $n \in \mathbb{Z}$ and any $i \in \mathbb{N}$. Hence, for any integer n the left $\mathcal{D}_\bullet$-module $\mathcal{F}^n_\bullet$ is ML-flasque (see 7.3.1.3). Hence, we get the first isomorphism $\underleftarrow{l}_{X*} \mathcal{F}^\bullet_\bullet \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*} \mathcal{F}^\bullet_\bullet \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*} \mathcal{E}^\bullet_\bullet$. Since $\mathcal{F}^n = 0$ for any $n \notin [a, b]$, since $\mathcal{F}^n_\bullet$ is $\mathcal{D}_\bullet$-flat for any $n \neq a$, since $\mathcal{F}^\bullet_\bullet$ has tor-amplitude in $[a, b]$, then it follows from [SGA6, Lemma I.5.1.1] that $\mathcal{F}^a_\bullet$ is $\mathcal{D}_\bullet$-flat. It follows from 7.3.1.20 that $\underleftarrow{l}_{X*} \mathcal{F}^\bullet_\bullet$ is a complex of flat left $\mathcal{D}$-modules of tor-amplitude in $[a, b]$. □

Corollary 7.3.2.16. *Let $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D})$. Then $\mathcal{E}^\bullet \in D_{\mathrm{qc,tdf}}(\mathcal{D})$ if and only if $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \in D_{\mathrm{qc,tdf}}(\mathcal{D}_\bullet)$.*

Proof. This is a consequence of 7.3.2.10 and 7.3.2.15. □

7.3.3 Coherent complexes

Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ satisfying the hypotheses of 7.3.2.

Proposition 7.3.3.1. *Let $\mathcal{E}^\bullet_\bullet \in D^-(\mathcal{D}_\bullet)$. The following conditions are equivalent:*

(a) $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$;

(b) For any affine open set $\mathfrak{U} = \mathrm{Spf}\, A \subset \mathfrak{X}$, there exists an inverse system of complexes $P^\bullet_\bullet \in C^-(\Gamma(\mathfrak{U}, \mathcal{D}_\bullet))$, endowed with an isomorphism $\mathcal{D}_\bullet|_{\mathfrak{U}} \otimes_{\Gamma(\mathfrak{U}, \mathcal{D}_\bullet)} P^\bullet_\bullet \xrightarrow{\sim} \mathcal{E}^\bullet_\bullet|_{\mathfrak{U}}$ in $D^-(\mathcal{D}_\bullet|_{\mathfrak{U}})$, such that P^n_i is a free of finite type left $\Gamma(\mathfrak{U}, \mathcal{D}_i)$-module, the transition maps $P^n_{i+1} \to P^n_i$ are surjective, and the canonical morphisms

$$\Gamma(\mathfrak{U}, \mathcal{D}_i) \otimes_{\Gamma(\mathfrak{U}, \mathcal{D}_{i+1})} P^\bullet_{i+1} \to P^\bullet_i \tag{7.3.3.1.1}$$

are quasi-isomorphisms.

Proof. It is clear that $(b) \Rightarrow (a)$. Now let us check $(a) \Rightarrow (b)$. Let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$. We can suppose $\mathfrak{X}$ is affine. Hence, it follows from 7.3.2.6 that we can suppose there exists $E^\bullet_\bullet \in D^-_{\mathrm{coh}}(D_\bullet)$ such that $\mathcal{E}^\bullet_\bullet = \mathcal{D}_\bullet \otimes_{D_\bullet} E^\bullet_\bullet$. Following 7.3.1.18, there exists $P^\bullet_\bullet \in C^-(D_\bullet)$, such that each P^n_i is a free left D_i-module of finite type and each map $P^n_i \to P^n_{i-1}$ is surjective, and an isomorphism of $D^-(\mathcal{D}_\bullet)$ of the form $P^\bullet_\bullet \xrightarrow{\sim} E^\bullet_\bullet$. Since P^n_i are flat left D_i-modules, then it follows the part 2) of the proof of 7.3.2.6, that the canonical morphisms 7.3.3.1.1 are isomorphisms. □

7.3.3.2. With notations and hypotheses of 7.3.2.4, let $P^\bullet_\bullet \in C^-(D_\bullet)$ such that P^n_i is a free of finite type left D_i-module, the transition maps $P^n_{i+1} \to P^n_i$ are surjective, and the canonical morphisms

$$D_i \otimes_{D_{i+1}} P^\bullet_{i+1} \to P^\bullet_i \tag{7.3.3.2.1}$$

are quasi-isomorphisms. Let $\mathcal{P}^\bullet_\bullet := \mathcal{D}_\bullet \otimes_{D_\bullet} P^\bullet_\bullet$. It follows from 7.3.3.2.1 that $H^s(P^\bullet_\bullet)$ is ML. Hence $H^s(\mathcal{P}^\bullet_\bullet) \xrightarrow{\sim} \mathcal{D}_\bullet \otimes_{D_\bullet} H^s(P^\bullet_\bullet)$ is ML-flasque. By considering the spectral sequence $R^r \underleftarrow{l}_{X*}(H^s(\mathcal{P}^\bullet_\bullet)) \Rightarrow H^n(\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{P}^\bullet_\bullet))$, this yields that the natural map

$$H^n(\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{P}^\bullet_\bullet)) \to \underleftarrow{l}_{X*}(H^n(\mathcal{P}^\bullet_\bullet)).$$

(coming from the above spectral sequence) is an isomorphism. It follows that $\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{P}^\bullet_\bullet) \in D^-_{\mathrm{coh}}(\mathcal{D})$.

Corollary 7.3.3.3. *We have the following properties.*

(a) The functors $\mathbb{R}\underleftarrow{l}_{X*}$ and $\mathbb{L}\underleftarrow{l}^*_X$ induce canonically quasi-inverse equivalences of categories between $D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$ (resp. $D_{\mathrm{perf}}(\mathcal{D}_\bullet)$) and $D^-_{\mathrm{coh}}(\mathcal{D})$ (resp. $D_{\mathrm{perf}}(\mathcal{D})$).

(b) Suppose $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$). Then the functors $\mathbb{R}\underleftarrow{l}_{X*}$ and $\mathbb{L}\underleftarrow{l}^*_X$ induce canonically quasi-inverse equivalences of categories between $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_\bullet)$ and $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D})$.

Proof. 0) The statement (b) follows from (a) and from the boundedness of the cohomological functor $\mathbb{R}\underleftarrow{l}_{X*}$ (see 7.3.1.2) and of $\mathbb{L}\underleftarrow{l}^*_X$ (see 7.3.2.9).

1) Let us check the non-respective case of (a). Since the pseudo-coherence is stable under derived pullbacks (see [Sta22, 08H4-21.42.3]), then we get the functor $\mathbb{L}\underleftarrow{l}^*_X \colon D^-_{\mathrm{coh}}(\mathcal{D}) \to D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$. Let $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}_\bullet)$. Let us check that $\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}^\bullet_\bullet) \in D^-_{\mathrm{coh}}(\mathcal{D})$. Since this is local, then this follows from 7.3.3.1 and 7.3.3.2. Hence, thanks to 7.3.1.16 and the first part of 7.3.2.10 we are done.

2) We check the respective cases by using the second part of 7.3.2.10 and furthermore 7.3.2.15 (and 7.1.3.12). □

Corollary 7.3.3.4. *Let $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D})$ and $\mathcal{E}^\bullet_\bullet := \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet)$.*

(a) Then $\mathcal{E}^\bullet \in D_{\mathrm{perf}}(\mathcal{D})$ (resp. $\mathcal{E}^\bullet \in D^-_{\mathrm{coh}}(\mathcal{D})$) if and only if $\mathcal{E}^\bullet_0 \in D_{\mathrm{perf}}(\mathcal{D}_0)$ (resp. $\mathcal{E}^\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}_0)$).

(b) Suppose $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$). Then $\mathcal{E}^\bullet \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D})$ if and only if $\mathcal{E}^\bullet_0 \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_0)$.

Proof. This is a consequence of 7.3.1.15, 7.3.2.10 and of 7.3.3.3. □

7.3.4 Derived completed tensor products and derived completed homomorphisms of complexes of (bi)modules

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Let $\mathcal{D}, \mathcal{D}', \mathcal{D}'', \mathcal{D}'''$ be four sheaves of rings on $\mathfrak{X}$ satisfying the hypotheses of 7.3.2 (or only 7.2.3 when the notion of quasi-coherence is not involved).

7.3.4.1 (Independence of $\mathbb{R}\underleftarrow{l}_{X*}$ and $\mathbb{L}\underleftarrow{l}^*_X$)**.** We have the topoi morphisms $\underleftarrow{l}_X \colon (X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$ and $\underleftarrow{l}_X \colon (X_\bullet, \mathbb{Z}_{X_\bullet}) \to (|\mathcal{X}|, \mathbb{Z}_{\mathfrak{X}})$.

(a) Both functors $\mathbb{R}\underleftarrow{l}_{X*}$ can be computed by taking the same flasque resolution (see 7.1.3.14) and therefore we have the canonical commutative diagram

$$\begin{array}{ccc} D^-(\mathcal{D}_\bullet) & \xrightarrow{\mathbb{R}\underleftarrow{l}_{X*}} & D^-(\mathcal{D}) \\ \downarrow & & \downarrow \\ D^-(\mathbb{Z}_{X_\bullet}) & \xrightarrow{\mathbb{R}\underleftarrow{l}_{X*}} & D^-(\mathbb{Z}_{\mathfrak{X}}), \end{array} \tag{7.3.4.1.1}$$

where the vertical maps are the forgetful functors. We have obviously the same property for the exact functor $\underleftarrow{l}^{-1}_X$.

(b) Following 7.3.2.3.d, the canonical morphism

$$\mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_X \mathcal{O}_{\mathfrak{X}}} \underleftarrow{l}^{-1}_X(\mathcal{E}^\bullet) \to \mathcal{D}_\bullet \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_X \mathcal{D}} \underleftarrow{l}^{-1}_X(\mathcal{E}^\bullet) =: \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet)$$

is an isomorphism for any $\mathcal{E}^\bullet \in D^-({}^l\mathcal{D})$, and similarly for right modules. Hence, the functor $\mathbb{L}\underleftarrow{l}^*_X$ does not depend, up to canonical forgetful functor, to the choice of such $\mathcal{D}$.

7.3.4.2. Let $\mathcal{E}^\bullet \in D^-({}^l\mathcal{D})$, $\mathcal{M}^\bullet \in D^-({}^r\mathcal{D})$ be two complexes of $\mathcal{D}$-modules, respectively to the left, to the right. Following 7.3.2.3.d, $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) = \mathcal{D}_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \in D^-({}^l\mathcal{D}_\bullet)$ and similarly for right modules. Hence, we can define their *completed tensor product* by setting

$$\mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet := \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{L}\underleftarrow{l}^*_X \mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathbb{L}\underleftarrow{l}^*_X \mathcal{E}^\bullet) \in D^-(\mathbb{Z}_{\mathfrak{X}}), \tag{7.3.4.2.1}$$

where $\mathbb{R}\underleftarrow{l}_{X*} \colon D^-(\mathbb{Z}_{X_\bullet}) \to D^-(\mathbb{Z}_{\mathfrak{X}})$ is the derived pushforward given by the topoi morphism $\underleftarrow{l}_X \colon (X_\bullet, \mathbb{Z}_{X_\bullet}) \to (|\mathcal{X}|, \mathbb{Z}_{\mathfrak{X}})$ and $\mathbb{L}\underleftarrow{l}^*_X \colon D^-({}^l\mathcal{D}) \to D^-({}^l\mathcal{D}_\bullet)$ is the derived pullback given by the topoi morphism $\underleftarrow{l}_X \colon (X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$.

Lemma 7.3.4.3. *Let $\mathcal{E}^\bullet \in D^-({}^l\mathcal{D})$, $\mathcal{M}^\bullet \in D^-({}^r\mathcal{D})$.*

(a) We have the morphism

$$\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \to \mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \tag{7.3.4.3.1}$$

which is an isomorphism when one of the two complexes belongs to $D^-_{\mathrm{coh}}(\mathcal{D})$ and the other to $D^-_{\mathrm{qc}}(\mathcal{D})$.

(b) When $\mathcal{D}$ is commutative, then we get the isomorphism of $D^-(\mathcal{D})$:

$$\mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*} \circ \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet), \tag{7.3.4.3.2}$$

where $\underleftarrow{l}_X$ is here the topoi morphism $\underleftarrow{l}_X \colon (X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$. Hence, we can consider $\mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet$ as an object of $D^-(\mathcal{D})$ and then the map 7.3.4.3.1 is the adjunction morphism.

Proof. a) Since $\mathbb{L}\underset{\leftarrow X}{l}^{-1}(\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) \xrightarrow{\sim} \mathbb{L}\underset{\leftarrow X}{l}^{-1}(\mathcal{M}^\bullet) \otimes^{\mathbb{L}}_{\underset{\leftarrow X}{l}^{-1}(\mathcal{D})} \mathbb{L}\underset{\leftarrow X}{l}^{-1}(\mathcal{E}^\bullet)$, then we get by adjunction the canonical morphism

$$\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \to \mathbb{R}\underset{\leftarrow X*}{l} \left(\underset{\leftarrow X}{l}^{-1}(\mathcal{M}^\bullet) \otimes^{\mathbb{L}}_{\underset{\leftarrow X}{l}^{-1}(\mathcal{D})} \underset{\leftarrow X}{l}^{-1}(\mathcal{E}^\bullet) \right) \tag{7.3.4.3.3}$$

By applying the functor $\mathbb{R}\underset{\leftarrow X*}{l}$ to the following canonical morphism

$$\underset{\leftarrow X}{l}^{-1}(\mathcal{M}^\bullet) \otimes^{\mathbb{L}}_{\underset{\leftarrow X}{l}^{-1}(\mathcal{D})} \underset{\leftarrow X}{l}^{-1}(\mathcal{E}^\bullet) \to \mathbb{L}\underset{\leftarrow X}{l}^{*} \mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathbb{L}\underset{\leftarrow X}{l}^{*} \mathcal{E}^\bullet \tag{7.3.4.3.4}$$

and by composing this latter with 7.3.4.3.3, we get 7.3.4.3.1. By using [Har66, I.7.1], when $\mathcal{M}^\bullet \in D^-_{\mathrm{coh}}(\mathcal{D})$ and $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D})$, to check that 7.3.4.3.1 is an isomorphism, we reduce to the case where $\mathcal{M}^\bullet = \mathcal{D}$, which follows from 7.3.2.10.

The part b) is obvious (use 7.3.4.1). □

7.3.4.4. Suppose there exists a homomorphism of sheaves of rings on $\mathfrak{X}$ of the form $\mathcal{D} \to \mathcal{D}'$ such that the composition of $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ with $\mathcal{D} \to \mathcal{D}'$ gives $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}'$.

(a) Let $* \in \{\mathrm{l}, \mathrm{r}\}$ and $\star \in \{-, \mathrm{b}\}$. By definition (see 7.3.1.12), the forgetful functor $D^\star({}^*\mathcal{D}') \to D^\star({}^*\mathcal{D})$ preserve the quasi-coherence, i.e. induces the functor

$$forg_{\mathcal{D},\mathcal{D}'} \colon D^\star_{\mathrm{qc}}({}^*\mathcal{D}') \to D^\star_{\mathrm{qc}}({}^*\mathcal{D}). \tag{7.3.4.4.1}$$

(b) With notation 7.3.1.12, since $\mathcal{D}_\bullet$ satisfied the condition 7.3.2.d for the left structure, then it follows from 7.3.1.14 that the functor $\mathcal{D}'_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} -$ preserves the quasi-coherence, i.e., induces the functor

$$\mathcal{D}'_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} - \colon D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet) \to D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_\bullet). \tag{7.3.4.4.2}$$

(c) Let $\mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D})$. Since $\mathcal{D}'_\bullet \xrightarrow{\sim} \mathbb{L}\underset{\leftarrow X}{l}^{*}\mathcal{D}'$ (see 7.3.2.3.b), then we get

$$\mathcal{D}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \xleftarrow{\sim} \mathbb{R}\underset{\leftarrow X*}{l}(\mathcal{D}'_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathbb{L}\underset{\leftarrow X}{l}^{*}\mathcal{E}^\bullet). \tag{7.3.4.4.3}$$

Using the preservation of the quasi-coherence under the functors $\mathbb{L}\underset{\leftarrow X}{l}^{*}$ and $\mathbb{R}\underset{\leftarrow X*}{l}$ of 7.3.2.10, the property 7.3.4.4.2 and the isomorphism 7.3.4.4.3, we get that the functor $\mathcal{D}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} -$ preserves the quasi-coherence, i.e. induces the functor

$$\mathcal{D}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} - \colon D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}) \to D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'). \tag{7.3.4.4.4}$$

In order to derive complexes of bimodules, we need further hypotheses on $\mathcal{D}$.

Definition 7.3.4.5. Let us introduce the following definition (compare with 4.6.3.2.b). A pair $(\mathcal{R}, \mathcal{K})$ consisting of a sheaf $\mathcal{R}$ of commutative rings on $\mathfrak{X}$ and an ideal $\mathcal{K}$ of $\mathcal{R}$ is said to be solving $(\mathcal{D}, \mathcal{D}', \mathcal{I})$ if it satisfies the following conditions:

(i) $\mathcal{R}$ is separated complete for the $\mathcal{K}$-adic topology and $\mathcal{O}_{\mathfrak{X}}$ is an $\mathcal{R}$-algebra such that $\mathcal{K}\mathcal{O}_{\mathfrak{X}} = \mathcal{I}$,

(ii) $\mathcal{R}$ is sent to the center of $\mathcal{D}$ and of $\mathcal{D}'$ via the composite ring homomorphisms $\mathcal{R} \to \mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ and $\mathcal{R} \to \mathcal{O}_{\mathfrak{X}} \to \mathcal{D}'$;

(iii) $\mathcal{D}$ and $\mathcal{D}'$ are flat on $\mathcal{R}$.

In that case, we say that $(\mathcal{D}, \mathcal{D}', \mathcal{I})$ is solvable by $(\mathcal{R}, \mathcal{K})$. We remark that if $(\mathcal{R}, \mathcal{K})$ a (left or right) solving pair of $(\mathcal{D}, \mathcal{D}', \mathcal{I})$, then $\mathcal{R}_\bullet$ is a (left or right) solving ring of $(\mathcal{D}_\bullet, \mathcal{D}'_\bullet)$ (see Definition 4.6.3.2.b), where $\mathcal{R}_\bullet := (\mathcal{R}/\mathcal{K}^{i+1}\mathcal{R})_{i \in \mathbb{N}}$, $\mathcal{D}_\bullet := (\mathcal{D}/\mathcal{I}^{i+1}\mathcal{D})_{i \in \mathbb{N}}$ and $\mathcal{D}'_\bullet := (\mathcal{D}'/\mathcal{I}^{i+1}\mathcal{D}')_{i \in \mathbb{N}}$. Finally, $(\mathcal{R}, \mathcal{K})$ (left or right) solves $(\mathcal{D}, \mathcal{I})$ means by definition that $(\mathcal{R}, \mathcal{K})$ (left or right) solves $(\mathcal{D}, \mathcal{D}, \mathcal{I})$.

Example 7.3.4.6. We will use essentially in this book the following cases. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension and is p-torsion free (see 3.3.1.12 for some example). For any integer $i \geq 0$, set $X_i^\sharp := X^\sharp \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$. Let Z be a divisor of X_0. Then it follows from 8.7.4.2 that we can choose $\mathcal{R} = \mathcal{V}, \mathcal{K} = \mathfrak{m}$, $\mathcal{I} = \mathfrak{m}\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{D} = \varprojlim_i \mathcal{B}_{X_i}(Z, r) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$.

Notation 7.3.4.7. Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$ is solved by $(\mathcal{R}, \mathcal{K})$. Let $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{D}', \mathcal{D}^{\mathrm{r}})$. The following condition

(iv) The structure of $\mathcal{R}$-modules induced on the $(\mathcal{D}', \mathcal{D})$-bimodules $\mathcal{E}^n$ by the structure of left $\mathcal{D}'$-module and of right $\mathcal{D}$-module coincide for any $n \in \mathbb{Z}$;

is equivalent to saying that $\mathcal{E}^\bullet \in K({}^{\mathrm{l}}\mathcal{D}' \otimes_{\mathcal{R}} \mathcal{D}^{\mathrm{o}})$.

We denote by $D({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ (resp. $D({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet \mathcal{D}_\bullet{}^{\mathrm{r}})$) the strictly full subcategory of $D({}^{\mathrm{l}}\mathcal{D}', {}^{\mathrm{r}}\mathcal{D})$ (resp. $D({}^{\mathrm{l}}\mathcal{D}'_\bullet, {}^{\mathrm{r}}\mathcal{D}_\bullet)$) consisting of complexes isomorphic to an object of $K({}^{\mathrm{l}}\mathcal{D}' \otimes_{\mathcal{R}} \mathcal{D}^{\mathrm{o}})$ (resp. $K({}^{\mathrm{l}}\mathcal{D}'_\bullet \otimes_{\mathcal{R}_\bullet} \mathcal{D}^{\mathrm{o}}_\bullet)$).

7.3.4.8. Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$ is solved by $(\mathcal{R}, \mathcal{K})$. The functor $\underset{\leftarrow}{l}_X^* = -\otimes_{\mathcal{R}} \mathcal{R}_\bullet \colon \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}', \mathcal{D}^{\mathrm{r}}) \to \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}'_\bullet, {}^{\mathrm{r}}\mathcal{D}_\bullet)$ is canonically isomorphic (modulo forgetful functors) to $-\otimes_{\mathcal{D}} \mathcal{D}_\bullet \colon \mathrm{Mod}(\mathcal{D}^{\mathrm{r}}) \to \mathrm{Mod}({}^{\mathrm{r}}\mathcal{D}_\bullet)$ and $\mathcal{D}'_\bullet \otimes_{\mathcal{D}'} - \colon \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}') \to \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}'_\bullet)$. Let $\star \in \{\emptyset, -\}$. Since $\mathcal{D}' \otimes_{\mathcal{R}} \mathcal{D}^{\mathrm{o}}$ is flat over $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{R}$, then by using resolutions by flat left $\mathcal{D}' \otimes_{\mathcal{R}} \mathcal{D}^{\mathrm{o}}$-modules we get the functor

$$\mathbb{L}\underset{\leftarrow}{l}_X^* \colon D^\star({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}}) \to D^\star({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}}) \tag{7.3.4.8.1}$$

which is canonically isomorphic (modulo some forgetful functors) to $\underset{\leftarrow}{l}_X^{-1}(-) \otimes^{\mathbb{L}}_{\underset{\leftarrow}{l}_X^{-1}(\mathcal{D})} \mathcal{D}_\bullet \colon D^\star(\mathcal{D}^{\mathrm{r}}) \to D^\star(\mathcal{D}_\bullet{}^{\mathrm{r}})$ and $\mathcal{D}'_\bullet \otimes^{\mathbb{L}}_{\underset{\leftarrow}{l}_X^{-1}(\mathcal{D}')} \underset{\leftarrow}{l}_X^{-1}(-) \colon D^\star({}^{\mathrm{l}}\mathcal{D}') \to D^\star({}^{\mathrm{l}}\mathcal{D}'_\bullet)$. Similarly, by using resolutions by K-injective complexes of left $\mathcal{D}' \otimes_{\mathcal{R}} (\mathcal{D})^o$-modules, we get the functor

$$\mathbb{R}\underset{\leftarrow}{l}_{X*} \colon D^\star({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}}) \to D^\star({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}}) \tag{7.3.4.8.2}$$

which is canonically isomorphic to the functors $\mathbb{R}\underset{\leftarrow}{l}_{X*} \colon D^\star(\mathcal{D}_\bullet{}^{\mathrm{r}}) \to D^\star(\mathcal{D}^{\mathrm{r}})$ and $\mathbb{R}\underset{\leftarrow}{l}_{X*} \colon : D^\star({}^{\mathrm{l}}\mathcal{D}_\bullet) \to D^\star({}^{\mathrm{l}}\mathcal{D}')$.

Let $\mathcal{E}_\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$. The property $\mathcal{E}_\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_\bullet)$ (resp. $\mathcal{E}_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}_\bullet{}^{\mathrm{r}})$) is satisfied if and only if both conditions hold:

(a) The image via the forgetful functor $D^-({}^{\mathrm{l}}\mathcal{D}'_0, \mathcal{D}_0{}^{\mathrm{r}}) \to D^-({}^{\mathrm{l}}\mathcal{D}'_0) \to D^-(\mathcal{O}_{X_0})$ (resp. $D^-({}^{\mathrm{l}}\mathcal{D}'_0, \mathcal{D}_0{}^{\mathrm{r}}) \to D^-(\mathcal{D}_0{}^{\mathrm{r}}) \to D^-(\mathcal{O}_{X_0})$)) of the complex $\mathcal{E}_0^\bullet$ is in $D^-_{\mathrm{qc}}(\mathcal{O}_{X_0})$.

(b) The canonical map

$$\mathcal{R}_i \overset{\mathbb{L}}{\otimes}_{\mathcal{R}_{i+1}} \mathcal{E}_{i+1}^\bullet \to \mathcal{E}_i^\bullet \tag{7.3.4.8.3}$$

is an isomorphism.

Indeed, following 7.3.1.14 and the flatness of $\mathcal{R} \to \mathcal{D}$ or of $\mathcal{R} \to \mathcal{D}'$, the condition 7.3.4.8.3 is equivalent to the condition 7.3.1.10.b.

Let $? \in \{\mathrm{qc}, \mathrm{coh}, \mathrm{tdf}, \mathrm{perf}\}$. Let $\sharp \in \{\emptyset, \mathrm{b}, +, -\}$. We denote respectively by $D^\sharp_{?,.}({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ and $D^\sharp_{.,?}({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ the full subcategory of $D^\sharp({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ consisting of complexes $\mathcal{E}_\bullet$ which belongs to $\mathcal{E}_\bullet \in D^\sharp_?({}^{\mathrm{l}}\mathcal{D}'_\bullet)$ (resp. $\mathcal{E}_\bullet \in D^\sharp_?(\mathcal{D}_\bullet{}^{\mathrm{r}})$). Beware that the property 7.3.4.7.(iv) is not necessarily satisfied for $\mathcal{O}_{\mathfrak{X}}$ (which is involved in the condition (a)) instead of $\mathcal{R}$, so we do have to distinguish the categories $D^\sharp_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ and $D^\sharp_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$.

Similarly, we denote by $D^\sharp_{?,.}({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{.,?}({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ the full subcategory of $D^\sharp({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ consisting of complexes $\mathcal{E}$ which belongs to $\mathcal{E} \in D^\sharp_?({}^{\mathrm{l}}\mathcal{D}')$ (resp. $\mathcal{E} \in D^\sharp_?(\mathcal{D}^{\mathrm{r}})$). The functors $\mathbb{L}\underset{\leftarrow}{l}_X^*$ of 7.3.4.8.1 and $\mathbb{R}\underset{\leftarrow}{l}_{X*}$ induce quasi-inverse equivalences of categories between $D^\sharp_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ and between $D^\sharp_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$.

7.3.4.9. Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$ and $(\mathcal{D}, \mathcal{D}'', \mathcal{I})$ are solved by $(\mathcal{R}, \mathcal{K})$. Let $*, ** \in \{r, l\}$. Following 4.6.3.2, we have the bifunctors

$$-\otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} - \colon D({}^*\mathcal{D}'_\bullet, \mathcal{R}, \mathcal{D}_\bullet{}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}, {}^{**}\mathcal{D}''_\bullet) \to D({}^*\mathcal{D}'_\bullet, \mathcal{R}, {}^{**}\mathcal{D}''_\bullet). \tag{7.3.4.9.1}$$

We have similar bifunctors by changing the indices l and r. With 7.3.4.8.1, we can check the bifunctor 7.3.4.2.1 induces

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} - \colon D({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^{**}\mathcal{D}'') \to D({}^*\mathcal{D}', \mathcal{R}, {}^{**}\mathcal{D}''). \tag{7.3.4.9.2}$$

Let $\mathcal{M}^\bullet \in D({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$, $\mathcal{E}^\bullet \in D({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^{**}\mathcal{D}'')$ be two complexes. Since $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}^\bullet) \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet)$ (remark we have three different functors $\mathbb{L}\underleftarrow{l}^*_X$ of the form 7.3.4.8.1), then we get by adjunction via the adjoint pair $(\mathbb{L}\underleftarrow{l}^*_X \dashv \mathbb{R}\underleftarrow{l}_{X*})$ (see the functors 7.3.4.8.1 and 7.3.4.8.2) the canonical morphism

$$\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \to \mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \tag{7.3.4.9.3}$$

which is an isomorphism when either $\mathcal{M}^\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}^{\mathrm{r}})$ and $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$, or $\mathcal{M}^\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}^{\mathrm{r}})$ and $\mathcal{E}^\bullet \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}')$. Indeed, for instance, suppose $\mathcal{M}^\bullet \in D^-_{\mathrm{coh}}(\mathcal{D}^{\mathrm{r}})$ and $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$. Then, since both functors $- \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet$ and $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet$ are way-out left (use 7.3.1.2 for the second functor), then following (the way-out left version of) [Har66, I.7.1 (ii) and (iv)], since this is local, we reduce to the case where $\mathcal{M}^\bullet = \mathcal{D}$, which is obvious.

Proposition 7.3.4.10. *Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$, $(\mathcal{D}, \mathcal{D}'', \mathcal{I})$ are solved by $(\mathcal{R}, \mathcal{K})$. Let $*, ** \in \{r, l\}$. The functors 7.3.4.9.2 and 7.3.4.9.1 preserve the quasi-coherence for bounded above complexes, i.e. they factor through the functor*

$$- \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} -\colon D_{\mathrm{qc},.}({}^*\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}}) \times D_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, {}^{**}\mathcal{D}''_\bullet) \to D_{\mathrm{qc},.}({}^*\mathcal{D}'_\bullet, \mathcal{R}_\bullet, {}^{**}\mathcal{D}''_\bullet), \tag{7.3.4.10.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} -\colon D^-_{\mathrm{qc},.}({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}}) \times D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^{**}\mathcal{D}'') \to D^-_{\mathrm{qc},.}({}^*\mathcal{D}', \mathcal{R}, {}^{**}\mathcal{D}''), \tag{7.3.4.10.2}$$

and similarly replacing the indexes "$\mathrm{qc},.$" by "$.,\mathrm{qc}$".

Proof. Since the functors $\mathbb{L}\underleftarrow{l}^*_X$ and $\mathbb{R}\underleftarrow{l}_{X*}$ induce quasi-inverse equivalences of categories between $D^\star_{\mathrm{qc},.}({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$ and $D^\star_{\mathrm{qc},.}({}^*\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ (see 7.3.4.8) and similarly for the other respective categories, then we reduce to check 7.3.4.10.1.

Let $\mathcal{M}^\bullet_\bullet \in D^-_{\mathrm{qc},.}({}^*\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$ and $\mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, {}^{**}\mathcal{D}''_\bullet)$. We have to check that $\mathcal{G}_\bullet := \mathcal{M}^\bullet_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathcal{E}^\bullet_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}'_\bullet)$. Since the property $\mathcal{G}_0 = \mathcal{M}^\bullet_0 \otimes^{\mathbb{L}}_{\mathcal{D}_0} \mathcal{E}^\bullet_0 \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_0)$ is local, we can suppose $\mathfrak{X}$ affine. By using [Har66, I.7.3.(iv)] (in fact, a left way-out version), since $\mathcal{M}^\bullet_0 \otimes^{\mathbb{L}}_{\mathcal{D}_0} -$ is way-out left, we reduce to the case where $\mathcal{E}^\bullet_0$ is a free left $\mathcal{D}_0$-module, which is obvious.

We conclude the proof via the isomorphisms (recall the computations of $\mathcal{E}^\bullet_\bullet$ and $\mathcal{M}^\bullet_\bullet$ to justify the associativity isomorphisms of the tensor products)

$$\begin{aligned} &\mathcal{D}'_i \otimes^{\mathbb{L}}_{\mathcal{D}'_{i+1}} \left(\mathcal{M}^\bullet_{i+1} \otimes^{\mathbb{L}}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1}\right) \xrightarrow{\sim} \left(\mathcal{D}'_i \otimes^{\mathbb{L}}_{\mathcal{D}'_{i+1}} \mathcal{M}^\bullet_{i+1}\right) \otimes^{\mathbb{L}}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1} \\ &\xrightarrow{\sim} \mathcal{M}^\bullet_i \otimes^{\mathbb{L}}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1} \xrightarrow{\sim} \mathcal{M}^\bullet_i \otimes^{\mathbb{L}}_{\mathcal{D}_i} \left(\mathcal{D}_i \otimes^{\mathbb{L}}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1}\right) \xrightarrow{\sim} \mathcal{M}^\bullet_i \otimes^{\mathbb{L}}_{\mathcal{D}_i} \mathcal{E}^\bullet_i. \end{aligned} \tag{7.3.4.10.3}$$

□

Example 7.3.4.11. When $\mathcal{D}$ is commutative, we get the factorisations

$$- \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} -\colon D_{\mathrm{qc}}(\mathcal{D}_\bullet) \times D_{\mathrm{qc}}(\mathcal{D}_\bullet) \to D_{\mathrm{qc}}(\mathcal{D}_\bullet), \tag{7.3.4.11.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} -\colon D^-_{\mathrm{qc}}(\mathcal{D}) \times D^-_{\mathrm{qc}}(\mathcal{D}) \to D^-_{\mathrm{qc}}(\mathcal{D}). \tag{7.3.4.11.2}$$

Proposition 7.3.4.12. *Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$, $(\mathcal{D}, \mathcal{D}'', \mathcal{I})$, $(\mathcal{D}'', \mathcal{D}''', \mathcal{I})$ are solved by $(\mathcal{R}, \mathcal{K})$. Let $*, ** \in \{r, l\}$. Let $\mathcal{E}^\bullet \in D^-_{\mathrm{qc},.}({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$, $\mathcal{F}^\bullet \in D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^{\mathrm{r}}\mathcal{D}'')$, $\mathcal{G}^\bullet \in D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'', \mathcal{R}, {}^{**}\mathcal{D}''')$. The associativity isomorphism of the derived complete tensor product of quasi-coherent complexes holds, i.e. we have the isomorphism in $D^-_{\mathrm{qc},.}({}^*\mathcal{D}', \mathcal{R}, {}^{**}\mathcal{D}''')$:*

$$\left(\mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{F}^\bullet\right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''} \mathcal{G}^\bullet \xrightarrow{\sim} \mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \left(\mathcal{F}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''} \mathcal{G}^\bullet\right). \tag{7.3.4.12.1}$$

Proof. Since $\mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{F}^\bullet \in D^-_{\mathrm{qc},.}({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}''^{\mathrm{r}})$ (see 7.3.4.10) then it follows from 7.3.4.8 that we have the canonical isomorphism

$$\mathbb{L}\underleftarrow{l}^*_X \left(\mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{F}^\bullet\right) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{F}^\bullet). \tag{7.3.4.12.2}$$

By associativity of the tensor product, we get the middle isomorphism:

$$\begin{aligned} \left(\mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{F}^\bullet\right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''} \mathcal{G}^\bullet &= \mathbb{R}\underleftarrow{l}_{X*} \left(\mathbb{L}\underleftarrow{l}^*_X \left(\mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{F}^\bullet\right) \otimes^{\mathbb{L}}_{\mathcal{D}''_\bullet} \mathbb{L}\underleftarrow{l}^*_X \mathcal{G}^\bullet\right) \\ &\underset{7.3.4.12.2}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{X*} \left(\left(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{F}^\bullet)\right) \otimes^{\mathbb{L}}_{\mathcal{D}''_\bullet} \mathbb{L}\underleftarrow{l}^*_X \mathcal{G}^\bullet\right) \end{aligned} \tag{7.3.4.12.3}$$

$$\underset{4.6.3.5}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{X*} \left(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \left(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{F}^\bullet) \otimes^{\mathbb{L}}_{\mathcal{D}''_\bullet} \mathbb{L}\underleftarrow{l}^*_X \mathcal{G}^\bullet\right)\right) \underset{7.3.4.12.2}{\xrightarrow{\sim}} \mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \left(\mathcal{F}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''} \mathcal{G}^\bullet\right). \tag{7.3.4.12.4}$$

□

Remark 7.3.4.13. Without quasi-coherence hypotheses, the associativity isomorphism 7.3.4.12.1 is false. This is the main reason why Berthelot introduced his notion of quasi-coherence.

In the case of extension by ring homomorphisms, contrary to 7.3.4.12, the associativity becomes straightforward:

Proposition 7.3.4.14. *Suppose there exists a homomorphism of sheaf of rings on $\mathfrak{X}$ of the form $\mathcal{D} \to \mathcal{D}' \to \mathcal{D}''$ such that the composition of $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ with $\mathcal{D} \to \mathcal{D}'$ gives $\mathcal{O}_{\mathfrak{X}}^{\sharp} \to \mathcal{D}'$ and the composition of $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}'$ with $\mathcal{D}' \to \mathcal{D}''$ gives $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}''$.*

(a) For any Let $\mathcal{E}_\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet)$, we have the associativity isomorphism in $D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}''_\bullet)$:

$$\mathcal{D}''_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathcal{E}_\bullet \xrightarrow{\sim} \left(\mathcal{D}''_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}'_\bullet} \mathcal{D}'\right) \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathcal{E}_\bullet \xrightarrow{\sim} \mathcal{D}''_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}'_\bullet} \left(\mathcal{D}'_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} \mathcal{E}_\bullet\right). \tag{7.3.4.14.1}$$

(b) For any $\mathcal{E} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$, we have the associativity isomorphism in $D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'')$:

$$\mathcal{D}''\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}\mathcal{E} \xrightarrow{\sim} \left(\mathcal{D}''\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}'}\mathcal{D}'\right)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}\mathcal{E} \xrightarrow{\sim} \mathcal{D}''\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}'}\left(\mathcal{D}'\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}\mathcal{E}\right). \tag{7.3.4.14.2}$$

Proof. By using a flat resolution of we get the isomorphism 7.3.4.14.1. By using Theorem 8.5.1.10, we get 7.3.4.14.2 from 7.3.4.14.1. □

7.3.4.15. Suppose $p\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$. Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$, $(\mathcal{D}, \mathcal{D}'', \mathcal{I})$ are solved by $(\mathcal{R}, \mathcal{K})$. It follows from the universal properties of categories of fractions, that the functor 7.3.4.9.2 induces

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}-\colon D_{\mathbb{Q}}({}^*\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}}) \times D_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^*\mathcal{D}'') \to D_{\mathbb{Q}}({}^*\mathcal{D}', \mathcal{R}, {}^*\mathcal{D}''). \tag{7.3.4.15.1}$$

7.3.4.16. Let $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^-({}^l\mathcal{D})$ be two complexes of left $\mathcal{D}$-modules. Following 7.3.2.3.d, $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) = \mathcal{D}_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}_\bullet)$ and similarly for right modules. Hence, we can define their *completed internal homomorphism* by setting

$$\mathbb{R}\widehat{\mathscr{H}om}_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) := \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathbb{L}\underleftarrow{l}^*_X\mathcal{E}^\bullet, \mathbb{L}\underleftarrow{l}^*_X\mathcal{F}^\bullet)), \tag{7.3.4.16.1}$$

where $\mathbb{R}\underleftarrow{l}_{X*}\colon D(\mathbb{Z}_{X_\bullet}) \to D(\mathbb{Z}_{\mathfrak{X}})$ is the derived pushforward given by the topoi morphism $\underleftarrow{l}_X\colon (X_\bullet, \mathbb{Z}_{X_\bullet}) \to (|\mathcal{X}|, \mathbb{Z}_{\mathfrak{X}})$ and $\mathbb{L}\underleftarrow{l}^*_X\colon D({}^{\mathrm{l}}\mathcal{D}) \to D({}^{\mathrm{l}}\mathcal{D}_\bullet)$ is the derived pullback given by the topoi morphism $\underleftarrow{l}_X\colon (X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$.

Suppose $(\mathcal{D}, \mathcal{D}, \mathcal{I})$ is solved by $(\mathcal{R}, \mathcal{K})$. Then we have the morphism we have the map of $D(\mathcal{R}_\bullet)$:

$$\underleftarrow{l}^{-1}_X\mathbb{R}\mathscr{H}om_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \overset{4.6.5.5.1}{\longrightarrow} \mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathbb{L}\underleftarrow{l}^*_X\mathcal{E}^\bullet, \mathbb{L}\underleftarrow{l}^*_X\mathcal{F}^\bullet)).$$

Hence, by adjunction we get the morphism of $D(\mathcal{R})$:

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathbb{R}\widehat{\mathscr{H}om}_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet). \tag{7.3.4.16.2}$$

Remark that if $\mathcal{D}$ is commutative, then $(\mathcal{D}, \mathcal{D}, \mathcal{I})$ is solved by $(\mathcal{D}, \mathcal{I})$ and the map 7.3.4.16.2 is a morphism of $D(\mathcal{D})$.

Proposition 7.3.4.17. *Suppose $\mathfrak{X}$ is quasi-compact. Suppose $(\mathcal{D}_\bullet, \mathcal{D}'_\bullet)$ and $(\mathcal{D}_\bullet, \mathcal{D}''_\bullet)$ are solved by $\mathcal{R}_\bullet$. Let $*, ** \in \{r, l\}$. Let $\star \in \{\mathrm{b}, -\}$. Let $\mathcal{E}^\bullet_\bullet \in D_{\mathrm{perf}}({}^*\mathcal{D}_\bullet)$, $\mathcal{F}^\bullet_\bullet \in D^\star_{\cdot,\mathrm{qc}}({}^*\mathcal{D}_\bullet, \mathcal{R}_\bullet, {}^{**}\mathcal{D}'_\bullet)$. Then $\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}^\bullet_\bullet, \mathcal{F}^\bullet_\bullet) \in D^\star_{\mathrm{qc}}({}^{**}\mathcal{D}'_\bullet)$. If moreover $\mathcal{F}^\bullet_\bullet \in D_{\mathrm{perf}}({}^{**}\mathcal{D}'_\bullet)$, then $\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}^\bullet_\bullet, \mathcal{F}^\bullet_\bullet) \in D_{\mathrm{perf}}({}^{**}\mathcal{D}'_\bullet)$.*

Proof. a) Suppose $\star = \mathrm{b}$. By using 7.1.3.23, we get $(\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}^\bullet_\bullet, \mathcal{F}^\bullet_\bullet))_0 \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_0}(\mathcal{E}^\bullet_0, \mathcal{F}^\bullet_0) \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{**}\mathcal{D}'_0)$. Hence, by using loc. cit., we reduce to check that the morphism

$$\mathcal{R}_j \otimes^{\mathbb{L}}_{\mathcal{R}_i} \mathbb{R}\mathcal{H}om_{\mathcal{D}_i}(\mathcal{E}^\bullet_i, \mathcal{F}^\bullet_i) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}^\bullet_j, \mathcal{F}^\bullet_j). \tag{7.3.4.17.1}$$

induced by 7.1.3.19.2 is an isomorphism. This morphism is equal to the composition

$$\mathcal{R}_j \otimes^{\mathbb{L}}_{\mathcal{R}_i} \mathbb{R}\mathcal{H}om_{\mathcal{D}_i}(\mathcal{E}^\bullet_i, \mathcal{F}^\bullet_i) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{R}_j \otimes^{\mathbb{L}}_{\mathcal{R}_i} \mathcal{E}^\bullet_i, \mathcal{R}_j \otimes^{\mathbb{L}}_{\mathcal{R}_i} \mathcal{F}^\bullet_i) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_j}(\mathcal{E}^\bullet_j, \mathcal{F}^\bullet_j).$$

Following 7.1.3.13, since $\mathcal{E}_i^\bullet \in D_{\mathrm{perf}}(\mathcal{D}_i)$, then we can check that the first morphism is an isomorphism. Since $\mathcal{E}_\bullet^\bullet$ and $\mathcal{F}_\bullet^\bullet$ are quasi-coherent, then so is the second morphism.

b) Let $\mathcal{F}_\bullet^\bullet \in D^-_{.,\mathrm{qc}}({}^*\mathcal{D}_\bullet, \mathcal{R}_\bullet, {}^{**}\mathcal{D}'_\bullet)$. Since the property $\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathcal{E}_\bullet^\bullet, \mathcal{F}_\bullet^\bullet) \in D^-_{\mathrm{qc}}({}^{**}\mathcal{D}'_\bullet)$ is Zariski local in $\mathfrak{X}$, we can suppose $\mathcal{E}_\bullet^\bullet \in C({}^*\mathcal{D}_\bullet)$ is a strictly perfect complex (see definition 7.1.3.10). Then this is checked similarly to the proof of a). □

7.3.4.18. Suppose $(\mathcal{D}, \mathcal{D}', \mathcal{I})$ and $(\mathcal{D}, \mathcal{D}'', \mathcal{I})$ are solved by $(\mathcal{R}, \mathcal{K})$. Since $(\mathcal{D}_\bullet, \mathcal{D}'_\bullet)$ and $(\mathcal{D}_\bullet, \mathcal{D}''_\bullet)$ are solved by $\mathcal{R}_\bullet$, then following 7.1.3.23.1, we get the bifunctor:

$$\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{D}_\bullet}(-,-)\colon D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}'^{\mathrm{r}}_\bullet) \times D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}_\bullet, \mathcal{D}''^{\mathrm{r}}_\bullet) \to D({}^{\mathrm{l}}\mathcal{D}'_\bullet, \mathcal{R}_\bullet, \mathcal{D}''^{\mathrm{r}}_\bullet). \tag{7.3.4.18.1}$$

By setting $\mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{D}}(-,-) := \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{R}\mathscr{H}om_{\mathcal{D}_\bullet}(\mathbb{L}\underleftarrow{l}^*_X -, \mathbb{L}\underleftarrow{l}^*_X -))$, this yields the bifunctor

$$\mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{D}}(-,-)\colon D({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, {}^{\mathrm{l}}\mathcal{D}) \times D({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^{\mathrm{l}}\mathcal{D}'') \to D({}^{\mathrm{r}}\mathcal{D}', \mathcal{R}, {}^{\mathrm{l}}\mathcal{D}''). \tag{7.3.4.18.2}$$

Let $\mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}', \mathcal{R}, \mathcal{D}^{\mathrm{r}})$, $\mathcal{F}^\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}, \mathcal{R}, {}^{\mathrm{l}}\mathcal{D}'')$ be two complexes. Similarly to 7.3.4.16.2 , we construct the canonical morphism

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \tag{7.3.4.18.3}$$

which is an isomorphism when $\mathcal{E}^\bullet \in D_{\mathrm{perf}}(\mathcal{D}^{\mathrm{r}})$ and $\mathcal{F}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$. Indeed, since this is local, we reduce to the case where $\mathcal{E}^\bullet = \mathcal{D}$, which is obvious.

7.4 Up to isogeny complexes

7.4.1 Quotient and localization of triangulated categories, general derived functors reminders

The purpose of this subsection is to fix some terminologies (which are not standard) and recall some properties.

Definition 7.4.1.1. Let $\mathfrak{D}$ be a pre-triangulated category. We say a full pre-triangulated subcategory $\mathfrak{D}'$ of $\mathfrak{D}$ is "saturated" if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathfrak{D}'$. Recall that this notion is equivalent to that of "épaisse triangulated subcategory" but is more convenient to handle (see [Sta22, 05RB-13.6.1]).

7.4.1.2. Let $\mathfrak{D}$ be a triangulated category. Let S be a multiplicative systems in $\mathfrak{D}$ compatible with the triangulated structure. Then we have the localisation functor $Q_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ (see [Sta22, 05R6-13.5.5]). Let $\mathfrak{B}(S) := \ker(Q_S) := \{X \in \mathrm{Ob}(\mathfrak{D}) \mid Q_S(X) \xrightarrow{\sim} 0\}$. Then $\mathfrak{B}(S)$ is a strictly full saturated pre-triangulated subcategory of $\mathfrak{D}$ (see [Sta22, 05RC-13.6.2]).

Let us recall the following Lemma.

Lemma 7.4.1.3. *Let $\mathfrak{D}$ be a triangulated category. Let $\mathfrak{B} \subset \mathfrak{D}$ be a full triangulated subcategory. Let $S(\mathfrak{B})$ be the family of morphisms f of $\mathfrak{D}$ such that there exists a distinguished triangle of $\mathfrak{D}$ of the form (X, Y, Z, f, g, h) with Z isomorphic to an object of $\mathfrak{B}$.*

(i) Then $S(\mathfrak{B})$ is a multiplicative system compatible with the triangulated structure on $\mathfrak{D}$.

(ii) In this situation the following are equivalent

  (a) $S(\mathfrak{B})$ is a saturated multiplicative system,

  (b) $\mathfrak{B}$ is a saturated triangulated subcategory.

Proof. See [Sta22, 05RG-13.6.6]. □

Proposition 7.4.1.4. *Let $\mathfrak{D}$ be a triangulated category. The operations described above have the following properties*

(a) For any multiplicative system S in $\mathfrak{D}$ compatible with the triangulated structure, $S_{\mathrm{sat}} := S(\mathfrak{B}(S))$ is the "saturation" of S, i.e., it is the smallest saturated multiplicative system in $\mathfrak{D}$ containing S.

(b) *For any full triangulated subcategory $\mathfrak{B} \subset \mathfrak{D}$, $\mathfrak{B}_{\text{sat}} := \mathfrak{B}(S(\mathfrak{B}))$ is the "saturation" of $\mathfrak{B}$, i.e., it is the smallest strictly full saturated triangulated subcategory of $\mathfrak{D}$ containing $\mathfrak{B}$.*

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in $\mathfrak{D}$ compatible with the triangulated structure on $\mathfrak{D}$ and the (partially ordered) set of strictly full saturated triangulated subcategories of $\mathfrak{D}$.

Proof. See [Sta22, 05RL]. □

Notation 7.4.1.5 (Quotient of triangulated category)**.** Let $\mathfrak{D}$ be a triangulated category.

Let $\mathfrak{B} \subset \mathfrak{D}$ be a full triangulated subcategory. We set $\mathfrak{D}/\mathfrak{B} := S(\mathfrak{B})^{-1}\mathfrak{D}$ and $Q_{\mathfrak{B}} : \mathfrak{D} \to \mathfrak{D}/\mathfrak{B}$ be the localization functor. Following [Sta22, 05RJ–13.6.8.(2)] the functor $Q_{\mathfrak{B}}$ satisfies the following universal property: for any exact functor of triangulated categories $F: \mathfrak{D} \to \mathfrak{D}'$ such that $\mathfrak{B} \subset \operatorname{Ker} F$, there exists a unique factorization $G: \mathfrak{D}/\mathfrak{B} \to \mathfrak{D}'$ such that $F = G \circ Q_{\mathfrak{B}}$. Since $\mathfrak{B} \subset \operatorname{Ker} Q_{\mathfrak{B}} = \mathfrak{B}_{\text{sat}} = \operatorname{Ker} Q_{\mathfrak{B}_{\text{sat}}}$, from this universal properties satisfies respectively by $Q_{\mathfrak{B}}$ and $Q_{\mathfrak{B}_{\text{sat}}}$, we can construct canonical quasi-inverse equivalences of categories $\mathfrak{D}/\mathfrak{B} \to \mathfrak{D}/\mathfrak{B}_{\text{sat}}$ and $\mathfrak{D}/\mathfrak{B}_{\text{sat}} \to \mathfrak{D}/\mathfrak{B}$.

Moreover, let S be a multiplicative systems in $\mathfrak{D}$ compatible with the triangulated structure. Since $S \subset S_{\text{sat}}$, by using the universal properties satisfied by $Q_{S_{\text{sat}}}$, we get the functor $S^{-1}\mathfrak{D} \to S_{\text{sat}}^{-1}\mathfrak{D}$. Since $S_{\text{sat}} = \{f \in \mathit{Arrow}(\mathfrak{D})$, such that $Q_S(f)$ is an isomorphism$\}$, then by using the universal properties satisfied respectively by Q_S we get the functor $S_{\text{sat}}^{-1}\mathfrak{D} \to S^{-1}\mathfrak{D}$, which is quasi-inverse to $S^{-1}\mathfrak{D} \to S_{\text{sat}}^{-1}\mathfrak{D}$. This yields $S^{-1}\mathfrak{D} \cong \mathfrak{D}/\mathfrak{B}(S)$. Hence, the difference between a localization of $\mathfrak{D}$ and a quotient of $\mathfrak{D}$ is just a matter of point of view. Moreover, both saturations appearing in 7.4.1.4 do not affect the associated quotient or localization of a triangulated category.

Lemma 7.4.1.6. *Let $\mathfrak{D}$ be a triangulated category. Let $\mathfrak{B}$ be a full triangulated subcategory of $\mathfrak{D}$ and $\mathfrak{B}'$ be a full triangulated subcategory of $\mathfrak{D}/\mathfrak{B}$. Then we have the equivalence of categories:*

$$\mathfrak{D}/\ker(Q_{\mathfrak{B}'} \circ Q_{\mathfrak{B}}) \cong (\mathfrak{D}/\mathfrak{B})/\mathfrak{B}'. \tag{7.4.1.6.1}$$

Proof. Since we are dealing with quotients, we can suppose that $\mathfrak{B} = \mathfrak{B}_{\text{sat}}$ and $\mathfrak{B}' = \mathfrak{B}'_{\text{sat}}$ (see 7.4.1.5). The, this can be easily checked by using the universal property of the quotient given at 7.4.1.5. □

Proposition 7.4.1.7. *Let $H: \mathfrak{D} \to \mathfrak{A}$ be a homological functor from a triangulated category $\mathfrak{D}$ to an abelian category $\mathfrak{A}$. The subcategory $Ker(H) := \{X \in \operatorname{Ob}(\mathfrak{D}) \mid H(X[n]) = 0$ for all $n \in \mathbb{Z}\}$ is a strictly full saturated triangulated subcategory of $\mathfrak{D}$ whose corresponding saturated multiplicative system (see Lemma 7.4.1.4) is the set $S = \{f \in \operatorname{Arrows}(\mathfrak{D}) \mid H^i(f)$ is an isomorphism for all $i \in Z\}$. The functor H factors through the quotient functor $Q: \mathfrak{D} \to \mathfrak{D}/Ker(H)$.*

Proof. See [Sta22, 05RM-13.6.11]. □

Definition 7.4.1.8. Let $\mathfrak{C}$ be a triangulated category. A null system in the sense of [KS06, 10.2.2] is a strictly full triangulated subcategories (beware in this latter book "saturatedness" means "strictness", i.e. the stability under isomorphisms property). Recall that when the null system is saturated (in the sense of 7.4.1.1) then it corresponds bijectively via 7.4.1.4 to a saturated multiplicative system of $\mathfrak{C}$.

Definition 7.4.1.9. Let $\mathfrak{C}$ be a category, S a right multiplicative system in $\mathfrak{C}$, $Q_S: \mathfrak{D} \to S^{-1}\mathfrak{D}$ be the localization functor and $F: \mathfrak{C} \to \mathfrak{A}$ be a functor.

(a) We say that F is "right localizable (with respect to S)" (in the sense of [KS06, 7.3.1]) if there exists a (unique up to isomorphism) functor $\mathbb{R}_S F: S^{-1}\mathfrak{C} \to \mathfrak{A}$ together with a morphism of functors $\tau: F \to \mathbb{R}_S F \circ Q_S$ such that for any functor $G: : S^{-1}\mathfrak{C} \to \mathfrak{A}$ the map induced by τ and the functor $\circ Q_S$
$$\operatorname{Hom}_{\operatorname{Fct}(S^{-1}\mathfrak{C},\mathfrak{A})}(\mathbb{R}_S F, G) \to \operatorname{Hom}_{\operatorname{Fct}(\mathfrak{C},\mathfrak{A})}(F, G \circ Q_S), \tag{7.4.1.9.1}$$
where $\operatorname{Fct}(S^{-1}\mathfrak{C}, \mathfrak{A})$ and $\operatorname{Fct}(\mathfrak{C}, \mathfrak{A})$ are the corresponding category of functors, is bijective.

(b) We say that F is "universally right localizable (with respect to S)" if for any functor $K: \mathfrak{A} \to \mathfrak{A}'$, the functor $K \circ F$ is localizable and $\mathbb{R}_S(K \circ F) \xrightarrow{\sim} K \circ \mathbb{R}_S F$.

(c) Let $X \in \mathfrak{C}$ and X/S be the filtered category of arrows $s\colon X \to X'$ in S with source X. Following definition [Sta22, 05S9-13.14.2] (or [KS06, 7.4.2]), we say the "right derived functor of F (with respect to S) is defined at X" if the ind-object

$$\alpha^X\colon (X/S) \to \mathfrak{A}, \qquad (s\colon X \to X') \mapsto F(X') \tag{7.4.1.9.2}$$

is essentially constant, i.e. "$\varinjlim$"α^X is representable by an object of $\mathfrak{A}$ which is denoted $\mathbb{R}_S F(X)$. The $\mathbb{R}F(X)$ is called the value of $\mathbb{R}_S F$ at X.

(d) Assume that for any $X \in \mathfrak{C}$, the category X/S is cofinally small. Following [KS06, 7.4.4], the two conditions below are equivalent (and both $\mathbb{R}_S F(X)$ coincide for each $X \in \mathfrak{C}$):

  (i) F is right localizable at each $X \in \mathfrak{C}$ (with respect to S),

  (ii) F is universally right localizable (with respect to S).

Definition 7.4.1.10. Let $F\colon \mathfrak{D} \to \mathfrak{D}'$ be a triangulated functor of triangulated categories. Let $\mathfrak{N}$ (resp. $\mathfrak{N}'$) be a null system of $\mathfrak{D}$ (resp. $\mathfrak{D}'$). We say that the functor F is "right localizable with respect to $(\mathfrak{N}, \mathfrak{N}')$" if $Q_{\mathfrak{N}'} \circ F$ is is universally right localizable with respect to the multiplicative system $S(\mathfrak{N})$ (see definition 7.4.1.9). When it exists, we denote the right localization with respect to $(\mathfrak{N}, \mathfrak{N}')$ by $\mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F\colon \mathfrak{D}/\mathfrak{N} \to \mathfrak{D}'/\mathfrak{N}'$. When $\mathfrak{N}' = 0$, we simply say F is "right localizable with respect to $\mathfrak{N}$" and we simply write $\mathbb{R}_{\mathfrak{N}} F$.

Remark 7.4.1.11. With notation 7.4.1.10, suppose there exists a functor $G\colon \mathfrak{D}/\mathfrak{N} \to \mathfrak{D}'/\mathfrak{N}'$ such that there exists an isomorphism of the form $\phi\colon Q_{\mathfrak{N}'} \circ F \xrightarrow{\sim} G \circ Q_{\mathfrak{N}}$. Remark that for any functors $G_i\colon \mathfrak{D}/\mathfrak{N} \to \mathfrak{D}'/\mathfrak{N}'$ pour $i = 1, 2$, the data of a functor $G_1 \to G_2$ is the same as the data of a functor $G_1 \circ Q_{\mathfrak{N}} \to G_2 \circ Q_{\mathfrak{N}}$. Using this remark, we can check the right localizable with respect to $(\mathfrak{N}, \mathfrak{N}')$ of F exists and we can choose $G = \mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F$ and the $Q_{\mathfrak{N}'} \circ F \to \mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F \circ Q_{\mathfrak{N}}$ equal to ϕ.

Remark 7.4.1.12. With notation 7.4.1.10, su suppose that $X/S(\mathfrak{N})$ is cofinally small, for any $X \in \mathfrak{D}$.

(a) Then if F is right localizable with respect to $(\mathfrak{N}, \mathfrak{N}')$, then $\mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F$ is a triangulated functor of triangulated categories. Indeed, following 7.4.1.9.d, this means that $Q_{\mathfrak{N}'} \circ F$ is right localizable with respect to $\mathfrak{N}$ at each $X \in \mathfrak{D}$ and we have $\mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F = \mathbb{R}_{S(\mathfrak{N})}(Q_{\mathfrak{N}'} \circ F)$. Hence, we conclude by using [Sta22, 05SE-13.14.8].

(b) It follows from a) that the triangulated functor $\mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F$ satisfies a similar universal property than 7.4.1.9.1 where we replace functors by triangulated functors. In the notion of "right localizable with respect to $(\mathfrak{N}, \mathfrak{N}')$" in the sense of [KS06, 10.3.1], the functor $\mathbb{R}^{\mathfrak{N}'}_{\mathfrak{N}} F$ is by definition a triangulated functor, i.e. F is "universally right localizable" in the sense that we restrict to triangulated functors. Since our functors will be right localizable with respect to $\mathfrak{N}$ at each $X \in \mathfrak{D}$, since we prefer to avoid confusion between two different notions of "universally right localizable", we stick with our definition 7.4.1.10 which is a priori stronger than [KS06, 10.3.1].

Lemma 7.4.1.13. *Let $\mathfrak{D}_1,\ \mathfrak{D}'_1,\ \mathfrak{D}$ be some triangulated categories endowed respectively with the null systems $\mathfrak{N}_1,\ \mathfrak{N}'_1, \mathfrak{N}$. Let $\mathfrak{N}_2,\ \mathfrak{N}'_2$ be some null systems of respectively $\mathfrak{D}_2 := \mathfrak{D}_1/\mathfrak{N}_1,\ \mathfrak{D}'_2 := \mathfrak{D}'_1/\mathfrak{N}'_1$ (see notations 7.4.1.5). Let $\mathfrak{N}_3,\ \mathfrak{N}'_3$ be some null systems of respectively $\mathfrak{D}_1,\ \mathfrak{D}'_1$ such that $\mathfrak{D}_1 \to \mathfrak{D}_2/\mathfrak{N}_2$ and $\mathfrak{D}'_1 \to \mathfrak{D}'_2/\mathfrak{N}'_2$ induce the equivalence of categories $\mathfrak{D}_1/\mathfrak{N}_3 \cong \mathfrak{D}_2/\mathfrak{N}_2$, $\mathfrak{D}'_1/\mathfrak{N}'_3 \cong \mathfrak{D}'_2/\mathfrak{N}'_2$.*

Let $F\colon \mathfrak{D}_1 \times \mathfrak{D}'_1 \to \mathfrak{D}$ a triangulated bifunctor. We assume that the right localization of F with respect to $(\mathfrak{N}_1 \times \mathfrak{N}'_1, \mathfrak{N})$ exists (see the definition 7.4.1.9) and is denoted by $\mathbb{R}^{\mathfrak{N}}_{\mathfrak{N}_1 \times \mathfrak{N}'_1} F$. If one of the following conditions:

(a) the right localization of F with respect to $(\mathfrak{N}_3 \times \mathfrak{N}'_3, \mathfrak{N})$ exists,

(b) the right localization of $\mathbb{R}^{\mathfrak{N}}_{\mathfrak{N}_1 \times \mathfrak{N}'_1} F$ with respect to $\mathfrak{N}_2 \times \mathfrak{N}'_2$ exists

is satisfied, then so is the second one and we have in this case the isomorphism of bifunctors

$$\mathbb{R}_{\mathfrak{N}_2 \times \mathfrak{N}'_2} \mathbb{R}^{\mathfrak{N}}_{\mathfrak{N}_1 \times \mathfrak{N}'_1} F \xrightarrow{\sim} \mathbb{R}^{\mathfrak{N}}_{\mathfrak{N}_3 \times \mathfrak{N}'_3} F. \tag{7.4.1.13.1}$$

Proof. This follows from the universal property of right localisations. □

7.4.2 Localising by isogenies, Serre subcategories

For any abelian sheaf $\mathcal{E}$, we write $\mathcal{E}_{\mathbb{Q}} = \mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Q}$.

7.4.2.1. Let $\mathfrak{C}$ be an additive category.

(a) We denote by $\mathfrak{C}_{\mathbb{Q}}$ the category of objects of $\mathfrak{C}$ up to isogeny. By definition $\mathrm{Ob}(\mathfrak{C}_{\mathbb{Q}}) = \mathrm{Ob}(\mathfrak{C})$ and for any objects E, F of $\mathrm{Ob}(\mathfrak{C}_{\mathbb{Q}})$ we set

$$\mathrm{Hom}_{\mathfrak{C}_{\mathbb{Q}}}(E, F) := \mathrm{Hom}_{\mathfrak{C}}(E, F) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{7.4.2.1.1}$$

Via the maps $\mathrm{Hom}_{\mathfrak{C}}(E, F) \to \mathrm{Hom}_{\mathfrak{C}_{\mathbb{Q}}}(E, F)$ given by $f \mapsto f \otimes 1$ and the identities on the objects, we get a canonical functor $\mathfrak{C} \to \mathfrak{C}_{\mathbb{Q}}$ which essentially surjective but not faithful. Remark that if $\mathfrak{D}$ is a full subcategory of $\mathfrak{C}$ then $\mathfrak{D}_{\mathbb{Q}}$ is a full subcategory of $\mathfrak{C}_{\mathbb{Q}}$.

(b) When $\mathfrak{C}$ is a equal to a category of the form $D^*(\mathfrak{A})$ where $* \in \{\emptyset, -, +, \mathrm{b}\}$ and $\mathfrak{A}$ is an abelian category, we often write $D^*_{\mathbb{Q}}(\mathfrak{A})$ instead of $D^*(\mathfrak{A})_{\mathbb{Q}}$.

(c) Let $f \in \mathrm{Hom}_{\mathfrak{C}}(E, F)$. We say that f is an "n-isogeny" if $n \geq 1$ is an integer so that there exists and a morphism $g \in \mathrm{Hom}_{\mathfrak{C}}(F, E)$ such that $f \circ g = n\,\mathrm{id}_F$ and $g \circ f = n\,\mathrm{id}_E$. We say that f is an "isogeny" if there exists an integer $n \geq 1$ such that f is an n-isogeny. Let $\Sigma \subset \mathit{Arrows}(\mathfrak{C})$ be the family of isogenies of $\mathfrak{C}$.

Lemma 7.4.2.2. *The family Σ of isogenies is a saturated multiplicative system. A morphism f of $\mathfrak{C}$ is an isogeny if and only if it is an isomorphism of $\mathfrak{C}_{\mathbb{Q}}$. Moreover, if $\mathfrak{C}_{\Sigma} := \Sigma^{-1}\mathfrak{C}$ is the localization category (see [KS06, 7.1.16]), then the canonical functor $\mathfrak{C} \to \mathfrak{C}_{\mathbb{Q}}$ factors into the equivalence of categories $\mathfrak{C}_{\Sigma} \xrightarrow{\sim} \mathfrak{C}_{\mathbb{Q}}$.*

Proof. 1) i) Let us check that Σ is a right multiplicative system, i.e. the properties RMS1–3 hold (see [Sta22, 04VC-4.27.1].

RMS1: the identity is an isogeny.

RMS2 Let $f : E \to F$ be an isogeny of $\mathfrak{C}$ and $h \in \mathrm{Hom}_{\mathfrak{C}}(G, F)$. Let $n \geq 1$ be an integer, $g \in \mathrm{Hom}_{\mathfrak{C}}(F, E)$ such that $f \circ g = n\,\mathrm{id}_F$ and $g \circ f = n\,\mathrm{id}_E$. Then $g \circ h \in \mathrm{Hom}_{\mathfrak{C}}(G, E)$ and $n_G \in G \to G$ is an isogeny such that $h \circ n_G = f \circ (g \circ h)$.

RMS3 Let $f, g \in \mathrm{Hom}_{\mathfrak{C}}(E, F)$, $h : F \to G$ be an isogeny such that $h \circ f = h \circ g$. Then there exists an integer $n \geq 1$ such that $n_F \circ f = n_F \circ g$. Hence, $f \circ n_E = g \circ n_E$.

ii) We check similarly the dual properties, i.e. Σ is a left multiplicative system.

2) An isogeny of $\mathfrak{C}$ is an isomorphism in $\mathfrak{C}_{\mathbb{Q}}$. More precisely, let $n \geq 1$ be an integer, $f \in \mathrm{Hom}_{\mathfrak{C}}(F', F)$, $g \in \mathrm{Hom}_{\mathfrak{C}}(F, F')$ such that $f \circ g = n\,\mathrm{id}_F$ and $g \circ f = n\,\mathrm{id}_{F'}$. Then $f \otimes 1$, the image of f in $\mathrm{Hom}_{\mathfrak{C}}(F, F') \otimes_{\mathbb{Z}} \mathbb{Q}$, has the inverse $g \otimes \frac{1}{n}$.

Conversely, let $f \in \mathrm{Hom}_{\mathfrak{C}}(F', F)$ such that $f \otimes 1 \in \mathrm{Hom}_{\mathfrak{C}_{\mathbb{Q}}}(F', F)$ is an isomorphism. Then there exists an integer $n \geq 1$, there exists $g \in \mathrm{Hom}_{\mathfrak{C}}(F, F')$ such that $(f \otimes 1) \circ (g \otimes \frac{1}{n}) = \frac{1}{1}$ and $(g \otimes \frac{1}{n}) \circ (f \otimes 1) = \frac{1}{1}$, which yields, increasing n if necessary, $f \circ g = n\,\mathrm{id}_F$ and $g \circ f = n\,\mathrm{id}_{F'}$.

3) It follows from 2), that Σ is saturated.

4) By using the universal property on localisation functor, we get the factorization $\mathfrak{C}_{\Sigma} \to \mathfrak{C}_{\mathbb{Q}}$. Let $E, F \in \mathrm{Ob}(\mathfrak{C})$, it remains to check that the map $\mathrm{Hom}_{\mathfrak{C}_{\Sigma}}(E, F) \to \mathrm{Hom}_{\mathfrak{C}}(E, F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a bijection. Injectivity: If the equivalence class of $E \xrightarrow{\phi} F' \xleftarrow{f} F$ with f an isogeny is sent to zero, then ϕ is sent to zero. Hence, there exists $n \geq 1$ such that $n_{F'} \circ \phi = 0$, which implies that $E \xrightarrow{\phi} F' \xleftarrow{f} F$ is the class of zero. Surjectivity: the equivalence class of $E \xrightarrow{f} F \xleftarrow{n_F} F$ is sent to $f \otimes \frac{1}{n}$. ☐

7.4.2.3. Let F be an object of $\mathfrak{C}_{\mathbb{Q}}$. We denote by $\mathbb{N}^{\star}$ the category whose objects are the integers $n \geq 1$ and $\mathrm{Hom}_{\mathbb{N}^{\star}}(n, m)$ is empty if n do not divide m otherwise it is a singleton. Consider the functor $\chi_F : \mathbb{N}^{\star} \to \mathfrak{C}$ given for any integers $n, d \geq 1$ by $\chi_F(n) = F$ and $\chi_F(n \to nd)$ it the multiplication by d. We denote by $n^*F := \chi_F(n)$. For any objects E, F of $\mathfrak{C}_{\mathbb{Q}}$, we get from 7.4.2.2 the equality:

$$\mathrm{Hom}_{\mathfrak{C}_{\mathbb{Q}}}(E, F) = \mathrm{Hom}_{\mathfrak{C}_{\Sigma}}(E, F) = \varinjlim_{n \in \mathbb{N}^{\star}} \mathrm{Hom}_{\mathfrak{C}}(E, n^*F), \tag{7.4.2.3.1}$$

Proposition 7.4.2.4. *Let $\mathfrak{C}$ be a triangulated category. There exists on $\mathfrak{C}_{\mathbb{Q}}$ a unique triangulated structure such that the functor $Q : \mathfrak{C} \to \mathfrak{C}_{\mathbb{Q}}$ is an exact functor.*

Proof. We apply 7.4.1.7 to the family of functors $\mathrm{Hom}_{\mathfrak{C}}(E,-)\otimes\mathbb{Q}\colon \mathfrak{C}\to\mathfrak{Ab}$. □

Let us now consider the case of an abelian category. First, let us collect few facts on Serre subcategories and localisations.

7.4.2.5 (Serre subcategories)**.** Let $\mathfrak{A}$ be an abelian category. Let S be a multiplicative system of $\mathfrak{A}$. Then $S^{-1}\mathfrak{A}$ is an abelian category and the localisation functor $Q_S\colon \mathfrak{A}\to S^{-1}\mathfrak{A}$ is exact (see [Sta22, 05QG]). It follows from [Sta22, 02MQ] that $\mathfrak{B}(S) := \mathrm{Ker}\, Q_S$, where $\mathrm{Ker}\, Q_S$ is the full subcategory of objects X of $\mathfrak{A}$ such that $Q_S(X)=0$, forms a Serre subcategory of $\mathfrak{A}$.

Let $\mathfrak{B}\subset\mathfrak{A}$ be a Serre subcategory (see [Sta22, 02MO]). Consider the set of arrows of $\mathfrak{A}$ defined by the following formula

$$S(\mathfrak{B}) := \{f\in \mathit{Arrows}(\mathfrak{A}) \mid \mathrm{Ker}(f),\, \mathrm{Coker}(f)\in \mathit{Ob}(\mathfrak{B})\}.$$

Then $S(\mathfrak{B})$ is a saturated multiplicative system (see the proof of [Sta22, 02MS]) such that $\mathfrak{B}(S(\mathfrak{B}))=\mathfrak{B}$ (this is a consequence of [Sta22, 06XK]). We set $\mathfrak{A}/\mathfrak{B} := (S(\mathfrak{B}))^{-1}\mathfrak{A}$. Following [Sta22, 02MS], the category $\mathfrak{A}/\mathfrak{B}$ and the localisation functor $F\colon\mathfrak{A}\to\mathfrak{A}/\mathfrak{B}$ are characterized by the following universal property: for any exact functor $G\colon\mathfrak{A}\to\mathfrak{C}$ such that $\mathfrak{B}\subset\mathrm{Ker}(G)$, there exists a factorization $G=H\circ F$ for a unique exact functor $H\colon\mathfrak{A}/\mathfrak{B}\to\mathfrak{C}$.

Let S be a multiplicative system of $\mathfrak{A}$. Then we easily see that $S(\mathfrak{B}(S))$ is equal to $\widehat{S}=\{f\in \mathit{Arrows}(\mathfrak{A})\mid Q_S(f)$ is an isomorphism$\}$, which is also the smallest saturated multiplicative system containing S (see [Sta22, 05Q9]). Hence, $S\mapsto\mathfrak{B}(S)$ and $\mathfrak{B}\mapsto S(\mathfrak{B})$ are reciprocal bijections of each other between the set of saturated multiplicative systems of $\mathfrak{A}$ and Serre subcategories of $\mathfrak{A}$.

Lemma 7.4.2.6. *Let $\mathfrak{A}$ be an abelian category, $n,m\in\mathbb{N}\setminus\{0\}$.*

(a) Let $X\xrightarrow{f}Y\xrightarrow{g}Z$ be two morphisms of $\mathfrak{A}$. If f is an n-isogeny of $\mathfrak{A}$ and g is an m-isogeny of $\mathfrak{A}$, then $g\circ f$ is an nm-isogeny of $\mathfrak{A}$.

(b) Let $0\to X\xrightarrow{f}Y\xrightarrow{g}Z\to 0$ be an exact sequence of $\mathfrak{A}$. If X is killed by n then g is an n-isogeny of $\mathfrak{A}$. If Z is killed by n then f is an n-isogeny of $\mathfrak{A}$.

(c) Let $f\colon X\to Y$ be an n-isogeny of $\mathfrak{A}$. Then, $\mathrm{Ker}\, f$ *and* $\mathrm{Coker}\, f$ *are killed by n.*

(d) Let $f\colon X\to Y$ be a morphism of $\mathfrak{A}$ such that $\mathrm{Ker}\, f$ *is killed by n and* $\mathrm{Coker}\, f$ *is killed by n, then f is an nm-isogeny of $\mathfrak{A}$*

Proof. Exercice. □

7.4.2.7. Let $\mathfrak{A}$ be an abelian category. Let $\mathfrak{B}$ be the full subcategory of $\mathfrak{A}$ consisting of objects $X\in\mathrm{Ob}\,\mathfrak{A}$ such that there exists $n\in\mathbb{N}\setminus\{0\}$ so that X is killed by n. Let Σ be the class of isogenies of $\mathfrak{A}$. We easily check from 7.4.2.2, 8.1.5.4 and 7.4.2.6 the following facts.

(a) The category $\mathfrak{B}$ is a Serre subcategory of $\mathfrak{A}$.

(b) The saturated multiplicative system of $\mathfrak{A}$ associated to the Serre subcategory $\mathfrak{B}$ of $\mathfrak{A}$ is equal to Σ.

(c) We have $\mathfrak{A}/\mathfrak{B}\cong\mathfrak{A}_{\mathbb{Q}}$. In particular, $\mathfrak{A}_{\mathbb{Q}}$ is an abelian category.

7.4.3 Commutation with tensorisation by $\mathbb{Q}$

7.4.3.1. Let $\mathfrak{X}$ be a topological space. Let $\mathcal{G}$ be a complex of abelian sheaves. Let $(\mathcal{G}_n)_{n\geq 1}$ be the inductive system such that $\mathcal{G}_n=\mathcal{G}$ and the transition maps are the maps $\mathcal{G}_n\to\mathcal{G}_{dn}$ which are the multiplication by d. Then $\varinjlim_n \mathcal{G}_n \xrightarrow{\sim} \mathcal{G}_{\mathbb{Q}}$, where the inductive limit is computed in the category of complexes of abelian sheaves.

Suppose now in the rest of the paragraph that $\mathfrak{X}$ is a coherent topological space (see [FK18, 0.2.2.1]) and that $\mathcal{G}$ is an abelian sheaf. It follows from [FK18, 0.3.1.8] that for any quasi-compact open subset $\mathfrak{U}$ (hence $\mathfrak{U}$ is coherent following [FK18, 0.2.2.3]) the canonical map $\varinjlim_n\Gamma(\mathfrak{U},\mathcal{G}_n)\to\Gamma(\mathfrak{U},\varinjlim_n\mathcal{G}_n)$ is an isomorphism. Hence, the canonical map $\Gamma(\mathfrak{U},\mathcal{G})_{\mathbb{Q}}\to\Gamma(\mathfrak{U},\mathcal{G}_{\mathbb{Q}})$ is an isomorphism. In particular, when $\mathfrak{X}$ is noetherian, then this means that $\mathfrak{U}\in\mathfrak{X}_{\mathrm{zar}}\mapsto\Gamma(\mathfrak{U},\mathcal{G})_{\mathbb{Q}}$ is a sheaf canonically isomorphic to $\mathcal{G}_{\mathbb{Q}}$.

Moreover, similarly, by using [FK18, 0.3.1.16] (and [FK18, 0.2.2.3]), for any complex $\mathcal{G}$ of abelian sheaves, for any coherent open subset $\mathfrak{U}$, we can check the canonical map

$$\mathbb{R}\Gamma(\mathfrak{U}, \mathcal{G})_{\mathbb{Q}} \to \mathbb{R}\Gamma(\mathfrak{U}, \mathcal{G}_{\mathbb{Q}}). \tag{7.4.3.1.1}$$

is an isomorphism.

Lemma 7.4.3.2. *Let $\mathfrak{X}$ be a coherent topological space, $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$. For any $\mathcal{E} \in D^-_{\mathrm{coh}}(\mathcal{D})$, $\mathcal{F} \in D^+(\mathcal{D})$, the canonical morphisms*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{E}, \mathcal{F})_{\mathbb{Q}} \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}(\mathcal{E}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}}), \tag{7.4.3.2.1}$$

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{F})_{\mathbb{Q}} \to \mathbb{R}\operatorname{Hom}_{\mathcal{D}_{\mathbb{Q}}}(\mathcal{E}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}}), \tag{7.4.3.2.2}$$

are isomorphisms.

Proof. Since $\mathfrak{X}$ is coherent, it follows from 7.4.3.1.1 that the canonical morphism $\mathbb{R}\Gamma(\mathfrak{X}, \mathcal{G})_{\mathbb{Q}} \to \mathbb{R}\Gamma(\mathfrak{X}, \mathcal{G}_{\mathbb{Q}})$ is an isomorphism for any complex $\mathcal{G}$ of abelian sheaves. Hence, the isomorphism 7.4.3.2.2 is a consequence of 4.6.2.7.6 and 7.4.3.2.1. To check this latter one, it is sufficient to prove this is an n-isomorphism for any integer $n \in \mathbb{N}$, i.e. the cone of 7.4.3.2.1 is acyclic in degree $\geq n$. Let $n \in \mathbb{N}$. Since this is local on $\mathfrak{X}$ and $\mathcal{E}$ is n-pseudo-coherent, then we can suppose there exists an n-isomorphism of the form $\mathcal{L} \to \mathcal{E}$ with $\mathcal{L}$ being a strictly perfect complex of $\mathcal{D}$-modules. Since $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{L}, \mathcal{F})_{\mathbb{Q}} \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}(\mathcal{L}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$ is an isomorphism (use [Har66, I.7.1.(i)]), then 7.4.3.2.1 is an n-isomorphism. □

7.4.4 Localisation by isogenies of derived categories of an abelian category

Let $\mathfrak{A}$ be an abelian category. Let $\mathfrak{B}$ be the full subcategory of $\mathfrak{A}$ consisting of objects $X \in \operatorname{Ob}\mathfrak{A}$ such that there exists $n \in \mathbb{N} \setminus \{0\}$ so that X is killed by n. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$. Let $S^\sharp$ be the class of isogenies of $D^\sharp_{\mathbb{Q}}(\mathfrak{A})$.

First, let us give below some link between isogenies and vanishing of the cone of the morphism. This will be useful later, e.g. at 9.1.2.1.

Lemma 7.4.4.1. *Let $n, m \in \mathbb{N} \setminus \{0\}$.*

(a) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms of $D(\mathfrak{A})$. If f is an n-isogeny of $D(\mathfrak{A})$ and g is an m-isogeny of $D(\mathfrak{A})$, then $g \circ f$ is an nm-isogeny of $D(\mathfrak{A})$.

(b) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle of $D(\mathfrak{A})$ such that Z is killed by n, i.e. $n \cdot \mathrm{id}_Z \colon Z \to Z$ is null. Then f is an n^2-isogeny of $D(\mathfrak{A})$.

(c) Let $f \colon X \to Y$ be an n-isogeny of $D(\mathfrak{A})$. Then, the cone $C(f)$ of f is killed by n^2.

(d) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle of $D(\mathfrak{A})$ such that X is killed by n and Y by m, then Z is killed by nm.

(e) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms of $D(\mathfrak{A})$. If f is an n-isogeny of $D(\mathfrak{A})$ and $g \circ f$ is an m-isogeny of $D(\mathfrak{A})$, then g is an n^2m-isogeny of $D(\mathfrak{A})$. If g is an n-isogeny of $D(\mathfrak{A})$ and $g \circ f$ is an m-isogeny of $D(\mathfrak{A})$, then f is an n^2m-isogeny of $D(\mathfrak{A})$.

(f) Let $\mathfrak{N}(\mathfrak{A})$ be the full subcategory of $D(\mathfrak{A})$ consisting in objects X which are killed by some integer $l \geq 1$. Then $\mathfrak{N}(\mathfrak{A})$ is a saturated triangulated subcategory of $D(\mathfrak{A})$ and the associated saturated multiplicative system $S(\mathfrak{N}(\mathfrak{A}))$ of $D(\mathfrak{A})$ (see notation of 7.4.1.3) consists of isogenies.

Proof. a) Straightforward: we compose the inverses up to isogeny.

b) Since (X, Y, Z, f, g, h) is a distinguished triangle, since $g \circ (n \cdot \mathrm{id}_Y) = (n \cdot \mathrm{id}_Z) \circ g = 0$, then there exists a morphism $\alpha \colon Y \to X$ of $D(\mathfrak{A})$ such that $f \circ \alpha = n \cdot \mathrm{id}_Y$. Since $(Z[-1], X, Y, -h[-1], f, g)$ is a distinguished triangle, since $f \circ (\alpha \circ f - n \cdot \mathrm{id}_X) = 0$, then $-h[-1] \circ \beta = \alpha \circ f - n \cdot \mathrm{id}_X$. Since $(n \cdot \mathrm{id}_X) \circ -h[-1] = -h[-1] \circ (n \cdot \mathrm{id}_{Z[-1]}) = 0$, then $(n \cdot \alpha) \circ f = n^2 \cdot \mathrm{id}_X$. Since $f \circ (n \cdot \alpha) = n^2 \cdot \mathrm{id}_Y$, then we are done.

c) We have the canonical distinguished triangle $(X, Y, C(f), f, g, h)$. Let $\alpha\colon Y \to X$ be a morphism of $D(\mathfrak{A})$ such that $f \circ \alpha = n \cdot \mathrm{id}_Y$ and $\alpha \circ f = n \cdot \mathrm{id}_X$. Since $h \circ n = n \circ h = \alpha[1] \circ f[1] \circ h = 0$, then there exits $\beta\colon Z \to Y$ a morphism of $D(\mathfrak{A})$ such that $g \circ \beta = n \cdot \mathrm{id}_Z$. Hence, $n^2 \cdot \mathrm{id}_Z = g \circ f \circ \alpha \circ \beta = 0$.

d) Since $h \circ n = n \circ h = 0$, then there exists $\alpha\colon Z \to Y$ such that $g \circ \alpha = n$. Hence, $mn \cdot \mathrm{id}_Z = m \cdot \mathrm{id}_Z \circ n \cdot \mathrm{id}_Z = m \cdot \mathrm{id}_Z \circ g \circ \alpha = g \circ m \cdot \mathrm{id}_Y \circ \alpha = 0$.

e) Since the second assertion is checked similarly, let us only treat the first one. By hypotheses, there exists $\alpha\colon Y \to X$ such that $f \circ \alpha = n$ and $\alpha \circ f = n$, there exists $\beta\colon Z \to X$ such that $g \circ f \circ \beta = m$ and $\beta \circ g \circ f = m$. Hence, $g \circ (n^2 \cdot f \circ \beta) = n^2 m$. We have the canonical distinguished triangle $(X, Y, C(f), f, a, b)$. Since $\beta \circ g \circ f = m$, then $(f \circ \beta \circ g) \circ f = m \circ f$. Hence, there exists $\gamma\colon C(f) \to Y$ such that $(f \circ \beta \circ g) - m = a \circ \gamma$. Since f is an n-isogeny, then $C(f)$ is killed by n^2. Hence, $(n^2 \cdot f \circ \beta) \circ g = n^2 m$.

f) This is a consequence of the previous statements. □

Lemma 7.4.4.2. *The canonical functor $\mathfrak{A}_{\mathbb{Q}} \to D_{\mathbb{Q}}(\mathfrak{A})$ is fully faithful.*

Proof. This comes from the fact that the application $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{E}, n^*\mathcal{F}) \to \mathrm{Hom}_{D(\mathfrak{A})}(\mathcal{E}, n^*\mathcal{F})$ is bijective for any $n \in \mathbb{N}$ and that we have the equalities 7.4.2.3.1 for $\mathfrak{C} = \mathfrak{A}$ and $\mathfrak{C} = D(\mathfrak{A})$. □

7.4.4.3. Since the functor $\mathcal{H}^n$ carries isogenies to isogenies, then we get the functor $\mathcal{H}^n$ making commutative (up to canonical equivalence)

$$\begin{array}{ccc} D_{\mathbb{Q}}(\mathfrak{A}) & \overset{\mathcal{H}^n}{\dashrightarrow} & \mathfrak{A}_{\mathbb{Q}} \\ \uparrow & & \uparrow \\ D(\mathfrak{A}) & \xrightarrow{\mathcal{H}^n} & \mathfrak{A}, \end{array} \tag{7.4.4.3.1}$$

where the vertical arrows are the localization functors.

Lemma 7.4.4.4. *The functor $\mathcal{H}^n\colon D_{\mathbb{Q}}(\mathfrak{A}) \to \mathfrak{A}_{\mathbb{Q}}$ defined at 7.4.4.3 is a cohomological functor.*

Proof. By construction (see the proof of [Sta22, 05R6 Proposition 13.5.5]), a distinguished triangle of $D_{\mathbb{Q}}(\mathfrak{A})$ is isomorphic in $D_{\mathbb{Q}}(\mathfrak{A})$ to the image of a distinguished triangle of $K(\mathfrak{A})$ by the canonical localization functor $K(\mathfrak{A}) \to D_{\mathbb{Q}}(\mathfrak{A})$. Since $\mathcal{H}^n\colon K(\mathfrak{A}) \to \mathfrak{A}$ is a cohomological functor, since the localization functor $\mathfrak{A} \to \mathfrak{A}_{\mathbb{Q}}$ is an exact functor between abelian categories (it follows from the properties of localisations by a subcategory of Serre and of 8.1.5.5), this implies the result.

□

7.4.4.5. Denote by $D^0_{\mathbb{Q}}(\mathfrak{A})$ the strictly full sub-category of $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$ consisting of complexes $\mathcal{E}$ such that for any integer $n \neq 0$ we have $\mathcal{H}^n(\mathcal{E}) \xrightarrow{\sim} 0$ in $\mathfrak{A}_{\mathbb{Q}}$.

Remark 7.4.4.6. Let $\mathcal{E} \in D_{\mathbb{Q}}(\mathfrak{A})$ such that $\mathcal{H}^n(\mathcal{E}) \xrightarrow{\sim} 0$ in $\mathfrak{A}_{\mathbb{Q}}$ for any $n \in \mathbb{Z}$. Then it seems false that $\mathcal{E} \xrightarrow{\sim} 0$ in $D_{\mathbb{Q}}(\mathfrak{A})$. When $\mathcal{E} \in D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$, this property becomes true (see 7.4.4.7), which explains why we have defined $D^0_{\mathbb{Q}}(\mathfrak{A})$ as a the strictly full subcategory of $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$ and not $D_{\mathbb{Q}}(\mathfrak{A})$ in 7.4.4.5.

Lemma 7.4.4.7. *The canonical functor*

$$\mathfrak{A}_{\mathbb{Q}} \to D^0_{\mathbb{Q}}(\mathfrak{A}) \tag{7.4.4.7.1}$$

is an equivalence of categories with quasi-inverse $H^0\colon D^0_{\mathbb{Q}}(\mathfrak{A}) \to \mathfrak{A}_{\mathbb{Q}}$.

Proof. The proof is similar to that of 8.1.5.10. □

Corollary 7.4.4.8. *Let $\phi\colon \mathcal{E} \to \mathcal{F}$ be a morphism in $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$. The morphism ϕ is an isomorphism in $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$ if and only if, for any integer $n \in \mathbb{Z}$, the morphism $\mathcal{H}^n(\phi)\colon \mathcal{H}^n(\mathcal{E}) \to \mathcal{H}^n(\mathcal{F})$ is an isomorphism of $\mathfrak{A}_{\mathbb{Q}}$.*

Proof. There exists a distinguished triangle in $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$ of the form $\mathcal{E} \underset{\phi}{\longrightarrow} \mathcal{F} \to \mathcal{G} \to \mathcal{E}[1]$. Following the properties concerning the triangulated categories, ϕ is an isomorphism if and only if $\mathcal{G} \xrightarrow{\sim} 0$ in $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$. Following 7.4.4.7, this is equivalent to saying that, for any integer $n \in \mathbb{Z}$, we have $\mathcal{H}^n(\mathcal{G}) \xrightarrow{\sim} 0$ in $\mathfrak{A}_{\mathbb{Q}}$. The lemma 7.4.4.4 allows us to conclude. □

7.4.4.9. Let us denote by $D^{\sharp}_{\mathfrak{B}}(\mathfrak{A})$ the saturated (in the sense of 7.4.1.1) full triangulated subcategory of $D^{\sharp}(\mathfrak{A})$ consisting of complexes whose cohomology spaces are in $\mathfrak{B}$ i.e. $D^{\sharp}_{\mathfrak{B}}(\mathfrak{A})$ is the kernel of the canonical functor $D^{\sharp}(\mathfrak{A}) \to D^{\sharp}(\mathfrak{A}_{\mathbb{Q}})$ induced by the localization functor $\mathfrak{A} \to \mathfrak{A}_{\mathbb{Q}}$.

With notation 7.4.1.3, let $S^{\sharp}_{N} := S(D^{\sharp}_{\mathfrak{B}}(\mathfrak{A}))$ be the saturated multiplicative system compatible with the triangulation of $D^{\sharp}(\mathfrak{A})$ which corresponds to $D^{\sharp}_{\mathfrak{B}}(\mathfrak{A})$. We can deduce from the theorem [Miy91, 3.2] that the canonical functor $D^{\sharp}(\mathfrak{A}) \to D^{\sharp}(\mathfrak{A}_{\mathbb{Q}})$ induces canonically the equivalence of categories

$$D^{\sharp}(\mathfrak{A})/D^{\sharp}_{\mathfrak{B}}(\mathfrak{A}) := S^{\sharp -1}_{N} D^{\sharp}(\mathfrak{A}) \cong D^{\sharp}(\mathfrak{A}_{\mathbb{Q}}). \tag{7.4.4.9.1}$$

By definition, a morphism f of $D^{\sharp}(\mathfrak{A})$ belongs to $S^{\sharp}_{N}$ if and only if, for all distinguished triangle in $D^{\sharp}(\mathfrak{A})$ of the form $\mathcal{E} \xrightarrow{f} \mathcal{F} \to \mathcal{G} \to \mathcal{E}[1]$, for all integer $n \in \mathbb{Z}$, we have $\mathcal{H}^{n}(\mathcal{G}) \in \mathfrak{B}$).

Lemma 7.4.4.10. *With notation 7.4.4.9, we have $S^{\mathrm{b}} = S^{\mathrm{b}}_{N}$. For $\sharp \in \{+, -, \mathrm{b}, \emptyset\}$, we have $S^{\sharp} \subset S^{\sharp}_{N}$.*

Proof. 1) First we show $S^{\mathrm{b}}_{N} \subset S^{\mathrm{b}}$. Take $f \in S^{\mathrm{b}}_{N}$ and a distinguished triangle in $D^{\mathrm{b}}(\mathfrak{A})$ of the form $\mathcal{E} \xrightarrow{f} \mathcal{F} \to \mathcal{G} \to \mathcal{E}[1]$. By definition, for all integer $n \in \mathbb{Z}$, $\mathcal{H}^{n}(\mathcal{G}) \in \mathfrak{B}$. That is, for all integer $n \in \mathbb{Z}$, we have $\mathcal{H}^{n}(\mathcal{G}) \xrightarrow{\sim} 0$ in $\mathfrak{A}_{\mathbb{Q}}$. By 7.4.4.7, this implies that $\mathcal{G} \xrightarrow{\sim} 0$ in $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$. According to the properties of triangulated categories, f is an isomorphism in $D^{\mathrm{b}}_{\mathbb{Q}}(\mathfrak{A})$, i.e. $f \in S^{\mathrm{b}}$.

2) Next we show that $S^{\sharp} \subset S^{\sharp}_{N}$. Let $f \colon \mathcal{E} \to \mathcal{F}$ be a morphism of $D^{\sharp}(\mathfrak{A})$. Since the cohomology space functor $H^{0} \colon D^{\sharp}(\mathfrak{A}) \to \mathfrak{A}_{\mathbb{Q}}$ is a cohomological functor, then we get at a long exact sequence in $\mathfrak{A}_{\mathbb{Q}}$ from the distinguished triangle in $D^{\sharp}(\mathfrak{A})$ of the form $\mathcal{E} \xrightarrow{f} \mathcal{F} \to \mathcal{G} \to \mathcal{E}[1]$. Looking at this long exact sequence, we can check that $f \in S^{\sharp}_{N}$ if and only if, for all integer $n \in \mathbb{Z}$, $\mathcal{H}^{n}(f)$ is an isomorphism in $\mathfrak{A}_{\mathbb{Q}}$ (which is an abelian category). But, if $f \in S^{\sharp}$, then its image in $D^{\sharp}_{\mathbb{Q}}(\mathfrak{A})$ is an isomorphism. As the functor $\mathcal{H}^{n} \colon D^{\sharp}(\mathfrak{A}) \to \mathfrak{A}_{\mathbb{Q}}$ factors through $D^{\sharp}_{\mathbb{Q}}(\mathfrak{A}) \to \mathfrak{A}_{\mathbb{Q}}$, we deduce the required inclusion $S^{\sharp} \subset S^{\sharp}_{N}$. ☐

Proposition 7.4.4.11. *For $\sharp \in \{+, -, \mathrm{b}, \emptyset\}$, the canonical functor $D^{\sharp}(\mathfrak{A}) \to D^{\sharp}(\mathfrak{A}_{\mathbb{Q}})$ of triangulated categories induced by the functor of abelian categories $\mathfrak{A} \to \mathfrak{A}_{\mathbb{Q}}$ induces the morphism of triangulated categories*

$$\mathfrak{Q} \colon D^{\sharp}_{\mathbb{Q}}(\mathfrak{A}) \to D^{\sharp}(\mathfrak{A}_{\mathbb{Q}}) \tag{7.4.4.11.1}$$

making commutative the diagram

$$\begin{array}{ccccc} & & S^{\sharp -1}_{N} D^{\sharp}(\mathfrak{A}) & \xrightarrow[7.4.4.9.1]{\cong} & D^{\sharp}(\mathfrak{A}_{\mathbb{Q}}) \\ & \nearrow^{Q_{S^{\sharp}_{N}}} & \uparrow & & \uparrow \mathfrak{Q} \\ D^{\sharp}(\mathfrak{A}) & \xrightarrow{Q_{S^{\sharp}}} & S^{\sharp -1} D^{\sharp}(\mathfrak{A}) & = & D^{\sharp}_{\mathbb{Q}}(\mathfrak{A}) \end{array} \tag{7.4.4.11.2}$$

When $\sharp = \mathrm{b}$, the morphism $\mathfrak{Q}$ is an equivalence of categories.

Proof. The left vertical arrow comes from the inclusion $S^{\sharp} \subset S^{\sharp}_{N}$ (see 7.4.4.10). When $\sharp = \mathrm{b}$, since this inclusion becomes an equality, both vertical arrow are equivalences of categories. ☐

7.4.4.12. The morphism 7.4.4.11.1 commutes with cohomological functors, i.e. we have for any $n \in \mathbb{N}$ the commutative diagram

$$\begin{array}{ccccc} D^{\sharp}(\mathfrak{A}) & \longrightarrow & D^{\sharp}_{\mathbb{Q}}(\mathfrak{A}) & \xrightarrow{\mathfrak{Q}} & D^{\sharp}(\mathfrak{A}_{\mathbb{Q}}) \\ \downarrow H^{n} & & \downarrow H^{n} & & \downarrow H^{n} \\ \mathfrak{A} & \longrightarrow & \mathfrak{A}_{\mathbb{Q}} & = & \mathfrak{A}_{\mathbb{Q}} \end{array} \tag{7.4.4.12.1}$$

where the middle vertical arrow is the one making commutative by definition the left square (see 7.4.4.3). Indeed, since the canonical functor $\mathfrak{A} \to \mathfrak{A}_{\mathbb{Q}}$ is exact, the outer of the large rectangle is commutative.

7.4.4.13. We have the commutative diagram up to canonical isomorphism

$$\begin{array}{ccc} \mathfrak{A}_{\mathbb{Q}} & \xrightarrow{7.4.4.2} & D_{\mathbb{Q}}(\mathfrak{A}) \\ & \searrow & \wr\downarrow 7.4.4.11.1 \\ & & D(\mathfrak{A}_{\mathbb{Q}}). \end{array} \tag{7.4.4.13.1}$$

Indeed, by using the universal property of the localisation functor, we reduce to check it after applying the functor $\mathfrak{A} \to \mathfrak{A}_{\mathbb{Q}}$, which is easy.

7.4.5 Coherent $\mathcal{D}_{\mathbb{Q}}$-modules, Cartan's theorems A and B

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ satisfying the conditions of 7.2.3. Recall following 7.2.3.3, $\mathcal{D}$ is left coherent. We suppose $p\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$.

Lemma 7.4.5.1. *Let $\mathcal{E}$ be a coherent $\mathcal{D}$-module, and $\mathcal{E}_t$ be the subsheaf of p-torsion sections of $\mathcal{E}$. Then $\mathcal{E}_t$ and $\mathcal{E}/\mathcal{E}_t$ are $\mathcal{D}$-coherent.*

Proof. Since this is local, we can suppose $\mathfrak{X}$ is affine. For any positif integer n, let $\mathcal{E}(n)$ be the coherent $\mathcal{D}$-module equal to the kernel of the multiplication by p^n on $\mathcal{E}$. It follows from 7.2.3.16.ii and the noetherianity of $\Gamma(\mathfrak{X}, \mathcal{D})$ that for n large enough, $\mathcal{E}_t = \mathcal{E}(n)$. Hence, we are done. □

Proposition 7.4.5.2. *Assume that $\mathfrak{X}$ is noetherian. The functor $\mathcal{M} \mapsto \mathcal{M}_{\mathbb{Q}}$ defines an equivalence of categories* $\mathrm{Coh}(\mathcal{D})_{\mathbb{Q}} \to \mathrm{Coh}(\mathcal{D}_{\mathbb{Q}})$.

Proof. See [Ber96c, 3.4.5]. □

Definition 7.4.5.3. Let $\mathcal{E}$ be a coherent $\mathcal{D}_{\mathbb{Q}}$-module. An integral model $\overset{\circ}{\mathcal{E}}$ of $\mathcal{E}$ is a coherent $\mathcal{D}$-module such that there exists an isomorphism of $\mathcal{D}_{\mathbb{Q}}$-modules of the form $\overset{\circ}{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. We say that $\overset{\circ}{\mathcal{E}}$ is a lattice is $\overset{\circ}{\mathcal{E}}$ is a sub $\mathcal{D}$-module of $\mathcal{E}$ such the induced morphism $\overset{\circ}{\mathcal{E}}_{\mathbb{Q}} \to \mathcal{E}$ is an isomorphism. Following 7.4.5.1 and 7.4.5.2, there exists such a integral model.

Proposition 7.4.5.4. *Suppose that $\mathfrak{X}$ is affine and put $D = \Gamma(\mathfrak{X}, \mathcal{D})$, $D_{\mathbb{Q}} = D \otimes \mathbb{Q} \cong \Gamma(\mathfrak{X}, \mathcal{D}_{\mathbb{Q}})$.*

(a) Let $\mathcal{M}$ be a left $\mathcal{D}_{\mathbb{Q}}$-module. The left $\mathcal{D}_{\mathbb{Q}}$-module $\mathcal{M}$ is coherent if and only if $\Gamma(\mathfrak{X}, \mathcal{M})$ is a $D_{\mathbb{Q}}$-module of finite type and the canonical morphism

$$\mathcal{D}_{\mathbb{Q}} \otimes_{D_{\mathbb{Q}}} \Gamma(\mathfrak{X}, \mathcal{M}) \to \mathcal{M} \tag{7.4.5.4.1}$$

is an isomorphism.

(b) For any coherent left $\mathcal{D}_{\mathbb{Q}}$-module $\mathcal{M}$ we have $H^q(\mathfrak{X}, \mathcal{M}) = 0$ for all $q \geq 1$.

(c) The functors $\mathcal{D}_{\mathbb{Q}} \otimes_{D_{\mathbb{Q}}} -$ and $\Gamma(\mathfrak{X}, -)$ induce canonically exact quasi-inverse equivalences of categories between $\mathrm{Coh}({}^{1}D_{\mathbb{Q}})$ and $\mathrm{Coh}({}^{1}\mathcal{D}_{\mathbb{Q}})$.

Proof. This is a consequence of 7.2.3.16, 7.4.5.2 and 7.4.3.1.1. □

7.4.6 Quasi-coherent and coherent $\mathcal{D}_{\mathbb{Q}}$-complexes

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ satisfying the hypotheses of 7.3.2 and we suppose $p\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$.

Definition 7.4.6.1. Let $\mathcal{E} \in D_{\mathbb{Q}}(\mathcal{D})$. The complex $\mathcal{E}$ is said to be quasi-coherent (resp. of finite tor dimension up, resp. perfect, resp. coherent) up to isogeny if there exists $\mathcal{F} \in D_{\mathrm{qc}}(\mathcal{D})$ (resp. $\mathcal{F} \in D_{\mathrm{perf}}(\mathcal{D})$, resp. $\mathcal{F} \in D_{\mathrm{coh}}(\mathcal{D})$) together with an isomorphism in $D_{\mathbb{Q}}(\mathcal{D})$ of the form $\mathcal{E} \xrightarrow{\sim} \mathcal{F}$. For $\star \in \{-, \mathrm{b}\}$, we denote by $D^{\star}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D})$ (resp $D^{\star}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D})$, resp $D^{\star}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D})$) the strictly full subcategory of $D_{\mathbb{Q}}(\mathcal{D})$ consisting quasi-coherent (resp. perfect, resp. coherent) up to isogeny.

It follows from the remark of 7.4.2.1.a that the natural functors $D^{\star}_{?}(\mathcal{D})_{\mathbb{Q}} \to D^{\star}_{\mathbb{Q},?}(\mathcal{D})$ with $? \in \{\mathrm{qc}, \mathrm{tdf}, \mathrm{perf}, \mathrm{coh}\}$ are equivalence of categories. Remark since we prefer to work with strict full subcategories, there is a slight difference with Berthelot's notation [Ber02, 3.3.2].

Proposition 7.4.6.2. *Suppose $\mathfrak{X}$ is affine and keep notation 7.2.3.8.*

a) The functors The functors $\mathcal{D}_{\mathbb{Q}} \otimes_{D_{\mathbb{Q}}} -$ and $\mathbb{R}\Gamma(\mathfrak{X}, -)$ induce canonically quasi-inverse equivalences of categories between $D^{-}_{\mathrm{coh}}(D_{\mathbb{Q}})$ and $D^{-}_{\mathrm{coh}}(\mathcal{D}_{\mathbb{Q}})$.

b) Let $\mathcal{E}^{\bullet} \in D^{-}(\mathcal{D}_{\mathbb{Q}})$. The following conditions are equivalent

(i) $H^n(\mathcal{E}^{\bullet})$ is a coherent left $\mathcal{D}_{\mathbb{Q}}$-module for any $n \in \mathbb{Z}$;

(ii) $\mathcal{E}^{\bullet}$ is pseudo-coherent ;

(iii) $\mathcal{E}^{\bullet}$ is quasi-isomorphic to a bounded above complex of finite free left $\mathcal{D}_{\mathbb{Q}}$-modules.

Proof. Since $\mathcal{D}_{\mathbb{Q}}$ satisfies Theorem A and B for coherent modules (see 7.4.5.4), than the proposition can be checked similarly to 4.6.1.7. □

Corollary 7.4.6.3. *Let $\mathcal{E}^{\bullet} \in D^{-}(\mathcal{D}_{\mathbb{Q}})$. The following conditions are equivalent*

(i) $H^n(\mathcal{E}^{\bullet})$ is a coherent left $\mathcal{D}_{\mathbb{Q}}$-module for any $n \in \mathbb{Z}$;

(ii) $\mathcal{E}^{\bullet}$ is pseudo-coherent.

Lemma 7.4.6.4. *Let $F \colon \mathfrak{C} \to \mathfrak{D}$ be an exact and full functor of triangulated categories. Then the essential image of F is a strictly full triangulated subcategory of $\mathfrak{D}$.*

Proof. Left to the reader. □

Notation 7.4.6.5. With notation 7.4.2.1.(b), we denote by $\mathbb{Q} \otimes - \colon D_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}) \to D({}^{\mathrm{l}}\mathcal{D}_{\mathbb{Q}})$ the natural functor given by $\mathcal{E} \mapsto \mathcal{E}_{\mathbb{Q}}$.

Proposition 7.4.6.6. *Suppose $\mathfrak{X}$ is noetherian of finite Krull dimension. With notation 1.4.3.27 and 7.4.6.1, the natural functor $\mathbb{Q} \otimes -$ (see 7.4.6.5) induces the equivalence of categories:*

$$\mathbb{Q} \otimes - \colon D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathbb{Q}}). \tag{7.4.6.6.1}$$

Proof. Following 7.4.3.2.2, we already know that the functor is fully faithful. From 7.4.6.4, since the functor is exact and full, then its essential image is a full triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathbb{Q}})$. Moreover, following 7.4.5.2, the essential image of the functor 7.4.6.6.1 contains coherent left $\mathcal{D}_{\mathbb{Q}}$-modules. Since the smallest strictly full triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathbb{Q}})$ containing coherent left $\mathcal{D}_{\mathbb{Q}}$-modules is $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathbb{Q}})$ we are done. □

7.4.7 Derived completed tensor products and derived completed homomorphisms of complexes of (bi)modules

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Suppose $p\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$. Let $\mathcal{D}, \mathcal{D}', \mathcal{D}'', \mathcal{D}'''$ be four sheaves of rings on $\mathfrak{X}$ satisfying the hypotheses of 7.3.2 (or only 7.2.3 when the notion of quasi-coherence is not involved).

7.4.7.1. We have the topoi morphisms $\underleftarrow{l}_X \colon (X_{\bullet}, \mathcal{D}_{\bullet}) \to (|\mathcal{X}|, \mathcal{D})$ and $\underleftarrow{l}_X \colon (X_{\bullet}, \mathbb{Z}_{X_{\bullet}}) \to (|\mathcal{X}|, \mathbb{Z}_{\mathfrak{X}})$.

(a) Both functors $\mathbb{R}\underleftarrow{l}_{X*}$ preserve isogenies and induce $\mathbb{R}\underleftarrow{l}_{X*} \colon D^{-}_{\mathbb{Q}}(\mathcal{D}_{\bullet}) \to D^{-}_{\mathbb{Q}}(\mathcal{D})$ and $\mathbb{R}\underleftarrow{l}_{X*} \colon D^{-}_{\mathbb{Q}}(\mathbb{Z}_{X_{\bullet}}) \to D^{-}_{\mathbb{Q}}(\mathbb{Z}_{\mathfrak{X}})$. We have obviously the same property for the exact functor $\underleftarrow{l}_X^{-1}$.

(b) Similarly, we get the functor

$$\mathbb{L}\underleftarrow{l}_X^{*} = \mathcal{D}_{\bullet} \otimes^{\mathbb{L}}_{\underleftarrow{l}_X^{-1}\mathcal{D}} \underleftarrow{l}_X^{-1} - \colon D^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}) \to D^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}_{\bullet}).$$

and similarly for right modules.

7.4.7.2. By preservation of the isogenies, we get the functor

$$-\otimes_{\mathcal{D}_\bullet}-\colon D^-_{\mathbb{Q}}({}^r\mathcal{D}_\bullet)\times D^-_{\mathbb{Q}}({}^l\mathcal{D}_\bullet)\to D^-_{\mathbb{Q}}(\mathbb{Z}_{X_\bullet}) \tag{7.4.7.2.1}$$

Since sends isogenies to isogenies, this induces the functor

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}-:=\mathbb{R}\underleftarrow{l}_{X*}\left(\mathbb{L}\underleftarrow{l}^*_X(-)\otimes^{\mathbb{L}}_{\mathcal{D}_\bullet}\mathbb{L}\underleftarrow{l}^*_X(-)\right):D^-_{\mathbb{Q}}({}^r\mathcal{D})\times D^-_{\mathbb{Q}}({}^l\mathcal{D})\to D^-_{\mathbb{Q}}(\mathbb{Z}_{\mathfrak{X}}). \tag{7.4.7.2.2}$$

which is also the functor induced by 7.3.4.2.1.

Lemma 7.4.7.3. *Let $\mathcal{E}^\bullet\in D^-_{\mathbb{Q}}({}^l\mathcal{D})$, $\mathcal{M}^\bullet\in D^-_{\mathbb{Q}}({}^r\mathcal{D})$.*

(a) We have the morphism

$$\mathcal{M}^\bullet\otimes^{\mathbb{L}}_{\mathcal{D}}\mathcal{E}^\bullet\to\mathcal{M}^\bullet\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}\mathcal{E}^\bullet \tag{7.4.7.3.1}$$

which is an isomorphism when one of the two complexes belongs to $D^-_{\mathbb{Q},\mathrm{coh}}(\mathcal{D})$ and the other to $D^-_{\mathbb{Q},\mathrm{qc}}(\mathcal{D})$.

(b) When $\mathcal{D}$ is commutative, then we get the isomorphism of $D^-_{\mathbb{Q}}(\mathcal{D})$:

$$\mathcal{M}^\bullet\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}\mathcal{E}^\bullet\xrightarrow{\sim}\mathbb{R}\underleftarrow{l}_{X*}\circ\mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}^\bullet\otimes^{\mathbb{L}}_{\mathcal{D}}\mathcal{E}^\bullet), \tag{7.4.7.3.2}$$

where $\underleftarrow{l}_X$ is here the topoi morphism $\underleftarrow{l}_X\colon(X_\bullet,\mathcal{D}_\bullet)\to(|\mathcal{X}|,\mathcal{D})$. Hence, we can consider $\mathcal{M}^\bullet\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}\mathcal{E}^\bullet$ as an object of $D^-_{\mathbb{Q}}(\mathcal{D})$ and then the map 7.4.7.3.1 is the adjunction morphism.

Proof. This follows from 7.3.4.3. □

7.4.7.4. Suppose there exists a homomorphism of sheaves of rings on $\mathfrak{X}$ of the form $\mathcal{D}\to\mathcal{D}'$ such that the composition of $\mathcal{O}_{\mathfrak{X}}\to\mathcal{D}$ with $\mathcal{D}\to\mathcal{D}'$ gives $\mathcal{O}_{\mathfrak{X}}\to\mathcal{D}'$.

(a) Let $*\in\{\mathrm{l},\mathrm{r}\}$ and $\star\in\{-,\mathrm{b}\}$. The functor 7.3.4.4.1 induces:

$$forg_{\mathcal{D},\mathcal{D}'}\colon D^\star_{\mathbb{Q},\mathrm{qc}}({}^*\mathcal{D}')\to D^\star_{\mathbb{Q},\mathrm{qc}}({}^*\mathcal{D}). \tag{7.4.7.4.1}$$

(b) With notation 7.3.1.12, we have from 7.3.4.4.2 the functor

$$\mathcal{D}'_\bullet\otimes^{\mathbb{L}}_{\mathcal{D}_\bullet}-\colon D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}_\bullet)\to D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_\bullet). \tag{7.4.7.4.2}$$

(c) We have from 7.3.4.4.4 the functor:

$$\mathcal{D}'\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}-\colon D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})\to D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'). \tag{7.4.7.4.3}$$

7.4.7.5. Suppose $(\mathcal{D},\mathcal{D}',\mathcal{I})$ is solved by $(\mathcal{R},\mathcal{K})$ (see definition 7.3.4.5). Let $\star\in\{\emptyset,-\}$. We set $D^\star_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}}):=D^\star({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})_{\mathbb{Q}}$ and $D^\star_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}}):=D^\star({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})_{\mathbb{Q}}$ (see notation 7.3.4.7) etc. The functors 7.3.4.8.1 and 7.3.4.8.2 preserve isogenies and induce therefore:

$$\mathbb{L}\underleftarrow{l}^*_X\colon D^\star_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})\to D^\star_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}}), \tag{7.4.7.5.1}$$

$$\mathbb{R}\underleftarrow{l}_{X*}\colon D^\star_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})\to D^\star_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}}). \tag{7.4.7.5.2}$$

Let $\sharp\in\{\emptyset,+,-,\mathrm{b}\}$. Let $?\in\{\mathrm{qc},\mathrm{coh},\mathrm{tdf},\mathrm{perf}\}$. We denote respectively by $D^\sharp_{\mathbb{Q},?,.}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ and $D^\sharp_{\mathbb{Q},.,?}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ the strictly full subcategory of $D^\sharp_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ consisting of complexes $\mathcal{E}_\bullet$ which are isomorphic to an object of $D^\sharp_{?,.}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ and $D^\sharp_{.,?}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ (see notation 7.3.4.8). In other words, $D^\sharp_{\mathbb{Q},?,.}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ is the essential image of the full faithful functor $D^\sharp_{?,.}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})\to D^\sharp_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ etc.

Similarly, we denote by $D^\sharp_{\mathbb{Q},?,.}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{\mathbb{Q},.,?}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$ the strictly full subcategory of $D^\sharp_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$ consisting of complexes $\mathcal{E}$ which are isomorphic to an object of $D^\sharp_{?,.}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{.,?}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$. The functors $\mathbb{L}\underleftarrow{l}^*_X$ of 7.3.4.8.1 and $\mathbb{R}\underleftarrow{l}_{X*}$ induce quasi-inverse equivalences of categories between $D^\sharp_{\mathbb{Q},.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{\mathbb{Q},.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$ and between $D^\sharp_{\mathbb{Q},\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$ and $D^\sharp_{\mathbb{Q},\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'_\bullet,\mathcal{R}_\bullet,\mathcal{D}_\bullet{}^{\mathrm{r}})$.

7.4.7.6. Suppose $(\mathcal{D},\mathcal{D}',\mathcal{I})$ and $(\mathcal{D},\mathcal{D}'',\mathcal{I})$ are solved by $(\mathcal{R},\mathcal{K})$. Let $*,** \in \{r,l\}$. The functors 7.3.4.9.1 and 7.3.4.9.2 preserve the isogenies and induce

$$-\otimes^{\mathbb{L}}_{\mathcal{D}_\bullet} -\colon D_{\mathbb{Q}}({}^*\mathcal{D}'_\bullet,\mathcal{R},\mathcal{D}_\bullet{}^{\mathrm{r}}) \times D_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}_\bullet,\mathcal{R},{}^{**}\mathcal{D}''_\bullet) \to D_{\mathbb{Q}}({}^*\mathcal{D}'_\bullet,\mathcal{R},{}^{**}\mathcal{D}''_\bullet), \tag{7.4.7.6.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}}-\colon D_{\mathbb{Q}}({}^*\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}}) \times D_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D},\mathcal{R},{}^{**}\mathcal{D}'') \to D_{\mathbb{Q}}({}^*\mathcal{D}',\mathcal{R},{}^{**}\mathcal{D}''). \tag{7.4.7.6.2}$$

We have similar bifunctors by changing the indices l and r. Let $\mathcal{M}^\bullet \in D_{\mathbb{Q}}({}^*\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$, $\mathcal{E}^\bullet \in D_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D},\mathcal{R},{}^{**}\mathcal{D}'')$ be two complexes. The canonical morphism 7.3.4.9.3 induces

$$\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \to \mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}^\bullet \tag{7.4.7.6.3}$$

which is an isomorphism when either $\mathcal{M}^\bullet \in D^-_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{\mathrm{r}})$ and $\mathcal{E}^\bullet \in D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D})$, or $\mathcal{M}^\bullet \in D^-_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{\mathrm{r}})$ and $\mathcal{E}^\bullet \in D^-_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}')$.

7.4.7.7. By adding some $\mathbb{Q}$ the properties of 7.3.4 are still valid. For instance, for any $\mathcal{E}^\bullet \in D^-_{\mathbb{Q},\mathrm{qc},.}({}^*\mathcal{D}',\mathcal{R},\mathcal{D}^{\mathrm{r}})$, $\mathcal{F}^\bullet \in D^-_{\mathbb{Q},\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D},\mathcal{R},{}^{\mathrm{r}}\mathcal{D}'')$, $\mathcal{G}^\bullet \in D^-_{\mathbb{Q},\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'',\mathcal{R},{}^{**}\mathcal{D}''')$, we have the associativity isomorphism in $D^-_{\mathbb{Q},\mathrm{qc},.}({}^*\mathcal{D}',\mathcal{R},{}^{**}\mathcal{D}''')$ of the form:

$$\left(\mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \mathcal{F}^\bullet\right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''} \mathcal{G}^\bullet \xrightarrow{\sim} \mathcal{E}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}} \left(\mathcal{F}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''} \mathcal{G}^\bullet\right). \tag{7.4.7.7.1}$$

7.5 Operations involving completed sheaves of differential operators of level m

7.5.1 Completed sheaves of differential operators of level m

Let $m \in \mathbb{N} \cup \{+\infty\}$. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian.

7.5.1.1. Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the following conditions.

(a) $\mathcal{B}$ is endowed with a structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra ;

(b) For any affine open $\mathfrak{U}$ of $\mathfrak{X}$, the ring $\Gamma(U,\mathcal{B})$ is noetherian ;

(c) For any $i \geq 0$, $\mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module and the canonical homomorphism $\mathcal{B} \to \varprojlim_{i\in\mathbb{N}} \mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}$ is an isomorphism.

We remark that the conditions (b) and (c) are equal to that of 7.2.3 in the case where $\mathcal{I} = \mathfrak{m}$. Hence, for instance we get following 7.2.3.3 that for any open immersion $\mathfrak{V} \subset \mathfrak{U}$ of affine opens, the homomorphism $\Gamma(\mathfrak{U},\mathcal{B}) \to \Gamma(\mathfrak{V},\mathcal{B})$ is flat.

Example 7.5.1.2. The sheaf $\mathcal{O}_{\mathfrak{X}}$ is endowed with a canonical structure of left $\widehat{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module, which induces a structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\mathcal{O}_{\mathfrak{X}}$. The conditions of 7.5.1.1 are satisfied by $\mathcal{O}_{\mathfrak{X}}$. It follows from 3.1.4.5.1 and 3.2.3.5.1 that the induced structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\mathcal{O}_{\mathfrak{X}}$ is given via the formula

$$P(f) := P \circ p^n_{1(m)}(f). \tag{7.5.1.2.1}$$

7.5.1.3. Suppose $m \in \mathbb{N}$. We keep the notations and hypotheses of 7.5.1.1. By using 4.1.2.17, the ring $\mathcal{D} := \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies the conditions of 7.2.2.3. Hence, with the remark 7.2.3.1, its p-adic completion denoted by $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \widehat{\mathcal{D}}$ satisfies the conditions of 7.2.3. For instance, we get the coherence of $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ as well as theorems of type A and B for the left or right coherent $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules (for more precisions, see 7.2.3.16). In the same way, it follows from 7.4.5.4 the coherence of $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} := (\widehat{\mathcal{D}})_{\mathbb{Q}}$ as well as theorems of type A and B for the left or right coherent $\mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules.

Proposition 7.5.1.4. *Let $\mathcal{B}$ be an $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the conditions of 7.5.1.1 for some $m \in \mathbb{N}$. Then the sheaf $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is right and left flat on $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and on $\mathcal{B}$.*

Proof. Following 4.1.2.17.(d), $\Gamma(\mathfrak{U}, \mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ is right and left noetherian for any affine open $\mathfrak{U}\subset\mathfrak{X}$. According to 7.2.2.2, $\Gamma(\mathfrak{U}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ is the p-adic completion of $\Gamma(\mathfrak{U}, \mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Hence, we are done. □

7.5.1.5 (Local description and notation)**.** Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the conditions of 7.5.1.1. Suppose in this paragraph that $\mathfrak{X}$ is an affine, noetherian formal scheme of finite Krull dimension and $\mathfrak{X}^\sharp\to\mathfrak{S}^\sharp$ is endowed with logarithmic coordinates $(u_\lambda)_{\lambda=1,\dots,r}$. Let $\mathfrak{Y}:=\mathfrak{X}^{\sharp*}$ be the open of $\mathfrak{X}$ where $M_{\mathfrak{X}^\sharp}$ is trivial and $j\colon\mathfrak{Y}\hookrightarrow\mathfrak{X}^\sharp$ be the canonical open immersion. Let $(t_\lambda)_{\lambda=1,\dots,r}$ be the induced coordinates of $\mathfrak{Y}/\mathfrak{S}$. Put $\tau_{\sharp\lambda(m)}:=\mu^n_{(m)}(u_\lambda)-1$ (or simply $\tau_{\sharp\lambda}$), where for any $a\in M_{\mathfrak{X}^\sharp}$ we denote $\mu^n_{(m)}(a)$ the unique section of $\ker(\mathcal{O}^*_{\Delta^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}}\to\mathcal{O}^*_{\mathfrak{X}})$ such that we get in $M^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}$ the equality $p_1^{n*}(a)=p_0^{n*}(a)\mu^n_{(m)}(a)$. We still denote by $\tau_{\sharp\lambda(m)}$ its image via the canonical morphism $\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}\to\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}$. The elements $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}\}_{|\underline{k}|\le n}$ form a $\mathcal{B}$-basis of $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}$. The corresponding dual basis of $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,n}$ is denoted by $\{\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}\}_{|\underline{k}|\le n}$. The $\mathcal{B}$-module $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is free and has the basis $\{\underline{\partial}^{\langle\underline{k}\rangle_{(m)}}\}_{\underline{k}\in\mathbb{N}^d}$. Let $\epsilon_1,\dots,\epsilon_r$ be the canonical basis of $\mathbb{N}^r$, i.e. the coordinates of ϵ_i are 0 except for the ith term which is 1. We put $\partial_{\sharp i}:=\underline{\partial}^{\langle\epsilon_i\rangle_{(m)}}$.

(a) A section $P\in\Gamma(\mathfrak{X}, \mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ can uniquely be written of the form

$$P=\sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}, \tag{7.5.1.5.1}$$

where $b_{\underline{k}}\in B$ and the sum is finite.

(b) A section $P\in\Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ can uniquely be written of the form

$$P=\sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}, \tag{7.5.1.5.2}$$

where $b_{\underline{k}}$ is a sequence of elements of $\Gamma(\mathfrak{X},\mathcal{B})$ converging to 0 for the p-adic topology when $|\underline{k}|$ goes to infinity.

(c) The ring $\Gamma(\mathfrak{X},\mathcal{B}_{\mathbb{Q}})=\Gamma(\mathfrak{X},\mathcal{B})_{\mathbb{Q}}$ (see 7.4.3.1.1) is a Tate algebra (i.e. is an Huber ring with a pseudo-uniformizer): a ring of definition is given by the image $\Gamma(\mathfrak{X},\mathcal{B})\to\Gamma(\mathfrak{X},\mathcal{B})_{\mathbb{Q}}$, the ring of definition is endowed with the p-adic topology and p is a pseudo-uniformizer (we will define in this context a p-adic norm later in 8.7.1.6). A section $P\in\Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ can uniquely be written of the form

$$P=\sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}, \tag{7.5.1.5.3}$$

where $b_{\underline{k}}$ is a sequence of elements of $\Gamma(\mathfrak{X},\mathcal{B}_{\mathbb{Q}})$ converging to 0 when $|\underline{k}|$ goes to infinity.

7.5.1.6 (Local description and notation: semi-nice coordinate)**.** Suppose $\mathfrak{X}\to\mathfrak{S}$ is a smooth morphism of $\mathcal{V}$-formal schemes and suppose there exists a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor $\mathfrak{D}$ such that $\mathfrak{X}^\sharp:=(\mathfrak{X},M(\mathfrak{D}))$. Let $f\colon X^\sharp\to X$ be the canonical morphism. Suppose there exist semi-nice coordinates $t_1,\dots,t_d$ of $\mathfrak{X}^\sharp/\mathfrak{S}$. Let $r\in\mathbb{N}$ be such that $\mathfrak{D}$ is empty (if $r=0$) or $\mathfrak{D}$ is cut out by $\prod_{1\le j\le r}t_j$ in $\mathfrak{X}$. With notation 4.5.1.4, we have the basis $\{\underline{\partial}_{(r)}^{\langle\underline{k}\rangle_{(m)}}:\underline{k}\in\mathbb{N}^d\}$ of $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}}$. Hence, we get the same description as 7.5.1.5.1, 7.5.1.5.2 and 7.5.1.5.3 by replacing $\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}$ by $\underline{\partial}_{(r)}^{\langle\underline{k}\rangle_{(m)}}$.

7.5.1.7 (p-adic norms)**.** Let M be a p-torsion free p-adically separated complete $\mathcal{V}$-module. We define on $M_K:=M\otimes_{\mathcal{V}}K$ the function $v_p\colon M_K\setminus\{0\}\setminus\to\mathbb{Z}$ by setting $v_p(x):=\max\{n\in\mathbb{Z}, x\in p^nM\}$ for any $x\in M_K$. This yields a norm on $\|-\|\colon M_K\to\mathbb{R}$ by setting $\|x\|:=p^{-v_p(x)}$. This norm is called the p-adic norm on M_K given by M.

When M is moreover a $\mathcal{V}$-algebra, then we have $\|xx'\| \leq \|x\|\|x'\|$ and then $v_p\colon M_K \to \mathbb{Z} \cup \{+\infty\}$ is a quasi-valuation. We call v_p to be the p-adic quasi-valuation of M_K induced by M. This yields a structure of Banach K-algebra on M_K.

7.5.1.8. Let $\mathfrak{U}^\sharp$ be an affine open of $\mathfrak{X}^\sharp$. For all $\in \mathbb{N}$, the p-free algebra $B = \Gamma(\mathfrak{U}, \mathcal{B})$ is naturally equipped with the p-adic topology, which defines the Tate algebra $\Gamma(\mathfrak{U}, \mathcal{B}_\mathbb{Q}) \xrightarrow{\sim} B_\mathbb{Q}$ (see 7.4.3.1.1): a ring of definition is given by B endowed with the p-adic topology and p is a pseudo-uniformizer. We get the Banach algebra $B_\mathbb{Q}$ (see 7.5.1.7). The topology obtained on this algebra is a Banach topology, because B is an algebra topologically of finite type, and $B_\mathbb{Q}$ can therefore be provided with a Banach norm which defines its topology. Likewise, if there exists a logarithmic coordinate system on $\mathfrak{U}^\sharp$, all operator $P \in \Gamma(\mathfrak{U}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$, can be written uniquely $P = \sum_{|\underline{k}|\leq n} b_{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}$, where the $b_{\underline{k}} \in B_\mathbb{Q}$ tend to 0 for $|\underline{k}| \to \infty$, and, choosing a Banach norm on $B_\mathbb{Q}$, we equip the algebra $D_\mathbb{Q} = \Gamma(\mathfrak{U}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ with a Banach norm defining its p-adic topology by setting $\| P \| = \sup_{\underline{k}} \| b_{\underline{k}} \|$. The ring of operator of $D_\mathbb{Q}$ of norm ≤ 1 is equal to D. Hence, a change of logarithmic coordinates does not change this norm. (Indeed, if $\underline{\partial}_\sharp'^{\langle \underline{k}\rangle}$ is the basis of $\Gamma(\mathfrak{U}, \mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ corresponding to another choice of logarithmic coordinates, the base change of $\Gamma(\mathfrak{U}, \mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ is given by a matrix with coefficients in $\Gamma(\mathfrak{U}, \mathcal{B})$.)

If $\mathcal{E}$ is a coherent $\mathcal{B}_\mathbb{Q}$-module (resp. a coherent $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module), then $E = \Gamma(\mathfrak{U}, \mathcal{E})$, is a finite type module on $B_\mathbb{Q}$ (resp. $D_\mathbb{Q}$) according to theorem A, and we equip E with a Banach norm by taking the quotient norm defined by any finite presentation ; as $B_\mathbb{Q}$ (resp. $D_\mathbb{Q}$) is a Noetherian Banach algebra, the norms defined by two presentations are equivalent (e.g. copy the proof of [BGR84, 3.7.3.3]). Denote by $\mathfrak{M}(D_\mathbb{Q})$ the category of all left $D_\mathbb{Q}$-modules of finite type endowed with a Banach norm given by a finite presentation with $D_\mathbb{Q}$-linear maps as morphisms. Any $D_\mathbb{Q}$-linear morphism between two objects of $\mathfrak{M}(D_\mathbb{Q})$ is strict and continuous (e.g. copy the proof of [BGR84, 3.7.3.3]). Moreover, any $D_\mathbb{Q}$-submodule of an object of $\mathfrak{M}(D_\mathbb{Q})$ is closed (e.g. copy the proof of [BGR84, 3.7.3.1])

7.5.1.9. Let $\mathcal{B}$ be a commutative $\mathcal{O}_\mathfrak{X}$-algebra satisfying the conditions of 7.5.1.1. Let $\mathcal{B}_t$ be the ideal of $\mathcal{B}$ consisting of p-torsion sections and $\mathcal{B}' := \mathcal{B}/\mathcal{B}_t$. Then $\mathcal{B}'$ is a coherent $\mathcal{B}$-module (see 7.4.5.1). With 7.2.3.16, this implies that $\mathcal{B}'$ satisfies the properties (b) and (c) of 7.5.1.1. Since p is in the center of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, then $\mathcal{B}_t$ is a sub-$\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module of $\mathcal{B}$ and then $\mathcal{B}'$ is endowed with a structure of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module such that the surjection $\mathcal{B} \to \mathcal{B}'$ is $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. Since this is a surjection, then the property (a) of 7.5.1.1 holds (check the Leibnitz formula). Moreover, with Lemma 4.1.2.2, we get the ring homomorphism $\mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B}' \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. It follows from 7.2.3.16 (and 1.4.5.2) that $\mathcal{B}' \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a coherent $\mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules (via the latter homomorphism).

Via 4.2.3.5, we get a structure of left $\mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\mathcal{B}_t \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \mathcal{B}_t \otimes_\mathcal{B} (\mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. By flatness of $\mathcal{O}_\mathfrak{X} \to \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, this yields the exact sequence of coherent $\mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules:

$$0 \to \mathcal{B}_t \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B}' \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to 0.$$

By p-adic completion, this yields the exact sequence of coherent $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules:

$$0 \to \mathcal{B}_t\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B}'\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to 0. \tag{7.5.1.9.1}$$

The following lemma will be useful to check 7.5.3.1.

Lemma 7.5.1.10. *With notation 7.5.1.9, the canonical homomorphism*

$$\mathcal{B}\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \mathcal{B}'\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$$

is an isomorphism.

Proof. Since this is local, we can suppose $\mathfrak{X}$ is noetherian. Hence, there exists an integer N such that $p^N \mathcal{B}_t = 0$. This implies that the canonical morphism $\mathcal{B}_t \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{B}_t\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is an isomorphism. Since $\mathcal{B}_t \otimes \mathbb{Q} = 0$, using 7.5.1.9.1 we are done. □

7.5.1.11. Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the conditions of 7.5.1.1. By p-adic completion, it follows from 4.3.5.1 that the sheaf $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is endowed with a canonical right $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure extending its structure of $\mathcal{B}$-module. Suppose $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d \in M_{\mathfrak{X}^\sharp}$. With the local description 7.5.1.5.2, we can check the logarithmic adjoint operator (see 3.4.1.2.3) extends to a map $\Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ given by $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}} \mapsto \widetilde{P} := \sum_{\underline{k}} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}\rangle_{(m)}} b_{\underline{k}}$, where $b_{\underline{k}}$ is a sequence of elements of $\Gamma(\mathfrak{X}, \mathcal{B})$ converging to 0 for the p-adic topology when $|\underline{k}|$ goes to infinity. Via the local description of 3.4.5.1.1, we can check the action of $P \in \Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ on the section $b\, d\log\, t_1 \wedge \cdots \wedge d\log\, t_d$, where b is section of $\mathcal{B}_X$ is given by the formula

$$(b\ d\log\, t_1 \wedge \cdots \wedge d\log\, t_d) \cdot P = \widetilde{P}(b) d\log\, t_1 \wedge \cdots \wedge d\log\, t_d. \tag{7.5.1.11.1}$$

7.5.1.12. Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the conditions of 7.5.1.1. Following 4.3.5.7, the functors $- \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} = \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -)$ and $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} -$ are exact and induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules.

7.5.1.13. Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the conditions of 7.5.1.1. Set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

(a) By p-adic completion, we get from 4.3.5.7 a structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-bimodule on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

(b) Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Via the canonical isomorphism of $\mathcal{B}$-modules:

$$\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E} \xrightarrow{\sim} \left(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\right) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E} \tag{7.5.1.13.1}$$

we get a structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E}$. Suppose $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d \in M_{\mathfrak{X}^\sharp}$. With the notation of 7.5.1.11, we compute the action of $P \in \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ on the section $d\log\, t_1 \wedge \cdots \wedge d\log\, t_d \otimes x$ of $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E}$, where x is a section of $\mathcal{E}$, is given by the formula

$$(d\log\, t_1 \wedge \cdots \wedge d\log\, t_d \otimes x) \cdot P = d\log\, t_1 \wedge \cdots \wedge d\log\, t_d \otimes \widetilde{P} \cdot x. \tag{7.5.1.13.2}$$

Hence, the structure of right $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E}$ given by 7.5.1.12 is equal to the one induced (via the canonical map $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$) by its structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module.

(c) Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Via the canonical isomorphism

$$\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M}) \xrightarrow{\sim} \mathscr{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M}), \tag{7.5.1.13.3}$$

we get a structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M})$. Suppose $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d \in M_{\mathfrak{X}^\sharp}$. With the notation of 7.5.1.5.2, we compute the action of $P \in \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ on the section $x \otimes (d\log\, t_1 \wedge \cdots \wedge d\log\, t_d)^*$ of $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M})$, where x is section of $\mathcal{M}$, is given by the formula

$$P \cdot (x \otimes (d\log\, t_1 \wedge \cdots \wedge d\log\, t_d)^*) = x \cdot \widetilde{P} \otimes (d\log\, t_1 \wedge \cdots \wedge d\log\, t_d)^*. \tag{7.5.1.13.4}$$

Hence, the induced structure of left $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module corresponds to that of 7.5.1.12.

(d) Since $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is locally free (of rank one), the canonical $\mathcal{B}_{\mathfrak{X}}$-linear morphism, then we have the $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear isomorphism

$$\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},\mathcal{M}) \xrightarrow{\sim} (\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},\mathcal{M}) \xrightarrow{\sim} \mathcal{M}, \tag{7.5.1.13.5}$$

where the last isomorphism is the evaluation one. We have the canonical $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear isomorphism:

$$\mathcal{E} \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}\mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E}). \tag{7.5.1.13.6}$$

(e) Similarly to 4.3.5.6, this yields that for any left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\mathcal{E}$ (resp. $\mathcal{M}$) , we have the following isomorphism of $\mathcal{O}_{\mathfrak{S}}$-modules:

$$\mathcal{M}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}\mathcal{E} \xrightarrow{\sim} (\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E})\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(\mathcal{M}\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}). \tag{7.5.1.13.7}$$

(f) As for 4.3.5.7, using the above results, we can check that the functors $-\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}=\mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},-)$ and $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}-$ are exact and induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules. Hence, for any $\star\in\{-,+,\mathrm{b},\emptyset\}$, the functors $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}-$ and $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},-)$ induce quasi-inverse equivalences of categories between $D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $D^\star({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Moreover, these equivalences preserve K-flat complexes and K-injective complexes.

7.5.2 Topological nilpotence and $\mathcal{B}$-coherence

We keep notation of 7.5.1.1. Let us start by considering the case of $\mathcal{B}$-coherent $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules:

Proposition 7.5.2.1. *Let $\mathcal{E}$ be a left $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module, coherent as $\mathcal{B}$-module.*

(a) If $\mathfrak{X}$ is affine then $\mathcal{E}$ is globally of finite presentation on $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

(b) The sheaf $\mathcal{E}$ is coherent as $\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module.

(c) The canonical homomorphism $\mathcal{E}\to\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}\mathcal{E}$ is an isomorphism and the sheaf $\mathcal{E}$ is coherent as $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module.

Proof. Suppose that $\mathfrak{X}$ is affine. Let $B=\Gamma(\mathfrak{X},\mathcal{B})$, $\mathcal{D}:=\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, $D=\Gamma(\mathfrak{X},\mathcal{D})$. As $\mathcal{E}$ is $\mathcal{B}$-coherent, then following theorem of type A, there exists a surjective $\mathcal{B}$-linear homomorphism $\mathcal{B}^n\to\mathcal{E}$, and so a surjective $\mathcal{D}$-linear homomorphism $\mathcal{D}^n\to\mathcal{E}$; let $\mathcal{N}$ be its kernel. Let $\mathcal{L}_i:=(\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,i})^n$ be the filtration on $\mathcal{L}:=(\mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})^n$ induced by the order filtration of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Let $\mathcal{N}_i:=\mathcal{N}\cap\mathcal{L}_i$, which yields the exact sequence $0\to\mathcal{N}_i\to\mathcal{L}_i\to\mathcal{E}\to 0$. Since the kernel of an homomorphism between coherent modules is coherent, then the submodules $\mathcal{N}_i$ are $\mathcal{B}$-coherent. Since filtered inductive limits are exact, since $\mathcal{L}\xrightarrow{\sim}\varinjlim_i\mathcal{L}_i$ then $\mathcal{N}\xrightarrow{\sim}\varinjlim_i\mathcal{N}_i$. As the functor $\Gamma(\mathfrak{X},-)$ commutes with filtered inductive limits (see [SGA4.2, VI.5.3]), by using theorem of type A for coherent $\mathcal{B}$-modules, this yields that the canonical homomorphisms $\mathcal{B}\otimes_B D\to\mathcal{D}$ and $\mathcal{B}\otimes_B\Gamma(\mathfrak{X},\mathcal{N})\to\mathcal{N}$ are isomorphisms. Thus $\mathcal{D}\otimes_D\Gamma(\mathfrak{X},\mathcal{N})\to\mathcal{N}$ is also an isomorphism. Since D is noetherian, then $\Gamma(\mathfrak{X},\mathcal{N})$ is a D-module of finite type. The coherence of $\mathcal{E}$ over $\mathcal{D}$ follows from Theorem A (see 4.1.3.19) Hence, we have checked a and therefore b). Let us treat now c). As $\mathcal{E}$ is a coherent $\mathcal{B}$-module, it is canonically isomorphic to its p-adic completion. Moreover, as $\mathcal{E}$ is a coherent $\mathcal{D}$-module, its p-adic completion is canonically isomorphic to $\widehat{\mathcal{D}}\otimes_{\mathcal{D}}\mathcal{E}$. Hence we are done. □

Corollary 7.5.2.2. *Let $\mathcal{B}'$ be a commutative $\mathcal{B}$-algebra satisfying the conditions of 7.5.1.1 and such that $\mathcal{B} \to \mathcal{B}'$ is $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. Let $\mathcal{E}$ be a left $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module, coherent as $\mathcal{B}$-module. The canonical morphism*

$$\mathcal{B}' \otimes_{\mathcal{B}} \mathcal{E} \to (\mathcal{B}' \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \otimes_{\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E} \tag{7.5.2.2.1}$$

is an isomorphism.

Proof. It follows from 7.5.2.1 that both terms of 7.5.2.2.1 are p-adically separated and complete. Hence, we reduce to check the map 7.5.2.2.1 is an isomorphism after reduction modulo π^{i+1}, which is 4.3.4.6.1. □

We have the following criterium to get $\mathcal{B}$-coherence which will be useful in order to prove Theorem 11.2.1.12.

Proposition 7.5.2.3. *We suppose $\mathfrak{X}$ affine. Let $\mathcal{E}$ be a coherent $\mathcal{B}\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Then $\mathcal{E}$ is $\mathcal{B}$-coherent if and only if $\Gamma(\mathfrak{X}, \mathcal{E})$ is a $\Gamma(\mathfrak{X}, \mathcal{B})$-module of finite type.*

Proof. Set $E := \Gamma(\mathfrak{X}, \mathcal{E})$, $B := \Gamma(\mathfrak{X}, \mathcal{B})$, $\mathcal{D} := \mathcal{B}\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $D := \Gamma(\mathfrak{X}, \mathcal{D})$. For any $i \in \mathbb{N}$, set $\mathcal{B}_i := \mathcal{B}/\pi^{i+1}\mathcal{B}$,$B_i := B/\pi^{i+1}B$,$\mathcal{D}_i := \mathcal{D}/\pi^{i+1}\mathcal{D}$, $D_i := D/\pi^{i+1}D$. Since we have theorem of type A for coherent $\mathcal{B}$-modules (see 7.2.3.16), then the $\mathcal{B}$-coherence of $\mathcal{E}$ implies that E is a B-module of finite type.

Conversely, suppose E is a B-module of finite type. Since the canonical morphism $B_i \otimes_B D \to D_i$ is an isomorphism, then so is $B_i \otimes_B E \to D_i \otimes_D E$. Since the canonical morphism $\mathcal{B}_i \otimes_{B_i} D_i \to \mathcal{D}_i$ is an isomorphism, then so is $\mathcal{B}_i \otimes_{B_i} (B_i \otimes_B E) \to \mathcal{D}_i \otimes_{D_i} (D_i \otimes_D E)$. This yields the canonical morphism

$$\mathcal{B}_i \otimes_{\mathcal{B}} (\mathcal{B} \otimes_B E) \to \mathcal{D}_i \otimes_{\mathcal{D}} (\mathcal{D} \otimes_D E).$$

is an isomorphism. Since E is of finite type over D and over B, the canonical morphisms $\varprojlim_i \mathcal{B}_i \otimes_{\mathcal{B}} (\mathcal{B} \otimes_B E) \to \mathcal{B} \otimes_B E$ and $\varprojlim_i \mathcal{D}_i \otimes_{\mathcal{D}} (\mathcal{D} \otimes_D E) \to \mathcal{D} \otimes_D E$ are isomorphisms. Hence, the canonical morphism

$$\mathcal{B} \otimes_B E \to \mathcal{D} \otimes_D E.$$

is an isomorphism. Since we have theorem of type A for coherent $\mathcal{D}$-modules, then the canonical morphism

$$\mathcal{D} \otimes_D E \to \mathcal{E}$$

is an isomorphism. Moreover, $\mathcal{B} \otimes_B E$ is $\mathcal{B}$-coherent. Hence, we are done. □

Definition 7.5.2.4. Set $\widehat{\mathcal{D}} := \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

(a) Let $\mathfrak{U}$ be an affine open of $\mathfrak{X}$ on which there exists a system of logarithmic coordinates. Let E be a left $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}_{\mathbb{Q}})$-module of finite type. We say that E is "topologically nilpotent" if, for all $e \in E$, we have $\underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m)}}(e) \to 0$ (for the Banach topology on E defined in 7.5.1.8) when $|\underline{k}| \to +\infty$.

(b) A "topologically nilpotent left $\widehat{\mathcal{D}}_{\mathbb{Q}}$-module" is a left $\widehat{\mathcal{D}}_{\mathbb{Q}}$-module $\mathcal{E}$ so that there exists a basis $\mathfrak{B}$ of affine open sets having logarithmic coordinates, such that for all $\mathfrak{U} \in \mathfrak{B}$, $\Gamma(\mathfrak{U}, \mathcal{E})$ is of finite type on $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}_{\mathbb{Q}})$, and is topologically nilpotent.

Remark 7.5.2.5. With notation 7.5.2.4, suppose $\mathfrak{X}$ affine and endowed with logarithmic coordinates.

(a) If $\mathcal{E}$ is a topologically nilpotent left $\widehat{\mathcal{D}}_{\mathbb{Q}}$-module, then this is not clear that $\mathcal{E}$ is a coherent $\widehat{\mathcal{D}}_{\mathbb{Q}}$-module.

(b) Let E be a left $\Gamma(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathbb{Q}})$-module of finite type. We get the coherent $\widehat{\mathcal{D}}_{\mathbb{Q}}$-modules $\mathcal{E} := \widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\Gamma(\mathfrak{X}, \widehat{\mathcal{D}}_{\mathbb{Q}})} E$. Then $\mathcal{E}$ is topologically nilpotent if and only if so is E. If $\mathring{\mathcal{E}}$ is a model of $\mathcal{E}$, and if $\mathring{\mathcal{E}}_i$ is the reduction of $\mathring{\mathcal{E}}$ module $\mathfrak{m}^{i+1}$, it amounts to the same thing to say that, for all $i \in \mathbb{N}$ $\mathcal{E}_i$ is nilpotent as left $(\mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$-module (see definition 4.2.1.11) ; we thus see that this condition does not depend on logarithmic coordinates.

Lemma 7.5.2.6. *Let $\rho\colon A \to B$ be a continuous homomorphism between Noetherian Banach K-algebras, and E a left B-module which is of finite type on A. Then the Banach norms on E induced by its structure of finite type A-module and by its structure of finite type B-module are equivalent. In particular, the action of B over E is continuous for the topology of B-module and the topology of A-module of E.*

Proof. Let $\| - \|_A$ and $\| - \|_B$ be the norms of A and B. As ρ is continuous, there exists $c \in \mathbb{R}$ such that, for all $a \in A$, we have $\| \rho(a) \|_B \leq c \| a \|_A$. Let's fix a surjective A-linear map of the form $u\colon A^r \to E$; we deduce a quotient norm $\| - \|_1$ on E. Let $u'\colon B^r \to E$ be the B-linear factorization of u; as u' is surjective, we obtain another quotient norm $\| - \|_2$ on E. Thanks to the previous relation, we see that $\| x \|_2 \leq c \| x \|_1$ for all $x \in E$, so that the map $\mathrm{id}_E\colon (E, \| - \|_1) \to (E, \| - \|_2)$ is continuous. Both norms making E a Banach space on K, the Banach theorem results in them defining the same topology on E. The last assertion follows from this. □

Lemma 7.5.2.7. *Suppose $\mathcal{B}$ is endowed with a structure of left $\mathcal{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra. Set $\widehat{\mathcal{D}}^{(m)} := \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widehat{\mathcal{D}}^{(m+1)} := \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Let $\mathcal{E}$ be a left $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-module such that*

(a) there exists a basis $\mathfrak{B}$ of affine open sets having logarithmic coordinates, such that for all $\mathfrak{U} \in \mathfrak{B}$, $\Gamma(\mathfrak{U}, \mathcal{E})$ is of finite type on $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}})$

(b) and the structure of $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-module of $\mathcal{E}$ is induced by a structure of $\widehat{\mathcal{D}}^{(m+1)}_{\mathbb{Q}}$-module.

Then $\mathcal{E}$ is topologically nilpotent as left $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-module.

Proof. Let $\mathfrak{U} \in \mathfrak{B}$. Set $D^{(m)} := \Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)})$ and similarly for $m+1$. Then $E := \Gamma(\mathfrak{U}, \mathcal{E})$ is of finite type on $D^{(m)}_{\mathbb{Q}}$. Hence, E is also of finite type on $D^{(m+1)}_{\mathbb{Q}}$ Lemma 7.5.2.6 implies that the topologies defined by these two structures of modules are the same. Now the elements $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}}$ form a bounded family of $D^{(m+1)}_{\mathbb{Q}}$, therefore the $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}}(e)$ form a bounded family of E. The assertion then follows from $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} = \frac{q^{(m)}_{\underline{k}}!}{q^{(m+1)}_{\underline{k}}!}\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}}$ and the fact that $\frac{q^{(m)}_{\underline{k}}!}{q^{(m+1)}_{\underline{k}}!} \to 0$ when $|\underline{k}| \to 0$. □

Proposition 7.5.2.8. *For any $0 \leq m' \leq m$, set $\mathcal{D}^{(m')} := \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and its p-adic completion $\widehat{\mathcal{D}}^{(m')} := \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Let $\mathcal{E}$ be a topologically nilpotent $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-module which is coherent over $\mathcal{B}_{\mathbb{Q}}$.*

(a) There exists a p-torsion free $\widehat{\mathcal{D}}^{(m)}$-module $\mathring{\mathcal{E}}$, coherent over $\mathcal{B}$ together with an $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$;

(b) The canonical homomorphism $\mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{\mathbb{Q}}} \mathcal{E}$ is an isomorphism and the sheaf $\mathcal{E}$ is coherent as $\mathcal{D}^{(m)}_{\mathbb{Q}}$-module or as $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-module ;

(c) For any $0 \leq m' \leq m$, the canonical homomorphism

$$\mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m')}_{\mathbb{Q}}} \mathcal{E} \tag{7.5.2.8.1}$$

is an isomorphism.

(d) Let $\mathcal{B}'$ be a commutative $\mathcal{B}$-algebra satisfying the conditions of 7.5.1.1 and such that $\mathcal{B} \to \mathcal{B}'$ is $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. The canonical homomorphism

$$\mathcal{B}'_{\mathbb{Q}} \otimes_{\mathcal{B}_{\mathbb{Q}}} \mathcal{E} \to (\mathcal{B}'\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}} \mathcal{E} \tag{7.5.2.8.2}$$

is an isomorphism.

Proof. 1) Let us check (a). We suppose first that $\mathfrak{X}^\sharp$ is an affine open, endowed with logarithmic coordinates and such the nilpotence condition of 7.5.2.4.(a) holds. Let $A := \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, $P_i := \Gamma(X_i, \mathcal{P}_{X_i^\sharp/S_i(m)})$, $B := \Gamma(\mathfrak{X}, \mathcal{B})$, $D^{(m')} := \Gamma(\mathfrak{X}, \mathcal{D}^{(m')})$, $E := \Gamma(\mathfrak{X}, \mathcal{E})$. We set $P' := \varprojlim_i B \otimes_A P_i$. Since $\mathcal{E}$ is coherent over $\mathcal{B}_{\mathbb{Q}}$, then E is a $B_{\mathbb{Q}}$-module of finite type. Let $E' \subset E$ be a sub B-module of finite type such that $E'_{\mathbb{Q}} \xrightarrow{\sim} E$. It follows from 7.2.1.4 that $E' \otimes_B P'$ is separated and complete for the p-adic topology. The nilpotence hypothesis means that, for any $e \in E$, $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(e) \to 0$ (for the Banach topology on E defined in 7.5.1.8 as $D^{(m)}_{\mathbb{Q}}$-module of finite type) when $|\underline{k}| \to +\infty$. It follows from 7.5.2.6 that $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(e) \to 0$ for the Banach topology on E defined as $B_{\mathbb{Q}}$-module of finite type. As the latter is none other than that defined by the p-adic topology of E', then $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(e) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}} \to 0$ in $E \otimes_B P'$ for the topology given by the p-adic topology of $E' \otimes_B P'$. We can therefore define an application $\theta \colon E \to E \otimes_B P'$ by setting

$$\theta(e) := \sum_{\underline{k} \in \mathbb{N}^d} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(e) \otimes \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}. \tag{7.5.2.8.3}$$

We have the algebra isomorphisms $B \otimes_A P_i \xrightarrow{\sim} P_i \otimes_A B$, which is given by the m-PD-stratification of $\mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}$. By p-adic completion, this yields the isomorphism $P' \xrightarrow{\sim} \varprojlim_i P_i \otimes_A B$, which gives a second B-module structure on P' by right multiplication. We then check easily that θ is B-linear for this B-module structure on P'. Hence, we get a p-torsion free B-module $\overset{\circ}{E} = \theta^{-1}(E' \otimes_B P')$ such that $\overset{\circ}{E} \otimes \mathbb{Q} = E$. Since $B \otimes_A P_i$ is free over $B/\mathfrak{m}^{i+1}B$ with the basis $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}\}_{\underline{k} \in \mathbb{N}^d}$. Hence, P' is the p-adic completion of a free B-module with the basis $\{\underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}\}_{\underline{k} \in \mathbb{N}^d}$. This yields that the canonical homomorphism

$$E' \otimes_B P' \to \left(\oplus_{\underline{k}} E'\right)^{\wedge}$$

is an isomorphism and any element $x \in E' \otimes_B P'$ can be written as a convergent series $x = \sum_{\underline{k}} e_{\underline{k}} \otimes \underline{\tau}_\sharp^{\{\underline{k}\}_{(m)}}$, with $e_{\underline{k}} \in E'$ and $e_{\underline{k}} \to 0$ as $|\underline{k}| \to \infty$. Hence, it follows from (7.5.2.8.3) that we have

$$\overset{\circ}{E} = \{e \in E' : \forall\, \underline{k}, \partial^{\langle \underline{k} \rangle} e \in E'\}.$$

In particular, we get $\overset{\circ}{E} \subset E'$. As B is noetherian and E' is of finite type over B then $\overset{\circ}{E}$ is of finite type over B. Moreover, it follows from 1.4.2.7 that $\overset{\circ}{E}$ is a sub-$D^{(m)}$-module of E. We set $\overset{\circ}{\mathcal{E}} := \mathcal{B} \otimes_B \overset{\circ}{E}$. Then using 7.5.2.1 we can check $\overset{\circ}{\mathcal{E}}$ satisfies every required properties. Finally, we move from the affine case to the general case by reasoning as in [Ber96c, 3.4.3].

2) By using 7.5.2.1 (resp. 7.5.2.2.1) with the module $\overset{\circ}{\mathcal{E}}$ and then tensoring with $\mathbb{Q}$, we get the assertion (b) (resp. (d)).

3) Since $\mathcal{E}$ be also a topologically nilpotent $(\mathcal{B} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathbb{Q}}$-module (or remark we can use again $\overset{\circ}{\mathcal{E}}$), we get from (*b*) that the canonical homomorphism $\mathcal{E} \to \widehat{\mathcal{D}}^{(m')}_{\mathbb{Q}} \otimes_{\mathcal{D}^{(m')}_{\mathbb{Q}}} \mathcal{E}$ is an isomorphism for any $m' \leq m$. Since the canonical map $\mathcal{D}^{(m')}_{\mathbb{Q}} \to \mathcal{D}^{(m)}_{\mathbb{Q}}$ is an isomorphism, this implies 7.5.2.8.1. □

Corollary 7.5.2.9. *Set* $\widehat{\mathcal{D}} := \mathcal{B} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. *Let* $\mathcal{E}$ *be a topologically nilpotent* $\widehat{\mathcal{D}}_{\mathbb{Q}}$*-module which is coherent over* $\mathcal{B}_{\mathbb{Q}}$. *Let* $\overset{\circ}{\mathcal{E}}$ *be a p-torsion free coherent* $\widehat{\mathcal{D}}$*-module together with a* $\widehat{\mathcal{D}}_{\mathbb{Q}}$*-linear isomorphism of the form* $\mathcal{E} \xrightarrow{\sim} \overset{\circ}{\mathcal{E}}_{\mathbb{Q}}$. *Then* $\overset{\circ}{\mathcal{E}}$ *is* $\mathcal{B}$*-coherent and* $\overset{\circ}{\mathcal{E}}/\pi^{i+1}\overset{\circ}{\mathcal{E}}$ *is a nilpotent* $\mathcal{D}/\pi^{i+1}\mathcal{D} = (\mathcal{B}/\pi^{i+1}\mathcal{B}) \otimes_{\mathcal{O}_{X_i}} \widehat{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}$*-module (see definition 4.2.1.11).*

Proof. Since this is local, we can suppose $\mathfrak{X}$ affine and that $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates. Following 7.5.2.8.(a), there exists a p-torsion free $\widehat{\mathcal{D}}$-module $\mathcal{F}$, coherent over $\mathcal{B}$ together with an $\widehat{\mathcal{D}}_{\mathbb{Q}}$-linear isomorphism $\mathcal{F}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. We get a homomorphism $\overset{\circ}{\mathcal{E}} \hookrightarrow \overset{\circ}{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{F}_{\mathbb{Q}}$. Multiplying this homomorphism by a power of p, we get the injective $\widehat{\mathcal{D}}$-linear homomorphism $\overset{\circ}{\mathcal{E}} \hookrightarrow \mathcal{F}$. Using 7.5.2.3, this yields the coherence of $\overset{\circ}{\mathcal{E}}$ over $\mathcal{B}$. Since $\mathcal{F}$ is $\mathcal{B}$-coherent, for r large enough we get $p^r\mathcal{F} \hookrightarrow \overset{\circ}{\mathcal{E}} \hookrightarrow \mathcal{F}$ whose composition is the canonical inclusion. For any $x \in \Gamma(\mathfrak{Y}, \overset{\circ}{\mathcal{E}})$, for any $\underline{k} \in \mathbb{N}^d$, we get in $\Gamma(\mathfrak{Y}, \mathcal{F})$ the formula

$$\underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m)}} x = \frac{q_{\underline{k}}^{(m)}!}{q_{\underline{k}}^{(m+1)}!} \underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m+1)}} x,$$

and $q_{\underline{k}}^{(m)}!/q_{\underline{k}}^{(m+1)}!$ converges p-adically to 0 when $|\underline{k}| \to \infty$. Hence, we are done. □

Corollary 7.5.2.10. *Set $\mathcal{D}^{(m)} := \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Let $\mathcal{E}, \mathcal{F}$ be two topologically nilpotent left $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-modules which are coherent over $\mathcal{B}_{\mathbb{Q}}$. Then $\mathcal{E} \otimes_{\mathcal{B}_{\mathbb{Q}}} \mathcal{F}$ and $\mathcal{H}om_{\mathcal{B}_{\mathbb{Q}}}(\mathcal{E}, \mathcal{F})$ are both topologically nilpotent $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-modules and a coherent $\mathcal{B}_{\mathbb{Q}}$-module.*

Proof. The $\mathcal{B}_{\mathbb{Q}}$-coherent is obvious. Moreover, following 7.5.2.8.(a), there exists a p-torsion free $\widehat{\mathcal{D}}^{(m)}$-module $\overset{\circ}{\mathcal{E}}$ (resp. $\overset{\circ}{\mathcal{F}}$), coherent over $\mathcal{B}$ together with an $\widehat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$-linear isomorphism $\overset{\circ}{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ (resp. $\overset{\circ}{\mathcal{F}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{F}$). Since $\overset{\circ}{\mathcal{E}} \otimes_{\mathcal{B}} \overset{\circ}{\mathcal{F}}$ and $\mathcal{H}om_{\mathcal{B}}(\overset{\circ}{\mathcal{E}}, \overset{\circ}{\mathcal{F}})$ are $\mathcal{B}$-coherent, then they are p-adically complete.

For any $i \in \mathbb{N}$, set $\mathcal{B}_i := \mathcal{B}/\pi^{i+1}\mathcal{B}$, $\mathcal{D}_i := \mathcal{D}/\pi^{i+1}\mathcal{D}$, $\overset{\circ}{\mathcal{E}}_i := \overset{\circ}{\mathcal{E}}_i \otimes_{\mathcal{B}} \mathcal{B}_i = \overset{\circ}{\mathcal{E}}_i/\pi^{i+1}\overset{\circ}{\mathcal{E}}_i$, and similarly for $\overset{\circ}{\mathcal{F}}_i$. We reduce to check that $(\overset{\circ}{\mathcal{E}} \otimes_{\mathcal{B}} \overset{\circ}{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{B}_i \xrightarrow{\sim} \overset{\circ}{\mathcal{E}}_i \widehat{\otimes}_{\mathcal{B}_i} \overset{\circ}{\mathcal{F}}_i$ and $\mathcal{H}om_{\mathcal{B}}(\overset{\circ}{\mathcal{E}}, \overset{\circ}{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{B}_i \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_I}(\overset{\circ}{\mathcal{E}}_i, \overset{\circ}{\mathcal{F}}_i)$ are nilpotent for any $i \in \mathbb{N}$, which follows from 4.2.3.6. □

7.5.3 Increasing the level: flatness with unchanged coefficients

Let $m \in \mathbb{N} \cup \{+\infty\}$. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian.

Let $\mathcal{B}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the following conditions.

(a) $\mathcal{B}$ is endowed with a structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra ;

(b) For any affine open $\mathfrak{U}$ of $\mathfrak{X}$, the ring $\Gamma(U, \mathcal{B})$ is noetherian ;

(c) For any $i \geq 0$, $\mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module and the canonical homomorphism $\mathcal{B} \to \varprojlim_{i\in\mathbb{N}} \mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}$ is an isomorphism.

Recall both conditions b and c are equal to that of 7.2.3 in the case where $\mathcal{I} = \mathfrak{m}$. For an example, see 7.5.1.2.1.

Theorem 7.5.3.1. *Suppose $m \in \mathbb{N}$. Let $\mathcal{B}$ be an $\mathcal{O}_{\mathfrak{X}}$-algebra satisfying the conditions of 7.5.3 for $m+1$. The canonical homomorphism $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ is right and left flat.*

Proof. Since the check is the same for the right flatness, we will only prove the left flatness. The assertion is local. We can therefore suppose $\mathfrak{X}$ affine and $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ is endowed with logarithmic coordinates $b_1, \ldots, b_d$ and we use the notation of 7.5.1.5. We denote by $D^{(m)} := \Gamma(\mathfrak{X}, \mathcal{B}\otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\widehat{D}^{(m)}$ its p-adic completion (which can be identified with $\Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and similarly for $m+1$. It is sufficient to prove that the extension $\widehat{D}^{(m)}_{\mathbb{Q}} \to \widehat{D}^{(m+1)}_{\mathbb{Q}}$ is flat. We put $A = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$, $B = \Gamma(\mathfrak{U}, \mathcal{B})$. With 7.5.1.10, we can suppose $\mathcal{B}$ p-torsion free .

For any $\underline{k} \in \mathbb{N}^d$, for any $i = 1, \ldots, d$, with notation 1.2.1.2 we have $k_i = p^m q^{(m)}_{k_i} + r^{(m)}_{k_i} = p^{m+1} q^{(m+1)}_{k_i} + r^{(m+1)}_{k_i}$. We have the formula $\underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m)}} = \frac{q_{\underline{k}}^{(m)}!}{q_{\underline{k}}^{(m+1)}!} \underline{\partial}_{\sharp}^{\langle \underline{k} \rangle_{(m+1)}}$. So, with the uniqueness of the writing 7.5.1.5.2 and 7.5.1.5.3, the canonical homomorphisms $D^{(m)} \to \widehat{D}^{(m)} \to \widehat{D}^{(m)}_{\mathbb{Q}}$ and $D^{(m)} \to D^{(m+1)}$ are injective. Let us then denote by D' the subgroup of $\widehat{D}^{(m)}_{\mathbb{Q}}$ generated by $\widehat{D}^{(m)}$ and $D^{(m+1)}$.

1) We check in this step that D' is in fact a subring of $\widehat{D}^{(m)}_{\mathbb{Q}}$,

Let $P \in \widehat{D}^{(m)}$ and $Q \in D^{(m+1)}$. It is sufficient to check $PQ \in D'$ and $QP \in D'$. Since Q has finite order, then there exists an integer $r \geq 0$ such that $p^r Q \in D^{(m)}$. By using the description 7.5.1.5.1 and

7.5.1.5.2, we can write $P = P_1 + p^r P_2$ with $P_1 \in D^{(m)}$ and $P_2 \in \widehat{D}^{(m)}$. We get $PQ = P_1 Q + P_2(p^r Q)$ with $P_1 Q \in D^{(m+1)}$ and $P_2(p^r Q) \in \widehat{D}^{(m)}$; $QP = QP_1 + (p^r Q)P_2$ with $QP_1 \in D^{(m+1)}$ and $(p^r Q)P_2 \in \widehat{D}^{(m)}$.

2) We claim that $D^{(m+1)}/p^i D^{(m+1)} \to D'/p^i D'$ is an isomorphism for all integer $i \geq 1$.

i) First we show surjectivity. Let $R \in D'$. Choose $P \in \widehat{D}^{(m)}$, $Q \in D^{(m+1)}$ such that $R = P + Q$ We can write $P = P_1 + p^i P_2$ with $P_1 \in D^{(m)}$, $P_2 \in \widehat{D}^{(m)}$. Hence, $R = P_1 + Q + p^i P_2$ with $P_1 + Q \in D^{(m+1)}$ and $P_2 \in \widehat{D}^{(m)} \subset D'$.

ii) For injectivity, take $R \in D^{(m+1)} \cap p^i D'$. Put $R = p^i(P + Q)$ with $P \in \widehat{D}^{(m)}$ and $Q \in D^{(m+1)}$. Then for some n we can write in $\widehat{D}^{(m+1)}$

$$R = \sum_{|\underline{k}| \leq n} a_{\underline{k}} \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}}, \qquad Q = \sum_{|\underline{k}| \leq n} c_{\underline{k}} \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}},$$

$$P = \sum_{\underline{k} \in \mathbb{N}^d} b_{\underline{k}} \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}} = \sum_{\underline{k} \in \mathbb{N}^d} \frac{q_{\underline{k}}^{(m)}!}{q_{\underline{k}}^{(m+1)}!} b_{\underline{k}} \otimes \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}}$$

with $a_{\underline{k}}, b_{\underline{k}}, c_{\underline{k}} \in B$. Since $\mathcal{B}$ is p-torsion free we get $b_{\underline{k}} = 0$ for $|\underline{k}| \rangle n$. This yields $P \in D^{(m)}$ and therefore $R \in p^i D^{(m+1)}$.

3) It follows from 2) that $\widehat{D}^{(m+1)} \to \widehat{D'}$ is an isomorphism. This implies $\widehat{D}_{\mathbb{Q}}^{(m+1)} \xrightarrow{\sim} \widehat{D'}_{\mathbb{Q}}$. On the other hand $D_{\mathbb{Q}}^{(m+1)} = D_{\mathbb{Q}}^{(m)} \subset \widehat{D}_{\mathbb{Q}}^{(m)}$ and thus the inclusion $\widehat{D}_{\mathbb{Q}}^{(m)} \hookrightarrow D'_{\mathbb{Q}}$ is an equality. To end the proof, it is sufficient therefore to establish that D' is noetherian (because this yields that $\widehat{D'}$ is right and left flat on D' and then similarly with the index $\mathbb{Q}$).

By using 4.1.2.17.b), we can check that $D^{(m+1)}$ is generated as left $D^{(m)}$-modules by the operators $(\underline{\partial}_\sharp^{[p^{m+1}]})^{\underline{q}}$, for $\underline{q} \in \mathbb{N}^d$. This yields that D' is generated as left $\widehat{D}^{(m)}$-module by the operator $(\underline{\partial}_\sharp^{[p^{m+1}]})^{\underline{q}}$, for $\underline{q} \in \mathbb{N}^d$. Let $1 \leq j \leq d$ be an integer. The operator $\partial_{\sharp j}^{[p^{m+1}]} = \partial_{\sharp j}^{\langle p^{m+1} \rangle_{(m+1)}}$ commutes with $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m+1)}}$ for any $\underline{k} \in \mathbb{N}^d$. For any $b \in B$, in $D_{\mathbb{Q}}^{(m)}$ following 4.1.2.2.3 we have

$$[\partial_{\sharp j}^{[p^{m+1}]}, b] = \sum_{i < p^{m+1}} \partial_{\sharp j}^{[p^{m+1} - i]}(b) \otimes \partial_{\sharp j}^{[i]}.$$

For $i < p^{m+1}$, we have $q_i^{(m+1)} < p$. Thus $q_i^{(m+1)}! \in \mathbb{Z}_{(p)}^*$ and $\partial_{\sharp j}^{[i]} = (q_i^{(m+1)}!)^{-1} \partial_{\sharp j}^{\langle i \rangle_{(m)}} \in D^{(m)}$ (see 3.2.3.5.2). It follows $[\partial_{\sharp j}^{[p^{m+1}]}, b] \in D^{(m)}$. Passing to limit, we get $[\partial_{\sharp j}^{[p^{m+1}]}, P] \in \widehat{D}^{(m)}$, for all $P \in \widehat{D}^{(m)}$. Since $[(\partial_{\sharp j}^{[p^{m+1}]})^k, P] = \sum_{l=0}^{k-1} (\partial_{\sharp j}^{[p^{m+1}]})^{k-1-l} [\partial_{\sharp j}^{[p^{m+1}]}, P] (\partial_{\sharp j}^{[p^{m+1}]})^l$ for all integer $k \geq 1$ and $P \in \widehat{D}^{(m)}$, by iteration we get

$$[(\partial_{\sharp j}^{[p^{m+1}]})^k, P] \in \sum_{i < k} \widehat{D}^{(m)} (\partial_{\sharp j}^{[p^{m+1}]})^i. \qquad (7.5.3.1.1)$$

For $j = 0, \ldots, d$, let D'_j be the subring of D' generated by $\widehat{D}^{(m)}$ and the $\partial_{\sharp k}^{[p^{m+1}]}$ for $k \leq j$. Let us prove by induction on $j \geq 0$ that D'_j is noetherian. This is already known when $j = 0$. Suppose $j \geq 1$ and D'_{j-1} is noetherian. Let $I \subset D'_j$ be a left ideal. Write $\partial' = \partial_{\sharp j}^{[p^{m+1}]}$. An element $P \in I$ can be written as $P = \sum_{i \leq r} A_i \partial'^i$ with $A_i \in D'_{j-1}$. Let $J \subset D'_{j-1}$ be the set of elements A such that there is $P \in I$ which can be written of the form

$$P = A\partial'^r + \sum_{i < r} A_i \partial'^i,$$

with $A_i \in D'_{j-1}$. Then J is closed under addition: if $P' \in I$ can be written $P' = A\partial'^{r'} + \sum_{i < r'} A_i \partial'^i$ with $r \leq r'$, then $\partial'^{r'-r} P + P' \in I$ and can be written thanks to 7.5.3.1.1 of the form

$$\partial'^{r'-r} P + P' = (A + A')\partial'^{r'} + \sum_{i < r'} A''_i \partial'^i$$

with $A''_i \in \mathcal{D}'_{j-1}$. Hence, J is a left ideal of D'_{j-1}. By induction hypothesis, J has a finite set of generators $A_1, \ldots, A_s$. For $1 \leq k \leq s$ let $I \ni P_k = A_k \partial'^{r_k} + \sum_{i < r_k} A_{k,i} \partial'^i$ with $A_{k,i} \in D'_{j-1}$. Put $r = \sup r_k$. Let

$M \subset D'_j$ be the left D'_{j-1}-module generated by $1, \partial', \ldots, \partial'^r$. Then M is left noetherian and $I \cap M$ is a finitely generated submodule. If $Q_1, \ldots, Q_t$ is a set of generators of $I \cap M$ as left D'_{j-1}-module then we can check by using 7.5.3.1.1 that $\{P_1, \ldots, P_s, Q_1, \ldots, Q_t\}$ generates I. This completes the proof of theorem 7.5.3.1.

□

Corollary 7.5.3.2. *The canonical homomorphism* $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ *is right and left flat.*

7.5.4 Derived completed tensor products, derived completed internal homomorphisms, swapping left and the right modules

Let $m \in \mathbb{N}$. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0,p)$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme, where $\mathfrak{X}^\sharp$ is a log smooth $\mathfrak{S}^\sharp$-log-formal scheme. Let $\mathcal{B}_{\mathfrak{X}}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible structure of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. We suppose moreover that $\mathcal{B}_{\mathfrak{X}}$ satisfies the hypotheses of 7.3.2 (remark when $\mathcal{X}$ is p-torsion free, the condition 7.3.2.d is equivalent to saying that $\mathcal{B}_{\mathfrak{X}}$ is p-torsion free). Recall following 7.3.2.3 that $\mathcal{B}_{\mathfrak{X}}$ is in particular quasi-coherent in the sense of 7.3.1.5.

We denote by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Recall also, following 7.3.2.1, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies the conditions of 7.3.2, i.e. in particular the projective systems $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} := (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i})_{i \in \mathbb{N}}$ with $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is left and right quasi-coherent in the sense of 7.3.1.10. Let $\mathfrak{U} := \mathfrak{X}^{\sharp *}$ be the open of $\mathfrak{X}$ where $M_{\mathfrak{X}^\sharp}$ is trivial and $j_{\mathfrak{U}} \colon \mathfrak{U} \hookrightarrow \mathfrak{X}^\sharp$ be the canonical open immersion. Moreover, we suppose $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}})/\mathfrak{S}^\sharp$ is a strongly quasi-flat morphism of ringed $\mathcal{V}$-log formal schemes (see Definition 4.4.1.3.b).

7.5.4.1. Since $\widetilde{\mathfrak{X}}^\sharp/\mathfrak{S}^\sharp$ and $\widetilde{\mathfrak{Y}}^\sharp/\mathfrak{T}^\sharp$ are strongly quasi-flat morphisms of ringed $\mathcal{V}$-log formal schemes, then there exists some integer d such that $\mathcal{B}_{X_0} \otimes_{\mathcal{B}_{\mathfrak{X}}} -$ and $\oplus_{i \in \mathbb{N}} \mathcal{B}_{X_i}/\mathcal{B}_{X_{i+1}} \otimes_{\mathcal{B}_{X_0}} -$ have cohomological dimension $\leq d$. Hence, it follows from 7.3.2.8 that $\mathcal{B}_{X_\bullet}$ has right tor dimension $\leq 2d$ on $\underleftarrow{l}_{\mathfrak{X}}^{-1}\mathcal{B}_{\mathfrak{X}}$ and $\mathcal{B}_{X_j}$ has right tor dimension $\leq 2d$ on $\mathcal{B}_{X_i}$, for any integers $0 \leq j \leq i$.

7.5.4.2 (Adjunction)**.** Let $* \in \{r, l\}$. Let $\mathcal{D} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\mathcal{D} := \mathcal{B}_{\mathfrak{X}}$) and $\mathcal{D}_\bullet = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ (resp. $\mathcal{D}_\bullet := \mathcal{B}_{X_\bullet}$). From [Sta22, 07A6], this yields the functors $\mathbb{R}\underleftarrow{l}_{X*} \colon D({}^*\mathcal{D}_\bullet) \to D({}^*\mathcal{D})$ and $\mathbb{L}\underleftarrow{l}^*_X \colon D({}^*\mathcal{D}) \to D({}^*\mathcal{D}_\bullet)$ which are adjoint, i.e., we have the bifunctorial bijections of the form

$$\mathrm{Hom}_{D({}^*\mathcal{D}_\bullet)}(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet), \mathcal{F}^\bullet_\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D({}^*\mathcal{D})}(\mathcal{E}^\bullet, \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{F}^\bullet_\bullet)) \tag{7.5.4.2.1}$$

for any $\mathcal{E}^\bullet \in D({}^*\mathcal{D})$ and any $\mathcal{F}^\bullet_\bullet \in D({}^*\mathcal{D}_\bullet)$. This is equivalent to saying that we have a morphism of functors $D({}^*\mathcal{D}_\bullet) \to D({}^*\mathcal{D}_\bullet)$ of the form $\mathrm{adj}_{RL} \colon \mathrm{id} \to \mathbb{R}\underleftarrow{l}_{X*} \circ \mathbb{L}\underleftarrow{l}^*_X$ and a morphism of functors $D({}^*\mathcal{D}) \to D({}^*\mathcal{D})$ of the form $\mathrm{adj}_{LR} \colon \mathbb{L}\underleftarrow{l}^*_X \circ \mathbb{R}\underleftarrow{l}_{X*} \to \mathrm{id}$ such that $(\mathrm{adj}_{LR} \circ \mathbb{L}\underleftarrow{l}^*_X) \circ (\mathbb{L}\underleftarrow{l}^*_X \circ \mathrm{adj}_{RL})$ is the identity $\mathbb{L}\underleftarrow{l}^*_X \to \mathbb{L}\underleftarrow{l}^*_X$ and $(\mathbb{R}\underleftarrow{l}_{X*} \circ \mathrm{adj}_{LR}) \circ (\mathrm{adj}_{RL} \circ \mathbb{R}\underleftarrow{l}_{X*})$ is the identity $\mathbb{R}\underleftarrow{l}_{X*} \to \mathbb{R}\underleftarrow{l}_{X*}$ (see notation [KS06, 1.3.4]).

We have the commutative up to canonical isomorphism diagram

$$\begin{array}{ccc} D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) & \xrightarrow{\mathbb{R}\underleftarrow{l}_{X*}} & D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \\ \downarrow & & \downarrow \\ D(\mathcal{B}_{X_\bullet}) & \xrightarrow{\mathbb{R}\underleftarrow{l}_{X*}} & D(\mathcal{B}_{\mathfrak{X}}) \end{array} , \qquad \begin{array}{ccc} D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) & \xrightarrow{\mathbb{L}\underleftarrow{l}^*_X} & D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \\ \downarrow & & \downarrow \\ D(\mathcal{B}_{\mathfrak{X}}) & \xrightarrow{\mathbb{L}\underleftarrow{l}^*_X} & D(\mathcal{B}_{X_\bullet}) \end{array} , \tag{7.5.4.2.2}$$

where the vertical maps are the forgetful functors. This yields that the adjunction morphisms adj_{LR} and adj_{RL} commute with the forgetful functors (indeed, the equalities satisfied by adj_{LR} and $\mathrm{adj}_{RL}$are still valid after applying the forgetful functors). This implies that we have the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})}(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet), \mathcal{F}^\bullet_\bullet) & \xrightarrow[7.5.4.2.1]{\sim} & \mathrm{Hom}_{D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})}(\mathcal{E}^\bullet, \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{F}^\bullet_\bullet)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D({}^*\mathcal{B}_{X_\bullet})}(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet), \mathcal{F}^\bullet_\bullet) & \xrightarrow[7.5.4.2.1]{\sim} & \mathrm{Hom}_{D({}^*\mathcal{B}_{\mathfrak{X}})}(\mathcal{E}^\bullet, \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{F}^\bullet_\bullet)). \end{array} \tag{7.5.4.2.3}$$

7.5.4.3. Let $* \in \{\mathrm{l}, \mathrm{r}\}$ and $\star \in \{-, \mathrm{b}\}$. Let $\mathcal{D} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\mathcal{D} := \mathcal{B}_{\mathfrak{X}}$) and $\mathcal{D}_\bullet = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ (resp. $\mathcal{D}_\bullet := \mathcal{B}_{X_\bullet}$). We have the morphism of ringed topoi $\underset{\leftarrow}{l}_X \colon (X_\bullet, \mathcal{D}_\bullet) \to (|\mathcal{X}|, \mathcal{D})$. Following 7.3.2.11.a, $\mathcal{D}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}$) has left and right finite tor dimension on $\mathcal{D}$ (resp. $\mathcal{D}_0$). Hence following 7.3.2.9 or 7.3.2.10, the functors $\mathbb{R}\underset{\leftarrow}{l}_{X*}$ and $\mathbb{L}\underset{\leftarrow}{l}^*_X$ induce canonically quasi-inverse equivalences of categories between $D^\star_{\mathrm{qc}}(^*\mathcal{D}_\bullet)$ and $D^\star_{\mathrm{qc}}(^*\mathcal{D})$. The adjunction morphisms adj_{LR} and adj_{RL} (of the paragraph 7.5.4.2) are isomorphisms when restricted to bounded above quasi-coherent complexes.

7.5.4.4. Let $\star \in \{-, \mathrm{b}\}$. Let $\mathcal{D}_\bullet = \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ or $\mathcal{D}_\bullet := \mathcal{B}_{X_\bullet}$. Let $\mathcal{E}^\bullet_\bullet \in D^\star(^{\mathrm{l}}\mathcal{D}_\bullet)$, $\mathcal{M}^\bullet_\bullet \in D^\star(^{\mathrm{r}}\mathcal{D}_\bullet)$. Then via the beginning of the paragraph 7.3.1.14 (still valid for right modules), we get that the property $\mathcal{E}^\bullet_\bullet \in D^\star_{\mathrm{qc}}(^{\mathrm{l}}\mathcal{D}_\bullet)$ (resp. $\mathcal{M}^\bullet_\bullet \in D^\star_{\mathrm{qc}}(^{\mathrm{r}}\mathcal{D}_\bullet)$) is satisfied if and only if both conditions hold:

(a) The complex $\mathcal{E}^\bullet_0$ (resp. $\mathcal{M}^\bullet_0$) is in $D^-_{\mathrm{qc}}(\mathcal{O}_{X_0})$.

(b) The canonical left (resp. right) map

$$\mathcal{D}_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{i+1}} \mathcal{E}^\bullet_{i+1} \to \mathcal{E}^\bullet_i, \qquad \mathcal{M}^\bullet_{i+1} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{i+1}} \mathcal{D}_i \to \mathcal{M}^\bullet_i \tag{7.5.4.4.1}$$

is an isomorphism.

7.5.4.5. Let $* \in \{r, l\}$.

(a) Since $\mathcal{B}_{X_\bullet} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ is flat, then a K-flat complex of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules is a K-flat complex of $\mathcal{B}_{X_\bullet}$-modules. Hence, similarly to 4.6.6.1.4, we get the top functor making commutative (up to a canonical isomorphism) the diagram:

$$\begin{array}{ccc} D(^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \times D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}}-} & D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \\ \downarrow & & \downarrow \\ D(\mathcal{B}_{X_\bullet}) \times D(\mathcal{B}_{X_\bullet}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}}-} & D(\mathcal{B}_{X_\bullet}) \end{array} \tag{7.5.4.5.1}$$

where the vertical maps are the forgeful functors. Moreover, the top morphism of 7.5.4.5.1 preserves the quasi-coherence, i.e. we get

$$-\otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}} - \colon D(^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \times D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}). \tag{7.5.4.5.2}$$

(b) Since $\mathcal{B}_{X_\bullet} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ is flat, then a K-injective complex of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules is a K-injective complex of $\mathcal{B}_{X_\bullet}$-modules. Hence, similarly to 4.6.6.1.4, we get the top functor making commutative (up to a canonical isomorphism) the diagram:

$$\begin{array}{ccc} D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \times D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) & \xrightarrow{\mathbb{R}\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(-,-)} & D(^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \\ \downarrow & & \downarrow \\ D(\mathcal{B}_{X_\bullet}) \times D(\mathcal{B}_{X_\bullet}) & \xrightarrow{\mathbb{R}\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(-,-)} & D(\mathcal{B}_{X_\bullet}) \end{array} \tag{7.5.4.5.3}$$

where the vertical maps are the forgeful functors. Suppose $\mathfrak{X}$ is quasi-compact. It follows from 7.3.4.17 that for any $\mathcal{E}_\bullet \in D(^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ such that $\mathcal{E}_\bullet \in D_{\mathrm{perf}}(\mathcal{B}_{X_\bullet})$, the functor $\mathbb{R}\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\mathcal{E}_\bullet, -)$ preserves the quasi-coherence and the perfectness.

7.5.4.6 (Swapping left and right $\mathcal{D}_\bullet$-module)**.** Following 7.5.1.13, the sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, is endowed with a canonical structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Moreover, $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies the conditions of 7.3.2, i.e. in particular the projective systems $\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} := (\widetilde{\omega}_{X^\sharp_i/S^\sharp_i})_{i\in\mathbb{N}}$ with $\widetilde{\omega}_{X^\sharp_i/S^\sharp_i} := \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is quasi-coherent in the sense of 7.3.1.10. Similarly to 4.3.5.7, we check that the functors $-\otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\omega}^{-1}_{X^\sharp_\bullet/S^\sharp_\bullet} = \mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}, -)$ and $\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} -$ are exact and induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules.

Since these functors preserve the quasi-coherence (see 7.5.4.5) and are exact, then for any $\star \in \{-, \mathrm{b}\}$, the functors $\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}, -)$ and $\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} -$ induce quasi-inverse equivalences between the categories $D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ and $D^\star_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$.

7.5.4.7. This paragraph is a variation of 7.3.4.2. Let $* \in \{r, l\}$. Let $\mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\mathcal{M}^\bullet \in D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Following 7.3.2.3.d, $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^\bullet) \xrightarrow{\sim} \mathcal{B}_{X^\sharp_\bullet} \otimes^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}^\sharp}} \mathcal{E}^\bullet \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ and similarly for right modules. Hence, we define an object of $D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ by setting

$$\mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}^\bullet := \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{L}\underleftarrow{l}^*_X \mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}} \mathbb{L}\underleftarrow{l}^*_X \mathcal{E}^\bullet), \tag{7.5.4.7.1}$$

which is called the *completed tensor product of $\mathcal{M}^\bullet$ over $\mathcal{B}_{\mathfrak{X}}$ of $\mathcal{E}^\bullet$.*

It follows from 7.3.4.11.1 and 7.5.4.5.1 that the functor 7.5.4.7.1 preserves the quasi-coherence and the isogenies, i.e., with notation 7.4.6.1 we get the functors

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} - \colon D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \times D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}), \tag{7.5.4.7.2}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} - \colon D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \times D^-_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D^-_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}). \tag{7.5.4.7.3}$$

Moreover, for any $\mathcal{E} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, for any $\mathcal{M} \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, by using a K-flat complex of left (resp. left or right according to the value of $*$), it follows from 7.5.4.3 that we get the bifunctorial canonical isomorphism of $D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ of the form

$$\mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}) \otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}). \tag{7.5.4.7.4}$$

7.5.4.8. This paragraph is a variation of 7.3.4.18. Let $* \in \{r, l\}$. Let $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. We define an object of $D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ by setting

$$\mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) := \mathbb{R}\underleftarrow{l}_{X*}(\mathbb{R}\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\mathbb{L}\underleftarrow{l}^*_X\mathcal{E}^\bullet, \mathbb{L}\underleftarrow{l}^*_X\mathcal{F}^\bullet)). \tag{7.5.4.8.1}$$

which is called *completed internal homomorphism over $\mathcal{B}_{\mathfrak{X}}$ of $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$.*

Suppose $\mathfrak{X}$ is quasi-compact. Let $\mathcal{E}^\bullet \in K({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ such that $\mathcal{E}^\bullet \in D_{\mathrm{perf}}(\mathcal{B}_{\mathfrak{X}})$. Let $\star \in \{\mathrm{b}, -\}$. Then it follows from 7.3.4.17 and 7.5.4.5.1 that 7.5.4.8.1 induces the functor:

$$\mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{E}^\bullet, -) \colon D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}), \tag{7.5.4.8.2}$$

$$\mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{E}^\bullet, -) \colon D^\star_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D^\star_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}). \tag{7.5.4.8.3}$$

Hence, for any $\mathcal{F} \in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, it follows from 7.5.4.3 that we get the canonical isomorphism of $D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ of the form

$$\mathbb{L}\underleftarrow{l}^*_X\mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}), \mathbb{L}\underleftarrow{l}^*_X(\mathcal{F}). \tag{7.5.4.8.4}$$

7.5.4.9. Following 4.6.6.1.4, we have the functors

$$- \otimes^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} - \colon D({}^{\mathrm{l}}\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \times D({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}), \tag{7.5.4.9.1}$$

$$\mathbb{R}\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(-,-) \colon D({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \times D({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}). \tag{7.5.4.9.2}$$

7.5.4.10. Let $\mathcal{E}^\bullet \in D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $\mathcal{M}^\bullet, \mathcal{N}^\bullet \in D^-({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Since $\mathcal{B}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ are flat morphisms, then a K-flat complex of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules is also a K-flat complex of left (resp. right) $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules and a K-flat complex of $\mathcal{B}_{\mathfrak{X}}$-modules. We construct by adjunction (similarly to 7.3.4.3) the following morphism of $D^-({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$

$$\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}^\bullet \to \mathcal{M}^\bullet \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}^\bullet. \tag{7.5.4.10.1}$$

By applying the forgetful functor $D^-({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D(\mathcal{B}_{\mathfrak{X}})$, we get the map 7.3.4.3. Hence, in the case where $\mathcal{M}^\bullet \in D^-_{\mathrm{coh}}(\mathcal{B}_{\mathfrak{X}})$ and $\mathcal{E}^\bullet \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, the map 7.5.4.10.1 is an isomorphism.

We construct (similarly to 7.3.4.16.2) the morphism of $D({}^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\mathbb{R}\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet). \tag{7.5.4.10.2}$$

When $\mathcal{M}^\bullet \in D_{\mathrm{perf}}(\mathcal{B}_{\mathfrak{X}})$ and $\mathcal{N}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ the map 7.5.4.10.2 is an isomorphism.

7.5.4.11 (Swapping left and right quasi-coherent complexes of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules)**.** Let $\star \in \{-, \mathrm{b}\}$. Since $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a projective $\mathcal{B}_{\mathfrak{X}}$-module, by using 7.5.1.13.(f) we construct similarly to 7.5.4.10 the isomorphisms of functors:

$$\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -) \xrightarrow{\sim} \mathbb{R}\mathscr{H}\widehat{om}_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -) \colon D^\star_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}), \tag{7.5.4.11.1}$$

$$\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} - \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} - \colon D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D^\star_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}), \tag{7.5.4.11.2}$$

so that the functors $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} -$ and $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -)$ induce quasi-inverse equivalences of categories between $D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $D^\star_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. These equivalences preserve K-flat complexes, K-injective complexes. Since these equivalences preserve also isogenies and coherence, they induce quasi-inverse equivalence between $D^\star_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $D^\star_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and between $D^\star_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $D^\star_{\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ etc.

7.5.4.12. Let $* \in \{r, l\}$. Let $\mathcal{E}_\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$. Let $\mathcal{M} \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. We get the isomorphisms in $D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$:

$$\mathbb{L}\underleftarrow{l}^*_X\left(\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}}\mathcal{M}\right) \xrightarrow[\sim]{7.5.4.7.4} \mathbb{L}\underleftarrow{l}^*_X \circ \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet) \otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}) \xrightarrow[\sim]{7.5.4.3} \mathcal{E}_\bullet \otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}) \tag{7.5.4.12.1}$$

Since $\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}}\mathcal{M}$ is quasi-coherent (see 7.5.4.7.2), then we get from 7.5.4.3 the isomorphisms of $D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}}\mathcal{M} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*}\mathbb{L}\underleftarrow{l}^*_X\left(\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}}\mathcal{M}\right) \xrightarrow[\sim]{7.5.4.12.1} \mathbb{R}\underleftarrow{l}_{X*}\left(\mathcal{E}_\bullet \otimes^{\mathbb{L}}_{\mathcal{B}_{X_\bullet}} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M})\right). \tag{7.5.4.12.2}$$

7.5.4.13. Since $\mathbb{L}\underleftarrow{l}^*_X(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}$, since $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a locally free $\mathcal{B}_{\mathfrak{X}}$-module of rank one, then with 7.5.1.13, then for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ we get by using some K-flat representation of $\mathcal{E}$ the canonical $D({}^r\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$:

$$\mathbb{L}\underleftarrow{l}^*_X(\mathcal{E} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}) \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}. \tag{7.5.4.13.1}$$

Moreover, for any $\mathcal{E}_\bullet \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, by using some K-injective resolution of $\mathcal{E}_\bullet$ we get the isomorphism of $D({}^r\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{E}_\bullet) \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*}\left(\mathcal{E}_\bullet \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}\right). \tag{7.5.4.13.2}$$

For any $\mathcal{M} \in D({}^r\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we get from 7.5.4.8.4 the isomorphism

$$\mathbb{L}\underleftarrow{l}^*_X(\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M})) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}, \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}). \tag{7.5.4.13.3}$$

For any $\mathcal{M}_\bullet \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, by using K-injective representation of $\mathcal{M}_\bullet$, we construct the isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ of the form:

$$\mathbb{R}\underleftarrow{l}_{X*}\mathscr{H}om_{\mathcal{B}_{X_\bullet}}(\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}, \mathcal{M}^\bullet_\bullet) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathbb{R}\underleftarrow{l}_{X*}(\mathcal{M}^\bullet_\bullet)). \tag{7.5.4.13.4}$$

In other words, the functors $\mathbb{L}\underleftarrow{l}^*_X$ and $\mathbb{R}\underleftarrow{l}_{X*}$ are compatible with the quasi-inverse equivalences of categories 7.5.4.6 and of 7.5.4.11 (or 7.5.1.13.(f)).

7.5.5 Extraordinary inverse images

Let $m \in \mathbb{N}$. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0,p)$ with maximal ideal $\mathfrak{m}$. Let

$$\begin{array}{ccc} \mathfrak{X}^\sharp & \xrightarrow{f} & \mathfrak{Y}^\sharp \\ \downarrow{\scriptstyle p_{\mathfrak{X}^\sharp}} & & \downarrow{\scriptstyle p_{\mathfrak{Y}^\sharp}} \\ \mathfrak{S}^\sharp & \xrightarrow{\phi} & \mathfrak{T}^\sharp, \end{array} \tag{7.5.5.0.1}$$

be a commutative diagram where $\mathfrak{S}^\sharp$ and $\mathfrak{T}^\sharp$ are nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10, where $\mathfrak{X}^\sharp$ is a log smooth $\mathfrak{S}^\sharp$-log-formal scheme and $\mathfrak{Y}^\sharp$ is a log smooth $\mathfrak{T}^\sharp$-log formal scheme. Let $\mathcal{B}_{\mathfrak{X}}$ (resp. $\mathcal{B}_{\mathfrak{Y}}$) be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra (resp. $\mathcal{O}_{\mathfrak{Y}}$-algebra) endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. left $\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$-module) and satisfying the hypotheses of 7.3.2. Recall following 7.3.2.3 that $\mathcal{B}_{\mathfrak{X}}$ and $\mathcal{B}_{\mathfrak{Y}}$ are in particular quasi-coherent in the sense of 7.3.1.5. The action of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $f^*\mathcal{B}_{\mathfrak{Y}}$ is compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra (see 3.4.4.6). We suppose that we have a morphism of algebras $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ which is moreover $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. We denote by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} := \mathcal{B}_{\mathfrak{Y}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}}\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$. Recall also, following 7.3.2.1, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ satisfy the conditions of 7.3.2, i.e. in particular the projective systems $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} := (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i})_{i\in\mathbb{N}}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} := (\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})_{i\in\mathbb{N}}$) with $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$) is left and right quasi-coherent in the sense of 7.3.1.10.

We denote by $\widetilde{\mathfrak{X}}^\sharp$ (resp. $\widetilde{\mathfrak{Y}}^\sharp$) the ringed $\mathcal{V}$-log formal scheme $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}})$ (resp. $(\mathfrak{Y}^\sharp, \mathcal{B}_{\mathfrak{Y}})$), and by $\widetilde{f}\colon \widetilde{\mathfrak{X}}^\sharp/\mathfrak{S}^\sharp \to \widetilde{\mathfrak{Y}}^\sharp/\mathfrak{T}^\sharp$ the morphism of relative ringed $\mathcal{V}$-log formal schemes induced by the diagram 7.5.5.0.1 and by $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$. When $\mathfrak{S}^\sharp \to \mathfrak{T}^\sharp$ in understood, by abuse of notation, we also denote by $\widetilde{f}$ the induced morphism $\widetilde{\mathfrak{X}}^\sharp \to \widetilde{\mathfrak{Y}}^\sharp$ of ringed $\mathcal{V}$-log formal schemes.

Let $\mathfrak{U} := \mathfrak{X}^{\sharp *}$ be the open of $\mathfrak{X}$ where $M_{\mathfrak{X}^\sharp}$ is trivial and $j_{\mathfrak{U}}\colon \mathfrak{U} \hookrightarrow \mathfrak{X}^\sharp$ be the canonical open immersion. Let $\mathfrak{V} := \mathfrak{Y}^{\sharp *}$ be the open of $\mathfrak{Y}$ where $M_{\mathfrak{Y}^\sharp}$ is trivial and $j_{\mathfrak{V}}\colon \mathfrak{V} \hookrightarrow \mathfrak{Y}^\sharp$ be the canonical open immersion.

Moreover, we suppose $\widetilde{f}$ is a strongly quasi-flat morphism of relative ringed $\mathcal{V}$-log formal schemes (see Definition 4.4.1.3.d). The quasi-flatness will be useful to get Lemma 7.5.5.5. The strongly quasi-flatness of $\widetilde{f}$ implies that $\widetilde{\mathfrak{X}}^\sharp/\mathfrak{S}^\sharp$ and $\widetilde{\mathfrak{Y}}^\sharp/\mathfrak{T}^\sharp$ are strongly quasi-flat morphisms of ringed $\mathcal{V}$-log formal schemes (in the sense of 4.4.1.3.b). Hence, we get for instance the finite tor-dimensions of 7.5.4.1 and we can apply 7.5.4.3 for both.

Lemma 7.5.5.1. *Suppose $f^{-1}\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ has tor dimension $\leq d$ for some integer d, i.e. suppose the functor $\widetilde{f}^* = \mathcal{B}_{\mathfrak{X}} \otimes_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}-$ from the category of $\mathcal{B}_{\mathfrak{Y}}$-modules to that of $\mathcal{B}_{\mathfrak{X}}$-modules has cohomological dimension $\leq d$. Then the functor $\widetilde{f}^*_\bullet = \mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}-$ has cohomological dimension $\leq d$.*

Proof. Since $\widetilde{f}^*$ is quasi-flat, then there exists a morphism $\mathfrak{Y} \to \mathfrak{U}$ of $\mathcal{V}$-formal schemes such that both induced morphisms of ringed spaces $\widetilde{\mathfrak{X}}^\sharp \to \mathfrak{U}$ and $\widetilde{\mathfrak{Y}}^\sharp \to \mathfrak{U}$ are flat. Hence $\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{U}}} \mathcal{B}_{\mathfrak{X}} \xrightarrow{\sim} \mathcal{B}_{X_i}$ and $\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{U}}} \mathcal{B}_{\mathfrak{Y}} \xrightarrow{\sim} \mathcal{B}_{Y_i}$ for any integer $i \geq 0$. This yields the canonical isomorphism $\mathcal{B}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{B}_{Y_i} \xrightarrow{\sim} \mathcal{B}_{X_i}$, for any integer $i \geq 0$. Hence, for any $i \in \mathbb{N}$ and any $\mathcal{B}_{Y_i}$-module $\mathcal{M}_i$, the canonical morphism $\mathcal{B}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{M}_i \to \mathcal{B}_{X_i} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{Y_i}} f^{-1}\mathcal{M}_i$ is an isomorphism and we are done. $\square$

7.5.5.2. Let $\star \in \{-, \mathrm{b}\}$. In the case where $\star = \mathrm{b}$, we suppose moreover $f^{-1}\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ has finite tor dimension, and therefore following 7.5.5.1 that so is the functor $\widetilde{f}^*_\bullet = \mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}-$. Hence, we get the functors

$$\mathbb{L}\widetilde{f}^*_{\mathrm{alg}} = \mathcal{B}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}-\colon D^\star(\mathcal{B}_{\mathfrak{Y}}) \to D^\star(\mathcal{B}_{\mathfrak{X}}), \qquad \mathbb{L}\widetilde{f}^*_\bullet = \mathcal{B}_{X_\bullet} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}-\colon D^\star(\mathcal{B}_{Y_\bullet}) \to D^\star(\mathcal{B}_{X_\bullet}). \tag{7.5.5.2.1}$$

Let $\mathcal{F}_\bullet \in D^\star(\mathcal{B}_{Y_\bullet})$. It follows from 7.1.3.6.1 that we have the isomorphism:

$$\left(\mathbb{L}\widetilde{f}^*_\bullet(\mathcal{F}_\bullet)\right)_i \xrightarrow{\sim} \mathbb{L}\widetilde{f}^*_i(\mathcal{F}_i). \tag{7.5.5.2.2}$$

Since $\mathcal{B}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{B}_{X_{i+1}}} \mathbb{L}\widetilde{f}^*_{i+1}(\mathcal{F}_{i+1}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^*_i(\mathcal{B}_{Y_i} \otimes^{\mathbb{L}}_{\mathcal{B}_{Y_{i+1}}} \mathcal{F}_{i+1})$, then it follows from 7.5.4.4 that the right functor of 7.5.5.2.1 preserves the quasi-coherence (in the sense of 7.3.1.10), i.e. induces the functor

$$\mathbb{L}\widetilde{f}^*_\bullet\colon D^\star_{\mathrm{qc}}(\mathcal{B}_{Y_\bullet}) \to D^\star_{\mathrm{qc}}(\mathcal{B}_{X_\bullet}). \tag{7.5.5.2.3}$$

Since $\mathcal{B}_{Y_\bullet} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}$ is flat, then a K-flat complex of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}$-modules is a K-flat complex of $\mathcal{B}_{Y_\bullet}$-modules. Hence, we get the functor $\mathbb{L}\widetilde{f}^*\colon D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \to D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ making commutative the diagram:

$$\begin{array}{ccc} D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}) & \xrightarrow{\mathbb{L}\widetilde{f}^*_\bullet} & D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \\ \downarrow & & \downarrow \\ D^\star_{\mathrm{qc}}(\mathcal{B}_{Y_\bullet}) & \xrightarrow{\mathbb{L}\widetilde{f}^*_\bullet} & D^\star_{\mathrm{qc}}(\mathcal{B}_{X_\bullet}) \end{array} \tag{7.5.5.2.4}$$

where the vertical functors are the forgetful ones.

7.5.5.3. Following 7.5.1.13, the sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} := \mathcal{B}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \omega_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$) is endowed with a canonical structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$-module). Moreover, $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ satisfy the conditions of 7.3.2, i.e. in particular the projective systems $\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} := (\widetilde{\omega}_{X^\sharp_i/S^\sharp_i})_{i\in\mathbb{N}}$ (resp. $\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet} := (\widetilde{\omega}_{Y^\sharp_i/T^\sharp_i})_{i\in\mathbb{N}}$) with $\widetilde{\omega}_{X^\sharp_i/S^\sharp_i} := \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\widetilde{\omega}_{Y^\sharp_i/T^\sharp_i} := \widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$) is quasi-coherent in the sense of 7.3.1.10.

Notation 7.5.5.4. We deduce by functoriality from 7.5.5.2 that we get a structure of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$-bimodule on $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet\to Y^\sharp_\bullet/T^\sharp_\bullet} := \widetilde{f}^*_\bullet\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} = \mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}$. When $\mathfrak{S} \to \mathfrak{T}$ is the identity, we can simply write $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet\to Y^\sharp_\bullet/S^\sharp_\bullet}$ and when moreover there is no doubt about S we write $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet\to Y^\sharp_\bullet}$. By functoriality from 7.5.5.2 and 7.5.4.6, we get a structure of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodule on

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet} := \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{f}^*_{\bullet r}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}\right),$$

where the index "r" means that we have chosen the right (i.e. the twisted) structure of left $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}$-module on $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}$ to compute the structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module via the functor $\widetilde{f}^*_\bullet$.

When $\mathfrak{S} \to \mathfrak{T}$ is the identity, we can simply write $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$ and when moreover there is no doubt about S we write $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet\leftarrow X^\sharp_\bullet}$.

We have the isomorphism of left $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodules

$$\widetilde{f}^*_{\bullet r}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}\right) \underset{\widetilde{f}^*_\bullet(4.2.5.6.3)}{\xrightarrow{\sim}} \widetilde{f}^*_{\bullet l}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}\right),$$

where the index "l" (resp. "r ") means that we have chosen the left (resp. right) structure to compute $\widetilde{f}^*_\bullet$. By tensoring this latter isomorphism with $\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} -$, we get the isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodules

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet} \xrightarrow{\sim} \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet\to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}\widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}. \tag{7.5.5.4.1}$$

Lemma 7.5.5.5. *Since $\widetilde{f}$ is quasi-flat (see Definition 4.4.1.3), then $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$) is solvable in the sense of 4.6.3.2.b as a complex of $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ (resp. $D(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$).*

Proof. By definition, there exists a morphism of $\mathcal{V}$-formal schemes $\mathfrak{T} \to \mathfrak{U}$ such that both induced morphisms of ringed spaces $g\colon (\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}) \to \mathfrak{U}$ and $h\colon (\mathfrak{Y}, \mathcal{B}_{\mathfrak{Y}}) \to \mathfrak{U}$ are flat. This yields that $g_\bullet\colon (X_\bullet, \mathcal{B}_{X_\bullet}) \to U_\bullet$ and $h_\bullet\colon (Y_\bullet, \mathcal{B}_{Y_\bullet}) \to U_\bullet$ are flat. Since $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}/\mathcal{B}_{Y_\bullet}$ is flat, since $h_\bullet$ is flat we get that $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}/h_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is flat. Moreover, $h_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is sent in the center of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}$. Hence, $f_\bullet^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}/g_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is flat and $g_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is sent in the center of $f_\bullet^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}$. Since $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}/\mathcal{B}_{X_\bullet}$ is flat, since $g_\bullet$ is flat, we get that $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}/g_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is flat. Hence, $g_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is a solving ring of $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}$. Similarly, we check that $g_\bullet^{-1}(\mathcal{O}_{U_\bullet})$ is a solving ring of $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$. □

Definition 7.5.5.6. We keep notation 7.5.5.4.

(a) The (left version of the) extraordinary inverse image functor of level m by $\widetilde{f}_\bullet$ is the functor $\widetilde{f}_\bullet^{(m)!}\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ which is defined by setting

$$\widetilde{f}_\bullet^{(m)!}(\mathcal{F}_\bullet) := \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes^{\mathbb{L}}_{f_\bullet^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f_\bullet^{-1}\mathcal{F}_\bullet\ [\delta_{X^\sharp_\bullet/S^\sharp_\bullet} - \delta_{Y^\sharp_\bullet/T^\sharp_\bullet} \circ f],$$

where $\mathcal{F}_\bullet \in D(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$, $\delta_{X^\sharp_\bullet/S^\sharp_\bullet}$, $\delta_{Y^\sharp_\bullet/T^\sharp_\bullet}$ are respectively the rank (as a locally constant function on X or Y respectively) of the locally free modules $\Omega_{X^\sharp_\bullet/S^\sharp_\bullet}$ and $\Omega_{Y^\sharp_\bullet/T^\sharp_\bullet}$.

(b) The (right version of the) extraordinary inverse image functor of level m by $\widetilde{f}$ is the functor $\widetilde{f}_\bullet^{(m)!}\colon D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \to D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ which is defined by setting

$$\widetilde{f}_\bullet^{(m)!}(\mathcal{M}_\bullet) := f_\bullet^{-1}\mathcal{M}_\bullet \otimes^{\mathbb{L}}_{f_\bullet^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}\ [\delta_{X^\sharp_\bullet/S^\sharp_\bullet} - \delta_{Y^\sharp_\bullet/T^\sharp_\bullet} \circ f],$$

where $\mathcal{M}_\bullet \in D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$.

(c) For any $* \in \{r, l\}$, the extraordinary inverse image functor $\widetilde{f}^{(m)!}\colon D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ of level m by $\widetilde{f}$ is defined by setting

$$\widetilde{f}^{(m)!}(\mathcal{F}) := \mathbb{R}\underleftarrow{l}_{X*} \circ \widetilde{f}_\bullet^{(m)!} \circ \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{F}) \tag{7.5.5.6.1}$$

where $\mathcal{F} \in D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. Since the functor $\widetilde{f}^{(m)!}$ preserves isogenies, then we get with notation 7.4.2.1.(b) the extraordinary inverse image functor $\widetilde{f}^{(m)!}\colon D_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to D_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

We set $\mathbb{L}\widetilde{f}_\bullet^{(m)*} := \widetilde{f}_\bullet^{(m)!}[-\delta_{X^\sharp_\bullet/S^\sharp_\bullet} + \delta_{Y^\sharp_\bullet/T^\sharp_\bullet} \circ f]$ and $\mathbb{L}\widetilde{f}^{(m)*}(\mathcal{F}) := \mathbb{R}\underleftarrow{l}_{X*} \circ \widetilde{f}_\bullet^{(m)*} \circ \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{F})$. The functor $\mathbb{L}\widetilde{f}_\bullet^{(m)*}$ is isomorphic in $D(\mathcal{B}_{\mathfrak{X}})$ to the functor $\mathbb{L}\widetilde{f}_\bullet^{*}$ of 7.5.5.2.1. When complexes are coherent, the functor $\mathbb{L}\widetilde{f}^{(m)*}$ is a left derived functor of some functor $\widetilde{f}_\bullet^{*}$ (more precisely, see 7.5.5.13.(c) below). Beware that this is not a priori the case in general and the notation is a bit misleading. The functor $\mathbb{L}\widetilde{f}^{(m)*}$ is a p-adically separated complete of the functor $\mathbb{L}\widetilde{f}^{*}_{\mathrm{alg}}$ of 7.5.5.2.1, which explain why we have distinguished them with the index alg.

7.5.5.7 (Left to right)**.** For any $\mathcal{M}_\bullet \in D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$, by copying the proof of 5.1.1.5.1 (replace the use of 5.1.1.2.1 by that of 7.5.5.4.1), we get the canonical isomorphism:

$$\widetilde{f}_\bullet^{(m)!}(\mathcal{M}_\bullet \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \xrightarrow{\sim} \widetilde{f}_\bullet^{(m)!}(\mathcal{M}_\bullet) \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\omega}^{-1}_{X^\sharp_\bullet/S^\sharp_\bullet}. \tag{7.5.5.7.1}$$

For any $\mathcal{E}_\bullet \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$, this yields the isomorphism

$$\widetilde{f}_\bullet^{(m)!}(\mathcal{E}_\bullet \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \xrightarrow{\sim} \widetilde{f}_\bullet^{(m)!}(\mathcal{E}_\bullet) \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}. \tag{7.5.5.7.2}$$

Hence, for any $\mathcal{E} \in D^-({}^l\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ we get the isomorphisms

$$\widetilde{f}^{(m)!}(\mathcal{E} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \underset{7.5.4.13.1}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{X*} \circ \widetilde{f}^{(m)!}_\bullet(\mathbb{L}\underleftarrow{l}^*_Y(\mathcal{E}) \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \underset{7.5.5.7.2}{\xrightarrow{\sim}}$$
$$\mathbb{R}\underleftarrow{l}_{X*}(\widetilde{f}^{(m)!}_\bullet \circ \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{E}) \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}) \underset{7.5.4.13.2}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{X*} \circ \widetilde{f}^{(m)!}_\bullet \circ \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{E}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} = \widetilde{f}^{(m)!}(\mathcal{E}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{7.5.5.7.3}$$

This yields

$$\widetilde{f}^{(m)!}(\widetilde{\omega}^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \mathcal{M}) \xrightarrow{\sim} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{f}^{(m)!}(\mathcal{M}). \tag{7.5.5.7.4}$$

Proposition 7.5.5.8. *Let $* \in \{r, l\}$, $\star \in \{-, \mathrm{b}\}$. In the case where $\star = \mathrm{b}$, we suppose moreover $f^{-1}\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ has finite tor dimension. For any $\mathcal{F}_\bullet \in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$, we have $\widetilde{f}^{(m)!}_\bullet(\mathcal{F}_\bullet) \in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$. For any $\mathcal{F} \in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, we have $\widetilde{f}^{(m)!}(\mathcal{F}) \in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. For any $\mathcal{F} \in D^\star_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, we have $\widetilde{f}^{(m)!}(\mathcal{F}) \in D^\star_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.*

Proof. By using the equivalence of categories 7.5.4.3 and the isomorphism 7.5.5.7.2, we reduce to check that for any $\mathcal{F}_\bullet \in D^\star_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$, we have $\widetilde{f}^{(m)!}_\bullet(\mathcal{F}_\bullet) \in D^\star_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$. Since the canonical morphism

$$\mathcal{B}_{X_\bullet} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}$$

is an isomorphism, then so is the canonical morphism

$$\mathbb{L}\widetilde{f}^*_\bullet(\mathcal{F}_\bullet) = \mathcal{B}_{X_\bullet} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}\mathcal{F}_\bullet \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}\mathcal{F}_\bullet. \tag{7.5.5.8.1}$$

Since the functor $\mathbb{L}\widetilde{f}^*_\bullet$ induces $\mathbb{L}\widetilde{f}^*_\bullet \colon D^\star_{\mathrm{qc}}(\mathcal{B}_{Y_\bullet}) \to D^\star_{\mathrm{qc}}(\mathcal{B}_{X_\bullet})$ (see 7.5.5.2.3), then we are done. □

Lemma 7.5.5.9. *For any $\mathcal{E}_\bullet, \mathcal{F}_\bullet \in D^-({}^l\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$, we have the isomorphism of $D^-({}^l\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$:*

$$\mathbb{L}\widetilde{f}^{(m)!}_\bullet(\mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{B}_X} \mathbb{L}\widetilde{f}^{(m)!}_\bullet(\mathcal{F}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^{(m)!}_\bullet(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}_Y} \mathcal{F})[d_f]. \tag{7.5.5.9.1}$$

For any $\mathcal{E}, \mathcal{F} \in D^-_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, we have the isomorphism of $D^-_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\mathbb{L}\widetilde{f}^{(m)!}(\mathcal{E}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} \mathbb{L}\widetilde{f}^{(m)!}(\mathcal{F}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^{(m)!}(\mathcal{E} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{Y}}} \mathcal{F})[d_f]. \tag{7.5.5.9.2}$$

Proof. We check 7.5.5.9.1 similarly to 7.5.5.9.1. Modulo the equivalence of categories 7.3.2.10 this yields 7.5.5.9.2. □

Notation 7.5.5.10. We have a structure of $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$-bimodule on

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} := \underleftarrow{l}_{X*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}.$$

We get a structure of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$-bimodule on

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \underleftarrow{l}_{X*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}.$$

By applying $\underleftarrow{l}_{X*}$ to 7.5.5.4.1 we get:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\widetilde{\omega}^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}. \tag{7.5.5.10.1}$$

Lemma 7.5.5.11. *Let $\mathcal{F} \in D^-_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. With notation 7.5.5.6, the canonical morphism*

$$\mathcal{B}_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{F} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F} = \mathbb{L}\widetilde{f}^{(m)*}(\mathcal{F}) \tag{7.5.5.11.1}$$

is an isomorphism.

Proof. This follows from 7.5.5.8.1. □

We will need later this following elementary lemma.

Lemma 7.5.5.12. *Let $\mathcal{F}$ be a pseudo-quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$-module (see definition 7.2.3.5) which is $\mathcal{B}_{\mathfrak{Y}}$-flat.*

(a) For all $r \neq 0$, $H^r(\mathbb{L}\widetilde{f}^{(m)*}(\mathcal{F})) = 0$. Setting $\widetilde{f}^{(m)*}(\mathcal{F}) := H^0(\mathbb{L}\widetilde{f}^{(m)*}(\mathcal{F}))$, $\widetilde{f}^{(m)*}(\mathcal{F})$ is $\mathcal{B}_{\mathfrak{X}}$-flat.

(b) Suppose $\mathfrak{Y}$ and $\mathfrak{X}$ are affine. We have the canonical isomorphism

$$\Gamma(\mathfrak{X}, \widetilde{f}^{(m)*}(\mathcal{F})) \xrightarrow{\sim} \Gamma(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}})\widehat{\otimes}_{\Gamma(\mathfrak{Y},\mathcal{B}_{\mathfrak{Y}})}\Gamma(\mathfrak{Y}, \mathcal{F}). \tag{7.5.5.12.1}$$

The module $\Gamma(\mathfrak{X}, \widetilde{f}^{(m)*}(\mathcal{F}))$ is $\Gamma(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}})$-flat and p-adically separated complete.

(c) If $\mathcal{F}$ is of the form $(\mathcal{B}^{(J)}_{\mathfrak{Y}})^\wedge$, i.e., is isomorphic to the p-adic completion of a free $\mathcal{B}_{\mathfrak{Y}}$-module, then $\widetilde{f}^{(m)*}(\mathcal{F})$ is isomorphic to $(\mathcal{B}^{(J)}_{\mathfrak{X}})^\wedge$.

Proof. a) As $\mathcal{F}$ is an $\mathcal{B}_{\mathfrak{Y}}$-flat module then $\mathcal{F}_\bullet := \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{F}) \xrightarrow{\sim} \mathcal{B}_{Y_\bullet} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \mathcal{F}$ is $\mathcal{B}_{Y_\bullet}$-flat. Hence $\mathbb{L}\widetilde{f}^*_\bullet \circ \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{F}) \xrightarrow{\sim} \mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{F}_\bullet$, which is an ML-flasque $\mathcal{B}_{X_\bullet}$-module and via Mittag-Leffler we obtain therefore the last isomorphism (see 7.3.1.3.(c)):

$$\mathbb{L}\widetilde{f}^{(m)*}(\mathcal{F}) \overset{7.5.5.11.1}{\xleftarrow{\sim}} \mathbb{R}\underleftarrow{l}_{X*}\mathbb{L}\widetilde{f}^*_\bullet \circ \mathbb{L}\underleftarrow{l}^*_Y(\mathcal{F}) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{X*}\left(\mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{F}\right) \xrightarrow{\sim} \underleftarrow{l}_{X*}\left(\mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{F}\right). \tag{7.5.5.12.2}$$

Since $\mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{Y_\bullet}} f^{-1}\mathcal{F}_\bullet$ is $\mathcal{B}_{X_\bullet}$-flat, then if follows from 7.3.1.20 that $\widetilde{f}^{(m)*}(\mathcal{F})$ is $\mathcal{B}_{\mathfrak{X}}$-flat.

b) Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are affine. Set $F := \Gamma(\mathfrak{Y}, \mathcal{F})$, $E := \Gamma(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}) \otimes_{\Gamma(\mathfrak{Y},\mathcal{B}_{\mathfrak{Y}})} \Gamma(\mathfrak{Y}, \mathcal{F})$. Following 7.2.3.13.1, we have the isomorphism $\Gamma(Y_i, \mathcal{B}_{Y_i} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \mathcal{F}) \xrightarrow{\sim} F/\pi^{i+1}F$. Since the $\mathcal{B}_{Y_i}$-module $\mathcal{B}_{Y_i} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \mathcal{F}$ is quasi-coherent, then this yields the canonical isomorphism $E/\pi^{i+1}E \xrightarrow{\sim} \Gamma(\mathfrak{X}, f_i^*(\mathcal{B}_{Y_i} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \mathcal{F}))$. By applying the functor $\Gamma(\mathfrak{X}, -)$ (which commutes to projective limits) to the isomorphism 7.5.5.12.2, we get the isomorphism 7.5.5.12.1. By using 7.2.1.4, it follows from 7.5.5.12.1 that $\Gamma(\mathfrak{X}, \widetilde{f}^{(m)*}(\mathcal{F}))$ is $\Gamma(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}})$-flat and p-adically separated complete.

c) If $\mathcal{F}$ is of the form $(\mathcal{B}^{(J)}_{\mathfrak{Y}})^\wedge$, then $\mathcal{B}_{X_\bullet} \otimes_{f^{-1}\mathcal{B}_{\mathfrak{Y}}} f^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{B}^{(J)}_{X_\bullet}$. Via 7.5.5.12.2, we are done. □

Proposition 7.5.5.13. *Set $d_{(f,\phi)} := \delta_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} - \delta_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \circ f$, where $\delta_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, $\delta_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ are respectively the rank (as a locally constant function on X' or X respectively) of the locally free modules $\Omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\Omega_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$. By abuse of notation, we can simply write d_f. We have the following properties.*

(a) For any $\mathcal{F} \in D({}^l\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, we have the canonical morphism

$$\widetilde{f}^{(m)!}(\mathcal{F})_{\mathrm{alg}} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F}\,[d_f] \to \widetilde{f}^{(m)!}(\mathcal{F}). \tag{7.5.5.13.1}$$

(b) For any $\mathcal{F} \in D({}^r\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, we have the canonical morphism

$$\widetilde{f}^{(m)!}(\mathcal{F})_{\mathrm{alg}} := f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\,[d_f] \to \widetilde{f}^{(m)!}(\mathcal{F}). \tag{7.5.5.13.2}$$

(c) For any $* \in \{r, l\}$, if $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, then the morphism 7.5.5.13.1 or 7.5.5.13.2 is an isomorphism.

Proof. Let us construct 7.5.5.13.1. Denoting by $\mathcal{G}$ the left term of 7.5.5.13.1, we have the adjoint morphism $\mathcal{G} \to \mathbb{R}\underleftarrow{l}_{X*} \circ \mathbb{L}\underleftarrow{l}^*_X(\mathcal{G})$. Since $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{G})$ is isomorphic to $\widetilde{f}^{(m)!}_\bullet(\mathbb{L}\underleftarrow{l}^*_Y\mathcal{F})$, we are done. Similarly, we construct 7.5.5.13.2. Finally, to check the last statement, we reduce to the case where $\mathcal{F} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$, which is obvious. □

Definition 7.5.5.14. We keep notation 7.5.5.10.

(a) The (left version of the) extraordinary inverse image functor of level m by $\widetilde{f}$ is the functor of the form $\widetilde{f}^{(m)!}\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ which is defined for any $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ by setting

$$\widetilde{f}^{(m)!}(\mathcal{F}) := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}} f^{-1}(\mathcal{F})\,[d_f]. \tag{7.5.5.14.1}$$

(b) The (right version of the) extraordinary inverse image functor of level m by $\widetilde{f}$ is the functor of the form $\widetilde{f}^{(m)!}\colon D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ which is defined for any $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ by setting

$$\widetilde{f}^{(m)!}(\mathcal{M}) := f^{-1}(\mathcal{M}) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}\,[d_f]. \tag{7.5.5.14.2}$$

Remark 7.5.5.15. The functors of 7.5.5.14 might have been written $\widetilde{f}^{(m)!}_{\mathrm{alg}}$ similarly to 7.5.5.13, but since no confusion is possible over categories of the form $D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ then the indication alg is useless and we have removed it.

Proposition 7.5.5.16. *We have the following properties. Let* $* \in \{r, l\}$.

(a) With notation 7.4.6.5, we have the canonical morphism of functors $D_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ *of the form*

$$\widetilde{f}^{(m)!} \circ (\mathbb{Q} \otimes -) \to (\mathbb{Q} \otimes -) \circ \widetilde{f}^{(m)!}. \tag{7.5.5.16.1}$$

(b) For any $* \in \{r, l\}$*, the restriction to* $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ *of the morphism 7.5.5.16.1 is an isomorphism.*

Proof. This easily follows from 7.5.5.13. □

Remark 7.5.5.17. Suppose $\mathfrak{X}$ is noetherian of finite Krull dimension. Let $* \in \{r, l\}$ and $(\mathbb{Q}\otimes -)^{-1}\colon D^{\mathrm{b}}_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ be a quasi-inverse functor of the natural equivalence of categories 7.4.6.6. Then the functor $(\mathbb{Q} \otimes -) \circ \widetilde{f}^{(m)!} \circ (\mathbb{Q} \otimes -)^{-1}\colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}) \to D^{\mathrm{b}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ is isomorphic to the functor $\widetilde{f}^{(m)!}$.

Proposition 7.5.5.18. *Let* $* \in \{\mathrm{r}, \mathrm{l}\}$*. Suppose one of the following conditions holds:*

(i) either $m = 0$,

(ii) or log-structures are trivial.

Then, we have the factorizations $\widetilde{f}^{(m)!}_{\bullet}\colon D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}) \to D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ *and* $\widetilde{f}^{(m)!}\colon D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

Proof. This is a consequence of 7.3.2.15 (and 7.1.3.6) and 5.3.2.6 (and the fact that the tor amplitude does not depend on i). □

7.5.5.19. It follows from 7.1.3.6.1 that we have the isomorphism for any $\mathcal{F}_\bullet \in D^\star({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$:

$$\left(\widetilde{f}^{(m)!}_{\bullet}(\mathcal{F}_\bullet)\right)_i \xrightarrow{\sim} \widetilde{f}^{(m)!}_i(\mathcal{F}_i). \tag{7.5.5.19.1}$$

By using the equivalences of categories 7.3.3.3, it follows from 7.5.5.8 that for any $\mathcal{F} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$, $\mathbb{L}\underleftarrow{l}^*_X \widetilde{f}^{(m)!}(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^{(m)!}_{\bullet}(\mathbb{L}\underleftarrow{l}^*_Y \mathcal{F})$. Since $(\mathbb{L}\underleftarrow{l}^*_Y \mathcal{F})_i \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/S^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}$ and since $\left(\mathbb{L}\underleftarrow{l}^*_X \widetilde{f}^{(m)!}(\mathcal{F})\right)_i \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{f}^{(m)!}(\mathcal{F})$, this yields

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{f}^{(m)!}(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^{(m)!}_i(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/S^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}), \tag{7.5.5.19.2}$$

and similarly for right modules.

7.5.5.20. To check that the extraordinary inverse image behaves well with respect to the composition, let

$$\begin{array}{ccc} \mathfrak{Y}^\sharp & \xrightarrow{g} & \mathfrak{Z}^\sharp \\ \downarrow & & \downarrow \\ \mathfrak{T}^\sharp & \longrightarrow & \mathfrak{U}^\sharp, \end{array} \tag{7.5.5.20.1}$$

be a commutative diagram where $\mathfrak{U}^\sharp$ is a nice fine $\mathcal{V}$-log formal scheme, where $\mathfrak{Z}^\sharp$ is a log smooth $\mathfrak{U}^\sharp$-log formal scheme. Moreover, let $\mathcal{B}_{\mathfrak{Z}}$ be an $\mathcal{O}_{\mathfrak{Z}}$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m}_{\mathfrak{Z}^\sharp/\mathfrak{U}^\sharp}$-module satisfying the hypotheses of 7.3.2 and with a morphism of algebras $g^*\mathcal{B}_{\mathfrak{Z}} \to \mathcal{B}_{\mathfrak{Y}}$ which is moreover $\mathcal{D}^{(m}_{\mathfrak{Z}^\sharp/\mathfrak{U}^\sharp}$-linear. We denote by $\widetilde{\mathfrak{Z}}^\sharp$ the ringed $\mathcal{V}$-log formal scheme $(\mathfrak{Z}^\sharp, \mathcal{B}_{\mathfrak{Z}})$, and by $\tilde{g}\colon \widetilde{\mathfrak{Y}}^\sharp/\mathfrak{T}^\sharp \to \widetilde{\mathfrak{Z}}^\sharp/\mathfrak{U}^\sharp$ the morphism of relative ringed $\mathcal{V}$-log formal schemes induced by the diagram 7.5.5.20.1 and by $g^*\mathcal{B}_{\mathfrak{Z}} \to \mathcal{B}_{\mathfrak{Y}}$. We suppose $\tilde{g}$ is a strongly quasi-flat morphism of relative ringed $\mathcal{V}$-log formal schemes.

Lemma 7.5.5.21. *We keep notation 7.5.5.20.*

(a) We have the canonical isomorphism of $D(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}, (g\circ f)^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet})$:

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes^{\mathbb{L}}_{f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet}. \tag{7.5.5.21.1}$$

(b) We have the canonical isomorphism of $D((g\circ f)^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$:

$$f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet \leftarrow Y^\sharp_\bullet/T^\sharp_\bullet} \otimes^{\mathbb{L}}_{f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}. \tag{7.5.5.21.2}$$

Proof. By quasi-flatness of $\widetilde{f}$, it follows from 5.1.1.3 that the morphisms are well defined. a) Let us check 7.5.5.21.1. i) We have the isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet} \xrightarrow{\sim} \widetilde{f}^*_\bullet \widetilde{g}^*_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet}$ and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \xrightarrow{\sim} (\widetilde{g}\circ \widetilde{f})^*_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet}$. Hence, it follows from 4.4.5.6 that we get the canonical isomorphism of left $\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet}.$$

We obtain by functoriality the fact that this isomorphism is an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}, (g\circ f)^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet})$-bimodules.

ii) Moreover, since $\widetilde{g}^*_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet}$ is $\mathcal{B}_{Y_\bullet}$-flat then $\mathbb{L}\widetilde{f}^*_\bullet\left(\widetilde{g}^*_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet}\right) \xrightarrow{\sim} \widetilde{f}^*_\bullet\left(\widetilde{g}^*_\bullet \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet}\right)$. Hence, it follows from 7.5.5.8.1 the isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes^{\mathbb{L}}_{f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}_\bullet \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \to Z^\sharp_\bullet/U^\sharp_\bullet}.$$

b) Finally, we get the isomorphism 7.5.5.21.2 from 7.5.5.21.1 by twisting (use 7.5.5.4.1 and 4.3.5.6.1). □

Proposition 7.5.5.22. *With notation 7.5.5.20, let* $* \in \{r, l\}$.

(a) For any $\mathcal{G}_\bullet \in D({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet})$*, we have the canonical isomorphism*

$$\widetilde{f}^{(m)!}_\bullet \circ \widetilde{g}^{(m)!}_\bullet(\mathcal{G}_\bullet) \xrightarrow{\sim} (\widetilde{g\circ f})^{(m)!}_\bullet(\mathcal{G}_\bullet). \tag{7.5.5.22.1}$$

(b) Let $\star \in \{-, \mathrm{b}\}$*. In the case where* $\star = \mathrm{b}$*, we suppose moreover* $f^{-1}\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ *and* $g^{-1}\mathcal{B}_{\mathfrak{Z}} \to \mathcal{B}_{\mathfrak{Y}}$ *have finite tor dimension. For any* $\mathcal{G} \in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{U}^\sharp})$*, we have the canonical isomorphism*

$$\widetilde{f}^{(m)!} \circ \widetilde{g}^{(m)!}(\mathcal{G}) \xrightarrow{\sim} (\widetilde{g\circ f})^{(m)!}(\mathcal{G}). \tag{7.5.5.22.2}$$

Proof. By using the equivalence of categories 7.5.4.3, we reduce to check the isomorphism 7.5.5.22.1. This latter one can be checked similarly to 5.1.1.13 by using 7.5.5.21. ☐

7.5.5.23 (Non-lifted case notation)**.** To highlight the "crystalline nature" of the operations such as $f^!$ and f_+, we can extend the context of 7.5.5 to the non-lifted case as follows. Let $m \in \mathbb{N}$. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0, p)$ with maximal ideal $\mathfrak{m}$. Let

$$\begin{array}{ccccc} X_0^\sharp & \longrightarrow & \mathfrak{X}^\sharp & \xrightarrow{p_{\mathfrak{X}^\sharp}} & \mathfrak{S}^\sharp \\ \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle \phi} \\ Y_0^\sharp & \longrightarrow & \mathfrak{Y}^\sharp & \xrightarrow{p_{\mathfrak{Y}^\sharp}} & \mathfrak{T}^\sharp, \end{array} \tag{7.5.5.23.1}$$

be a commutative diagram where $\mathfrak{S}^\sharp$ and $\mathfrak{T}^\sharp$ are nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10, where $\mathfrak{X}^\sharp$ is a log smooth $\mathfrak{S}^\sharp$-log-formal scheme and $\mathfrak{Y}^\sharp$ is a log smooth $\mathfrak{T}^\sharp$-log formal scheme. Let $\mathcal{B}_{\mathfrak{X}}$ (resp. $\mathcal{B}_{\mathfrak{Y}}$) be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra (resp. $\mathcal{O}_{\mathfrak{Y}}$-algebra) endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. left $\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$-module) and satisfying the hypotheses of 7.3.2. We suppose that we have a morphism of algebras $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ which is moreover $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. We denote by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} := \mathcal{B}_{\mathfrak{Y}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$. Following 4.4.5.11, we have the $(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}, f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$-bimodule by setting

$$\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} := \widetilde{f}_0^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}. \tag{7.5.5.23.2}$$

By p-adic completion we get a $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$-bimodule by setting

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} := \underleftarrow{l}_{X*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}. \tag{7.5.5.23.3}$$

According to 5.1.1.17.2, we get a structure of $(f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodule on

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet} := \widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{f}_{0r}^{*}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}\right), \tag{7.5.5.23.4}$$

where the index "r" means that we have chosen the right (i.e. the twisted) structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ on $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{\omega}^{-1}_{Y^\sharp_\bullet/T^\sharp_\bullet}$ to compute structure of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ via the functor $\widetilde{f}_0^{*}$. This yields the $(f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$-bimodule by setting

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \underleftarrow{l}_{X*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}, \tag{7.5.5.23.5}$$

For any $* \in \{\mathrm{l}, \mathrm{r}\}$, via these above bimodules we can define the pullback $\widetilde{f}_0^{(m)!} \colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ as in 7.5.5.6. The properties of the subsection extends to this context.

7.5.6 Base change

We keep notation 7.5.5. Suppose the diagram 7.5.5.0.1 is cartesian and the morphism $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ is an isomorphism.

In that case, we say that $\tilde{f}^*$ is the base change by $\phi \colon \mathfrak{S}^\sharp \to \mathfrak{T}^\sharp$.

Proposition 7.5.6.1. *We have the following properties.*

(a) The canonical morphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}. \tag{7.5.6.1.1}$$

is an isomorphism.

(b) The composite map

$$\rho_{\widetilde{f}} \colon f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{7.5.6.1.2}$$

is a homomorphism of sheaves of rings which fits into the commutative diagram

$$\begin{array}{ccc} f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} & \xrightarrow{7.5.6.1.2} & \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{B}_{\mathfrak{Y}} & \longrightarrow & \mathcal{B}_{\mathfrak{X}}. \end{array} \qquad (7.5.6.1.3)$$

The canonical morphism of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$*-modules 7.5.6.1.1 is in fact an isomorphism of* $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$*-bimodules, where the structure of right* $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$*-module on* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ *is given via 7.5.6.1.2.*

Proof. This follows by completion of 4.4.4.1. □

Lemma 7.5.6.2. *The canonical homomorphism of* $\mathcal{B}_{\mathfrak{X}}$*-modules:*

$$\widetilde{f}^*(\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \qquad (7.5.6.2.1)$$

is an isomorphism. Let us denote by $\rho^{\omega}_{\widetilde{f}}\colon f^{-1}(\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \to \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ *the canonical homomorphism. With 7.5.6.1.2, this yields the map*

$$f^{-1}\left(\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}\right) \to \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \qquad (7.5.6.2.2)$$

given by $\omega \otimes P \mapsto \rho^{\omega}_{\widetilde{f}}(\omega) \otimes \rho_{\widetilde{f}}(P)$*. The map 7.5.6.2.2 is a homomorphism of right* $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$*-bimodules, where the right structure (resp. the left structure i.e. the twisted one) of right* $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$*-module of* $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ *comes from its right structure (resp. left structure) of right* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$*-module via the ring homomorphism 7.5.6.1.2.*

Proof. The lemma follows by p-adic completion from 4.4.4.2. □

7.5.6.3. By p-adic completion, we get from 4.2.5.5.1 the transposition isomorphism

$$\delta\colon \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \qquad (7.5.6.3.1)$$

exchanging the two structures of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules and such that, for each section x of $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, $\delta_{\mathcal{M}}(x \otimes 1) = x \otimes 1$. It follows by p-adic completion of 4.4.4.3.1 that we get the commutative diagram:

$$\begin{array}{ccc} f^{-1}\left(\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}\right) & \xrightarrow{7.5.6.2.2} & \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \\ \downarrow{\scriptstyle 7.5.6.3.1} & & \downarrow{\scriptstyle 7.5.6.3.1} \\ f^{-1}\left(\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}\right) & \xrightarrow{7.5.6.2.2} & \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \end{array} \qquad (7.5.6.3.2)$$

where the vertical maps are the transposition isomorphisms, is commutative. This yields the isomorphism of right $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$-bimodules

$$f^{-1}\left(\widetilde{\omega}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}\right)_l \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \qquad (7.5.6.3.3)$$

where the index l means that in the tensor product we use the left structure of right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module, where the structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. right $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module) of $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is its left structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. comes from its structure of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module via the ring homomorphism $\rho_{\widetilde{f}}$ of 7.5.6.1.2).

7.5.6.4. Let $\mathcal{F} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. By p-adic completion of 4.4.4.1.4, we get the canonical isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{F}), \tag{7.5.6.4.1}$$

where $f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ the ring homomorphism of 7.5.6.1.2. Remark that if $p^{-1}_{X^\sharp_\bullet}\mathcal{O}_{S_\bullet}$ and $f^{-1}\mathcal{B}_{Y_\bullet}$ are tor independent over $f^{-1}p^{-1}_{Y^\sharp_\bullet}\mathcal{O}_{T_\bullet}$, then by p-adic completion of 5.1.1.15.3 we get the canonical morphism

$$p^{-1}_{\mathfrak{X}^\sharp}\mathcal{O}_{\mathfrak{S}}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}p^{-1}_{\mathfrak{Y}^\sharp}\mathcal{O}_{\mathfrak{T}}} f^{-1}\mathcal{F} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F} \tag{7.5.6.4.2}$$

is an isomorphism. By abuse of notation, under this flatness condition we can simply denote by $\mathcal{O}_{\mathfrak{S}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}}\mathcal{F} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F}$.

Let $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. It follows by p-adic completion from 5.1.1.15.4 that we have the canonical isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$f^{-1}\mathcal{M}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{M}). \tag{7.5.6.4.3}$$

By functoriality, this implies that we have the canonical isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$-bimodules of the form $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. The isomorphism 7.5.5.7.4 applied in the case where $\mathcal{M} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ gives the second isomorphism:

$$\begin{aligned}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{Y}}}\widetilde{\omega}^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})_{\mathrm{r}} &\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{Y}}}\widetilde{\omega}^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})_{\mathrm{r}}\\ \xrightarrow{\sim}\left(f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\right)\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} &\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\end{aligned} \tag{7.5.6.4.4}$$

where the index r means that we take the right structure of left $\mathcal{D}$-module.

As for the left case, when if $p^{-1}_{X^\sharp_\bullet}\mathcal{O}_{S_\bullet}$ and $f^{-1}\mathcal{B}_{Y_\bullet}$ are tor independent over $f^{-1}p^{-1}_{Y^\sharp_\bullet}\mathcal{O}_{T_\bullet}$, the canonical morphism

$$f^{-1}\mathcal{M}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}p^{-1}_{\mathfrak{Y}^\sharp}\mathcal{O}_{\mathfrak{T}}}p^{-1}_{\mathfrak{X}^\sharp}\mathcal{O}_{\mathfrak{T}} \to f^{-1}\mathcal{M}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$$

is an isomorphism and we can simply set in this case $\mathcal{O}_{\mathfrak{S}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}}\mathcal{M} := f^{-1}\mathcal{M}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

7.5.6.5 (Preservation of the coherence)**.** Let $\mathcal{F} \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. The canonical morphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F} \tag{7.5.6.5.1}$$

is an isomorphism. Hence, with 7.5.6.5.2 we get the isomorphism of $D^-_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{F}). \tag{7.5.6.5.2}$$

Let $\mathcal{M} \in D^-_{\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. We have the isomorphisms of $D^-_{\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$f^{-1}\mathcal{M}\otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} f^{-1}\mathcal{M}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \underset{7.5.6.4.3}{\xrightarrow{\sim}} \widetilde{f}^{(m)!}(\mathcal{M}). \tag{7.5.6.5.3}$$

By functoriality, this implies that we have the canonical isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$-bimodules of the form $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. The isomorphism 7.5.5.7.4 applied in the case where $\mathcal{M} =$

$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ gives the second isomorphism:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\omega}^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})_{\mathrm{r}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Y}}} \widetilde{\omega}^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})_{\mathrm{r}}$$

$$\xrightarrow{\sim} \left(f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}) \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \right) \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \tag{7.5.6.5.4}$$

where the index r means that we take the right structure of left $\mathcal{D}$-module.

7.5.6.6 (Preservation of the coherence)**.** Let $\mathcal{G} \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$. Tensoring with $\mathbb{Q}$ the results of the proposition 7.5.6.1, we can check we have the canonical isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}} f^{-1}\mathcal{G} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{G}), \tag{7.5.6.6.1}$$

where the right term is defined at 7.5.5.14.

Let $\mathcal{M} \in D^-_{\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$. We have the isomorphisms of $D^-_{\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$:

$$f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \xrightarrow{\sim} \widetilde{f}^{(m)!}(\mathcal{M}). \tag{7.5.6.6.2}$$

Remark 7.5.6.7. We can split the diagram 5.1.1.1.1 as follows

$$\begin{array}{ccccc}
\mathfrak{X}^\sharp & \xrightarrow{g} & \mathfrak{Z}^\sharp & \xrightarrow{h} & \mathfrak{Y}^\sharp \\
\downarrow{\scriptstyle p_{\mathfrak{X}^\sharp}} & & \downarrow{\scriptstyle p_{\mathfrak{Z}^\sharp}} \ \square & & \downarrow{\scriptstyle p_{\mathfrak{Y}^\sharp}} \\
\mathfrak{S}^\sharp & = & \mathfrak{S}^\sharp & \longrightarrow & \mathfrak{T}^\sharp,
\end{array} \tag{7.5.6.7.1}$$

where the right square is cartesian. Let $\mathcal{B}_{\mathfrak{Z}} := h^*\mathcal{B}_{\mathfrak{Y}}$, $\widetilde{\mathfrak{Z}}^\sharp$ be the ringed logarithmic $\mathcal{V}$-formal scheme $(\mathfrak{Z}^\sharp, \mathcal{B}_{\mathfrak{Z}})$, and $\widetilde{h}\colon \widetilde{\mathfrak{Z}}^\sharp/\mathfrak{T}^\sharp \to \widetilde{\mathfrak{Y}}^\sharp/\mathfrak{T}^\sharp$ be the morphism of relative ringed logarithmic $\mathcal{V}$-formal schemes induced by the cartesian square of the diagram 7.5.6.7.1 and by $\mathcal{B}_{\mathfrak{Z}} = h^*\mathcal{B}_{\mathfrak{Y}}$. Hence, since $\widetilde{g}^{(m)!} \circ \widetilde{h}^{(m)!} \xrightarrow{\sim} \widetilde{f}^{(m)!}$ over quasi-coherent complexes, then to study the extraordinary pullback functor (for instance) over quasi-coherent complexes, we reduce to the case of the base change or to the case where $\phi = \mathrm{id}$.

7.5.7 Projection formula

We keep notation 7.5.5.

Proposition 7.5.7.1. Let $\mathcal{F}_\bullet \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ and $\mathcal{G}_\bullet \in D({}^{\mathrm{l}}f_\bullet^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$.

(i) We have the canonical morphism in $D(\mathbb{Z}_{Y_\bullet})$:

$$\mathcal{F}_\bullet \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} \mathbb{R}f_{\bullet *}(\mathcal{G}_\bullet) \to \mathbb{R}f_{\bullet *}\left(f_\bullet^{-1}\mathcal{F}_\bullet \otimes^{\mathbb{L}}_{f_\bullet^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} \mathcal{G}_\bullet \right). \tag{7.5.7.1.1}$$

Let $\mathcal{D}_\bullet$ be a sheaf of rings on $Y_\bullet$ such that $(\mathcal{D}_\bullet, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ is solvable by $\mathcal{R}_\bullet$ and $\mathcal{F}_\bullet \in D(\mathcal{D}_\bullet, \mathcal{R}_\bullet, \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ (see definition and notation 4.6.3.2). Then the morphism 7.5.7.1.1 can also be viewed as a morphism of $D(\mathcal{D}_\bullet)$.

(ii) Suppose f is quasi-compact and quasi-separated. Suppose moreover for any $i \in \mathbb{Z}$ one of the following conditions hold:

(a) either $\mathcal{F}_i \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$, and $\mathcal{G}_i \in D({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$,

(b) or T_i is a noetherian scheme of finite Krull dimension, and $\mathcal{F}_i \in D^-_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$, and $\mathcal{G}_i \in D^-({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$.

Then the morphism 7.5.7.1.1 is an isomorphism.

Proof. We construct 7.5.7.1.1 as 5.1.2.5.1. The second statement is a consequence of 5.1.2.5.1. □

Remark 7.5.7.2. Inverting r and l, we get the morphism

$$\mathbb{R}f_{\bullet *}(\mathcal{G}_\bullet) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} \mathcal{F}_\bullet \to \mathbb{R}f_{\bullet *}\left(\mathcal{G}_\bullet \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}} f^{-1}\mathcal{F}_\bullet\right), \tag{7.5.7.2.1}$$

which is an isomorphism when we invert the corresponding hypotheses.

Corollary 7.5.7.3. *Let* $*, ** \in \{\mathrm{l}, \mathrm{r}\}$ *such that both are not equal to* r. *Suppose* f *is quasi-compact and quasi-separated.*

(a) either $\mathcal{F}_i \in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$*, and* $\mathcal{G}_i \in D({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$*,*

(b) or T_i *is a noetherian scheme of finite Krull dimension, and* $\mathcal{F}_i \in D^{-}_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$*, and* $\mathcal{G}_i \in D^{-}({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$*.*

Then we have the following isomorphism of $D^{-}({}^{**}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i})$*:*

$$\mathcal{F}_\bullet \otimes^{\mathbb{L}}_{\mathcal{B}_{Y_\bullet}} \mathbb{R}f_{\bullet *}(\mathcal{G}_\bullet) \xrightarrow{\sim} \mathbb{R}f_{\bullet *}\left(f^{-1}\mathcal{F}_\bullet \otimes^{\mathbb{L}}_{f^{-1}\mathcal{B}_{Y_\bullet}} \mathcal{G}_\bullet\right). \tag{7.5.7.3.1}$$

Proof. As 5.1.2.8, this is a consequence of 7.5.7.1. □

7.5.8 Direct image

We keep notation 7.5.5. Assume that S and T are noetherian scheme sof finite Krull dimension, f is quasi-compact and quasi-separated.

Definition 7.5.8.1. We keep notation 7.5.5.4.

(a) The (left version of the) direct image functor of level m by $\widetilde{f}_\bullet$ is the functor $\widetilde{f}^{(m)}_{\bullet +} \colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ which is defined by setting

$$\widetilde{f}^{(m)}_{\bullet +}(\mathcal{E}_\bullet) := \mathbb{R}f_{\bullet *}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \mathcal{E}_\bullet\right),$$

where $\mathcal{E}_\bullet \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$.

(b) The (right version of the) direct image functor of level m by $\widetilde{f}_\bullet$ is the functor $\widetilde{f}^{(m)}_{\bullet +} \colon D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \to D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ which is defined by setting

$$\widetilde{f}^{(m)}_{\bullet +}(\mathcal{M}_\bullet) := \mathbb{R}f_{\bullet *}\left(\mathcal{M}_\bullet \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}\right),$$

where $\mathcal{M}_\bullet \in D({}^{r}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$.

(c) For any $* \in \{r, l\}$, the direct image functor of level m by $\widetilde{f}$ of the form $\widetilde{f}^{(m)}_{+} \colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ is defined by setting

$$\widetilde{f}^{(m)}_{+}(\mathcal{E}) := \mathbb{R}\underset{\leftarrow}{l}_{Y*}(\widetilde{f}^{(m)}_{\bullet +}(\mathbb{L}\underset{\leftarrow}{l}^*_X\mathcal{E}))$$

where $\mathcal{E} \in D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Since this functor preserves the isogenies, this yields the functor $\widetilde{f}^{(m)}_{+} \colon D_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$.

If there is no ambiguity with the level, we might simply write $\widetilde{f}_{\bullet +}$ and $\widetilde{f}_{+}$.

Proposition 7.5.8.2. *We have the following properties.*

(a) For any $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we have the canonical morphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$:

$$\mathbb{R}f_* \left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \right) \to \widetilde{f}^{(m)}_{+}(\mathcal{M}). \tag{7.5.8.2.1}$$

(b) For any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we have the canonical morphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$

$$\mathbb{R}f_* \left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E} \right) \to \widetilde{f}^{(m)}_{+}(\mathcal{E}). \tag{7.5.8.2.2}$$

(c) For any $* \in \{r, l\}$, if $\mathcal{F} \in D^{-}_{\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, then the morphism 7.5.8.2.1 or respectively 7.5.8.2.2 is an isomorphism of $D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$.

Proof. Let us construct the morphism 7.5.8.2.1. By adjointness, we get the first morphism:

$$\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \overset{7.3.4.9.3}{\longrightarrow} \mathbb{R}\underleftarrow{l}_{X*} \left(\mathbb{L}\underleftarrow{l}^*_{X}(\mathcal{M}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet} \right) \tag{7.5.8.2.3}$$

It follows from 7.1.3.15.3 that we have the canonical isomorphism $\mathbb{R}f_* \circ \mathbb{R}\underleftarrow{l}_{X*} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{Y*} \circ \mathbb{R}f_{\bullet *}$. Hence, by applying the functor $\mathbb{R}f_*$ to 7.5.8.2.3, we get 7.5.8.2.1 up to canonical isomorphism.

Similarly, we construct 7.5.8.2.2. Finally, to check the last statement, follows from the fact that 7.5.8.2.3 becomes an isomorphism (see again 7.3.4.9.3). □

Definition 7.5.8.3. We keep notation 7.5.5.10.

(a) The (left version of the) direct image functor of level m by $\widetilde{f}$ of the form $\widetilde{f}^{(m)}_{+} \colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ which is defined by setting for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$

$$\widetilde{f}^{(m)}_{+}(\mathcal{E}) := \mathbb{R}f_* \left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} (\mathcal{E}) \right). \tag{7.5.8.3.1}$$

(b) The (right version of the) direct image functor of level m by $\widetilde{f}$ of the form $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ which is defined by setting for any $\mathcal{M} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$

$$\widetilde{f}^{(m)}_{+}(\mathcal{M}) := \mathbb{R}f_* \left(\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}} \right). \tag{7.5.8.3.2}$$

Proposition 7.5.8.4. *We have the following properties.*

(a) With notation 7.4.6.5, we have the canonical morphism of functors $D_{\mathbb{Q}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ of the form

$$\widetilde{f}^{(m)}_{+} \circ (\mathbb{Q} \otimes -) \to (\mathbb{Q} \otimes -) \circ \widetilde{f}^{(m)}_{+}. \tag{7.5.8.4.1}$$

(b) For any $* \in \{r, l\}$, the restriction to $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ of the morphism 7.5.8.4.1 is an isomorphism.

Proof. This easily follows from 7.5.8.2. □

Remark 7.5.8.5. Let $* \in \{r, l\}$ and $(\mathbb{Q} \otimes -)^{-1} \colon D^{\mathrm{b}}_{\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ be a quasi-inverse functor of the natural equivalence of categories 7.4.6.6. Then the functor $(\mathbb{Q} \otimes -) \circ \widetilde{f}^{(m)}_{+} \circ (\mathbb{Q} \otimes -)^{-1} \colon D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ is isomorphic to $\widetilde{f}^{(m)}_{+}$.

7.5.8.6. We have the following boundedness preservation results.

(a) Since T is a noetherian scheme of finite Krull dimension, then it follows from 5.1.2.4.i that we get the factorization $\widetilde{f}^{(m)}_{\bullet +}\colon D^{-}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \to D^{-}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$.

(b) Suppose either $m=0$ or log-structures are trivial. Then, by copying the proof of 5.3.2.4 (remark the estimate is geometrical and do not depend on i or m), we can check the right (resp. left) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet \to Y^\sharp_\bullet/T^\sharp_\bullet}$) has finite tor-dimension. Hence, we get the induced functor

$$\widetilde{f}^{(m)}_{\bullet +}\colon D^{\mathrm{b}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \to D^{\mathrm{b}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}). \tag{7.5.8.6.1}$$

Proposition 7.5.8.7. *Let $* \in \{r,l\}$, and $\star \in \{-, \mathrm{b}\}$ such that one of the following conditions holds:*

(i) either $\star = -$,

(ii) or $m=0$,

(iii) or log structures are trivial.

Then, we have the following properties.

(a) For any $\mathcal{E}_\bullet \in D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, $\widetilde{f}^{(m)}_{\bullet +}(\mathcal{E}_\bullet) \in D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$.

(b) For any $\mathcal{E} \in D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\widetilde{f}^{(m)}_{+}(\mathcal{E}) \in D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$. For any $\mathcal{E} \in D^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\widetilde{f}^{(m)}_{+}(\mathcal{E}) \in D^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$.

Proof. By using the equivalences of categories 7.5.4.3, we reduce to check (a). It follows from 7.5.8.6.1 that we reduce to check the case (i). By similarity, we can suppose $* = l$. It follows from 5.1.3.5 that $\widetilde{f}^{(m)}_{0+}(\mathcal{E}_0) \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_0/T^\sharp_0})$. It follows from 7.5.4.1 that we can apply 5.3.3.3, i.e. we have the base change isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_{i+1}/T^\sharp_{i+1}}} \widetilde{f}^{(m)}_{i+1+}(\mathcal{E}_{i+1}) \xrightarrow{\sim} \widetilde{f}^{(m)}_{i+}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_{i+1}/S^\sharp_{i+1}}} \mathcal{E}_{i+1}).$$

Hence, we are done. □

7.5.8.8. For any $* \in \{r,l\}$, for any $\mathcal{E}_\bullet \in D({}^{*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, it follows from 7.1.3.15.2 and 7.1.3.6.1 that we have

$$\left(\widetilde{f}^{(m)}_{\bullet +}(\mathcal{E}_\bullet)\right)_i \xrightarrow{\sim} \widetilde{f}^{(m)}_{i+}(\mathcal{E}_i). \tag{7.5.8.8.1}$$

By using the equivalence of categories 7.3.3.3, it follows from 7.5.8.7 that, for any $\mathcal{E} \in D^{-}_{\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we get

$$\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/T^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m)}_{+}(\mathcal{E}) \xrightarrow{\sim} \widetilde{f}^{(m)}_{i+}(\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E}), \tag{7.5.8.8.2}$$

and similarly for right modules.

7.5.8.9. Suppose the bottom morphism ϕ of 7.5.5.0.1 is the identity and f is log-étale. Then, following 5.1.3.6, the canonical morphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} \to \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet \to Y^\sharp_\bullet/S^\sharp_\bullet}$ is an isomorphism. Hence, $\widetilde{f}^{(m)}_{\bullet +}(\mathcal{M}_\bullet) \xrightarrow{\sim} \mathbb{R}f_{\bullet *}(\mathcal{M}_\bullet)$. Similarly, the canonical morphism of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} \to \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$ is an isomorphism.

7.5.8.10. The functors of 7.5.8.1 are compatible with the quasi-inverse functors $-\otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp_\bullet/S^\sharp_\bullet}$ and $\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} -$ exchanging left and right $\mathcal{D}^{(m)}_X$-module structures. More precisely, for any $\mathcal{E}_\bullet \in D({}^{1}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ we have the canonical isomorphism

$$\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet} \otimes_{\mathcal{B}_{Y_\bullet}} \widetilde{f}^{(m)}_{\bullet +}(\mathcal{E}_\bullet) \xrightarrow{\sim} \widetilde{f}^{(m)}_{\bullet +}(\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} \mathcal{E}_\bullet), \tag{7.5.8.10.1}$$

which is constructed as follows:

$$\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet}\otimes_{\mathcal{B}_{Y_\bullet}}\mathbb{R}f_{\bullet*}\left(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{E}_\bullet\right)\underset{7.5.7.3.1}{\xrightarrow{\sim}}\mathbb{R}f_{\bullet*}\left(f^{-1}\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet}\otimes_{f^{-1}\mathcal{B}_{Y_\bullet}}(\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{E}_\bullet)\right)$$

$$\xrightarrow{\sim}\mathbb{R}f_{\bullet*}\left((f^{-1}\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet}\otimes_{f^{-1}\mathcal{B}_{Y_\bullet}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet})\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{E}_\bullet\right)$$

$$\overset{4.3.5.6.1}{\xrightarrow{\sim}}\mathbb{R}f_{\bullet*}\left((\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}\otimes_{\mathcal{B}_{X_\bullet}}\mathcal{E}_\bullet)\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}(f^{-1}\widetilde{\omega}_{Y^\sharp_\bullet/T^\sharp_\bullet}\otimes_{f^{-1}\mathcal{B}_{Y_\bullet}}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet}\otimes_{\mathcal{B}_{X_\bullet}}\widetilde{\omega}^{-1}_{X^\sharp_\bullet/S^\sharp_\bullet})\right)$$

$$\underset{7.5.5.4.1}{\xrightarrow{\sim}}\mathbb{R}f_{\bullet*}\left((\widetilde{\omega}_{X^\sharp_\bullet/S^\sharp_\bullet}\otimes_{\mathcal{B}_{X_\bullet}}\mathcal{E}_\bullet)\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet\to Y^\sharp_\bullet/T^\sharp_\bullet}\right).$$

Proposition 7.5.8.11. *With notation 7.5.5.20, we suppose U is a noetherian scheme of finite Krull dimension and g is quasi-compact and quasi-separated. Let $*\in\{r,l\}$, and $\star\in\{-,\mathrm{b}\}$ such that one of the following conditions holds:*

(i) either $\star=-$

(ii) or $m=0$,

(iii) or log structures are trivial.

In each cases, we have the following properties.

1. *For any $\mathcal{E}_\bullet\in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, we have the canonical isomorphism of $D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/U^\sharp_\bullet})$:*

$$\tilde{g}^{(m)}_{\bullet+}\circ\tilde{f}^{(m)}_{\bullet+}(\mathcal{E}_\bullet)\xrightarrow{\sim}\widetilde{(g\circ f)}^{(m)}_{\bullet+}(\mathcal{E}_\bullet).\tag{7.5.8.11.1}$$

2. *For any $\mathcal{E}\in D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we have the canonical isomorphism of $D^\star_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{U}^\sharp})$:*

$$\tilde{g}^{(m)}_{+}\circ\tilde{f}^{(m)}_{+}(\mathcal{E})\xrightarrow{\sim}\widetilde{(g\circ f)}^{(m)}_{+}(\mathcal{E}).\tag{7.5.8.11.2}$$

Proof. By using 7.5.7.1 and 7.5.5.21.2, we can check 7.5.8.11.1 by copying the proof of 5.1.3.8. By using the equivalence of categories 7.5.4.3, this yields 7.5.8.11.2. □

Proposition 7.5.8.12. *Suppose f is proper, $f^*\mathcal{B}_{\mathfrak{Y}}\to\mathcal{B}_{\mathfrak{X}}$ is an isomorphism. Suppose $\mathcal{B}_{Y_0}$ is an $\mathcal{O}_{Y_0}$-algebra of finite type. Let $\star\in\{-,\mathrm{b}\}$ and $*\in\{\mathrm{r},\mathrm{l}\}$. Suppose moreover one of the following conditions:*

(i) either $\star=-$,

(ii) or $m=0$,

(iii) or log structures are trivial.

In each cases we have the following properties.

(a) The functor $\widetilde{f}^{(m)}_{\bullet+}$ sends $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ to $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/S^\sharp_\bullet})$.

(b) The functor $\widetilde{f}^{(m)}_{+}$ sends $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ (resp. $D^\star_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, resp. $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$) to $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ (resp. $D^\star_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$, resp. $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$).

Proof. By using the equivalence of categories 7.3.3.3, we reduce to check (a). It follows from 5.3.2.11 that $\widetilde{f}^{(m)}_{0+}$ sends $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_0/S^\sharp_0})$ to $D^\star_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_0/S^\sharp_0})$. We conclude by using 7.3.3.4 and 7.5.8.7. □

Notation 7.5.8.13 (Varying m notation)**.** We keep notation and hypotheses 7.5.5. Fix $m \geq m' \geq 0$ a second integer. Let $\mathcal{B}'_{\mathfrak{X}}$ (resp. $\mathcal{B}'_{\mathfrak{Y}}$) be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra (resp. $\mathcal{O}_{\mathfrak{Y}}$-algebra) endowed with a compatible structure of left $\mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. left $\mathcal{D}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$-module) and satisfying the hypotheses of 7.3.2. We suppose that we have algebras morphisms $\mathcal{B}'_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}}$, $\mathcal{B}'_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{Y}}$, $f^*\mathcal{B}'_{\mathfrak{Y}} \to \mathcal{B}'_{\mathfrak{X}}$ which are respectively $\mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear (resp. $\mathcal{D}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$-linear, resp. $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear) and inducing the commutative diagram

$$\begin{array}{ccc} f^*\mathcal{B}_{\mathfrak{Y}} & \longrightarrow & \mathcal{B}_{\mathfrak{X}} \\ \uparrow & & \uparrow \\ f^*\mathcal{B}'_{\mathfrak{Y}} & \longrightarrow & \mathcal{B}'_{\mathfrak{X}}. \end{array}$$

We denote by $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}'_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} := \mathcal{B}'_{\mathfrak{Y}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{D}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$. We keep similar to 7.5.5.4 or 7.5.5.10 notation by replacing m by m'.

Proposition 7.5.8.14. *With notation 7.5.8.13, $f^*\mathcal{B}'_{\mathfrak{Y}} \to \mathcal{B}'_{\mathfrak{X}}$ and $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ are isomorphism. Let $\star \in \{-, \mathrm{b}\}$ and $* \in \{\mathrm{r}, \mathrm{l}\}$. Suppose moreover one of the following conditions holds:*

(a) either $\star = -$,

(b) or log structures are trivial.

Suppose moreover either $\mathcal{B}'_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}}$ is an isomorphism or $f^{-1}\mathcal{O}_{Y_0}$ and $\mathcal{O}_{X_0}$ are tor independent over $f^{-1}\mathcal{B}_{Y_0}$. For any $\mathcal{F} \in D^{\star}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we have in $D^{\star}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ the canonical isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right). \tag{7.5.8.14.1}$$

Proof. We construct the morphism 7.5.8.14.1 similarly to 5.3.2.15.1. To check that this is an isomorphism, since both terms of 7.5.8.14.1 are quasi-coherent, it follows from 7.3.2.14 that we reduce to check that the following induced morphism

$$\widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) \to \widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right) \tag{7.5.8.14.2}$$

is an isomorphism. On one hand, we have the isomorphisms

$$\begin{aligned} \widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right) &\overset{7.5.8.8.2}{\xrightarrow{\sim}} \widetilde{f}^{(m)}_{0+}\left(\widetilde{\mathcal{D}}^{(m)}_{X_0^\sharp/S_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right) \\ &\xrightarrow{\sim} \widetilde{f}^{(m)}_{0+}\left(\widetilde{\mathcal{D}}^{(m)}_{X_0^\sharp/S_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right). \end{aligned} \tag{7.5.8.14.3}$$

On the other hand, we get:

$$\begin{aligned} \widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) &\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{Y_0^\sharp/T_0^\sharp}} \widetilde{\mathcal{D}}^{(m')}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) \\ &\overset{7.5.8.8.2}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{Y_0^\sharp/T_0^\sharp}} \widetilde{f}^{(m')}_{0+}\left(\widetilde{\mathcal{D}}^{(m')}_{X_0^\sharp/S_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right). \end{aligned} \tag{7.5.8.14.4}$$

Hence, via 7.5.8.14.3 and 7.5.8.14.4, the morphism 7.5.8.14.2 is canonically isomorphic to

$$\widetilde{\mathcal{D}}^{(m)}_{Y_0^\sharp/T_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{Y_0^\sharp/T_0^\sharp}} \widetilde{f}^{(m')}_{0+}\left(\widetilde{\mathcal{D}}^{(m')}_{X_0^\sharp/S_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right) \to \widetilde{f}^{(m)}_{0+}\left(\widetilde{\mathcal{D}}^{(m)}_{X_0^\sharp/S_0^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right), \tag{7.5.8.14.5}$$

which is an isomorphism following 5.3.2.15.

□

Corollary 7.5.8.15. *With notation and hypotheses of 7.5.8.14, suppose moreover $\mathcal{B}_{Y_0}$ is an $\mathcal{O}_{Y_0}$-algebra of finite type, f is proper and $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ is an isomorphism. For any $\mathcal{F} \in D^{\star}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, we have in $D^{\star}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ the canonical isomorphism*

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}\right). \tag{7.5.8.15.1}$$

For any $\mathcal{F} \in D^{\star}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$, we have in $D^{\star}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}})$ the canonical isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp,\mathbb{Q}}} \widetilde{f}^{(m')}_{+}(\mathcal{F}) \xrightarrow{\sim} \widetilde{f}^{(m)}_{+}\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{F}\right). \tag{7.5.8.15.2}$$

Proof. We get the isomorphism 7.5.8.15.1 from 7.5.8.12 and 7.5.8.14. Moreover, we construct the morphism 7.5.8.15.2 similarly to 5.3.2.15.1. To check that this is an isomorphism, since the both side functors are wayout left, then we reduce to the case where $\mathcal{F}$ is a coherent $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module. Since it has a coherent $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-model (see 7.4.5.2), then 7.5.8.15.2 is a consequence of 7.5.8.15.1 and 7.5.8.4. □

7.5.8.16 (Non-lifted case notation)**.** In the constext of 7.5.5.23, we can define as in 7.5.8.1 the functors $\widetilde{f}^{(m)}_{0+}\colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet})$ and $\widetilde{f}^{(m)}_{0+}\colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$ for any $* \in \{\mathrm{r}, \mathrm{l}\}$.

7.5.9 Exterior tensor products and commutation with pullbacks and push-forwards

Let $m \in \mathbb{N}$. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0, p)$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal schemes as defined in 3.3.1.10, let $\mathfrak{X}^\sharp$ and $\mathfrak{Y}^\sharp$ be two log smooth $\mathfrak{S}^\sharp$-log-formal scheme. Let $\mathcal{B}_{\mathfrak{X}}$ (resp. $\mathcal{B}_{\mathfrak{Y}}$) be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra (resp. $\mathcal{O}_{\mathfrak{Y}}$-algebra) endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. left $\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module) and satisfying the hypotheses of 7.3.2. Recall following 7.3.2.3 that $\mathcal{B}_{\mathfrak{X}}$ and $\mathcal{B}_{\mathfrak{Y}}$ are in particular quasi-coherent in the sense of 7.3.1.5. The action of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $f^*\mathcal{B}_{\mathfrak{Y}}$ is compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra (see 3.4.4.6). We suppose that we have a morphism of algebras $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ which is moreover $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. We denote by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}_{\mathfrak{Y}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}}\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$. Recall also, following 7.3.2.1, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ satisfy the conditions of 7.3.2, i.e. in particular the projective systems $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} := (\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i})_{i\in\mathbb{N}}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/S^\sharp_\bullet} := (\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/S^\sharp_i})_{i\in\mathbb{N}}$) with $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_i/S^\sharp_i} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}/\mathfrak{m}^{i+1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$) is left and right quasi-coherent in the sense of 7.3.1.10.

Let $\mathfrak{Z}^\sharp := \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Y}^\sharp$, $p\colon \mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp$, $q\colon \mathfrak{Z}^\sharp \to \mathfrak{Y}^\sharp$ be the projections. Set $\mathcal{B}_{\mathfrak{Z}} := p^*\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{Z}}} q^*\mathcal{B}_{\mathfrak{Y}}$.

We denote by $\widetilde{\mathfrak{X}}^\sharp$ (resp. $\widetilde{\mathfrak{Y}}^\sharp$, resp. $\widetilde{\mathfrak{Z}}^\sharp$) the ringed $\mathcal{V}$-log formal scheme $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}})$ (resp. $(\mathfrak{Y}^\sharp, \mathcal{B}_{\mathfrak{Y}})$, resp. $(\mathfrak{Z}^\sharp, \mathcal{B}_{\mathfrak{Z}})$), and by $\tilde{p}\colon \widetilde{\mathfrak{Z}}^\sharp/\mathfrak{S}^\sharp \to \widetilde{\mathfrak{X}}^\sharp/\mathfrak{S}^\sharp$, $\tilde{q}\colon \widetilde{\mathfrak{Z}}^\sharp/\mathfrak{S}^\sharp \to \widetilde{\mathfrak{Y}}^\sharp/\mathfrak{S}^\sharp$ the induced morphism of relative ringed $\mathcal{V}$-log formal schemes. By abuse of notation, we also denote by $\tilde{p}, \tilde{q}$ the induced morphism of ringed $\mathcal{V}$-log formal schemes. We suppose that $\widetilde{\mathfrak{X}}^\sharp/\mathfrak{S}^\sharp$ and $\widetilde{\mathfrak{Y}}^\sharp/\mathfrak{T}^\sharp$ are strongly quasi-flat morphisms of ringed $\mathcal{V}$-log formal schemes (in the sense of 4.4.1.3.b). Hence, we get for instance the finite tor-dimensions of 7.5.4.1 and we can apply 7.5.4.3 for both.

7.5.9.1. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. With the notation 7.5.5.6, we define the external tensor product

$$-\widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}-\colon D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \times D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/S^\sharp_\bullet}) \to D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/S^\sharp_\bullet}), \tag{7.5.9.1.1}$$

defined as follows: for any $\mathcal{E}_\bullet \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, $\mathcal{F}_\bullet \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/S^\sharp_\bullet})$, we set

$$\mathcal{E}_\bullet\widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}\mathcal{F}_\bullet := \mathbb{L}\widetilde{p}^{(m)!}_\bullet(\mathcal{E}_\bullet) \otimes^{\mathbb{L}}_{\mathcal{B}_{Z_\bullet}} \mathbb{L}\widetilde{q}^{(m)!}_\bullet(\mathcal{F}_\bullet)[d_Z].$$

When $* = \mathrm{l}$, using 5.1.5.4.3, this is compatible to the external tensor product defined at 5.1.5.4.5. By using 5.1.5.9 and 7.5.5.7, this yields when $* = \mathrm{r}$ that 7.5.9.1.1 is compatible to the external tensor product defined at 5.1.5.4.5. When $\mathcal{B}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{B}_{\mathfrak{Y}} = \mathcal{O}_{\mathfrak{Y}}$, we simply write $-\boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}-$.

7.5.9.2. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. Using the tensor product defined in 7.5.9.1.1, we get the bifunctor

$$- \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} - \colon D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \times D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \to D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}), \tag{7.5.9.2.1}$$

defined as follows: for any $\mathcal{E} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\mathcal{F} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$, we set

$$\mathcal{E} \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{F} := \mathbb{R} \underleftarrow{l}_{Z*} \left(\mathbb{L} \underleftarrow{l}^{*}_{X}(\mathcal{E}) \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}} \mathbb{L} \underleftarrow{l}^{*}_{Y}(\mathcal{F}) \right) \xrightarrow{\sim} \widetilde{p}^{(m)!}(\mathcal{E}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{Z}}} \widetilde{q}^{(m)!}(\mathcal{F})[d_Z], \tag{7.5.9.2.2}$$

where the isomorphism is checked by using the definition 7.5.5.6 and the fact that $\mathbb{R}\underleftarrow{l}_{Z*}$ is an equivalence of categories between quasi-coherent complexes. When $\mathcal{B}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{B}_{\mathfrak{Y}} = \mathcal{O}_{\mathfrak{Y}}$, we simply write $-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} -$.

Proposition 7.5.9.3. *Let $* \in \{\mathrm{l}, \mathrm{r}\}$. Suppose $? \in \{\mathrm{qc}, \mathrm{tdf}, \mathrm{perf}\}$ (resp. suppose $? = \mathrm{coh}$ and S is locally noetherian). The functor 7.5.9.2.1 preserves the finite tor-dimension and perfectness (resp. bounded above coherent complexes), i.e. induces*

$$- \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} - \colon D^{-}_{?}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \times D^{-}_{?}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \to D^{-}_{?}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}). \tag{7.5.9.3.1}$$

Proof. This is a consequence of 5.1.5.5.1 and of the equivalence of categories of 7.3.2.15 and 7.3.3.3. □

Proposition 7.5.9.4. *Let $f\colon \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ and $g\colon : \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ be two morphisms of log smooth $\mathfrak{S}^\sharp$-log formal schemes. Set $\mathfrak{Z}'^\sharp := \mathfrak{X}'^\sharp \times_{\mathfrak{S}'^\sharp} \mathfrak{Y}'^\sharp$ and $h = f \times g \colon \mathfrak{Z}'^\sharp \to \mathfrak{Z}^\sharp$ the induced morphism. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. Let $\mathcal{E} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\mathcal{F} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. We have the canonical homomorphism of $D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}'^\sharp/\mathfrak{S}^\sharp})$:*

$$\widetilde{f}^{(m)!}(\mathcal{E}') \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \widetilde{g}^{(m)!}(\mathcal{F}') \to \widetilde{h}^{(m)!}(\mathcal{E}' \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{F}'). \tag{7.5.9.4.1}$$

Proof. This follows from 7.5.5.9.2. □

Theorem 7.5.9.5. *We suppose the underlying scheme $\mathfrak{S}$ is noetherian of finite Krull dimension. Let $f\colon \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ and $g\colon : \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ be two quasi-separated and quasi-compact morphisms of log smooth $\mathfrak{S}^\sharp$-log formal schemes. Set $\mathfrak{Z}'^\sharp := \mathfrak{X}'^\sharp \times_{\mathfrak{S}'^\sharp} \mathfrak{Y}'^\sharp$ and $h = f \times g \colon \mathfrak{Z}'^\sharp \to \mathfrak{Z}^\sharp$ the induced morphism. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. Let $\mathcal{E}' \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp})$, $\mathcal{F}' \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$.*

(a) We have the canonical homomorphism of $D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{f}^{(m)}_{+}(\mathcal{E}') \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \widetilde{g}^{(m)}_{+}(\mathcal{F}') \to \widetilde{h}^{(m)}_{+}(\mathcal{E}' \widetilde{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{F}'). \tag{7.5.9.5.1}$$

(b) When $\mathcal{B}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{B}_{\mathfrak{Y}} = \mathcal{O}_{\mathfrak{Y}}$, the homomorphism 7.5.9.5.1 is therefore an isomorphism.

Proof. This is a consequence of 5.3.5.13. □

7.5.10 Log smooth morphisms: Spencer resolutions, stability of the coherence and varying the level of pullbacks, pushforwards as relative de Rham complexes complexes

We keep notation and hypotheses 7.5.8.13. We suppose f is a log-smooth and $\phi = \mathrm{id}$. We suppose T is a noetherian scheme of finite Krull dimension.

We set $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ and $\widetilde{\omega}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} := \mathcal{B}'_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} := \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} := \mathcal{B}'_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$.

Lemma 7.5.10.1. *We have the following properties.*

a) We have the canonical isomorphism

$$\mathcal{B}_{\mathfrak{X},\mathbb{Q}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp,\mathbb{Q}}} \mathcal{B}'_{\mathfrak{X}^\sharp,\mathbb{Q}}, \tag{7.5.10.1.1}$$

$$\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp,\mathbb{Q}} \xrightarrow{\sim} \widetilde{\omega}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp,\mathbb{Q}}. \tag{7.5.10.1.2}$$

b) We have the canonical isomorphisms

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}, \tag{7.5.10.1.3}$$

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}. \tag{7.5.10.1.4}$$

Proof. a) i) For all $m'' \in \{m', m\}$, since the morphism

$$\mathcal{B}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m'')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m'')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}} \mathcal{B}_{\mathfrak{X}}$$

is an isomorphism (this is checked similarly to 11.1.1.5), since $(\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp})_{\mathbb{Q}} = (\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp})_{\mathbb{Q}}$ then we get the canonical morphism

$$\mathcal{B}_{\mathfrak{X},\mathbb{Q}} \to (\mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp})_{\mathbb{Q}} \otimes_{(\mathcal{B}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp})_{\mathbb{Q}}} \mathcal{B}_{\mathfrak{X},\mathbb{Q}}$$

is an isomorphism. Moreover, since the canonical morphism

$$\mathcal{B}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{B}'_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}} \mathcal{B}'_{\mathfrak{X}}$$

is an is an isomorphism (use the arguments of the proof of [Ber96c, 4.4.8]: write the right version). Hence, we are done.

ii) We proceed similarly to check 7.5.10.1.2.

b) i) Let us prove 7.5.10.1.3. From 5.3.2.1.1, we get by projective limit the isomorphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}} \mathcal{B}_{\mathfrak{X}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}, \qquad \widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}} \mathcal{B}'_{\mathfrak{X}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}. \tag{7.5.10.1.5}$$

Hence, using 7.5.10.1.1, we get the isomorphism 7.5.10.1.3.

ii) To check 7.5.10.1.4, we proceed similarly than i) by using this time the isomorphism 5.3.2.2.1 (resp. 7.5.10.1.2) instead of 5.3.2.1.1 (resp. 7.5.10.1.1). □

7.5.10.2. We suppose $m' = 0$ and $\mathcal{B}'_{\mathfrak{X}} = \mathcal{B}_{\mathfrak{X}}$. Since the extension $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is flat, then by extension from 5.3.2.1.3 we get the exact sequence

$$0 \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \wedge^d \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \cdots \underset{\delta}{\longrightarrow} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \underset{\delta}{\longrightarrow} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \to 0. \tag{7.5.10.2.1}$$

Let us denote by $\widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ the complex of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules given by

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \wedge^d \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \cdots \underset{\delta}{\longrightarrow} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \underset{\delta}{\longrightarrow} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$$

whose maps are obtained from 7.5.10.2.1 by applying the functor $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} -$. When $\mathcal{B}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$, we simply write $\mathrm{Sp}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$. With 4.7.3.8.1, remark $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathrm{Sp}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$. We call it the *Spencer complex of level m with coefficient in $\mathcal{B}_{\mathfrak{X}}$.*

By taking projective limits, it follows from 5.3.2.7.a that $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ is a coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. By using 7.5.10.1.3 we get from 7.5.10.2.1 the morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}. \tag{7.5.10.2.2}$$

Since the morphism $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ is flat (see 7.5.3.1), then, by applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$ to 7.5.10.2.2, we get the exact sequence of coherent left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules:

$$0 \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \wedge^d \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \cdots \underset{\delta}{\longrightarrow} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \underset{\delta}{\longrightarrow} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to 0. \tag{7.5.10.2.3}$$

is exact. Via the equivalence of categories of 7.4.6.6.1 of the form $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$, the morphism 7.5.10.2.2 is therefore an isomorphism in $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

Let $\mathcal{M} \in D^{-}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. The canonical map

$$\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \to \mathcal{M} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \tag{7.5.10.2.4}$$

is an isomorphism and will be denoted by $\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}^{\bullet}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$.

7.5.10.3. We suppose $m' = 0$ and $\mathcal{B}'_{\mathfrak{X}} = \mathcal{B}_{\mathfrak{X}}$. Since the extension $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is flat, then by extension from 5.3.2.2.3 we get the exact sequence of right $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules:

$$0 \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow[d]{} \Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow[d]{} \cdots \xrightarrow[d]{} \omega_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to 0. \tag{7.5.10.3.1}$$

Let us denote by $\widetilde{\mathrm{DR}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ the complex

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow[d]{} \Omega^1_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow[d]{} \cdots \xrightarrow[d]{} \omega_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},$$

where $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is the 0th term. By using the similar to 7.5.10.2 arguments, we get by extension from 7.5.10.3.1 that the canonical isomorphism of $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ of the form

$$\widetilde{\mathrm{DR}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}[d_f] \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{7.5.10.3.2}$$

Let $\mathcal{E} \in D^{-}_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. The canonical map

$$\widetilde{\mathrm{DR}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}'^\sharp}} \mathcal{E} \to \widetilde{\mathrm{DR}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}'^\sharp}} \mathcal{E} \tag{7.5.10.3.3}$$

is an isomorphism and will be denoted by $\Omega^{\bullet}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$.

Example 7.5.10.4. Suppose $\mathfrak{Y}^\sharp = \mathfrak{S}^\sharp$. Since in that case $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} = \mathcal{B}_{\mathfrak{X},\mathbb{Q}}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} = \widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$, then 7.5.10.2.3 and of 7.5.10.3.2 can be reformulated as follows.

(a) We have the exact sequence of coherent left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules:

$$0 \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \wedge^d \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \cdots \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \mathcal{B}_{\mathfrak{X},\mathbb{Q}} \to 0, \tag{7.5.10.4.1}$$

i.e. the canonical complex morphism

$$\widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \mathcal{B}_{\mathfrak{X},\mathbb{Q}}. \tag{7.5.10.4.2}$$

is a quasi-isomorphism. In particular, we get

$$\mathcal{B}_{\mathfrak{X},\mathbb{Q}} \in D^{\mathrm{b}}_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}). \tag{7.5.10.4.3}$$

(b) The map $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \xrightarrow{\beta} \widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ given by the structure of a right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module on $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ induces a $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-linear resolution $\mathrm{DR}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})[d_{\mathfrak{X}/\mathfrak{S}}] \xrightarrow{\sim} \widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ of $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$.

When $m = 0$, we can remove $\mathbb{Q}$, i.e. we have similar results by replacing $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ with $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

Proposition 7.5.10.5. *We suppose the rank of $\Omega^1_{X^\sharp/S^\sharp}$ is constant and equal to d.*

(a) We have $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}}(\mathcal{B}_{\mathfrak{X},\mathbb{Q}}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) = 0$ for $i \neq d$.

(b) There is a canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$*-modules*

$$\mathcal{E}xt^d_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}}(\mathcal{B}_{\mathfrak{X},\mathbb{Q}}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \xrightarrow{\sim} \widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}.$$

When $m = 0$*, we can remove* $\mathbb{Q}$*, i.e. we have similar results by replacing* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ *with* $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$*,* $\mathcal{B}_{\mathfrak{X},\mathbb{Q}}$ *with* $\mathcal{B}_{\mathfrak{X}}$ *and* $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ *with* $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$*.*

Proof. By using the flatness of the extensions $\mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$, we get the level m case from the level 0 one of 4.7.3.14. □

Proposition 7.5.10.6 (Berthelot)**.** *Let* $\mathfrak{S}^\sharp$ *be a nice fine* $\mathcal{V}$*-log formal scheme (see definition 3.3.1.10). Moreover, let* $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ *be a log smooth morphism of log formal schemes. We suppose* $\mathfrak{X}$ *is locally noetherian,* $\mathfrak{X}$ *is* $\mathcal{V}$*-flat,* S *is regular and the rank of* $\Omega^1_{X^\sharp/S^\sharp}$ *is constant and equal to* d*. Let* $r := \sup_{s\in f(X)} \dim \mathcal{O}_{S,s}$*. Set* $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ *and* $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} := \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$*.*

(a) The ring $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ *has homological global dimension equal to* $2d + r + 1$*.*

(b) Let E *be a p-torsion free left* $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$*-module of finite type. Then* E *admits a resolution by projective of finite type left* $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$*-modules of length* $\le 2d + r$*.*

(c) Let E *be a left* $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$*-module of finite type. Then* E *admits a resolution by projective of finite type left* $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$*-modules of length* $\le 2d + r$*.*

(d) We have the inequalities $d \le \mathrm{gl.dim}(\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \le 2d + r$*.*

Proof. 1) Let us exhibit a left $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module E such that $\mathrm{Ext}^{2d+r+1}_{\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(E, \widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \ne 0$. We use more or less the same example than 2.3.4.5. Let $s \in f(X)$ such that $\dim \mathcal{O}_{S,s} = r$, $\overline{s}_1, \dots, \overline{s}_r \in \mathfrak{m}_{S,x} \subset \mathcal{O}_{S,s}$ be a regular sequence of generators and let $s_1, \dots, s_r \in\subset \mathcal{O}_{\mathfrak{S}}$ be some lifting. Since $\pi, s_1, \dots, s_r, t_1^p, \dots, t_d^p$ are in the center of $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ then the sub-$\mathcal{O}_{\mathfrak{X}}$-module of $\mathcal{O}_{\mathfrak{X}}$ generated by $(\pi, s_1, \dots, s_r, t_1^p, \dots, t_d^p)$ is in fact a sub-$\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module of $\mathcal{O}_{\mathfrak{X}}$. Hence, we get a left $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (resp. a left $\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module) by setting $\mathcal{E} := \mathcal{O}_{\mathfrak{X}}/(\pi, s_1, \dots, s_r, t_1^p, \dots, t_d^p)$ (resp. $E = \Gamma(X, \mathcal{E})$). By using the level 0 case (without $\mathbb{Q}$) of 7.5.10.5, we conclude similarly to 2.3.4.5 that $\mathrm{Ext}^{2d+r+1}_{\widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(E, \widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \ne 0$.

2) The inequality $d \le \mathrm{gl.dim}\, \widehat{D}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ follows from 7.5.10.5.

3) By using 1.4.3.31 and 2.3.4.5, we get the rest of the proposition. □

Proposition 7.5.10.7. *Assume* f *is quasi-compact and quasi-separated morphism.*

(a) For any $\mathcal{E} \in D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$*, we have the isomorphism:*

$$f^{(m)}_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_*\left(\Omega^\bullet_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}\right)[d_f]; \tag{7.5.10.7.1}$$

For any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$*, we have the isomorphism:*

$$f_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_*\left(\Omega^\bullet_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}\right)[d_f]; \tag{7.5.10.7.2}$$

(b) For any $\mathcal{M} \in D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$*, we have the isomorphism:*

$$f^{(m)}_+(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}f_*\left(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}^\bullet_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}\right). \tag{7.5.10.7.3}$$

For any $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$*, we have the isomorphism:*

$$f_+(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}f_*\left(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}^\bullet_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}\right). \tag{7.5.10.7.4}$$

When $m=0$, we can remove $\mathbb{Q}$.

Proof. By using 9.2.4.2.1, the isomorphism 7.5.10.7.1 (resp. 7.5.10.7.3) is a consequence of 9.4.1.2.1 (resp. 7.5.10.3.2). Recalling the definition 7.5.8.3, we get the other isomorphisms by using the isomorphism equal to the image via $\underrightarrow{l}^*_{\mathbb{Q}}$ of 9.4.1.2.1 (resp. 7.5.10.3.2) □

Proposition 7.5.10.8. *Let $* \in \{\mathrm{l}, \mathrm{r}\}$.*

(a) For $\mathcal{E} \in D^-_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$, we have $\widetilde{f}^{(m)!}(\mathcal{E}) \in D^-_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

(b) For $\mathcal{E}' \in D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$, we have the isomorphism of $D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{f}^{!(m')}(\mathcal{E}') \xrightarrow{\sim} \widetilde{f}^{(m)!}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E}'),$$

and similarly for complexes of right modules.

(c) For $\mathcal{E}' \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$, we have the isomorphism of $D^-_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \widetilde{f}^{!(m')}(\mathcal{E}') \xrightarrow{\sim} \widetilde{f}^{(m)!}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}'),$$

and similarly for complexes of right modules.

Proof. a) The first part is a consequence of 5.3.2.7.

b) We denote by $\widetilde{\mathrm{Sp}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ the Spencer complex of level m' with coefficient in $\mathcal{B}'_X$ and by $\widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ the Spencer complex of level m with coefficient in $\mathcal{B}_X$. We have the isomorphisms $D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{f}^{!(m')}(\mathcal{E}') \underset{7.5.10.2.2}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \left(\widetilde{\mathrm{Sp}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{E}' \right) [d_f]$$

The third and forth parts are a consequence of the previous ones. Since $\widetilde{\mathrm{Sp}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, then we get the first isomorphism of $D^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathrm{Sp}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathrm{Sp}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \xrightarrow{\sim} \widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp},$$

the last isomorphism is a consequence of the fact that the terms of $\widetilde{\mathrm{Sp}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp}$ are locally free left $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules. Hence,

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{f}^{!(m')}(\mathcal{E}') \xrightarrow{\sim} \widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{E}'[d_f]$$

$$\xrightarrow{\sim} \widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{Y}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E}')[d_f] \xrightarrow{\sim} \widetilde{f}^{(m)!}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E}').$$

c) Follows from b). □

7.5.11 Pushforwards: way-out properties, stability of the coherence, tor dimension finiteness, perfectness

We keep notation and hypotheses 7.5.5. We prove in this subsection when log structure are not trivial and $m \neq 0$ the proposition 7.5.8.12 remains true up to isogeny (see 9.4.2.3).

Lemma 7.5.11.1. *We suppose f is a (not necessary exact) closed immersion and $\phi = \mathrm{id}$.*

(a) The left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet \to Y^\sharp_\bullet/S^\sharp_\bullet}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet \leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$) is locally free.

(b) The left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \to \mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp \leftarrow \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$) is flat.

Proof. Since $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \mathbb{R}l_{\leftarrow\mathfrak{X}*}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet\to Y^\sharp_\bullet/S^\sharp_\bullet}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \mathbb{R}l_{\leftarrow\mathfrak{X}*}\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$, using 7.2.1.4, we reduce to check the first assertion. We can copy the proof of 5.2.3.1. □

Proposition 7.5.11.2. We have the following properties:

(a) We have $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet\to Y^\sharp_\bullet/T^\sharp_\bullet} \in D^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ and $\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet} \in D^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$.

(b) We have $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \in D^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \in D^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

Proof. 0) Since $\mathbb{L}l^*_{\leftarrow\mathfrak{X}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/T^\sharp_\bullet\to Y^\sharp_\bullet/S^\sharp_\bullet}$ and $\mathbb{L}l^*_{\leftarrow\mathfrak{X}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\leftarrow\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet\leftarrow X^\sharp_\bullet/S^\sharp_\bullet}$, then by using 7.5.5.10.1, we reduce to check $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp} \in D^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

1) We prove in this step the proposition in the case where $\phi = \mathrm{id}$. Following 9.2.1.1, since the assertion is étale locally on X, we can suppose there exists an exact closed immersion $v\colon \mathfrak{X}^\sharp \hookrightarrow \mathfrak{Y}'^\sharp$ and a log etale morphism $\mathfrak{Y}'^\sharp \to \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Y}^\sharp$ whose composite map gives $\mathfrak{X}^\sharp \hookrightarrow \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Y}^\sharp$, the graph of f. Let g be the composite morphism $\mathfrak{Y}'^\sharp \to \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Y}^\sharp \to \mathfrak{Y}^\sharp$, $\mathcal{B}_{\mathfrak{Y}'} := g^*(\mathcal{B}_{\mathfrak{Y}})$ $\widetilde{g}\colon (\mathfrak{Y}', \mathcal{B}_{\mathfrak{Y}'}) \to \widetilde{\mathfrak{Y}}$ and $\widetilde{v}\colon \widetilde{\mathfrak{X}} \to (\mathfrak{Y}', \mathcal{B}_{\mathfrak{Y}'})$ be the induced morphisms. Set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} := \widetilde{v}^*(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} := \widetilde{g}^*(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. It follows from 7.5.10.8.(a) and 7.5.5.13.(c) that we have the isomorphism of $D^{\mathrm{b}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, {}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{v^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} v^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}.$$

Since g is log smooth, then following 7.5.10.2.a) we have the canonical isomorphism

$$\widetilde{\mathrm{Sp}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{Y}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$$

in $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$. Hence, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ is isomorphic in $D^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$ to a bounded complex whose terms are of the form $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{v^{-1}\mathcal{O}_{\mathfrak{Y}'}} v^{-1}\wedge^i \mathcal{T}_{\mathfrak{Y}'^\sharp/\mathfrak{Y}^\sharp}$. It follows from 7.5.11.1 that such terms are flat $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules. Hence, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ is bounded complex with up to isogeny tor amplitude in $[0, d_{p_{\mathfrak{X}^\sharp}}]$.

2) When the diagram 7.5.5.0.1 is cartesian and the morphism $f^*\mathcal{B}_Y \to \mathcal{B}_X$ is an isomorphism (such case is called the base change one), $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}$ is an isomorphism of $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$-bimodules (see 7.5.6.4).

3) Hence, by using the splitting of the remark 7.5.6.7, we conclude from the above cases 1) and 2) by using 7.5.5.13.(c). □

Corollary 7.5.11.3. *Let $* \in \{\mathrm{r}, \mathrm{l}\}$. Assume that S and T are noetherian schemes of finite Krull dimension, f is quasi-compact and quasi-separated.*

$$\widetilde{f}^{(m)}_{\bullet+}\colon D^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}) \to D^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/T^\sharp_\bullet}), \tag{7.5.11.3.1}$$

$$\widetilde{f}^{(m)}_{+}\colon D^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{T}^\sharp}) \to D^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}). \tag{7.5.11.3.2}$$

Proof. By hypotheses, the functor $\mathbb{R}f_{\bullet*}$ has finite cohomological dimension. Hence, the functor 7.5.11.3.1 is well defined thanks to 7.5.11.2.(a) (recall definition 7.5.8.1). By construction, since the functors of the form $\mathbb{L}l^*_{\leftarrow\mathfrak{X}}$ and $\mathbb{L}l_{\leftarrow\mathfrak{X}*}$ preserve bounded quasi-coherent complexes, then 7.5.11.3.2 is a consequence of 7.5.11.3.1. □

Proposition 7.5.11.4. *Suppose f is proper, $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ is an isomorphism and $\mathcal{B}_{Y_0}$ is an $\mathcal{O}_{Y_0}$-algebra of finite type. Let $\star \in \{-, \mathrm{b}\}$ and $* \in \{\mathrm{r}, \mathrm{l}\}$. The functor $f^{(m)}_{+}$ sends $D^{\star}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$ (resp. $D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$) to $D^{\star}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ (resp. $D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$).*

Proof. The non-respective case is a consequence of 7.5.8.12 and 7.5.11.3. The respective case follows from 7.4.6.6. □

Proposition 7.5.11.5. *Suppose $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ is an isomorphism and the bottom arrow of 7.5.5.0.1 is the identity. Suppose $f^{-1}\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ has finite tor dimension, S is a noetherian scheme of finite Krull dimension, f is quasi-compact and quasi-separated. Let $* \in \{\mathrm{r}, \mathrm{l}\}$. The functor $\widetilde{f}^{(m)}_{\bullet +}$ sends $D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ to $D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/S^\sharp_\bullet})$. The functor $\widetilde{f}^{(m)}_{+}$ sends $D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ to $D_{\mathrm{qc,tdf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$.*

Proof. This is a consequence of 7.3.2.15 (and 7.1.3.6) and 5.3.2.12 (and the fact that the tor amplitude does not depend on i). □

Proposition 7.5.11.6. *Suppose $f^*\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ is an isomorphism and the bottom arrow of 7.5.5.0.1 is the identity. Suppose $\mathcal{B}_Y$ is an $\mathcal{O}_Y$-algebra of finite type. Suppose $f^{-1}\mathcal{B}_{\mathfrak{Y}} \to \mathcal{B}_{\mathfrak{X}}$ has finite tor dimension, S is a noetherian scheme of finite Krull dimension, f is proper. Let $* \in \{\mathrm{r}, \mathrm{l}\}$. The functor $\widetilde{f}^{(m)}_{\bullet +}$ sends $D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ to $D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{Y^\sharp_\bullet/S^\sharp_\bullet})$. The functor $\widetilde{f}^{(m)}_{+}$ sends $D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ to $D_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$.*

Proof. Following [Sta22, 08G8], a complex is perfect if and only if it is pseudo-coherent and locally has finite tor dimension. Hence, this is a consequence of 7.5.11.4 and 7.5.11.5. □

Chapter 8

Localisation of derived categories of inductive systems of arithmetic $\mathcal{D}$-modules

8.1 Localisation of derived categories of inductive systems

8.1.1 Topos of inductive systems of sheaves on a topological space

Notation 8.1.1.1. Let I be a partially ordered set and let X be a topological space. An inductive system indexed by I can be viewed as a projective system indexed by I° the partially ordered set equal to I as a set but equipped with the ordering opposite to that of I (see 7.1.1.1). We set $X^{(I)} := \mathrm{Top}(X)_{I^{\circ}}$ the topos of inductive system indexed by I of sheaves on X.

8.1.1.2. Let I be a partially ordered set, $\{*\}$ be some one element set and let X be a topological space. The results of 7.1 can be translated for $X^{(I)}$. Let us fix some notations.

(a) Let $u\colon I \to I'$ be an increasing map of partially ordered sets. Following 7.1.2.4.1, since $(I^{\circ})_{i'} = (I^{i'})^{\circ}$, then we get the morphism of topos

$$\underrightarrow{u}_X := u^{\circ}_X \colon X^{(I)} \to X^{(I')} \tag{8.1.1.2.1}$$

given for any $\mathcal{F}'^{(\bullet)} \in X^{(I')}$ by $\underrightarrow{u}_X^{-1}(\mathcal{F}'^{(\bullet)})^{(i)} = \mathcal{F}'^{(u(i))}$ and for any $\mathcal{F}^{(\bullet)} \in X^{(I)}$ by

$$\underrightarrow{u}_{X*}(\mathcal{F}^{(\bullet)})^{(i')} = \varprojlim_{u(i)\geq i'} \mathcal{F}^{(i)},$$

where the projective limit indexed by $(I^{i'})^{\circ}$ is that of the functor $\mathrm{Cat}((I^{i'})^{\circ})^{\mathrm{op}} \to \mathrm{Sh}(X_{Zar})$ induced by $\mathcal{F}^{(\bullet)}$. Moreover, we have a left adjoint $\underrightarrow{u}_{X!}$ of $\underrightarrow{u}_X^{-1}$ which is given for any $\mathcal{F}^{(\bullet)} \in X^{(I)}$ by

$$\underrightarrow{u}_{X!}(\mathcal{F}^{(\bullet)})^{(i')} = \varinjlim_{u(i)\leq i'} \mathcal{F}^{(i)}, \tag{8.1.1.2.2}$$

where the inductive limit is that of the inductive system $\mathrm{Cat}(I_{i'}) \to \mathrm{Sh}(X_{Zar})$ induced by $\mathcal{F}^{(\bullet)}$.

(b) In the case where u is a map of the form $u\colon I \to \{*\}$, we get the morphism of topos

$$\underrightarrow{l}^{(I)}_X := \underrightarrow{u}_X = (\underrightarrow{u}_X^{-1} \dashv \underrightarrow{u}_{X*})\colon X^{(I)} \to X. \tag{8.1.1.2.3}$$

Suppose I is a filtered set. Then $u^{\circ}\colon I^{\circ} \to \{*\}^{\circ}$ is cofiltered (see definition 7.1.2.2). Hence, following 7.1.2.4.3, we get the morphism of topos

$$\underrightarrow{l}_{X,I} := (\underrightarrow{u}_{X!} \dashv \underrightarrow{u}_X^{-1})\colon X \to X^{(I)}. \tag{8.1.1.2.4}$$

Following 7.1.2.4.2, we have the formula $\underrightarrow{l}_{X,I}^{-1}(\mathcal{F}^{(\bullet)}) = \underrightarrow{u}_{X!}(\mathcal{F}^{(\bullet)}) = \varinjlim_{i\in I} \mathcal{F}^{(i)}$. Moreover, $\underrightarrow{l}_{X,I*}(\mathcal{F}) = \underrightarrow{u}_X^{-1}(\mathcal{F})$ is the constant inductive system with value $\mathcal{F}$.

(c) Fix $i \in I$ and let $i\colon \{*\} \to I$ be the map sending $*$ to i. The morphism of 8.1.1.2.1 is in this case

$$\underrightarrow{i}_X = (\underrightarrow{i}_X^{-1} \dashv \underrightarrow{i}_{X*})\colon X \to X^{(I)}, \tag{8.1.1.2.5}$$

where X means by abuse of notation $\mathrm{Top}(X)$. We have $\underrightarrow{i}_X^{-1}(\mathcal{F}^{(\bullet)}) = \mathcal{F}^{(i)}$ for any $\mathcal{F}^{(\bullet)} \in X^{(I)}$ and we compute

$$(\underrightarrow{i}_{X*}(\mathcal{F}))^{(j)} = \begin{cases} \mathcal{F} & \text{if } j \le i \\ e & \text{otherwise} \end{cases}, \qquad (\underrightarrow{i}_{X!}(\mathcal{F}))^{(j)} = \begin{cases} \mathcal{F} & \text{if } j \ge i \\ \emptyset & \text{otherwise} \end{cases} \tag{8.1.1.2.6}$$

where e (resp. $\emptyset$) is the final (resp. initial) object of $\mathrm{Top}(X)$ for any $\mathcal{F} \in \mathrm{Top}(X)$.

(d) For any continuous map $f\colon X \to X'$ of topological spaces, it follows from 7.1.2.9.1 we have the topos morphism

$$\underrightarrow{f}_I := f_{I^\circ} = (f_{I^\circ}^{-1} \dashv f_{I^\circ *})\colon X^{(I)} \to X'^{(I)} \tag{8.1.1.2.7}$$

defined by setting $\underrightarrow{f}_I^{-1}(\mathcal{G}^{(\bullet)})\colon I \to \mathrm{Top}(X)$ is the functor $i \mapsto f^{-1}\mathcal{G}^{(i)}$ and $\underrightarrow{f}_{I*}(\mathcal{F}^{(\bullet)})\colon I \to \mathrm{Top}(X)$ is the functor $i \mapsto f_*\mathcal{F}^{(i)}$.

Remark 8.1.1.3 (When I has a smallest element). Let I be a partially ordered set and let X be a topological space.

1. Following 8.1.1.2.5, we have the topoi morphism $\underrightarrow{i}_X\colon X \to X^{(I)}$, following 8.1.1.2.3 we have the topoi morphism $\underrightarrow{l}_X^{(I)}\colon X^{(I)} \to X$. When I has a smallest element i, we notice $(\underrightarrow{l}_X^{(I)})^{-1} = \underrightarrow{i}_{X!}$ and $(\underrightarrow{l}_X^{(I)})_* = \underrightarrow{i}_X^{-1}$. Hence, we get the adjoint functors $(\underrightarrow{l}_X^{(I)})^{-1} = \underrightarrow{i}_{X!} \dashv (\underrightarrow{l}_X^{(I)})_* = \underrightarrow{i}_X^{-1} \dashv \underrightarrow{i}_{X*}$.
2. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. We have the ringed topoi morphism $\underrightarrow{i}_X\colon (X, \mathcal{D}^{(i)}) \to (X^{(I)}, \mathcal{D}^{(\bullet)})$. Suppose I has a smallest element i. Then we get the canonical adjunction ring morphism $(\underrightarrow{l}_X^{(I)})^{-1}(\mathcal{D}^{(i)}) \to \mathcal{D}^{(\bullet)}$ which induces the ringed topoi morphism $\underrightarrow{l}_X^{(I)}\colon (X^{(I)}, \mathcal{D}^{(\bullet)}) \to (X, \mathcal{D}^{(i)})$. Moreover, we have the equality of functors $(\underrightarrow{l}_X^{(I)})_* = \underrightarrow{i}_X^{-1} = \underrightarrow{i}_X^{*}\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(i)})$ given by $\mathcal{E}^{(\bullet)} \mapsto \mathcal{E}^{(i)}$. They have the left adjoint $(\underrightarrow{l}_X^{(I)})^* = \underrightarrow{i}_{X!}\colon \mathrm{Mod}(\mathcal{D}^{(i)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$ (beware there are two different functors $\underrightarrow{i}_{X!}$) given by $\mathcal{E}^{(i)} \mapsto \mathcal{D}^{(\bullet)} \otimes_{(\underrightarrow{l}_X^{(I)})^{-1}\mathcal{D}^{(i)}} (\underrightarrow{l}_X^{(I)})^{-1}\mathcal{E}^{(i)}$.

Notation 8.1.1.4. We suppose I has a smallest element i_0. Let $\mathcal{G}^{(i_0)}$ be a left $\mathcal{D}^{(i_0)}$-module and $\mathcal{F}^{(i_0)} \in D(\mathcal{D}^{(i_0)})$. We set

$$\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)} := \mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{(\underrightarrow{l}_X^{(I)})^{-1}\mathcal{D}^{(i_0)}} (\underrightarrow{l}_X^{(I)})^{-1}\mathcal{F}^{(i_0)}, \qquad \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{G}^{(i_0)} := \mathcal{D}^{(\bullet)} \otimes_{(\underrightarrow{l}_X^{(I)})^{-1}\mathcal{D}^{(i_0)}} (\underrightarrow{l}_X^{(I)})^{-1}\mathcal{G}^{(i_0)}, \tag{8.1.1.4.1}$$

Notation 8.1.1.5. Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. A left $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)} = (\mathcal{E}^{(i)}, \alpha^{(j,i)})$ is the data for any $i \in I$ of some $\mathcal{D}^{(i)}$-module $\mathcal{E}^{(i)}$ equipped with transition morphisms $\alpha^{(j,i)}\colon \mathcal{E}^{(i)} \to \mathcal{E}^{(j)}$ which are semi-linear with respect to the homomorphism $\mathcal{D}^{(i)} \to \mathcal{D}^{(j)}$ for any elements $i \le j$ of I.

(a) The category of left $\mathcal{D}^{(\bullet)}$-modules will be denoted by $\mathrm{Mod}(\mathcal{D}^{(\bullet)})$ or simply by $M(\mathcal{D}^{(\bullet)})$.

(b) Let $\mathcal{E}^{(\bullet),\bullet} \in C(\mathcal{D}^{(\bullet)})$. We get

$$\cdots \mathcal{E}^{(\bullet)n-1} \to \mathcal{E}^{(\bullet)n} \to \mathcal{E}^{(\bullet)n+1} \cdots \tag{8.1.1.5.1}$$

the corresponding complexes of left $\mathcal{D}^{(\bullet)}$-modules. If no confusion is possible, we simply write $\mathcal{E}^{(\bullet)}$.

(c) Let U be an open subset of X and $j\colon U \hookrightarrow X$ is the induced open immersion. With notation 8.1.1.2.7 we set $\mathcal{D}^{(\bullet)}|U := \underrightarrow{j}_I^{-1}(\mathcal{D}^{(\bullet)})$. This yields the ringed topos morphism

$$\underrightarrow{j}_I =\colon (U^{(I)}, \mathcal{D}^{(\bullet)}|U) \to (X^{(I)}, \mathcal{D}^{(\bullet)}). \tag{8.1.1.5.2}$$

We denote by $\underrightarrow{j}_I^*\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)}|U)$ and $\underrightarrow{j}_{I*}\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}|U) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$ the induced morphism. We can also simply simply denote by $|U$ the functor $\underrightarrow{j}_I^*$. The functor $\underrightarrow{j}_I^*$ has a left adjoint that we denote by $\underrightarrow{j}_{I!}\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}|U) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$, which is defined by setting $\underrightarrow{j}_{I!}(\mathcal{G}^{(\bullet)})\colon I \to \mathrm{Top}(X)$ is the functor $i \mapsto j_!\mathcal{G}^{(i)}$ (the extension by zero functor) for any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{G}^{(\bullet)}$.

(d) Let $i \in I$ and U be an open set of X. By definition, $X^{(I)} = \mathrm{Sh}((I^{\mathrm{o}})^{\natural} \times X_{Zar})$. For any $(i, U) \in (I^{\mathrm{o}})^{\natural} \times X_{Zar}$, from 7.1.2.18.1 we get,

$$\begin{array}{ccc} ((I_{\geq i})^{\mathrm{o}})^{\natural} \times U_{Zar} & \xrightarrow{\sim} & (I^{\mathrm{o}})^{\natural} \times X_{Zar}/(i, U) \\ & \searrow & \downarrow{\scriptstyle j_{(i,U)}} \\ & & (I^{\mathrm{o}})^{\natural} \times X_{Zar} \end{array} \tag{8.1.1.5.3}$$

where the top right term is the localization of the site $(I^{\mathrm{o}})^{\natural} \times X_{Zar}$ at the object (i, U), $j_{(i,U)}$ is the forgetful functor (see 4.6.2.3) and the horizontal morphism is an isomorphism of sites. Following 7.1.2.18.3, this yields the open immersion of topoi

$$j_{(i,U)} \colon U^{(I_{\geq i})} \to X^{(I)}. \tag{8.1.1.5.4}$$

We denote by $\mathcal{D}^{(\bullet)}|_{(i,U)} := j^{-1}_{(i,U)} \mathcal{D}^{(\bullet)}$ and by $\mathcal{D}^{(\bullet)}|_i := j^{-1}_{(i,X)} \mathcal{D}^{(\bullet)}$. This yields the ringed topoi morphism $j_{(i,U)} \colon (U^{(I_{\geq i})}, \mathcal{D}^{(\bullet)}|_{(i,U)}) \to (X^{(I)}, \mathcal{D}^{(\bullet)})$ which induces the exact functor $j^{-1}_{(i,U)} \colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)}|_{(i,U)})$. We get the functor $j^{-1}_{(i,U)} \colon D(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^{(\bullet)}|_{(i,U)})$ which will be simply denoted by $|_{(i,U)}$. Following 7.1.3.3.2, the functor $j^{-1}_{(i,U)} = j^{*}_{(i,U)} \colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)}|_{(i,U)})$ has a left adjoint $j_{(i,X)!} \colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}|_{(i,U)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$ which is such that for any $\mathcal{F}^{(\bullet)} \in \mathrm{Mod}(\mathcal{D}^{(\bullet)}|_{(i,U)})$ we have

$$(j_{(i,U)!}(\mathcal{F}^{(\bullet)}))^{(j)} = \begin{cases} j_{U!}(\mathcal{F}^{(j)}) & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases} \tag{8.1.1.5.5}$$

Modulo the equivalence of topoi 7.1.2.13.1, the functor $j_{(i,U)!}$ corresponds to the extension by zero via the open immersion $I_{\geq i} \times U \subset I \times X$ (for the canonical topology).

The functor $|_{(i,X)}$ is equal to $\underset{\to}{u}_{i,X}$ where u_i is the inclusion $I_{\geq i} \subset I$ of 8.1.1.2.1. In this case, we simply write $|_i := |_{(i,X)}$. We remark that the functor $j_{(i,U)}$ is the composition of the functor $|_i$ with the functor $\underset{\to I}{j}$ where j is the inclusion $U \subset X$.

8.1.1.6. With notation 8.1.1.5, let $\mathcal{E}^{(\bullet)} \in K({}^{1}\mathcal{D}^{(\bullet)})$. Following 7.1.3.6, $\mathcal{E}^{(\bullet)}$ is a K-flat complex of $K(\mathcal{D}^{(\bullet)})$ if and only if $\mathcal{E}^{(i)}$ is a K-flat complex of $K(\mathcal{D}^{(i)})$ for any $i \in I$.

8.1.2 Ind-isogenies

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.

Notation 8.1.2.1. Let $M(I)$ be the set of increasing maps $\chi \colon I \to \mathbb{N}$. It is endowed with the following order: $\chi \leq \chi'$ if and only if $\chi(i) \leq \chi'(i)$ for an $i \in I$. The partially ordered set $M(I)$ is filtered.

8.1.2.2. We have the following notations and definitions.

(a) For any map $\chi \in M(I)$, for any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)} = (\mathcal{E}^{(i)}, \alpha^{(j,i)})$ (also simply denoted by $\mathcal{E}$), we set

$$\chi^{*}(\mathcal{E}^{(\bullet)}) := (\mathcal{E}^{(i)}, p^{\chi(j)-\chi(i)} \alpha^{(j,i)}).$$

We obtain the functor $\chi^{*} \colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$ as follows: if $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ is a morphism of $\mathrm{Mod}(\mathcal{D}^{(\bullet)})$, then $\chi^{*}(f^{(\bullet)}) \colon \chi^{*}(\mathcal{E}^{(\bullet)}) \to \chi^{*}(\mathcal{F}^{(\bullet)})$ is the morphism of left $\mathcal{D}^{(\bullet)}$-modules such that $(\chi^{*}(f^{(\bullet)}))^{(i)} = f^{(i)}$. Since the functor $\chi^{*} \colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$ is exact, then this induces the functor $\chi^{*} \colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\mathcal{D}^{(\bullet)})$.

(b) If $\chi_1, \chi_2 \in M(I)$, we compute $\chi_1^{*} \circ \chi_2^{*} = (\chi_1 + \chi_2)^{*}$, and in particular χ_1^{*} and χ_2^{*} commute.

(c) Let $\chi_1, \chi_2 \in M(I)$ such that $\chi_1 \leq \chi_2$. For any $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$, we denote by

$$\theta_{\mathcal{E},\chi_2,\chi_1} \colon \chi_1^{*}(\mathcal{E}^{(\bullet)}) \to \chi_2^{*}(\mathcal{E}^{(\bullet)}) \tag{8.1.2.2.1}$$

be the morphism defined by $p^{\chi_2(i)-\chi_1(i)} \colon \mathcal{E}^{(i)} \to \mathcal{E}^{(i)}$ for any $i \in I$.

(d) Let $\chi_1, \chi_2, \chi_3 \in M(I)$ such that $\chi_1 \leq \chi_2$. For any $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$, we have

$$\chi_3^*(\theta_{\mathcal{E},\chi_2,\chi_1}) = \theta_{\mathcal{E},\chi_3+\chi_2,\chi_3+\chi_1}. \tag{8.1.2.2.2}$$

If furthermore $\chi_2 \leq \chi_3$ then

$$\theta_{\mathcal{E},\chi_3,\chi_2} \circ \theta_{\mathcal{E},\chi_2,\chi_1} = \theta_{\mathcal{E},\chi_3,\chi_1}. \tag{8.1.2.2.3}$$

(e) For any $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$, it follows from 8.1.2.2.3 that we get a functor $\theta_{\mathcal{E}}\colon M(I) \to D(\mathcal{D}^{(\bullet)})$ given by $\chi \mapsto \chi^*(\mathcal{E}^{(\bullet)})$, where $M(I)$ is the category associated with its structure of partially ordered set (see 7.1.2.1). This gives a meaning of the notion of functoriality with respect to χ.

(f) For any map $\chi \in M(I)$, we write $\theta_{\mathcal{E},\chi} := \theta_{\mathcal{E},\chi,0}\colon \mathcal{E}^{(\bullet)} \to \chi(\mathcal{E}^{(\bullet)})$. For instance, when $\chi = n$ is the constant map with value n then $\theta_{\mathcal{E},\chi}$ is the multiplication by p^n on $\mathcal{E}^{(i)}$ for each $i \in I$.

(g) A morphism $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^\sharp(\mathcal{D}^{(\bullet)})$ is an "ind-isogeny of $D^\sharp(\mathcal{D}^{(\bullet)})$" if there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. We denote by $\Xi(\mathcal{D}^{(\bullet)})$ the set of ind-isogenies of $D^\sharp(\mathcal{D}^{(\bullet)})$. If no confusion is possible with respect to $\mathcal{D}^{(\bullet)}$, we simply write $\Xi^\sharp := \Xi^\sharp(\mathcal{D}^{(\bullet)})$. For instance, the morphisms of the form 8.1.2.2.1 are ind-isogenies (see 8.1.4.11).

Lemma 8.1.2.3. *Let $\mathfrak{Ab}$ be the category of abelian groups. For any $\mathcal{G}^{(\bullet)} \in D^\sharp(\mathcal{D}^{(\bullet)})$, let $H_{\mathcal{G}^{(\bullet)}}\colon D^\sharp(\mathcal{D}^{(\bullet)}) \to \mathfrak{Ab}$ be the homological functor defined by setting for any $\mathcal{E}^{(\bullet)} \in D^\sharp(\mathcal{D}^{(\bullet)})$*

$$H_{\mathcal{G}^{(\bullet)}}(\mathcal{E}^{(\bullet)}) := \varinjlim_{\chi \in M(I)} \operatorname{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{G}^{(\bullet)}, \chi^*\mathcal{E}^{(\bullet)}).$$

Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$. The morphism $f^{(\bullet)}$ is an ind-isogeny if and only if $H_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is an isomorphism for any $\mathcal{G}^{(\bullet)} \in D^\sharp(\mathcal{D}^{(\bullet)})$.

Proof. 1) Suppose that $f^{(\bullet)} \in \Xi^\sharp$. Hence there exists $\chi_0 \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi_0^*(\mathcal{E}^{(\bullet)})$ of $D(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi_0^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$

a) Let us check that $H_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is injective. Let $u^{(\bullet)}\colon \mathcal{G}^{(\bullet)} \to \chi^*(\mathcal{E}^{(\bullet)})$ be a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$ such that the image of $\chi^*(f^{(\bullet)}) \circ u^{(\bullet)}$ in $H_{\mathcal{G}^{(\bullet)}}(\mathcal{F}^{(\bullet)})$ is null. It is a question of checking that the image of $u^{(\bullet)}$ in $H_{\mathcal{G}^{(\bullet)}}(\mathcal{E}^{(\bullet)})$ is null. By increasing χ_0 and χ if necessary, we can suppose that $\chi = \chi_0$ and that $\chi^*(f^{(\bullet)}) \circ u^{(\bullet)} = 0$ in $D^\sharp(\mathcal{D}^{(\bullet)})$. By using the formula 8.1.2.2.2, we get the first equality $\theta_{\mathcal{E},2\chi,\chi} = \chi^*(\theta_{\mathcal{E},\chi}) = \chi^*(g^{(\bullet)}) \circ \chi^*(f^{(\bullet)})$. This yields $\theta_{\mathcal{E},2\chi,\chi} \circ u^{(\bullet)} = 0$. Hence, we are done.

b) Let us check the surjectivity of $H_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$. Let $v^{(\bullet)}\colon \mathcal{G}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ be a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$. By increasing χ_0 and χ if necessary, we can suppose $\chi = \chi_0$. Set $u^{(\bullet)} := \chi^*(g^{(\bullet)}) \circ v^{(\bullet)}\colon \mathcal{G}^{(\bullet)} \to (2\chi)^*\mathcal{E}^{(\bullet)}$. We compute $(2\chi^*)(f^{(\bullet)}) \circ u^{(\bullet)} = \theta_{\mathcal{F},2\chi,\chi} \circ v^{(\bullet)}$. Hence, the class of $u^{(\bullet)}$ is sent to the class of $v^{(\bullet)}$ via $H_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$.

2) Suppose now that $H_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is an isomorphism for any $\mathcal{G}^{(\bullet)} \in D^\sharp(\mathcal{D}^{(\bullet)})$. Since $H_{\mathcal{F}^{(\bullet)}}(f^{(\bullet)})$ is in particular surjective, then there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. The functor $H_{\mathcal{E}^{(\bullet)}}(f^{(\bullet)})$ sends the class of $g^{(\bullet)} \circ f^{(\bullet)}$ to the class of $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{F},\chi} \circ f^{(\bullet)}$, which is the class of $f^{(\bullet)}$. Since $H_{\mathcal{E}^{(\bullet)}}(f^{(\bullet)})$ is in particular injective, the class of $g^{(\bullet)} \circ f^{(\bullet)}$ is equal to the class of the identity of $\mathcal{E}^{(\bullet)}$. Increasing χ if necessary, this yields that the morphism $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$. □

Notation 8.1.2.4. The subset of ind-isogenies is a saturated multiplicative system compatible with its triangulated structure (this follows from Proposition 7.4.1.7 and Lemma 8.1.2.3). The localisation of $D^\sharp(\mathcal{D}^{(\bullet)})$ with respect to ind-isogenies is denoted by $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

8.1.2.5. A morphism $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ can be represented by a morphism $\phi^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$, i.e. by $\mathcal{E}^{(\bullet)} \stackrel{\phi^{(\bullet)}}{\to} \chi^*\mathcal{F}^{(\bullet)} \stackrel{\theta_{\mathcal{F},\chi}}{\leftarrow} \mathcal{F}^{(\bullet)}$ for some $\chi \in M(I)$. Moreover, two morphisms $\phi_1^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_1^*\mathcal{F}^{(\bullet)}$ and $\phi_2^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_2^*\mathcal{F}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ induce the same arrow $\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ if and only if there exists $\chi \geq \chi_1, \chi_2$, such that both composite arrows $\mathcal{E}^{(\bullet)} \stackrel{\phi_1^{(\bullet)}}{\longrightarrow} \chi_1^*\mathcal{F}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ and $\mathcal{E}^{(\bullet)} \stackrel{\phi_2^{(\bullet)}}{\longrightarrow} \chi_2^*\mathcal{F}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ are equal. To sum-up, for any $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, we have the formula

$$\operatorname{Hom}_{\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) = \varinjlim_{\chi \in M(I)} \operatorname{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \chi^*\mathcal{F}^{(\bullet)}). \tag{8.1.2.5.1}$$

Remark 8.1.2.6. Let a be an integer and $\chi \in M(I)$. The truncation functors $\tau^{\leq a}\colon D(\mathcal{D}^{(\bullet)}) \to D^{\leq a}(\mathcal{D}^{(\bullet)})$ and $\tau^{\geq a}\colon D(\mathcal{D}^{(\bullet)}) \to D^{\geq a}(\mathcal{D}^{(\bullet)})$ (we have chosen to use notation [KS06, 12.3.1], this is not the stupid truncation and we prefer to put the index above and not below contrary to for example [Sta22, 0118]) commute with the functors $\chi^*\colon D(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^{(\bullet)})$, i.e., we compute we have the equality

$$\chi^* \tau^{\leq a} \mathcal{E}^{(\bullet)} = \tau^{\leq a} \chi^* \mathcal{E}^{(\bullet)}. \tag{8.1.2.6.1}$$

Similarly, we have the equality

$$\tau^{\geq a} \chi^* \mathcal{E}^{(\bullet)} = \chi^* \tau^{\geq a} \mathcal{E}^{(\bullet)}. \tag{8.1.2.6.2}$$

Lemma 8.1.2.7. *The canonical functors $\underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ are fully faithful.*

Proof. The proof is divided into the following two (non-symmetric) parts. i) With the notation 8.1.1.5.1, let $a \leq b$ be two integers and let $f^{(\bullet)}\colon \mathcal{E}^{(\bullet),\bullet} \to \mathcal{F}^{(\bullet),\bullet}$ be a morphism of $D(\mathcal{D}^{(\bullet)})$ such that $\mathcal{E}^{(\bullet)n} = 0$ for any $n < a$ (resp. $\mathcal{F}^{(\bullet)n} = 0$ for any $n > b$ and $\mathcal{E}^{(\bullet)n} = 0$ for any $n \notin [a, b]$). Suppose there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^* \mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. We denote by $\phi^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \tau^{\geq a} \mathcal{F}^{(\bullet)}$ the composition of the canonical morphism $\mathcal{F}^{(\bullet)} \to \tau^{\geq a} \mathcal{F}^{(\bullet)}$ with $f^{(\bullet)}$. Since the canonical morphism $\chi^* \mathcal{E}^{(\bullet)} \to \tau^{\geq a} \chi^* \mathcal{E}^{(\bullet)}$ is an isomorphism, we get the morphism $\psi^{(\bullet)}\colon \tau^{\geq a} \mathcal{F}^{(\bullet)} \to \chi^* \mathcal{E}^{(\bullet)}$ whose composition with the canonical morphism $\mathcal{F}^{(\bullet)} \to \tau^{\geq a} \mathcal{F}^{(\bullet)}$ is $g^{(\bullet)}$. We get $\psi^{(\bullet)} \circ \phi^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(\phi^{(\bullet)}) \circ \psi^{(\bullet)} = \theta_{\tau^{\leq a}\mathcal{F},\chi}$, i.e. $\phi^{(\bullet)}$ is an ind-isogeny. Hence, by using [Har66, I.3.3.(ii)], we get that the canonical functor $\underrightarrow{D}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ are fully faithful.

ii) Let $a \leq b$ be two integers and let $f^{(\bullet)}\colon \mathcal{E}^{(\bullet),\bullet} \to \mathcal{F}^{(\bullet),\bullet}$ be a morphism of $D(\mathcal{D}^{(\bullet)})$ such that $\mathcal{F}^{(\bullet)n} = 0$ for any $n \rangle b$ (resp. $\mathcal{E}^{(\bullet)n} = 0$ for any $n < a$ and $\mathcal{F}^{(\bullet)n} = 0$ for any $n \notin [a, b]$). Suppose there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^* \mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. We denote by $\phi^{(\bullet)}\colon \tau^{\leq b} \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ the composition of the canonical morphism $\tau^{\leq b} \mathcal{E}^{(\bullet)} \to \mathcal{E}^{(\bullet)}$ with $f^{(\bullet)}$. Since the canonical morphism $\tau^{\leq b} \chi^* \mathcal{F}^{(\bullet)} \to \chi^* \mathcal{F}^{(\bullet)}$ is an isomorphism, we get a (unique) morphism $\psi^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \tau^{\leq b} \chi^* \mathcal{E}^{(\bullet)} = \chi^* \tau^{\leq b} \mathcal{E}^{(\bullet)}$ whose composition with the canonical morphism $\tau^{\leq b} \chi^* \mathcal{E}^{(\bullet)} \to \chi^* \mathcal{E}^{(\bullet)}$ is $g^{(\bullet)}$. We compute $\psi^{(\bullet)} \circ \phi^{(\bullet)} = \theta_{\tau^{\leq a}\mathcal{E},\chi}$ and $\chi^*(\phi^{(\bullet)}) \circ \psi^{(\bullet)} = \theta_{\mathcal{F},\chi}$, i.e. $\phi^{(\bullet)}$ is an ind-isogeny. Hence, by using [Har66, I.3.3.(i)], we can prove that the canonical functor $\underrightarrow{D}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ are fully faithful. □

Remark 8.1.2.8. Beware that the images of the functors of 8.1.2.7 are not equal to the respective essential images, i.e. we do not have strictly full subcategories but only full subcategories.

8.1.2.9. Since the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}\colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$ sends ind-isogenies to isomorphisms we get the factorization $- \otimes_{\mathbb{Z}} \mathbb{Q}\colon \underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$.

8.1.3 Lim-isomorphisms

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.

Notation 8.1.3.1. Let $L(I)$ be the set of increasing maps $\lambda\colon I \to I$ such that $\lambda(i) \geq i$. The set $L(I)$ is closed under composition and is endowed with the following order: $\lambda \leq \mu$ if and only if $\lambda(i) \leq \mu(i)$ for an $i \in I$. The partially ordered set $L(I)$ is filtered.

8.1.3.2. We have the following notations and definitions.

(a) For any map $\lambda \in L(I)$, for any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)} = (\mathcal{E}^{(i)}, \alpha^{(j,i)})$ (also simply denoted by $\mathcal{E}$), we set

$$\lambda^*(\mathcal{E}^{(\bullet)}) := (\mathcal{E}^{(\lambda(i))}, \alpha^{(\lambda(j),\lambda(i))})_{i \leq j}.$$

We obtain the functor $\lambda^*\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\lambda^*\mathcal{D}^{(\bullet)})$ as follows: if $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ is a morphism of $\mathrm{Mod}(\mathcal{D}^{(\bullet)})$, then $\lambda^*(f^{(\bullet)})\colon \lambda^*(\mathcal{E}^{(\bullet)}) \to \lambda^*(\mathcal{F}^{(\bullet)})$ is the morphism of left $\lambda^*\mathcal{D}^{(\bullet)}$-modules such that $(\lambda^*(f^{(\bullet)}))^{(i)} = f^{(\lambda(i))}$. Since the functor $\lambda^*\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\lambda^*\mathcal{D}^{(\bullet)})$ is exact, then this induces the functor $\lambda^*\colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\lambda^*\mathcal{D}^{(\bullet)})$.

(b) When $\lambda_1, \lambda_2 \in L(I)$, we compute $\lambda_1^* \circ \lambda_2^* = (\lambda_2 \circ \lambda_1)^*$.

(c) Let $\chi \in M(I)$ and $\lambda \in L(I)$. The functors χ^* and λ^* defined respectively at 8.1.2.2.a and 8.1.3.2.a do not commute. However, we have $\chi \circ \lambda \in M(I)$ and for any $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$ we have the canonical equalities

$$\lambda^* \circ \chi^*(\mathcal{E}^{(\bullet)}) = (\chi \circ \lambda)^* \circ \lambda^*(\mathcal{E}^{(\bullet)}), \qquad \lambda^*(\theta_{\mathcal{E},\chi}) = \theta_{\lambda^*\mathcal{E},\chi\circ\lambda}. \tag{8.1.3.2.1}$$

This yields that the functor $\lambda^*\colon D^\sharp(\mathcal{D}^{(\bullet)}) \to D^\sharp(\lambda^*\mathcal{D}^{(\bullet)})$ the sends the ind-isogenies to the ind-isogenies. Hence, we get the δ-functor $\lambda^*\colon \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$.

(d) Let $\lambda_1, \lambda_2 \in L(I)$. When $\lambda_1 \leq \lambda_2$, for any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)} = (\mathcal{E}^{(i)}, \alpha^{(j,i)})$ we have the canonical morphism $\lambda_1^*(\mathcal{E}^{(\bullet)}) \to \lambda_2^*(\mathcal{E}^{(\bullet)})$ defined by the morphism $\alpha^{(\lambda_2(i),\lambda_1(i))}\colon \mathcal{E}^{(\lambda_1(i))} \to \mathcal{E}^{(\lambda_2(i))}$. The morphism $\lambda_1^*\mathcal{D}^{(\bullet)} \to \lambda_2^*\mathcal{D}^{(\bullet)}$ is in fact a ring homomorphism and the morphism $\lambda_1^*(\mathcal{E}^{(\bullet)}) \to \lambda_2^*(\mathcal{E}^{(\bullet)})$ is $\lambda_1^*\mathcal{D}^{(\bullet)}$-linear. This yields the morphisms of functors $\rho_{\lambda_1,\lambda_2}\colon D^\sharp(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\lambda_1^*\mathcal{D}^{(\bullet)})$ (resp. $\rho_{\lambda_2,\lambda_1}\colon \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\lambda_1^*\mathcal{D}^{(\bullet)})$) of the form $\lambda_1^* \to \lambda_2^*$. For any $\mathcal{E}^{(\bullet)} \in \operatorname{Ob} D^\sharp(\mathcal{D}^{(\bullet)}) = \operatorname{Ob} \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, we set $\rho_{\mathcal{E},\lambda_2,\lambda_1} := \rho_{\lambda_2,\lambda_1}(\mathcal{E}^{(\bullet)})\colon \lambda_1^*(\mathcal{E}^{(\bullet)}) \to \lambda_2^*(\mathcal{E}^{(\bullet)})$. When $\lambda_1 = \mathrm{id}$, we set $\rho_{\mathcal{E},\lambda_2} := \rho_{\mathcal{E},\lambda_2,\mathrm{id}}$.

(e) Let $\lambda_1, \lambda_2, \lambda_3 \in L(I)$ such that $\lambda_1 \leq \lambda_2$. For any $\mathcal{E}^{(\bullet)} \in \operatorname{Ob} \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, we have

$$\lambda_3^*(\rho_{\mathcal{E},\lambda_2,\lambda_1}) = \rho_{\mathcal{E},\lambda_2\circ\lambda_3,\lambda_1\circ\lambda_3}. \tag{8.1.3.2.2}$$

If furthermore $\lambda_2 \leq \lambda_3$ then

$$\rho_{\mathcal{E},\lambda_3,\lambda_2} \circ \rho_{\mathcal{E},\lambda_2,\lambda_1} = \rho_{\mathcal{E},\lambda_3,\lambda_1}. \tag{8.1.3.2.3}$$

(f) For any $\mathcal{E}^{(\bullet)} \in \operatorname{Ob} \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, it follows from 8.1.3.2.3 that we get a functor $\rho_{\mathcal{E}}\colon L(I) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ given by $\rho \mapsto \rho^*(\mathcal{E}^{(\bullet)})$, where $L(I)$ is the category associated with its structure of partially ordered set (see 7.1.2.1). This gives a meaning of the notion of functoriality with respect to λ.

(g) We denote by $\Lambda^\sharp(\mathcal{D}^{(\bullet)})$ the set of morphisms $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that there exist $\lambda \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{E},\lambda}$ and $\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \rho_{\mathcal{F},\lambda}$ in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. If no confusion is possible with respect to $\mathcal{D}^{(\bullet)}$, we simply write $\Lambda^\sharp := \Lambda^\sharp(\mathcal{D}^{(\bullet)})$. The morphisms belonging to Λ are called "lim-isomorphisms". For instance, the morphisms of the form $\rho_{\lambda_1,\lambda_2}$ are lim-isomorphisms (see 8.1.4.11).

Remark 8.1.3.3. Let $\lambda \in L(I)$. By definition, we have the equality of functors $\lambda^* = \underrightarrow{\lambda}_X^{-1}$ (see notation 8.1.1.2.1).

Lemma 8.1.3.4. *For any $\mathcal{G}^{(\bullet)} \in \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, let $I_{\mathcal{G}^{(\bullet)}}\colon \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \mathfrak{Ab}$ the cohomological functor with value in the category of abelian groups defined by setting for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$*

$$I_{\mathcal{G}^{(\bullet)}}(\mathcal{E}^{(\bullet)}) := \varinjlim_{\lambda \in L(I)} \operatorname{Hom}_{\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{G}^{(\bullet)}, \lambda^*\mathcal{E}^{(\bullet)}).$$

Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$. The morphism $f^{(\bullet)}$ is a lim-isomorphism if and only if $I_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is an isomorphism for any $\mathcal{G}^{(\bullet)} \in \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Proof. 1) Suppose that $f^{(\bullet)} \in \Lambda^\sharp$. Hence there exists $\lambda_0 \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \lambda_0^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{E},\lambda_0}$ and $\lambda_0^*(f^{(\bullet)}) \circ g^{(\bullet)} = \rho_{\mathcal{F},\lambda_0}$.

a) Let us check that $I_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is injective. Let $\lambda \in L(I)$ and $u^{(\bullet)}\colon \mathcal{G}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ be a morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that the image of $\lambda^*(f^{(\bullet)}) \circ u^{(\bullet)}$ in $I_{\mathcal{G}^{(\bullet)}}(\mathcal{F}^{(\bullet)})$ is 0. By increasing λ_0 and λ if necessary, we can suppose that $\lambda = \lambda_0$ and $\lambda^*(f^{(\bullet)}) \circ u^{(\bullet)} = 0$ in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We have $\rho_{\mathcal{E},\lambda\circ\lambda,\lambda} \circ u^{(\bullet)} = \lambda^*(\rho_{\mathcal{E},\lambda}) \circ u^{(\bullet)} = \lambda^*(g^{(\bullet)}) \circ \lambda^*(f^{(\bullet)}) \circ u^{(\bullet)} = 0$. Hence, the image of $u^{(\bullet)}$ in $I_{\mathcal{G}^{(\bullet)}}(\mathcal{F}^{(\bullet)})$ is 0.

b) Let us check the surjectivity of $I_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$. Let $\lambda \in L(I)$ and $v^{(\bullet)}\colon \mathcal{G}^{(\bullet)} \to \lambda^*\mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. By increasing λ_0 or λ if necessary, we can suppose that $\lambda = \lambda_0$. Set $u^{(\bullet)} := \lambda^*(g^{(\bullet)}) \circ v^{(\bullet)}\colon \mathcal{G}^{(\bullet)} \to (\lambda \circ \lambda)^*\mathcal{E}^{(\bullet)}$. Then, the image of $u^{(\bullet)}$ in $I_{\mathcal{G}^{(\bullet)}}(\mathcal{E}^{(\bullet)})$ is sent to the image of $v^{(\bullet)}$ in $I_{\mathcal{G}^{(\bullet)}}(\mathcal{F}^{(\bullet)})$ via $I_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$.

2) Suppose now that $I_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is an isomorphism for any $\mathcal{G}^{(\bullet)} \in D^\sharp(\mathcal{D}^{(\bullet)})$. Since $I_{\mathcal{F}^{(\bullet)}}(f^{(\bullet)})$ is in particular surjective, then there exists $\lambda \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such

that $\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \rho_{\mathcal{F},\lambda}$. The functor $I_{\mathcal{E}^{(\bullet)}}(f^{(\bullet)})$ sends the class of $g^{(\bullet)} \circ f^{(\bullet)}$ in $I_{\mathcal{G}^{(\bullet)}}(\mathcal{E}^{(\bullet)})$ to the class of $\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{F},\lambda} \circ f^{(\bullet)}$ in $I_{\mathcal{G}^{(\bullet)}}(\mathcal{F}^{(\bullet)})$, which is also the class of $f^{(\bullet)}$. Since $I_{\mathcal{E}^{(\bullet)}}(f^{(\bullet)})$ is in particular injective, the class of $g^{(\bullet)} \circ f^{(\bullet)}$ is equal to the class of the identity of $\mathcal{E}^{(\bullet)}$. By increasing λ if necessary, this yields $g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{E},\lambda}$. □

Notation 8.1.3.5. It follows from Lemma 8.1.3.4 and Proposition 7.4.1.7 that $\Lambda^\sharp$ is a saturated multiplicative system of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ compatible with its triangulated structure. By localizing $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ with respect to lim-isomorphisms we get a category denoted by $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Lemma 8.1.3.6. *The canonical functors $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ are fully faithful.*

Proof. We can copy the proof of 8.1.2.7. □

Remark 8.1.3.7. Beware that the images of the functors of 8.1.4.6 are not equal to the respective essential images, i.e. we do not have strictly full subcategories.

Definition 8.1.3.8. We have the following definitions.

(a) Given an increasing map $u\colon I \to I'$ of partially ordered sets, we say that u is an "L-equivalence" if there exists an increasing map $v\colon I' \to I$ such that $u \circ v \in L(I')$, $v \circ u \in L(I)$.

(b) We say a subset $J \subset I$ is "strictly cofinal" if the inclusion of J in I is an L-equivalence.

(c) We say I is "strictly filtered" if it is non-empty and if for all $i \in I$ the subset $I_{\geq i}$ of I is strictly cofinal.

Remark 8.1.3.9. (a) If I is strictly filtered then I is filtered, which justifies the terminology.

(b) Suppose I satisfies the following property: for any $i, j \in I$, the set $\{k \in I \text{ such that } i \leq k, j \leq k\}$ has a smallest element that we denote by $\sup\{i,j\}$ (i.e. I^o satisfies 7.1.2.15). Then I is strictly filtered. Indeed, for any $i \in I$ we get the map $\psi\colon I \to I_{\geq i}$ given by $j \mapsto \sup\{i,j\}$. Denoting by $\phi\colon I_{\geq i} \to I$ the inclusion, we get $\phi \circ \psi \in L(I)$, $\psi \circ \phi \in L(I_{\geq i})$.

(c) Let $u\colon I \to I'$ be an increasing map of partially ordered sets which is an L-equivalence. Then I is filtered if and only if I' is filtered.

8.1.3.10. Let $u\colon I \to I'$ be an increasing map of partially ordered sets. For any $\mathcal{F}'^{(\bullet)} \in X^{(I')}$, we have the canonical morphism

$$\varinjlim_{i \in I} \mathcal{E}^{(u(i))} \to \varinjlim_{i' \in I'} \mathcal{E}^{(i')} \tag{8.1.3.10.1}$$

induced by the functorial in $i \in I$ maps $\mathcal{E}^{(u(i))} \to \varinjlim_{i' \in I'} \mathcal{E}^{(i')}$.

Lemma 8.1.3.11. *When u is an L-equivalence then the morphism 8.1.3.10.1 is an isomorphism.*

Proof. a) Let $\lambda \in L(I)$. Since $\lambda(i) \geq i$ for any $i \in I$, we get the canonical morphism $\varinjlim_{i \in I} \mathcal{E}^{(i)} \to \varinjlim_{i \in I} \mathcal{E}^{(\lambda(i))}$ which is induced by the functorial in $i \in I$ maps $\mathcal{E}^{(i)} \to \mathcal{E}^{(\lambda(i))} \to \varinjlim_{i \in I} \mathcal{E}^{(\lambda(i))}$. By using the universal property of the inductive limits, we can check that this morphism $\varinjlim_{i \in I} \mathcal{E}^{(i)} \to \varinjlim_{i \in I} \mathcal{E}^{(\lambda(i))}$ is the inverse of 8.1.3.10.1 (when $u = \lambda$).

b) Let us check the general case. By hypothesis, there exists $v\colon I' \to I$ such that $u \circ v \in L(I')$, $v \circ u \in L(I)$. Since $u \circ v \in L(I')$, from the part a) of the proof, the canonical morphism $\varinjlim_{i' \in I'} \mathcal{E}^{(u \circ v(i'))} \to \varinjlim_{i' \in I'} \mathcal{E}^{(i')}$ of 8.1.3.10.1 is an isomorphism. We have the canonical morphism $\varinjlim_{i' \in I'} \mathcal{E}^{(u \circ v(i'))} \to \varinjlim_{i \in I} \mathcal{E}^{(u(i))}$ given by the functorial in $i' \in I'$ maps $\mathcal{E}^{(u \circ v(i'))} \to \varinjlim_{i \in I} \mathcal{E}^{(u(i))}$. This yields the morphism $\varinjlim_{i' \in I'} \mathcal{E}^{(i')} \xleftarrow{\sim} \varinjlim_{i' \in I'} \mathcal{E}^{(u \circ v(i'))} \to \varinjlim_{i \in I} \mathcal{E}^{(u(i))}$, which is the inverse of 8.1.3.10.1 (again use the universal property of the inductive limits). □

8.1.4 Lim-ind-isogenies

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.

Notation 8.1.4.1. We endow the set $L(I) \times M(I)$ with the order product, i.e. $(\lambda_1, \chi_1) \leq (\lambda_2, \chi_2)$ in $L(I) \times M(I)$ if and only if $\chi_1 \leq \chi_2$ in $M(I)$ and $\lambda_1 \leq \lambda_2$ in $L(I)$. Let $(\lambda_1, \chi_1) \leq (\lambda_2, \chi_2)$ in $L(I) \times M(I)$. Let $\mathcal{E}^{(\bullet)} \in D^{\sharp}(\mathcal{D}^{(\bullet)})$. We get the canonical morphism $\chi_1^* \lambda_1^* \mathcal{E}^{(\bullet)} \to \chi_2^* \lambda_2^* \mathcal{E}^{(\bullet)}$ which is given by

$$\sigma_{\mathcal{E},(\lambda_2,\chi_2),(\lambda_1,\chi_1)} := \theta_{\lambda_2^*\mathcal{E},(\chi_2,\chi_1)} \circ \chi_1^*(\rho_{\mathcal{E},(\lambda_2,\lambda_1)}) = \chi_2^*(\rho_{\mathcal{E},(\lambda_2,\lambda_1)}) \circ \theta_{\lambda_1^*\mathcal{E},(\chi_2,\chi_1)} \tag{8.1.4.1.1}$$

We set $\sigma_{\mathcal{E},(\lambda_2,\chi_2)} := \sigma_{\mathcal{E},(\lambda_2,\chi_2),(\mathrm{id},0)}$. For any $(\lambda_1, \chi_1) \leq (\lambda_2, \chi_2) \leq (\lambda_3, \chi_3)$ in $L(I) \times M(I)$, using 8.1.2.2.3 and 8.1.3.2.3 we get

$$\sigma_{\mathcal{E},(\lambda_3,\chi_3),(\lambda_1,\chi_1)} = \sigma_{\mathcal{E},(\lambda_3,\chi_3),(\lambda_2,\chi_2)} \circ \sigma_{\mathcal{E},(\lambda_2,\chi_2),(\lambda_1,\chi_1)}. \tag{8.1.4.1.2}$$

For any $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$, it follows from 8.1.4.1.2 that we get a functor $\sigma_{\mathcal{E}} \colon L(I) \times M(I) \to D(\mathcal{D}^{(\bullet)})$ given by $(\lambda, \chi) \mapsto \chi^* \rho^* (\mathcal{E}^{(\bullet)})$, where $L(I) \times M(I)$ is the category associated with its structure of partially ordered set (see 7.1.2.1). This gives a meaning of the notion of functoriality with respect to (ρ, χ).

8.1.4.2. Let (λ_1, χ_1) and (λ_2, χ_2) be two elements of $L(I) \times M(I)$. Using 8.1.2.2.a and 8.1.3.2.c, we get the equalities

$$\chi_2^* \lambda_2^* \chi_1^* \lambda_1^* \mathcal{E}^{(\bullet)} = \chi_2^* (\chi_1 \circ \lambda_2)^* \lambda_2^* \lambda_1^* \mathcal{E}^{(\bullet)} = (\chi_2 + \chi_1 \circ \lambda_2)^* (\lambda_1 \circ \lambda_2)^* \mathcal{E}^{(\bullet)} \tag{8.1.4.2.1}$$

Via 8.1.4.2.1, the canonical morphism $\sigma_{\chi_1^*\lambda_1^*\mathcal{E},(\lambda_2,\chi_2)} \colon \chi_1^* \lambda_1^* \mathcal{E}^{(\bullet)} \to \chi_2^* \lambda_2^* \chi_1^* \lambda_1^* \mathcal{E}^{(\bullet)}$ satisfies the equality

$$\sigma_{\chi_1^*\lambda_1^*\mathcal{E},(\lambda_2,\chi_2)} = \sigma_{\mathcal{E},(\lambda_1\circ\lambda_2,\chi_2+\chi_1\circ\lambda_2),(\lambda_1,\chi_1)}. \tag{8.1.4.2.2}$$

Moreover, via 8.1.4.2.1, the canonical morphism $\chi_2^* \lambda_2^* \sigma_{\mathcal{E},(\lambda_1,\chi_1)} \colon \chi_2^* \lambda_2^* \mathcal{E}^{(\bullet)} \to \chi_2^* \lambda_2^* \chi_1^* \lambda_1^* \mathcal{E}^{(\bullet)}$ satisfies the equality

$$\chi_2^* \lambda_2^* \sigma_{\mathcal{E},(\lambda_1,\chi_1)} = \sigma_{\mathcal{E},(\lambda_1\circ\lambda_2,\chi_2+\chi_1\circ\lambda_2),(\lambda_2,\chi_2)}. \tag{8.1.4.2.3}$$

In particular, we see that the morphisms of the form 8.1.4.1.1 are closed under the functors of the form χ^* or λ^*.

Definition 8.1.4.3. Let $S^{\sharp}(\mathcal{D}^{(\bullet)})$ be the collection of morphisms $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^{\sharp}(\mathcal{D}^{(\bullet)})$ such that there exist $\chi \in M(I)$, $\lambda \in L(I)$ and a morphism $g^{(\bullet)} \colon \mathcal{F}^{(\bullet)} \to \chi^* \lambda^* \mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \sigma_{\mathcal{E},(\lambda,\chi)}$ and $\chi^* \lambda^* (f^{(\bullet)}) \circ g^{(\bullet)} = \sigma_{\mathcal{F},(\lambda,\chi)}$. If no confusion is possible with respect to $\mathcal{D}^{(\bullet)}$, we simply write $S^{\sharp} := S^{\sharp}(\mathcal{D}^{(\bullet)})$. The morphisms of $S^{\sharp}$ are called "lim-ind-isogenies". For instance, the morphisms of the form 8.1.4.1.1 are lim-ind-isomorphisms (see 8.1.4.11).

Lemma 8.1.4.4. *For any $\mathcal{G}^{(\bullet)} \in D^{\sharp}(\mathcal{D}^{(\bullet)})$, let $J_{\mathcal{G}^{(\bullet)}} \colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to \mathfrak{Ab}$ be the cohomological functor with value in the category of abelian groups defined by setting for any $\mathcal{E}^{(\bullet)} \in D^{\sharp}(\mathcal{D}^{(\bullet)})$*

$$J_{\mathcal{G}^{(\bullet)}}(\mathcal{E}^{(\bullet)}) := \varinjlim_{\lambda \in L} \varinjlim_{\chi \in M} \mathrm{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{G}^{(\bullet)}, \chi^* \lambda^* \mathcal{E}^{(\bullet)}).$$

Let $f \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $D^{\sharp}(\mathcal{D}^{(\bullet)})$. The morphism $f^{(\bullet)}$ is a lim-ind-isogeny if and only if $J_{\mathcal{G}^{(\bullet)}}(f^{(\bullet)})$ is an isomorphism for any $\mathcal{G}^{(\bullet)} \in D^{\sharp}(\mathcal{D}^{(\bullet)})$.

Proof. By using 8.1.4.2, we can copy the proof of 8.1.3.4. □

8.1.4.5. It follows from 8.1.4.4 and 7.4.1.7 that $S^{\sharp}$ is a saturated multiplicative system of $D^{\sharp}(\mathcal{D}^{(\bullet)})$ compatible with the triangulated structure. We get the localised triangle category $(S^{\sharp})^{-1} D^{\sharp}(\mathcal{D}^{(\bullet)})$.

Lemma 8.1.4.6. *The canonical functors $(S^{\sharp})^{-1} D^{\sharp}(\mathcal{D}^{(\bullet)}) \to S^{-1} D(\mathcal{D}^{(\bullet)})$ and $(S^{\mathrm{b}})^{-1} D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to (S^{\sharp})^{-1} D^{\sharp}(\mathcal{D}^{(\bullet)})$ are fully faithful.*

Proof. We can copy the proof of 8.1.2.7. □

8.1.4.7. A morphism $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $(S^\sharp)^{-1}D^\sharp(\mathcal{D}^{(\bullet)})$ can be represented by a morphism $\phi^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$, i.e. by $\mathcal{E}^{(\bullet)} \overset{\phi^{(\bullet)}}{\to} \chi^*\lambda^* d\mathcal{F}^{(\bullet)} \overset{\sigma_{\mathcal{F},(\lambda,\chi)}}{\leftarrow} \mathcal{F}^{(\bullet)}$ for some $\chi \in M(I)$ and $\lambda \in L(I)$. Moreover, for any $\chi_1, \chi_2 \in M(I)$ and $\lambda_1, \lambda_2 \in L(I)$, two morphisms $\phi_1^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_1^*\lambda_1^*\mathcal{F}^{(\bullet)}$ and $\phi_2^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_2^*\lambda_2^*\mathcal{F}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ induce the same arrow $\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $(S^\sharp)^{-1}D^\sharp(\mathcal{D}^{(\bullet)})$ if and only if there exists $\chi \geq \chi_1, \chi_2$ and $\lambda \geq \lambda_1, \lambda_2$, such that both composite arrows $\mathcal{E}^{(\bullet)} \overset{\phi_1^{(\bullet)}}{\longrightarrow} \chi_1^*\lambda_1^*\mathcal{F}^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$ and $\mathcal{E}^{(\bullet)} \overset{\phi_2^{(\bullet)}}{\longrightarrow} \chi_2^*\lambda_2^*\mathcal{F}^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$ are equal. This yields the formula

$$\mathrm{Hom}_{S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) = \varinjlim_{(\lambda,\chi)\in L(I)\times M(I)} \mathrm{Hom}_{D^\sharp(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}). \quad (8.1.4.7.1)$$

Remark that for any $\chi \in M(I)$ and $\lambda \in L(I)$, $\mathcal{F}^{(\bullet)} \in D^\sharp(\mathcal{D}^{(\bullet)})$, since $\chi\circ\lambda \geq \chi$ in $M(I)$, we have the morphism $\sigma_{\mathcal{E},(\lambda,\chi\circ\lambda),(\lambda,\chi)}\colon \chi^*\lambda^*\mathcal{F}^{(\bullet)} \to (\chi\circ\lambda)^*\lambda^*\mathcal{F}^{(\bullet)} = \lambda^*\chi^*\mathcal{F}^{(\bullet)}$ (see the formula 8.1.3.2.c). This yields

$$\mathrm{Hom}_{S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) = \varinjlim_{(\lambda,\chi)\in L(I)\times M(I)} \mathrm{Hom}_{D^\sharp(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \lambda^*\chi^*\mathcal{F}^{(\bullet)}) \quad (8.1.4.7.2)$$

Lemma 8.1.4.8. *(a) Let $\chi_1 \in M(I)$ and $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_1^*\mathcal{F}^{(\bullet)}$ be a morphism of $D(\mathcal{D}^{(\bullet)})$. If there exist $\chi_2 \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi_2^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that we have the equalities in $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (not necessarily in $D(\mathcal{D}^{(\bullet)})$): $\chi_1^*(g^{(\bullet)}) \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi_2+\chi_1}$ and $\chi_2^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{E},\chi_2+\chi_1}$, then $f^{(\bullet)}$ is an ind-isogeny.*

(b) Let $\chi_1 \in M(I)$, $\lambda_1 \in L(I)$ and $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_1^*\lambda_1^*\mathcal{F}^{(\bullet)}$ be a morphism of $D(\mathcal{D}^{(\bullet)})$. If there exist $\chi_2 \in M(I)$, $\lambda_2 \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi_2^*\lambda_2^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that we have the equalities in $S^{-1}D(\mathcal{D}^{(\bullet)})$ (not necessarily in $D(\mathcal{D}^{(\bullet)})$): $\chi_1^*\lambda_1^*(g^{(\bullet)}) \circ f^{(\bullet)} = \sigma_{\mathcal{E},(\lambda_2\circ\lambda_1,\chi_1+\chi_2\circ\lambda_1)}$ and $\chi_2^*\lambda_2^*(f^{(\bullet)}) \circ g^{(\bullet)} = \sigma_{\mathcal{E},(\lambda_1\circ\lambda_2,\chi_2+\chi_1\circ\lambda_2)}$ (i.e. are the canonical morphisms following 8.1.4.2), then $f^{(\bullet)}$ is a lim-ind-isogeny.

Proof. Proof of (a). i) First we suppose that $\chi_1 = 0$, i.e. $\chi_1^* = \mathrm{id}$. By hypothesis, there exists $\chi \geq \chi_2$ such that $g^{(\bullet)} \circ f^{(\bullet)}$ composed with the canonical morphism $\chi_2^*\mathcal{E}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ is the canonical morphism in $D(\mathcal{D}^{(\bullet)})$ and such that $\chi_2^*(f^{(\bullet)}) \circ g^{(\bullet)}$ composed with the canonical morphism $\chi_2^*\mathcal{F}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ is the canonical morphism in $D(\mathcal{D}^{(\bullet)})$. Let us write $h^{(\bullet)}$ for the composition of $g^{(\bullet)}$ with the canonical morphism $\chi_2^*\mathcal{E}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$, we then check that $h^{(\bullet)} \circ f^{(\bullet)}$ and $\chi^*(f^{(\bullet)}) \circ h^{(\bullet)}$ are the canonical morphisms in $D(\mathcal{D}^{(\bullet)})$.

ii) Now we deal with the general case. Composing two consecutive arrows of the sequence $\mathcal{E}^{(\bullet)} \underset{f^{(\bullet)}}{\longrightarrow} \chi_1^*\mathcal{F}^{(\bullet)} \underset{\chi_1^*(g^{(\bullet)})}{\longrightarrow} \chi_1^*\chi_2^*\mathcal{E}^{(\bullet)} \underset{\chi_1^*\chi_2^*(f^{(\bullet)})}{\longrightarrow} \chi_1^*\chi_2^*(\chi_1^*\mathcal{F}^{(\bullet)})$ we obtain the canonical morphisms in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. This yields the result according to case a) treated above.

Proof of (b). We proceed in the analogius manner: first we treat the case where $\lambda_1 = \mathrm{id}$ and $\chi_1 = 0$, then we handle the general case (we replace χ_i^* by $\chi_i^*\lambda_i^*$ for $i = 1, 2$). □

Lemma 8.1.4.9. *We have the canonical equivalence of categories*

$$S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)}) \cong \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$$

which is the identity over the objects.

Proof. a) Since the canonical functor $D^\sharp(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ carries a lim-ind-isogeny to a lim-isomorphism, then the canonical functor $D^\sharp(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ factors canonically through the functor $S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

b) On the other hand, since an ind-isogeny is in particular a lim-ind-isogeny, then the canonical functor $D^\sharp(\mathcal{D}^{(\bullet)}) \to S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)})$ factors canonically through $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)})$. Let $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that there exist $\lambda_2 \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \lambda_2^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{E},\lambda}$ and $\lambda_2^*(f^{(\bullet)}) \circ g^{(\bullet)} = \rho_{\mathcal{F},\lambda}$. There exists $\chi_1 \in M(I)$ such that $f^{(\bullet)}$ is represented by a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$ of the form $\phi^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi_1^*\mathcal{F}^{(\bullet)}$. There exists $\chi_2 \in M(I)$ such that $g^{(\bullet)}$ is represented by a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$ of the form $\psi^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi_2^*\lambda_2^*\mathcal{E}^{(\bullet)}$. We check therefore that $\chi_1^*(\psi^{(\bullet)}) \circ \phi^{(\bullet)}$ and $\chi_2^*\lambda_2^*(\phi^{(\bullet)}) \circ \psi^{(\bullet)}$ are the canonical morphisms in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (and thus in $S^{-1}D^\sharp(\mathcal{D}^{(\bullet)})$). According to lemma 8.1.4.8.b, this implies that $\phi^{(\bullet)}$ is a lim-ind-isogeny. Hence, $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)})$ factors canonically through $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)})$ □

8.1.4.10. Let $\mathcal{D}'^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$.

A morphism $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ is an "ind-isogeny of $D^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$" if there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. We denote by $\Xi^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ the set of ind-isogenies of $D^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$. The localisation of $D^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ with respect to ind-isogenies is denoted by $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$.

We denote by $\Lambda^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ the set of morphisms $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ such that there exist $\lambda \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{E},\lambda}$ and $\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \rho_{\mathcal{F},\lambda}$ in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$. The morphisms belonging to $\Lambda^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ are called "lim-isomorphisms". By localising $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ with respect to lim-isomorphisms, we get the category $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$.

Let $S^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ be the collection of morphisms $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ such that there exist $\chi \in M(I)$, $\lambda \in L(I)$ and a morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \sigma_{\mathcal{E},(\lambda,\chi)}$ and $\chi^*\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \sigma_{\mathcal{F},(\lambda,\chi)}$. The morphisms of $S^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ are called "lim-ind-isogenies". Similarly to 8.1.4.9, we have the canonical equivalence of categories

$$S^{\sharp-1}D^\sharp(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}) \cong \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$$

which is the identity over the objects. We define also the abelian category $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$. Similarly to 8.1.5.3 we establish that the canonical functor $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ is fully faithful.

Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ is left or right solved by $\mathcal{R}^{(\bullet)}$ (see definition 4.6.3.2). Then we denote by $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ (resp. $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$) the strictly full subcategory of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ (resp. $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$) consisting of complexes isomorphic to a complex of $K(\mathcal{D}^{(\bullet)} \otimes_{\mathcal{R}^{(\bullet)}} \mathcal{D}'^{(\bullet)})$.

8.1.4.11. We have the following properties.

(a) Let $\phi^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $D^\sharp(\mathcal{D}^{(\bullet)})$. Since $\Xi^\sharp$ (resp. $S^\sharp$) is saturated, then using [Sta22, 05Q9] the morphism $\phi^{(\bullet)}$ is an isomorphism in $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$) if and only if $\phi^{(\bullet)}$ is an ind-isogeny (resp. a lim-ind-isogeny).

(b) It follows from (a) that the morphisms of the form 8.1.2.2.1 (resp. 8.1.4.1.1) are ind-isogenies (resp. lim-ind-isomorphisms). Similarly, the morphisms of the form $\rho_{\lambda_2,\lambda_1}$ (see 8.1.3.2) are lim-isomorphisms.

(c) The part (a) implies also that a complex $\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ is isomorphic to 0 in $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$) if and only if there exists $\chi \in M(I)$ (resp. $\chi \in M(I)$ and $\lambda \in L(I)$) such that canonical map $\mathcal{E}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ (resp. $\mathcal{E}^{(\bullet)} \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$) is the null morphism.

8.1.4.12. It follows from 8.1.4.7.1 and 8.1.4.9 that for any $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, we have the equality

$$\mathrm{Hom}_{\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) = \varinjlim_{\lambda \in L} \varinjlim_{\chi \in M} \mathrm{Hom}_{D^\sharp(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}). \qquad (8.1.4.12.1)$$

8.1.5 Point of view of a derived category of an abelian category

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.

8.1.5.1. We denote by $M(\mathcal{D}^{(\bullet)})$ the category of $\mathcal{D}^{(\bullet)}$-modules. Replacing D by M in 8.1.2.2, 8.1.3.2 and 8.1.4.5, we define the notion of ind-isogenies (resp. of lim-ind-isogenies) of $M(\mathcal{D}^{(\bullet)})$ and we denote by $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $S^{-1}M(\mathcal{D}^{(\bullet)})$) the localization by ind-isogenies (resp. by lim-ind-isogenies). We define also the multiplicative system of lim-isomorphisms of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and we denote by $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ the corresponding localized category.

8.1.5.2. Since the functors of the form λ^* and χ^* are exact for any $\lambda \in L(I)$ and $\chi \in M(I)$, the previous results in the subsections 8.1.2, 8.1.3, 8.1.4 are still valid by replacing complexes by modules, i.e. by replacing the letter D (for derived categories) by M (for modules). For instance, similarly to 8.1.4.9, we can check the following canonical equivalence of categories

$$S^{-1}M(\mathcal{D}^{(\bullet)}) \cong \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}). \tag{8.1.5.2.1}$$

This yields, for any $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$

$$\mathrm{Hom}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) = \varinjlim_{\lambda \in L(I)} \varinjlim_{\chi \in M(I)} \mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}). \tag{8.1.5.2.2}$$

Lemma 8.1.5.3. *The canonical functors $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ are fully faithful.*

Proof. Since the check of the full faithfulness of the first functor is similar, let us prove it for the second one. This comes from the fact that the application $\mathrm{Hom}_{M(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \mathrm{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)})$ is bijective for any $\lambda \in L(I)$ and $\chi \in M(I)$ and that we have the equalities 8.1.4.12.1 and 8.1.5.2.2. □

8.1.5.4 (Serre subcategories)**.** Let us collect few facts on Serre subcategories and localisations. Let $\mathfrak{A}$ be an abelian category.

Let S be a multiplicative system of $\mathfrak{A}$. Then $S^{-1}\mathfrak{A}$ is an abelian category and the localisation functor $Q_S\colon \mathfrak{A} \to S^{-1}\mathfrak{A}$ is exact (see [Sta22, 05QG]). It follows from [Sta22, 02MQ] that $\mathfrak{B}(S) := \mathrm{Ker}\, Q_S$, where $\mathrm{Ker}\, Q_S$ is the full subcategory of objects X of $\mathfrak{A}$ such that $Q_S(X) = 0$, forms a Serre subcategory of $\mathfrak{A}$.

Let $\mathfrak{B} \subset \mathfrak{A}$ be a Serre subcategory (see [Sta22, 02MO]). Consider the set of arrows of $\mathfrak{A}$ defined by the following formula

$$S(\mathfrak{B}) := \{f \in \mathit{Arrows}(\mathfrak{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathit{Ob}(\mathfrak{B})\}.$$

Then $S(\mathfrak{B})$ is a saturated multiplicative system (see the proof of [Sta22, 02MS]) such that $\mathfrak{B}(S(\mathfrak{B})) = \mathfrak{B}$ (this is a consequence of [Sta22, 06XK]). We set $\mathfrak{A}/\mathfrak{B} := (S(\mathfrak{B}))^{-1}\mathfrak{A}$. Following [Sta22, 02MS], the category $\mathfrak{A}/\mathfrak{B}$ and the localisation functor $F\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{B}$ are characterized by the following universal property: for any exact functor $G\colon \mathfrak{A} \to \mathfrak{C}$ such that $\mathfrak{B} \subset \mathrm{Ker}(G)$, there exists a factorization $G = H \circ F$ for a unique exact functor $H\colon \mathfrak{A}/\mathfrak{B} \to \mathfrak{C}$.

Let S be a multiplicative system of $\mathfrak{A}$. Then we easily see that $S(\mathfrak{B}(S))$ is equal to $\widehat{S} = \{f \in \mathit{Arrows}(\mathfrak{A}) \mid Q_S(f)$ is an isomorphism$\}$, which is also the smallest saturated multiplicative system containing S (see [Sta22, 05Q9]). Hence, $S \mapsto \mathfrak{B}(S)$ and $\mathfrak{B} \mapsto S(\mathfrak{B})$ are reciprocal bijections of each other between the set of saturated multiplicative systems of $\mathfrak{A}$ and Serre subcategories of $\mathfrak{A}$.

Lemma 8.1.5.5. *Let $N(\mathcal{D}^{(\bullet)})$ be the full subcategory of $M(\mathcal{D}^{(\bullet)})$ consisting of modules which are null in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.*

1. *The category $N(\mathcal{D}^{(\bullet)})$ is a Serre subcategory of $M(\mathcal{D}^{(\bullet)})$.*
2. *The multiplicative system associated to the Serre subcategory $N(\mathcal{D}^{(\bullet)})$ is equal to S. In particular, S is saturated.*
3. *We have $M(\mathcal{D}^{(\bullet)})/N(\mathcal{D}^{(\bullet)}) = S^{-1}M(\mathcal{D}^{(\bullet)}) \cong \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. In particular, $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is an abelian category.*

Proof. Apply 8.1.5.4.

□

8.1.5.6. For all $n \in \mathbb{Z}$, we have the n-th cohomology space functor $\mathcal{H}^n\colon D(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^{(\bullet)})$ which is defined for $\mathcal{E}^{(\bullet)} = (\mathcal{E}^{(i)}, \alpha^{(j,i)}) \in D(\mathcal{D}^{(\bullet)})$ by $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) = (\mathcal{H}^n(\mathcal{E}^{(i)}), \mathcal{H}^n(\alpha^{(j,i)}))$. We have the canonical isomorphism $\mathcal{H}^n\chi^*\lambda^*(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \chi^*\lambda^*\mathcal{H}^n(\mathcal{E}^{(\bullet)})$ which is right arrow of the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^n(\mathcal{E}^{(\bullet)}) & \xrightarrow{\mathcal{H}^n(\sigma_{\mathcal{E},(\lambda,\chi)})} & \mathcal{H}^n\chi^*\lambda^*(\mathcal{E}^{(\bullet)}) \\ \| & & \downarrow\wr \\ \mathcal{H}^n(\mathcal{E}^{(\bullet)}) & \xrightarrow{\sigma_{\mathcal{H}^n\mathcal{E},(\lambda,\chi)}} & \chi^*\lambda^*\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \end{array}$$

where the arrows of the form σ is the canonical arrow defined at 8.1.4.1.1. We deduce from this that $\mathcal{H}^n$ carries lim-ind-isogenies to lim-ind-isogenies. Then we get the functor $\mathcal{H}^n$ making commutative (up to canonical equivalence) the diagram

$$\begin{array}{ccc} \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \overset{\mathcal{H}^n}{\dashrightarrow} & \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \\ \uparrow & & \uparrow \\ D(\mathcal{D}^{(\bullet)}) & \xrightarrow{\mathcal{H}^n} & M(\mathcal{D}^{(\bullet)}), \end{array} \tag{8.1.5.6.1}$$

where the vertical arrows are the localization functors.

Likewise, replacing lim-ind-isogenies by ind-isogenies we get the functor $\mathcal{H}^n$ making commutative (up to canonical equivalence) the diagram

$$\begin{array}{ccc} \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \overset{\mathcal{H}^n}{\dashrightarrow} & \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \\ \uparrow & & \uparrow \\ D(\mathcal{D}^{(\bullet)}) & \xrightarrow{\mathcal{H}^n} & M(\mathcal{D}^{(\bullet)}), \end{array} \tag{8.1.5.6.2}$$

where the vertical arrows are the localization functors.

Lemma 8.1.5.7. *The functors* $\mathcal{H}^n : \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ *and* $\mathcal{H}^n : \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ *defined at 8.1.5.6 are cohomological functors.*

Proof. Let us treat the first functor. By construction (see the proof of [Sta22, 05R6 Proposition 13.5.5]), a distinguished triangle of $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ to the image of a distinguished triangle of $K(\mathcal{D}^{(\bullet)})$ by the canonical localization functor $K(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Since $\mathcal{H}^n : K(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^{(\bullet)})$ is a cohomological functor, since the localization functor $M(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is an exact functor between abelian categories (it follows from the properties of localisations by a subcategory of Serre and of 8.1.5.5), this implies the result of the first functor. Likewise, we check the second one. ☐

8.1.5.8. Denote by $\underrightarrow{LD}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{D}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$) the strictly full sub-category of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$) consisting of complexes $\mathcal{E}^{(\bullet)}$ such that for any integer $n \neq 0$ we have $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Remark 8.1.5.9. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ for any $n \in \mathbb{Z}$. Then it seems false that $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} 0$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. When $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, this property becomes true (see 8.1.5.10), which explains why we have defined $\underrightarrow{LD}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ as a the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and not $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ in 8.1.5.8. We have the same remark concerning $\underrightarrow{D}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Lemma 8.1.5.10. *The canonical functors*

$$\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}), \qquad \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \tag{8.1.5.10.1}$$

is an equivalence of categories with respective quasi-inverse $H^0 : \underrightarrow{LD}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, $H^0 : \underrightarrow{D}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Proof. Since the other case is similar, let us treat the first one. It follows from 8.1.5.3 that the functor 8.1.5.10.1 is fully faithful. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^0_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. It remains to check there exists in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ an isomorphism of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} H^0(\mathcal{E}^{(\bullet)})$. Since $\mathcal{E}^{(\bullet)}$ is an object of $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$, there exists an integer $N \geq 1$ large enough such that, for any $j \notin [-N, N] \cap \mathbb{Z}$ we have $\mathcal{E}^{j(\bullet)} = 0$ in $M(\mathcal{D}^{(\bullet)})$. For any integer $n \in \mathbb{Z}$, we denote by $\tau^{\geq n} : D(\mathcal{D}^{(\bullet)}) \to D^{\geq n}(\mathcal{D}^{(\bullet)})$, $\tau^{\leq n} : D(\mathcal{D}^{(\bullet)}) \to D^{\leq n}(\mathcal{D}^{(\bullet)})$. the cohomological canonical truncation functors (see 8.1.2.6)

1) *Let us check* $\tau^{\leq -1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ *in* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

a) Set $\mathcal{F}^{(\bullet)} := \tau^{\leq -1}(\mathcal{E}^{(\bullet)})$. We have therefore the isomorphism $\tau^{\leq -N-1}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \tau^{\leq -N-1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $D(\mathcal{D}^{(\bullet)})$. By using the exact triangle $\tau^{\leq -N-1}(\mathcal{F}^{(\bullet)}) \to \mathcal{F}^{(\bullet)} \to \tau^{\geq -N}(\mathcal{F}^{(\bullet)}) \to +1$ of $D(\mathcal{D}^{(\bullet)})$, this implies that the canonical morphism $\mathcal{F}^{(\bullet)} \to \tau^{\geq -N}(\mathcal{F}^{(\bullet)})$ is an isomorphism of $D(\mathcal{D}^{(\bullet)})$.

b) For any $j \in [-N, -1] \cap \mathbb{Z}$, we have the distinguished triangle in $D(\mathcal{D}^{(\bullet)})$ (and then in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$):

$$\mathcal{H}^j(\mathcal{F}^{(\bullet)}) \to \tau^{\geq j}(\mathcal{F}^{(\bullet)}) \to \tau^{\geq j+1}(\mathcal{F}^{(\bullet)}) \to +1.$$

Since, for any $j \in [-N, -1] \cap \mathbb{Z}$, $\mathcal{H}^j(\mathcal{F}^{(\bullet)}) = \mathcal{H}^j(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, this yields that the arrow $\tau^{\geq j}(\mathcal{F}^{(\bullet)}) \to \tau^{\geq j+1}(\mathcal{F}^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Hence we get $\tau^{\geq -N}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \tau^{\geq 0}(\mathcal{F}^{(\bullet)})$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

c) Since we have in $D(\mathcal{D}^{(\bullet)})$ the isomorphism $\tau^{\geq 0}(\mathcal{F}^{(\bullet)}) = \tau^{\geq 0}(\tau^{\leq -1}(\mathcal{E}^{(\bullet)})) \xrightarrow{\sim} 0$, it follows from the steps a) and b) that $\mathcal{F}^{(\bullet)} \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

2) *Let us now prove that the canonical morphism $H^0(\mathcal{E}^{(\bullet)}) \to \tau^{\geq 0}(\mathcal{E}^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.*

Set $\mathcal{G}^{(\bullet)} := \tau^{\geq 0}(\mathcal{E}^{(\bullet)})$. Similarly to the step 1.b), we check that the canonical morphism $\tau^{\geq 1}(\mathcal{E}^{(\bullet)}) = \tau^{\geq 1}(\mathcal{G}^{(\bullet)}) \to \tau^{\geq N+1}(\mathcal{G}^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Since $\tau^{\geq N+1}(\mathcal{G}^{(\bullet)}) \xrightarrow{\sim} \tau^{\geq N+1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $D(\mathcal{D}^{(\bullet)})$, this yields that $\tau^{\geq 1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. By using the distinguished triangle of $D(\mathcal{D}^{(\bullet)})$ (see [Sta22, 08J5 Remark 13.12.4])

$$H^0(\mathcal{E}^{(\bullet)}) \to \tau^{\geq 0}(\mathcal{E}^{(\bullet)}) \to \tau^{\geq 1}(\mathcal{E}^{(\bullet)}) \to +1,$$

this implies the result.

3) Via 1) and 2), we conclude by using the exact sequence $0 \to \tau^{\leq -1}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to \tau^{\geq 0}(\mathcal{E}^{(\bullet)}) \to 0$ in $C(\mathcal{D}^{(\bullet)})$.

□

Corollary 8.1.5.11. *Let $\phi\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$). The morphism ϕ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$) if and only if, for any integer $n \in \mathbb{Z}$, the morphism $\mathcal{H}^n(\phi)\colon \mathcal{H}^n(\mathcal{E}^{(\bullet)}) \to \mathcal{H}^n(\mathcal{F}^{(\bullet)})$ is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$).*

Proof. Let us treat the non-respective case. There exists a distinguished triangle in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \underset{\phi}{\longrightarrow} \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}^{(\bullet)}[1]$. Following the properties concerning the triangulated categories, ϕ is an isomorphism if and only if $\mathcal{G}^{(\bullet)} \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Following 8.1.5.10, this is equivalent to saying that, for any integer $n \in \mathbb{Z}$, we have $\mathcal{H}^n(\mathcal{G}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The lemma 8.1.5.7 allows us to conclude the non-respective case. Likewise we check the respective one. □

8.1.5.12. Let us denote by $D^{\sharp}_{N(\mathcal{D}^{(\bullet)})}(\mathcal{D}^{(\bullet)})$ the saturated (in the sense of 7.4.1.1) full triangulated subcategory of $D^{\sharp}(\mathcal{D}^{(\bullet)})$ consisting of complexes whose cohomology spaces are in $N(\mathcal{D}^{(\bullet)})$ i.e. $D^{\sharp}_{N(\mathcal{D}^{(\bullet)})}(\mathcal{D}^{(\bullet)})$ is the kernel of the canonical functor $D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ induced by the localization functor $M(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

With notation 7.4.1.3, let $S^{\sharp}_{Nqi} := S(D^{\sharp}_{N(\mathcal{D}^{(\bullet)})}(\mathcal{D}^{(\bullet)}))$ be the saturated multiplicative system compatible with the triangulation of $D^{\sharp}(\mathcal{D}^{(\bullet)})$ which corresponds to $D^{\sharp}_{N(\mathcal{D}^{(\bullet)})}(\mathcal{D}^{(\bullet)})$. We can deduce from the theorem [Miy91, 3.2] that the canonical functor $D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ induces canonically the equivalence of categories

$$D^{\sharp}(\mathcal{D}^{(\bullet)})/D^{\sharp}_{N(\mathcal{D}^{(\bullet)})}(\mathcal{D}^{(\bullet)}) := S^{\sharp -1}_{Nqi} D^{\sharp}(\mathcal{D}^{(\bullet)}) \cong D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})). \qquad (8.1.5.12.1)$$

By definition, a morphism $f^{(\bullet)}$ of $D^{\sharp}(\mathcal{D}^{(\bullet)})$ belongs to $S^{\sharp}_{Nqi}$ if and only if, for all distinguished triangle in $D^{\sharp}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{f^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}^{(\bullet)}[1]$, for all integer $n \in \mathbb{Z}$, we have $\mathcal{H}^n(\mathcal{G}^{(\bullet)}) \in N(\mathcal{D}^{(\bullet)})$).

Lemma 8.1.5.13. *With notation 8.1.5.12, we have $S^{\mathrm{b}} = S^{\mathrm{b}}_{Nqi}$. For $\sharp \in \{+, -, \mathrm{b}, \emptyset\}$, we have $S^{\sharp} \subset S^{\sharp}_{Nqi}$.*

Proof. 1) First we show $S^{\mathrm{b}}_{Nqi} \subset S^{\mathrm{b}}$. Take $f^{(\bullet)} \in S^{\mathrm{b}}_{Nqi}$ and a distinguished triangle in $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{f^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}^{(\bullet)}[1]$. By definition, for all integer $n \in \mathbb{Z}$, $\mathcal{H}^n(\mathcal{G}^{(\bullet)}) \in N(\mathcal{D}^{(\bullet)})$. That is, for all integer $n \in \mathbb{Z}$, we have $\mathcal{H}^n(\mathcal{G}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. By 8.1.5.10, this implies that $\mathcal{G}^{(\bullet)} \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. According to the properties of triangulated categories, $f^{(\bullet)}$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. By 8.1.4.11, we get that $f^{(\bullet)} \in S^{\mathrm{b}}$.

2) Next we show that $S^{\sharp} \subset S^{\sharp}_{Nqi}$. Let $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $D^{\sharp}(\mathcal{D}^{(\bullet)})$. Since the cohomology space functor $H^0 \colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is a cohomological functor, then we get at a long exact sequence in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ from the distinguished triangle in $D^{\sharp}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{f^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}^{(\bullet)}[1]$. Looking at this long exact sequence, we can check that $f^{(\bullet)} \in S^{\sharp}_{Nqi}$ if and only if, for all integer $n \in \mathbb{Z}$, $\mathcal{H}^n(f^{(\bullet)})$ is an isomorphism in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (which is an abelian category). But, if $f^{(\bullet)} \in S^{\sharp}$, then its image in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is an isomorphism. As the functor $\mathcal{H}^n \colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ factors through $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, we deduce the required inclusion $S^{\sharp} \subset S^{\sharp}_{Nqi}$. □

Proposition 8.1.5.14. *For $\sharp \in \{+, -, \mathrm{b}, \emptyset\}$, the canonical functor $D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ of triangulated categories induced by the functor of abelian categories $M(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ induces the morphism of triangulated categories*

$$\mathfrak{e} \colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \tag{8.1.5.14.1}$$

making commutative the diagram

$$\begin{array}{ccccc} & & S^{\sharp -1}_{Nqi} D^{\sharp}(\mathcal{D}^{(\bullet)}) & \xrightarrow[8.1.5.12.1]{\cong} & D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \\ & \nearrow^{Q_{S^{\sharp}_{Nqi}}} & \uparrow & & \uparrow \mathfrak{e} \\ D^{\sharp}(\mathcal{D}^{(\bullet)}) & \xrightarrow{Q_{S^{\sharp}}} & S^{\sharp -1} D^{\sharp}(\mathcal{D}^{(\bullet)}) & \xrightarrow[8.1.4.9]{\cong} & \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \end{array} \tag{8.1.5.14.2}$$

When $\sharp = \mathrm{b}$, the morphism $\mathfrak{e}$ is an equivalence of categories.

Proof. The left vertical arrow comes from the inclusion $S^{\sharp} \subset S^{\sharp}_{Nqi}$ (see 8.1.5.13). When $\sharp = \mathrm{b}$, since this inclusion becomes an equality, both vertical arrow are equivalences of categories. □

8.1.5.15. The morphism 8.1.5.14.1 commutes with cohomological functors, i.e. we have for any $n \in \mathbb{N}$ the commutative diagram

$$\begin{array}{ccccc} D^{\sharp}(\mathcal{D}^{(\bullet)}) & \longrightarrow & \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\mathfrak{e}} & D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \\ \downarrow H^n & & \vdots\, H^n & & \downarrow H^n \\ M(\mathcal{D}^{(\bullet)}) & \longrightarrow & \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & = & \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \end{array} \tag{8.1.5.15.1}$$

where the middle vertical arrow is the one making commutative by definition the left square (see 8.1.5.6). Indeed, since the canonical functor $M(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is exact, the outer of the large rectangle is commutative.

8.1.5.16. We have the commutative diagram up to canonical isomorphism

$$\begin{array}{ccc} \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{8.1.5.3} & \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \\ & \searrow & \downarrow \mathfrak{e}\ 8.1.5.14.1 \\ & & D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})). \end{array} \tag{8.1.5.16.1}$$

Indeed, by using the universal property of the localisation functor, we reduce to check it after applying the functor $M(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, which is easy.

8.2 Homomorphism bifunctor over $LD_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.

8.2.1 Homomorphism bifunctors of $\mathcal{D}^{(\bullet)}$-modules

8.2.1.1. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}$ be two left $\mathcal{D}^{(\bullet)}$-modules.

(a) We have the abelian sheaf on $X^{(I)}$ that we will denote by $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})$ which is characterized by the property: for any object $K^{(\bullet)}$ of $X^{(I)}$,

$$\mathrm{Hom}_{X^{(I)}}(K^{(\bullet)}, \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) = \mathrm{Hom}_{\mathcal{D}^{(\bullet)}|_{K^{(\bullet)}}}(\mathcal{E}^{(\bullet)}|_{K^{(\bullet)}}, \mathcal{F}^{(\bullet)}|_{K^{(\bullet)}})).$$

With notation 4.6.2.7.1, in the case where $K^{(\bullet)}$ is a final object, we have

$$\Gamma(X^{(I)}, \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) = \mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})). \tag{8.2.1.1.1}$$

Moreover, for any $(i, U) \in (I^{\circ})^{\natural} \times X_{Zar}$, we have

$$\mathrm{Hom}_{X^{(I)}}((i, U), \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) = \mathrm{Hom}_{\mathcal{D}^{(\bullet)}|_{(i,U)}}(\mathcal{E}^{(\bullet)}|_{(i,U)}, \mathcal{F}^{(\bullet)}|_{(i,U)})). \tag{8.2.1.1.2}$$

Via 8.1.1.5.3, a morphism of this latter abelian group is therefore a compatible family of $\mathcal{D}^{(j)}|_U$-linear homomorphisms $\mathcal{E}^{(j)}|_U \to \mathcal{F}^{(j)}|_U$ for any $j \geq i$.

(b) We define the abelian sheaf $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})$ on X by setting, for any open set U of X,

$$\Gamma(U, \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) = \mathrm{Hom}_{\mathcal{D}^{(\bullet)}|_U}(\mathcal{E}^{(\bullet)}|_U, \mathcal{F}^{(\bullet)}|_U). \tag{8.2.1.1.3}$$

8.2.1.2. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}$ be two left $\mathcal{D}^{(\bullet)}$-modules. With notation 8.1.1.2.5 and of 8.1.1.5.(d), with the description of 8.2.1.1.2, we have the equality of abelian sheaves on X:

$$\underset{\rightarrow X}{i}^{-1}(\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) = \mathcal{H}om_{\mathcal{D}^{(\bullet)}|_i}(\mathcal{E}^{(\bullet)}|_i, \mathcal{F}^{(\bullet)}|_i), \tag{8.2.1.2.1}$$

i.e. $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})$ is the projective system $(\mathcal{H}om_{\mathcal{D}^{(\bullet)}|_i}(\mathcal{E}^{(\bullet)}|_i, \mathcal{F}^{(\bullet)}|_i))_{i\in I}$ whose transition maps are the forgetful maps. This yields the isomorphism of abelian sheaves on X:

$$\underset{\rightarrow X*}{l}^{(I)} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}), \tag{8.2.1.2.2}$$

where following notation 8.1.1.2.3 we have $\underset{\rightarrow X*}{l}^{(I)} = \varprojlim_{i\in I^{\circ}}$.

8.2.1.3. Let $\mathcal{E}^{(\bullet)}$ be a $\mathcal{D}^{(\bullet)}$-module. Let $\mathcal{I}^{(\bullet)}$ be an injective $\mathcal{D}^{(\bullet)}$-module. Then it follows from [SGA4.2, V.4.10.2)] that $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)})$ is a flasque abelian sheaf on $X_{\bullet}$. Since the functor $\underset{\rightarrow X*}{l}^{(I)}$ preserves the flasqueness (see [SGA4.2, V.4.9)]) then we deduce from 8.2.1.2.2 that $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)})$ is a flasque abelian sheaf on X.

Notation 8.2.1.4. Denote by $\mathrm{Mod}(\mathbb{Z}_X)$ the abelian category of sheaves of abelian groups on X. Let $\mathcal{E}^{(\bullet),\bullet}$, $\mathcal{F}^{(\bullet),\bullet} \in K(\mathcal{D}^{(\bullet)})$ (exceptionally, we indicate the second $\bullet$ to clarify the following notations). With notation 8.2.1.1.b, we define the bifunctor

$$\mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,-) \colon K(\mathcal{D}^{(\bullet)}) \times K(\mathcal{D}^{(\bullet)}) \to K(\mathbb{Z}_X)$$

whose nth term for any integer $n \in \mathbb{Z}$ is defined by setting:

$$\mathcal{H}om^{n}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{F}^{(\bullet),\bullet}) := \prod_{p\in\mathbb{Z}} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),p},\ \mathcal{F}^{(\bullet),\ p+n}) \tag{8.2.1.4.1}$$

and the transition morphisms are given by the formula $d = d_{\mathcal{E}} + (-1)^n d_{\mathcal{F}}$.

Notation 8.2.1.5. We denote by $\mathbb{Z}_{X^{(\bullet)}}$ or $\mathbb{Z}_X^{(\bullet)}$ the constant inductive system of rings of X indexed by I with value $\mathbb{Z}_X$. Denote by $\mathrm{Mod}(\mathbb{Z})$ (resp. $\mathrm{Mod}(\mathbb{Z}_{X^{(\bullet)}})$) the category of abelian groups (resp. of abelian groups on $X^{(I)}$). Replacing "$\mathcal{H}om$" by "Hom" (resp. "$\mathscr{H}om$"), we define similarly to the construction of 8.2.1.4 the bifunctor (which are in fact the classical bifunctors of homomorphisms of the abelian category $\mathrm{Mod}(\mathcal{D}^{(\bullet)})$):

$$\mathrm{Hom}^\bullet_{\mathcal{D}^{(\bullet)}}(-,-)\colon K(\mathcal{D}^{(\bullet)}) \times K(\mathcal{D}^{(\bullet)}) \to K(\mathbb{Z}),$$
$$\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(-,-)\colon K(\mathcal{D}^{(\bullet)}) \times K(\mathcal{D}^{(\bullet)}) \to K(\mathbb{Z}_{X^{(\bullet)}}).$$

By construction (see respectively 8.2.1.1.1 and 8.2.1.1.3), for any $\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$, for any $(i,U) \in (I^\circ)^\natural \times X_{Zar}$, we have we have the canonical isomorphism of bifunctors

$$\Gamma((i,U),\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}|_{(i,U)}}(\mathcal{E}^{(\bullet)}|_{(i,U)},\ \mathcal{F}^{(\bullet)}|_{(i,U)}) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{D}^{(\bullet)}|_{U^{(\bullet)}}}(\mathcal{E}^{(\bullet)}|_{U^{(\bullet)}},\ \mathcal{F}^{(\bullet)}|_{U^{(\bullet)}}), \tag{8.2.1.5.1}$$
$$\Gamma(U,\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{D}^{(\bullet)}|_U}(\mathcal{E}^{(\bullet)}|_U,\ \mathcal{F}^{(\bullet)}|_U). \tag{8.2.1.5.2}$$

8.2.1.6. We recall the following facts.

(a) An object $\mathcal{I}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$ is K-injective if for every acyclic complex $\mathcal{M}^{(\bullet)}$ we have $\mathrm{Hom}_{K(\mathcal{D}^{(\bullet)})}(\mathcal{M}^{(\bullet)},\mathcal{I}^{(\bullet)}) = 0$ (see [Sta22, 070H]).

(b) Products in the derived category $D(\mathcal{D}^{(\bullet)})$ of K-injective objects are obtained by taking termwise products and products of K-injective complexes are K-injective (see [Sta22, 0BK6]).

8.2.1.7. Let U be an open subset of X and $j\colon U \hookrightarrow X$ is the induced open immersion.

(a) Let $\mathcal{I}^{(\bullet)}$ be an injective left $\mathcal{D}^{(\bullet)}$-module. Since $\underset{\to I!}{j}$ is exact (see notation 8.1.1.5), then $\mathcal{I}^{(\bullet)}|U$ is an injective $\mathcal{D}^{(\bullet)}|U$-module. This yields that the functor $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,\mathcal{I}^{(\bullet)})\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathbb{Z}_X)$ is exact.

(b) Since $j^{-1}_{(i,U)} = j^*_{(i,U)}\colon \mathrm{Mod}({}^*\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}({}^*\mathcal{D}^{(\bullet)}|_{(i,U)})$ has an exact left adjoint functor (see 7.1.3.3.2), then it follows from [Sta22, 08BJ] that this functor $j^{-1}_{(i,U)}$ preserves the K-injectivity.

(c) Since $\underset{\to I!}{j}$ is exact (see notation 8.1.1.5), it follows from [Sta22, 08BJ] that for any K-injective complex $\mathcal{I}^{(\bullet)}$ of $K(\mathcal{D}^{(\bullet)})$, the complex $\mathcal{I}^{(\bullet)}|U$ is K-injective as object of $K(\mathcal{D}^{(\bullet)}|U)$.

8.2.1.8. Let $i \in I$, U be an open subset of X and $j\colon U \hookrightarrow X$ be the induced open immersion. It follows by construction from respectively 8.2.1.1.2 and 8.2.1.1.1 that for any $\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$, we have

$$H^n(\Gamma((i,U),\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}))) = \mathrm{Hom}_{K(\mathcal{D}^{(\bullet)}|_{(i,U)})}(\mathcal{E}^{(\bullet)}|_{(i,U)},\mathcal{F}^{(\bullet)}|_{(i,U)}[n])), \tag{8.2.1.8.1}$$
$$H^n(\Gamma(X^{(\bullet)},\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}))) = \mathrm{Hom}_{K(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}[n])). \tag{8.2.1.8.2}$$

Similarly, we check that for any $\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$, we have

$$H^n(\Gamma(U,\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)})) = \mathrm{Hom}_{K(\mathcal{D}^{(\bullet)}|_U)}(\mathcal{E}^{(\bullet)}|U,\mathcal{F}^{(\bullet)}|_U[n])). \tag{8.2.1.8.3}$$

8.2.2 Derived homomorphism bifunctors of $\mathcal{D}^{(\bullet)}$-modules

8.2.2.1. Similarly to [Sta22, 0A95], we check that for any quasi-isomorphism $\mathcal{I}^{(\bullet)} \xrightarrow{\sim} \mathcal{I}'^{(\bullet)}$ of K-injective complexes of $K(\mathcal{D}^{(\bullet)})$, for any quasi-isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$ of complexes of $K(\mathcal{D}^{(\bullet)})$, the morphism of $K(\mathbb{Z}_X)$

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}'^{(\bullet)},\mathcal{I}^{(\bullet)}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{I}'^{(\bullet)}) \tag{8.2.2.1.1}$$

is a quasi-isomorphism.

Let $\mathcal{E}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$ and $F_{\mathcal{E}}\colon K(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_X)$ be the functor defined by $\mathcal{F}^{(\bullet)} \mapsto \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)})$. By using the isomorphism 8.2.2.1.1 (in the case where $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$ is the identity of $\mathcal{E}^{(\bullet)}$) then it follows from [Sta22, 06XN] that $\mathbb{R}F_{\mathcal{E}}$ is everywhere defined and every K-injective complex computes $\mathbb{R}F_{\mathcal{E}}$. Since this is functorial in $\mathcal{E}^{(\bullet)}$, we get the bifunctor

$$\mathbb{R}_{\mathrm{II}}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,-)\colon K(\mathcal{D}^{(\bullet)}) \times D(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_X)$$

which is given by $\mathbb{R}_{\mathrm{II}}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) = \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{I}^{(\bullet)})$ where $\mathcal{I}^{(\bullet)}$ is a K-injective complex representing $\mathcal{F}^{(\bullet)}$. Using again 8.2.2.1.1 (this time by varying $\mathcal{E}$), we get the bifunctor

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,-)\colon D(\mathcal{D}^{(\bullet)}) \times D(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_X), \tag{8.2.2.1.2}$$

which is the unique functor satisfying $\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(Q(-),-) = \mathbb{R}_{\mathrm{II}}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,-)$, where $Q\colon K(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^{(\bullet)})$ is the localisation morphism.

Notation 8.2.2.2. Similarly to 8.2.2.1.2, we get the bifunctors

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(-,-)\colon D(\mathcal{D}^{(\bullet)}) \times D((\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}).$$
$$\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,-)\colon D(\mathcal{D}^{(\bullet)}) \times D(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_{X^{(\bullet)}}).$$

which are respectively given for any $\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$ by the formulas

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) = \mathrm{Hom}^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{I}^{(\bullet)}), \qquad \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) = \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{I}^{(\bullet)})$$

where $\mathcal{I}^{(\bullet)}$ is a K-injective complex representing $\mathcal{F}^{(\bullet)}$.

Remark 8.2.2.3. Let $\mathcal{R}$ be a sheaf on X of commutative rings endowed with a homomorphism of rings $\underleftarrow{l}_{X,I*}(\mathcal{R}) \to \mathcal{D}^{(\bullet)}$ such that the image of $\mathcal{R}$ in $\mathcal{D}^{(i)}$ are sent into the center of $\mathcal{D}^{(i)}$ for any $i \in I$. For instance, we can always take $\mathcal{R} = \mathbb{Z}_X$. Then we get the functor

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,-)\colon D(\mathcal{D}^{(\bullet)}) \times D(\mathcal{D}^{(\bullet)}) \to D(\mathcal{R}), \tag{8.2.2.3.1}$$

whose composition with the forgetful functor $D(\mathcal{R}) \to D(\mathbb{Z}_X)$ is isomorphic to 8.2.2.1.2. We have similar facts for the functors $\mathbb{R}\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(-,-)$ and $\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,-)$.

Lemma 8.2.2.4. *Let $\mathcal{R}$ be a sheaf on X of commutative rings endowed with a homomorphism of rings $\underleftarrow{l}_{X,I*}(\mathcal{R}) \to \mathcal{D}^{(\bullet)}$ such that the image of $\mathcal{R}$ in $\mathcal{D}^{(i)}$ are sent into the center of $\mathcal{D}^{(i)}$ for any $i \in I$. Set $\mathcal{R}^{(\bullet)} := \underleftarrow{l}_{X,I*}(\mathcal{R})$.*

Let $\mathcal{P}^{(\bullet)}$ be a complex of $K(\mathcal{D}^{(\bullet)})$ which is K-flat as an object of $K(\mathcal{R}^{(\bullet)})$. Let $\mathcal{I}^{(\bullet)}$ be a K-injective complex of $K(\mathcal{D}^{(\bullet)})$.

(a) The object $\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet)})$ of $K(\mathcal{R}^{(\bullet)})$ is K-injective.

(b) The object $\mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet)})$ of $K(\mathcal{R})$ is K-injective.

Proof. We prove (a) similarly to [Sta22, 0A96] (our lemma is a non-commutative version): for any acyclic complex $\mathcal{F} \in K(\mathcal{R}^{(\bullet)})$, we get the isomorphisms

$$\mathrm{Hom}_{K(\mathcal{R}^{(\bullet)})}\left(\mathcal{F},\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet)})\right) \xrightarrow[8.2.1.8.2]{\sim} H^0\Gamma\left(X^{(\bullet)},\mathscr{H}om^{\bullet}_{K(\mathcal{R}^{(\bullet)})}\left(\mathcal{F},\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet)})\right)\right)$$
$$\xrightarrow{\sim} H^0\Gamma\left(X^{(\bullet)},\mathscr{H}om^{\bullet}_{K(\mathcal{D}^{(\bullet)})}\left(\mathcal{F}\otimes_{\mathcal{R}^{(\bullet)}}\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet)}\right)\right) \xrightarrow[8.2.1.8.2]{\sim} \mathrm{Hom}_{K(\mathcal{D}^{(\bullet)})}\left(\mathcal{F}\otimes_{\mathcal{R}^{(\bullet)}}\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet)}\right) = 0,$$

which proves (a). Since $\underrightarrow{l}^{(I)}_{X*}$ has an exact left adjoint functor, then the functor $\underrightarrow{l}^{(I)}_{X*}\colon K(\mathcal{R}^{(\bullet)}) \to K(\mathcal{R})$ preserves K-injectivity (see [Sta22, 08BJ]). By using 8.2.1.2.2 this yields (b). □

Proposition 8.2.2.5. *Suppose there exists a sheaf $\mathcal{R}$ on X of commutative rings endowed with a flat homomorphism of rings $\underleftarrow{l}_{X,I*}(\mathcal{R}) \to \mathcal{D}^{(\bullet)}$ such that the image of $\mathcal{R}$ in $\mathcal{D}^{(i)}$ are sent into the center of $\mathcal{D}^{(i)}$ for any $i \in I$. Then for any $\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$, we have the isomorphisms*

$$\mathbb{R}\Gamma(X^{(\bullet)},-)\circ\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}), \tag{8.2.2.5.1}$$

$$\mathbb{R}\underrightarrow{l}^{(I)}_{X*}\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}), \tag{8.2.2.5.2}$$

$$\mathbb{R}\Gamma(X,-)\circ\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}). \tag{8.2.2.5.3}$$

Proof. Let us prove 8.2.2.5.1. Let $\mathcal{P}^{(\bullet)}$ be a K-flat complex of $K(\mathcal{D}^{(\bullet)})$ representing $\mathcal{E}^{(\bullet)}$. Let $\mathcal{I}^{(\bullet)}$ be a K-injective complex of $K(\mathcal{D}^{(\bullet)})$ representing $\mathcal{F}^{(\bullet)}$. We have $\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)})$. Set $\mathcal{R}^{(\bullet)} := \underset{\leftarrow}{l}_{X,I*}(\mathcal{R})$. Since $\mathcal{D}^{(\bullet)}$ is a $\mathcal{R}^{(\bullet)}$-flat module, then $\mathcal{P}^{(\bullet)}$ is K-flat as an object of $K(\mathcal{R}^{(\bullet)})$. Hence, following 8.2.2.4 $\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)})$ is a K-injective object of $K(\mathcal{R}^{(\bullet)})$. This yields the second quasi-isomorphism

$$\mathbb{R}\Gamma(X^{(\bullet)}, \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\Gamma(X^{(\bullet)}, \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)})) \xrightarrow{\sim} \Gamma(X^{(\bullet)}, \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)}))$$
$$= \mathrm{Hom}^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}).$$

Replacing $\mathbb{R}\Gamma(X^{(\bullet)}, -)$ by $\mathbb{R}\underset{\rightarrow}{l}^{(I)}_{X*}$, we get similarly 8.2.2.5.2 from 8.2.1.2.2. Since $\mathbb{R}\Gamma(X, -) \circ \mathbb{R}\underset{\rightarrow}{l}^{(I)}_{X*} \xrightarrow{\sim} \mathbb{R}\Gamma(X^{(\bullet)}, -)$, we get 8.2.2.5.3 from both 8.2.2.5.1 and 8.2.2.5.2. □

8.2.2.6. Let $\mathscr{F}$ be the collection of flasque $\mathbb{Z}_{X^{(\bullet)}}$-modules. We denote by $K^+(\mathscr{F})$ the full subcategory of $K^+(\mathbb{Z}_{X^{(\bullet)}})$ whose objects consist in complexes of terms in $\mathscr{F}$. Since an object of $\mathscr{F}$ is right acyclic for the functors $\underset{\rightarrow}{l}^{(I)}_{X*}$ and $\Gamma(X^{(\bullet)}, -)$, then from Leray's acyclicity lemma (see [Sta22, 015E-Lemma 13.16.7]), the complexes of $K^+(\mathscr{F})$ compute $\mathbb{R}\underset{\rightarrow}{l}^{(I)}_{X*}$ and $\mathbb{R}\Gamma(X^{(\bullet)}, -)$.

For bounded complexes, we can remove the flatness hypothesis of 8.2.2.5:

Proposition 8.2.2.7. *For any $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)}), \mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)})$, we have the isomorphisms*

$$\mathbb{R}\Gamma(X^{(\bullet)}, -) \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}), \tag{8.2.2.7.1}$$
$$\mathbb{R}\underset{\rightarrow}{l}^{(I)}_{X*}\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}), \tag{8.2.2.7.2}$$
$$\mathbb{R}\Gamma(X, -) \circ \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}). \tag{8.2.2.7.3}$$

Proof. Let $\mathcal{I}^{(\bullet)}$ be an injective resolution in $K^+(\mathcal{D}^{(\bullet)})$ of $\mathcal{F}^{(\bullet)}$. We have $\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)})$. Since a product of flasque sheaves is flasque, then by using 8.2.1.3, we get that $\mathscr{H}om^{n}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)})$ is a flasque abelian sheaf on $X^{(\bullet)}$ for any integer n, i.e. $\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)}) \in K^+(\mathscr{F})$. Hence, following 8.2.2.6, we get the first isomorphism

$$\mathbb{R}\Gamma(X^{(\bullet)}, -) \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \Gamma(X^{(\bullet)}, \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)}))$$
$$\underset{8.2.1.1.1}{\xrightarrow{\sim}} \mathrm{Hom}^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}).$$

We proceed similarly to check 8.2.2.7.2. This yields 8.2.2.7.3 by composition. □

8.2.2.8. Let $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$. It follows from 8.2.1.7.(b) and from 8.2.1.2.1 that we have the isomorphisms

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})|_{(i,U)} \cong \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}|_{(i,U)}}(\mathcal{E}^{(\bullet)}|_{(i,U)}, \mathcal{F}^{(\bullet)}|_{(i,U)}), \tag{8.2.2.8.1}$$
$$\underset{\rightarrow}{i}^{-1}_{X}(\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}|_i}(\mathcal{E}^{(\bullet)}|_i, \mathcal{F}^{(\bullet)}|_i). \tag{8.2.2.8.2}$$

8.2.3 From I to $I^{\emptyset}$

8.2.3.1. We denote by $I^{\emptyset}$ the partially ordered set whose elements are equal to that of I and with the coarse order, i.e. $i \leq j$ if and only if $i = j$. We get the increasing map $\rho_I : I^{\emptyset} \to I$ of partially ordered sets. Following 8.1.1.2.1, this yields the morphism of topos $\underset{\rightarrow}{\rho}_{I,X} : X^{(I^{\emptyset})} \to X^{(I)}$. We set $\mathcal{D}^{(\bullet)}_{\emptyset} := \underset{\rightarrow}{\rho}^{-1}_{I,X}(\mathcal{D}^{(\bullet)})$, i.e. $\mathcal{D}^{(\bullet)}_{\emptyset}$ is the family $(\mathcal{D}^{(i)})_{i \in I}$ without transition maps (except the identities given by $i \leq i$). We get the morphism of ringed topoi $\underset{\rightarrow}{\rho}_{I,X} : (X^{(I^{\emptyset})}, \mathcal{D}^{(\bullet)}_{\emptyset}) \to (X^{(I)}, \mathcal{D}^{(\bullet)})$. The functor $\underset{\rightarrow}{\rho}^{-1}_{I,X} : \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)}_{\emptyset})$ is the forgetful functor (of the transition maps). It has a left adjoint functor that we will denote by $\underset{\rightarrow}{\rho}_{I,X,!} : \mathrm{Mod}(\mathcal{D}^{(\bullet)}_{\emptyset}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)})$ (even if the notation is the same, beware this is different from the left adjoint of 8.1.1.2.2 computed in the category of sheaves of sets, but we will only consider the modules case). Let $\mathcal{E}^{(\bullet)}_{\emptyset}$ be a left $\mathcal{D}^{(\bullet)}_{\emptyset}$-module. We denote by $\mathcal{F}^{(i)} = \oplus_{h \leq i} \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(h)}} \mathcal{E}^{(h)}_{\emptyset}$ and for any $i \leq j$ and by $\alpha^{j,i} : \mathcal{F}^{(i)} \to \mathcal{F}^{(j)}$ the canonical transition maps induced for any elements $h \leq i$ of I by the canonical maps $\mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(h)}} \mathcal{E}^{(h)}_{\emptyset} \to \mathcal{D}^{(j)} \otimes_{\mathcal{D}^{(h)}} \mathcal{E}^{(h)}_{\emptyset} \to \mathcal{F}^{(j)}$. Then $\underset{\rightarrow}{\rho}_{I,X,!}(\mathcal{E}^{(\bullet)}_{\emptyset}) = (\mathcal{F}^{(i)}, \alpha^{(j,i)})$.

8.2.3.2. Let U be an open subset of X and $j\colon U \hookrightarrow X$ be the induced open immersion. With notation 8.1.1.2.7, we have

$$\underrightarrow{j}_{I}^{-1} \circ \underrightarrow{\rho}_{I,X,!} \xrightarrow{\sim} \underrightarrow{\rho}_{I,U,!} \circ \underrightarrow{j}_{I^{\emptyset}}^{-1}. \tag{8.2.3.2.1}$$

8.2.3.3. Let $\mathscr{P}_{\emptyset}$ be the collection of flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-modules. We denote by $K^{-}(\mathscr{P}_{\emptyset})$ the full subcategory of $K^{-}(\mathcal{D}_{\emptyset}^{(\bullet)})$ whose objects consist in complexes of terms in $\mathscr{P}_{\emptyset}$.

Let $0 \to \mathcal{P}_{\emptyset}^{\prime(\bullet)} \to \mathcal{P}_{\emptyset}^{(\bullet)} \to \mathcal{P}_{\emptyset}^{\prime\prime(\bullet)} \to 0$ be a short exact sequence of flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-modules. For any $i \in I$, $\mathcal{P}_{\emptyset}^{\prime\prime(i)}$ is a flat $\mathcal{D}^{(i)}$-module. Hence, for any $i \in I$, we get the short exact sequence of flat left $\mathcal{D}^{(i)}$-modules

$$0 \to \oplus_{h \leq i} \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(h)}} \mathcal{P}_{\emptyset}^{\prime(h)} \to \oplus_{h \leq i} \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(h)}} \mathcal{P}_{\emptyset}^{(h)} \to \oplus_{h \leq i} \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(h)}} \mathcal{P}_{\emptyset}^{\prime\prime(h)} \to 0,$$

i.e. we have the short exact sequence $0 \to \underrightarrow{\rho}_{I,X,!}(\mathcal{P}_{\emptyset}^{\prime(\bullet)}) \to \underrightarrow{\rho}_{I,X,!}(\mathcal{P}_{\emptyset}^{(\bullet)}) \to \underrightarrow{\rho}_{I,X,!}(\mathcal{P}_{\emptyset}^{\prime\prime(\bullet)}) \to 0$ of flat left $\mathcal{D}^{(\bullet)}$-modules. Since any left $\mathcal{D}_{\emptyset}^{(\bullet)}$-modules is a quotient of a flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module, it follows from [Sta22, 05T9] that the objects of $\mathscr{P}_{\emptyset}$ are acyclic for the functor $\mathbb{L}\underrightarrow{\rho}_{I,X,!}$. Hence, from Leray's acyclicity lemma (see the dual version of [Sta22, 015E]), the complexes of $K^{-}(\mathscr{P}_{\emptyset})$ compute $\mathbb{L}\underrightarrow{\rho}_{I,X,!}$.

Let $\mathcal{E}_{\emptyset}^{(\bullet)} \in K^{-}(\mathcal{D}_{\emptyset}^{(\bullet)})$. By using [Sta22, 05T7], there exists a quasi-isomorphism of $\mathcal{P}_{\emptyset}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}_{\emptyset}^{(\bullet)}$ of $K^{-}(\mathcal{D}_{\emptyset}^{(\bullet)})$ with $\mathcal{P}_{\emptyset}^{(\bullet)} \in K^{-}(\mathscr{P}_{\emptyset})$. Hence, $\mathbb{L}\underrightarrow{\rho}_{I,X,!}$ is defined at $\mathcal{E}_{\emptyset}^{(\bullet)}$ and we have

$$\underrightarrow{\rho}_{I,X,!}(\mathcal{P}_{\emptyset}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\underrightarrow{\rho}_{I,X,!}(\mathcal{E}_{\emptyset}^{(\bullet)}). \tag{8.2.3.3.1}$$

8.2.3.4. Let $\mathcal{E}_{\emptyset}^{(\bullet)}$ be a left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module and $\mathcal{F}^{(\bullet)}$ be a left $\mathcal{D}^{(\bullet)}$-module. By adjointness (and also use 8.2.3.2.1 for the first isomorphism of the second line) we have

$$\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\underrightarrow{\rho}_{I,X,!}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{E}_{\emptyset}^{(\bullet)}, \underrightarrow{\rho}_{I,X}^{-1}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \prod_{i \in I} \mathrm{Hom}_{\mathcal{D}^{(i)}}(\mathcal{E}_{\emptyset}^{(i)}, \mathcal{F}^{(i)}), \tag{8.2.3.4.1}$$

$$\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underrightarrow{\rho}_{I,X,!}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{E}_{\emptyset}^{(\bullet)}, \underrightarrow{\rho}_{I,X}^{-1}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \prod_{i \in I} \mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_{\emptyset}^{(i)}, \mathcal{F}^{(i)}). \tag{8.2.3.4.2}$$

8.2.3.5. Let $\mathcal{E}_{\emptyset}^{(\bullet)}$ be a left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module and $\mathcal{F}^{(\bullet)}$ be a left $\mathcal{D}^{(\bullet)}$-module.

(a) The object $\mathscr{H}om_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{E}_{\emptyset}^{(\bullet)}, \underrightarrow{\rho}_{I,X}^{-1}\mathcal{F}^{(\bullet)})$ is an abelian sheaf on $X^{(I^{\emptyset})}$. With notation 8.1.1.2.5, for any $i \in I^{\emptyset}$, we compute

$$\underrightarrow{i}_{X}^{-1}\left(\mathscr{H}om_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{E}_{\emptyset}^{(\bullet)}, \underrightarrow{\rho}_{I,X}^{-1}\mathcal{F}^{(\bullet)})\right) = \mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_{\emptyset}^{(i)}, \mathcal{F}^{(i)}). \tag{8.2.3.5.1}$$

(b) On the other hand, the object $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underrightarrow{\rho}_{I,X,!}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)})$ an abelian sheaf on $X^{(I)}$. By construction (see 8.2.1.1.2), $\mathrm{Hom}_{X^{(I)}}((i,U), \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underrightarrow{\rho}_{I,X,!}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)}))$ corresponds to a compatible family of $\mathcal{D}^{(j)}|_U$-linear homomorphisms $\oplus_{h \leq j} \mathcal{D}^{(j)} \otimes_{\mathcal{D}^{(h)}} \mathcal{E}_{\emptyset}^{(h)}|_U \to \mathcal{F}^{(j)}|_U$ for any $j \geq i$, which is equal to a compatible family of $\mathcal{D}^{(h)}|_U$-linear homomorphisms $\mathcal{E}_{\emptyset}^{(h)}|_U \to \mathcal{F}^{(j)}|_U$ for any $j \geq i$ and any $h \leq j$, which is equal to a family of $\mathcal{D}^{(h)}|_U$-linear homomorphisms $\mathcal{E}_{\emptyset}^{(h)}|_U \to \mathcal{F}^{(i)}|_U$ any $h < i$ and of a family of $\mathcal{D}^{(j)}|_U$-linear homomorphisms $\mathcal{E}_{\emptyset}^{(j)}|_U \to \mathcal{F}^{(j)}|_U$ any $j \geq i$.

$$\underrightarrow{i}_{X}^{-1}\left(\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underrightarrow{\rho}_{I,X,!}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)})\right) = \prod_{j \geq i} \mathcal{H}om_{\mathcal{D}^{(j)}}(\mathcal{E}_{\emptyset}^{(j)}, \mathcal{F}^{(j)}) \times \prod_{h < i} \mathcal{H}om_{\mathcal{D}^{(h)}}(\mathcal{E}_{\emptyset}^{(h)}, \mathcal{F}^{(i)}). \tag{8.2.3.5.2}$$

8.2.3.6. Let $\mathrm{Mod}(\mathbb{Z})$ be the category of abelian groups. Since products are exact, then following [Sta22, 07KC] we can check the following properties

(a) $D(\mathbb{Z})$ has products,

(b) products $\prod_{i\in I} K_i$ in $D(\mathbb{Z})$ are obtained by taking termwise products of any complexes representing the K_i, and

(c) $H^n(\prod_{i\in I} K_i) = \prod_{i\in I} H^n(K_i)$ for any integer $n \in \mathbb{Z}$.

Lemma 8.2.3.7. *We have the following properties.*

(a) If $\mathcal{I}_\emptyset^{(\bullet)}$ is an injective left $\mathcal{D}_\emptyset^{(\bullet)}$-module then $\mathcal{I}_\emptyset^{(i)}$ is an injective left $\mathcal{D}_\emptyset^{(i)}$-module for any $i \in I$.

(b) A complex $\mathcal{E}_\emptyset^{(\bullet)} \in K({}^{\mathrm{l}}\mathcal{D}_\emptyset^{(\bullet)})$ is a K-injective if and only if $\mathcal{E}_\emptyset^{(i)}$ is a K-injective complex of $K(\mathcal{D}^{(i)})$ for any $i \in I$.

Proof. This is a consequence of 7.1.3.21.c-d. □

8.2.3.8. Let $\mathcal{E}_\emptyset^{(\bullet)}, \mathcal{F}_\emptyset^{(\bullet)} \in D({}^{\mathrm{l}}\mathcal{D}_\emptyset^{(\bullet)})$. Let $\mathcal{R}$ be a sheaf on X of commutative rings endowed with a homomorphism of rings $\underleftarrow{l}_{X,I,*}(\mathcal{R}) \to \mathcal{D}^{(\bullet)}$ such that the image of $\mathcal{R}$ in $\mathcal{D}^{(i)}$ are sent into the center of $\mathcal{D}^{(i)}$ for any $i \in I$. Let $R := \Gamma(X, \mathcal{R})$.

(a) Let $\mathcal{I}_\emptyset^{(\bullet)}$ be a K-injective complex of $K(\mathcal{D}_\emptyset^{(\bullet)})$ representing $\mathcal{F}_\emptyset^{(\bullet)}$. Following 8.2.3.7, for any $i \in I$, the object $\mathcal{I}_\emptyset^{(i)}$ is a K-injective complex of $K(\mathcal{D}^{(i)})$ representing $\mathcal{F}_\emptyset^{(i)}$. We have therefore in $D(R)$ the isomorphisms:

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)}, \mathcal{F}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \operatorname{Hom}^\bullet_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)}, \mathcal{I}_\emptyset^{(\bullet)}) = \prod_{i\in I} \operatorname{Hom}^\bullet_{\mathcal{D}^{(i)}}(\mathcal{E}_i^{(\bullet)}, \mathcal{I}_\emptyset^{(i)}) \underset{8.2.3.6}{\xrightarrow{\sim}} \prod_{i\in I} \mathbb{R}\operatorname{Hom}_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)}, \mathcal{F}_\emptyset^{(i)}). \tag{8.2.3.8.1}$$

(b) Suppose $\mathcal{D}^{(\bullet)}$ is a flat $\underleftarrow{l}_{X,I,*}(\mathcal{R})$-module, which is equivalent to saying that $\mathcal{D}^{(i)}$ is a flat $\mathcal{R}^{(i)}$-module for any $i \in I$. Set $\mathcal{R}^{(\bullet)} := \underleftarrow{l}_{X,I*}(\mathcal{R})$ and $\mathcal{R}_\emptyset^{(\bullet)} := \underleftarrow{l}_{X,I^\emptyset,*}(\mathcal{R})$. Let $\mathcal{P}_\emptyset^{(\bullet)}$ be a K-flat complex of $K(\mathcal{D}_\emptyset^{(\bullet)})$ representing $\mathcal{E}_\emptyset^{(\bullet)}$. Let $\mathcal{I}_\emptyset^{(\bullet)}$ be a K-injective complex of $K(\mathcal{D}_\emptyset^{(\bullet)})$ representing $\mathcal{F}_\emptyset^{(\bullet)}$. This yields $\mathbb{R}\mathcal{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)}, \mathcal{F}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \mathcal{H}om^\bullet_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{P}_\emptyset^{(\bullet)}, \mathcal{I}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \prod_{i\in I} \mathcal{H}om^\bullet_{\mathcal{D}^{(i)}}(\mathcal{P}_\emptyset^{(i)}, \mathcal{I}_\emptyset^{(i)})$, where the product is computed in $K(\mathcal{R})$. For any $i \in I$, it follows from respectively 8.1.1.6 and 8.2.3.7 that $\mathcal{P}_\emptyset^{(i)}$ is a K-flat complex of $K(\mathcal{D}^{(i)})$ representing $\mathcal{E}_\emptyset^{(i)}$ and $\mathcal{I}_\emptyset^{(i)}$ is a K-injective complex of $K(\mathcal{D}^{(i)})$ representing $\mathcal{F}_\emptyset^{(i)}$. Since $\mathcal{D}^{(i)}$ is a flat $\mathcal{R}$-module, then $\mathcal{P}_\emptyset^{(\bullet)}$ is K-flat as an object of $K(\mathcal{R})$. Hence, following 8.2.2.4 $\mathcal{H}om^\bullet_{\mathcal{D}^{(i)}}(\mathcal{P}_\emptyset^{(i)}, \mathcal{I}_\emptyset^{(i)})$ is a K-injective object of $K(\mathcal{R})$. Using 8.2.1.6.b, we get in $D(\mathcal{R})$ the isomorphism of complexes $\mathcal{H}om^\bullet_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{P}_\emptyset^{(\bullet)}, \mathcal{I}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \prod_{i\in I} \mathcal{H}om^\bullet_{\mathcal{D}^{(i)}}(\mathcal{P}_\emptyset^{(i)}, \mathcal{I}_\emptyset^{(i)}) \xrightarrow{\sim} \prod_{i\in I} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)}, \mathcal{F}_\emptyset^{(i)})$, where the products are computed in $D(\mathcal{R})$. By composition, we get the isomorphism in $D(\mathcal{R})$:

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)}, \mathcal{F}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \prod_{i\in I} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)}, \mathcal{F}_\emptyset^{(i)}). \tag{8.2.3.8.2}$$

Similarly, for any $i \in I$, by using 8.2.3.5.1, 8.2.2.4 and 8.2.3.7, we can check the isomorphism in $D(\mathcal{R})$:

$$\underrightarrow{i}_X^{-1}\left(\mathbb{R}\mathscr{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)}, \mathcal{F}_\emptyset^{(\bullet)})\right) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)}, \mathcal{F}_\emptyset^{(i)}). \tag{8.2.3.8.3}$$

8.2.3.9. Let $\mathscr{F}_\emptyset$ be the collection of flasque $\mathbb{Z}_{X^{(I^\emptyset)}}$-modules. Recall, with the definition 7.1.3.14, following 7.1.2.19.(a), a $\mathbb{Z}_{X^{(I^\emptyset)}}$-module $\mathcal{E}^{(\bullet)}$ is flasque if and only if $\mathcal{E}^{(i)}$ is a flasque sheaf of sets on X for any $i \in I$. We denote by $K^+(\mathscr{F}_\emptyset)$ the full subcategory of $K^+(\mathbb{Z}_{X^{(I^\emptyset)}})$ whose objects consist in complexes of terms in $\mathscr{F}_\emptyset$.

(a) Since the objects of $\mathscr{F}_\emptyset$ are acyclic for the functor $\underleftarrow{l}_{X,I^\emptyset,*}$, then from Leray's acyclicity lemma (see [Sta22, 015E]), the complexes of $K^+(\mathscr{F}_\emptyset)$ compute $\mathbb{R}\underleftarrow{l}_{X,I^\emptyset,*}$ (beware, the functor $\underleftarrow{l}_{X,I^\emptyset,*} = \prod_{i\in I}$ is the product computed in $K^+(\mathbb{Z}_X)$ and is not exact). Following 7.1.3.14, there exists a quasi-isomorphism of $\mathcal{E}_\emptyset^{(\bullet)} \xrightarrow{\sim} \mathcal{F}_\emptyset^{(\bullet)}$ of $K^+(\mathbb{Z}_{X^{(I^\emptyset)}})$ with $\mathcal{F}_\emptyset^{(\bullet)} \in K^+(\mathscr{F}_\emptyset)$ and we have

$$\mathbb{R}\underleftarrow{l}_{X,I^\emptyset,*}(\mathcal{E}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \underleftarrow{l}_{X,I^\emptyset,*}(\mathcal{F}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \prod_{i\in I} \mathcal{F}_\emptyset^{(i)}, \tag{8.2.3.9.1}$$

where $\prod_{i\in I}\mathcal{F}_\emptyset^{(i)}$ is the product computed in $K^+(\mathbb{Z}_X)$ (i.e. it is computed by taking termwise products).

(b) Let $\mathcal{F}_\emptyset^{(\bullet)}$ be an object of $K^+(\mathscr{F}_\emptyset)$, i.e. let $\mathcal{F}_\emptyset^{(i)}$ be a collection of flasque abelian sheaves on X for any $i\in I$. Let $\mathcal{I}_\emptyset^{(\bullet)}$ be a complex of injective $\mathbb{Z}_{X^{(I^\emptyset)}}$-modules together with a quasi-isomorphism $\mathcal{F}_\emptyset^{(\bullet)}\xrightarrow{\sim}\mathcal{I}_\emptyset^{(\bullet)}$. Since this latter map is a quasi-isomorphism between two objects of $K^+(\mathscr{F}_\emptyset)$, then it follows from 8.2.3.9.1 that the induced morphism $\prod_{i\in I}\mathcal{F}_\emptyset^{(i)}\to\prod_{i\in I}\mathcal{I}_\emptyset^{(i)}$ is a quasi-isomorphism of $K^+(\mathbb{Z}_X)$. Since $\prod_{i\in I}\mathcal{I}_\emptyset^{(i)}$ is also equal to the product computed in $D^+(\mathbb{Z}_X)$ (see 8.2.1.6.b), this yields that $\prod_{i\in I}\mathcal{F}_\emptyset^{(i)}$ is also the product computed in $D^+(\mathbb{Z}_X)$. By default, a product $\prod_{i\in I}$ will means a product in the derived category.

8.2.3.10. Let $\mathcal{E}_\emptyset^{(\bullet)}\in D^-({}^{\mathrm{l}}\mathcal{D}_\emptyset^{(\bullet)})$, $\mathcal{F}_\emptyset^{(\bullet)}\in D^+({}^{\mathrm{l}}\mathcal{D}_\emptyset^{(\bullet)})$. Let $\mathcal{R}$ be a sheaf on X of commutative rings endowed with a homomorphism of rings $\underset{\leftarrow}{l}_{X,I,*}(\mathcal{R})\to\mathcal{D}^{(\bullet)}$ such that the image of $\mathcal{R}$ in $\mathcal{D}^{(i)}$ are sent into the center of $\mathcal{D}^{(i)}$ for any $i\in I$. Let $R:=\Gamma(X,\mathcal{R})$. Thanks to our boundedness hypotheses on $\mathcal{E}_\emptyset^{(\bullet)}$ and $\mathcal{F}_\emptyset^{(\bullet)}$, we can check the above isomorphism 8.2.3.8.2 without the hypotheses (even if this is in general harmless when $\mathcal{D}^{(\bullet)}$ comes from the theory of $\mathcal{D}$-modules) that $\mathcal{D}^{(\bullet)}$ is a flat $\underset{\leftarrow}{l}_{X,I,*}(\mathcal{R})$-module (therefore we can always choose $\mathcal{R}=\mathbb{Z}_X$) as follows:

Let $\mathcal{I}_\emptyset^{(\bullet)}$ be an injective resolution in $K^+(\mathcal{D}_\emptyset^{(\bullet)})$ of $\mathcal{F}_\emptyset^{(\bullet)}$. This yields the quasi-isomorphisms in $K(\mathbb{Z}_X)$:

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)},\mathcal{F}_\emptyset^{(\bullet)})\xrightarrow{\sim}\mathcal{H}om^\bullet_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)},\mathcal{I}_\emptyset^{(\bullet)})\xrightarrow{\sim}\prod_{i\in I}\mathcal{H}om^\bullet_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)},\mathcal{I}_\emptyset^{(i)}),\tag{8.2.3.10.1}$$

where the product is computed in $K(\mathbb{Z}_X)$. Since a product of flasque sheaves is flasque, then by using 8.2.1.3, we get that $\mathcal{H}om^n_{\mathcal{D}_\emptyset^{(i)}}(\mathcal{E}_\emptyset^{(i)},\mathcal{I}_\emptyset^{(i)})$ is a flasque sheaf for any integer n. Hence following 8.2.3.9.(b), the product appearing at 8.2.3.10.1 is also a product in $D(\mathbb{Z}_X)$. Since $\mathcal{I}_\emptyset^{(i)}$ is an injective resolution in $K^+(\mathcal{D}_\emptyset^{(i)})$ of $\mathcal{F}_\emptyset^{(i)}$ for any $i\in I$, this yields the isomorphism in $D(\mathbb{Z}_X)$

$$\prod_{i\in I}\mathcal{H}om^\bullet_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)},\mathcal{I}_\emptyset^{(i)})\xrightarrow{\sim}\prod_{i\in I}\mathbb{R}\mathcal{H}om^\bullet_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)},\mathcal{F}_\emptyset^{(i)}),\tag{8.2.3.10.2}$$

where the product of the right term is the product in $D(\mathbb{Z}_X)$. By composing 8.2.3.10.1 and 8.2.3.10.2, we get the isomorphism in $D^+(\mathbb{Z}_X)$:

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)},\mathcal{F}_\emptyset^{(\bullet)})\xrightarrow{\sim}\prod_{i\in I}\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)},\mathcal{F}_\emptyset^{(i)}).\tag{8.2.3.10.3}$$

Similarly, for any $i\in I$, we get the isomorphism in $D^+(\mathbb{Z}_X)$:

$$\underset{\rightarrow}{i}{}_X^{-1}\left(\mathbb{R}\mathscr{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)},\mathcal{F}_\emptyset^{(\bullet)})\right)\xrightarrow{\sim}\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}_\emptyset^{(i)},\mathcal{F}_\emptyset^{(i)}).\tag{8.2.3.10.4}$$

8.2.3.11. Let $\mathcal{I}^{(\bullet)}$ be an injective left $\mathcal{D}^{(\bullet)}$-module, $\mathcal{Q}_\emptyset^{(\bullet)}$ be a flat left $\mathcal{D}_\emptyset^{(\bullet)}$-module. Since $\mathcal{Q}_\emptyset^{(i)}$ is a flat left $\mathcal{D}^{(i)}$-module for any $i\in I$ (see 7.1.3.6), then $\mathcal{I}^{(i)}$ is acyclic for the functors $\mathrm{Hom}_{\mathcal{D}^{(i)}}(\mathcal{Q}_\emptyset^{(i)},-)$ and $\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{Q}_\emptyset^{(i)},-)$ (see 7.1.3.21.b). This means that $\underset{\rightarrow}{\rho}{}_{I,X}^{-1}(\mathcal{I}^{(\bullet)})$ is acyclic for the functor $\mathrm{Hom}_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{Q}_\emptyset^{(\bullet)},-)$ and $\mathcal{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{Q}_\emptyset^{(\bullet)},-)$. Moreover, $\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{Q}^{(i)},\mathcal{I}^{(i)})$ is flasque for any $i\in I$ (see 7.1.3.21.a).

8.2.3.12. Let $\mathcal{E}_\emptyset^{(\bullet)}\in D^-(\mathcal{D}_\emptyset^{(\bullet)})$ and $\mathcal{F}^{(\bullet)}\in D^+(\mathcal{D}^{(\bullet)})$.

(a) By taking a flat resolution of $\mathcal{E}^{(\bullet)}$ and an injective resolution of $\mathcal{F}^{(\bullet)}$, by using 8.2.3.3.1, 8.2.3.4.1 (resp. 8.2.3.8.1) we get the following first (resp. second) isomorphism:

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathbb{L}\underset{\rightarrow}{\rho}_{I,X,!}(\mathcal{E}_\emptyset^{(\bullet)}),\mathcal{F}^{(\bullet)})\xrightarrow{\sim}\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{E}_\emptyset^{(\bullet)},\underset{\rightarrow}{\rho}{}_{I,X}^{-1}\mathcal{F}^{(\bullet)})\xrightarrow{\sim}\prod_{i\in I}\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(i)}}(\mathcal{E}^{(i)},\mathcal{F}^{(i)}).\tag{8.2.3.12.1}$$

(b) Similarly, by using 8.2.3.3.1, 8.2.3.4.2, (resp. 8.2.3.10.3) the following first (resp. second) isomorphisms

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathbb{L}\underset{\to I,X,!}{\rho}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{E}_{\emptyset}^{(\bullet)}, \underset{\to I,X}{\rho}^{-1}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \prod_{i\in I} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}^{(i)}, \mathcal{F}^{(i)}). \tag{8.2.3.12.2}$$

(c) We have the isomorphism

$$\underset{\to X}{i}^{-1}\left(\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathbb{L}\underset{\to I,X,!}{\rho}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)})\right) = \prod_{j\geq i} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(j)}}(\mathcal{E}_{\emptyset}^{(j)}, \mathcal{F}^{(j)}) \times \prod_{h<i} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(h)}}(\mathcal{E}_{\emptyset}^{(h)}, \mathcal{F}^{(i)}). \tag{8.2.3.12.3}$$

Indeed, let $\mathcal{P}_{\emptyset}^{(\bullet)\bullet} \in K^-(\mathscr{P}_{\emptyset})$ together with a quasi-isomorphism of $K^-(\mathcal{D}_{\emptyset}^{(\bullet)})$ of the form $\mathcal{P}_{\emptyset}^{(\bullet),\bullet} \xrightarrow{\sim} \mathcal{E}_{\emptyset}^{(\bullet)}$ (see notation 8.2.3.3). Let $\mathcal{I}^{(\bullet)\bullet}$ be an injective resolution in $K^+(\mathcal{D}^{(\bullet)})$ of $\mathcal{F}^{(\bullet)}$. Following 8.2.3.3.1, we get

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathbb{L}\underset{\to I,X,!}{\rho}(\mathcal{E}_{\emptyset}^{(\bullet)}), \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{P}_{\emptyset}^{(\bullet),\bullet}), \mathcal{I}^{(\bullet),\bullet}).$$

Since $\underset{\to X}{i}^{-1}$ commutes with finite products, we get for any $n \in \mathbb{Z}$ the first equality:

$$\begin{aligned}\underset{\to X}{i}^{-1}\left(\mathscr{H}om^{n}_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{P}_{\emptyset}^{(\bullet),\bullet}), \mathcal{I}^{(\bullet),\bullet})\right) &= \prod_{p\in\mathbb{Z}} \underset{\to X}{i}^{-1}\left(\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{P}_{\emptyset}^{(\bullet),p}), \mathcal{I}^{(\bullet),n+p})\right)\\ &\overset{8.2.3.5.2}{=} \prod_{p\in\mathbb{Z}}\left(\prod_{j\geq i}\mathcal{H}om_{\mathcal{D}^{(j)}}(\mathcal{P}_{\emptyset}^{(j),p}, \mathcal{I}^{(j),n+p}) \times \prod_{h<i}\mathcal{H}om_{\mathcal{D}^{(h)}}(\mathcal{P}_{\emptyset}^{(h),p}, \mathcal{I}^{(i),n+p})\right)\\ &\overset{8.2.1.4.1}{=} \prod_{j\geq i}\mathcal{H}om^{n}_{\mathcal{D}^{(j)}}(\mathcal{P}_{\emptyset}^{(j),\bullet}, \mathcal{I}^{(j),\bullet}) \times \prod_{h<i}\mathcal{H}om^{n}_{\mathcal{D}^{(h)}}(\mathcal{P}_{\emptyset}^{(h),\bullet}, \mathcal{I}^{(i),\bullet}).\end{aligned}$$

This yields

$$\underset{\to X}{i}^{-1}\left(\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{P}_{\emptyset}^{(\bullet),\bullet}), \mathcal{I}^{(\bullet),\bullet})\right) = \prod_{j\geq i}\mathcal{H}om^{\bullet}_{\mathcal{D}^{(j)}}(\mathcal{P}_{\emptyset}^{(j),\bullet}, \mathcal{I}^{(j),\bullet}) \times \prod_{h<i}\mathcal{H}om^{\bullet}_{\mathcal{D}^{(h)}}(\mathcal{P}_{\emptyset}^{(h),\bullet}, \mathcal{I}^{(i),\bullet}), \tag{8.2.3.12.4}$$

where the products are computed in $K(\mathbb{Z}_X)$. Since a product of flasque sheaves is flasque, then by using 8.2.3.11, we get that the sheaf $\mathcal{H}om^{n}_{\mathcal{D}^{(h)}}(\mathcal{I}_{\emptyset}^{(h),\bullet}, \mathcal{P}^{(j),\bullet})$ is flasque for any $h \leq j$ and any n. Hence following 8.2.3.9.(b), the product appearing at 8.2.3.12.4 is also a product in $D(\mathbb{Z}_X)$. From Leray's acyclicity lemma (see [Sta22, 015E]), it follows from 8.2.3.11 that for any $h \leq j$, any p we have

$$\mathcal{H}om^{\bullet}_{\mathcal{D}^{(h)}}(\mathcal{P}_{\emptyset}^{(h),\bullet}, \mathcal{I}^{(i),\bullet}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(h)}}(\mathcal{E}_{\emptyset}^{(h),\bullet}, \mathcal{F}^{(i),\bullet}).$$

Hence, we are done.

Proposition 8.2.3.13. *Let $u\colon I \to I'$ be increasing map of partially ordered sets, $\mathcal{D}'^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I')}$ endowed with a homomorphism of sheaf of rings $\mathcal{D}^{(\bullet)} \to \underset{\to X}{u}^{-1}(\mathcal{D}'^{(\bullet)})$. Let $\mathcal{F}'^{(\bullet)}$ be a left $\mathcal{D}'^{(\bullet)}$-module. We consider $\underset{\to X}{u}^{-1}(\mathcal{F}'^{(\bullet)})$ as a left $\mathcal{D}^{(\bullet)}$-module via the homomorphism $\mathcal{D}^{(\bullet)} \to \underset{\to X}{u}^{-1}(\mathcal{D}'^{(\bullet)})$.*

(a) Suppose $\underset{\to I',X}{\rho}^{-1}(\mathcal{F}'^{(\bullet)})$ is acyclic for the functors $\mathrm{Hom}_{\mathcal{D}'^{(\bullet)}_{\emptyset}}(\mathcal{Q}'^{(\bullet)}_{\emptyset}, -)$ for any flat left $\mathcal{D}'^{(\bullet)}_{\emptyset}$-module $\mathcal{Q}'^{(\bullet)}_{\emptyset}$. Then, for any flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module $\mathcal{Q}_{\emptyset}^{(\bullet)}$, we have:

(i) the left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module $\underset{\to I,X}{\rho}^{-1}(\underset{\to X}{u}^{-1}(\mathcal{F}'^{(\bullet)}))$ is acyclic for the functor $\mathrm{Hom}_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{Q}_{\emptyset}^{(\bullet)}, -)$,

(ii) the module $\underset{\to X}{u}^{-1}(\mathcal{F}'^{(\bullet)})$ is acyclic for the functor $\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}), -)$.

(b) Suppose $\mathcal{I}'^{(\bullet)} := \mathcal{F}'^{(\bullet)}$ is an injective left $\mathcal{D}'^{(\bullet)}$-module and let $\mathcal{Q}_{\emptyset}^{(\bullet)}$ be a flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module. Then:

(i) The left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module $\underset{\rightarrow I,X}{\rho}^{-1}(\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)}))$ is right acyclic for both functors $\mathrm{Hom}_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{Q}_{\emptyset}^{(\bullet)},-)$ *and* $\mathcal{H}om_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{Q}_{\emptyset}^{(\bullet)},-)$ *;*

(ii) The module $\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)})$ is right acyclic for the three functors of the form $\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),-)$, $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),-)$ *and* $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),-)$ *;*

(iii) The abelian sheaves $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)}))$ *and* $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)}))$ *are flasque.*

Proof. Let us check a). Since the non-respective case is checked similarly, let us prove the respective case. Let $\mathcal{Q}_{\emptyset}^{(\bullet)}$ be a flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module. We get $\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)})$ (see 8.2.3.3). By using the isomorphism

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),\underset{\rightarrow X}{u}^{-1}(\mathcal{F}'^{(\bullet)})) \underset{8.2.3.12.1}{\xrightarrow{\sim}} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{Q}_{\emptyset}^{(\bullet)},\underset{\rightarrow I,X}{\rho}^{-1}(\underset{\rightarrow X}{u}^{-1}(\mathcal{F}'^{(\bullet)}))) \tag{8.2.3.13.1}$$

we reduce to check (i).

Following 8.2.3.6 and 8.2.3.8.1, the hypothesis on $\underset{\rightarrow I',X}{\rho}^{-1}(\mathcal{F}'^{(\bullet)})$ is equivalent to saying that for any $i' \in I'$ the module $\mathcal{F}'^{(i')}$ is acyclic for the functors $\mathrm{Hom}_{\mathcal{D}'^{(i')}}(\mathcal{Q}'^{(i')},-)$ for any flat left $\mathcal{D}'^{(i')}$-module $\mathcal{Q}'^{(i')}$. This yields the last isomorphisms in $D^+(\mathbb{Z})$ (recall products are easily computed in $D^+(\mathbb{Z})$: see 8.2.3.6):

$$\begin{gathered}\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{Q}_{\emptyset}^{(\bullet)},\underset{\rightarrow I,X}{\rho}^{-1}(\underset{\rightarrow X}{u}^{-1}(\mathcal{F}'^{(\bullet)}))) \underset{8.2.3.8.1}{\xrightarrow{\sim}} \prod_{i\in I}\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{(i)}}(\mathcal{Q}_{\emptyset}^{(i)},\mathcal{F}'^{(u(i))}) \\ \xrightarrow{\sim} \prod_{i\in I}\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}'^{(u(i))}}(\mathcal{D}'^{(u(i))}\otimes^{\mathbb{L}}_{\mathcal{D}^{(i)}}\mathcal{Q}_{\emptyset}^{(i)},\mathcal{F}'^{(u(i))}) \xrightarrow{\sim} \prod_{i\in I}\mathrm{Hom}_{\mathcal{D}'^{(u(i))}}(\mathcal{D}'^{(u(i))}\otimes_{\mathcal{D}^{(i)}}\mathcal{Q}_{\emptyset}^{(i)},\mathcal{F}'^{(u(i))}).\end{gathered} \tag{8.2.3.13.2}$$

Let us check b). It follows from 8.2.3.11 that $\mathcal{I}'^{(\bullet)}$ satisfies the condition of the part (a) of the Lemma and therefore we get the acyclicity concerning the functors involving Hom. By replacing 8.2.3.8.1 by 8.2.3.10.3 and by using the acyclicity of 8.2.3.11, we get the isomorphisms 8.2.3.13.2 where Hom is replaced by $\mathcal{H}om$ and $\mathcal{F}'^{(\bullet)}$ is replaced by $\mathcal{I}'^{(\bullet)}$, which yields:

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_{\emptyset}^{(\bullet)}}(\mathcal{Q}_{\emptyset}^{(\bullet)},\underset{\rightarrow I,X}{\rho}^{-1}(\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)}))) \xrightarrow{\sim} \prod_{i\in I}\mathcal{H}om_{\mathcal{D}'^{(u(i))}}(\mathcal{D}'^{(u(i))}\otimes_{\mathcal{D}^{(i)}}\mathcal{Q}_{\emptyset}^{(i)},\mathcal{I}'^{(u(i))}),$$

where the product in computed in $D^+(\mathbb{Z}_X)$ (hence, beware this is not obvious that the right term is a module). Since $\mathcal{I}'^{(u(i))}$ is injective, then using again 8.2.3.11 the sheaf $\mathcal{G}^{(i)} := \mathcal{H}om_{\mathcal{D}'^{(u(i))}}(\mathcal{D}'^{(u(i))}\otimes_{\mathcal{D}^{(i)}}\mathcal{Q}_{\emptyset}^{(i)},\mathcal{I}'^{(u(i))})$ is flasque. By using 8.2.3.9.(b), the product $\prod_{i\in I}\mathcal{G}^{(i)}$ computed in the category of flasque abelian sheaves on X is also a product in $D^+(\mathbb{Z}_X)$. Hence, we get the acyclicity of (i). By replacing 8.2.3.12.1 by 8.2.3.12.2, we get the isomorphisms 8.2.3.13.1 where Hom is replaced by $\mathcal{H}om$ and $\mathcal{F}'^{(\bullet)}$ is replaced by $\mathcal{I}'^{(\bullet)}$. This implies the acyclity of (ii) from (i) for the first two functors. Concerning the third one,

$$\begin{gathered}\underset{\rightarrow X}{i}^{-1}\left(\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)}))\right) \overset{8.2.3.12.3}{\xrightarrow{\sim}} \prod_{j\geq i}\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(j)}}(\mathcal{Q}_{\emptyset}^{(j)},\mathcal{I}'^{(u(j))})\times\prod_{h<i}\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(h)}}(\mathcal{Q}_{\emptyset}^{(h)},\mathcal{I}'^{(u(i))}) \\ \overset{8.2.3.11}{\xrightarrow{\sim}} \prod_{j\geq i}\mathcal{H}om_{\mathcal{D}^{(j)}}(\mathcal{Q}_{\emptyset}^{(j)},\mathcal{I}'^{(u(j))})\times\prod_{h<i}\mathcal{H}om_{\mathcal{D}^{(h)}}(\mathcal{Q}_{\emptyset}^{(h)},\mathcal{I}'^{(u(i))}),\end{gathered} \tag{8.2.3.13.3}$$

where the last product is equal to the product in $K(\mathbb{Z}_X)$ (use again 8.2.3.9.(b), and is therefore a $\mathbb{Z}_X$-sheaf. Hence, we get the required acyclicity.

Finally, we get by composition the isomorphism

$$\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\rightarrow I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)}),\underset{\rightarrow X}{u}^{-1}(\mathcal{I}'^{(\bullet)})) \xrightarrow{\sim} \prod_{i\in I}\mathcal{H}om_{\mathcal{D}'^{(u(i))}}(\mathcal{D}'^{(u(i))}\otimes_{\mathcal{D}^{(i)}}\mathcal{Q}_{\emptyset}^{(i)},\mathcal{I}'^{(u(i))})$$

of flasque sheaves (recall a product of flasque sheaves is flasque), which yields (iii) for the first sheaf. Concerning the second sheaf, this is a consequence of 8.2.3.13.3.

□

Remark 8.2.3.14. With notation 8.2.3.13, the functor (denoted slightly abusively) $\underset{\to}{u}_X^{-1}$ is the composition of $\mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\underset{\to}{u}_X^{-1}\mathcal{D}^{(\bullet)})$ induced by the inverse image by the topos morphism $\underset{\to}{u}_X \colon X^{(I')} \to X^{(I)}$ with the forgetful functor $\mathrm{Mod}(\underset{\to}{u}_X^{-1}\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}'^{(\bullet)})$. Hence, we have checked above a unification of both cases.

Corollary 8.2.3.15. *Let $\lambda \in L(I)$ and $\chi \in M(I)$. Let $\mathcal{I}^{(\bullet)}$ be an injective left $\mathcal{D}^{(\bullet)}$-module. We consider $\chi^*\lambda^*(\mathcal{I}^{(\bullet)})$ as a left $\mathcal{D}^{(\bullet)}$-module via the canonical homomorphism $\mathcal{D}^{(\bullet)} \to \lambda^*(\mathcal{D}^{(\bullet)})$. Let $\mathcal{Q}_\emptyset^{(\bullet)}$ be a flat left $\mathcal{D}_\emptyset^{(\bullet)}$-module.*

(a) The left $\mathcal{D}_\emptyset^{(\bullet)}$-module $\underset{\to I,X}{\rho}^{-1}(\chi^*\lambda^*(\mathcal{I}^{(\bullet)}))$ is acyclic for the functors $\mathrm{Hom}_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{Q}_\emptyset^{(\bullet)}, -)$ and $\mathcal{H}om_{\mathcal{D}_\emptyset^{(\bullet)}}(\mathcal{Q}_\emptyset^{(\bullet)}, -)$.

(b) The module $\chi^*\lambda^*(\mathcal{I}^{(\bullet)})$ is acyclic for the following three functors $\mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), -)$, $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), -)$ and $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), -)$.

(c) The sheaves $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), \chi^*\lambda^*(\mathcal{I}^{(\bullet)}))$ and $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), \chi^*\lambda^*(\mathcal{I}^{(\bullet)}))$ are flasque.

Proof. Since $\underset{\to I,X}{\rho}^{-1}(\chi^*\lambda^*(\mathcal{I}^{(\bullet)})) = \underset{\to I,X}{\rho}^{-1}(\lambda^*(\mathcal{I}^{(\bullet)}))$, we get (a) by applying 8.2.3.13.(b) in the case where $u = \lambda$ (recall $\lambda^*(\mathcal{I}^{(\bullet)}) = \underset{\to}{u}_X^{-1}(\mathcal{I}^{(\bullet)})$). Since $\mathcal{Q}_\emptyset^{(\bullet)}$ is flat, then $\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)})$. Hence, we get b) and c) for the first functor from 8.2.3.13.(b) and via the isomorphisms:

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), \chi^*\lambda^*\mathcal{I}^{(\bullet)}) &\underset{8.2.3.12.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}_\emptyset}(\mathcal{Q}_\emptyset^{(\bullet)}, \underset{\to I,X}{\rho}^{-1}(\chi^*\lambda^*\mathcal{I}^{(\bullet)}))\\ = \mathbb{R}\mathcal{H}om_{\mathcal{D}_\emptyset}(\mathcal{Q}_\emptyset^{(\bullet)}, \underset{\to I,X}{\rho}^{-1}(\lambda^*\mathcal{I}^{(\bullet)})) &\underset{8.2.3.12.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), \lambda^*\mathcal{I}^{(\bullet)}).\end{aligned}$$

Similarly, we get b) and c) for the second functor from 8.2.3.13.(b) and via the isomorphisms (which is a consequence of 8.2.3.13.3):

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), \chi^*\lambda^*\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\underset{\to I,X,!}{\rho}(\mathcal{Q}_\emptyset^{(\bullet)}), \lambda^*\mathcal{I}^{(\bullet)}).$$

□

Lemma 8.2.3.16. *Let $\mathscr{P}$ be a collection of left $\mathcal{D}^{(\bullet)}$-modules. We denote by $\mathscr{I}(\mathscr{P})$ the collection of left $\mathcal{D}^{(\bullet)}$-modules which are acyclic for the functors $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet)}, -)$ (resp. $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet)}, -)$) for any $\mathcal{Q}^{(\bullet)} \in \mathscr{P}$. We denote by $\mathscr{P}^+$ the collection of left $\mathcal{D}^{(\bullet)}$-modules which are acyclic for the functors $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-, \mathcal{F}^{(\bullet)})$ (resp. $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-, \mathcal{F}^{(\bullet)})$) for any $\mathcal{F}^{(\bullet)} \in \mathscr{I}(\mathscr{P})$.*

Let $\mathcal{Q}_1^{(\bullet),\bullet} \to \mathcal{Q}_2^{(\bullet),\bullet}$ be a quasi-isomorphism of $K^-(\mathcal{D}^{(\bullet)})$ such that $\mathcal{Q}_i^{(\bullet),n} \in \mathscr{P}^+$, for any $i = 1, 2$, $n \in \mathbb{Z}$. Let $\mathcal{F}_1^{(\bullet),\bullet} \to \mathcal{F}_2^{(\bullet),\bullet}$ be a quasi-isomorphism of $K^+(\mathcal{D}^{(\bullet)})$ such that $\mathcal{F}_i^{(\bullet),n} \in \mathscr{I}(\mathscr{P})$, for any $i = 1, 2$, $n \in \mathbb{Z}$. Then the morphism of $K^+(\mathbb{Z}_X)$ (resp. $K^+(\mathbb{Z}_X^{(\bullet)})$)

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}_2^{(\bullet),\bullet}, \mathcal{F}_1^{(\bullet),\bullet}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}_1^{(\bullet),\bullet}, \mathcal{F}_2^{(\bullet),\bullet})$$
$$(\textit{resp. } \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}_2^{(\bullet),\bullet}, \mathcal{F}_1^{(\bullet),\bullet}) \to \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}_1^{(\bullet),\bullet}, \mathcal{F}_2^{(\bullet),\bullet})\ )$$

is a quasi-isomorphism.

Proof. Since the respective case is checked identically, let us only prove the non-respective one. 0) We remark that $\mathscr{I}(\mathscr{P})$ satisfies the following property:

(a) for any $\mathcal{F}^{(\bullet)}, \mathcal{G}^{(\bullet)} \in \mathscr{I}(\mathscr{P})$, we have $\mathcal{F}^{(\bullet)} \oplus \mathcal{G}^{(\bullet)} \in \mathscr{I}(\mathscr{P})$;

(b) for any exact sequence of left $\mathcal{D}^{(\bullet)}$-modules $0 \to \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to 0$, if $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \mathscr{I}(\mathscr{P})$, then $\mathcal{G}^{(\bullet)} \in \mathscr{I}(\mathscr{P})$.

and $\mathscr{P}^+$ satisties the dual property of $\mathscr{I}(\mathscr{P})$:

(a*) for any $\mathcal{F}^{(\bullet)}, \mathcal{G}^{(\bullet)} \in \mathscr{P}^+$, we have $\mathcal{F}^{(\bullet)} \oplus \mathcal{G}^{(\bullet)} \in \mathscr{P}^+$;

(b*) for any exact sequence of left $\mathcal{D}^{(\bullet)}$-modules $0 \to \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to 0$, if $\mathcal{F}^{(\bullet)}, \mathcal{G}^{(\bullet)} \in \mathscr{P}^+$, then $\mathcal{E}^{(\bullet)} \in \mathscr{P}^+$.

Hence, the cone of $\mathcal{Q}_1^{(\bullet),\bullet} \to \mathcal{Q}_2^{(\bullet),\bullet}$ is a complex of left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{P}^+$ and the cone of $\mathcal{F}_1^{(\bullet),\bullet} \to \mathcal{F}_2^{(\bullet),\bullet}$ is a complex of left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{I}(\mathscr{P})$. Hence we reduce to check the following two facts. Let $\mathcal{F}^{(\bullet),\bullet} \in K^+(\mathcal{D}^{(\bullet)})$ such that $\mathcal{F}^{(\bullet),n} \in \mathscr{I}(\mathscr{P})$, for any $n \in \mathbb{Z}$. Let $\mathcal{Q}^{(\bullet),\bullet} \in K^-(\mathcal{D}^{(\bullet)})$ such that $\mathcal{Q}^{(\bullet),n} \in \mathscr{P}^+$, for any $n \in \mathbb{Z}$.

1) In this step we prove that if $\mathcal{Q}^{(\bullet),\bullet}$ is acyclic then $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})$ is acyclic.

i) Remark that if $\mathcal{Q}^{(\bullet),n} = 0$ for any integer $n > a$ and $\mathcal{F}^{(\bullet),n} = 0$ for any integer $n < b$ then $H^n\left(\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})\right) = 0$ for any $n < b - a$.

ii) Since $\mathcal{Q}^{(\bullet),\bullet}$ is bounded above, by using the property (b^*) satisfied by $\mathscr{P}^+$, we can prove by decreasing induction on the integer $n \in \mathbb{Z}$ that $\ker d^n$ are left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{P}^+$, where $d^n \colon \mathcal{Q}^{(\bullet),n} \to \mathcal{Q}^{(\bullet),n+1}$ are the transition maps. For any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{F}^{(\bullet)} \in \mathscr{I}(\mathscr{P})$, the functor $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-, \mathcal{F}^{(\bullet)})$ sends short exact sequences of left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{P}^+$ to short exact sequences of abelian sheaves on X. Hence, by splitting $\mathcal{Q}^{(\bullet),\bullet}$ into short exact sequences, we can check that the complex $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),j})$ is acyclic for any integer j.

iii) We can suppose $\mathcal{F}^{(\bullet),n} = 0$ for any integer $n < 0$ and $\mathcal{Q}^{(\bullet),n} = 0$ for any integer $n > 0$. By applying the δ-functor $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, -)$ to the naive exact sequence

$$0 \to \mathcal{F}^{(\bullet),0} \to \mathcal{F}^{(\bullet),\bullet} \to \mathcal{F}^{(\bullet),\geq 1} \to 0,$$

we get the exact triangle

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),0}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\geq 1}) \to +1.$$

By using i) and ii), this yields that $H^0\left(\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})\right) = 0$. Similarly, by induction on $n \geq 0$ we check that $H^n\left(\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})\right) = 0$.

2) It remains to prove that if $\mathcal{F}^{(\bullet),\bullet}$ is acyclic then so is $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})$.

i) Since $\mathcal{F}^{(\bullet),\bullet}$ is bounded below, by using the property (b) satisfied by $\mathscr{I}(\mathscr{P})$, we can prove by induction on the integer $n \in \mathbb{Z}$ that $\operatorname{Im} d^n$ are left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{I}(\mathscr{P})$, where $d^n \colon \mathcal{F}^{(\bullet),n} \to \mathcal{F}^{(\bullet),n+1}$ are the transition maps. For any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{Q}^{(\bullet)} \in \mathscr{P}^+$, the functor $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet)}, -)$ sends short exact sequences of left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{I}(\mathscr{P})$ to short exact sequences of abelian sheaves on X. Hence, by splitting $\mathcal{F}^{(\bullet),\bullet}$ into short exact sequences, this implies that $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),j}, \mathcal{F}^{(\bullet),\bullet})$ is acyclic for any integer j.

ii) We can suppose $\mathcal{F}^{(\bullet),n} = 0$ for any integer $n < 0$ and $\mathcal{Q}^{(\bullet),n} = 0$ for any integer $n > 0$. By applying the δ-functor $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(-, \mathcal{F}^{(\bullet),\bullet})$ to the naive exact sequence

$$0 \to \mathcal{Q}^{(\bullet),0} \to \mathcal{Q}^{(\bullet),\bullet} \to \mathcal{Q}^{(\bullet),\leq -1} \to 0,$$

we get the exact triangle

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\leq -1}, \mathcal{F}^{(\bullet),\bullet}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),0}, \mathcal{F}^{(\bullet),\bullet}) \to +1.$$

By using 1.i.) and 2.ii), this yields that $H^0\left(\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})\right) = 0$. Similarly, by induction on $n \geq 0$ we check that $H^n\left(\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet),\bullet}, \mathcal{F}^{(\bullet),\bullet})\right) = 0$.

□

8.2.4 Derived homomorphism bifunctor over $LD_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$

8.2.4.1. Let $\mathcal{E}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$. The family $(\chi^*\lambda^*\mathcal{E}^{(\bullet)})_{(\lambda,\chi)\in L(I)\times M(I)}$ can be considered as an inductive system of objects of $X^{(I)}$ indexed by $L(I) \times M(I)$ with transition morphisms given by $\sigma_{\mathcal{E},(\lambda,\chi),(\lambda',\chi')}$ for any $(\lambda', \chi') \leq (\lambda, \chi)$ (see notation 8.1.4.1.1).

Notation 8.2.4.2. We denote by $\mathscr{I}(\mathcal{D}^{(\bullet)})$ or simply $\mathscr{I}$ the category of injective left $\mathcal{D}^{(\bullet)}$-modules, by $K^+(\mathscr{I})$ the full subcategory of $K^+(\mathcal{D}^{(\bullet)})$ whose objects consist in complexes of injective left $\mathcal{D}^{(\bullet)}$-modules. We denote by $\mathscr{P}(\mathcal{D}^{(\bullet)})$ or simply $\mathscr{P}$ the collection of objects of the form $\underset{\to I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)})$ where $\mathcal{Q}_{\emptyset}^{(\bullet)}$ is a flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module. We denote by $K^-(\mathscr{P})$ the full subcategory of $K^-(\mathcal{D}^{(\bullet)})$ whose objects consist in complexes of terms in $\mathscr{P}$.

We will need the following notation for some proofs. We denote by $\mathscr{I}(\mathscr{P})$ the collection of left $\mathcal{D}^{(\bullet)}$-modules which are acyclic for the functors $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet)},-)$ and $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{Q}^{(\bullet)},-)$ for any $\mathcal{Q}^{(\bullet)}\in\mathscr{P}$. We denote by $K^-(\mathscr{I}(\mathscr{P}))$ the full subcategory of $K^-(\mathcal{D}^{(\bullet)})$ whose objects consist in complexes of terms in $\mathscr{I}(\mathscr{P})$. Moreover, we denote by $\mathscr{P}^+$ the collection of left $\mathcal{D}^{(\bullet)}$-modules which are acyclic for the functors $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,\mathcal{F}^{(\bullet)})$ and $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\mathcal{F}^{(\bullet)})$ for any $\mathcal{F}^{(\bullet)}\in\mathscr{I}(\mathscr{P})$.

8.2.4.3. We have the following properties.

(a) Let $\mathcal{E}_{\emptyset}^{(\bullet)}$ be a flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module, i.e. $\mathcal{E}_{\emptyset}^{(i)}$ is a flat left $\mathcal{D}^{(i)}$-module for any $i\in I$. Then $\underset{\to I,X,!}{\rho}(\mathcal{E}_{\emptyset}^{(\bullet)})$ is a flat left $\mathcal{D}^{(\bullet)}$-modules, i.e. $\underset{\to X}{i}^{-1}(\underset{\to I,X,!}{\rho}(\mathcal{E}_{\emptyset}^{(\bullet)}))$ is a left $\mathcal{D}^{(i)}$-module for any $i\in I$.

(b) Let $\mathcal{E}^{(\bullet)}$ be a left $\mathcal{D}^{(\bullet)}$-module. Choose $\mathcal{Q}_{\emptyset}^{(\bullet)}$ a flat left $\mathcal{D}_{\emptyset}^{(\bullet)}$-module endowed with an epimorphism of left $\mathcal{D}_{\emptyset}^{(\bullet)}$-modules $\mathcal{Q}_{\emptyset}^{(\bullet)}\to\underset{\to I,X}{\rho}^{-1}(\mathcal{E}^{(\bullet)})$. Since $\underset{\to I,X,!}{\rho}$ is right exact (but not left exact when the transition homomorphisms of $\mathcal{D}^{(\bullet)}$ are not flat), this yields an epimorphism of left $\mathcal{D}^{(\bullet)}$-modules $\underset{\to I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)})\to\underset{\to I,X,!}{\rho}\circ\underset{\to I,X}{\rho}^{-1}(\mathcal{E}^{(\bullet)})$. Since the adjoint morphism $\underset{\to I,X,!}{\rho}\circ\underset{\to I,X}{\rho}^{-1}(\mathcal{E}^{(\bullet)})\to\mathcal{E}^{(\bullet)}$ is an epimorphism, this implies by composition the epimorphism of the form $\underset{\to I,X,!}{\rho}(\mathcal{Q}_{\emptyset}^{(\bullet)})\to\mathcal{E}^{(\bullet)}$.

(c) Let $\mathcal{E}^{(\bullet)}\in K^-(\mathcal{D}^{(\bullet)})$. It follows from (b) and [Sta22, 05T7], that there exists $\mathcal{P}^{(\bullet)}\in K^-(\mathscr{P})$ endowed with a quasi-isomorphism $\mathcal{P}^{(\bullet)}\xrightarrow{\sim}\mathcal{E}^{(\bullet)}$.

(d) Let $(\lambda,\chi)\in L(I)\times M(I)$, $\mathcal{P}^{(\bullet)}\in K^-(\mathscr{P})$, and $\mathcal{I}^{(\bullet)}\in K^+(\mathscr{I})$. It follows from 8.2.3.15 that $\chi^*\lambda^*\mathcal{I}^{(\bullet)}\in K^+(\mathscr{I}(\mathscr{P}))$ (see the notation of 8.2.4.2). Hence, since $\mathscr{P}\subset\mathscr{P}^+$, it follows from 8.2.3.16 that the complexes $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\chi^*\lambda^*\mathcal{I}^{(\bullet)})$ and $\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\chi^*\lambda^*\mathcal{I}^{(\bullet)})$ are acyclic if either $\mathcal{P}^{(\bullet)}$ or $\mathcal{I}^{(\bullet)}$ is acyclic.

Notation 8.2.4.4. Remark that since $L(I)\times M(I)$ is filtered, the functor $\varinjlim_{(\lambda,\chi)\in L(I)\times M(I)}$ is exact on the category of abelian sheaves on X. If no confusion is possible, let us simply write $\varinjlim_{\lambda,\chi}:=\varinjlim_{(\lambda,\chi)\in L(I)\times M(I)}$.

Consider the bifunctor

$$\varinjlim_{\lambda,\chi}\mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon K^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}}\times K^+(\mathcal{D}^{(\bullet)})\to K^+(\mathbb{Z}_X)\tag{8.2.4.4.1}$$

whose nth term for any integer $n\in\mathbb{Z}$ is defined for any $\mathcal{E}^{(\bullet),\bullet}\in K^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet),\bullet}\in K^+(\mathcal{D}^{(\bullet)})$ by setting

$$\varinjlim_{\lambda,\chi}\mathcal{H}om^{n}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \chi^*\lambda^*\mathcal{F}^{(\bullet),\bullet}):=\varinjlim_{\lambda,\chi}\prod_{p\in\mathbb{Z}}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),p},\ \chi^*\lambda^*\mathcal{F}^{(\bullet),\ p+n})\tag{8.2.4.4.2}$$

and the transition morphisms are given by the formula $d=d_{\mathcal{E}}+(-1)^{n+1}d_{\mathcal{F}}$.

8.2.4.5. Let $\mathcal{I}^{(\bullet)}\in K^+(\mathscr{I})$. Consider the functor

$$\phi_{\mathcal{I}^{(\bullet)}}\colon\varinjlim_{\lambda,\chi}\mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*\mathcal{I}^{(\bullet)})\colon K^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}}\to D^+(\mathbb{Z}_X).$$

Following 8.2.4.3.d, for any acyclic complex $\mathcal{P}^{(\bullet)}\in K^-(\mathscr{P})$, $\phi_{\mathcal{I}^{(\bullet)}}(\mathcal{P}^{(\bullet)})$ is acyclic. By using 8.2.4.3.c and [Sta22, 06XN], we get therefore the right derived functor $\mathbb{R}\phi_{\mathcal{I}^{(\bullet)}}\colon:D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}}\to D^+(\mathbb{Z}_X)$. This yields by functoriality the bifunctor

$$\mathbb{R}\varinjlim_{\lambda,\chi}\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}}\times K^+(\mathscr{I})\to D^+(\mathbb{Z}_X)$$

defined by setting

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{I}^{(\bullet)}) := \mathbb{R}\phi_{\mathcal{I}^{(\bullet)}}(\mathcal{E}^{(\bullet)})$$

for any $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$, $\mathcal{I}^{(\bullet)} \in K^+(\mathscr{I})$. Since the canonical functor $K^+(\mathscr{I}) \to D^+(\mathcal{D}^{(\bullet)})$ is an equivalence of categories, then we get the bifunctor

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D^+(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}_X). \tag{8.2.4.5.1}$$

For any $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)})$, by construction

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi^*\lambda^*\mathcal{I}^{(\bullet)})$$

where $\mathcal{I}^{(\bullet)} \in K^+(\mathscr{I})$ is a complex representing $\mathcal{F}^{(\bullet)}$ and $\mathcal{P}^{(\bullet)} \in K^-(\mathscr{P})$ is a complex representing $\mathcal{E}^{(\bullet)}$.

Proposition 8.2.4.6. *Let $Q^{\sharp}_{\mathrm{qi}}\colon K^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\mathcal{D}^{(\bullet)})$ be the localisation functor and $\mathcal{N}^{\sharp}_{\mathrm{qi}}(\mathcal{D}^{(\bullet)}) := \operatorname{Ker} Q^{\sharp}_{\mathrm{qi}}$ be the saturated null system of acyclic complexes of $K(\mathcal{D}^{(\bullet)})$; and similarly with $\mathbb{Z}_X$ instead of $\mathcal{D}^{(\bullet)}$.*

The functor 8.2.4.4.1 (remark that a bifunctor can be viewed as a functor) is right localizable with respect to $(\mathfrak{N}^-_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \mathfrak{N}^+_{\mathrm{qi}}(\mathcal{D}^{(\bullet)}), \mathfrak{N}^+_{\mathrm{qi}}(\mathbb{Z}_X))$ (in the sense of 7.4.1.10). Moreover, its right localization is the functor 8.2.4.5.1.

Proof. Denote by F the functor 8.2.4.4.1 and by Q_{qi} with the localization functor $K^+(\mathbb{Z}_X) \to D^+(\mathbb{Z}_X)$. Let $S^{\sharp}_{\mathrm{qi}}(\mathcal{D}^{(\bullet)}) := S(\mathfrak{N}^{\sharp}_{\mathrm{qi}}(\mathcal{D}^{(\bullet)}))$ be the saturated system of quasi-isomorphisms of $K^{\sharp}(\mathcal{D}^{(\bullet)})$ (we keep notation 7.4.1.2). Following 7.4.1.9.d and by definition (see 7.4.1.10), we have to check that the right derived functor of $Q_{\mathrm{qi}} \circ F$ with respect to $(S^-_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times S^+_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})$ is defined at any object of $K^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K^+(\mathcal{D}^{(\bullet)})$. By using 8.2.4.3.(c)-(d), we can checked that both conditions are satisfied: i) for any $(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \in K^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K^+(\mathcal{D}^{(\bullet)})$, there exists $(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)}) \in K^-(\mathscr{P})^{\mathrm{op}} \times K^+(\mathscr{I})$ and a morphism $(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to (\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)})$ in $(S^-_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times S^+_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})$ and ii) for any arrow $s\colon (\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)}) \to (\mathcal{P}'^{(\bullet)}, \mathcal{I}'^{(\bullet)})$ of $(S^-_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times S^+_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})$ with $(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet)}), (\mathcal{P}'^{(\bullet)}, \mathcal{I}'^{(\bullet)}) \in K^-(\mathscr{P})^{\mathrm{op}} \times K^+(\mathscr{I})$, its image by $Q_{\mathrm{qi}} \circ F$ is an isomorphism. Hence, we conclude by using [Sta22, 06XN]. ☐

8.2.4.7. Endowing the set $I \times L(I) \times M(I)$ with the order product, we can see $X^{(I\times L(I)\times M(I))}$ as the topos of inductive systems of objects of $X^{(I)}$ indexed by $L(I) \times M(I)$. We denote by $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$ the constant inductive system of rings of $X^{(I)}$ indexed by $L(I) \times M(I)$ with value $\mathcal{D}^{(\bullet)}$. We denote by $\mathbb{Z}_X^{(\bullet,\bullet)}$ the constant inductive system of rings of X indexed by $L(I) \times M(I)$ with value $\mathbb{Z}_X$.

Consider the bifunctor

$$\mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ -)\colon K(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K(\mathcal{D}^{(\bullet),(\bullet,\bullet)}) \to K(\mathbb{Z}_X^{(\bullet,\bullet)}) \tag{8.2.4.7.1}$$

which is defined for any $\mathcal{E}^{(\bullet),\bullet} \in K^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet),(\bullet,\bullet),\bullet} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$, for any $(\lambda,\chi) \in L(I) \times M(I)$ and $n \in \mathbb{Z}$ by setting

$$\mathcal{H}om^{(\lambda,\chi),n}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{F}^{(\bullet),(\bullet,\bullet),\bullet}) := \prod_{p\in\mathbb{Z}} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),p},\ \mathcal{F}^{(\bullet),(\lambda,\chi),\ p+n}) \tag{8.2.4.7.2}$$

and the transition morphisms are given by the formula $d = d_{\mathcal{E}} + (-1)^{n+1} d_{\mathcal{F}}$.

8.2.4.8. By construction, for any $\mathcal{E}^{(\bullet)} \in K^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet),(\bullet,\bullet)} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$, we can see the object $\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{F}^{(\bullet),(\bullet,\bullet)})$ as the following inductive system of $K(\mathbb{Z}_X)$ indexed by $L(I) \times M(I)$:

$$\mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{F}^{(\bullet),(\bullet,\bullet)}) = \left(\mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet),(\lambda,\chi)})\right)_{(\lambda,\chi)\in L(I)\times M(I)}, \tag{8.2.4.8.1}$$

where the right hand side is defined at 8.2.1.4.

8.2.4.9. The bifunctor 8.2.4.7.1 can be right derived as follows.

(a) Following 7.1.3.21.c (via the identification between inductive systems indexed by J and projective systems indexed by J° where $J = L(I) \times M(I)$, and by using the fact that the section 7.1 is still valid by replacing the topos $\mathrm{Top}(X)$ by any topos and in particular by the topos $X^{(I)}$), if $\mathcal{I}^{(\bullet),(\bullet,\bullet)}$ is an injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$-module, then for any $(\lambda,\chi) \in L(I) \times M(I)$ the left $\mathcal{D}^{(\bullet)}$-module $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ is injective.

(b) Similarly to (a), following 7.1.3.21.d if $\mathcal{I}^{(\bullet),(\bullet,\bullet)}$ is K-injective complex of $K({}^l\mathcal{D}^{(\bullet),(\bullet,\bullet)})$-module, then $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ is a K-injective complex of $K({}^l\mathcal{D}^{(\bullet)})$ for any $(\lambda,\chi) \in L(I) \times M(I)$.

(c) A morphism $\mathcal{G}^{(\bullet,\bullet)} \to \mathcal{G}'^{(\bullet,\bullet)}$ of $K(\mathbb{Z}_X^{(\bullet,\bullet)})$ is a quasi-isomorphism if and only if for any $(\lambda,\chi) \in L(I) \times M(I)$, the induced morphisms $\mathcal{G}^{(\lambda,\chi)} \to \mathcal{G}'^{(\lambda,\chi)}$ are quasi-isomorphisms.

(d) By using b) and c), by using the quasi-isomorphisms 8.2.2.1.1 and the equality 8.2.4.8.1, we can check that for any quasi-isomorphism $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \xrightarrow{\sim} \mathcal{I}'^{(\bullet),(\bullet,\bullet)}$ of K-injective complexes of $K(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$, for any quasi-isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$ of complexes of $K(\mathcal{D}^{(\bullet)})$, the morphism of $K(\mathbb{Z}_X^{(\bullet,\bullet)})$

$$\mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}'^{(\bullet)}, \mathcal{I}^{(\bullet),(\bullet,\bullet)}) \to \mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\bullet,\bullet)}) \tag{8.2.4.9.1}$$

is a quasi-isomorphism. Hence it follows from [Sta22, 06XN] that the bifunctor 8.2.4.7.1 is right localizable with respect to $(\mathfrak{N}_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \mathfrak{N}_{\mathrm{qi}}(\mathcal{D}^{(\bullet),(\bullet,\bullet)}), \mathfrak{N}_{\mathrm{qi}}(\mathbb{Z}_X))$. Its right localisation will be denoted by

$$\mathbb{R}\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(-,\ -)\colon D(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D(\mathcal{D}^{(\bullet),(\bullet,\bullet)}) \to D(\mathbb{Z}_X^{(\bullet,\bullet)}) \tag{8.2.4.9.2}$$

and is computed by taking K-injective representation of objects in $D(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$.

(e) The exact functor $\varinjlim_{\lambda,\chi}\colon \mathrm{Mod}(\mathbb{Z}_X^{(\bullet,\bullet)}) \to \mathrm{Mod}(\mathbb{Z}_X)$ induces $\varinjlim_{\lambda,\chi}\colon D(\mathbb{Z}_X^{(\bullet,\bullet)}) \to D(\mathbb{Z}_X)$. By composing 8.2.4.9.2 with this latter functor, we get the bifunctor

$$\varinjlim_{\lambda,\chi} \circ \mathbb{R}\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(-,\ -)\colon D(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D(\mathcal{D}^{(\bullet),(\bullet,\bullet)}) \to D(\mathbb{Z}_X). \tag{8.2.4.9.3}$$

8.2.4.10. We define the functor $\mathfrak{c}\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ by setting, for any left $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$, $\mathfrak{c}(\mathcal{E}^{(\bullet)})^{(\lambda,\chi)} := \chi^*\lambda^*(\mathcal{E}^{(\bullet)})$ and where for any $(\lambda_1,\chi_1) \le (\lambda_2,\chi_2)$ the transition maps $\chi_1^*\lambda_1^*\mathcal{E}^{(\bullet)} \to \chi_2^*\lambda_2^*\mathcal{E}^{(\bullet)}$ are the canonical ones i.e. are equal to $\sigma_{\mathcal{E},(\lambda_2,\chi_2),(\lambda_1,\chi_1)}$ (see 8.1.4.1.1). Since $\mathfrak{c}$ is exact, this yields the functor $\mathfrak{c}\colon D^\sharp(\mathcal{D}^{(\bullet)}) \to D^\sharp(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$, with $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.

Lemma 8.2.4.11. *For any $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)})$, we have the isomorphism*

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} \circ \mathbb{R}\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathfrak{c}(\mathcal{F}^{(\bullet)})), \tag{8.2.4.11.1}$$

where the right bifunctor is constructed at 8.2.4.9.3 and the left one at 8.2.4.5.1.

Proof. Let $\mathcal{P}^{(\bullet)} \in K^-(\mathscr{P})$ be a complex endowed with a quasi-isomorphism $\mathcal{P}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$ and $\mathcal{I}^{(\bullet)} \in K^+(\mathscr{I})$ be a complex endowed with a quasi-isomorphism $\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{I}^{(\bullet)}$. Since $\mathfrak{c}$ is exact, we get the quasi-isomorphism $\mathfrak{c}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathfrak{c}(\mathcal{I}^{(\bullet)})$. Let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$-modules endowed with a quasi-isomorphism $\mathfrak{c}(\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \mathcal{I}^{(\bullet),(\bullet,\bullet)}$.

Let $(\lambda,\chi) \in L(I) \times M(I)$. Since $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ is a bounded below complex of injective left $\mathcal{D}^{(\bullet)}$-modules (see 8.2.4.9.(a)), then we get the quasi-isomorphism

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) \xleftarrow{\sim} \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}). \tag{8.2.4.11.2}$$

Since $\mathfrak{c}(\mathcal{I}^{(\bullet)})^{(\lambda,\chi)} = \chi^*\lambda^*(\mathcal{F}^{(\bullet)})$, we get a quasi-isomorphism $\chi^*\lambda^*(\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \mathcal{I}^{(\bullet),(\lambda,\chi)}$ of $K^+(\mathcal{D}^{(\bullet)})$. With notation 8.2.4.2, it follows from 8.2.3.15 that $\chi^*\lambda^*(\mathcal{I}^{(\bullet)}) \in K(\mathscr{I}(\mathscr{P}))$. Since the term of $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ are injective, then we have also $\mathcal{I}^{(\bullet),(\lambda,\chi)} \in K(\mathscr{I}(\mathscr{P}))$. Hence, by using 8.2.3.16, we get that the morphism

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi^*\lambda^*\mathcal{I}^{(\bullet)}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) \tag{8.2.4.11.3}$$

is a quasi-isomorphism. By composing 8.2.4.11.2 and 8.2.4.11.3 and by taking the filtered inductive limit, we get the quasi-isomorphism of $K(\mathbb{Z}_X)$

$$\varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi^*\lambda^*\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}), \tag{8.2.4.11.4}$$

which is our isomorphism 8.2.4.11.1 in $D(\mathbb{Z}_X)$. □

8.2.4.12. Replacing everywhere $\mathcal{H}om$ by Hom, similarly to 8.2.4.5.1 and 8.2.4.9.2 we get the bifunctors

$$\mathbb{R}\varinjlim_{\lambda,\chi} \operatorname{Hom}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D^+(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}), \tag{8.2.4.12.1}$$

$$\mathbb{R}\operatorname{Hom}^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(-,\ -)\colon D(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D(\mathcal{D}^{(\bullet),(\bullet,\bullet)}) \to D(\mathbb{Z}^{(\bullet,\bullet)}). \tag{8.2.4.12.2}$$

Similarly to 8.2.4.11, for any $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)})$, we have the isomorphism

$$\mathbb{R}\varinjlim_{\lambda,\chi} \operatorname{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} \circ \mathbb{R}\operatorname{Hom}^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathfrak{c}(\mathcal{F}^{(\bullet)})). \tag{8.2.4.12.3}$$

When f is the canonical map of topological spaces $f\colon X \to \{*\}$ and $J = L(I) \times M(I)$, the functor $\underset{\to}{f}_{J*}$ (see notation 8.1.1.2.7) will simply be denoted by $\Gamma(X^{(\bullet,\bullet)}, -)$. If $\mathcal{G}^{(\bullet,\bullet)} \in D^+(\mathbb{Z}_X^{(\bullet,\bullet)})$, we have by definition

$$\Gamma(X^{(\bullet,\bullet)}, \mathcal{G}^{(\bullet,\bullet)}) = \left(\Gamma(X, \mathcal{G}^{(\lambda,\chi)})\right)_{(\lambda,\chi)\in L(I)\times M(I)}. \tag{8.2.4.12.4}$$

8.2.4.13. Replacing everywhere $\mathcal{H}om$ by $\mathscr{H}om$, similarly to 8.2.4.5.1 and 8.2.4.9.2 we get the bifunctors

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D^+(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}_X^{(\bullet)}), \tag{8.2.4.13.1}$$

$$\mathbb{R}\mathscr{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(-,\ -)\colon D(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D(\mathcal{D}^{(\bullet),(\bullet,\bullet)}) \to D(\mathbb{Z}_X^{(\bullet),(\bullet,\bullet)}). \tag{8.2.4.13.2}$$

Similarly to 8.2.4.11, for any $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)})$, we have the isomorphism

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} \circ \mathbb{R}\mathscr{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathfrak{c}(\mathcal{F}^{(\bullet)})). \tag{8.2.4.13.3}$$

When f is the canonical map of topological spaces $f\colon X \to \{*\}$ and $J = I \times L(I) \times M(I)$, the functor $\underset{\to}{f}_{J*}$ (see notation 8.1.1.2.7) will simply be denoted by $\Gamma(X^{(\bullet),(\bullet,\bullet)}, -)$. If $\mathcal{G}^{(\bullet),(\bullet,\bullet)} \in D^+(\mathbb{Z}_X^{(\bullet),(\bullet,\bullet)})$, we have by definition

$$\Gamma(X^{(\bullet),(\bullet,\bullet)}, \mathcal{G}^{(\bullet),(\bullet,\bullet)}) = \left(\Gamma(X^{(\bullet)}, \mathcal{G}^{(\bullet),(\lambda,\chi)})\right)_{(\lambda,\chi)\in L(I)\times M(I)} = \left(\Gamma(X, \mathcal{G}^{(i),(\lambda,\chi)})\right)_{(i,\lambda,\chi)\in I\times L(I)\times M(I)}. \tag{8.2.4.13.4}$$

Lemma 8.2.4.14. *We have the following properties.*

1. *If X is coherent, then for any $\mathcal{E}^{(\bullet,\bullet)} \in D^+(\mathbb{Z}_X^{(\bullet,\bullet)})$ we have the functorial isomorphism*

$$\varinjlim_{\lambda,\chi} \circ \mathbb{R}\Gamma(X^{(\bullet,\bullet)}, \mathcal{E}^{(\bullet,\bullet)}) \xrightarrow{\sim} \mathbb{R}\Gamma(X, \varinjlim_{\lambda,\chi} \mathcal{E}^{(\bullet,\bullet)}). \tag{8.2.4.14.1}$$

2. *If I^o is coherent (for the topology 7.1.2.13) and X is coherent, then for any $\mathcal{E}^{(\bullet),(\bullet,\bullet)} \in D^+(\mathbb{Z}_X^{(\bullet),(\bullet,\bullet)})$ we have the functorial isomorphism*

$$\varinjlim_{\lambda,\chi} \circ \mathbb{R}\Gamma(X^{(\bullet),(\bullet,\bullet)}, \mathcal{E}^{(\bullet),(\bullet,\bullet)}) \xrightarrow{\sim} \mathbb{R}\Gamma(X^{(\bullet)}, \varinjlim_{\lambda,\chi} \mathcal{E}^{(\bullet),(\bullet,\bullet)}). \tag{8.2.4.14.2}$$

Proof. Since the first isomorphism is checked similarly, let us only check 8.2.4.14.2. Let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^+(\mathbb{Z}_X^{(\bullet),(\bullet,\bullet)})$ be a complex of injective abelian sheaves on $X^{(\bullet),(\bullet,\bullet)}$ quasi-isomorphic to $\mathcal{E}^{(\bullet),(\bullet,\bullet)}$. Then, for any $(\lambda,\chi) \in L(I) \times M(I)$, $\mathcal{I}^{(\lambda,\chi)}$ is a complex of flasque abelian sheaves on $X^{(\bullet)}$. Since I^o and X are coherent, then filtered inductive limits of flasque abelian sheaves on $X^{(\bullet)}$ are flasque (see 7.1.2.19.(c)) and since $X^{(\bullet)}$ is coherent then filtered inductive limits commute of abelian sheaves on $X^{(\bullet)}$ with the global section functor $\Gamma(X^{(\bullet)},-)$ (see 7.1.2.16). This yields

$$\varinjlim_{\lambda,\chi} \circ \mathbb{R}\Gamma(X^{(\bullet),(\bullet,\bullet)}, \mathcal{E}^{(\bullet),(\bullet,\bullet)}) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} \left(\Gamma(X^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)})\right)_{\lambda,\chi} \tag{8.2.4.14.3}$$

$$\xrightarrow{\sim} \Gamma(X^{(\bullet)}, \varinjlim_{\lambda,\chi} \mathcal{I}^{(\bullet),(\lambda,\chi)}) \xrightarrow{\sim} \mathbb{R}\Gamma(X^{(\bullet)}, \varinjlim_{\lambda,\chi} \mathcal{E}^{(\bullet),(\bullet,\bullet)}). \tag{8.2.4.14.4}$$

□

Proposition 8.2.4.15. *Let $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)})$, $\mathcal{G}^{(\bullet),(\bullet,\bullet)} \in D^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$.*

(a) We have the bifunctorial isomorphisms

$$\mathbb{R}\Gamma(X^{(\bullet,\bullet)}, \mathbb{R}\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)})) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)}), \tag{8.2.4.15.1}$$

$$\mathbb{R}\Gamma(X^{(\bullet),(\bullet,\bullet)}, \mathbb{R}\mathscr{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)})) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)}). \tag{8.2.4.15.2}$$

(b) If X is coherent, then we have the bifunctorial isomorphisms

$$\mathbb{R}\Gamma(X, \mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\varinjlim_{\lambda,\chi} \operatorname{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}). \tag{8.2.4.15.3}$$

(c) If I^o is coherent (for the topology 7.1.2.13) and X is coherent, then we have the bifunctorial isomorphisms

$$\mathbb{R}\Gamma(X^{(\bullet)}, \mathbb{R}\varinjlim_{\lambda,\chi} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\varinjlim_{\lambda,\chi} \operatorname{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}). \tag{8.2.4.15.4}$$

Proof. Let us check 8.2.4.15.1. Let $\mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$-modules representing $\mathcal{G}^{(\bullet),(\bullet,\bullet)}$. By construction,

$$\mathbb{R}\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)}) \xrightarrow{\sim} \mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet}).$$

For any integers r,s, since $\mathcal{I}^{(\bullet),(\bullet,\bullet),s}$ is injective, then $\mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),r},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),s}))$ is a flasque sheave on X (see 8.2.1.3). Hence, $\mathcal{H}om^{(\lambda,\chi),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet})$ is a complex of flasque abelian sheaves on X for any $(\lambda,\chi) \in L(I) \times M(I)$. Following 7.1.3.15.2 (recall an inductive system can be viewed as a projective system), since $\mathcal{H}om^{(\lambda,\chi),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet}))$ is acyclic for the functor $\Gamma(X,-)$, this yields that $\mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet}))$ is acyclic for the functor $\Gamma(X^{(\bullet,\bullet)},-)$. Hence, we get

$$\mathbb{R}\Gamma(X^{(\bullet,\bullet)}, \mathbb{R}\mathcal{H}om^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)}))) \xrightarrow{\sim} \Gamma(X^{(\bullet,\bullet)}, \mathcal{H}om^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet}))$$

$$\xrightarrow[8.2.1.1.3]{\sim} \operatorname{Hom}^{(\bullet,\bullet),\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{I}^{(\bullet),(\bullet,\bullet),\bullet}) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}^{(\bullet,\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet),(\bullet,\bullet)})), \tag{8.2.4.15.5}$$

where for the second isomorphism we also use 8.2.4.8.1 and 8.2.4.12.4. Similarly, we prove 8.2.4.15.2 by using 8.2.1.1.1, 8.2.4.14.2 and 8.2.4.13.3.

Finally, by applying the functor $\varinjlim_{\lambda,\chi}$ to 8.2.4.15.1 (resp. 8.2.4.15.2) in the case where $\mathcal{G}^{(\bullet),(\bullet,\bullet)} = \mathfrak{c}(\mathcal{F}^{(\bullet)})$, we get 8.2.4.15.3 (resp. 8.2.4.15.4), modulo the isomorphisms of 8.2.4.14, 8.2.4.11.1 (resp.8.2.4.13.3) and 8.2.4.12.3. □

8.2.4.16. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$, $\lambda \in L(I)$, $\chi \in M(I)$ and $f \colon \chi^*\lambda^*\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $K(\mathcal{D}^{(\bullet)})$. With notation 8.1.4.1, consider the following diagram

$$\begin{array}{ccccc} \mathcal{E}^{(\bullet)} & \xrightarrow{\sigma_{\mathcal{E},(\lambda,\chi)}} & \chi^*\lambda^*\mathcal{E}^{(\bullet)} & \xrightarrow{f} & \mathcal{F}^{(\bullet)} \\ \downarrow{\scriptstyle \sigma_{\mathcal{E},(\lambda,\chi)}} & & \downarrow{\scriptstyle \sigma_{\chi^*\lambda^*\mathcal{E},(\lambda,\chi)}} & & \downarrow{\scriptstyle \sigma_{\mathcal{F},(\lambda,\chi)}} \\ \chi^*\lambda^*\mathcal{E}^{(\bullet)} & \xrightarrow{\chi^*\lambda^*(\sigma_{\mathcal{E},(\lambda,\chi)})} & \chi^*\lambda^*\chi^*\lambda^*\mathcal{E}^{(\bullet)} & \xrightarrow{\chi^*\lambda^*(f)} & \chi^*\lambda^*\mathcal{F}^{(\bullet)} \end{array} \tag{8.2.4.16.1}$$

which is commutative by functoriality. Concerning the left square, we compute more precisely that $\chi^*\lambda^*(\sigma_{\mathcal{E},(\lambda,\chi)}) = \sigma_{\chi^*\lambda^*\mathcal{E},(\lambda,\chi)}$.

If moreover $\mathcal{F}^{(\bullet)} \in K(\lambda^*\mathcal{D}^{(\bullet)})$ and $f \colon \chi^*\lambda^*\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ is a morphism of $K(\lambda^*\mathcal{D}^{(\bullet)})$, then the right square of 8.2.4.16.1 is therefore a commutative square of $K(\lambda^*\mathcal{D}^{(\bullet)})$.

8.2.4.17. Let $u \colon E' \to E$ be a topoi morphism. Let $\mathcal{D}$ be a sheaf of rings on E. Then, since (u^{-1}, u_*) is a adjoint paire, it follows from [SGA4.1, IV.13.4.2] that for any $\mathcal{D}$-modules $\mathcal{E}$ and $\mathcal{F}$ we have the canonical bifunctorial morphism $u^{-1}\mathscr{H}om_{\mathcal{D}}(\mathcal{E}, \mathcal{F}) \to \mathscr{H}om_{u^{-1}\mathcal{D}}(\mathcal{E}, \mathcal{F})$.

8.2.4.18. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$.

(a) Let $\chi \in M(I)$. The functor $\chi^* \colon K(\mathcal{D}^{(\bullet)}) \to K(\mathcal{D}^{(\bullet)})$ induces

$$\mathrm{Hom}^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathrm{Hom}^\bullet_{\mathcal{D}^{(\bullet)}}(\chi^*\mathcal{E}^{(\bullet)}, \chi^*\mathcal{F}^{(\bullet)}), \tag{8.2.4.18.1}$$

which is $f^{(\bullet)} \mapsto f^{(\bullet)}$ (the identity). By using 8.2.1.1.3 (resp. 8.2.1.2.1), we get from 8.2.4.18.1 a similar to 8.2.4.18.1 morphism with $\mathcal{H}om$ (resp. $\mathscr{H}om$) instead of Hom.

(b) Let $\lambda \colon I \to I$ be a morphism of $L(I)$. Remark than λ is continuous for the canonical topology (see 7.1.2.13) and also that $\lambda^* = \underrightarrow{\lambda}_X^{-1}$, where $\underrightarrow{\lambda}_X \colon X^{(I)} \to X^{(I)}$ is the topoi morphism induced by λ (see 8.1.1.2.1) We get a ringed topoi morphism $\underrightarrow{\lambda}_X \colon (X^{(I)}, \lambda^*\mathcal{D}^{(\bullet)}), \to (X^{(I)}, \mathcal{D}^{(\bullet)})$. This yields from 8.2.4.17 the canonical bifunctorial morphism of abelian sheaves on $X^{(I)}$:

$$\lambda^*\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathscr{H}om^\bullet_{\lambda^*\mathcal{D}^{(\bullet)}}(\lambda^*\mathcal{E}^{(\bullet)}, \lambda^*\mathcal{F}^{(\bullet)}). \tag{8.2.4.18.2}$$

Modulo 8.2.1.2.1, it corresponds to the functorial on $i \in I$ (the transition maps are the forgetful ones) family of morphisms

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}|_{\lambda(i)}}(\mathcal{E}^{(\bullet)}|_{\lambda(i)}, \mathcal{F}^{(\bullet)}|_{\lambda(i)}) \to \mathcal{H}om^\bullet_{(\lambda^*\mathcal{D}^{(\bullet)})|_i}((\lambda^*\mathcal{E}^{(\bullet)})|_i, (\lambda^*\mathcal{F}^{(\bullet)})|_i),$$

which is given by the forgetful map. The map λ induces also the morphism of abelian sheaves on X:

$$\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathcal{H}om^\bullet_{\lambda^*\mathcal{D}^{(\bullet)}}(\lambda^*\mathcal{E}^{(\bullet)}, \lambda^*\mathcal{F}^{(\bullet)}). \tag{8.2.4.18.3}$$

8.2.4.19. Let $\lambda \colon I \to I$ be a morphism of $L(I)$. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in K(\lambda^*\mathcal{D}^{(\bullet)})$. We have the forgetful morphisms of abelian sheaves:

$$\mathcal{H}om^\bullet_{\lambda^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}), \tag{8.2.4.19.1}$$

$$\mathscr{H}om^\bullet_{\lambda^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \lambda^*\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}), \tag{8.2.4.19.2}$$

8.2.4.20. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$, $\lambda \in L(I)$, $\chi \in M(I)$, $\mathcal{G}^{(\bullet)} \in K(\lambda^*\mathcal{D}^{(\bullet)})$. Consider the diagram

$$\begin{array}{ccccc} \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) & \xrightarrow{(3)} & \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) & \xrightarrow{(2)} & \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \\ \downarrow{\scriptstyle (1)} & & {\scriptstyle (\star)}\downarrow & & {\scriptstyle (4)}\uparrow \\ \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) & \longleftarrow & \mathcal{H}om^\bullet_{\lambda^*\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}) & \longrightarrow & \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)}, \chi^*\lambda^*\mathcal{F}^{(\bullet)}), \end{array} \tag{8.2.4.20.1}$$

where the numbered arrows are given by functoriality of the bifunctor $\mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(-,-)$ via the morphisms $\sigma_{\mathcal{E},(\lambda,\chi)}$ for the top horizontal maps (and by composition with the forgetful map $\mathcal{H}om^\bullet_{\lambda^*\mathcal{D}^{(\bullet)}}(-,-) \to$

$\mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,-)$ for the bottom morphism) and $\sigma_{\mathcal{F},(\lambda,\chi)}$ for the vertical maps and where the arrow $(\star)$ is defined by functoriality of the functor $\chi^*\lambda^*$ (see 8.2.4.18). Since $\chi^*\lambda^*(\sigma_{\mathcal{E},(\lambda,\chi)}) = \sigma_{\chi^*\lambda^*\mathcal{E},(\lambda,\chi)}$, then by using the commutativity of the diagram 8.2.4.16.1 we get the commutativity of the left square of 8.2.4.20.1. The commutativity of the right triangle is checked by functoriality in $\mathcal{E}$ of $\sigma_{\mathcal{E},(\lambda,\chi)}$.

Consider the diagram

$$\begin{tikzcd}
\lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \arrow[r,"(3)"] \arrow[d,"(1)"] & \lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \arrow[r,"(2)"] \arrow[d,"(\star)"] & \lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) \\
\lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) & \mathscr{H}om^{\bullet}_{\lambda^*\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) \arrow[l] \arrow[r] & \lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}), \arrow[u,"(4)"']
\end{tikzcd} \tag{8.2.4.20.2}$$

where both bottom horizontal arrows are equal to the composition morphism of 8.2.4.19.2, where the arrow $(\star)$ is defined by the functor $\chi^*\lambda^*$ (see 8.2.4.18), where the numbered arrows are given by functoriality of the bifunctor $\lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,-)$. We easily check its commutativity.

From the diagram 8.2.4.16.1 (where $\mathcal{F}$ is replaced by $\mathcal{G}$ and f is a morphism of $K(\lambda^*\mathcal{D}^{(\bullet)})$) we check similarly the commutativity of the left square of the diagram

$$\begin{tikzcd}
\mathscr{H}om^{\bullet}_{\lambda^*\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\mathcal{G}^{(\bullet)}) \arrow[r] \arrow[d] \arrow[rr, bend left] & \lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\mathcal{G}^{(\bullet)}) \arrow[d] & \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\mathcal{G}^{(\bullet)}) \arrow[l] \arrow[d] \\
\mathscr{H}om^{\bullet}_{\lambda^*\mathcal{D}^{(\bullet)}}(\chi^*\lambda^*\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{G}^{(\bullet)}) & \lambda^*\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{G}^{(\bullet)}) \arrow[l,"(\star)"'] & \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{G}^{(\bullet)}), \arrow[l]
\end{tikzcd} \tag{8.2.4.20.3}$$

where the curvy arrow is the first morphism of 8.2.4.19.2, where the left top horizontal arrow is the composition of 8.2.4.19.2 (i.e. the top "triangle" is commutative by definition), where the arrow $(\star)$ is defined by the functor $\chi^*\lambda^*$ (see 8.2.4.18), where the vertical arrows are given by functoriality from the canonical morphisms $\sigma_{\mathcal{E},(\lambda,\chi)}$ or $\sigma_{\mathcal{F},(\lambda,\chi)}$. We easily check its commutativity.

Lemma 8.2.4.21. *Let $f\colon \mathcal{E}'^{(\bullet)} \to \mathcal{E}^{(\bullet)}$ be a morphism belonging to $S^-(\mathcal{D}^{(\bullet)})$ and $g\colon \mathcal{F}^{(\bullet)} \to \mathcal{F}'^{(\bullet)}$ be a morphism belonging to $S^+(\mathcal{D}^{(\bullet)})$, i.e. f and g are lim-ind-isogenies (see 8.1.4.3). Then the canonical morphisms*

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \mathbb{R}\varinjlim_{\lambda,\chi} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}'^{(\bullet)},\chi^*\lambda^*\mathcal{F}'^{(\bullet)}) \tag{8.2.4.21.1}$$

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}'^{(\bullet)},\chi^*\lambda^*\mathcal{F}'^{(\bullet)}) \tag{8.2.4.21.2}$$

$$\mathbb{R}\varinjlim_{\lambda,\chi} \mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \mathbb{R}\varinjlim_{\lambda,\chi} \mathrm{Hom}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}'^{(\bullet)},\chi^*\lambda^*\mathcal{F}'^{(\bullet)}) \tag{8.2.4.21.3}$$

are isomorphisms.

Proof. Since the proofs are similar (replace $\mathscr{H}om$ by Hom or $\mathcal{H}om$), let us prove 8.2.4.21.1. We easily reduce to the following two cases.

1) Suppose there exists $\chi_0 \in M(I)$ and $\lambda_0 \in L(I)$ such that $g = \sigma_{\mathcal{F},(\lambda_0,\chi_0)}$ and $f = \mathrm{id}$ (see notation 8.1.4.1). Let $\mathcal{I}^{(\bullet)} \in K^+(\mathscr{I})$ be a complex representing $\mathcal{F}^{(\bullet)}$ and $\mathcal{P}^{(\bullet)} \in K^-(\mathscr{P})$ is a complex representing $\mathcal{E}^{(\bullet)}$. Using 8.1.2.2 and 8.1.3.2, we get

$$\chi^*\lambda^*\chi_0^*\lambda_0^*(\mathcal{I}^{(\bullet)}) = \chi^*(\chi_0\circ\lambda)^*\lambda^*\lambda_0^*(\mathcal{I}^{(\bullet)}) = (\chi + (\chi_0\circ\lambda))^*(\lambda_0\circ\lambda)^*(\mathcal{I}^{(\bullet)}). \tag{8.2.4.21.4}$$

Since the map $\lambda \mapsto \lambda_0\circ\lambda$ is an element of $L(L(I))$, since the map $\chi \mapsto \chi + (\chi_0\circ\lambda)$ is an element of $L(M(I))$, this yields the morphism

$$\varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\chi^*\lambda^*\mathcal{I}^{(\bullet)}) \to \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\chi^*\lambda^*\chi_0^*\lambda_0^*(\mathcal{I}^{(\bullet)})) \tag{8.2.4.21.5}$$

is an isomorphism.

Let $\mathcal{I}'^{(\bullet)} \in K^+(\mathscr{J})$ be a complex representing $\chi_0^*\lambda_0^*\mathcal{I}^{(\bullet)}$. For any $\chi \in M(I)$ and $\lambda \in L(I)$, the functors χ^* and λ^* are exact, Hence, we get the quasi-isomorphism $\chi^*\lambda^*\chi_0^*\lambda_0^*(\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \chi^*\lambda^*\mathcal{I}'^{(\bullet)}$. Following 8.2.3.15, the modules of the complex $\chi^*\lambda^*\mathcal{I}'^{(\bullet)}$ belong to $\mathscr{J}(\mathscr{P})$ (see notation 8.2.4.2). Using 8.2.4.21.4, we check similarly that the modules of the complex $\chi^*\lambda^*\chi_0^*\lambda_0^*(\mathcal{I}^{(\bullet)})$ belong to $\mathscr{J}(\mathscr{P})$. Hence, it follows from 8.2.3.16 that the morphism

$$\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi^*\lambda^*\chi_0^*\lambda_0^*(\mathcal{I}^{(\bullet)})) \to \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi^*\lambda^*\mathcal{I}'^{(\bullet)}) \tag{8.2.4.21.6}$$

is an isomorphism. By applying the filtered limit $\varinjlim_{\lambda,\chi}$ to 8.2.4.21.6 and by composing this latter morphism with 8.2.4.21.5, we get the isomorphism 8.2.4.21.3.

2) Suppose there exists $\chi_0 \in M(I)$ and $\lambda_0 \in L(I)$ such that $f = \sigma_{\mathcal{E},(\lambda_0,\chi_0)}$ and $g = \mathrm{id}$.

i) Since the category $\mathscr{P}$ is not stable under the functor of the form $\chi_0^*\lambda_0^*$, via the isomorphism 8.2.4.13.3 this is more convenient to check that the canonical morphism

$$\varinjlim_{\lambda,\chi} \circ \mathbb{R}\mathscr{H}om^{(\bullet;\bullet)}_{\mathcal{D}^{(\bullet)}}(\chi_0^*\lambda_0^*\mathcal{E}^{(\bullet)},\ \mathfrak{c}(\mathcal{F}^{(\bullet)})) \to \varinjlim_{\lambda,\chi} \circ \mathbb{R}\mathscr{H}om^{(\bullet;\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathfrak{c}(\mathcal{F}^{(\bullet)})) \tag{8.2.4.21.7}$$

is an isomorphism.

ii) Let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$-modules representing $\mathfrak{c}(\mathcal{F}^{(\bullet)})$. Let $\mathcal{I}'^{(\bullet),(\bullet,\bullet)} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$-modules endowed with a quasi-isomorphism $\chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\bullet,\bullet)} \to \mathcal{I}'^{(\bullet),(\bullet,\bullet)}$ of $K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ where $\chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\bullet,\bullet)} := (\chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)})_{\lambda,\chi}$. Consider the following diagram

$$\begin{array}{ccc}
\varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\chi_0^*\lambda_0^*\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(5)} & \varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) \\
\downarrow{\scriptstyle (1)} & \swarrow{\scriptstyle (\star)} & \downarrow{\scriptstyle (3)} \\
\varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\chi_0^*\lambda_0^*\mathcal{E}^{(\bullet)}, \chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(6)} & \varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) \\
\downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (4)} \\
\varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\chi_0^*\lambda_0^*\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(7)} & \varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)}),
\end{array} \tag{8.2.4.21.8}$$

where the numbered arrows are given by functoriality of the bifunctor $\varinjlim_{\lambda,\chi} \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(-,-)$ and where the arrow $(\star)$ is defined by functoriality from the functor $\chi_0^*\lambda_0^*$ (i.e. is the composition of 8.2.4.18.3 with the first morphism of 8.2.4.19.1). From the $\mathscr{H}om$ version of the commutative diagram 8.2.4.20.1, we get the commutativity of both triangles of the upper square of the diagram 8.2.4.21.8. Since the lower square is commutative by fonctoriality, the whole diagram 8.2.4.21.8 is commutative. Remark that the top morphism (5) represents 8.2.4.21.7. We now check this morphism (5) is an isomorphism via the following steps.

iii) Let us check that the composition morphism $(4)\circ(3)$ is a quasi-isomorphism. Let $\mathcal{P}^{(\bullet)} \in K^+(\mathcal{D}^{(\bullet)})$ be a complex of left $\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{P}$ representing $\mathcal{E}^{(\bullet)}$. Since $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ and $\mathcal{I}'^{(\bullet),(\lambda,\chi)}$ are bounded below complexes of injective left $\mathcal{D}^{(\bullet)}$-modules, then we reduce to check that $(4)\circ(3)$ is a quasi-isomorphism when $\mathcal{E}^{(\bullet)}$ is replaced by $\mathcal{P}^{(\bullet)}$. Let $\mathcal{I}^{(\bullet)} \in K^+(\mathcal{D}^{(\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet)}$-modules representing $\mathcal{F}^{(\bullet)}$. Since $\mathfrak{c}$ is exact, then $\mathfrak{c}(\mathcal{I}^{(\bullet)})$ is quasi-isomorphic to $\mathfrak{c}(\mathcal{F}^{(\bullet)})$. Hence, there exists a quasi-isomorphim of $K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ of the form $\mathfrak{c}(\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \mathcal{I}^{(\bullet),(\bullet,\bullet)}$. This induces the quasi-isomorphisms of $K^+(\mathcal{D}^{(\bullet)})$ of the form $\chi^*\lambda^*\mathcal{I}^{(\bullet)} \xrightarrow{\sim} \mathcal{I}^{(\bullet),(\lambda,\chi)}$ for any $\lambda \in L(I)$, $\chi \in M(I)$. Consider the diagram

$$\begin{array}{ccccc}
\varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)}) \\
\uparrow & & \uparrow & & \\
\varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi^*\lambda^*\mathcal{I}^{(\bullet)}) & \longrightarrow & \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)}, \chi_0^*\lambda_0^*\chi^*\lambda^*\mathcal{I}^{(\bullet)}), & &
\end{array} \tag{8.2.4.21.9}$$

whose morphisms are given by functoriality of the bifunctor $\varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,-)$.

Since $\chi^*\lambda^*\mathcal{I}^{(\bullet)} \xrightarrow{\sim} \mathcal{I}^{(\bullet),(\lambda,\chi)}$ and $\chi_0^*\lambda_0^*\chi^*\lambda^*\mathcal{I}^{(\bullet)} \xrightarrow{\sim} \chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}$ are quasi-isomorphisms of complexes of $K^+(\mathcal{D}^{(\bullet)})$ whose terms belong to $\mathscr{I}(\mathscr{P})$ (see Proposition 8.2.3.15) for any $\lambda \in L(I)$, $\chi \in M(I)$, then it follows from 8.2.3.16 that both vertical morphisms of 8.2.4.21.9 are quasi-isomorphisms. Similarly, we get from the quasi-isomorphism $\chi_0^*\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)} \xrightarrow{\sim} \mathcal{I}'^{(\bullet),(\lambda,\chi)}$, from 8.2.3.15 and 8.2.3.16 that the right top horizontal morphism of 8.2.4.21.9 is a quasi-isomorphism.

Using some formulas of 8.1.2.2 and 8.1.3.2, we compute

$$\chi_0^*\lambda_0^*\chi^*\lambda^*(\mathcal{I}^{(\bullet)}) = \chi_0^*(\chi\circ\lambda_0)^*\lambda_0^*\lambda^*(\mathcal{I}^{(\bullet)}) = (\chi_0 + (\chi\circ\lambda_0))^*(\lambda\circ\lambda_0)^*(\mathcal{I}^{(\bullet)}). \tag{8.2.4.21.10}$$

Since the map $\lambda \mapsto \lambda\circ\lambda_0$ is an element of $L(L(I)$, since the map $\chi \mapsto \chi_0 + (\chi\circ\lambda_0)$ is an element of $L(M(I))$, this yields the bottom horizontal morphism of 8.2.4.21.9 is an isomorphism. Hence, the morphisms of 8.2.4.21.9 are quasi-isomorphisms, in particular the composition of the top horizontal ones, i.e. $(4)\circ(3)$ is a quasi-isomorphism.

iv) Replacing $\mathcal{E}^{(\bullet)}$ by $\chi_0^*\lambda_0^*\mathcal{E}^{(\bullet)}$, it follows from iii) that $(2)\circ(1)$ is a quasi-isomorphism. Hence, we get from the commutativity of 8.2.4.21.8 that $(2)\circ(\star)$, (5) and (7) are quasi-isomorphisms. In particular, (5) is a quasi-isomorphism and then we are done. □

Definition 8.2.4.22. It follows from 8.2.4.21 that the bifunctor $\mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)$ of 8.2.4.5.1 factors through the localisation of the categories with respect to lim-ind-isogenies, i.e. via the Lemma 8.1.4.9 induces the following bifunctor denoted by

$$\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}_X). \tag{8.2.4.22.1}$$

Similarly, it follows from 8.2.4.21 that the bifunctor of 8.2.4.12.1 induces the bifunctor

$$\mathbb{R}\,\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}). \tag{8.2.4.22.2}$$

Similarly, it follows from 8.2.4.21 that the bifunctor of 8.2.4.13.1 induces the composite bifunctor

$$\mathbb{R}\mathscr{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}_X^{(\bullet)}) \to D^+(\mathbb{Z}_X^{(\bullet)}). \tag{8.2.4.22.3}$$

Remark 8.2.4.23. We have a $\mathscr{H}om$ or Hom version of the following which we leave to the reader to specify. Denote by $F(-,-)$ the bounded version of the bifunctor 8.2.4.4.1:

$$F(-,-) := \varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to K^+(\mathbb{Z}_X) \tag{8.2.4.23.1}$$

(a) Following 8.2.4.6 and with its notation, we have

$$\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}}\times\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\mathcal{D}^{(\bullet)})} F(-,-) = \mathbb{R}\varinjlim_{\lambda,\chi} \mathcal{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-)\colon D^{\mathrm{b}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}_X), \tag{8.2.4.23.2}$$

which is the restricted functor of 8.2.4.5.1.

(b) Since the composition of the localization morphism $Q_{LD}\colon D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ with the equivalence of categories $\mathfrak{e}$ of 8.1.5.14.1 is the canonical functor $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, then with notation 8.1.5.12 we have $\mathfrak{N}_{LD:=}\operatorname{Ker} Q_{LD} = D^{\mathrm{b}}_{N(\mathcal{D}^{(\bullet)})}(\mathcal{D}^{(\bullet)})$. Moreover, $\mathfrak{N}_{\mathrm{LD}}$ is a saturated (in the sense of 7.4.1.1) strictly full triangulated subcategory of $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$. Let us denote by $G(-,-)$ the bifunctor 8.2.4.23.2. With the remark 7.4.1.11, the right localizable with respect to $\mathfrak{N}^{\mathrm{op}}_{LD} \times \mathfrak{N}_{LD}$ of G exists and

$$\mathbb{R}_{\mathfrak{N}^{\mathrm{op}}_{LD}\times\mathfrak{N}_{LD}} G(-,-) = \mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^{+}(\mathbb{Z}_X). \quad (8.2.4.23.3)$$

(c) Let $Q^{\mathrm{qi}}_{LD}\colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ be the localization morphism, let $\mathfrak{N}^{\mathrm{qi}}_{LD} := \operatorname{Ker} Q^{\mathrm{qi}}_{LD}$ be the saturated strictly full triangulated subcategory of $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$. It follows from 7.4.1.6 and 7.4.1.13.1 that

$$\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{(\mathfrak{N}^{\mathrm{qi}}_{LD})^{\mathrm{op}}\times\mathfrak{N}^{\mathrm{qi}}_{LD}} F(-,-) = \mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^{+}(\mathbb{Z}_X). \quad (8.2.4.23.4)$$

Proposition 8.2.4.24. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We have the isomorphism*

$$H^0\mathbb{R}\operatorname{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \operatorname{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}). \quad (8.2.4.24.1)$$

Proof. Let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^{+}(\mathcal{D}^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$-modules representing $\mathfrak{c}(\mathcal{F}^{(\bullet)})$. By using (the Hom version of) 8.2.4.8.1, we get the isomorphism

$$H^0\mathbb{R}\operatorname{Hom}^{(\bullet;\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathfrak{c}(\mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \left(H^0\operatorname{Hom}^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{I}^{(\bullet),(\lambda,\chi)})\right)_{(\lambda,\chi)\in L(I)\times M(I)}. \quad (8.2.4.24.2)$$

Since $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ is a bounded below complex of injective left $\mathcal{D}^{(\bullet)}$-module and is quasi-isomorphic to $\chi^*\lambda^*\mathcal{F}^{(\bullet)}$, we get

$$\begin{aligned} H^0\operatorname{Hom}^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathcal{I}^{(\bullet),(\lambda,\chi)}) &\xrightarrow{\sim} \operatorname{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{I}^{(\bullet),(\lambda,\chi)}) \\ &\xrightarrow{\sim} \operatorname{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}). \end{aligned} \quad (8.2.4.24.3)$$

Since $L(I)\times M(I)$ is filtered, taking the inductive limit on 8.2.4.24.2 and 8.2.4.24.3 we get

$$H^0\varinjlim_{\lambda,\chi}\mathbb{R}\operatorname{Hom}^{(\bullet;\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathfrak{c}(\mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \varinjlim_{\lambda,\chi} H^0\mathbb{R}\operatorname{Hom}^{(\bullet;\bullet)}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\mathfrak{c}(\mathcal{F}^{(\bullet)})) \quad (8.2.4.24.4)$$

$$\xrightarrow{\sim} \varinjlim_{\lambda,\chi}\operatorname{Hom}_{D(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\chi^*\lambda^*\mathcal{F}^{(\bullet)}) \underset{8.1.4.12.1}{\xrightarrow{\sim}} \operatorname{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}). \quad (8.2.4.24.5)$$

We conclude the proof with 8.2.4.12.3. □

Proposition 8.2.4.25. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$,*

1. *If X is coherent, then we have the bifunctorial isomorphism*

$$\mathbb{R}\Gamma(X,\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}). \quad (8.2.4.25.1)$$

2. *If I^o and X are coherent, then we have the bifunctorial isomorphism*

$$\mathbb{R}\Gamma(X^{(\bullet)},\mathbb{R}\mathscr{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}). \quad (8.2.4.25.2)$$

Proof. By definition (see 8.2.4.22), this is a consequence of respectively 8.2.4.15.3 and 8.2.4.15.4. □

8.3 Homomorphism bifunctor of the derived category of $LM_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$

8.3.1 Invariance of $LD_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ via L-equivalences

Lemma 8.3.1.1. *Let $u\colon I' \to I$ be an increasing map of partially ordered sets, $\mathcal{D}'^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I')}$ endowed with a homomorphism of sheaf of rings $\mathcal{D}'^{(\bullet)} \to \underrightarrow{u}_X^{-1}(\mathcal{D}^{(\bullet)})$. Let $\mathcal{F}^{(\bullet)}$ be a left $\mathcal{D}^{(\bullet)}$-module. We consider $\underrightarrow{u}_X^{-1}(\mathcal{F}^{(\bullet)})$ as a left $\mathcal{D}'^{(\bullet)}$-module via the homomorphism $\mathcal{D}'^{(\bullet)} \to \underrightarrow{u}_X^{-1}(\mathcal{D}^{(\bullet)})$, which yields the exact functor $\underrightarrow{u}_X^{-1}\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}'^{(\bullet)})$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.*

(a) The functor $\underrightarrow{u}_X^{-1}$ extends to the exact functor $\underrightarrow{u}_X^{-1}\colon \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ and the δ-functor $\underrightarrow{u}_X^{-1}\colon \underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ making commutative the diagram

$$\begin{array}{ccc} \underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}} & \underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)}) \\ {\scriptstyle 8.1.5.3}\uparrow & & {\scriptstyle 8.1.5.3}\uparrow \\ \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}} & \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)}). \end{array} \tag{8.3.1.1.1}$$

(b) If $u\colon I' \to I$ is an L-equivalence (see definition 8.1.3.8) then the functor $\underrightarrow{u}_X^{-1}$ extends to the exact functor $\underrightarrow{u}_X^{-1}\colon \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ and the δ-functor $\underrightarrow{u}_X^{-1}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ making commutative the diagram

$$\begin{array}{ccc} \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}} & \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)}) \\ {\scriptstyle 8.1.5.3}\uparrow & & {\scriptstyle 8.1.5.3}\uparrow \\ \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}} & \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)}). \end{array} \tag{8.3.1.1.2}$$

Proof. a) Since $\underrightarrow{u}_X^{-1}$ is an exact functor, we get the δ-functor $\underrightarrow{u}_X^{-1}\colon D^{\sharp}(\mathcal{D}^{(\bullet)}) \to D^{\sharp}(\mathcal{D}'^{(\bullet)})$. Let $\chi \in M(I)$ and $\mathcal{E}^{(\bullet)} \in D^{\sharp}(\mathcal{D}^{(\bullet)})$. To prove the factorizations of (a), it is sufficient to check that $\underrightarrow{u}_X^{-1}(\theta_{\mathcal{E},\chi})$ is invertible in $\underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$. Since $\chi \circ u \in M(I')$, this is a consequence of the commutative diagram:

$$\begin{array}{ccc} \underrightarrow{u}_X^{-1}(\mathcal{E}^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}(\theta_{\mathcal{E},\chi})} & \underrightarrow{u}_X^{-1}(\chi^*\mathcal{E}^{(\bullet)}) \\ & \searrow^{\theta_{\underrightarrow{u}_X^{-1}\mathcal{E},\chi\circ u}} & \| \\ & & (\chi \circ u)^*\underrightarrow{u}_X^{-1}(\mathcal{E}^{(\bullet)}). \end{array} \tag{8.3.1.1.3}$$

b) Let $v\colon I \to I'$ such that $v \circ u \in L(I')$ and $u \circ v \in L(I)$. Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\Lambda^{\sharp}(\mathcal{D}^{(\bullet)})$. Choose $\lambda \in L(I)$ and a morphism $g\colon \mathcal{F}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that $g \circ f = \rho_{\mathcal{E},\lambda}$ and $\lambda^*(f) \circ g = \rho_{\mathcal{F},\lambda}$ in $\underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We easily check that $v \circ \lambda \circ u \in L(I')$, $u \circ v \circ \lambda \in L(I)$ and we have the commutative diagram:

$$\begin{array}{ccc} \underrightarrow{u}_X^{-1}(\mathcal{E}^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}(\rho_{\mathcal{E},u\circ v\circ\lambda})} & \underrightarrow{u}_X^{-1}(u \circ v \circ \lambda)^*\mathcal{E}^{(\bullet)} \\ & \searrow^{\rho_{\underrightarrow{u}_X^{-1}\mathcal{E},v\circ\lambda\circ u}} & \| \\ & & (v \circ \lambda \circ u)^*\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}. \end{array} \tag{8.3.1.1.4}$$

Let g' be the morphism given by the following composition:

$$g'\colon \underrightarrow{u}_X^{-1}(\mathcal{F}^{(\bullet)}) \xrightarrow{\underrightarrow{u}_X^{-1}(g)} \underrightarrow{u}_X^{-1}\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\underrightarrow{u}_X^{-1}\lambda^*(\rho_{\mathcal{E},u\circ v})} \underrightarrow{u}_X^{-1}\lambda^*(u \circ v)^*\mathcal{E}^{(\bullet)} = (v \circ \lambda \circ u)^*\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}.$$

Via 8.3.1.1.4, we can check $g' \circ \underrightarrow{u}_X^{-1}(f) = \rho_{\underrightarrow{u}_X^{-1}\mathcal{E},v\circ\lambda\circ u}$ and $(v \circ \lambda \circ u)^*(\underrightarrow{u}_X^{-1}(f)) \circ g' = \rho_{\underrightarrow{u}_X^{-1}\mathcal{F},v\circ\lambda\circ u}$ in $\underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$. This yields that $\underrightarrow{u}_X^{-1}(f) \in \Lambda^{\sharp}(\mathcal{D}'^{(\bullet)})$ and we are done. □

Remark 8.3.1.2. With notation 8.3.1.1, the functor (denoted slightly abusively) $\underrightarrow{u}_X^{-1}$ is the composition of $\mathrm{Mod}(\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$ induced by the inverse image by the topos morphism $\underrightarrow{u}_X\colon X^{(I')} \to X^{(I)}$ with the forgetful functor $\mathrm{Mod}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}) \to \mathrm{Mod}(\mathcal{D}'^{(\bullet)})$. Hence, we have checked above a unification of both cases.

Proposition 8.3.1.3. *Let $u\colon I' \to I$ be an L-equivalence between two partially ordered sets (see definition 8.1.3.8). Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$.*

(a) The following functors

$$\underrightarrow{u}_X^{-1}\colon \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}), \qquad \underrightarrow{u}_X^{-1}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}). \tag{8.3.1.3.1}$$

defined at 8.3.1.1 are equivalences of categories.

Suppose $I' = I$ and $u \in L(I)$, denoting by $\lambda_0 := u$, the forgetful functors $\underrightarrow{LM}_{\mathbb{Q}}(\lambda_0^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda_0^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, induced by the canonical morphism $\rho_{\mathcal{D},\lambda_0}\colon \mathcal{D}^{(\bullet)} \to \lambda_0^*\mathcal{D}^{(\bullet)}$, are quasi-inverse equivalences of 8.3.1.3.1.

(b) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Then $\underrightarrow{u}_X^{-1}$ induces the isomorphisms

$$\mathbb{R}\operatorname{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})}(\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \underrightarrow{u}_X^{-1}\mathcal{F}^{(\bullet)}), \tag{8.3.1.3.2}$$

$$\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})}(\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \underrightarrow{u}_X^{-1}\mathcal{F}^{(\bullet)}), \tag{8.3.1.3.3}$$

$$\underrightarrow{u}_X^{-1}\mathbb{R}\mathscr{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})}(\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \underrightarrow{u}_X^{-1}\mathcal{F}^{(\bullet)}). \tag{8.3.1.3.4}$$

Proof. a) Since the case of modules is checked similarly let us only treat the case of complexes. Since the second part of (a) induces the first one, we can suppose $I' = I$ and $u \in L(I)$. In that case, we prefer to write $\lambda_0 := u$. Recall that by definition $\lambda_0^* = \underrightarrow{\lambda}_{0X}^{-1}$ (see 8.1.3.3). Let us denote by $\lambda_{0*}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda_0^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, the forgetful functor induced by the canonical morphism $\rho_{\mathcal{D},\lambda_0}\colon \mathcal{D}^{(\bullet)} \to \lambda_0^*\mathcal{D}^{(\bullet)}$. For any $\mathcal{E}^{(\bullet)} \in K(\mathcal{D}^{(\bullet)})$ the morphism $\rho_{\mathcal{E},\lambda_0}$ is an isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. By functoriality in $\mathcal{E}$ of $\rho_{\mathcal{E},\lambda_0}$, this induces the isomorphism of functors $\mathrm{id} \to \lambda_{0*} \circ \lambda_0^*$ on $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Conversely, for any $\mathcal{F}^{(\bullet)} \in K(\lambda_0^*\mathcal{D}^{(\bullet)})$ the morphism $\rho_{\mathcal{F},\lambda_0}$ constructed in $K(\mathcal{D}^{(\bullet)})$ is in fact the same morphism constructed in $K(\lambda_0^*\mathcal{D}^{(\bullet)})$. Hence, $\rho_{\mathcal{F},\lambda_0}$ is an isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda_0^*\mathcal{D}^{(\bullet)})$. By functoriality in $\mathcal{F}$ of $\rho_{\mathcal{F},\lambda_0}$, this yields the isomorphism of functors $\mathrm{id} \xrightarrow{\sim} \lambda_0^* \circ \lambda_{0*}$ on $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda_0^*\mathcal{D}^{(\bullet)})$.

b) i) Since the proof of 8.3.1.3.2 or 8.3.1.3.3 are similar, let us only prove 8.3.1.3.4. Let $\mathcal{I}^{(\bullet)} \in K^+(\mathscr{I}(\mathcal{D}^{(\bullet)}))$ be a complex endowed with a quasi-isomorphism $\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{I}^{(\bullet)}$, let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^+(\mathscr{I}(\mathcal{D}^{(\bullet),(\bullet,\bullet)}))$ together with a quasi-isomorphism $\mathfrak{c}(\mathcal{I}^{(\bullet)}) \xrightarrow{\sim} \mathcal{I}^{(\bullet),(\bullet,\bullet)}$, and $\mathcal{I}'^{(\bullet),(\bullet,\bullet)} \in K^+(\mathscr{I}((\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})^{(\bullet,\bullet)}))$ together with a quasi-isomorphism $\underrightarrow{u}_X^{-1}\mathcal{I}^{(\bullet),(\bullet,\bullet)} \xrightarrow{\sim} \mathcal{I}'^{(\bullet),(\bullet,\bullet)}$. Modulo the isomorphism 8.2.4.13.3, we construct the morphism as being equal to the composition of

$$\underrightarrow{u}_X^{-1} \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) \to \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}}(\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \underrightarrow{u}_X^{-1}\mathcal{I}^{(\bullet),(\lambda,\chi)}) \to \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}}(\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)}), \tag{8.3.1.3.5}$$

where the first morphism follows from the commutation of the functor $\underrightarrow{u}_X^{-1}$ with inductive limits and of 8.2.4.17. We check easily that this is, up to isomorphism, independent of the choices of $\mathcal{I}^{(\bullet),(\bullet,\bullet)}$ and $\mathcal{I}'^{(\bullet),(\bullet,\bullet)}$. It remains to check this is a quasi-isomorphism.

ii) The construction of 8.3.1.3.4 is transitive with respect to composition of the map u, i.e. if $v\colon I'' \to I'$ is an L-equivalence of partially ordered sets, if $\mathcal{I}''^{(\bullet),(\bullet,\bullet)}$ is an injective resolution of $\underrightarrow{v}_X^{-1}\mathcal{I}'^{(\bullet),(\bullet,\bullet)}$ in $K^+((\underrightarrow{v}_X^{-1}\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})^{(\bullet,\bullet)})$ then the composition

$$\begin{aligned}\underrightarrow{v}_X^{-1}\underrightarrow{u}_X^{-1} \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) &\to \underrightarrow{v}_X^{-1} \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}}(\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)})\\ &\to \varinjlim_{\lambda,\chi} \mathscr{H}om^{\bullet}_{\underrightarrow{v}_X^{-1}\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}}(\underrightarrow{v}_X^{-1}\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)}, \mathcal{I}''^{(\bullet),(\lambda,\chi)}).\end{aligned} \tag{8.3.1.3.6}$$

is equal (up to isomorphism) to the morphism constructed at 8.3.1.3.5 in the case where u is replaced by $u \circ v$. Hence, we reduce to the case where $\lambda_0 := u$.

iii) Take a resolution $\mathcal{P}^{(\bullet)}$ in $K^-(\mathcal{D}^{(\bullet)})$ of $\mathcal{E}^{(\bullet)}$ with terms in $\mathscr{P}$. Since for any $\lambda \in L(I)$, $\chi \in M(I)$, $\mathcal{I}^{(\bullet),(\lambda,\chi)}$ is a bounded above complex of injective $\mathcal{D}^{(\bullet)}$-modules, $\mathcal{I}'^{(\bullet),(\lambda,\chi)}$ is a bounded above complex of injective $\lambda_0^*\mathcal{D}^{(\bullet)}$-modules, we reduce to check that the morphism 8.3.1.3.5 is a quasi-isomorphism when $\mathcal{E}^{(\bullet)}$ is replaced by $\mathcal{P}^{(\bullet)}$.

iv) Take an injective resolution $\mathcal{I}''^{(\bullet),(\bullet,\bullet)}$ of $\lambda_0^*\mathcal{I}'^{(\bullet,\bullet)}$ in $K^+((\lambda_0^*\mathcal{D}^{(\bullet)})^{(\bullet,\bullet)})$. Take a resolution $\mathcal{P}'^{(\bullet)}$ in $K^-(\lambda_0^*\mathcal{D}^{(\bullet)})$ of $\lambda_0^*\mathcal{P}^{(\bullet)}$ with terms in $\mathscr{P}(\lambda_0^*\mathcal{D}^{(\bullet)})$. Consider the following diagram

$$\begin{array}{ccccc}
\lambda_0^*\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \lambda_0^*\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \lambda_0^*\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}'^{(\bullet),(\lambda,\chi)}) \\
\downarrow{\scriptstyle (\star)} & \nearrow & & \nearrow & \downarrow{\scriptstyle (\star)} \\
\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{P}^{(\bullet)},\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \rightarrow & \mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{P}^{(\bullet)},\mathcal{I}'^{(\bullet),(\lambda,\chi)}) & \rightarrow & \mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{P}^{(\bullet)},\lambda_0^*\mathcal{I}'^{(\bullet),(\lambda,\chi)}) \\
\downarrow & = & & \swarrow & \downarrow \\
\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{P}^{(\bullet)},\mathcal{I}'^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{P}'^{(\bullet)},\lambda_0^*\mathcal{I}'^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{P}'^{(\bullet)},\mathcal{I}''^{(\bullet),(\lambda,\chi)})
\end{array} \tag{8.3.1.3.7}$$

where the arrows $(\star)$ are defined by functoriality of λ_0^* (see 8.2.4.18.2) and where the other ones are induced by functoriality of some homomorphism bifunctor. The bottom part of the diagram 8.3.1.3.7 is commutative by definition of the arrows, the commutativity of its left top triangle is induced by that of the right square of 8.2.4.20.2 (in the case where $\chi = \mathrm{id}$), the commutativity of its right top triangle is induced by that of the left square of 8.2.4.20.3 (in the case where $\chi = \mathrm{id}$). Since the trapeze is commutative by functoriality, then we have checked the commutativity of the whole diagram 8.3.1.3.7.

v) Let us check that the images under the functor $\varinjlim_{\lambda,\chi}$ of the top horizontal morphisms of the diagram 8.3.1.3.7 are quasi-isomorphisms. Since the morphism $\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)} \to \mathcal{I}'^{(\bullet),(\lambda,\chi)}$ is a quasi-isomorphism of bounded above complexes whose modules belong to $\mathscr{I}(\mathscr{P})$ (use see 8.2.3.13 in the case where $u = \mathrm{id}$ and $\mathcal{D}'^{(\bullet)} = \lambda_0^*\mathcal{D}^{(\bullet)}$ for the left term), then it follows from 8.2.3.16 that the right top horizontal morphism is an isomorphism.

It follows from 8.2.3.15 and 8.2.3.16 that the vertical morphisms

$$\begin{array}{ccc}
\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\chi^*\lambda^*\mathcal{I}^{(\bullet)}) & \longrightarrow & \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\lambda_0^*\chi^*\lambda^*\mathcal{I}^{(\bullet)}) \\
\downarrow & & \downarrow \\
\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \longrightarrow & \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{P}^{(\bullet)},\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)})
\end{array} \tag{8.3.1.3.8}$$

are quasi-isomorphism. Using 8.1.2.2 and 8.1.3.2, we get $\lambda_0^*\chi^*\lambda^*(\mathcal{I}^{(\bullet)}) = (\chi\circ\lambda_0)^*\lambda_0^*\lambda(\mathcal{I}^{(\bullet)}) = (\chi\circ\lambda_0)^*(\lambda\circ\lambda_0)^*(\mathcal{I}^{(\bullet)})$. Hence, by applying the functor $\varinjlim_{\lambda,\chi}$ to the top morphism of 8.3.1.3.8 we get an isomorphism. Hence, so is its bottom morphism, which is the top left horizontal morphism of 8.3.1.3.7. Hence, we are done.

vi) The morphism $\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{P}^{(\bullet)},\mathcal{I}'^{(\bullet),(\lambda,\chi)}) \to \mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{P}'^{(\bullet)},\mathcal{I}'^{(\bullet),(\lambda,\chi)})$ induced by the quasi-isomorphism $\mathcal{P}'^{(\bullet)} \xrightarrow{\sim} \lambda_0^*\mathcal{P}^{(\bullet)}$ is a quasi-isomorphism. Hence, to check that the bottom horizontal morphisms of the diagram 8.3.1.3.7 are quasi-isomorphisms, we reduce to the case where $\lambda_0^*\mathcal{P}^{(\bullet)}$ is replaced by $\mathcal{P}'^{(\bullet)}$ at the source. Replacing in the step (v) $\mathcal{D}^{(\bullet)}$ by $\lambda_0^*\mathcal{D}^{(\bullet)}$, $\mathcal{P}^{(\bullet)}$ by $\mathcal{P}'^{(\bullet)}$, $\mathcal{I}^{(\bullet),(\lambda,\chi)})$ by $\mathcal{I}'^{(\bullet),(\lambda,\chi)}$ and $\mathcal{I}'^{(\bullet),(\lambda,\chi)}$ by $\mathcal{I}''^{(\bullet),(\lambda,\chi)}$, we check that the images under the functor $\varinjlim_{\lambda,\chi}$ of these bottom horizontal morphisms are isomorphism.

vii) Since we have in the diagram 8.3.1.3.7 a morphism from the bottom left term to the top right term, it follows from v) and vi) that the image under the functor $\varinjlim_{\lambda,\chi}$ of the composition of the left vertical morphisms of the diagram 8.3.1.3.7 is a quasi-isomorphism. Since this latter quasi-isomorphism is equal to the composition 8.3.1.3.5, then we are done. □

8.3.1.4. With notation and hypotheses of 8.3.1.3, consider the following diagram:

$$\begin{array}{ccccc} D^{\sharp}(\mathcal{D}^{(\bullet)}) & \xrightarrow{Q} & \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow[8.1.5.6.1]{H^n} & \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \\ \downarrow {\scriptstyle \underrightarrow{u}_X^{-1}} & & \cong \downarrow {\scriptstyle \underrightarrow{u}_X^{-1}} & & \cong \downarrow {\scriptstyle \underrightarrow{u}_X^{-1}} \\ D^{\sharp}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}) & \xrightarrow{Q} & \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}) & \xrightarrow[8.1.5.6.1]{H^n} & \underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}) \end{array} \quad \text{(top and bottom curved arrows } H^n\text{)} \tag{8.3.1.4.1}$$

where Q are the localization morphisms, where the functor 8.1.5.6.1 are making commutative (up to isomorphism) by definition the "curvy triangles" of the top and bottom. Since the functor $\underrightarrow{u}_X^{-1} : M(\mathcal{D}^{(\bullet)}) \to M(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$ is exact; then the big rectangle (i.e. the composition of both squares) is commutative (up to isomorphism). The left square is commutative (up to isomorphism) by construction (see 8.3.1.1) of the equivalence of categories 8.3.1.3.1. By unicity of the factorization of the top functor 8.1.5.6.1, we get the commutativity (up to isomorphism) of the right square. Hence, so is the diagram 8.3.1.4.1.

Notation 8.3.1.5 (Bimodules case in LD categories)**.** So that the categories of the $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ form (see notation 8.1.4.10) are more easily manipulated we will make the following assumptions: Let $\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}$ be two sheaves of rings on the topos $X^{(I)}$. Let $\mathcal{R}$ be a sheaf of commutative rings on $\mathfrak{X}$. We still denote by $\mathcal{R}$ the constant inductive system of rings of X indexed by I with value $\mathcal{R}$. Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ is solved by $\mathcal{R}$ (see definition 4.6.3.2).

Lemma 8.3.1.6. *Let $u \colon \widetilde{I} \to I$ be an increasing map of partially ordered sets which is an L-equivalence (see definition 8.1.3.8). With notation and hypotheses 8.3.1.5, let $\widetilde{\mathcal{D}}^{(\bullet)}$ (resp. $\widetilde{\mathcal{D}}'^{(\bullet)}$) be a sheaf of rings on the topos $X^{(\widetilde{I})}$ endowed with a homomorphism of sheaf of rings $\widetilde{\mathcal{D}}^{(\bullet)} \to \underrightarrow{u}_X^{-1}(\mathcal{D}^{(\bullet)})$ (resp. $\widetilde{\mathcal{D}}'^{(\bullet)} \to \underrightarrow{u}_X^{-1}(\mathcal{D}'^{(\bullet)})$). We still denote by $\mathcal{R}$ the constant inductive system of rings of X indexed by I' (or by I) with value $\mathcal{R}$. Suppose $(\widetilde{\mathcal{D}}^{(\bullet)}, \widetilde{\mathcal{D}}'^{(\bullet)})$ is solved by $\mathcal{R}$ (see definition 8.5.2.4).*

The functor $\underrightarrow{u}_X^{-1}$ extends to the exact functor $\underrightarrow{u}_X^{-1} : \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}, \mathcal{R}, \widetilde{\mathcal{D}}'^{(\bullet)})$ and the δ-functor $\underrightarrow{u}_X^{-1} : \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}, \mathcal{R}, \widetilde{\mathcal{D}}'^{(\bullet)})$ making commutative the diagram

$$\begin{array}{ccc} \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}} & \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}, \mathcal{R}, \widetilde{\mathcal{D}}'^{(\bullet)}) \\ \uparrow & & \uparrow \\ \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) & \xrightarrow{\underrightarrow{u}_X^{-1}} & \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}, \mathcal{R}, \widetilde{\mathcal{D}}'^{(\bullet)}). \end{array} \tag{8.3.1.6.1}$$

Proof. We can copy 8.3.1.1. □

Proposition 8.3.1.7. *We keep notation and hypotheses of 8.3.1.5. Let $u \colon \widetilde{I} \to I$ be an increasing map of partially ordered sets which is an L-equivalence (see definition 8.1.3.8). The following functors*

$$\begin{aligned} \underrightarrow{u}_X^{-1} &: \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}, \mathcal{R}, \underrightarrow{u}_X^{-1}\mathcal{D}'^{(\bullet)}), \\ \underrightarrow{u}_X^{-1} &: \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}, \mathcal{R}, \underrightarrow{u}_X^{-1}\mathcal{D}'^{(\bullet)}). \end{aligned} \tag{8.3.1.7.1}$$

defined at 8.3.1.6 are equivalences of categories.

Suppose $\widetilde{I} = I$ and $u \in L(I)$, denoting by $\lambda := u$, the forgetful functors $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \lambda^*\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ and $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ are quasi-inverse equivalences of 8.3.1.7.1.

Proof. We proceed similarly to the proofs 8.3.1.3. □

8.3.2 Varying $\mathcal{D}^{(\bullet)}$ in $LD_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$

8.3.2.1. Let λ, μ be two elements of $L(I)$, let $\mathcal{F}^{(\bullet)}$ be a left $\lambda^*\mathcal{D}^{(\bullet)}$-module. Then $\mu^*\mathcal{F}^{(\bullet)}$ is canonically endowed with a structure of left $\mu^*\lambda^*\mathcal{D}^{(\bullet)}$-module. Via the canonical morphisms of rings $\lambda^*\mathcal{D}^{(\bullet)} \to$

$\mu^*\lambda^*\mathcal{D}^{(\bullet)}$, this yields a structure of $\lambda^*\mathcal{D}^{(\bullet)}$-module on $\mu^*\mathcal{F}^{(\bullet)}$. We easily compute that the morphism $\rho_{\mathcal{F},\mu}\colon \mathcal{F}^{(\bullet)} \to \mu^*\mathcal{F}^{(\bullet)}$ is $\lambda^*\mathcal{D}^{(\bullet)}$-linear. On the other hand, via the canonical morphism $\mu^*\mathcal{D}^{(\bullet)} \to \mu^*\lambda^*\mathcal{D}^{(\bullet)}$, we get also a structure of left $\mu^*\mathcal{D}^{(\bullet)}$-module on $\mu^*\mathcal{F}^{(\bullet)}$.

When $\lambda \le \mu$, both structures of $\mu^*\mathcal{D}^{(\bullet)}$-module and $\lambda^*\mathcal{D}^{(\bullet)}$-module on $\mu^*\mathcal{F}^{(\bullet)}$ are compatible because of the commutativity of the following diagram

$$\begin{array}{ccc} \mu^*\mathcal{D}^{(\bullet)} & \xrightarrow{\mu^*\sigma_{\mathcal{D},\lambda}} & \mu^*\lambda^*\mathcal{D}^{(\bullet)} \\ & \nwarrow^{\sigma_{\mathcal{D},\mu,\lambda}} & \uparrow{\scriptstyle \sigma_{\lambda^*\mathcal{D},\mu}} \\ & & \lambda^*\mathcal{D}^{(\bullet)}. \end{array} \qquad (8.3.2.1.1)$$

We get similar properties when $\mathcal{F}^{(\bullet)} \in C(\lambda^*\mathcal{D}^{(\bullet)})$.

Lemma 8.3.2.2. *Suppose $\sharp \in \{\emptyset, -\}$.*

(a) Let $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ be a homomorphism of rings on the topos $X^{(I)}$. The extension functor $\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon D^\sharp(\mathcal{D}^{(\bullet)}) \to D^\sharp(\mathcal{D}'^{(\bullet)})$ induces by localisation the functor $\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ and the functor $\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon D^\sharp(\mathcal{D}^{(\bullet)}) \to D^\sharp(\mathcal{D}'^{(\bullet)})$ induces the functor $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$.

(b) Let $\lambda \le \mu$ be two elements of $L(I)$, and $\mathcal{F}^{(\bullet)} \in D^\sharp(\lambda^*\mathcal{D}^{(\bullet)})$.

(i) The canonical morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$

$$\mu^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \mu^*\mathcal{F}^{(\bullet)} \qquad (8.3.2.2.1)$$

belongs to $\Lambda(\mu^*\mathcal{D}^{(\bullet)})$.

(ii) The functors $\mu^*\colon D^\sharp(\lambda^*\mathcal{D}^{(\bullet)}) \to D^\sharp(\mu^*\mathcal{D}^{(\bullet)})$ (resp. $\mu^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} -\colon D^\sharp(\lambda^*\mathcal{D}^{(\bullet)}) \to D^\sharp(\mu^*\mathcal{D}^{(\bullet)})$) induces by localisation a functor $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$, that we still denote by μ^* (resp. $\mu^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} -$).

(c) Let $\lambda \in L(I)$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. The following assertions are equivalent:

(i) There exists an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$;

(ii) There exists an isomorphism $\lambda^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$;

(iii) There exists an isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$.

Proof. a) i) Set $\Xi := \Xi^\sharp(\mathcal{D}^{(\bullet)})$ and $\Xi' := \Xi^\sharp(\mathcal{D}'^{(\bullet)})$ (see notation 8.1.2.2). To get the factorisation $\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$, it is sufficient to check that the morphisms of the form $\mathrm{id} \otimes \theta_{\mathcal{E},\chi}$ for any $\mathcal{E}^{(\bullet)} \in C^\sharp(\mathcal{D}^{(\bullet)})$ and $\chi \in M(I)$ belong to Ξ'. Let $\mathcal{P}^{(\bullet)}$ be a K-flat complex of $C^\sharp(\mathcal{D}^{(\bullet)})$ which is quasi-isomorphic to $\mathcal{E}^{(\bullet)}$. It follows from 8.1.1.6 that $\chi^*\mathcal{P}^{(\bullet)}$ is a K-flat complex of $C^\sharp(\mathcal{D}^{(\bullet)})$ which is quasi-isomorphic to $\chi^*\mathcal{E}^{(\bullet)}$. Hence, the morphism $\mathrm{id} \otimes \theta_{\mathcal{P},\chi}\colon \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)} \to \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \chi^*\mathcal{P}^{(\bullet)}$ of $C^\sharp(\mathcal{D}'^{(\bullet)})$ represents the morphism $\mathrm{id} \otimes \theta_{\mathcal{E},\chi}$ in the derived category $D^\sharp(\mathcal{D}'^{(\bullet)})$. Since $\mathbb{Z}$ is in the center of $\mathcal{D}^{(\bullet)}$, we compute $\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \chi^*\mathcal{P}^{(\bullet)} = \chi^*\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)}$ and that $\mathrm{id} \otimes \theta_{\mathcal{P},\chi} = \theta_{\mathcal{P}',\chi}$, where $\mathcal{P}'^{(\bullet)} := \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)}$.

a) ii)We have to check that the morphism of the form $\mathrm{id} \otimes^{\mathbb{L}} \sigma_{\mathcal{G},\chi,\lambda}$ for any $\mathcal{G}^{(\bullet)} \in C^\sharp(\mathcal{D}^{(\bullet)})$ and $\chi \in M(I)$ and $\lambda \in L(I)$ belong to S'. Choose a K-flat complex $\mathcal{P}^{(\bullet)}$ of left $\mathcal{D}^{(\bullet)}$-module repreesenting $\mathcal{G}^{(\bullet)}$. Choose a K-flat complex $\widetilde{\mathcal{P}}^{(\bullet)}$ of left $\mathcal{D}^{(\bullet)}$-module together with a quasi-isomorphism $s\colon \widetilde{\mathcal{P}}^{(\bullet)} \to \chi^*\lambda^*\mathcal{G}^{(\bullet)}$ of $K(\mathcal{D}^{(\bullet)})$. Let $g\colon \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \chi^*\lambda^*\mathcal{P}^{(\bullet)} = \chi^*\left(\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \lambda^*\mathcal{P}^{(\bullet)}\right) \to \chi^*\left(\lambda^*\mathcal{D}'^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} \lambda^*\mathcal{P}^{(\bullet)}\right) = \chi^*\lambda^*\left(\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)}\right)$ be the canonical morphism. We conclude via the commutative diagram of

$K(\mathcal{D}'^{(\bullet)})$:

$$\begin{array}{ccccc}
\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{G}^{(\bullet)} & \xrightarrow{=} & \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)} & \xrightarrow{\sigma_{\chi,\lambda}} & \chi^*\lambda^*\left(\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)}\right) \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes^{\mathbb{L}}\sigma_{\mathcal{G},\chi,\lambda}} & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes\sigma_{\mathcal{P},\chi,\lambda}} & \nearrow {\scriptstyle g} & \Big\downarrow {\scriptstyle \chi^*\lambda^*(\mathrm{id}\otimes\sigma_{\mathcal{P},\chi,\lambda})} \\
 & & \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \chi^*\lambda^*\mathcal{P}^{(\bullet)} & \xrightarrow[\sigma_{\chi,\lambda}]{} & \chi^*\lambda^*\left(\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \chi^*\lambda^*\mathcal{P}^{(\bullet)}\right) \\
 & & \Big\uparrow {\scriptstyle \sim\, s} & & \Big\uparrow {\scriptstyle \sim\, \chi^*\lambda^*(s)} \\
\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \chi^*\lambda^*\mathcal{G}^{(\bullet)} & \xrightarrow{=} & \mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \widetilde{\mathcal{P}}^{(\bullet)} & \xrightarrow[\sigma_{\chi,\lambda}]{} & \chi^*\lambda^*\left(\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \widetilde{\mathcal{P}}^{(\bullet)}\right).
\end{array} \qquad (8.3.2.2.2)$$

b) i) Let us prove that 8.3.2.2.1 belongs to $\Lambda(\mu^*\mathcal{D}^{(\bullet)})$. Let $\mathcal{P}^{(\bullet)}$ be a K-flat complex of $C^\sharp(\lambda^*\mathcal{D}^{(\bullet)})$ which is quasi-isomorphic to $\mathcal{F}^{(\bullet)}$. The canonical morphism $\mathcal{P}^{(\bullet)} \to \mu^*\mathcal{P}^{(\bullet)}$ is a morphism in $C^\sharp(\lambda^*\mathcal{D}^{(\bullet)})$ (see 8.3.2.1). Since moreover $\mu^*\mathcal{P}^{(\bullet)} \in C^\sharp(\mu^*\mathcal{D}^{(\bullet)})$, with the commutativity of 8.3.2.1.1, this yields the morphism $f\colon \mathcal{P}'^{(\bullet)} := \mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)} \to \mu^*\mathcal{P}^{(\bullet)}$ of $C^\sharp(\mu^*\mathcal{D}^{(\bullet)})$. The morphism f represents the morphism 8.3.2.2.1. By applying the functor μ^* to the canonical morphism $\mathcal{P}^{(\bullet)} \to \mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{P}^{(\bullet)} = \mathcal{P}'^{(\bullet)}$, we get $g\colon \mu^*\mathcal{P}^{(\bullet)} \to \mu^*(\mathcal{P}'^{(\bullet)})$. We compute that $g \circ f = \rho_{\mathcal{P}',\mu}$ and $\mu^*(f) \circ g = \rho_{\mu^*\mathcal{P},\mu}$. Hence, $f \in \Lambda'$ and we are done.

ii) Set $S := S^\sharp(\lambda^*\mathcal{D}^{(\bullet)})$ and $S' := S^\sharp(\mu^*\mathcal{D}'^{(\bullet)})$ (see notation 8.1.4.3). To get the functor $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$, it is sufficient to check that the morphisms of the form $\mu^*(\sigma_{\mathcal{G},\chi,\nu})$ (resp. $\mathrm{id}\otimes\sigma_{\mathcal{G},\chi,\nu}$) for any $\mathcal{G}^{(\bullet)} \in C^\sharp(\mathcal{D}^{(\bullet)})$, $\chi \in M(I)$ and $\nu \in L(I)$ belong to S'. Since $\mu^*(\sigma_{\mathcal{G},\chi,\nu})\colon \mu^*\mathcal{G}^{(\bullet)} \to \mu^*\chi^*\nu^*\mathcal{G}^{(\bullet)} = (\chi\circ\mu)^*(\nu\circ\mu)^*\mathcal{G}^{(\bullet)}$ is a lim-ind-isogeny (see 8.1.4.11), then we get the factorization concerning the functor μ^*. By using b.i) and also 8.1.4.11.(a), this yields the factorization of the functor $\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda^*\mathcal{D}^{(\bullet)}} -$.

c) Since we have from 8.3.2.2.1 the isomorphism

$$\lambda^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$$

in $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$, then ii) and iii) are equivalent. Moreover, following 8.3.1.3.1 the functors λ^* and the forgetful functor induces canonically quasi-inverse equivalences of categories between $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. In particular, the canonical morphism $\mathcal{E}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ (resp. $\mathcal{F}^{(\bullet)} \to \lambda^*\mathcal{F}^{(\bullet)}$) is an isomorphism of $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$). This yields the equivalence between i) and iii). □

Notation 8.3.2.3. Let $\lambda \le \mu$ be two elements of $L(I)$. The forgetful functor induced by the canonical morphism of rings $\lambda^*\mathcal{D}^{(\bullet)} \to \mu^*\mathcal{D}^{(\bullet)}$ will be denoted by

$$\mathrm{forg}_{\lambda,\mu}\colon M(\mu^*\mathcal{D}^{(\bullet)}) \to M(\lambda^*\mathcal{D}^{(\bullet)}).$$

When $\mu = \mathrm{id}$, we simply note forg_λ. Since $\mathrm{forg}_{\lambda,\mu}$ is exact, it follows from 8.3.1.1 (and the remark 8.3.1.2) that we have the factorizations $\mathrm{forg}_{\lambda,\mu}\colon \underrightarrow{M}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$ and $\mathrm{forg}_{\lambda,\mu}\colon \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. We use similar notation by replacing modules by complexes, i.e. by replacing the letters M appearing in the notation of the categories by D.

Proposition 8.3.2.4. *Let $\sharp \in \{\emptyset, -\}$ and $\lambda \le \mu$ be two elements of $L(I)$.*

1. *The functors $\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda^*\mathcal{D}^{(\bullet)}} -$, $\mu^*\colon \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ are isomorphic.*

2. *The functors $\mathrm{forg}_{\lambda,\mu}$ and $\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda^*\mathcal{D}^{(\bullet)}} -$ are quasi-inverse equivalences of categories between $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$.*

Proof. The first statement is 8.3.2.2.1. Similarly to 8.3.1.3.a, we prove that $\mathrm{forg}_{\lambda,\mu}$ and μ^* are quasi-inverse equivalences of categories between $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. This yields the second statement. □

Let us now consider the case of modules.

Lemma 8.3.2.5. *We have the following properties.*

(a) *Let $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ be a homomorphism of rings on the topos $X^{(I)}$. The extension functor $\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} -: M(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}'^{(\bullet)})$ induces by localisation the functor $\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} -: \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ and the functor $\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} -: \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$.*

(b) *Let $\lambda \le \mu$ be two elements of $L(I)$, and $\mathcal{F}^{(\bullet)} \in M(\lambda^*\mathcal{D}^{(\bullet)})$.*

   (i) *The canonical morphism of $\underrightarrow{M}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$*

$$\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \mu^*\mathcal{F}^{(\bullet)} \tag{8.3.2.5.1}$$

   belongs to $\Lambda(\mu^*\mathcal{D}^{(\bullet)})$.

   (ii) *The functors $\mu^*: M(\lambda^*\mathcal{D}^{(\bullet)}) \to M(\mu^*\mathcal{D}^{(\bullet)})$ (resp. $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -: M(\lambda^*\mathcal{D}^{(\bullet)}) \to M(\mu^*\mathcal{D}^{(\bullet)})$) induces by localization the functors $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$, that we still denote by μ^* (resp. $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -$).*

(c) *Let $\lambda \in L(I)$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. The following assertions are equivalent:*

   (i) *There exists an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$;*

   (ii) *There exists an isomorphism $\lambda^*\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$;*

   (iii) *There exists an isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$.*

Proof. We can copy the proof of 8.3.2.2 (without taking K-flat resolution). □

8.3.2.6 (Adjointness)**.** Let $\lambda \le \mu$ be two elements of $L(I)$. The functor $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -: M(\lambda^*\mathcal{D}^{(\bullet)}) \to M(\mu^*\mathcal{D}^{(\bullet)})$ is a left adjoint to the forgetful functor $\mathrm{forg}_{\lambda,\mu}: M(\mu^*\mathcal{D}^{(\bullet)}) \to M(\lambda^*\mathcal{D}^{(\bullet)})$. Since the forgetful functor $\mathrm{forg}_{\lambda,\mu}$ commutes canonically to the functors of the form μ^* and χ^*, then it follows from 8.1.5.2.2 that $\mathrm{forg}_{\lambda,\mu}: \underrightarrow{M}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$ is a right adjoint to $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -: \underrightarrow{M}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{M}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ and $\mathrm{forg}_{\lambda,\mu}: \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$ is a right adjoint to $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -: \underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$.

Proposition 8.3.2.7. *Let $\lambda \le \mu$ be two elements of $L(I)$.*

1. *The functors $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -$, $\mu^*: \underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ are isomorphic.*
2. *The functors $\mathrm{forg}_{\lambda,\mu}$ and $\mu^*\mathcal{D}^{(\bullet)} \otimes_{\lambda^*\mathcal{D}^{(\bullet)}} -$ are exact quasi-inverse equivalences of categories between $\underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$.*

Proof. The first statement is 8.3.2.5.1. By using 8.3.2.6, it is a question of checking that the adjunction morphisms are isomorphisms. Then this follows from 8.3.2.5. □

Corollary 8.3.2.8. *Let $\lambda \le \mu$ be two elements of $L(I)$. The forgetful functor*

$$\underrightarrow{M}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \xrightarrow{\mathrm{forg}_{\lambda,\mu}} \underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \tag{8.3.2.8.1}$$

is exact.

Proof. Since this a quotient by a Serre subcategory, the localisation functor $\underrightarrow{M}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ is exact. We conclude by using 8.3.2.7. □

Lemma 8.3.2.9. *We keep notation and hypotheses 8.3.1.5.*

(a) *Let $\lambda_1, \lambda_2 \le \mu$ be three elements of $L(I)$, and $\mathcal{F}^{(\bullet)} \in D^\sharp(\lambda_1^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda_2^*\mathcal{D}'^{(\bullet)})$. The canonical morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}, \mu^*\mathcal{R}, \mu^*\mathcal{D}'^{(\bullet)})$*

$$\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda_1^*\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda_2^*\mathcal{D}'^{(\bullet)}} \mu^*\mathcal{D}'^{(\bullet)} \to \mu^*\mathcal{F}^{(\bullet)} \tag{8.3.2.9.1}$$

belongs to $\Lambda(\mu^*\mathcal{D}^{(\bullet)}, \mu^*\mathcal{D}'^{(\bullet)})$.

The canonical morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$

$$\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \mu^*\mathcal{F}^{(\bullet)} \tag{8.3.2.9.2}$$

belongs to $\Lambda(\mu^*\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$.

(b) Let $\lambda \in L(I)$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$. The following assertions are equivalent:

- *(i) There exists an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$;*
- *(ii) There exists an isomorphism $\lambda^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$;*
- *(iii) There exists an isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$.*

(c) Let $\lambda \in L(I)$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)})$. The following assertions are equivalent:

- *(i) There exists an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$;*
- *(ii) There exists an isomorphism $\lambda^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}'^{(\bullet)}} \lambda^*\mathcal{D}'^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)})$;*
- *(iii) There exists an isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)})$.*

Proof. This is checked similarly to the proof of 8.3.2.2. □

8.3.3 Local properties on X and $X^{(\bullet)}$

We suppose X coherent.

Definition 8.3.3.1. Put $\mathfrak{C} \in \{M, \underrightarrow{M}_{\mathbb{Q}}, \underrightarrow{LM}_{\mathbb{Q}}, \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}\}$. Let P be a property on the objects (resp. on the arrows) of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$, where X, I, $\mathcal{D}^{(\bullet)}$ can vary. We keep notation 8.1.1.5.c,d.

(a) We say that the property P for the objects $\mathcal{E}^{(\bullet)}$ (resp. for the arrows f) of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$ is local on X (with X and I fixed) if for any open subset U of X the following properties are equivalent:

- (i) The property P holds for $\mathcal{E}^{(\bullet)}|_U$ (resp. $f|_U$) ;
- (ii) For any open set V of U, the property P holds for $\mathcal{E}^{(\bullet)}|_V$ (resp. of $f|_V$) ;
- (iii) There exist an open covering $(U_\alpha)_{\alpha\in A}$ of U such that the property P holds for $\mathcal{E}^{(\bullet)}|_{U_\alpha}$ (resp. of $f|_{U_\alpha}$) for any $\alpha \in A$.

(b) We say that the property P for the objects $\mathcal{E}^{(\bullet)}$ (resp. for the arrows f) of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$ is local on I (with X and I fixed) if both properties are equivalent ;

- (i) The property P holds for $\mathcal{E}^{(\bullet)}$ (resp. f) ;
- (ii) For any $i \in I$, the property P holds for $\mathcal{E}^{(\bullet)}|_{(i,X)}$ (resp. of $f|_{(i,X)}$).

(c) We say that the property P for the objects $\mathcal{E}^{(\bullet)}$ (resp. for the arrows f) of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$ is local on $X^{(I)}$ if for any open subset U of X the following properties are equivalent:

- (i) The property P holds for $\mathcal{E}^{(\bullet)}|_U$ (resp. $f|_U$) ;
- (ii) For any $i \in I$, for any open subset V of U, the property P holds for $\mathcal{E}^{(\bullet)}|_{(i,V)}$ (resp. of $f_{(i,V)}$) ;
- (iii) For any $i \in I$, there exists an open covering $(U_\alpha)_{\alpha\in A}$ of U such that the property P holds for $\mathcal{E}^{(\bullet)}|_{(i,U_\alpha)}$ (resp. of $f|_{(i,U_\alpha)}$) for any $\alpha \in A$.

(d) We say that a property P for the objects and the arrows f of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$ is quasi-local on X (resp. quasi-local on I, resp. quasi-local on $X^{(\bullet)}$), if we have the implication $(i) \Rightarrow (ii)$ of (a) (resp. (b), resp. (c)).

(e) We say that a property is closed under L-equivariance if for any L-equivalence $u\colon I' \to I$ between two partially ordered sets (see definition 8.1.3.8), for any object $\mathcal{E}^{(\bullet)}$ (resp. for any arrow f) of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$, both properties are equivalent

- (i) The property P holds for $\mathcal{E}^{(\bullet)}$ (resp. f) ;
- (ii) The property P holds for $\underrightarrow{u}_X^{-1}(\mathcal{E}^{(\bullet)})$ (resp. $\underrightarrow{u}_X^{-1}(f)$).

Remark 8.3.3.2. We keep notation 8.3.3.1. We have the following links between above local properties.

(a) When I has a smallest element, the property P is local on I if and only if it is quasi-local on I.

(b) Assume I is strictly filtered. In that case, we have the following properties.

   (i) If the property P is quasi-local on I and is closed under L-equivariance, then P is local on I.

   (ii) If the property P is quasi-local on I for the objects (resp. the arrows) of $\mathfrak{C}(\mathcal{D}^{(\bullet)})$, closed under L-equivariance and local on X for the objects (resp. the arrows) of $\mathfrak{C}(\mathcal{D}^{(\bullet)}_{|(i,X)})$ for some $i \in I$, then P is local on $X^{(I)}$.

Lemma 8.3.3.3. *The property that "an object of $M(\mathcal{D}^{(\bullet)})$ is an object of $N(\mathcal{D}^{(\bullet)})$" (see notation 8.1.5.5) is local on X. When I is strictly filtered, it is local on $X^{(\bullet)}$.*

Proof. Let $\mathcal{E}^{(\bullet)}$ be a $\mathcal{D}^{(\bullet)}$-module. Following 8.1.4.11.(c) (still valid by replacing complexes by modules), the fact that $\mathcal{E}^{(\bullet)}$ is an object of $N(\mathcal{D}^{(\bullet)})$ is equivalent to saying that there exist $\chi \in M(I)$ and $\lambda \in L(I)$ such that the canonical arrow $\mathcal{E}^{(\bullet)} \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$ is the zero morphism (in the category $M(\mathcal{D}^{(\bullet)})$).

Let us check the property "an object of $M(\mathcal{D}^{(\bullet)})$ is an object of $N(\mathcal{D}^{(\bullet)})$" say P is local on X. Let $(X_i)_{i\in I}$ be an open covering of X such that, for any $i \in I$, there exist $\chi_i \in M(I)$ and $\lambda_i \in L(I)$ such that the canonical arrow $\mathcal{E}^{(\bullet)}|X_i \to \chi_i^*\lambda_i^*\mathcal{E}^{(\bullet)}|X_i$ is the zero morphism. Since X is quasi-compact, we can suppose I finite. Hence, there exist $\chi \in M(I)$ and $\lambda \in L(I)$ such that, for any $i \in I$, we have $\chi \geq \chi_i$ and $\lambda \geq \lambda_i$. For any $i \in I$, the canonical morphism $\mathcal{E}^{(\bullet)}|X_i \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}|X_i$ is therefore the null morphism. Hence, P is local on X

It is straightforward that the property P is quasi-local on $X^{(I)}$ (e.g. on I) and closed under L-equivariance (use the equivalence of category 8.3.1.3.a). Hence, when I is strictly filtered, this yields that P is local on $X^{(\bullet)}$ by using the remark 8.3.3.2.b.(ii). □

Lemma 8.3.3.4. *Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The property that the morphism f is a monomorphism (resp. an epimorphism, resp. an isomorphism) of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is local on X. When I is strictly filtered, it is local on $X^{(\bullet)}$.*

Proof. We already know that $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is an abelian category (see 8.1.5.5). Hence, f is a monomorphism (resp. an epimorphism, resp. an isomorphism) of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ if and only if its kernel (resp. cokernel, resp. kernel and cokernel) is null in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We conclude by using 8.3.3.3. □

Proposition 8.3.3.5. Let $\phi\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The property that the morphism ϕ is an isomorphism is local on X. When I is strictly filtered, it is local on $X^{(\bullet)}$.

Proof. It follows from 8.3.3.4, 8.1.5.11 that the property is local on X. By using the equivalence of categories 8.3.1.3.1, it is also closed under L-equivariance. Since it is also quasi-local on $X^{(I)}$, then we conclude by using the remark 8.3.3.2.b.(ii). □

Lemma 8.3.3.6. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The property $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{0}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is local on X. When I is strictly filtered, it is local on $X^{(\bullet)}$.*

Proof. Following 8.3.3.4, by using the commutativity (up to isomorphism) of the diagram 8.3.1.4.1, for any integer $n \in \mathbb{Z}$, the property $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is local on X and closed under L-equivalence. Since it is obviously quasi-local, then we conclude by using the remark 8.3.3.2.b.(ii). □

Lemma 8.3.3.7. *Let $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$ be two objects of $M(\mathcal{D}^{(\bullet)})$. Let $(X_\alpha)_{\alpha\in A}$ be an open covering of X. Then there is a canonical bijection between the following data:*

(a) A morphism $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$;

(b) A set $(f_\alpha)_{\alpha\in A}$ of morphisms $f_\alpha\colon \mathcal{E}^{(\bullet)}|X_\alpha \to \mathcal{F}^{(\bullet)}|X_\alpha$ of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|X_\alpha)$ such that $f_\alpha|X_\alpha \cap X_\beta = f_\beta|X_\alpha \cap X_\beta$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|X_\alpha \cap X_\beta)$ for any $\alpha, \beta \in A$.

When I is strictly filtered, then we have a bijection between the data (a) and the following one:

(c) A set $(f_i)_{i\in I}$ of morphisms $f_i\colon \mathcal{E}^{(\bullet)}|_{(i,X)} \to \mathcal{F}^{(\bullet)}|_{(i,X)}$ of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,X)})$ such that for any element $i \leq j$ of I we have $f_i|_{(j,X)} = f_j$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(j,X)})$.

These properties can be interpreted by saying that the existence of a morphism in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is local on X, I and on $X^{(I)}$.

Proof. The implications $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ are clear. When I is strictly filtered, $(c) \Rightarrow (a)$ follows from the fully faithfulness of the restriction functor $|X^{(I_{\geq i})}$ (see 8.3.1.3.1). Let us prove $(b) \to (a)$. Let $(f_\alpha)_{\alpha\in A}$ be a set morphisms satisfying the condition of (b). Since X is quasi-compact, we can suppose A finite. Since a finite family of elements of $L(I)$ (resp. of $M(I)$) is bounded above by an element of $L(I)$ (resp. of $M(I)$), by increasing if necessary the elements of $L(I)$ or $M(I)$ which appear in the choices of the morphisms representing f_α, we can suppose there exist $\lambda \in L(I)$ and $\chi \in M(I)$, some morphisms $a_\alpha\colon \mathcal{E}^{(\bullet)}|X_\alpha \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}|X_\alpha$ in $M(\mathcal{D}^{(\bullet)}|X_\alpha)$ representing f_α. Since $X_\alpha \cap X_\beta$ is quasi-compact, by increasing again λ and χ if necessary, we can moreover assume that $a_\alpha|X_\alpha \cap X_\beta = a_\beta|X_\alpha \cap X_\beta$ in $M(\mathcal{D}^{(\bullet)}|X_\alpha \cap X_\beta)$. Then we get a morphism $a\colon \mathcal{E}^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$ in $M(\mathcal{D}^{(\bullet)})$ such that $a_\alpha = a|X_\alpha$. Hence we are done.

□

Lemma 8.3.3.8. *Let $f,\ g\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be two morphisms of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The equality $f = g$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is local on X. When I is strictly filtered, it is local on $X^{(\bullet)}$.*

Proof. The equality $f = g$ is equivalent to saying that the canonical morphism $\ker(f-g) \to \mathcal{E}^{(\bullet)}$ is an isomorphism. Hence, this follows from 8.3.3.4. □

8.3.4 Derived homomorphism bifunctor over complexes of $LM_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, comparison

Suppose X is coherent.

Proposition 8.3.4.1. Let $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$ be two objects of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

(a) Denote by $\mathcal{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)})$ the presheaf of abelian groups on X defined by

$$U \mapsto \mathrm{Hom}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|U)}(\mathcal{E}^{(\bullet)}|U,\ \mathcal{F}^{(\bullet)}|U), \tag{8.3.4.1.1}$$

where U running over Z_{Zar}. Then $\mathcal{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)})$ is a sheaf.

(b) Denote by $\mathscr{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)})$ the presheaf of abelian groups on $X^{(\bullet)}$ defined by

$$(i,U) \mapsto \mathrm{Hom}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,U)})}(\mathcal{E}^{(\bullet)}|_{(i,U)},\ \mathcal{F}^{(\bullet)}|_{(i,U)}), \tag{8.3.4.1.2}$$

where (i,U) running over $(I^{\circ})^{\natural} \times X_{Zar}$. When I is strictly filtered, $\mathscr{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)})$ is a sheaf.

Proof. The proposition follows from the lemmas 8.3.3.7 and 8.3.3.8. □

8.3.4.2. With notation 8.3.4.1, we get by construction the isomorphism of abelian presheaves on X:

$$\underrightarrow{l}^{(I)}_{X,*}\mathscr{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}), \tag{8.3.4.2.1}$$

where $\underrightarrow{l}^{(I)}_{X,*}$ is the projective limit indexed by I^o (see notation 8.1.1.2.3).

Notation 8.3.4.3. Let $\mathcal{E}^{(\bullet),\bullet}$, $\mathcal{F}^{(\bullet),\bullet} \in K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ (exceptionally, we indicate the second $\bullet$ to clarify the following notations). With the notations of the proposition 8.3.4.1, we have the bifunctor

$$\mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \times K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K(\mathbb{Z}_X)$$

whose the n-th term for any integer $n \in \mathbb{Z}$ is defined by setting:

$$\mathcal{H}om^{n}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet),\bullet},\ \mathcal{F}^{(\bullet),\bullet}) := \prod_{p\in\mathbb{Z}} \mathcal{H}om_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet),p},\ \mathcal{F}^{(\bullet),\ p+n}) \tag{8.3.4.3.1}$$

and the transition morphisms are given by the formula $d = d_{\mathcal{E}} + (-1)^{n+1} d_{\mathcal{F}}$.

Notation 8.3.4.4. Similarly to the construction of 8.3.4.3, by replacing everywhere "$\mathcal{H}om$" by "Hom" or "$\mathscr{H}om$", we have the following results.

(a) We define the bifunctor (which is by the way the standard homomorphisms bifunctor associated to the abelian category $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$):

$$\mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \times K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K(\mathbb{Z}).$$

By construction (see 8.3.4.1.1), for any $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, for any open subset U of X, we have the bifunctorial isomorphism of $K(\mathbb{Z})$:

$$\Gamma(U, \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}). \tag{8.3.4.4.1}$$

(b) We define the bifunctor:

$$\mathscr{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)\colon K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \times K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K(\mathbb{Z}^{(\bullet)}_{X}).$$

By construction (see 8.3.4.1.2), for any $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, for any open subset U of X, for any $(i, U) \in (I^{\mathrm{o}})^{\natural} \times X_{Zar}$, we have the bifunctorial isomorphism of $K(\mathbb{Z})$:

$$\Gamma((i,U), \mathscr{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,U)})}(\mathcal{E}^{(\bullet)}|_{(i,U)},\ \mathcal{F}^{(\bullet)}|_{(i,U)}). \tag{8.3.4.4.2}$$

8.3.4.5. Suppose $\sharp \in \{-,\ +, \mathrm{b}\}$, let $Q^{\sharp}_{N}\colon K^{\sharp}(\mathcal{D}^{(\bullet)}) \to K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ be the canonical functor. Following the remark after the theorem [Miy91, 3.2], putting $\mathcal{N}^{\sharp}_{N}(\mathcal{D}^{(\bullet)}) := \ker Q^{\sharp}_{N}$ the associated saturated null system, the functor $Q^{\sharp}_{N}$ induces the equivalence of categories $K^{\sharp}(\mathcal{D}^{(\bullet)})/\mathcal{N}^{\sharp}_{N}(\mathcal{D}^{(\bullet)}) \cong K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, which is synthesized by saying $Q^{\sharp}_{N}$ is a quotient.

With the notations 8.2.4.4.1 (where we replace $\mathcal{H}om$ by Hom) and 8.3.4.4, it follows from 8.1.5.2.2 that we have the isomorphism of bifunctors $K^{-}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K^{+}(\mathcal{D}^{(\bullet)}) \to K^{+}(\mathbb{Z})$:

$$\varinjlim_{\lambda\in L(I),\ \chi\in M(I)} \mathrm{Hom}^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^{*}\lambda^{*}-) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q_N(-), Q_N(-)). \tag{8.3.4.5.1}$$

Since X is coherent, it has an open basis consisting of coherent open subsets. Since $\Gamma(U,-)$ commutes with filtered inductive limits when U is coherent (see 7.1.2.16), since $L(I) \times M(I)$ is filtered, then with notations 8.2.4.4.1 and 8.3.4.3 it follows from 8.2.1.5.2, 8.3.4.4.1 and 8.3.4.5.1 that we have the isomorphism of bifunctors $K^{-}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K^{+}(\mathcal{D}^{(\bullet)}) \to K^{+}(\mathbb{Z}_X)$:

$$\varinjlim_{\lambda\in L(I),\ \chi\in M(I)} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^{*}\lambda^{*}-) \xrightarrow{\sim} \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q_N(-), Q_N(-)). \tag{8.3.4.5.2}$$

Following the remark 7.4.1.11 and with notation 8.2.4.6, this implies that the right localisation with respect to $(\mathfrak{N}_N \times \mathfrak{N}_N, \mathfrak{N}_{qi})$ of the bifunctor of 8.2.4.4.1 exists and that we have the isomorphism of bifunctors $K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \times K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X)$:

$$\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{-}_{N}(\mathcal{D}^{(\bullet)})\times\mathfrak{N}^{+}_{N}(\mathcal{D}^{(\bullet)})} \varinjlim_{\lambda\in L,\ \chi\in M} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^{*}\lambda^{*}-) \xrightarrow{\sim} Q^{+}_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-). \tag{8.3.4.5.3}$$

8.3.4.6. Following notation 8.2.4.23 and 8.3.4.5, we denote by $Q^{\mathrm{b}}_N\colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ the canonical functor, $\mathcal{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)}) := \ker Q^{\mathrm{b}}_N$, $Q_{LD}\colon D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $Q^{\mathrm{qi}}_{LD}\colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ the localization morphisms, $\mathfrak{N}^{\mathrm{qi}}_{LD} := \operatorname{Ker} Q^{\mathrm{qi}}_{LD}$. Set $\widetilde{Q}^{\mathrm{qi}}_{LD} := \mathfrak{e}\circ Q^{\mathrm{qi}}_{LD}\colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, where $\mathfrak{e}$ is the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \cong D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ (see 8.1.5.14.1) and $Q_{\mathrm{qi}}\colon K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ is the localisation morphism. We notice $\operatorname{Ker} \widetilde{Q}^{\mathrm{qi}}_{LD} = \mathfrak{N}^{\mathrm{qi}}_{LD}$. To avoid confusion, the right localisation $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{qi}}_{LD}\times\mathfrak{N}^{\mathrm{qi}}_{LD}} F(-,-)$ of $F(-,-)$ (the functor defined at 8.2.4.23) will be denoted by $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{Q^{\mathrm{qi}}_{LD}\times Q^{\mathrm{qi}}_{LD}} F(-,-)$ as a functor $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\circ}\times\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_X)$ and by $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD}\times \widetilde{Q}^{\mathrm{qi}}_{LD}} F(-,-)$ as a functor of $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X)$. Since the right derived bifunctor $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{Q^{\mathrm{qi}}_{LD}\times Q^{\mathrm{qi}}_{LD}} F(-,-)$ exists (see 8.2.4.23.4), then so is $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD}\times \widetilde{Q}^{\mathrm{qi}}_{LD}} F(-,-)$ and we have the canonical isomorphism of functors

$$\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) = \mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{Q^{\mathrm{qi}}_{LD}\times Q^{\mathrm{qi}}_{LD}} F(-,-) \xrightarrow{\sim} \mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD}\times \widetilde{Q}^{\mathrm{qi}}_{LD}} F(\mathfrak{e}(-),\mathfrak{e}(-)). \tag{8.3.4.6.1}$$

Hence, since the right derived bifunctor $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)})\times\mathfrak{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)})} F(-,-)$ exists (see 8.3.4.5), since $\widetilde{Q}^{\mathrm{qi}}_{LD} = Q_{\mathrm{qi}}\circ Q^{\mathrm{b}}_N$ then it follows from Lemma 7.4.1.13 that $\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)})\times\mathfrak{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)})} F(-,-)$ is right localizable with respect to $(\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})), \mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})))$ and that we have the isomorphism of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X)$ of the form

$$\mathbb{R}_{\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))\times\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}\mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)})\times\mathfrak{N}^{\mathrm{b}}_N(\mathcal{D}^{(\bullet)})} F(-,-) \xrightarrow{\sim} \mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD}\times \widetilde{Q}^{\mathrm{qi}}_{LD}} F(-,-). \tag{8.3.4.6.2}$$

By using the (bounded version of the) isomorphism 8.3.4.5.3, this yields that the right localisation of the bifunctor $Q^{+}_{\mathrm{qi}}\circ\mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)$ with respect to $(\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})), \mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})))$ exists, which will be denoted by

$$\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-)\colon D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X), \tag{8.3.4.6.3}$$

and is endowed with the isomorphism of bifunctors

$$\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \xrightarrow{\sim} \mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD}\times \widetilde{Q}^{\mathrm{qi}}_{LD}} F(-,-). \tag{8.3.4.6.4}$$

Via 8.3.4.6.1 and 8.3.4.6.4 we have the isomorphism of bifunctors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\circ}\times\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_X)$ of the form

$$\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathfrak{e}(-),\mathfrak{e}(-)). \tag{8.3.4.6.5}$$

Moreover, by using the universal property of the right localization of a functor, using $Q_{\mathrm{qi}}\circ Q^{\mathrm{b}}_N = \widetilde{Q}^{\mathrm{qi}}_{LD}$, and $\widetilde{Q}^{\mathrm{qi}}_{LD} = \mathfrak{e} \circ Q^{\mathrm{qi}}_{LD}$, we get the commutative up to isomorphism diagram

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q^{\mathrm{qi}}_{LD}(-), Q^{\mathrm{qi}}_{LD}(-)) & \xrightarrow[\sim]{8.3.4.6.5} & \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathfrak{e} \circ Q^{\mathrm{qi}}_{LD}, \mathfrak{e} \circ Q^{\mathrm{qi}}_{LD}) \\
{\scriptstyle 8.3.4.6.1}\uparrow{\scriptstyle \sim} & & \Vert \\
\mathbb{R}^{\mathfrak{N}^+_{\mathrm{qi}}(\mathbb{Z}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD} \times \widetilde{Q}^{\mathrm{qi}}_{LD}} F(\widetilde{Q}^{\mathrm{qi}}_{LD}(-), \widetilde{Q}^{\mathrm{qi}}_{LD}(-)) & \xrightarrow[\sim]{8.3.4.6.4} & \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\widetilde{Q}^{\mathrm{qi}}_{LD}(-), \widetilde{Q}^{\mathrm{qi}}_{LD}(-)) \\
{\scriptstyle \sim}\uparrow{\scriptstyle 8.3.4.6.2} & & \Vert \\
\mathbb{R}_{\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}} \times \mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}} \mathbb{R}^{\mathfrak{N}^+_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{b}}_N \times \mathfrak{N}^{\mathrm{b}}_N} F(\widetilde{Q}^{\mathrm{qi}}_{LD}(-), \widetilde{Q}^{\mathrm{qi}}_{LD}(-)) & \xrightarrow{\sim} & \mathbb{R}_{\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}} \times \mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}} Q^+_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\widetilde{Q}^{\mathrm{qi}}_{LD}(-), \widetilde{Q}^{\mathrm{qi}}_{LD}(-)) \\
{\scriptstyle adj}\uparrow & & \uparrow{\scriptstyle adj} \\
\mathbb{R}^{\mathfrak{N}^+_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{b}}_N \times \mathfrak{N}^{\mathrm{b}}_N} F(Q^{\mathrm{b}}_N(-), Q_N n^{\mathrm{b}}(-)) & \xrightarrow[8.3.4.5.3]{\sim} & Q^+_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q^{\mathrm{b}}_N(-), Q^{\mathrm{b}}_N(-)) \\
{\scriptstyle adj}\uparrow & \nearrow{\scriptstyle \sim\ 8.3.4.5.2} & \\
Q^+_{\mathrm{qi}} \circ F(-,-) & &
\end{array} \tag{8.3.4.6.6}$$

where the “adj” morphisms are given by adjunction from 7.4.1.9.1.

8.3.4.7. Suppose I^o is coherent, which implies $X^{(\bullet)}$ is a coherent topos. By copying the proof of 8.3.4.5.3 (i.e. by replacing everywhere “$\mathcal{H}om$” by “$\mathscr{H}om$” and by replacing the coherence of $I^o \times X$ instead of that of X), we get the isomorphism of bifunctors $K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \times K(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}^{(\bullet)}_X)$:

$$\mathbb{R}^{\mathfrak{N}^+_{\mathrm{qi}}(\mathbb{Z}^{(\bullet)}_X)}_{\mathfrak{N}^-_N(\mathcal{D}^{(\bullet)}) \times \mathfrak{N}^+_N(\mathcal{D}^{(\bullet)})} \varinjlim_{\lambda \in L,\ \chi \in M} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^* \lambda^* -) \xrightarrow{\sim} Q^+_{\mathrm{qi}} \circ \mathscr{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-). \tag{8.3.4.7.1}$$

Similarly to 8.3.4.6, it follows from the (bounded version of the) isomorphism 8.3.4.7.1, that the right localisation of the bifunctor $Q^+_{\mathrm{qi}} \circ \mathscr{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)$ with respect to $(\mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})), \mathfrak{N}^{\mathrm{b}}_{\mathrm{qi}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})))$ exists, which will be denoted by

$$\mathbb{R}\mathscr{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \colon D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D^+(\mathbb{Z}^{(\bullet)}_X), \tag{8.3.4.7.2}$$

and is endowed with the isomorphism of bifunctors

$$\mathbb{R}\mathscr{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \xrightarrow{\sim} \mathbb{R}^{\mathfrak{N}^+_{\mathrm{qi}}(\mathbb{Z}^{(\bullet)}_X)}_{\widetilde{Q}^{\mathrm{qi}}_{LD} \times \widetilde{Q}^{\mathrm{qi}}_{LD}} \varinjlim_{\lambda, \chi} \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,\ \chi^* \lambda^* -). \tag{8.3.4.7.3}$$

Via the $\mathscr{H}om$ version of 8.3.4.6.1 and via 8.3.4.7.3, we have the isomorphism of bifunctors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\circ} \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^+(\mathbb{Z}^{(\bullet)}_X)$ of the form

$$\mathbb{R}\mathscr{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathfrak{e}(-), \mathfrak{e}(-)). \tag{8.3.4.7.4}$$

8.3.4.8. The standar bifunctor $\mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)$ is right localisable with respect to quasi-isomorphism and its right derivated functor is denoted by

$$\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \colon D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}). \tag{8.3.4.8.1}$$

Recall, $\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})$ is computed by taking an injective resolution of $\mathcal{F}^{(\bullet)}$. Similarly to 8.3.4.6, we have moreover the isomorphism of bifunctors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\circ} \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z})$ of the form

$$\mathbb{R}\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathfrak{e}(-), \mathfrak{e}(-)). \tag{8.3.4.8.2}$$

Hence it follows from 8.2.4.24.1 that we have the isomorphism of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))\to \mathrm{Mod}(\mathbb{Z})$ of the form:

$$H^{0}(\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-))\xrightarrow{\sim}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-). \tag{8.3.4.8.3}$$

It follows from 8.2.4.25.1, 8.3.4.6.5 and 8.3.4.8.2 that we have the isomorphism of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))\to D(\mathbb{Z})$ of the form

$$\mathbb{R}\Gamma(X,-)\circ\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-)\xrightarrow{\sim}\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-). \tag{8.3.4.8.4}$$

Suppose I^{o} is coherent. Similarly, it follows from 8.2.4.25.2 that we have that the isomorphism of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))\to D(\mathbb{Z})$ of the form

$$\mathbb{R}\Gamma(X^{(\bullet)},-)\circ\mathbb{R}\mathscr{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-)\xrightarrow{\sim}\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-). \tag{8.3.4.8.5}$$

8.4 Coherence

8.4.1 Coherence in $LD_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, comparison with the $\mathcal{D}^{\dagger}_{\mathbb{Q}}$-coherence, theorems A and B

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}$ be a sheaf of rings on the topos $X^{(I)}$.

Definition 8.4.1.1. Let $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The complex $\mathcal{E}^{(\bullet)}$ is said to be coherent if there exist $\lambda\in L(I)$ and $\mathcal{F}^{(\bullet)}\in D_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathcal{F}^{(\bullet)}$.

Remark 8.4.1.2. Let $\lambda\in L(I)$. Following 7.1.3.13, $\mathcal{F}^{(\bullet)}\in D_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ means that $\mathcal{F}^{(\bullet)}$ satisfies the following conditions:

(a) For any $i\in I$, $\mathcal{F}^{(i)}\in D_{\mathrm{coh}}(\mathcal{D}^{(\lambda(i))})$;

(b) For any $i,j\in I$ such that $i\leq j$, the canonical homomorphism

$$\mathcal{D}^{(\lambda(j))}\overset{\mathbb{L}}{\otimes}_{\mathcal{D}^{(\lambda(i))}}\mathcal{F}^{(\lambda(i))}\to\mathcal{F}^{(\lambda(j))} \tag{8.4.1.2.1}$$

is an isomorphism.

Notation 8.4.1.3. Let $\sharp\in\{\emptyset,-,\mathrm{b},0\}$. We denote by $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ the strictly full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ consisting of coherent complexes.

Lemma 8.4.1.4. *Let $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Let $\lambda\leq\mu$ be two elements of $L(I)$. Consider the following two properties.*

(a) There exists $\mathcal{F}^{(\bullet)}\in D_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathcal{F}^{(\bullet)}$.

(b) There exists $\mathcal{G}^{(\bullet)}\in D_{\mathrm{coh}}(\mu^{*}\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathcal{G}^{(\bullet)}$.

Then (a) $\Rightarrow$ (b).

Proof. Suppose there exists $\mathcal{F}^{(\bullet)}\in D_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathcal{F}^{(\bullet)}$. Following 8.3.2.2.c, we get the isomorphism $\underrightarrow{LD}_{\mathbb{Q}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ of the form $\lambda^{*}\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathcal{F}^{(\bullet)}$. This yields the isomorphism $\mu^{*}\mathcal{D}^{(\bullet)}\overset{\mathbb{L}}{\otimes}_{\lambda^{*}\mathcal{D}^{(\bullet)}}\lambda^{*}\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mu^{*}\mathcal{D}^{(\bullet)}\overset{\mathbb{L}}{\otimes}_{\lambda^{*}\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mu^{*}\mathcal{D}^{(\bullet)})$. Since $\mathcal{G}^{(\bullet)}:=\mu^{*}\mathcal{D}^{(\bullet)}\overset{\mathbb{L}}{\otimes}_{\lambda^{*}\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)}\in D_{\mathrm{coh}}(\mu^{*}\mathcal{D}^{(\bullet)})$, since $\mu^{*}\mathcal{D}^{(\bullet)}\overset{\mathbb{L}}{\otimes}_{\lambda^{*}\mathcal{D}^{(\bullet)}}\lambda^{*}\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mu^{*}\mathcal{E}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mu^{*}\mathcal{D}^{(\bullet)})$ (see 8.3.2.2.1), we are done by using again 8.3.2.2.c. □

Proposition 8.4.1.5. *Let $u\colon I' \to I$ be an L-equivalence between two partially ordered sets (see definition 8.1.3.8). Let $\sharp \in \{\emptyset, -, \mathrm{b}, 0\}$. The equivalence of categories $\underrightarrow{u}_X^{-1}$ of 8.3.1.3.1 preserves the coherence, i.e. it induces the equivalence of categories*

$$\underrightarrow{u}_X^{-1}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}). \tag{8.4.1.5.1}$$

Suppose $I' = I$ and $u \in L(I)$. Denoting by $\lambda := u$, we have the forgetful functor $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}\,\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}\,\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ which is a quasi-inverse equivalence of 8.4.1.5.1.

Proof. 0) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We already know that the proposition without the coherence hypotheses (see 8.3.1.3). Hence, we have to check $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ if and only if $\underrightarrow{u}_X^{-1}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$.

1) First suppose $I' = I$ and $\lambda := u \in L(I)$.

i) Suppose $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Let $\mu \in L(I)$ and $\mathcal{F}^{(\bullet)} \in D_{\mathrm{coh}}(\mu^*\mathcal{D}^{(\bullet)})$ together with an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Since $\lambda \circ \mu \geq \mu$, it follows from 8.4.1.4 that there exists $\mathcal{G}^{(\bullet)} \in D_{\mathrm{coh}}(\mu^*\lambda^*\mathcal{D}^{(\bullet)})$ together with an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. By using 8.3.2.2.c, this yields the isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. Hence, $\lambda^*(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$.

ii) Conversely, suppose $\lambda^*(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$. Let $\mu \in L(I)$ and $\mathcal{G}^{(\bullet)} \in D_{\mathrm{coh}}(\mu^*\lambda^*\mathcal{D}^{(\bullet)})$ together with an isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. By composing with the canonical isomorphism $\sigma_{\mathcal{E},\lambda}\colon \mathcal{E}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, we conclude $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$.

2) Let us come back to the general case. By definition, there exists an increasing map $v\colon I \to I'$ such that $u \circ v \in L(I)$, $v \circ u \in L(I')$.

i) Suppose $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Let $\lambda \in L(I)$ and $\mathcal{F}^{(\bullet)} \in D_{\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$ together with an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We have $\mu := u \circ (v \circ \lambda) \in L(I)$ and $\mu' := (v \circ \lambda) \circ u \in L(I')$. Since $\mu = u \circ v \circ \lambda \geq \lambda$, it follows from 8.4.1.4 that there exists $\mathcal{G}^{(\bullet)} \in D_{\mathrm{coh}}(\mu^*\mathcal{D}^{(\bullet)})$ together with an isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. With 8.3.2.2.c, this means we have the isomorphism $\mu^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$. Since $\mu'^*\underrightarrow{u}_X^{-1} = \underrightarrow{u}_X^{-1}\mu^*$, by applying the functor $\underrightarrow{u}_X^{-1}$ to this latter isomorphism, we get the isomorphism $\mu'^*\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \underrightarrow{u}_X^{-1}\mathcal{G}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mu'^*\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$, with $\underrightarrow{u}_X^{-1}\mathcal{G}^{(\bullet)} \in D_{\mathrm{coh}}(\mu'^*\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$. Hence, $\underrightarrow{u}_X^{-1}\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$.

ii) Since v is also an L-equivalence, we get from 2.i) that the functor $\underrightarrow{v}_X^{-1}$ preserve the coherence. Hence, by using 1), we get the converse of 2.i). □

Lemma 8.4.1.6. *We suppose I has a smallest element i_0. With notation 8.1.1.4, for any $\mathcal{E}^{(\bullet)} \in D_{\mathrm{coh}}(\mathcal{D}^{(\bullet)})$, we have $\mathcal{E}^{(i_0)} \in D_{\mathrm{coh}}(\mathcal{D}^{(i_0)})$ and the canonical isomorphism of $D(\mathcal{D}^{(\bullet)})$*

$$\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}. \tag{8.4.1.6.1}$$

Proof. First let us construct the morphism. Let $\mathcal{P}^{(\bullet)}$ be a K-flat complex of $K(\mathcal{D}^{(\bullet)})$ endowed with a quasi-isomorphism $\mathcal{P}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. Then $\mathcal{P}^{(i_0)}$ is a K-flat complex of $K(\mathcal{D}^{(i_0)})$ endowed with a quasi-isomorphism $\mathcal{P}^{(i_0)} \xrightarrow{\sim} \mathcal{E}^{(i_0)}$. The morphism 8.4.1.6.1 corresponds to the canonical map $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{P}^{(i_0)} \to \mathcal{P}^{(\bullet)}$.

Finally, since a morphism $\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ in $D(\mathcal{D}^{(\bullet)})$ is an isomorphism if and only if the induced morphisms $\mathcal{E}^{(i)} \to \mathcal{F}^{(i)}$ in $D(\mathcal{D}^{(i)})$ are isomorphisms for any $i \in I$, then the fact that the morphism 8.4.1.6.1 is an isomorphism is a consequence of 7.1.3.13. □

Proposition 8.4.1.7. *Suppose I has a smallest element or is strictly filtered (see 8.1.3.8). Let $\sharp \in \{\emptyset, -, \mathrm{b}\}$. Then the subcategory $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ of $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is a triangulated subcategory.*

Proof. The cases where $\sharp \in \{-, \mathrm{b}\}$ is a consequence of the case where $\sharp$ is empty.

Let $i_0 \in I$ and $j\colon I^{i_0} \subset I$ be the inclusion. Since j is an L-equivalence, then following 8.4.1.5 (and use also 8.3.1.3), a complex $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ belongs to $\underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ if and only if $\underrightarrow{j}_X^{-1}(\mathcal{E}^{(\bullet)})$ belongs to $\underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\underrightarrow{j}_X^{-1}\mathcal{D}^{(\bullet)})$. Hence, we reduce to the case where I has a smallest element i_0.

Let $\mathcal{E}^{(\bullet)} \xrightarrow{f} \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}^{(\bullet)}[1]$ be a triangle of $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ such that $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. By using 8.4.1.4 (and also 8.3.2.2.c), there exist $\mu \in L(I)$, $\mathcal{E}'^{(\bullet)}, \mathcal{F}'^{(\bullet)} \in D_{\mathrm{coh}}(\mu^*\mathcal{D}^{(\bullet)})$ together with the isomorphisms in $\underrightarrow{LD}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$ of the form $\mu^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$, $\mu^*\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}'^{(\bullet)}$. We get the triangle $\mathcal{E}'^{(\bullet)} \to \mathcal{F}'^{(\bullet)} \to \mu^*\mathcal{G}^{(\bullet)} \to \mathcal{E}'^{(\bullet)}[1]$ of $\underrightarrow{LD}_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$. Using again 8.3.1.3 and 8.4.1.5, if $\mu^*(\mathcal{G}^{(\bullet)}) \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mu^*\mathcal{D}^{(\bullet)})$ then $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Hence, replacing $\mathcal{D}^{(\bullet)}$ by $\mu^*\mathcal{D}^{(\bullet)}$ if necesary, we can suppose that $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in D_{\mathrm{coh}}(\mathcal{D}^{(\bullet)})$.

The morphism $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is represented by a morphism $g\colon \mathcal{E}^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ for some $(\lambda, \chi) \in L(I) \times M(I)$. Set $\mathcal{E}'^{(\bullet)} := \lambda^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)}$ and let $h\colon \mathcal{E}'^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$ be the induced morphism of $D(\lambda^*\mathcal{D}^{(\bullet)})$. Since the canonical morphisms $\mathcal{E}'^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ and $\mathcal{E}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ are morphism of $\Lambda(\mathcal{D}^{(\bullet)})$ (see 8.3.2.2.1 for the first one), then the canonical morphism $\mathcal{E}^{(\bullet)} \to \mathcal{E}'^{(\bullet)}$ is an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Hence, we get the distinguished triangle $\mathcal{E}'^{(\bullet)} \xrightarrow{h} \chi^*\lambda^*\mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}'^{(\bullet)}[1]$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Set $\mathcal{D}'^{(\bullet)} := \lambda^*\mathcal{D}^{(\bullet)}$, $\mathcal{F}'^{(\bullet)} := \lambda^*\mathcal{F}^{(\bullet)}$. We have $\mathcal{E}'^{(\bullet)}, \mathcal{F}'^{(\bullet)} \in D_{\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$ (see 8.4.1.6). Let $\chi' \in M(I)$ be the map defined by $\chi'(i) := \chi(i) - \chi(i_0)$ for any $i \in I$. Remark that $\chi^* = \chi'^*$ and that the morphism $\mathcal{F}'^{(i_0)} \to \mathcal{F}'^{(i_0)}$ induced by $\theta_{\mathcal{F}',\chi'}$ is the identity (but the one induced by $\theta_{\mathcal{F}',\chi}$ is the multiplication by $p^{\chi(i_0)}$. With notation 8.1.1.4, this implies the commutativity of the outline rectangle of the diagram

$$\begin{array}{ccccccc}
\mathcal{E}'^{(i_0)} & \longrightarrow & \mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}'^{(i_0)}} \mathcal{E}'^{(i_0)} & & \xrightarrow{\qquad\sim\qquad} & & \mathcal{E}'^{(\bullet)} \\
\downarrow{\scriptstyle h^{(i_0)}} & & \downarrow{\scriptstyle \mathrm{id} \otimes h^{(i_0)}} & & & & \downarrow{\scriptstyle h} \\
\mathcal{F}'^{(i_0)} & \longrightarrow & \mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}'^{(i_0)}} \mathcal{F}'^{(i_0)} & \xrightarrow{\sim} & \mathcal{F}'^{(\bullet)} & \xrightarrow{\theta_{\mathcal{F}',\chi'}} & \chi^*\mathcal{F}'^{(\bullet)}
\end{array} \tag{8.4.1.7.1}$$

in $D(\mathcal{D}'^{(\bullet)})$, where the horizontal isomorphisms follows from the fact that $\mathcal{E}'^{(\bullet)}, \mathcal{F}'^{(\bullet)} \in D_{\mathrm{coh}}(\mathcal{D}'^{(\bullet)})$ and where to lighten we have removed $(\underset{\to}{l}{}^{(I)}_{X})^{-1}$ in the notation (e.g. $\mathcal{F}'^{(i_0)} = (\underset{\to}{l}{}^{(I)}_{X})^{-1}(\mathcal{F}'^{(i_0)})$). By universal property of the extension, this yields the commutativity of the right square. Hence, we get a morphism $h'\colon \mathcal{E}'^{(\bullet)} \to \mathcal{F}'^{(\bullet)}$ in $D(\mathcal{D}'^{(\bullet)})$ making commutative the diagram, in particular such that $h = \theta_{\mathcal{F}',\chi'} \circ h'$. Since $\mathcal{E}'^{(\bullet)}, \mathcal{F}'^{(\bullet)} \in D_{\mathrm{coh}}(\mathcal{D}'^{(\bullet)})$, then the cone $\mathcal{G}'^{(\bullet)}$ of h' in $D(\mathcal{D}'^{(\bullet)})$ belongs to $D_{\mathrm{coh}}(\mathcal{D}'^{(\bullet)})$. Since $\theta_{\mathcal{F}',\chi'}$ is an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then $\mathcal{G}'^{(\bullet)}$ and $\mathcal{G}^{(\bullet)}$ are isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. □

Notation 8.4.1.8. Suppose I is a filtered set. Let us denote by $\mathcal{D}^\dagger := \varinjlim_{i \in I} \mathcal{D}^{(i)}$. From the topos morphism 8.1.1.2.4, we get the ringed topos morphism:

$$\underset{\to}{l}_{X,I}\colon (X, \mathcal{D}^\dagger) \to (X^{(I)}, \mathcal{D}^{(\bullet)}). \tag{8.4.1.8.1}$$

Recall $\underset{\to}{l}{}^{-1}_{X,I}(\mathcal{F}^{(\bullet)}) = \varinjlim_{i \in I} \mathcal{F}^{(i)}$ and $\underset{\leftarrow}{l}_{X,I*}(\mathcal{F})$ is the constant inductive system with value $\mathcal{F}$. We have $\underset{\to}{l}{}^*_{X,I} = \underset{\to}{l}{}^{-1}_{X,I}\colon M(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^\dagger)$, which is exact. It induces the functor $\underset{\to}{l}{}^*_{X,I}\colon D(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^\dagger)$. By composition with the tensor product $- \otimes_{\mathbb{Z}} \mathbb{Q}\colon D(\mathcal{D}^\dagger) \to D(\mathcal{D}^\dagger_{\mathbb{Q}})$ we get a functor $D(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^\dagger_{\mathbb{Q}})$ that we will denote by $\underset{\to}{l}{}^*_{X,I,\mathbb{Q}}$ or simply by $\underset{\to}{l}{}^*_{\mathbb{Q}}$ if there is no ambiguity with I and X.

Proposition 8.4.1.9. *Suppose that I is filtered.*

(a) The functor $\underset{\to}{l}{}^*_{\mathbb{Q}}\colon D(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^\dagger_{\mathbb{Q}})$ (see 8.4.1.8) factors through $\underset{\to}{l}{}^*_{\mathbb{Q}}\colon \underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^\dagger_{\mathbb{Q}})$ and $\underset{\to}{l}{}^*_{\mathbb{Q}}\colon \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^\dagger_{\mathbb{Q}})$.

(b) Let $u\colon I' \to I$ be an L-equivalence between two partially ordered sets (see definition 8.1.3.8). Set $\mathcal{D}'^\dagger := \underset{\to}{l}{}^{-1}_{X,I'}(\underset{\to}{u}{}^{-1}_{X}\mathcal{D}^{(\bullet)})$. Then we have a canonical isomorphism $\mathcal{D}'^\dagger \xrightarrow{\sim} \mathcal{D}^\dagger$ of sheaves of rings on X making commutative (up to equivalence of categories) the diagram

$$\begin{array}{ccc}
\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \underset{\underset{\to}{u}{}^{-1}_{X}}{\xrightarrow{\cong}} & \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\underset{\to}{u}{}^{-1}_{X}\mathcal{D}^{(\bullet)}) \\
\downarrow{\scriptstyle \underset{\to}{l}{}^*_{X,I,\mathbb{Q}}} & & \downarrow{\scriptstyle \underset{\to}{l}{}^*_{X,I',\mathbb{Q}}} \\
D(\mathcal{D}^\dagger_{\mathbb{Q}}) & \underset{\cong}{\longleftarrow} & D(\mathcal{D}'^\dagger_{\mathbb{Q}})
\end{array} \tag{8.4.1.9.1}$$

where the top horizontal functor is the equivalence of categories of 8.3.1.3.1,

(c) *Suppose I has a smallest element or is strictly filtered (see 8.1.3.8). Then, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$, we have $\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in D^{\sharp}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$.*

Proof. a) The $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$. Let $\lambda \in L(I)$, $\chi \in M(I)$. Then $\theta_{\mathcal{E},\chi} \otimes \mathrm{id}\colon \mathcal{E}^{(\bullet)} \otimes_{\mathbb{Z}} \mathbb{Q} \to \chi^{*}\mathcal{E}^{(\bullet)} \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism in $C(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$. Hence, $\underrightarrow{l}^{*}_{\mathbb{Q}}(\theta_{\mathcal{E},\chi})$ is an isomorphism in $D(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ and we get the first factorization. Following the part a) of the proof of 8.1.3.11, the canonical morphism $\underrightarrow{l}^{-1}_{X,I}(\rho_{\mathcal{E},\lambda})\colon \underrightarrow{l}^{-1}_{X,I}(\mathcal{E}^{(\bullet)}) \to \underrightarrow{l}^{-1}_{X,I}(\lambda^{*}\mathcal{E}^{(\bullet)})$ is an isomorphism in $C(\mathcal{D}^{(\bullet)})$. This yields the second factorization.

b) The property (b) follows from 8.1.3.11.

c) Since $\underrightarrow{l}^{*}_{X,I,\mathbb{Q}}$ is exact, we can suppose $\sharp$ is empty. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Let $i_0 \in I$. Since the inclusion $j\colon I^{i_0} \to I$ is an L-equivalence, following 8.4.1.5.1 (and 8.3.1.3.1), $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ if and only if $\underrightarrow{j}^{-1}_{X}\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\underrightarrow{j}^{-1}_{X}\mathcal{D}^{(\bullet)})$. Hence, by using the commutative diagram 8.4.1.9.1, we reduce to the case where $I^{i_0} = I$. By definition and by using 8.3.2.2.c, there exist $\lambda \in L(I)$, $\mathcal{E}'^{(\bullet)} \in D_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ together with the isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$. Set $\mathcal{D}'^{(\bullet)} = \lambda^{*}\mathcal{D}^{(\bullet)}$. From 8.4.1.6, the canonical morphism

$$\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}'^{(i_0)}} \mathcal{E}'^{(i_0)} \to \mathcal{E}'^{(\bullet)} \tag{8.4.1.9.2}$$

is an isomorphism in $D(\mathcal{D}'^{(\bullet)})$. Let $\mathcal{P}'^{(\bullet)}$ be a K-flat complex of $K(\mathcal{D}'^{(\bullet)})$ endowed with a quasi-isomorphism $\mathcal{P}'^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$. By construction of 8.4.1.9.2, we get that the canonical morphism $\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}'^{(i_0)}} \mathcal{P}'^{(i_0)} \to \mathcal{P}'^{(\bullet)}$ of $K(\mathcal{D}'^{(\bullet)})$ is a quasi-isomorphism. This yields the last quasi-isomorphism:

$$\underrightarrow{l}^{-1}_{X,I}(\mathcal{D}'^{(\bullet)}) \otimes_{\mathcal{D}'^{(i_0)}} \mathcal{P}'^{(i_0)} \xrightarrow{\sim} \underrightarrow{l}^{-1}_{X,I}(\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{D}'^{(i_0)}} \mathcal{P}'^{(i_0)}) \xrightarrow{\sim} \underrightarrow{l}^{-1}_{X,I}(\mathcal{P}'^{(\bullet)})$$

of $K(\underrightarrow{l}^{-1}_{X,I}(\mathcal{D}'^{(\bullet)}))$. Since $\underrightarrow{l}^{-1}_{X,I}(\mathcal{D}'^{(\bullet)}) \xrightarrow{\sim} \mathcal{D}^{\dagger}$, this means that the canonical morphism $\mathcal{D}^{\dagger} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}'^{(i_0)}} \mathcal{E}'^{(i_0)} \to \underrightarrow{l}^{-1}_{X,I}(\mathcal{E}'^{(\bullet)})$ is an isomorphism in $D(\mathcal{D}^{\dagger})$. Hence $\underrightarrow{l}^{-1}_{X,I}(\mathcal{E}'^{(\bullet)}) \in D_{\mathrm{coh}}(\mathcal{D}^{\dagger})$, and then $\underrightarrow{l}^{*}_{X,I,\mathbb{Q}}(\mathcal{E}'^{(\bullet)}) = (\underrightarrow{l}^{-1}_{X,I}(\mathcal{E}'^{(\bullet)}))_{\mathbb{Q}} \in D_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$. Since $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then we are done.

□

8.4.1.10. The proposition 8.4.1.9.a is still valid by replacing complexes by modules, i.e. by replacing in the notation of categories the letter D by M. More precisely, the exact functor $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon M(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ factors through $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon \underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ and even $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ making commutative the diagram

$$\begin{array}{ccccc} \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \longrightarrow & \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\mathcal{H}^n} & \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \\ \downarrow{\scriptstyle \underrightarrow{l}^{*}_{\mathbb{Q}}} & & \downarrow{\scriptstyle \underrightarrow{l}^{*}_{\mathbb{Q}}} & & \downarrow{\scriptstyle \underrightarrow{l}^{*}_{\mathbb{Q}}} \\ M(\mathcal{D}^{\dagger}_{\mathbb{Q}}) & \longrightarrow & D(\mathcal{D}^{\dagger}_{\mathbb{Q}}) & \xrightarrow{\mathcal{H}^n} & M(\mathcal{D}^{\dagger}_{\mathbb{Q}}). \end{array} \tag{8.4.1.10.1}$$

Proposition 8.4.1.11. *Suppose I is filtered. Let $i_0 \in I$, $\mathcal{E}^{(i_0)}$, $\mathcal{F}^{(i_0)}$ be two left $\mathcal{D}^{(i_0)}$-modules. Put $\mathcal{E}^{(i)} := \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)}$ and $\mathcal{F}^{(i)} := \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)}$ for any $i \geq i_0$. Put $\mathcal{D}^{\dagger} := \varinjlim_{i\in I} \mathcal{D}^{(i)}$, $\mathcal{E}^{\dagger} := \mathcal{D}^{\dagger} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)}$ and $\mathcal{F}^{\dagger} := \mathcal{D}^{\dagger} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)}$. We suppose $\mathcal{E}^{(i_0)}$ is of finite presentation.*

(a) *The canonical morphism*

$$\varinjlim_{i\in I^{i_0}} \mathcal{H}om_{\mathcal{D}^{(i)}}(\mathcal{E}^{(i)},\ \mathcal{F}^{(i)}) \to \mathcal{H}om_{\mathcal{D}^{\dagger}}(\mathcal{E}^{\dagger},\ \mathcal{F}^{\dagger}) \tag{8.4.1.11.1}$$

is an isomorphism.

(b) *Suppose X is coherent.*

(i) *The canonical morphism*

$$\varinjlim_{i\in I^{i_0}} \mathrm{Hom}_{\mathcal{D}^{(i)}}(\mathcal{E}^{(i)},\ \mathcal{F}^{(i)}) \to \mathrm{Hom}_{\mathcal{D}^{\dagger}}(\mathcal{E}^{\dagger},\ \mathcal{F}^{\dagger}) \tag{8.4.1.11.2}$$

is an isomorphism.

(ii) *Let $\mathcal{G}$ be a left $\mathcal{D}^\dagger$-module of finite presentation. Then there exists $j_0 \in I$ and a left $\mathcal{D}^{(j_0)}$-module $\mathcal{G}^{(j_0)}$ of finite presentation together with a $\mathcal{D}^\dagger$-linear isomorphism $\epsilon\colon \mathcal{D}^\dagger \otimes_{\mathcal{D}^{(j_0)}} \mathcal{G}^{(j_0)} \xrightarrow{\sim} \mathcal{G}$. Moreover, if $(j_0', \mathcal{G}'^{(j_0')}, \epsilon')$ satisfies the same property, then there exists $j \in I^{j_0} \cap I^{j_0'}$, a $\mathcal{D}^{(j)}$-linear isomorphism $\epsilon_j\colon \mathcal{D}^{(j)} \otimes_{\mathcal{D}^{(j_0)}} \mathcal{G}^{(j_0)} \xrightarrow{\sim} \mathcal{D}^{(j)} \otimes_{\mathcal{D}^{(j_0')}} \mathcal{G}'^{(j_0')}$ such that $\epsilon' \circ (\mathrm{id} \otimes \epsilon_j) = \epsilon$.*

Proof. a) The fact that the morphism 8.4.1.11.1 is an isomorphism is local on X. Hence, we can suppose that $\mathcal{E}^{(i_0)}$ has globally finite presentation. Since I^{i_0} is filtered, both functors $\varinjlim_{i \in I^{i_0}} \mathcal{H}om_{\mathcal{D}^{(i)}}(-,\ \mathcal{F}^{(i)})$ and $\mathcal{H}om_{\mathcal{D}^\dagger}(\mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(i_0)}} -,\ \mathcal{F}^\dagger)$ are left exact. Hence, by using the 5 lemma, we reduce to the case where $\mathcal{E}^{(i_0)} = \mathcal{D}^{(i_0)}$, which is obvious.

b) i) When X is coherent, filtered inductive limits commute with the global section functor $\Gamma(X, -)$ (see [SGA4.2, VI.5.1-2]). Hence, we get 8.4.1.11.2 from 8.4.1.11.1 by applying $\Gamma(X, -)$.

ii) Case 1): suppose $\mathcal{G}$ has globally finite presentation, i.e. is the cokernel of a morphism of $\mathcal{D}^\dagger$-modules of the form $f\colon (\mathcal{D}^\dagger)^r \to (\mathcal{D}^\dagger)^s$. It follows from b.i), that f comes from a morphism of $\mathcal{D}^{(i)}$-modules of the form $f^{(i)}\colon (\mathcal{D}^{(i)})^r \to (\mathcal{D}^{(i)})^s$. Hence, we can choose $\mathcal{G}^{(i)}$ as equal to the cokernel of $f^{(i)}$.

Case 2) In general, by glueing (and by using again the coherence of X), to construct the left $\mathcal{D}^{(j_0)}$-module $\mathcal{G}^{(j_0)}$ together with a $\mathcal{D}^\dagger$-linear isomorphism $\epsilon\colon \mathcal{D}^\dagger \otimes_{\mathcal{D}^{(j_0)}} \mathcal{G}^{(j_0)} \xrightarrow{\sim} \mathcal{G}$ we reduce to the case 1).

□

Remark 8.4.1.12. Let $f\colon (\mathcal{D}^\dagger)^r \to (\mathcal{D}^\dagger)^s$ be a morphism of left $\mathcal{D}^\dagger$-modules. Without the coherence hypothesis on X, this is not clear that f comes by extension from a morphism of $\mathcal{D}^{(i)}$-modules of the form $f^{(i)}\colon (\mathcal{D}^{(i)})^r \to (\mathcal{D}^{(i)})^s$. Indeed, the data of f is equivalent to that of rs elements of the set $\Gamma(X, \mathcal{D}^\dagger)$; the data of $f^{(i)}$ is equivalent to that of rs elements of the set $\Gamma(X, \mathcal{D}^{(i)})$. When X is not coherent, since $\Gamma(X, \mathcal{D}^\dagger) \xrightarrow{\sim} \varinjlim \Gamma(X, \mathcal{D}^{(i)})$ is not necessarily true, then this is not clear that an element of $\Gamma(X, \mathcal{D}^\dagger)$ comes from by extension an element of $\Gamma(X, \mathcal{D}^{(i)})$.

Corollary 8.4.1.13. *Suppose I is filtered, X is locally coherent, $\mathcal{D}^{(i)}$ is left (resp. right) coherent on X and $\mathcal{D}^{(i)} \to \mathcal{D}^{(j)}$ is flat for any $i \leq j$. Then $\mathcal{D}^\dagger := \varinjlim_{i \in I} \mathcal{D}^{(i)}$ is left (resp. right) coherent.*

Proof. Since this is local, we can suppose X is coherent. Let $f\colon (\mathcal{D}^\dagger)^r \to \mathcal{D}^\dagger$ be a morphism of left $\mathcal{D}^\dagger$-modules. Following 8.4.1.11, there exists $i \in I$, a morphism of left $\mathcal{D}^{(i)}$-modules $f^{(i)}\colon (\mathcal{D}^{(i)})^r \to \mathcal{D}^{(i)}$ such that $f = \mathcal{D}^\dagger \otimes_{\mathcal{D}^{(i)}} (f^{(i)})$. Since the functor $\mathcal{D}^\dagger \otimes_{\mathcal{D}^{(i)}} -$ is exact, $\ker f \xrightarrow{\sim} \mathcal{D}^\dagger \otimes_{\mathcal{D}^{(i)}} \ker f^{(i)}$ and we are done. □

Theorem 8.4.1.14. *Suppose X is the topological space associated to a noetherien affine $\mathcal{V}$-formal scheme $\mathfrak{X}$, Suppose I is a filtered set and put $\mathcal{D}^\dagger := \varinjlim_{i \in I} \mathcal{D}^{(i)}$, $D^\dagger := \Gamma(\mathfrak{X}, \mathcal{D}^\dagger)$. Suppose moreover that for any $i \in I$, $\mathcal{D}^{(i)}$ is a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}^{(i)}$ satisfying the conditions of 7.2.3 (for the p-adic topology).*

(a) *We have theorems A and B for $\mathcal{D}^\dagger$-modules of finite presentation:*

   (i) *The extension $D^\dagger \to \mathcal{D}^\dagger$ is flat ;*

   (ii) *For any $\mathcal{D}^\dagger$-module $\mathcal{E}$ of finite presentation, for any $q \geq 1$, we have $H^q(X, \mathcal{E}) = 0$;*

   (iii) *The functors $\Gamma(X, -)$ and $\mathcal{D}^\dagger \otimes_{D^\dagger} -$ induces canonically quasi-inverse exact equivalences between the category of $\mathcal{D}^\dagger$-modules of finite presentation and that of $D^\dagger$-modules of finite presentation.*

(b) *If moreover the transition maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any elements $i \leq j$ of I, then the sheaf of rings $\mathcal{D}^\dagger_{\mathbb{Q}}$ satisfies theorems A and B for coherent modules in the sense of 1.4.3.14.*

Proof. (a) Set $D^{(i)} := \Gamma(X, \mathcal{D}^{(i)})$. Since the topological space X is coherent, then for any $q \geq 0$, the functors $H^q(X, -)$ commute with filtered inductive limits and then for $q = 0$ we get $D^\dagger \xrightarrow{\sim} \varinjlim_{i \in I} D^{(i)}$. Moreover, following 7.2.3.16, the sheaves of rings $\mathcal{D}^{(i)}$ satisfy theorems A and B for coherent modules. In particular $D^{(i)} \to \mathcal{D}^{(i)}$ is flat then $D^\dagger \to \mathcal{D}^\dagger$ is flat.

Let $\mathcal{E}$ be a $\mathcal{D}^\dagger$-module of finite presentation. Following 8.4.1.11.b.ii), since X is coherent then there exists $i_0 \in I$ and a left $\mathcal{D}^{(i_0)}$-module $\mathcal{E}^{(i_0)}$ of finite presentation together with a $\mathcal{D}^\dagger$-linear isomorphism

$\epsilon\colon \mathcal{D}^{\dagger} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \xrightarrow{\sim} \mathcal{E}$. Set $\mathcal{E}^{(i)} := \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)}$ for any $i \geq i_0$. Following 7.2.3.3, $\mathcal{D}^{(i)}$ is therefore left coherent for any $i \in I$. Hence, $\mathcal{E}^{(i)}$ is a coherent $\mathcal{D}^{(i)}$-module. This yields

$$H^q(X, \mathcal{E}) \xrightarrow{\sim} H^q(X, \varinjlim_{i\in I^{i_0}} \mathcal{E}^{(i)}) \xrightarrow{\sim} \varinjlim_{i\in I^{i_0}} H^q(X, \mathcal{E}^{(i)}) \overset{7.2.3.16}{=} \varinjlim_{i\in I^{i_0}} 0 = 0.$$

For any $i \geq i_0$, set $E^{(i)} := \Gamma(X, \mathcal{E}^{(i)})$ and $E := \Gamma(X, \mathcal{E})$. It follows from Theorem A of 7.2.3.16.(i)-(ii), the canonical morphism

$$\mathcal{D}^{(i)} \otimes_{D^{(i_0)}} E^{(i_0)} \xrightarrow{\sim} \mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} = \mathcal{E}^{(i)} \tag{8.4.1.14.1}$$

is an isomorphism. By applying $\mathcal{D}^{(i)} \otimes_{D^{(i)}} -$ to the canonical morphism of coherent $D^{(i)}$-modules $D^{(i)} \otimes_{D^{(i_0)}} E^{(i_0)} \to E^{(i)}$, using again 7.2.3.16.(i)-(ii), we get 8.4.1.14.1 which is an isomorphism. Hence, $D^{(i)} \otimes_{D^{(i_0)}} E^{(i_0)} \to E^{(i)}$ is an isomorphism and is the image (up to isomorphism) by $\Gamma(X, -)$ of 8.4.1.14.1. Since filtered inductive limits commute with the functor $\Gamma(X, -)$ and with tensor products, we get $D^{\dagger} \otimes_{D^{(i_0)}} E^{(i_0)} \xrightarrow{\sim} E$. Hence, E is a $D^{\dagger}$-module of finite presentation.

Consider now the commutative diagram

$$\begin{array}{ccc} \varinjlim_{i\in I^{i_0}} \mathcal{D}^{(i)} \otimes_{D^{(i)}} E^{(i)} & \xrightarrow{\sim} & \varinjlim_{i\in I^{i_0}} \mathcal{E}^{(i)} \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{D}^{\dagger} \otimes_{D^{\dagger}} E, & \longrightarrow & \mathcal{E} \end{array}$$

where vertical isomorphisms come from the commutation of filtered inductive limits with the functor $\Gamma(X, -)$ and with tensor products. Since the sheaves of rings $\mathcal{D}^{(i)}$ satisfy theorems A and B for coherent modules, then the top morphism is an isomorphism. This implies that so is the bottom one. Using the same arguments, we check that for any $D^{\dagger}$-module E of finite presentation, the canonical morphism $E \to \Gamma(X, \mathcal{D}^{\dagger} \otimes_{D^{\dagger}} E)$ is an isomorphism.

(b) The last part is a consequence of the first one and of the fact that following the corollary 8.4.1.13 the sheaf of rings $\mathcal{D}^{\dagger}_{\mathbb{Q}} := \varinjlim_{i\in I} \mathcal{D}^{(i)}_{\mathbb{Q}}$ is left (resp. right) coherent. □

Theorem 8.4.1.15. *Suppose that either I is filtered and has a smallest element or I is strictly filtered. Suppose moreover that X is coherent.*

(a) For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, the functor $\underrightarrow{l}^{*}_{X,I,\mathbb{Q}}$ induces an isomorphism

$$\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{D}^{\dagger}_{\mathbb{Q}})}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})). \tag{8.4.1.15.1}$$

(b) Suppose the following conditions are satisfied

- *(i) The rings $\mathcal{D}^{(i)}$ are coherent sheaves for any $i \in I$*
- *(ii) The transition maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any elements $i \leq j$ of I;*
- *(iii) There exists an integer d such that, for any elements $i \leq j$ of I, the ring $\mathcal{D}^{(j)}$ is of tor-dimension $\leq d$ on $\mathcal{D}^{(i)}$;*
- *(iv) For any $i \in I$, for any coherent $\mathcal{D}^{(i)}_{\mathbb{Q}}$-module $\mathcal{E}$, there exists a coherent $\mathcal{D}^{(i)}$-module $\mathcal{E}'$ together with an isomorphism $\mathcal{E}'_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ of $\mathcal{D}^{(i)}_{\mathbb{Q}}$-modules.*

Under these hypotheses, the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ induces an equivalence of categories between $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$.

Proof. 0) Since the proof reduces (or the argument are the same) to the case where I is filtered and has a smallest element, let us suppose I is strictly filtered.

a) i) First let us construct canonically the morphism 8.4.1.15.1. For this purpose, let us construct for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, respectively in $D^{+}(\mathbb{Z}_X)$ and $D^{+}(\mathbb{Z})$ the following morphisms:

$$\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})), \tag{8.4.1.15.2}$$

$$\mathbb{R}\,\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})). \tag{8.4.1.15.3}$$

Set $\mathcal{E}^{\dagger}_{\mathbb{Q}} := \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$, $\mathcal{F}^{\dagger}_{\mathbb{Q}} := \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})$. The functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ induces the canonical morphism of bifunctors $K^{-}(\mathcal{D}^{(\bullet)}) \times K^{+}(\mathcal{D}^{(\bullet)}) \to K^{+}(\mathbb{Z}_X)$:

$$\varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-, \chi^*\lambda^* -) \to \varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(-), \underrightarrow{l}^{*}_{\mathbb{Q}}(\chi^*\lambda^*(-))). \tag{8.4.1.15.4}$$

Moreover, it follows from the proof of 8.4.1.9.a that the morphism $\underrightarrow{l}^{*}_{\mathbb{Q}}(\sigma_{\mathcal{F},\lambda,\chi}) \colon \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)}) \to \underrightarrow{l}^{*}_{\mathbb{Q}}(\chi^*\lambda^*\mathcal{F}^{(\bullet)})$ is an isomorphism in $C(\mathcal{D}^{\dagger}_{\mathbb{Q}})$. This yields that the canonical morphism of bifunctors $K^{-}(\mathcal{D}^{(\bullet)}) \times K^{+}(\mathcal{D}^{(\bullet)}) \to K(\mathbb{Z}_X)$:

$$\mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(-), \underrightarrow{l}^{*}_{\mathbb{Q}}(-)) \to \varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(-), \underrightarrow{l}^{*}_{\mathbb{Q}}(\chi^*\lambda^*(-))) \tag{8.4.1.15.5}$$

is an isomorphism. Hence, with notation 8.3.4.6, there exists a unique morphism

$$\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(-), \underrightarrow{l}^{*}_{\mathbb{Q}}(-)) \tag{8.4.1.15.6}$$

of bifunctors $\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \times \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D^{+}(\mathbb{Z}_X)$ making commutative the diagram of bifunctors $K^{-}(\mathcal{D}^{(\bullet)}) \times K^{+}(\mathcal{D}^{(\bullet)}) \to D^{+}(\mathbb{Z}_X)$:

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q^{\mathrm{qi}}_{LD}(-), Q^{\mathrm{qi}}_{LD}(-)) & \xrightarrow{8.4.1.15.6} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}} \circ Q^{\mathrm{qi}}_{LD}(-), \underrightarrow{l}^{*}_{\mathbb{Q}} \circ Q^{\mathrm{qi}}_{LD}(-)) \\
 & & \| \\
\uparrow{\scriptstyle adj} & & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(Q_{\mathrm{qi}} \circ \underrightarrow{l}^{*}_{\mathbb{Q}}(-), Q_{\mathrm{qi}} \circ \underrightarrow{l}^{*}_{\mathbb{Q}}(-)) \\
 & & \uparrow{\scriptstyle adj} \\
Q_{\mathrm{qi}} \circ \varinjlim_{\lambda,\chi} \mathcal{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-, \chi^*\lambda^*(-)) & \longrightarrow & Q_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(-), \underrightarrow{l}^{*}_{\mathbb{Q}}(-))
\end{array} \tag{8.4.1.15.7}$$

where the "adj" morphisms are given by adjunction from 7.4.1.9.1, where the bottom morphism is constructed by composing 8.4.1.15.4 and 8.4.1.15.5, where the right top arrow comes from the equality $\underrightarrow{l}^{*}_{\mathbb{Q}} \circ Q^{\mathrm{qi}}_{LD} = Q_{\mathrm{qi}} \circ \underrightarrow{l}^{*}_{\mathbb{Q}}$ as functors $K(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ with $\underrightarrow{l}^{*}_{\mathbb{Q}} \colon \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ for the left side and $\underrightarrow{l}^{*}_{\mathbb{Q}} \colon K(\mathcal{D}^{(\bullet)}) \to K(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ for the right one. By using Lemma 8.2.4.25 and the following similar isomorphism

$$\mathbb{R}\Gamma(X, \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)}))) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})), \tag{8.4.1.15.8}$$

by applying the functor $\mathbb{R}\Gamma(X,-)$ to 8.4.1.15.2, we get the functor 8.4.1.15.3 (or one can construct this latter morphism directly similarly to 8.4.1.15.2). By applying the functor H^0 to 8.4.1.15.3, it follows from 8.2.4.24 (and similarly for the right term), that we get the construction of the arrow 8.4.1.15.1.

ii) Suppose from now $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. We check 8.4.1.15.1, 8.4.1.15.2, 8.4.1.15.3 are isomorphisms.

1) By construction it is sufficient to prove it for 8.4.1.15.2. Let $\lambda \in L(I)$ and $\mathcal{E}'^{(\bullet)} \in D^{-}_{\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$ together with an isomorphism $\lambda^*\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$ in $\underrightarrow{LD}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$ (by definition of the coherence and use 8.3.2.2.c). Set $\mathcal{D}'^{\dagger}_{\mathbb{Q}} := \underrightarrow{l}^{*}_{X,I,\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$. Then we have a canonical isomorphism $\mathcal{D}'^{\dagger}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathbb{Q}}$ of sheaves of rings on X inducing the equivalence of categories of the right vertical arrow of the diagram:

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) & \xrightarrow{8.4.1.15.2} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})) \\
{\scriptstyle 8.3.1.3.3}\downarrow\cong & & \downarrow\cong \\
\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})}(\lambda^*\mathcal{E}^{(\bullet)}, \lambda^*\mathcal{F}^{(\bullet)}) & \xrightarrow{8.4.1.15.2} & \mathbb{R}\mathcal{H}om_{\mathcal{D}'^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}(\lambda^*\mathcal{E}^{(\bullet)}), \underrightarrow{l}^{*}_{\mathbb{Q}}(\lambda^*\mathcal{F}^{(\bullet)})).
\end{array} \tag{8.4.1.15.9}$$

By using the commutative diagram 8.4.1.9.1, we check that the diagram 8.4.1.15.9 is commutative up to canonical equivalence of categories. Hence, replacing $\mathcal{D}^{(\bullet)}$ by $\lambda^*\mathcal{D}^{(\bullet)}$ if necessary, by using the stability of the coherence of 8.4.1.5, we can suppose that $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Similarly, since I is strictly filtered, we can suppose that I has a smallest element i_0. Following 8.4.1.6, $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{coh}}(\mathcal{D}^{(i_0)})$ and we have the canonical isomorphism of $D^-(\mathcal{D}^{(\bullet)})$

$$\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}. \tag{8.4.1.15.10}$$

2) We reduce in the step to the case where $\mathcal{E}^{(\bullet)} = \mathcal{D}^{(\bullet)}$. Let us denote $f(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})$ the morphism 8.4.1.15.2. The morphism $f(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})$ is an isomorphism in $D^+(\mathbb{Z}_X)$ is equivalent to the fact that $H^k(f(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}))$ is an isomorphism of sheaves on X for any $k \in \mathbb{Z}$. Let $k \in \mathbb{Z}$. Since $\mathcal{F}^{(\bullet)}$ is bounded below, there exists n large enough (depending on k and on $\mathcal{F}^{(\bullet)}$ but not on $\mathcal{E}^{(\bullet)}$) such that the canonical morphism $H^k(f(\tau^{\geq n}\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \to H^k(f(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}))$ is an isomorphism. Since the fact that $H^k(f(\tau^{\geq n}\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}))$ is an isomorphism is local on X, since $\mathcal{E}^{(i_0)} \in D^-_{\mathrm{coh}}(\mathcal{D}^{(i_0)})$, we can suppose there exists a bounded complex $\mathcal{L}^{(i_0)}$ of $C^{\mathrm{b}}(\mathcal{D}^{(i_0)})$ which is strictly perfect together with a morphism of complexes $\mathcal{L}^{(i_0)} \to \mathcal{E}^{(i_0)}$ which is an n-isomorphism (i.e. its cone is acyclic in degree $\geq n$). With 8.4.1.15.10, we get the n-isomorphism $\mathcal{L}^{(\bullet)} := \mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(i_0)}} \mathcal{L}^{(i_0)} \to \mathcal{E}^{(\bullet)}$, i.e. $\tau^{\geq n}\mathcal{L}^{(\bullet)} \xrightarrow{\sim} \tau^{\geq n}\mathcal{E}^{(\bullet)}$. Since $H^k(f(\mathcal{L}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \to H^k(f(\tau^{\geq n}\mathcal{L}^{(\bullet)}, \mathcal{F}^{(\bullet)}))$ is an isomorphism, this yields $H^k(f(\tau^{\geq n}\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)})) \to H^k(f(\mathcal{L}^{(\bullet)}, \mathcal{F}^{(\bullet)}))$ is an isomorphism. Hence, we reduce to the case where $\mathcal{E}^{(\bullet)}$ is strictly perfect and we are done by additivity.

3) It remains to check the case $\mathcal{E}^{(\bullet)} = \mathcal{D}^{(\bullet)}$. Let $\mathcal{I}^{(\bullet)} \in K^+(\mathscr{J})$ endowed with a quasi-isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{I}^{(\bullet)}$ in $K^+(\mathcal{D}^{(\bullet)})$. It remains to check that $H^k(f(\mathcal{D}^{(\bullet)}, \mathcal{I}^{(\bullet)}))$ is an isomorphism for any $k \in \mathbb{Z}$. Since $\mathcal{D}^{(\bullet)} \in K^-(\mathscr{P})$, then the left arrow of 8.4.1.15.7 evaluated at $(\mathcal{D}^{(\bullet)}, \mathcal{I}^{(\bullet)})$ is an isomorphism. Moreover, since $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathbb{Q}}$, then the bottom right morphism of 8.4.1.15.7 is an isomorphism. Hence, we reduce to check that the canonical morphism

$$\varinjlim_{\lambda,\chi} \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{D}^{(\bullet)}, \chi^*\lambda^*(\mathcal{I}^{(\bullet)})) \to \mathcal{H}om^\bullet_{\mathcal{D}^\dagger_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}), \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{I}^{(\bullet)})), \tag{8.4.1.15.11}$$

is an isomorphism. Recall, the morphism 8.4.1.15.11 is constructed by composing 8.4.1.15.4 and 8.4.1.15.5.

We remark the following: for any complex $\mathcal{G}^{(\bullet)} \in C(\mathcal{D}^{(\bullet)})$, since i_0 is the smallest element of I, then we get the left canonical isomorphism of $C(\mathbb{Z}_X)$:

$$\mathcal{G}^{(i_0)} \xrightarrow{\sim} \mathcal{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{D}^{(\bullet)}, \mathcal{G}^{(\bullet)}), \qquad \mathcal{H}om^\bullet_{\mathcal{D}^\dagger_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}), \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{G}^{(\bullet)})) \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{G}^{(\bullet)}), \tag{8.4.1.15.12}$$

the right one is obvious. Hence, the morphism 8.4.1.15.11 is canonically isomorphic to the composition morphism

$$\varinjlim_{\lambda,\chi} (\chi^*\lambda^*\mathcal{I}^{(\bullet)})^{(i_0)} \underset{8.4.1.15.4}{\longrightarrow} \varinjlim_{\lambda,\chi} \underrightarrow{l}^*_{\mathbb{Q}}(\chi^*\lambda^*\mathcal{I}^{(\bullet)}) \underset{8.4.1.15.5}{\xleftarrow{\sim}} \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{I}^{(\bullet)}), \tag{8.4.1.15.13}$$

where the left arrow is induced by the canonical morphism $(\chi^*\lambda^*\mathcal{I})^{(i_0)} \to \underrightarrow{l}^*_{\mathbb{Q}}(\chi^*\lambda^*\mathcal{I}^{(\bullet)})$. Let $F_\lambda \colon M(I) \to \mathrm{Mod}(\mathbb{Z}_X)$ be the functor given by $\chi \mapsto (\chi^*\lambda^*\mathcal{I}^{(\bullet)})^{(i_0)}$. Let $\alpha \colon \mathbb{N} \to M(I)$ be the increasing map which associates to n the constant map with value n. Let $\beta \colon M(I) \to \mathbb{N}$ be the increasing map defined by $\chi \mapsto \chi(i_0)$. Then, $F_\lambda = F_\lambda \circ \alpha \circ \beta$. Moreover, $F_\lambda \circ \alpha$ is the inductive system $F_\lambda \circ \alpha(n) = \mathcal{I}^{(\lambda(i_0))}$ with transition map $F_\lambda \circ \alpha(n \to n+1)$ given by the multiplication with p. Hence, $\varinjlim F_\lambda \circ \alpha \xrightarrow{\sim} \mathcal{I}^{(\lambda(i_0))}_{\mathbb{Q}}$. Since β is surjective, then $\varinjlim F_\lambda = \varinjlim F_\lambda \circ \alpha \circ \beta \xrightarrow{\sim} \varinjlim F_\lambda \circ \alpha$. Hence, we have the isomorphisms

$$\varinjlim_{\lambda,\chi} (\chi^*\lambda^*\mathcal{I}^{(\bullet)})^{(i_0)} \xrightarrow{\sim} \varinjlim_{\lambda} \varinjlim F_\lambda \xrightarrow{\sim} \varinjlim_{\lambda \in L(I)} \mathcal{I}^{(\lambda(i_0))}_{\mathbb{Q}}. \tag{8.4.1.15.14}$$

Let $i \in I$. Since I is strictly filtered, then the canonical inclusion $j_i \colon I^i \subset I$ is an L-equivalence. Hence, there exists an increasing map $u_i \colon I \to I^i$ such that $\lambda_i := j_i \circ u_i \in L(I)$ and $u_i \circ j_i \in L(I^i)$. In particular, we get $\lambda_i(i_0) \geq i$. This yields a canonical morphism $\varinjlim_{i \in I} \mathcal{I}^{(i)}_{\mathbb{Q}} \to \varinjlim_{\lambda \in L(I)} \mathcal{I}^{(\lambda(i_0))}_{\mathbb{Q}}$ which is an inverse of $\varinjlim_{\lambda \in L(I)} \mathcal{I}^{(\lambda(i_0))}_{\mathbb{Q}} \to \varinjlim_{i \in I} \mathcal{I}^{(i)}_{\mathbb{Q}}$. Hence, we have the isomorphisms

$$\varinjlim_{\lambda,\chi} (\chi^*\lambda^*\mathcal{I}^{(\bullet)})^{(i_0)} \underset{8.4.1.15.14}{\xrightarrow{\sim}} \varinjlim_{\lambda \in L(I)} \mathcal{I}^{(\lambda(i_0))}_{\mathbb{Q}} \xrightarrow{\sim} \varinjlim_{i \in I} \mathcal{I}^{(i)}_{\mathbb{Q}} = \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{I}^{(\bullet)}). \tag{8.4.1.15.15}$$

b) It follows from a) that the functor $\underset{\rightarrow\mathbb{Q}}{l}^*\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger})$ is fully faithful. It remains to check its essential surjectivity. From 8.4.1.13, we get from the property (i) and (ii) that $\mathcal{D}^{\dagger}_{\mathbb{Q}}$ is a coherent sheaf of rings. Hence, $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ is generated as triangulated category by $\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (i.e. the smallest triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ containing $\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ is $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$). Hence, by using some distinguished triangles of truncation and by induction hypothesis on the cardinal of nonzero cohomological spaces, with [BGK+87, I.2.18], by fully faithfulness of functor $\underset{\rightarrow\mathbb{Q}}{l}^*$, we reduce therefore to check that an object $\mathcal{G} \in \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ is in the essential image of $\underset{\rightarrow\mathbb{Q}}{l}^*$. Following 8.4.1.11.b.(ii), since our rings are coherent (after tensorisation by $-\otimes\mathbb{Q}$), there exists $i \in I$ and a coherent left $\mathcal{D}^{(i)}_{\mathbb{Q}}$-module $\mathcal{G}^{(i)}$ together with a $\mathcal{D}^{\dagger}_{\mathbb{Q}}$-linear isomorphism $\epsilon\colon \mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{\mathcal{D}^{(i)}_{\mathbb{Q}}} \mathcal{G}^{(i)} \xrightarrow{\sim} \mathcal{G}$. Using the hypothesis (iv), there exists a coherent $\mathcal{D}^{(i)}$-module $\mathcal{F}^{(i)}$ together with an isomorphism $\mathcal{F}^{(i)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}^{(i)}$ of $\mathcal{D}^{(i)}_{\mathbb{Q}}$-modules. Let $j\colon I^i \subset I$ be the canonical inclusion. From the hypothesis (iii), we get $\mathcal{F}^{(\bullet)} := (\underset{\rightarrow X}{j}^{-1}\mathcal{D}^{(\bullet)}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(i)}} \mathcal{F}^{(i)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\underset{\rightarrow X}{j}^{-1}\mathcal{D}^{(\bullet)})$. Since j is an L-equivalence, with 8.4.1.9.b, we get $\underset{\rightarrow X,I^i,\mathbb{Q}}{l}^*(\underset{\rightarrow X}{j}^{-1}\mathcal{D}^{(\bullet)}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathbb{Q}}$ and then the isomorphisms $\underset{\rightarrow X,I^i,\mathbb{Q}}{l}^*(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \underset{\rightarrow X,I^i,\mathbb{Q}}{l}^*(\underset{\rightarrow X}{j}^{-1}\mathcal{D}^{(\bullet)}) \otimes_{\mathcal{D}^{(i)}_{\mathbb{Q}}} \mathcal{F}^{(i)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}$. It follows from the equivalence of categories 8.4.1.5.1 that there exists $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ endowed with an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\underset{\rightarrow X}{j}^{-1}\mathcal{D}^{(\bullet)})$ of the form $\underset{\rightarrow X}{j}^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$. From the commutative diagram 8.4.1.9.1, we get the first isomorphism $\underset{\rightarrow X,I,\mathbb{Q}}{l}^*(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \underset{\rightarrow X,I^i,\mathbb{Q}}{l}^*(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{G}$.

□

8.4.1.16. Suppose X is the topological space associated to a noetherien affine $\mathcal{V}$-formal scheme $\mathfrak{X}$ of finite Krull dimension. Set $D^{(\bullet)} := \Gamma(\mathfrak{X}, \mathcal{D}^{(\bullet)})$. Following 8.1.1.2.7, the structural morphism $f\colon \mathfrak{X} \to \mathrm{Spf}\,\mathcal{V}$ induces the ringed topoi morphism

$$\underset{\rightarrow I}{f}\colon (X^{(I)}, \mathcal{D}^{(\bullet)}) \to (\{*\}^{(I)}, D^{(\bullet)}), \tag{8.4.1.16.1}$$

where $\{*\}$ is the topological space underlying to $\mathrm{Spf}\,\mathcal{V}$. The functor $\underset{\rightarrow I*}{f}$ can also be denoted by $\Gamma(\mathfrak{X}^{(\bullet)}, -)$ and $\underset{\rightarrow I}{f}^* = \mathcal{D}^{(\bullet)} \otimes_{D^{(\bullet)}} -$. Let $\chi \in M(I)$. For any $\mathcal{G}^{(\bullet)\bullet} \in C(\mathcal{D}^{(\bullet)})$, we have the isomorphism of $C(D^{(\bullet)})$:

$$\chi^* \circ \underset{\rightarrow I*}{f}(\mathcal{G}^{(\bullet)\bullet}) \xrightarrow{\sim} \underset{\rightarrow I*}{f} \circ \chi^*(\mathcal{G}^{(\bullet)\bullet}). \tag{8.4.1.16.2}$$

Following 7.1.3.15.2, a $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is $\underset{\rightarrow I*}{f}$-acyclic if and only if $\mathcal{E}^{(i)}$ is a $\underset{\rightarrow I*}{f}$-acyclic $\mathcal{D}^{(\bullet)}$-module for any $i \in I$. Hence, the functor χ^* sends $\underset{\rightarrow I*}{f}$-acyclic modules to $\underset{\rightarrow I*}{f}$-acyclic modules. Since $\mathfrak{X}$ has finite Krull dimension then f_* has bounded cohomological dimension. Hence so is $\underset{\rightarrow I*}{f}$ (see 7.1.3.16). This yields that, for any $\mathcal{E}^{(\bullet)\bullet} \in D(\mathcal{D}^{(\bullet)})$, by using a resolution of $\mathcal{E}^{(\bullet)\bullet}$ by $\underset{\rightarrow I*}{f}$-acyclic modules (following 4.6.1.6.b, such resolutions exist because of the boundedness of the cohomological dimension of $\underset{\rightarrow I*}{f}$), we get the functorial in χ isomorphism of $D(\mathcal{D}^{(\bullet)})$:

$$\chi^* \circ \mathbb{R}\underset{\rightarrow I*}{f}(\mathcal{E}^{(\bullet)\bullet}) \xrightarrow{\sim} \chi^* \circ \underset{\rightarrow I*}{f}(\mathcal{F}^{(\bullet)\bullet}) \overset{8.4.1.16.2}{\xrightarrow{\sim}} \underset{\rightarrow I*}{f} \circ \chi^*(\mathcal{F}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{R}\underset{\rightarrow I*}{f} \circ \chi^*(\mathcal{E}^{(\bullet)\bullet}). \tag{8.4.1.16.3}$$

Let $E^{(\bullet)\bullet} \in D(D^{(\bullet)})$. Choose $P^{(\bullet)\bullet}$ a K-flat complex of $K(D^{(\bullet)})$ together with a quasi-isomorphism $P^{(\bullet)\bullet} \to E^{(\bullet)\bullet}$ of $K(D^{(\bullet)})$. Following 7.1.3.6, $P^{(\bullet)\bullet}$ is a K-flat complex means $P^{(i)\bullet}$ is a K-flat complex of $K(D^{(i)})$. Hence, using again 7.1.3.6 and the exactness of χ^*, we get that $\chi^*(P^{(\bullet)\bullet})$ is a K-flat complex of $K(D^{(\bullet)})$ together with a quasi-isomorphism $\chi^* P^{(\bullet)\bullet} \to \chi^* E^{(\bullet)\bullet}$ of $K(D^{(\bullet)})$. This yields the functorial in χ and $E^{(\bullet)\bullet}$ isomorphism in $D(\mathcal{D}^{(\bullet)})$:

$$\chi^* \circ \mathbb{L}\underset{\rightarrow I}{f}^*(E^{(\bullet)\bullet}) \xleftarrow{\sim} \chi^* \circ \underset{\rightarrow I}{f}^*(P^{(\bullet)\bullet}) \xrightarrow{\sim} \underset{\rightarrow I}{f}^* \circ \chi^*(P^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{L}\underset{\rightarrow I}{f}^* \circ \chi^*(E^{(\bullet)\bullet}). \tag{8.4.1.16.4}$$

Let $\lambda \in L(I)$. Let $E^{(\bullet)\bullet} \in D(D^{(\bullet)})$. Let $\mathcal{E}^{(\bullet)\bullet} \in D(\mathcal{D}^{(\bullet)})$. Similarly, we get the isomorphisms:

$$\lambda^* \circ \mathbb{R}\underset{\rightarrow I*}{f}(\mathcal{E}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{R}\underset{\rightarrow I*}{f} \circ \lambda^*(\mathcal{E}^{(\bullet)\bullet}), \qquad \lambda^* \circ \mathbb{L}\underset{\rightarrow I}{f}^*(E^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{L}\underset{\rightarrow I}{f}^* \circ \lambda^*(E^{(\bullet)\bullet}) \tag{8.4.1.16.5}$$

It follows from 8.4.1.16.3, 8.4.1.16.4, 8.4.1.16.5, that we get the factorisations:

$$\mathbb{L}\underset{\rightarrow I*}{f}\colon \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q}}(D^{(\bullet)}), \qquad \mathbb{L}\underset{\rightarrow I}{f}^*\colon \underrightarrow{LD}_{\mathbb{Q}}(D^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \tag{8.4.1.16.6}$$

Corollary 8.4.1.17. *Suppose X is the topological space associated to a noetherien affine $\mathcal{V}$-formal scheme $\mathfrak{X}$. Suppose that either I is filtered and has a smallest element or I is strictly filtered. Put $\mathcal{D}^{\dagger} := \varinjlim_{i\in I} \mathcal{D}^{(i)}$, $D^{(i)} := \Gamma(\mathfrak{X}, \mathcal{D}^{(i)})$, $D^{\dagger} := \Gamma(\mathfrak{X}, \mathcal{D}^{\dagger})$. Suppose moreover the following conditions holds*

(i) The sheaves of rings $\mathcal{D}^{(i)}$ on $\mathfrak{X}$ are equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}^{(i)}$ satisfying the conditions of 7.2.3 (for the p-adic topology) for any $i \in I$;

(ii) The transition maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any elements $i \le j$ of I;

(iii) There exists an integer d such that, for any elements $i \le j$ of I, the ring $\mathcal{D}^{(j)}$ is of tor-dimension $\le d$ on $\mathcal{D}^{(i)}$;

(iv) For any $i \in I$, for any coherent $\mathcal{D}^{(i)}_{\mathbb{Q}}$-module $\mathcal{E}$, there exists a coherent $\mathcal{D}^{(i)}$-module $\mathcal{E}'$ together with an isomorphism $\mathcal{E}'_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ of $\mathcal{D}^{(i)}_{\mathbb{Q}}$-modules.

With notation 8.4.1.16, the functors $\underset{\rightarrow I}{f}^{*} = \mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathbb{Q}}} -$ and $\mathbb{R}\underset{\rightarrow I*}{f} = \mathbb{R}\Gamma(\mathfrak{X}^{(\bullet)}, -)$ of 8.4.1.16.6 induce quasi-inverse equivalence between the categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(D^{(\bullet)})$. We have moreover the commutative (up to canonical isomorphism) diagram of categories

$$\begin{array}{ccc}
\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) & \xrightarrow[8.4.1.15]{\underset{\rightarrow}{l}^{*}_{\mathbb{Q}}} & D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathbb{Q}}) \\
{\scriptstyle \mathcal{D}^{(\bullet)}\otimes_{D^{(\bullet)}}-}\uparrow\ \downarrow{\scriptstyle \mathbb{R}\Gamma(\mathfrak{X}^{(\bullet)},-)} & & {\scriptstyle \mathcal{D}^{\dagger}_{\mathbb{Q}}\otimes_{D^{\dagger}_{\mathbb{Q}}}-}\uparrow\ \downarrow{\scriptstyle \mathbb{R}\Gamma(\mathfrak{X},-)} \\
\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(D^{(\bullet)}) & \xrightarrow[8.4.1.15]{\underset{\rightarrow}{l}^{*}_{\mathbb{Q}}} & D^{\mathrm{b}}_{\mathrm{coh}}(D^{\dagger}_{\mathbb{Q}})
\end{array} \tag{8.4.1.17.1}$$

whose functors are all equivalence of categories.

Proof. With our hypotheses, the functor $\underset{\rightarrow I}{f}^{*} = \mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathbb{Q}}} -$ is exact, which justifies the notation. Let $E^{(\bullet)} \in D(D^{(\bullet)})$, $\mathcal{E}^{(\bullet)} \in D(\mathcal{D}^{(\bullet)})$. Since the functors $\mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathbb{Q}}} -$ and $\Gamma(\mathfrak{X}, -)$ commute with filtrant inductive limites and with the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$, we get the following first (resp. second) isomorphism of $D(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (resp. $D(D^{\dagger}_{\mathbb{Q}})$):

$$\underset{\rightarrow}{l}^{*}_{\mathbb{Q}}(\mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathbb{Q}}} (E^{(\bullet)})) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathbb{Q}}} \underset{\rightarrow}{l}^{*}_{\mathbb{Q}}(E^{(\bullet)}), \qquad \underset{\rightarrow}{l}^{*}_{\mathbb{Q}}(\Gamma(\mathfrak{X}, \mathcal{E}^{(\bullet)})) \xrightarrow{\sim} \Gamma(\mathfrak{X}, \underset{\rightarrow}{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})).$$

With our hypotheses, the functors $\mathbb{R}\Gamma(\mathfrak{X}, -)$ and $\mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathbb{Q}}} -$ (the left part of the diagram 8.4.1.17.1) are quasi-inverse equivalence between $D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}D^{\dagger}_{\mathbb{Q}})$ and $D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (see Theorem 8.4.1.14). Following 8.4.1.15, both horizontal functors of the diagram 8.4.1.17.1 are equivalence of categories. Hence, we are done. □

Example 8.4.1.18. Both above theorems and corollary will essentially be useful in the case of differential operators with overconvergent singularities (see later 8.7.5.4) or also in the case of the sheaf of functions with overconvergent singularities (see 8.7.4.6).

8.4.2 Coherent modules up to ind-isogeny on formal scheme

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. We suppose $p\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$. Let I be a partially ordered set. Let $\mathcal{D}^{(\bullet)} = (\mathcal{D}^{(i)}, \alpha^{(j,i)}_{\mathcal{D}})$ be a sheaf of rings on the topos $\mathfrak{X}^{(I)}$. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$. For any $i \in I$, we suppose $\mathcal{D}^{(i)}$ is a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}^{(i)}$ satisfying the conditions of 7.2.3. Recall following 7.2.3.3, $\mathcal{D}^{(i)}$ is left coherent.

Lemma 8.4.2.1. *Assume I has a smallest element i_0. Then we have the following properties.*

1. *The functors $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} -$ and $\mathcal{E}^{(\bullet)} \mapsto \mathcal{E}^{(i_0)}$ induce quasi-inverse equivalences between the category of coherent $\mathcal{D}^{(i_0)}$-modules (resp. of $\mathcal{D}^{(i_0)}$-modules of global finite presentation) and that of $\mathcal{D}^{(\bullet)}$-modules of finite presentation (resp. of $\mathcal{D}^{(\bullet)}$-modules of global finite presentation).*

2. *Let $\mathcal{E}^{(\bullet)}$ be a $\mathcal{D}^{(\bullet)}$-module. The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is of finite presentation if and only if there exists an open covering $(\mathfrak{X}_\alpha)_{\alpha\in A}$ of $\mathfrak{X}$ such that, for any $\alpha \in A$, we have an exact sequence in $M(\mathcal{D}^{(\bullet)})$ of the form:*
$$\left(\mathcal{D}^{(\bullet)}|_{\mathfrak{X}_\alpha}\right)^{r_\alpha} \to \left(\mathcal{D}^{(\bullet)}|_{\mathfrak{X}_\alpha}\right)^{s_\alpha} \to \mathcal{E}^{(\bullet)}|_{\mathfrak{X}_\alpha} \to 0,$$
where $r_\alpha, s_\alpha \in \mathbb{N}$.

Proof. The respective first statement is obvious (see 7.1.3.7). Since $\mathcal{D}^{(i_0)}$ is coherent , the non-respective first statement follows from 7.1.3.8. This yields the second one. □

Remark 8.4.2.2. Assume I has a smallest element i_0. We complete Lemma 8.4.2.1 with the following facts.

(a) Since the extensions $\mathcal{D}^{(i)} \to \mathcal{D}^{(j)}$ are not flat, the category of $\mathcal{D}^{(\bullet)}$-modules of finite presentation is not stable under kernel.

(b) If $\mathfrak{X}$ is affine, following the theorem of type A, the notions of coherent $\mathcal{D}^{(i_0)}$-modules and of $\mathcal{D}^{(i_0)}$-modules having a global finite presentation are then equal. Following 8.4.2.1, this yields that the notions of $\mathcal{D}^{(\bullet)}$-modules of finite presentation and of $\mathcal{D}^{(\bullet)}$-modules having a global finite presentation are then equal. Moreover, the functor $\mathcal{D}^{(\bullet)} \otimes_{\Gamma(\mathfrak{X},\mathcal{D}^{(i_0)})} -$ from the category of $\Gamma(\mathfrak{X}, \mathcal{D}^{(i_0)})$-modules of finite type into that of $\mathcal{D}^{(\bullet)}$-modules of finite presentation is an equivalence of categories such that a quasi-inverse functor is given by $\mathcal{G}^{(\bullet)} \mapsto \Gamma(\mathfrak{X}, \mathcal{G}^{(i_0)})$.

Lemma 8.4.2.3. *Assume I has a smallest element i_0. Let $\mathcal{E}^{(\bullet)}$ be a $\mathcal{D}^{(\bullet)}$-module. The following conditions are equivalent.*

(a) The module $\mathcal{E}^{(\bullet)}$ is isomorphic in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ to a $\mathcal{D}^{(\bullet)}$-module of finite presentation ;

(b) The module $\mathcal{E}^{(i_0)}$ is isogenic to a coherent $\mathcal{D}^{(i_0)}$-module and the canonical morphism $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{E}^{(\bullet)}$ is an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$ (see notation 8.1.1.4).

Proof. i) Let $\mathcal{F}^{(i_0)}$ be a $\mathcal{D}^{(i_0)}$-module, $f^{(i_0)} : \mathcal{E}^{(i_0)} \to \mathcal{F}^{(i_0)}$ and $g^{(i_0)} : \mathcal{F}^{(i_0)} \to \mathcal{E}^{(i_0)}$ be two $\mathcal{D}^{(i_0)}$-linear morphisms such that $f^{(i_0)} \circ g^{(i_0)}$ and $g^{(i_0)} \circ f^{(i_0)}$ are the multiplications by p^n for some integer $n \geq 0$. We get by extension the morphisms $f^{(\bullet)} : \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)}$ and $g^{(\bullet)} : \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)} \to \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)}$. Taking $\chi \in M(I)$ equal to the constant fonction with value n, by composing $g^{(\bullet)}$ with the canonical morphism $id \to \chi^*$, we get the arrow $h^{(\bullet)} : \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)} \to \chi^*(\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)})$. This yields that $\chi^*(f^{(\bullet)}) \circ h^{(\bullet)} = \theta_\chi$ and $h^{(\bullet)} \circ f^{(\bullet)} = \theta_\chi$ are the canonical morphisms, i.e. $f^{(\bullet)}$ is an ind-isogeny.

ii) When $\mathcal{F}^{(i_0)}$ is a coherent $\mathcal{D}^{(i_0)}$-module, $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)}$ is a $\mathcal{D}^{(\bullet)}$-module of finite presentation (see 8.4.2.1). This yields the implication $(b) \Rightarrow (a)$.

iii) Let us prove $(a) \Rightarrow (b)$. Suppose the condition (a) is satisfied, i.e. assume that $\mathcal{E}^{(\bullet)}$ is isomorphic in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ to a $\mathcal{D}^{(\bullet)}$-module $\mathcal{F}^{(\bullet)}$ of finite presentation. In this case, there exists $\chi \in M(I)$ and an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$. This yields an isogeny $\mathcal{E}^{(i_0)} \to \mathcal{F}^{(i_0)}$. Hence, the induced map by extension $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)}$ is an ind-isogeny (see part i) of the proof). Following 8.4.2.1, $\mathcal{F}^{(i_0)}$ is a coherent $\mathcal{D}^{(i_0)}$-module and the canonical morphism $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)} \to \mathcal{F}^{(\bullet)}$ is an isomorphism. This yields the ind-isogeny of $M(\mathcal{D}^{(\bullet)})$:
$$\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{F}^{(i_0)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}.$$
Since this composition is equal to the composition $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{E}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$, since $\mathcal{E}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ is an ind-isogeny this implies that so is $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{E}^{(\bullet)}$.

□

Definition 8.4.2.4. Let $\mathcal{E}^{(\bullet)}$ be a left (resp. right) $\mathcal{D}^{(\bullet)}$-module.

(a) The module $\mathcal{E}^{(\bullet)}$ is "generated by finitely many global sections up to an ind-isogeny" if there exists a surjective morphism in $\underrightarrow{M}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)})$ of the form
$$(\mathcal{D}^{(\bullet)})^N \to \mathcal{E}^{(\bullet)},$$
for some positive integer N.

(b) The module $\mathcal{E}^{(\bullet)}$ has "a global presentation up to an ind-isogeny" (resp. "a global finite presentation up to an ind-isogeny", resp. is "free up to an ind-isogeny", resp. is "finite free up to an ind-isogeny") if there exists an exact sequence in $\underrightarrow{M}_{\mathbb{Q}}({}^{*}\mathcal{D}^{(\bullet)})$ of the form

$$\oplus_{i\in I}\mathcal{D}^{(\bullet)} \to \oplus_{j\in J}\mathcal{D}^{(\bullet)} \to \mathcal{E}^{(\bullet)} \to 0,$$

for some sets I and J (resp. some finite sets I and J, resp. I is the empty set and J is a set resp. I is the empty set and J is a finite set).

(c) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is of finite type up to an ind-isogeny (resp. has local presentation up to an ind-isogeny, resp. is of finite presentation up to an ind-isogeny, resp. is locally free up to an ind-isogeny, resp. is locally finite free up to an ind-isogeny) if for any object $(i,\mathfrak{U}) \in I^{\natural} \times \mathfrak{X}_{Zar}$, there exists a covering $\{(i,\mathfrak{U}_{\alpha}) \to (i,\mathfrak{U})\}_{\alpha\in A}$ such that for any $\alpha \in A$ the object $\mathcal{E}^{(\bullet)}|_{(i,\mathfrak{U}_{\alpha})}$ of $\underrightarrow{M}_{\mathbb{Q}}({}^{*}\mathcal{D}^{(\bullet)}|_{(i,\mathfrak{U}_{\alpha})})$ is generated by finitely many global sections (resp. has global presentation, resp. has global finite presentation, resp. is free, resp. is finite free) up to an ind-isogeny.

(d) The module $\mathcal{E}^{(\bullet)}$ is a " coherent up to an ind-isogeny $\mathcal{D}^{(\bullet)}$-module" if it is of finite type up to an ind-isogeny and if for any $i \in I$ and for any open set $\mathfrak{U}$ of $\mathfrak{X}$, for any homomorphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,\mathfrak{U})})$ of the form $u^{(\bullet)} \colon \left(\mathcal{D}^{(\bullet)}|_{(i,\mathfrak{U})}\right)^{r} \to \mathcal{E}^{(\bullet)}|_{(i,\mathfrak{U})}$, the kernel of $u^{(\bullet)}$ is of finite type up to an ind-isogeny.

Lemma 8.4.2.5. *Assume I has a smallest element i_0. Let $\mathcal{E}^{(\bullet)}$ a $\mathcal{D}^{(\bullet)}$-module. Then the following properties are equivalent.*

(a) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ has a global finite presentation up to an ind-isogeny.

(b) There exists a $\mathcal{D}^{(\bullet)}$-module $\mathcal{F}^{(\bullet)}$ having a global finite presentation together with an isomorphism in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$.

(c) The module $\mathcal{E}^{(i_0)}$ is isogenic to a $\mathcal{D}^{(i_0)}$-module having a global finite presentation and the canonical morphism $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)} \to \mathcal{E}^{(\bullet)}$ is an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$ (see notation 8.1.1.4).

Proof. To check the equivalence between (b) and (c), it is sufficient to copy the proof of 8.4.2.3. The implication $(b) \Rightarrow (a)$ is trivial. Conversely, let us prove $(a) \Rightarrow (b)$. Let's assume that $\mathcal{E}^{(\bullet)}$ is the cokernel in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of a arrow of the form $f^{(\bullet)} \colon \left(\mathcal{D}^{(\bullet)}\right)^{r} \to \left(\mathcal{D}^{(\bullet)}\right)^{s}$. Let $\chi \in M(I)$ and $\phi^{(\bullet)} \colon \left(\mathcal{D}^{(\bullet)}\right)^{r} \to \chi^{*}\left(\mathcal{D}^{(\bullet)}\right)^{s}$ be a morphism of $M(\mathcal{D}^{(\bullet)})$ representing $f^{(\bullet)}$. Put $\chi' \in M(I)$ such that for any $i \in I$, $\chi'(i) := \chi(i) - \chi(i_0)$. Hence, we get the commutative diagram

$$\begin{array}{ccccccc}
\left(\mathcal{D}^{(i_0)}\right)^{r} & \xrightarrow{\phi^{(i_0)}} & \left(\mathcal{D}^{(i_0)}\right)^{s} & \longrightarrow & \mathrm{Coker}\ \phi^{(i_0)} & \longrightarrow & 0 \\
\downarrow \alpha_{\mathcal{D}}^{(i,i_0)} & & \downarrow p^{\chi'(i)}\alpha_{\mathcal{D}}^{(i,i_0)} & & \vdots\downarrow & & \\
\left(\mathcal{D}^{(i)}\right)^{r} & \xrightarrow{\phi^{(i)}} & \left(\mathcal{D}^{(i)}\right)^{s} & \longrightarrow & \mathrm{Coker}\ \phi^{(i)} & \longrightarrow & 0
\end{array}$$

whose horizontal sequences are exact. This yields by extension the commutative diagram of $M(\mathcal{D}^{(\bullet)})$

$$\begin{array}{ccccccc}
\left(\mathcal{D}^{(\bullet)}\right)^{r} & \xrightarrow{\mathcal{D}^{(\bullet)}\otimes\phi^{(i_0)}} & \left(\mathcal{D}^{(\bullet)}\right)^{s} & \longrightarrow & \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathrm{Coker}\ \phi^{(i_0)} & \longrightarrow & 0 \\
\downarrow id & & \downarrow \theta_{\chi'} & & \downarrow & & \\
\left(\mathcal{D}^{(\bullet)}\right)^{r} & \xrightarrow{\phi^{(\bullet)}} & \chi'^{*}\left(\mathcal{D}^{(\bullet)}\right)^{s} & \longrightarrow & \mathrm{Coker}\ \phi^{(\bullet)} & \longrightarrow & 0
\end{array} \tag{8.4.2.5.1}$$

whose horizontal sequences are exact. By using the five lemma applied to the diagram 8.4.2.5.1 of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, since both left vertical morphisms of 8.4.2.5.1 are isomorphisms of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then so is $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathrm{Coker}\ \phi^{(i_0)} \to \mathrm{Coker}\ \phi^{(\bullet)}$. Since $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathrm{Coker}\ \phi^{(i_0)}$ is a $\mathcal{D}^{(\bullet)}$-module having a global finite presentation, this implies the required result. □

Lemma 8.4.2.6. *Assume I has a smallest element i_0. Let $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ be two $\mathcal{D}^{(\bullet)}$-modules having a global finite presentation up to an ind-isogeny. Let $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Then* $\mathrm{Coker}\, f^{(\bullet)}$ *has a global finite presentation up to an ind-isogeny.*

Proof. By using 8.4.2.5, we can suppose that $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ are two $\mathcal{D}^{(\bullet)}$-modules having a global finite presentation. Let $\chi \in M(I)$ and $\phi^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi^*\mathcal{F}^{(\bullet)}$ be a morphism of $M(\mathcal{D}^{(\bullet)})$ representing $f^{(\bullet)}$. By copying the proof of the implication $(a) \Rightarrow (b)$ of 8.4.2.5, we can check that $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathrm{Coker}\ \phi^{(i_0)} \to \mathrm{Coker}\ \phi^{(\bullet)}$ is an ind-isogeny. Since $\mathcal{D}^{(i_0)}$ is coherent, then $\mathrm{Coker}\ \phi^{(i_0)}$ is a coherent $\mathcal{D}^{(i_0)}$-module. Hence, we conclude by using again 8.4.2.5; □

Remark 8.4.2.7. Suppose I is a countable well ordered set. For any map $\alpha\colon I \to \mathbb{N}$ there exists $\chi \in M(I)$ such that $\alpha(i) \le \chi(i)$ for any $i \in I$.

Lemma 8.4.2.8. *Suppose I is a countable well ordered set and $\mathfrak{X}$ is noetherian. Let $f^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $M(\mathcal{D}^{(\bullet)})$ such that, for any $i \in I$, the $\mathcal{D}^{(i)}$-modules $\mathcal{E}^{(i)}$ and $\mathcal{F}^{(i)}$ are coherent. The following properties are equivalent.*

(a) The morphism $f^{(\bullet)}$ is an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$;

(b) The morphism $f^{(\bullet)}_{\mathbb{Q}}\colon \mathcal{E}^{(\bullet)}_{\mathbb{Q}} \to \mathcal{F}^{(\bullet)}_{\mathbb{Q}}$ induced by tensorisation with $\mathbb{Q}$ is an isomorphism of $\mathcal{D}^{(\bullet)}_{\mathbb{Q}}$-modules.

Proof. 0) Since the canonical morphisms of the form $\mathcal{E}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ with $\chi \in M(I)$ becomes isomorphisms after applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$, the implication $(a) \Rightarrow (b)$ is obvious. Conversely let's assume that $f^{(\bullet)}_{\mathbb{Q}}$ is an isomorphism.

1) Following 7.4.5.1, if we denote by $\mathcal{E}^{(i)}_t$ the subsheaf of p-torsion sections of $\mathcal{E}^{(i)}$, then $\mathcal{E}^{(i)}_t$ is a coherent sub-$\mathcal{D}^{(i)}$-module of $\mathcal{E}^{(i)}$. Denote by $\alpha\colon \mathcal{E}^{(\bullet)} \to \mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)}_t$ the canonical projection. Let us check that this is an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$. Since $\mathfrak{X}$ is noetherian, then there exists $\chi \in M(I)$ such that $p^{\chi(i)}\mathcal{E}^{(i)}_t = 0$ for any $i \in I$ (see the proof of 7.4.5.1 and use also the remark 8.4.2.7). Hence the map $p^{\chi(i)}\colon \mathcal{E}^{(i)} \to \mathcal{E}^{(i)}$ factors uniquely through $\beta^{(i)}\colon \mathcal{E}^{(i)}/\mathcal{E}^{(i)}_t \to \mathcal{E}^{(i)}$. This yields that the canonical map $\theta_{\mathcal{E},\chi}\colon \mathcal{E}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ factors through $\beta^{(\bullet)}\colon \mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)}_t \to \chi^*\mathcal{E}^{(\bullet)}$ So, the morphism $\beta^{(\bullet)}\colon \mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)}_t \to \chi^*\mathcal{E}^{(\bullet)}$of $M(\mathcal{D}^{(\bullet)})$ is such that $\beta^{(\bullet)} \circ \alpha^{(\bullet)}$ and $\chi^*(\alpha^{(\bullet)}) \circ \beta^{(\bullet)}$ are the canonical morphisms.

2) By using the step 1), we reduce to check that the canonical morphism $\mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)}_t \to \mathcal{F}^{(\bullet)}/\mathcal{F}^{(\bullet)}_t$ induced by $f^{(\bullet)}$ is an ind-isogeny, i.e. we can suppose that, for any $i \in I$, $\mathcal{E}^{(i)}$ and $\mathcal{F}^{(i)}$ are p-torsion free. For any $\chi \in M(I)$, let us denote by $h^{(\bullet)}_{\chi} := \theta_{\mathcal{E}_{\mathbb{Q}},\chi} \circ (f^{(\bullet)}_{\mathbb{Q}})^{-1}\colon \mathcal{F}^{(\bullet)}_{\mathbb{Q}} \to \chi^*\mathcal{E}^{(\bullet)}_{\mathbb{Q}}$, the morphism of $\mathcal{D}^{(\bullet)}_{\mathbb{Q}}$-modules. We can suppose that $\chi \in M(I)$ is large enough so that for any $i \in I$ the canonical isomorphism $h^{(i)}_{\chi} = p^{\chi(i)}(f^{(i)}_{\mathbb{Q}})^{-1}\colon \mathcal{F}^{(i)}_{\mathbb{Q}} \to \mathcal{E}^{(i)}_{\mathbb{Q}}$ factors uniquely through a morphism of the form $g^{(i)}\colon \mathcal{F}^{(i)} \to \mathcal{E}^{(i)}$ (use again the remark 8.4.2.7). Since $\chi^*\mathcal{E}^{(\bullet)} \subset \chi^*\mathcal{E}^{(\bullet)}_{\mathbb{Q}}$ and $\chi^*\mathcal{F}^{(\bullet)} \subset \chi^*\mathcal{F}^{(\bullet)}_{\mathbb{Q}}$, we compute in fact that the morphism $h^{(\bullet)}_{\chi}$ induces the morphism $g^{(\bullet)}\colon \mathcal{F}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ (i.e., the $g^{(i)}$ commute with transition morphisms) and satisfies the equalities $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. □

Since this is almost straightforward, let us give the following corollary of the above lemma which will be useful later:

Corollary 8.4.2.9. *Suppose I is a countable well ordered set and $\mathfrak{X}$ is noetherian. Let $f\colon \mathcal{E}^{(\bullet)\bullet} \to \mathcal{F}^{(\bullet)\bullet}$ be a morphism of $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ such that, for any $i \in I$, the complexes $\mathcal{E}^{(i)\bullet}$ and $\mathcal{F}^{(i)\bullet}$ are objects of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(i)})$. The following properties are equivalent.*

(a) The morphism f is an isomorphism of $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$;

(b) The induced morphism $f_{\mathbb{Q}}\colon \mathcal{E}^{(\bullet)\bullet}_{\mathbb{Q}} \to \mathcal{F}^{(\bullet)\bullet}_{\mathbb{Q}}$ is an isomorphism of $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Conversely let's assume that $f_{\mathbb{Q}}$ is an isomorphism. Let $\mathcal{G}^{(\bullet)}$ be a cone of f. It follows from 8.1.5.11 that we reduce to check that for any $n \in \mathbb{N}$ we have the isomorphism $\mathcal{H}^n\mathcal{G}^{(\bullet)} \xrightarrow{\sim} 0$ in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, i.e. the canonical map $\mathcal{H}^n\mathcal{G}^{(\bullet)} \to 0$ is an ind-isogeny. By hypothesis, since $\mathcal{G}^{(\bullet)}_{\mathbb{Q}} \xrightarrow{\sim} 0$ in $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$ then $\mathcal{H}^n\mathcal{G}^{(\bullet)}_{\mathbb{Q}} \xrightarrow{\sim} 0$ in $M(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$. Hence, we conclude by using 8.4.2.8 to the canonical map $\mathcal{H}^n\mathcal{G}^{(\bullet)} \to 0$. □

Definition 8.4.2.10. We say that I is locally countably well ordered if $I_{\ge i}$ is countable and well ordered for any $i \in I$.

Lemma 8.4.2.11. *Assume I is locally countably well ordered (see 8.4.2.10) and that the maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any $i \leq j$ elements of I. Let $\mathcal{E}^{(\bullet)}$ a $\mathcal{D}^{(\bullet)}$-module. Then the following properties are equivalent.*

(a) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is of finite presentation up to an ind-isogeny ;

(b) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is coherent up to an ind-isogeny.

In particular, the sheaf of rings $\mathcal{D}^{(\bullet)}$ is therefore coherent up to an ind-isogeny $\mathcal{D}^{(\bullet)}$-module.

Proof. The implication $(b) \Rightarrow (a)$ is obvious. Conversely, suppose $\mathcal{E}^{(\bullet)}$ is of finite presentation up to an ind-isogeny. Since both properties (a) and (b) are local on $\mathfrak{X}^{(\bullet)}$ (see definition 8.3.3.1) and stable by isomorphism in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, via the lemma 8.4.2.1 and the equivalence between (a) and (b) of 8.4.2.5, we can suppose I is countably well ordered with smallest element i_0, $\mathcal{E}^{(i_0)}$ is a $\mathcal{D}^{(i_0)}$-module having a global finite presentation and $\mathcal{E}^{(\bullet)} = \mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)}$. Let $f^{(\bullet)} \colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \mathcal{E}^{(\bullet)}$ be a morphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. It is a question of checking that $\ker f^{(\bullet)}$ is of finite type up to an ind-isogeny. Let $\chi \in M(I)$ such that there exists $a^{(\bullet)} \colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \chi^* \mathcal{E}^{(\bullet)}$ a morphism of $M(\mathcal{D}^{(\bullet)})$ representing $f^{(\bullet)}$. Since $\mathcal{D}^{(i)}$ is a coherent ring, since the extensions $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat, then it follows from the lemma 8.4.2.8 that the canonical morphism $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \ker a^{(i_0)} \to \ker a^{(\bullet)}$ is an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$. Hence we are done. □

Remark 8.4.2.12. With the notations of the lemma 8.4.2.11, since the extensions $\mathcal{D}^{(i)} \to \mathcal{D}^{(j)}$ are not flat, it seems false that the sheaf of rings $\mathcal{D}^{(\bullet)}$ is coherent (as object of $M(\mathcal{D}^{(\bullet)})$).

Lemma 8.4.2.13. *If $\mathcal{P}^{(\bullet)}$ is a projective object of $M(\mathcal{D}^{(\bullet)})$, then $\mathcal{P}^{(\bullet)}$ is a projective object of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.*

Proof. Let $g^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be an epimorphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ and $f^{(\bullet)} \colon \mathcal{P}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ a morphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Choose $\chi \in M(I)$, $\phi^{(\bullet)} \colon \mathcal{P}^{(\bullet)} \to \chi^* \mathcal{F}^{(\bullet)}$ and $\psi^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \chi^* \mathcal{F}^{(\bullet)}$ some morphisms of $M(\mathcal{D}^{(\bullet)})$ representing respectively $f^{(\bullet)}$ and $g^{(\bullet)}$. Denote by $\mathcal{G}^{(\bullet)} := \chi^* \mathcal{F}^{(\bullet)}$, $\mathcal{H}^{(\bullet)} := \operatorname{im} \psi^{(\bullet)}$ (computed in $M(\mathcal{D}^{(\bullet)})$), $\varpi^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{H}^{(\bullet)}$ the canonical epimorphism and by $\alpha^{(\bullet)} \colon \mathcal{H}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ the canonical monomorphism (both are morphisms of $M(\mathcal{D}^{(\bullet)})$). Since $g^{(\bullet)}$ is an epimorphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then $\alpha^{(\bullet)}$ is an ind-isogeny, i.e. there exists $\chi_1 \in M(I)$ and a morphism $\beta^{(\bullet)} \colon \mathcal{G}^{(\bullet)} \to \chi_1^* \mathcal{H}^{(\bullet)}$ such that $\beta^{(\bullet)} \circ \alpha^{(\bullet)} = \theta_{\mathcal{H},\chi_1}$ and $\chi_1^*(\alpha^{(\bullet)}) \circ \beta^{(\bullet)} = \theta_{\mathcal{G},\chi_1}$. Since $\mathcal{P}^{(\bullet)}$ is a projective object of $M(\mathcal{D}^{(\bullet)})$, since $\chi_1^*(\varpi^{(\bullet)})$ is an epimorphism of $M(\mathcal{D}^{(\bullet)})$, then there exists a morphism $\vartheta^{(\bullet)} \colon \mathcal{P}^{(\bullet)} \to \chi_1^* \mathcal{E}^{(\bullet)}$ of $M(\mathcal{D}^{(\bullet)})$ such that $\chi_1^*(\varpi^{(\bullet)}) \circ \vartheta^{(\bullet)} = \beta^{(\bullet)} \circ \phi^{(\bullet)}$. We notice moreover that $\chi_1^*(\psi^{(\bullet)}) \circ \vartheta^{(\bullet)} = \theta_{\mathcal{G},\chi_1} \circ \phi^{(\bullet)}$. We get the morphism $h^{(\bullet)} := (\theta_{\mathcal{E},\chi_1})^{-1} \circ \vartheta^{(\bullet)} \colon \mathcal{P}^{(\bullet)} \to \mathcal{E}^{(\bullet)}$ of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We have $g^{(\bullet)} \circ h^{(\bullet)} = f^{(\bullet)}$. □

Remark 8.4.2.14. It is not clear that if $\mathcal{I}^{(\bullet)}$ is an injective object of $M(\mathcal{D}^{(\bullet)})$, then $\mathcal{I}^{(\bullet)}$ is an injective object of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

Notation 8.4.2.15. Denote by $\underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ the full subcategory of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ consisting of coherent $\mathcal{D}^{(\bullet)}$-modules up to an ind-isogeny. Likewise, when $\mathfrak{X}$ is affine, taking all the above arguments into this context, we define similarly the full subcategory $\underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}(D^{(\bullet)})$ of $\underrightarrow{M}_{\mathbb{Q}}(D^{(\bullet)})$ consisting of coherent $D^{(\bullet)}$-modules up to an ind-isogeny, where $D^{(\bullet)} := \Gamma(\mathfrak{X}, \mathcal{D}^{(\bullet)})$.

Proposition 8.4.2.16. Assume I is locally countably well ordered (see 8.4.2.10) and that the maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any $i \leq j$ elements of I. The full subcategory $\underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is stable by isomorphism, kernel, cokernel, extension.

Proof. The coherence up to ind-isogeny being a local property on $\mathfrak{X}^{(\bullet)}$ we can suppose that I is countably well ordered. The stability of the coherence up to ind-isogeny by isomorphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is trivial. Let us check now the stability by kernel and cokernel. Let $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. By localness on $\mathfrak{X}^{(\bullet)}$, we can suppose $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ are two $\mathcal{D}^{(\bullet)}$-modules having a global finite presentation up to ind-isogeny. By using 8.4.2.5, we reduce to the case where $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ are two $\mathcal{D}^{(\bullet)}$-modules having a global finite presentation. From 8.4.2.6 and 8.4.2.11, we get the stability under cokernels. It remains the stability under kernels. Let $\chi \in M(I)$ and $\phi^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \chi^* \mathcal{F}^{(\bullet)}$ be a morphism of $M(\mathcal{D}^{(\bullet)})$ representing $f^{(\bullet)}$. We get the composite morphism $\psi^{(\bullet)} \colon \mathcal{E}^{(\bullet)}_{\mathbb{Q}} \xrightarrow{\phi^{(\bullet)} \otimes \mathrm{id}} (\chi^* \mathcal{F}^{(\bullet)})_{\mathbb{Q}} \xleftarrow{\sim} \mathcal{F}^{(\bullet)}_{\mathbb{Q}}$ of $M(\mathcal{D}^{(\bullet)}_{\mathbb{Q}})$.

Following 8.4.2.1, since $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ have global finite presentation, then $\psi^{(\bullet)} = \mathcal{D}^{(\bullet)}_{\mathbb{Q}} \otimes_{\mathcal{D}^{(i_0)}_{\mathbb{Q}}} \psi^{(i_0)}$. Since the extensions $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat, then $\operatorname{Ker} \psi^{(\bullet)} = \mathcal{D}^{(\bullet)}_{\mathbb{Q}} \otimes_{\mathcal{D}^{(i_0)}_{\mathbb{Q}}} \operatorname{Ker} \psi^{(i_0)}$. Moreover, since $\mathbb{Z} \to \mathbb{Q}$ is flat, we have $(\ker \phi^{(i_0)})_{\mathbb{Q}} = \ker \psi^{(i_0)}$ and $\ker \psi^{(\bullet)} = (\ker \phi^{(\bullet)})_{\mathbb{Q}}$. Since $\mathcal{D}^{(i)}$ is a coherent ring, then $\mathcal{D}^{(i)} \otimes_{\mathcal{D}^{(i_0)}} \ker \phi^{(i_0)}$ and $\ker \phi^{(i)}$ are coherent $\mathcal{D}^{(i)}$-modules for any $i \in I$. Hence, it follows from the lemma 8.4.2.8 that the canonical morphism $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(i_0)}} \ker \phi^{(i_0)} \to \ker \phi^{(\bullet)}$ is an ind-isogeny of $M(\mathcal{D}^{(\bullet)})$. We conclude by using 8.4.2.5.

Now let us treat the stability by extension. Let $0 \to \mathcal{E}^{(\bullet)} \underset{f^{(\bullet)}}{\longrightarrow} \mathcal{F}^{(\bullet)} \underset{g^{(\bullet)}}{\longrightarrow} \mathcal{G}^{(\bullet)} \to 0$ be an exact sequence in the category $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ with $\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet)} \in \underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Since the coherence of $\mathcal{F}^{(\bullet)}$ is local on $\mathfrak{X}^{(\bullet)}$, we can suppose there exists some epimorphisms in $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $a^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \mathcal{E}^{(\bullet)}$ and $b^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^s \to \mathcal{G}^{(\bullet)}$. Following 8.4.2.13, there exists $h^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^s \to \mathcal{F}^{(\bullet)}$ such that $g^{(\bullet)} \circ h^{(\bullet)} = b^{(\bullet)}$. This implies that $\left(\mathcal{D}^{(\bullet)}\right)^r \oplus \left(\mathcal{D}^{(\bullet)}\right)^s \to \mathcal{F}^{(\bullet)}$ defined by $f^{(\bullet)} \circ a^{(\bullet)} + h^{(\bullet)}$ is an epimorphism. We have therefore checked that $\mathcal{F}^{(\bullet)}$ is of finite type up to an ind-isogeny. Let $\alpha^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^t \to \mathcal{F}^{(\bullet)}$ a morphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. It remains to check that $\ker \alpha^{(\bullet)}$ is of finite type up to an ind-isogeny. Since $\mathcal{G}^{(\bullet)}$ is coherent up to an ind-isogeny, $\ker(g^{(\bullet)} \circ \alpha^{(\bullet)})$ is of finite type up to an ind-isogeny. Since what we have to check is local on $\mathfrak{X}^{(\bullet)}$, we can suppose there exists an epimorphism of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\beta^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^u \to \ker(g^{(\bullet)} \circ \alpha^{(\bullet)})$. Since $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is an abelian category, we have the canonical isomorphism $\alpha^{(\bullet)}(\ker(g^{(\bullet)} \circ \alpha^{(\bullet)})) \xrightarrow{\sim} \operatorname{im}(\alpha^{(\bullet)}) \cap \ker g^{(\bullet)}$. In particular, if we denote by $\delta^{(\bullet)}\colon \ker(g^{(\bullet)} \circ \alpha^{(\bullet)}) \to \mathcal{F}^{(\bullet)}$ the composition of $\alpha^{(\bullet)}$ with the monomorphism $\ker(g^{(\bullet)} \circ \alpha^{(\bullet)}) \subset \left(\mathcal{D}^{(\bullet)}\right)^t$, we get the canonical factorization $\epsilon^{(\bullet)}\colon \ker(g^{(\bullet)} \circ \alpha^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ of $\delta^{(\bullet)}$ by $f^{(\bullet)}$. Since $\mathcal{E}^{(\bullet)}$ is coherent up to an ind-isogeny, $\ker(\epsilon^{(\bullet)} \circ \beta^{(\bullet)})$ is of finite type up to an ind-isogeny. Since $f^{(\bullet)}$ is a monomorphism, this yields that $\ker(f^{(\bullet)} \circ \epsilon^{(\bullet)} \circ \beta^{(\bullet)})$ is of finite type up to an ind-isogeny. Since $f^{(\bullet)} \circ \epsilon^{(\bullet)} \circ \beta^{(\bullet)} = \delta^{(\bullet)} \circ \beta^{(\bullet)}$, then $\ker(\delta^{(\bullet)} \circ \beta^{(\bullet)})$ is of finite type up to an ind-isogeny. Moreover, we have the isomorphisms $\beta^{(\bullet)}(\ker(\delta^{(\bullet)} \circ \beta^{(\bullet)})) \xrightarrow{\sim} \operatorname{im}(\beta^{(\bullet)}) \cap \ker(\delta^{(\bullet)}) \xrightarrow{\sim} \ker(\delta^{(\bullet)}) = \ker(\alpha^{(\bullet)})$, which is therefore of finite type up to an ind-isogeny. □

8.4.3 Coherent modules up to lim-ind-isogeny on formal scheme

We keep notation and hypotheses of 8.4.2.

Definition 8.4.3.1 (Coherence up to lim-ind-isogeny)**.** Let $\mathcal{E}^{(\bullet)}$ be a left (resp. right) $\mathcal{D}^{(\bullet)}$-module.

(a) The module $\mathcal{E}^{(\bullet)}$ is "generated by finitely many global sections up to a lim-ind-isogeny " if there exists a surjective morphism in $\underrightarrow{LM}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)})$ of the form

$$(\mathcal{D}^{(\bullet)})^N \to \mathcal{E}^{(\bullet)},$$

for some positive integer N.

(b) The module $\mathcal{E}^{(\bullet)}$ has "a global presentation up to a lim-ind-isogeny " (resp. "a global finite presentation up to a lim-ind-isogeny ", resp. is "free up to a lim-ind-isogeny ", resp. is "finite free up to a lim-ind-isogeny ") if there exists an exact sequence in $\underrightarrow{LM}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)})$ of the form

$$\oplus_{i\in I}\mathcal{D}^{(\bullet)} \to \oplus_{j\in J}\mathcal{D}^{(\bullet)} \to \mathcal{E}^{(\bullet)} \to 0,$$

for some sets I and J (resp. some finite sets I and J, resp. I is the empty set and J is a set resp. I is the empty set and J is a finite set).

(c) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is of finite type up to a lim-ind-isogeny (resp. has local presentation up to a lim-ind-isogeny , resp. is of finite presentation up to a lim-ind-isogeny , resp. is locally free up to a lim-ind-isogeny , resp. is locally finite free up to a lim-ind-isogeny) if for any object $(i, \mathfrak{U}) \in I^{\natural} \times \mathfrak{X}_{Zar}$, there exists a covering $\{(i, \mathfrak{U}_\alpha) \to (i, \mathfrak{U})\}_{\alpha\in A}$ such that for any $\alpha \in A$ the object $\mathcal{E}^{(\bullet)}|_{(i,\mathfrak{U}_\alpha)}$ of $\underrightarrow{LM}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)}|_{(i,\mathfrak{U}_\alpha)})$ is generated by finitely many global sections (resp. has global presentation, resp. has global finite presentation, resp. is free, resp. is finite free) up to a lim-ind-isogeny .

(d) The module $\mathcal{E}^{(\bullet)}$ is a " coherent up to a lim-ind-isogeny $\mathcal{D}^{(\bullet)}$-module" if it is of finite type up to a lim-ind-isogeny and if for any $i \in I$ and for any open set $\mathfrak{U}$ of $\mathfrak{X}$, for any homomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,\mathfrak{U})})$ of the form $u^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}|_{(i,\mathfrak{U})}\right)^r \to \mathcal{E}^{(\bullet)}|_{(i,\mathfrak{U})}$, the kernel of $u^{(\bullet)}$ is of finite type up to a lim-ind-isogeny .

The following lemma will be improved via 8.4.5.10 with more hypotheses satisfied by $\mathcal{D}^{(\bullet)}$.

Lemma 8.4.3.2. *Assume I has a smallest element i_0. Let $\mathcal{E}^{(\bullet)}$ a $\mathcal{D}^{(\bullet)}$-module. Then the following properties are equivalent.*

(a) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ has a global finite presentation up to a lim-ind-isogeny.

(b) There exists $\lambda \in L(I)$ and a $\lambda^*\mathcal{D}^{(\bullet)}$-module $\mathcal{F}^{(\bullet)}$ having a global finite presentation such that $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ are isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

.

Proof. Since $(\mathcal{D}^{(\bullet)})^r \to (\lambda^*\mathcal{D}^{(\bullet)})^r$ is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then we get the implication $(b) \Rightarrow (a)$. Conversely, let us prove $(a) \Rightarrow (b)$. Let's assume that $\mathcal{E}^{(\bullet)}$ is the cokernel in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of a arrow of the form $f^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \left(\mathcal{D}^{(\bullet)}\right)^s$. Let $\chi \in M(I)$, $\lambda \in L(I)$ and $\phi^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \chi^*\left(\lambda^*\mathcal{D}^{(\bullet)}\right)^s$ be a morphism of $M(\mathcal{D}^{(\bullet)})$ representing $f^{(\bullet)}$. Since $\chi^*\left(\lambda^*\mathcal{D}^{(\bullet)}\right)^s$ is endowed with a canonical structure of $\lambda^*\mathcal{D}^{(\bullet)}$-module, this implies by adjointness (see 8.3.2.6) that the morphism $\phi^{(\bullet)}$ factors uniquely through a morphism of $M(\lambda^*\mathcal{D}^{(\bullet)})$ of the form $\psi^{(\bullet)}\colon \left(\lambda^*\mathcal{D}^{(\bullet)}\right)^r \to \chi^*\left(\lambda^*\mathcal{D}^{(\bullet)}\right)^s$. Let Coker $\psi^{(\bullet)}$ be the cokernel of $\psi^{(\bullet)}$ computed in $M(\lambda^*\mathcal{D}^{(\bullet)})$. From the proof $(a) \to (b)$ of the lemma 8.4.2.5 ("replace" $\mathcal{D}^{(\bullet)}$ by $\lambda^*\mathcal{D}^{(\bullet)}$), there exists a $\lambda^*\mathcal{D}^{(\bullet)}$-module $\mathcal{F}^{(\bullet)}$ having a global finite presentation together with an isomorphism in $\underrightarrow{M}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$ (and then in $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$) of the form $\mathcal{F}^{(\bullet)} \xrightarrow{\sim}$ Coker $\psi^{(\bullet)}$. Since Coker $\psi^{(\bullet)}$ is isomorphic to $\mathcal{E}^{(\bullet)}$ in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then we are done. □

Lemma 8.4.3.3. *Assume I is locally countably well ordered (see 8.4.2.10) and that the maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any $i \leq j$ elements of I. Let $\mathcal{E}^{(\bullet)}$ be a $\mathcal{D}^{(\bullet)}$-module. Then the following properties are equivalent.*

(a) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is of finite presentation up to a lim-ind-isogeny ;

(b) The $\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is coherent up to a lim-ind-isogeny.

In particular, the sheaf of rings $\mathcal{D}^{(\bullet)}$ is therefore a coherent up to a lim- ind-isogeny $\mathcal{D}^{(\bullet)}$-module.

Proof. The implication $(b) \Rightarrow (a)$ is trivial. Conversely, let us prove $(a) \Rightarrow (b)$. Let $\mathcal{E}^{(\bullet)}$ be a $\mathcal{D}^{(\bullet)}$-module of finite presentation up to lim-ind-isogeny. Since the coherence up to lim-ind-isogeny is local on $\mathfrak{X}^{(\bullet)}$, we can suppose I is countably well ordered and that $\mathcal{E}^{(\bullet)}$ is a $\mathcal{D}^{(\bullet)}$-module having a global finite presentation up to lim-ind-isogeny. Following 8.4.3.2, by stability under isomorphisms of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the property of coherence up to lim-ind-isogeny, we reduce to the case where $\mathcal{E}^{(\bullet)}$ is a $\lambda^*\mathcal{D}^{(\bullet)}$-module having a global finite presentation. Let $f^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \mathcal{E}^{(\bullet)}$ be morphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. It is a question of checking that $\ker f^{(\bullet)}$ is of finite type up to a lim-ind-isogeny. Let $\chi \in M(I)$, $\lambda \in L(I)$, $a^{(\bullet)}\colon \left(\mathcal{D}^{(\bullet)}\right)^r \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$ be a morphism of $M(\mathcal{D}^{(\bullet)})$ representing $f^{(\bullet)}$. By adjointness (see 8.3.2.6), the morphism $a^{(\bullet)}$ factors uniquely through a morphism $b^{(\bullet)}\colon \left(\lambda^*\mathcal{D}^{(\bullet)}\right)^r \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$ of $M(\lambda^*\mathcal{D}^{(\bullet)})$. It follows from propositions 8.4.2.11 and 8.4.2.16 that $\ker b^{(\bullet)}$ is a coherent $\lambda^*\mathcal{D}^{(\bullet)}$-module up to an ind-isogeny. Hence, $\ker b^{(\bullet)}$ is a $\lambda^*\mathcal{D}^{(\bullet)}$-module of finite type up to an ind-isogeny, and therefore is a $\mathcal{D}^{(\bullet)}$-module of finite type up to a lim-ind-isogeny (indeed, we can use the exactness of the functor $\underrightarrow{M}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of 8.3.2.8.1). Since $\ker b^{(\bullet)}$ and $\ker f^{(\bullet)}$ are isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then we conclude the proof. □

Notation 8.4.3.4. We denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ consisting of coherent $\mathcal{D}^{(\bullet)}$-modules up to lim-ind-isogeny. Similarly, when $\mathfrak{X}$ is affine and $D^{(\bullet)} = \Gamma(\mathfrak{X}, \mathcal{D}^{(\bullet)})$, we denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(D^{(\bullet)})$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q}}(D^{(\bullet)})$ consisting of coherent $\mathcal{D}^{(\bullet)}$-modules up to lim-ind-isogeny (we replace "$\mathcal{D}$" by "D").

Remark 8.4.3.5. Let $\lambda \in L(I)$. We have also the full subcategory $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$ of $\underrightarrow{LM}_{\mathbb{Q}}(\lambda^*\mathcal{D}^{(\bullet)})$ of coherent up to a lim-ind-isogeny $\lambda^*\mathcal{D}^{(\bullet)}$-modules. With the lemma 8.3.2.7 and with the characterization of the global finite presentation given in 8.4.3.2, the functors forg_λ and $\lambda^*\mathcal{D}^{(\bullet)}\otimes_{\mathcal{D}^{(\bullet)}} -$ induce quasi-inverse equivalences of categories between $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$ and $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$.

Lemma 8.4.3.6. *If $\mathcal{P}^{(\bullet)}$ is a projective object of $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then $\mathcal{P}^{(\bullet)}$ is a projective object of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.*

Proof. The proof is identical to that of 8.4.2.13 (we replace χ by λ). □

Proposition 8.4.3.7. *Assume I is locally countably well ordered (see 8.4.2.10) and that the maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any $i \leq j$ elements of I. Then the full subcategory $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is stable by isomorphisms, kernels, cokernels, extensions.*

Proof. We can copy the proof of 8.4.2.16 (we use 8.4.3.2, 8.4.3.6 and the fact that the category $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is abelian). □

8.4.4 Passage to the limits on $D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$

We keep notation and hypotheses of 8.4.2. Let $\sharp \in \{+,-,\mathrm{b},\emptyset\}$.

8.4.4.1. Suppose I is filtered. Then we have the exact functor $\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to M(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (see 8.4.1.10). We set $\mathcal{D}^{\dagger}_{\mathbb{Q}} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We get the functor $\underrightarrow{l}^*_{\mathbb{Q}} \colon K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K^{\sharp}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$. Since the functor $\underrightarrow{l}^*_{\mathbb{Q}} \colon K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K^{\sharp}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ sends acyclic complexes to acyclic complexes, hence it factors through the functor

$$\underrightarrow{l}^*_{\mathbb{Q}} \colon D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D^{\sharp}(\mathcal{D}^{\dagger}_{\mathbb{Q}}) \tag{8.4.4.1.1}$$

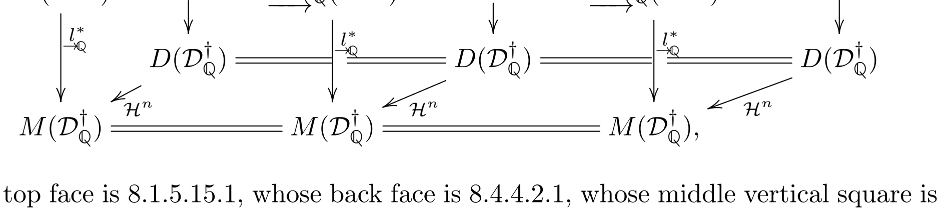
which commutes to the localization functors $Q_{\mathrm{qi}} \colon K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ and $Q_{\mathrm{qi}} \colon K^{\sharp}(\mathcal{D}^{\dagger}_{\mathbb{Q}}) \to D^{\sharp}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$, i.e. we have the equality $Q_{\mathrm{qi}} \circ \underrightarrow{l}^*_{\mathbb{Q}} = \underrightarrow{l}^*_{\mathbb{Q}} \circ Q_{\mathrm{qi}}$ of functors of $K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (up to canonical isomorphism), with $\underrightarrow{l}^*_{\mathbb{Q}} \colon K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ for the left side and $\underrightarrow{l}^*_{\mathbb{Q}} \colon D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ for the right one.

Lemma 8.4.4.2. *Suppose I is filtered. We have the commutative up to canonical isomorphism diagram:*

$$\begin{array}{ccc}
\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) & \xrightarrow[\mathfrak{e}]{8.1.5.14.1} & D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \\
\uparrow & \searrow^{\underrightarrow{l}^*_{\mathbb{Q}}}_{8.4.1.9} & \downarrow{\scriptstyle 8.4.4.1.1}\ \underrightarrow{l}^*_{\mathbb{Q}} \\
D^{\sharp}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\underrightarrow{l}^*_{\mathbb{Q}}} & D^{\sharp}(\mathcal{D}^{\dagger}_{\mathbb{Q}}).
\end{array} \tag{8.4.4.2.1}$$

Proof. The left triangle is commutative by definition of the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ of the diagonal (see the proof of 8.4.1.9). By using the universal property of localisation, to check the commutativity of the right triangle it is sufficient to check the commutativity of the outline square, which is easy. □

8.4.4.3. We have the canonical commutative diagram



(8.4.4.3.1)

whose top face is 8.1.5.15.1, whose back face is 8.4.4.2.1, whose middle vertical square is the right square of 8.4.1.10.1, whose commutativity of the right and left squares comes from the exactness of $\underrightarrow{l}^*_{\mathbb{Q}}$.

8.4.4.4. Suppose I is filtered. For any $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$ two objects of $K^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ induces the following morphism of $K(\mathbb{Z}_{\mathfrak{X}})$:

$$\mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)}) \to \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}), \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})). \tag{8.4.4.4.1}$$

Recall that $\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-)$ is the right localisation of $\mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-)$ (beware its existence is not obvious, see 8.3.4.6.3). By universal property of the right localisation of a (bi)functor, this yields there exists a canonical morphism

$$\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(-), \underrightarrow{l}^*_{\mathbb{Q}}(-)) \tag{8.4.4.4.2}$$

of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X)$ making commutative (by definition) the diagram of bifunctors $K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \times K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \to D(\mathbb{Z}_{\mathfrak{X}})$:

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(Q_{\mathrm{qi}}(-), Q_{\mathrm{qi}}(-)) & \xrightarrow{8.4.4.4.2} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}} \circ Q_{\mathrm{qi}}(-), \underrightarrow{l}^*_{\mathbb{Q}} \circ Q_{\mathrm{qi}}(-)) \\
 & & \| \\
\uparrow{\scriptstyle adj} & & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(Q_{\mathrm{qi}} \circ \underrightarrow{l}^*_{\mathbb{Q}}(-), Q_{\mathrm{qi}} \circ \underrightarrow{l}^*_{\mathbb{Q}}(-)) \\
 & & \uparrow{\scriptstyle adj} \\
Q_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) & \xrightarrow[8.4.4.4.1]{} & Q_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(-), \underrightarrow{l}^*_{\mathbb{Q}}(-))
\end{array} \tag{8.4.4.4.3}$$

where the "adj" morphisms are given by adjunction from 7.4.1.9.1 and where the top arrow is the composition of 8.4.4.4.2 with Q_{qi}.

Similarly, by universal property of the right localisation of a (bi)functor, we check there exists a canonical morphism

$$\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \to \mathbb{R}\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(-), \underrightarrow{l}^*_{\mathbb{Q}}(-)) \tag{8.4.4.4.4}$$

of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z})$.

Lemma 8.4.4.5. *We have the commutative diagram of bifunctors* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to D(\mathbb{Z}_X)$

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) & \underset{\cong}{\xrightarrow{8.3.4.6.5}} & \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathfrak{e}(-), \mathfrak{e}(-)) \\
\downarrow{\scriptstyle 8.4.1.15.2} & & \downarrow{\scriptstyle 8.4.4.4.2} \\
\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(-), \underrightarrow{l}^*_{\mathbb{Q}}(-) & \xrightarrow{\cong} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(\mathfrak{e}(-)), \underrightarrow{l}^*_{\mathbb{Q}}(\mathfrak{e}(-)).
\end{array} \tag{8.4.4.5.1}$$

Proof. By using the universal property of right localisation functor, with notation 8.2.4.23, we reduce to check it after composition with the adjunction map

$$Q^{+}_{\mathrm{qi}} \circ F(-,-) \to \mathbb{R}^{\mathfrak{N}^{+}_{\mathrm{qi}}(\mathbb{Z}_X)}_{\mathfrak{N}^{\mathrm{qi}}_{LD} \times \mathfrak{N}^{\mathrm{qi}}_{LD}} F(Q^{\mathrm{qi}}_{LD}(-), (Q^{\mathrm{qi}}_{LD}(-)) = \mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q^{\mathrm{qi}}_{LD}(-), (Q^{\mathrm{qi}}_{LD}(-)).$$

With notation 8.3.4.6, consider the commutative diagram of bifunctors $K^{\mathrm{b}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to D^{+}(\mathbb{Z}_X)$:

$$\begin{array}{ccc}
Q^{+}_{\mathrm{qi}} \circ F(-,-) & \xrightarrow{8.3.4.6.6} & Q^{+}_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(Q^{\mathrm{b}}_{N}(-), Q^{\mathrm{b}}_{N}(-)) \\
\downarrow{\scriptstyle 8.4.1.15.7} & & \downarrow{\scriptstyle 8.4.4.4.3} \\
Q_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(-), \underrightarrow{l}^*_{\mathbb{Q}}(-)) & \xrightarrow{\sim} & Q_{\mathrm{qi}} \circ \mathcal{H}om^{\bullet}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}} \circ Q^{\mathrm{b}}_{N}(-), \underrightarrow{l}^*_{\mathbb{Q}} \circ Q^{\mathrm{b}}_{N}(-))
\end{array} \tag{8.4.4.5.2}$$

where the left, top, right arrow are the canonical ones and appears respectively at the bottom diagram of 8.4.1.15.7, 8.3.4.6.6, 8.4.4.4.3, where the bottom isomorphism is given by the canonical isomorphism $\underrightarrow{l}^*_{\mathbb{Q}} \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}} \circ Q^{\mathrm{b}}_N$, where $\underrightarrow{l}^*_{\mathbb{Q}} \colon K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to K^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ for the left side and $\underrightarrow{l}^*_{\mathbb{Q}} \colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to K^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ for the right side. Therefore, by using the commutative diagrams 8.4.1.15.7, 8.3.4.6.6, 8.4.4.4.3, we are done. □

Proposition 8.4.4.6. *Let $u \colon I' \to I$ be an L-equivalence between two partially ordered sets (see definition 8.1.3.8). Then $\underrightarrow{u}_X^{-1}$ induces respectively the isomorphisms of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z})$ and $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X)$:*

$$\mathbb{R}\operatorname{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}))}(\underrightarrow{u}_X^{-1}(-), \underrightarrow{u}_X^{-1}(-)), \tag{8.4.4.6.1}$$

$$\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}))}(\underrightarrow{u}_X^{-1}(-), \underrightarrow{u}_X^{-1}(-)). \tag{8.4.4.6.2}$$

Proof. Let $\mathfrak{f} \colon D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \cong \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ be a quasi-inverse functor of $\mathfrak{e} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \cong D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ of 8.1.5.14.1. The isomorphism 8.4.4.6.2 is the one making commutative the diagram of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D(\mathbb{Z}_X)$:

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(-,-) & \dashrightarrow & \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}))}(\underrightarrow{u}_X^{-1}(-), \underrightarrow{u}_X^{-1}(-)) \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathfrak{e}\circ\mathfrak{f}(-), \mathfrak{e}\circ\mathfrak{f}(-)) & & \mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}))}(\mathfrak{e}\circ\underrightarrow{u}_X^{-1}\circ\mathfrak{f}(-), \mathfrak{e}\circ\underrightarrow{u}_X^{-1}\circ\mathfrak{f}(-)) \\
\uparrow{\scriptstyle\sim\ 8.3.4.6.5} & & \uparrow{\scriptstyle\sim\ 8.3.4.6.5} \\
\mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})}(\mathfrak{f}(-), \mathfrak{f}(-)) & \xrightarrow[\sim]{8.3.1.3.3} & \mathbb{R}\mathcal{H}om_{\underrightarrow{LD}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})}(\underrightarrow{u}_X^{-1}\circ\mathfrak{f}(-), \underrightarrow{u}_X^{-1}\circ\mathfrak{f}(-)),
\end{array} \tag{8.4.4.6.3}$$

where the top right (resp. top left) vertical isomorphism comes from the isomorphism $\mathfrak{e}\circ\underrightarrow{u}_X^{-1}\circ\mathfrak{f} \xrightarrow{\sim} \underrightarrow{u}_X^{-1}$ (resp. $\mathfrak{e}\circ\mathfrak{f} \xrightarrow{\sim} \mathrm{id}$). By using 8.3.1.3.2 instead of 8.3.1.3.3, we construct similarly the isomorphism 8.4.4.6.1.

□

8.4.4.7. Let $u \colon I' \to I$ be an L-equivalence between two partially ordered sets (see definition 8.1.3.8). Set $\mathcal{D}'^{\dagger} := \underrightarrow{l}^{-1}_{X,I'}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$. By composing 8.4.1.15.9 (still valid replacing λ^* by $\underrightarrow{u}_X^{-1}$) with 8.4.4.6.3, by using the commutative diagram 8.4.4.5.1, we get the commutative up to canonical isomorphism diagram of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))^{\mathrm{op}} \times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D^{+}(\mathbb{Z}_{\mathfrak{X}})$:

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(-,-) & \xrightarrow{8.4.4.4.2} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}(-), \underrightarrow{l}^*_{\mathbb{Q}}(-) \\
{\scriptstyle 8.4.4.6.2}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}))}(\underrightarrow{u}_X^{-1}(-), \underrightarrow{u}_X^{-1}(-)) & \xrightarrow{8.4.4.4.2} & \mathbb{R}\mathcal{H}om_{\mathcal{D}'^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^*_{\mathbb{Q}}\circ\lambda^*(-), \underrightarrow{l}^*_{\mathbb{Q}}\circ\lambda^*(-)).
\end{array} \tag{8.4.4.7.1}$$

8.4.5 Coherence in $D(LM_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, coherent modules, comparison

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. We suppose $p\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$. Let I be a partially ordered set which is strictly filtered and locally countably well ordered (see 8.4.2.10). Let $\mathcal{D}^{(\bullet)} = (\mathcal{D}^{(i)}, \alpha^{(j,i)}_{\mathcal{D}})$ be a sheaf of rings on the topos $\mathfrak{X}^{(I)}$. Suppose moreover the following conditions are satisfied

(i) For any $i \in I$, we suppose $\mathcal{D}^{(i)}$ is a sheaf of rings on $\mathfrak{X}$ equipped with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}^{(i)}$ satisfying the conditions of 7.2.3. Recall following 7.2.3.3, $\mathcal{D}^{(i)}$ is therefore left coherent.

(ii) The transition maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any elements $i \leq j$ of I;

(iii) There exists an integer d such that, for any elements $i \leq j$ of I, the ring $\mathcal{D}^{(j)}$ is of tor-dimension $\leq d$ on $\mathcal{D}^{(i)}$;

(iv) For any $i \in I$, for any coherent $\mathcal{D}_{\mathbb{Q}}^{(i)}$-module $\mathcal{E}$, there exists a coherent $\mathcal{D}^{(i)}$-module $\mathcal{E}'$ together with an isomorphism $\mathcal{E}'_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ of $\mathcal{D}_{\mathbb{Q}}^{(i)}$-modules.

Notation 8.4.5.1. For any $\sharp \in \{0, +, -, \mathrm{b}, \emptyset\}$, we denote by $D^{\sharp}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ the full subcategory of $D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that, for any $n \in \mathbb{Z}$, $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ (see notation 8.4.3.4). These objects are called coherent complexes of $D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$.

8.4.5.2. By definition, the property that an object of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is an object of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ is local on $\mathfrak{X}^{(\bullet)}$. This yields that the notion of coherence of 8.4.5.1 is local on $\mathfrak{X}^{(\bullet)}$, i.e. the fact that a complex $\mathcal{E}^{(\bullet)}$ of $D^{\sharp}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ is coherent is local on $\mathfrak{X}^{(\bullet)}$.

Theorem 8.4.5.3. *For any $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, $\mathcal{F}^{(\bullet)} \in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, the morphism of $D(\mathbb{Z}_{\mathfrak{X}})$ defined in 8.4.4.4.2 and the morphism of $D(\mathbb{Z})$ defined in 8.4.4.4.4:*

$$\mathbb{R}\mathcal{H}om_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)}) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}^{(\bullet)}, \underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{F}^{(\bullet)}), \tag{8.4.5.3.1}$$

$$\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)}) \to \mathbb{R}\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathbb{Q}}}(\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}^{(\bullet)}, \underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{F}^{(\bullet)}) \tag{8.4.5.3.2}$$

are isomorphisms.

Proof. i) Let us treat the case of 8.4.5.3.1. By using lemma [Har66, I.7.1.(ii)] (that we can use thanks to 8.4.3.7), we reduce to the case where $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. By using 8.4.1.5 and 8.4.4.7.1, since I is strictly filtered then we reduce to the case where I is countably well ordered. Let i_0 be the smallest element of I. Since this is local on $\mathfrak{X}$, we can suppose $\mathfrak{X}$ affine and $\mathcal{E}^{(\bullet)}$ has global finite presentation up to lim-ind-isogeny. Following the lemma 8.4.3.2, there exist $\lambda \in L(I)$ and a $\lambda^{*}\mathcal{D}^{(\bullet)}$-module $\mathcal{G}^{(\bullet)}$ having a global finite presentation such that $\mathcal{E}^{(\bullet)}$ and $\mathcal{G}^{(\bullet)}$ are isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Hence we reduce to check the theorem when $\mathcal{E}^{(\bullet)}$ has a global finite presentation as $\mathcal{D}^{(\bullet)}$-module. Following the second remark of 8.4.2.2, $E^{(i_0)} := \Gamma(\mathfrak{X}, \mathcal{E}^{(i_0)})$ is a coherent $D^{(i_0)} := \Gamma(\mathfrak{X}, \mathcal{D}^{(i_0)})$-module and the canonical morphism $\mathcal{D}^{(\bullet)} \otimes_{D^{(i_0)}} E^{(i_0)} \to \mathcal{E}^{(\bullet)}$ is an isomorphism. We get a left resolution of $E^{(i_0)}$ by free $D^{(i_0)}$-modules of finite type. It induces a left resolution of $\mathcal{E}^{(\bullet)}$ by free $\mathcal{D}^{(\bullet)}$-modules of finite type. Hence, by copying the proof of [Har66, I.7.1.(iv)] (i.e. by using stupid truncation exact sequences), we reduce to check the theorem when $\mathcal{E}^{(\bullet)}$ is a free $\mathcal{D}^{(\bullet)}$-module of finite type. By additivity, we can suppose $\mathcal{E}^{(\bullet)} = \mathcal{D}^{(\bullet)}$. Since $\mathcal{D}^{(\bullet)} \in D_{\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ (use the first remark of 8.4.1.2), then we conclude the proof by using Theorem 8.4.4.5 and the isomorphism 8.4.1.15.1.

ii) We check 8.4.5.3.2 similarly to 8.4.5.3.1 or we can also notice that by applying the functor $\mathbb{R}\Gamma(\mathfrak{X}, -)$ to the morphism 8.4.5.3.1, we get the arrow 8.4.5.3.2 by using the isomorphism 8.3.4.8.4. □

Lemma 8.4.5.4. *For any integer $n \in \mathbb{Z}$, the functor $\mathcal{H}^n \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ defined at 8.1.5.6.1 has the factorization $\mathcal{H}^n \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$.*

Proof. Let $n \in \mathbb{Z}$ and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. By using the right commutative diagram of 8.3.1.4.1, since I is strictly filtered then we reduce to the case where I is countably well ordered. Let i_0 be the smallest element of I. Since $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ is closed in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ under isomorphisms (see 8.4.3.7), then by definition (see 8.4.1.1) we can suppose there exists $\lambda \in L(I)$ such that $\mathcal{F}^{(\bullet)} \in D_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$. We set $\mathcal{G}^{(\bullet)} := \mathcal{H}^n(\mathcal{F}^{(\bullet)})$, $\mathcal{E}^{(i_0)} := \mathcal{H}^n(\mathcal{F}^{(i_0)})$ and $\mathcal{E}^{(\bullet)} := \lambda^{*}\mathcal{D}^{(\bullet)} \otimes_{\mathcal{D}^{(\lambda(i_0))}} \mathcal{E}^{(i_0)}$. Following 7.1.3.13 $\mathcal{F}^{(i_0)} \in D_{\mathrm{coh}}(\mathcal{D}^{(\lambda(i_0))})$. Hence, with the lemma 8.4.2.1, the $\lambda^{*}\mathcal{D}^{(\bullet)}$-module $\mathcal{E}^{(\bullet)}$ is of finite presentation. We have the canonical morphism $\mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ of $M(\lambda^{*}\mathcal{D}^{(\bullet)})$. Since the homomorphisms $\mathcal{D}_{\mathbb{Q}}^{(\lambda(i_0))} \to \mathcal{D}_{\mathbb{Q}}^{(\lambda(i))}$ are flat, we check that the arrow $\mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ becomes an isomorphism after applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$. For any $i \in I$, $\mathcal{E}^{(i)}, \mathcal{G}^{(i)}$ are coherent $\mathcal{D}^{(\lambda(i))}$-modules. Since I is countably well ordered, by using lemma 8.4.2.8, this yields that $\mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ is an ind-isogeny. Hence we are done. □

Lemma 8.4.5.5. *The equivalences of the lemma 8.1.5.10 factors through the canonical quasi-inverse equivalences of categories of the form $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \cong \underrightarrow{LD}^{0}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and $H^0 \colon \underrightarrow{LD}^{0}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \cong \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$.*

Proof. By using lemma 8.1.5.10, it is sufficient to check that the factorization of both functors. By using lemma 8.4.5.4, this is already known for the functor H^0. It remains to check that if $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ then $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^0_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. By using 8.3.1.1.2 and 8.4.1.5, since I is strictly filtered then we reduce to the case where I is countably well ordered. Let i_0 be the smallest element of I. Since this is local on $\mathfrak{X}$, we can suppose $\mathfrak{X}$ affine and $\mathcal{E}^{(\bullet)}$ has global finite presentation up to lim-ind-isogeny. Following the lemma 8.4.3.2, there exist $\lambda \in L(I)$ and a $\lambda^*\mathcal{D}^{(\bullet)}$-module $\mathcal{G}^{(\bullet)}$ having a global finite presentation such that $\mathcal{E}^{(\bullet)}$ and $\mathcal{G}^{(\bullet)}$ are isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Hence we reduce to the case where $\mathcal{E}^{(\bullet)}$ has a global finite presentation as $\mathcal{D}^{(\bullet)}$-module. Following 8.4.2.1, $\mathcal{E}^{(i_0)}$ is a coherent $\mathcal{D}^{(i_0)}$-module. We set $\mathcal{F}^{(\bullet)} := \mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(i_0)}} \mathcal{E}^{(i_0)}$. By using the hypothesis 8.4.5.iii, we get that the complex $\mathcal{F}^{(\bullet)}$ has bounded cohomology and therefore $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Let $n \in \mathbb{Z} \setminus \{0\}$. Moreover, since the extensions $\mathcal{D}^{(i_0)}_{\mathbb{Q}} \to \mathcal{D}^{(i)}_{\mathbb{Q}}$ are flat, we get $(\mathcal{H}^n(\mathcal{F}^{(\bullet)}))_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{H}^n(\mathcal{F}^{(\bullet)}_{\mathbb{Q}}) \xrightarrow{\sim} 0$. Since $\mathcal{H}^n(\mathcal{F}^{(i)})$ is a coherent $\mathcal{D}^{(i)}$-module for any $i \in I$, it follows from 8.4.2.8 that $\mathcal{H}^n(\mathcal{F}^{(\bullet)})$ is ind-isogenic to 0. Hence, we have checked $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^0_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. With the lemma 8.1.5.10, this implies the isomorphism $\mathcal{F}^{(\bullet)} \xrightarrow{\sim} H^0(\mathcal{F}^{(\bullet)})$ in $\underrightarrow{LD}^0_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Since $H^0(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$, this yields $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^0_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. □

Theorem 8.4.5.6. *We have the commutative diagram whose functors are equivalences of categories:*

$$\begin{array}{ccc} \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) & \xrightarrow[\mathfrak{e}]{8.1.5.14.1} & D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \\ & \searrow^{8.4.1.15}_{\underrightarrow{l}^*_{\mathbb{Q}}} & \downarrow {\scriptstyle 8.4.4.1.1}\ \underrightarrow{l}^*_{\mathbb{Q}} \\ & & D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}}). \end{array} \tag{8.4.5.6.1}$$

Proof. It follows from lemmas 8.1.5.15.1 and 8.4.5.4 that the equivalence of categories $\mathfrak{e} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}) \cong D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ of 8.1.5.14.1 factors through the (fully faithful) functor

$$\mathfrak{e} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})). \tag{8.4.5.6.2}$$

It follows from 8.4.5.5 and 8.4.1.15.b (and from the commutativity of the left square of 8.4.1.10.1) that we have the factorization

$$\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathbb{Q}}). \tag{8.4.5.6.3}$$

Moreover, since $\underrightarrow{l}^*_{\mathbb{Q}}$ commutes with H^n (more precisely, by using the commutative right square of 8.4.4.3.1), then the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ of 8.4.4.1.1 factors through

$$\underrightarrow{l}^*_{\mathbb{Q}} \colon D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}}). \tag{8.4.5.6.4}$$

Hence, we get the commutative diagram 8.4.5.6.1 from that of the top triangle of 8.4.4.2.1. It follows from Theorem 8.4.5.3.2 that the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ of 8.4.5.6.4 is fully faithful. Since the composition of the functors 8.4.5.6.2 and 8.4.5.6.4 is an equivalence of categories (see Theorem 8.4.1.15) then so is the functor 8.4.5.6.4. Hence, the functor 8.4.5.6.2 is also an equivalence.

□

8.4.5.7. Using the commutativity of the diagram 8.1.5.15.1 and the stability of the coherence of Lemma 8.4.5.4 and Theorem 8.4.5.6, we get the commutative (up to canonical isomorphism) diagram:

$$\begin{array}{ccccc} D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})) & \longrightarrow & D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) & \overset{\mathfrak{e}}{\underset{\cong}{\longleftarrow}} & \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \\ \downarrow {\scriptstyle H^n} & & \downarrow {\scriptstyle H^n} & & \downarrow {\scriptstyle H^n} \\ \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) & = & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) & = & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}). \end{array} \tag{8.4.5.7.1}$$

8.4.5.8. We have the following properties

(a) Using 8.4.3.7, we get that $D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ is a saturated triangulated subcategory (some authors say thick or épaisse) of $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, i.e. is a strict triangulated subcategory closed under direct summands. Hence, using the equivalences of categories 8.1.5.14.1 and 8.4.5.6.2, we get that $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ is a saturated triangulated subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

(b) Using again using the equivalences of categories 8.1.5.14.1 and 8.4.5.6.2, it follows from 8.4.5.2 the following local property: the fact that a complex of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ belongs to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ (resp. $\underrightarrow{LD}^{0}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$) is local on $\mathfrak{X}$.

Lemma 8.4.5.9. *Denote by $D^0_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathbb{Q}})$ the full subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathbb{Q}})$ of the complexes $\mathcal{E}$ such that, for any integer $n \neq 0$, we have $\mathcal{H}^n(\mathcal{E}) = 0$.*

(a) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. The property $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{0}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ is equivalent to $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in D^0_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathbb{Q}})$.

(b) We have the diagram of functors commutative up to canonical equivalence

$$\begin{array}{ccccc} \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) & \xrightarrow{\cong} & \underrightarrow{LD}^{0}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) & \xrightarrow[H^0]{\cong} & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \\ \cong\downarrow \underrightarrow{l}^*_{\mathbb{Q}} & & \cong\downarrow \underrightarrow{l}^*_{\mathbb{Q}} & & \cong\downarrow \underrightarrow{l}^*_{\mathbb{Q}} \\ \mathrm{Coh}(\mathcal{D}^\dagger_{\mathbb{Q}}) & \xrightarrow{\cong} & D^0_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathbb{Q}}) & \xrightarrow[H^0]{\cong} & \mathrm{Coh}(\mathcal{D}^\dagger_{\mathbb{Q}}), \end{array} \tag{8.4.5.9.1}$$

whose every functors are equivalences of categories.

Proof. It follows from the commutativity of the diagram 8.4.1.10.1 and from the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}}$ of Theorem 8.4.5.6, that we have the fully faithful functor $\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LD}^{0}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to D^0_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathbb{Q}})$. We have the functor 8.4.5.6.3 making commutative the diagram 8.4.5.9.1. It remains to check that the functors of the diagram 8.4.5.9.1 are equivalences of categories. Following Lemma 8.4.5.5, this is known for the top horizontal morphisms. For the bottom horizontal ones, this is obvious. By using the commutativity of the left square of the diagram 8.4.5.9.1 this yields that the left vertical functor of 8.4.5.9.1 is fully faithful. Hence, by using the commutativity of the diagram 8.4.1.10.1, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and for any $n \in \mathbb{Z}$, this yields $H^n(\mathcal{E}^{(\bullet)}) = 0$ if and only if $H^n(\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})) = 0$. This implies the property (a) and the fact that the middle vertical functor of 8.4.5.9.1 is an equivalence of categories. Hence, we are done. □

With the hypotheses of the subsection 8.4.5, the following proposition improves 8.4.3.2:

Lemma 8.4.5.10. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Suppose $\mathfrak{X}$ quasi-compact. Hence there exists $i \in I$, such that for any affine open U of X, $\mathcal{E}^{(\bullet)}|_{(i,U)}$ is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,U)})$ to a $\mathcal{D}^{(\bullet)}|_{(i,U)}$-module having a global finite presentation.*

Proof. We can suppose X affine. Let $\mathcal{G} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in \mathrm{Coh}(\mathcal{D}^\dagger_{\mathbb{Q}})$. Following 8.4.1.11.b.(ii), since our rings are coherent (after tensorisation with $\mathbb{Q}$), there exists $i \in I$ and a coherent left $\mathcal{D}^{(i)}_{\mathbb{Q}}$-module $\mathcal{G}^{(i)}$ together with a $\mathcal{D}^\dagger_{\mathbb{Q}}$-linear isomorphism $\epsilon \colon \mathcal{D}^\dagger_{\mathbb{Q}} \otimes_{\mathcal{D}^{(i)}_{\mathbb{Q}}} \mathcal{G}^{(i)} \xrightarrow{\sim} \mathcal{G}$. Using the hypothesis (8.4.5.iv), there exists a coherent $\mathcal{D}^{(i)}$-module $\mathcal{F}^{(i)}$ together with an isomorphism $\mathcal{F}^{(i)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}^{(i)}$ of $\mathcal{D}^{(i)}_{\mathbb{Q}}$-modules. Since X is affine, then $\mathcal{F}^{(i)}$ is a $\mathcal{D}^{(i)}$-module having a global finite presentation. This yields that $\mathcal{F}^{(\bullet)} := (\mathcal{D}^{(\bullet)}|_{(i,X)}) \otimes_{\mathcal{D}^{(i)}} \mathcal{F}^{(i)}$ is a $\mathcal{D}^{(\bullet)}|_{(i,X)}$-module having a global finite presentation. Let $j \colon I_{\geq i} \subset I$ be the canonical inclusion. Since j is an L-equivalence, with 8.4.1.9.b, we get $\underrightarrow{l}^*_{X,I_{\geq i},\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,X)}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathbb{Q}}$ and then the isomorphisms $\underrightarrow{l}^*_{X,I_{\geq i},\mathbb{Q}}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \underrightarrow{l}^*_{X,I_{\geq i},\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,X)}) \otimes_{\mathcal{D}^{(i)}_{\mathbb{Q}}} \mathcal{F}^{(i)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}$. By using the commutative diagram 8.4.1.9.1 (recall the restriction functor $|_{(i,X)}$ is equal by definition to $\underrightarrow{j}^{-1}_X$), we get the isomorphism $\mathcal{G} = \underrightarrow{l}^*_{X,I,\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \underrightarrow{l}^*_{X,I_{\geq i},\mathbb{Q}}(\mathcal{E}^{(\bullet)}|_{(i,X)})$. By using the full faithfulness of the functor $\underrightarrow{l}^*_{X,I_{\geq i},\mathbb{Q}}$ (see 8.4.5.9.1), this yields that $\mathcal{E}^{(\bullet)}|_{(i,X)}$ and $\mathcal{F}^{(\bullet)}$ are isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}|_{(i,X)})$. Hence, we are done. □

Corollary 8.4.5.11. *We suppose $\mathfrak{X}$ affine. We have the commutative canonical diagram*

$$\begin{array}{ccccc} D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})) & \longrightarrow & D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})) & \xleftarrow{\mathfrak{c}} & \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \\ \downarrow \underrightarrow{l}^*_{X,\mathbb{Q}} & & \searrow \underrightarrow{l}^*_{X,\mathbb{Q}} & 8.4.1.15 & \downarrow \underrightarrow{l}^*_{X,\mathbb{Q}} \\ D^{\mathrm{b}}(\mathrm{Coh}(\mathcal{D}^\dagger_{\mathbb{Q}})) & & \longrightarrow & & D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathbb{Q}}), \end{array} \tag{8.4.5.11.1}$$

whose functors are equivalences of categories.

Proof. By universal property of the localisation, to check that the left trapeze is commutative, we reduce to check it after composing with the functor $K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})) \to D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$, which is easy. Hence, we get the commutativity of the diagram 8.4.5.11.1 from that of 8.4.5.6.1. Moreover, if follows from 4.6.1.9 (resp. 8.4.5.6) that the bottom functor (resp. the functors of the right triangle) of the diagram 8.4.5.11.1 is an equivalence of categories. Since the exact functor $\underrightarrow{l}^*_{X,\mathbb{Q}}$ induces an equivalence of categories between $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and $\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (see 8.4.5.9.1), then so is the left functor. Hence, we are done. □

8.4.5.12. We have the following fact.

(a) Let $\mathcal{E}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$, Let $\mathcal{E}^{\bullet} := \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)\bullet}$, where $\underrightarrow{l}^*_{\mathbb{Q}}$ is induced by the equivalence of categories of 8.4.5.6.3. Choose m_0 large enough so that there exists p-torsion free coherent $\mathcal{D}^{(m_0)}$-modules $\mathcal{F}^{(0)n}$ such that $\mathcal{D}^{\dagger}_{\mathbb{Q}} \otimes_{\mathcal{D}^{(m_0)}} \mathcal{F}^{(0)n} \xrightarrow{\sim} \mathcal{E}^n$. For any $m \geq 0$, let $\mathcal{F}^{(m)}$ and be the quotient of $\mathcal{D}^{(m+m_0)} \otimes_{\mathcal{D}^{(m_0)}} \mathcal{F}^{(0)}$ by its p-torsion part. We get the complex $\mathcal{F}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$ such that $\underrightarrow{l}^*_{\mathbb{Q}}\mathcal{F}^{(\bullet)\bullet} \xrightarrow{\sim} \mathcal{E}^{\bullet}$. Hence, we obtain the isomorphism $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ of $K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$.

(b) Let $\mathcal{F}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$. It follows from 8.1.5.2.2 that there exists $\mathcal{G}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ together with an isomorphism of $K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$ of the form $\mathfrak{e}(\mathcal{G}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathcal{F}^{(\bullet)\bullet}$ such that $\mathcal{G}^{(\bullet)n}$ for any integer n is of the form $\lambda^*\chi^*\mathcal{F}^{(\bullet)n}$.

Corollary 8.4.5.13. *Suppose $\mathfrak{X}$ affine. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. There exist a representant $\mathcal{E}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ of $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)n}$ is p-torsion free and $\mathfrak{e}(\mathcal{E}^{(\bullet)n}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ for any $n \in \mathbb{Z}$.*

Proof. Let $\mathcal{E}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ be a representant of $\mathcal{E}^{(\bullet)}$. Following 8.4.5.11, there exists $\mathcal{F}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$ endowed with an isomorphism $\mathcal{F}^{(\bullet)\bullet} \xrightarrow{\sim} \mathfrak{e}(\mathcal{E}^{(\bullet)\bullet})$ of $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$, where $\mathfrak{e}\colon K^{\mathrm{b}}(\mathcal{D}^{(\bullet)}) \to K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}))$. It follows from 8.4.5.12 that there exists $\mathcal{G}^{(\bullet)\bullet} \in K^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ endowed with an isomorphism $\mathfrak{e}(\mathcal{G}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathcal{F}^{(\bullet)\bullet}$ of $K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}))$ and such that $\mathcal{G}^{(\bullet)n}$ is p-torsion free (and $\mathfrak{e}(\mathcal{G}^{(\bullet)n}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$) for any $n \in \mathbb{Z}$. Since $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ is a triangulated category, then we get by induction that $\mathcal{G}^{(\bullet)\bullet}$ is an object of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Hence, since $\mathfrak{e}(\mathcal{G}^{(\bullet)\bullet})$ and $\mathfrak{e}(\mathcal{E}^{(\bullet)\bullet})$ are isomorphic in $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$, then it follows from 8.4.5.11 that $\mathcal{G}^{(\bullet)\bullet}$ and $\mathcal{E}^{(\bullet)\bullet}$ are isomorphic in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. □

8.5 Quasi-coherence

8.5.1 Quasi-coherence in $D(\mathcal{D}^{(\bullet)})$ or $D(\mathcal{D}^{(\bullet)}_{\bullet})$

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Let I be a partially ordered set. Let $\mathcal{D}^{(\bullet)} = (\mathcal{D}^{(i)}, \alpha^{j,i}_{\mathcal{D}})$ be a sheaf of rings on the topos $\mathfrak{X}^{(I)}$. We suppose there exists a homomorphism $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}^{(\bullet)}$ such that $\mathcal{D}^{(i)}$ equipped with the induced $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}^{(i)}$ is a sheaf of rings on $\mathfrak{X}$ satisfying the conditions of 7.3.2.

Example 8.5.1.1. We will use essentially in this book the following cases. Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^{\sharp} \to \mathfrak{S}^{\sharp}$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension.

(a) Let $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}$ be a commutative $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$-algebra endowed with a compatible structure of left $\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$-module such that for any $m \in \mathbb{N}$, $\mathcal{B}^{(m)}_{\mathfrak{X}}$ satisfies the hypotheses of 7.3.2. Since $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ is $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}$-flat (for both structures), then $\mathcal{D}^{(\bullet)} = \mathcal{B}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ satisfies the conditions of 8.5.1 (in the case where $I = \mathbb{N}$).

(b) Suppose $\mathfrak{X}$ is moreover p-torsion free (see 3.3.1.12 for some example). Let $\lambda \in L(\mathbb{N})$. Let Z be a divisor of $\mathfrak{X}^{\sharp} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi\mathcal{V})$. For any $m \in \mathbb{N}$, we set $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z) := \mathcal{B}^{(\lambda(m))}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z) := \mathcal{B}^{(\lambda(m))}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$. Then following 8.7.4.2 the sheaves $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)$ satisfies the conditions of 8.5.1.

Notation 8.5.1.2. Following 8.1.1.1, with notation 7.1.2.1, the topos of inductive system indexed by I of sheaves on X is by definition

$$\mathfrak{X}^{(I)} := \mathrm{Top}(X)_{I^{\circ}} := \mathrm{Sh}(I^{\circ \natural} \times X_{Zar}),$$

where I° the partially ordered set equal to I as a set but equipped with the ordering opposite to that of I (see 7.1.1.1).

We endow the set $\mathbb{N} \times I^{\circ}$ with the order product i.e. $(n_1, i_1) \leq (n_2, i_2)$ in $\mathbb{N} \times I^{\circ}$ if and only if $n_1 \leq n_2$ in $\mathbb{N}$ and $i_1 \geq i_2$ in I. We set

$$X_{\bullet}^{(I)} := \mathrm{Top}(X)_{\mathbb{N} \times I^{\circ}} := \mathrm{Sh}((\mathbb{N} \times I^{\circ})^{\natural} \times X_{Zar}).$$

The object of $X_{\bullet}^{(I)}$ are denoted by $\mathcal{E}_{\bullet}^{(\bullet)} = (\mathcal{E}_n^{(i)}, \alpha_{m,n}^{j,i})_{i \leq j, m \leq n}$ where $\alpha_{m,n}^{j,i} \colon \mathcal{E}_n^{(i)} \to \mathcal{E}_m^{(j)}$ is a functorial in $i \leq j$ in I and $m \leq n$ in $\mathbb{N}$ morphism of sheaves on X. Remark this yields the morphism of $\mathrm{Top}(X)_{\mathbb{N}}$ (resp. $\mathrm{Top}(X)_{I^{\circ}}$) of the form $\alpha^{j,i} \colon \mathcal{E}_{\bullet}^{(i)} \to \mathcal{E}_{\bullet}^{(j)}$ (resp. $\alpha_{m,n} \colon \mathcal{E}_n^{(\bullet)} \to \mathcal{E}_m^{(\bullet)}$). We denote by $\mathcal{D}_{\bullet}^{(\bullet)} = (\mathcal{D}^{(i)}/\mathcal{I}^{n+1}\mathcal{D}^{(i)}, \alpha_{m,n}^{j,i})$ the sheaf of rings on the topos $X_{\bullet}^{(I)}$ so that $\alpha_{m,n}^{j,i} \colon \mathcal{D}^{(i)}/\mathcal{I}^{n+1}\mathcal{D}^{(i)} \to \mathcal{D}^{(j)}/\mathcal{I}^{m+1}\mathcal{D}^{(j)}$ are the morphisms induced by $\alpha_{\mathcal{D}}^{j,i}$ for any $i \leq j$ in I and $m \leq n$ in $\mathbb{N}$.

1) Fix $i \in I$. Let $i \colon \{i\} \to I$ be the inclusion map sending i to i and $i \colon \mathbb{N} \times \{i\} \to \mathbb{N} \times I^{\circ}$ be the inclusion map sending (n, i) to (n, i). The morphisms defined at 7.1.2.4.1 are in this case denoted by

$$\underrightarrow{i}_{\mathfrak{X}} = (\underrightarrow{i}_{\mathfrak{X}}^{-1} \dashv \underrightarrow{i}_{\mathfrak{X}*}) \colon \mathfrak{X} \to \mathfrak{X}^{(I)}, \qquad \underrightarrow{i}_{X_{\bullet}} = (\underrightarrow{i}_{X_{\bullet}}^{-1} \dashv \underrightarrow{i}_{X_{\bullet}*}) \colon X_{\bullet} \to X_{\bullet}^{(I)}, \tag{8.5.1.2.1}$$

where $\mathfrak{X}$ means by abuse of notation $\mathrm{Top}(\mathfrak{X})$. We have $\underrightarrow{i}_{X_{\bullet}}^{-1}(\mathcal{E}_{\bullet}^{(\bullet)}) = \mathcal{E}_{\bullet}^{(i)}$ for any $\mathcal{E}_{\bullet}^{(\bullet)} \in X_{\bullet}^{(I)}$ and $\underrightarrow{i}_{\mathfrak{X}}^{-1}(\mathcal{E}^{(\bullet)}) = \mathcal{E}^{(i)}$ for any $\mathcal{E}^{(\bullet)} \in \mathfrak{X}$ (with identities as transition maps). Similarly to 8.1.1.2.6, for any $\mathcal{E}_{\bullet} \in X_{\bullet}$, we compute

$$(\underrightarrow{i}_{X_{\bullet}*}(\mathcal{E}_{\bullet}))^{(j)} = \begin{cases} \mathcal{E}_{\bullet} & \text{if } j \leq i \\ e & \text{otherwise} \end{cases}, \qquad (\underrightarrow{i}_{X_{\bullet}!}(\mathcal{E}_{\bullet}))^{(j)} = \begin{cases} \mathcal{E}_{\bullet} & \text{if } j \geq i \\ \emptyset & \text{otherwise} \end{cases} \tag{8.5.1.2.2}$$

where e (resp. $\emptyset$) is the final (resp. initial) object of $X_{\bullet}$.

2) Let $\{*\}$ be some one element set. Let $u \colon \mathbb{N} \times I^{\circ} \to \{*\} \times I^{\circ}$ be the canonical projection. Then the morphism of 7.1.2.4.1 induces in this case the following ringed topos morphism

$$\underleftarrow{l}_{\mathfrak{X}^{(I)}} = (\underleftarrow{l}_{\mathfrak{X}^{(I)}}^{-1} \dashv \underleftarrow{l}_{\mathfrak{X}^{(I)},*}) \colon (X_{\bullet}^{(I)}, \mathcal{D}_{\bullet}^{(\bullet)}) \to (\mathfrak{X}^{(I)}, \mathcal{D}^{(\bullet)}). \tag{8.5.1.2.3}$$

We have $\underleftarrow{l}_{\mathfrak{X}^{(I)}}^{-1}(\mathcal{E}^{(\bullet)})_n = \mathcal{E}^{(\bullet)}$ for any $\mathcal{E}^{(\bullet)} \in \mathfrak{X}^{(I)}$ and any $n \in \mathbb{N}$; transition morphisms $\underleftarrow{l}_{\mathfrak{X}^{(I)}}^{-1}(\mathcal{E}^{(\bullet)})_{n+1} \to \underleftarrow{l}_{\mathfrak{X}^{(I)}}^{-1}(\mathcal{E}^{(\bullet)})_n$ are the identities. Moreover, for any $\mathcal{E}_{\bullet}^{(\bullet)} = (\mathcal{E}_n^{(i)}, \alpha_{m,n}^{j,i})_{i \leq j, m \leq n} \in X_{\bullet}^{(I)}$ and any $i \in I$, we have

$$\underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{E}_{\bullet}^{(\bullet)}) = (\varprojlim_{n \in \mathbb{N}} \mathcal{E}_n^{(i)}, \beta^{j,i}), \tag{8.5.1.2.4}$$

where the projective limit $\varprojlim_{n \in \mathbb{N}} \mathcal{E}_n^{(i)}$ is precisely that of the functor $\mathrm{Cat}(\mathbb{N})^{\mathrm{op}} \to \mathrm{Sh}(X_{Zar})$ induced by the object $\mathcal{E}_{\bullet}^{(i)}$, where $\beta^{j,i} \colon \varprojlim_{n \in \mathbb{N}} \mathcal{E}_n^{(i)} \to \varprojlim_{m \in \mathbb{N}} \mathcal{E}_m^{(j)}$ for $i \leq j$ in I is the morphism given by functoriality from the family of morphisms $\alpha_{m,n}^{j,i}$. Hence, we have

$$\underrightarrow{i}_{\mathfrak{X}}^{-1} \underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{E}_{\bullet}^{(\bullet)}) = \varprojlim_{n \in \mathbb{N}} \mathcal{E}_n^{(i)} = \underleftarrow{l}_{\mathfrak{X}*} \underrightarrow{i}_{X_{\bullet}}^{-1}(\mathcal{E}_{\bullet}^{(\bullet)}). \tag{8.5.1.2.5}$$

8.5.1.3. Let $\mathcal{E}_{\bullet}^{(\bullet)\bullet} \in K^{+}(\mathcal{D}_{\bullet}^{(\bullet)})$. Following 7.1.3.14, there exists $\mathcal{F}_{\bullet}^{(\bullet)\bullet} \in K^{+}(\mathcal{D}_{\bullet}^{(\bullet)})$ such that $\mathcal{F}_{\bullet}^{(\bullet)n}$ is a flasque $\mathcal{D}_{\bullet}^{(\bullet)}$-module for any $n \in \mathbb{Z}$ and a quasi-isomorphism $\mathcal{E}_{\bullet}^{(\bullet)\bullet} \to \mathcal{F}_{\bullet}^{(\bullet)\bullet}$ of $K^{+}(\mathcal{D}_{\bullet}^{(\bullet)})$.

Recall, following 7.1.2.19.a, a $\mathcal{D}_{\bullet}^{(\bullet)}$-module $\mathcal{G}_{\bullet}^{(\bullet)}$ is flasque if and only if $\mathcal{G}_n^{(i)}$ is a flasque sheaf on X for any $i \in I$ and $n \in \mathbb{N}$ and the morphism $\mathcal{G}_n^{(i)} \to \mathcal{G}_m^{(j)}$ is surjective in the category of presheaves on X for any $i \leq j, m \leq n$. For any $i \in I$, this yields that $\mathcal{F}_{\bullet}^{(i)\bullet} = \underrightarrow{i}_{X_{\bullet}}^{-1}(\mathcal{F}_{\bullet}^{(\bullet)\bullet})$ is a complex of flasque $\mathcal{D}_{\bullet}^{(i)}$-modules which is endowed with a quasi-isomorphism $\mathcal{E}_\bullet^{(i)\bullet} \to \mathcal{F}_\bullet^{(i)\bullet}$ of $K^+(\mathcal{D}_\bullet^{(i)})$. Hence we get the isomorphisms of $D^+(\mathcal{D}^{(i)})$:

$$\begin{aligned} \underset{\rightarrow}{i}_{\mathfrak{X}}^{-1} \circ \mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}(I)*}(\mathcal{E}_\bullet^{(\bullet)\bullet}) &\xrightarrow{\sim} \underset{\rightarrow}{i}_{\mathfrak{X}}^{-1} \circ \underset{\leftarrow}{l}_{\mathfrak{X}(I)*}(\mathcal{F}_\bullet^{(\bullet)\bullet}) \overset{8.5.1.2.5}{\xrightarrow{\sim}} \underset{\leftarrow}{l}_{\mathfrak{X}*}\underset{\rightarrow}{i}_{X_\bullet}^{-1}(\mathcal{F}_\bullet^{(\bullet)\bullet}) \\ &= \underset{\leftarrow}{l}_{\mathfrak{X}*}(\mathcal{F}_\bullet^{(i)\bullet}) \xleftarrow{\sim} \mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}*}(\mathcal{E}_\bullet^{(i)\bullet}) = \mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}*}\underset{\rightarrow}{i}_{X_\bullet}^{-1}(\mathcal{E}_\bullet^{(\bullet)\bullet}). \end{aligned} \tag{8.5.1.3.1}$$

It follows from 8.5.1.3.1 that a $\mathcal{D}_\bullet^{(\bullet)}$-module $\mathcal{G}_\bullet^{(\bullet)}$ is $\underset{\leftarrow}{l}_{\mathfrak{X}(I),*}$-acyclic if and only if the $\mathcal{D}_\bullet^{(i)}$-module $\mathcal{G}_\bullet^{(i)}$ is $\underset{\leftarrow}{l}_{\mathfrak{X},*}$-acyclic for any $i \in I$.

8.5.1.4 (Bounded cohomological dimension)**.** Following 7.3.1.2, the functor $\underset{\leftarrow}{l}_{\mathfrak{X}*}$ has cohomological dimension bounded by $d+1$. By using 8.5.1.3.1 this yields that the functor $\underset{\leftarrow}{l}_{\mathfrak{X}(I)*}\colon \operatorname{Mod}(\mathcal{D}_\bullet^{(\bullet)}) \to \operatorname{Mod}(\mathcal{D}^{(\bullet)})$ has also cohomological dimension bounded by $d+1$ and then the functor $\underset{\leftarrow}{l}_{\mathfrak{X}(I)*}$ is way-out in both directions.

Following 4.6.1.6.b, for any $\mathcal{E}_\bullet^{(\bullet)\bullet} \in K(\mathcal{D}_\bullet^{(\bullet)})$ (resp. for any $\mathcal{E}_\bullet^{(\bullet)} \in K^-(\mathcal{D}_\bullet^{(\bullet)})$), there exist a complex $\mathcal{I}_\bullet^{(\bullet)\bullet} \in K(\mathcal{D}_\bullet^{(\bullet)})$ (resp. $\mathcal{I}_\bullet^{(\bullet)\bullet} \in K^-(\mathcal{D}_\bullet^{(\bullet)})$) of $\underset{\leftarrow}{l}_{\mathfrak{X}(I),*}$-acyclic left $\mathcal{D}_\bullet^{(\bullet)}$-modules and a quasi-isomorphism $\mathcal{E}_\bullet^{(\bullet)\bullet} \xrightarrow{\sim} \mathcal{I}_\bullet^{(\bullet)\bullet}$. Moreover, we have the isomorphism $\mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}(I),*}\mathcal{E}_\bullet^{(\bullet)\bullet} \xrightarrow{\sim} \underset{\leftarrow}{l}_{\mathfrak{X}(I),*}\mathcal{I}_\bullet^{(\bullet)\bullet}$.

It follows from 8.5.1.3.1 that the functor $\underset{\rightarrow}{i}_{X_\bullet}^{-1}$ sends an $\underset{\leftarrow}{l}_{\mathfrak{X},*}$-acyclic module from a $\underset{\leftarrow}{l}_{\mathfrak{X}(I),*}$-acyclic module. Hence, similarly to 8.5.1.3.1, for any $\mathcal{E}_\bullet^{(\bullet)\bullet} \in D(\mathcal{D}_\bullet^{(\bullet)})$ we get the isomorphism

$$\underset{\rightarrow}{i}_{\mathfrak{X}}^{-1} \circ \mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}(I)*}(\mathcal{E}_\bullet^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}*} \circ \underset{\rightarrow}{i}_{X_\bullet}^{-1}(\mathcal{E}_\bullet^{(\bullet)\bullet}). \tag{8.5.1.4.1}$$

8.5.1.5. Let $\mathcal{E}^{(\bullet)\bullet} \in K^-(\mathcal{D}^{(\bullet)})$. There exists $\mathcal{F}^{(\bullet)\bullet} \in K^-(\mathcal{D}^{(\bullet)})$ such that $\mathcal{F}^{(\bullet)n}$ is a flat $\mathcal{D}^{(\bullet)}$-module for any $n \in \mathbb{Z}$ together with a quasi-isomorphism $\mathcal{E}^{(\bullet)\bullet} \to \mathcal{F}^{(\bullet)\bullet}$ of $K^-(\mathcal{D}^{(\bullet)})$. Let $i \in I$. It follows from 7.1.3.6 that $\mathcal{F}^{(i)n}$ is a flat $\mathcal{D}^{(i)}$-module for any $n \in \mathbb{Z}$ and that the induced morphism $\mathcal{E}^{(i)\bullet} \to \mathcal{F}^{(i)\bullet}$ is a quasi-isomorphism of $K^-(\mathcal{D}^{(i)})$. This yields the canonical isomorphisms

$$\begin{aligned} \underset{\rightarrow}{i}_{X_\bullet}^{-1} \circ \mathbb{L}\underset{\leftarrow}{l}_{\mathfrak{X}(I)}^{*}(\mathcal{E}^{(\bullet)\bullet}) &= \underset{\rightarrow}{i}_{X_\bullet}^{-1}(\mathcal{D}_\bullet^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)\bullet}) \xleftarrow{\sim} \underset{\rightarrow}{i}_{X_\bullet}^{-1}(\mathcal{D}_\bullet^{(\bullet)} \otimes_{\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)\bullet}) \\ &= \mathcal{D}_\bullet^{(i)} \otimes_{\mathcal{D}^{(i)}} \mathcal{F}^{(i)\bullet} \xrightarrow{\sim} \mathcal{D}_\bullet^{(i)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(i)}} \mathcal{E}^{(i)\bullet} = \mathbb{L}\underset{\leftarrow}{l}_{\mathfrak{X}}^{*}\underset{\rightarrow}{i}_{\mathfrak{X}}^{-1}(\mathcal{E}^{(\bullet)\bullet}), \end{aligned} \tag{8.5.1.5.1}$$

where $\underset{\leftarrow}{l}_{\mathfrak{X}(I)}$ is the ringed topos morphism 8.5.1.2.3.

Similarly to 7.3.1.5, we have the following definition:

Definition 8.5.1.6. Let $\mathcal{O}_{\mathfrak{X}}^{(\bullet)} := \underset{\rightarrow}{l}_{\mathfrak{X},I*}(\mathcal{O}_{\mathfrak{X}})$ be the constant inductive system with value $\mathcal{O}_{\mathfrak{X}}$ (see notation 8.1.1.2.4). Let $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{O}_{\mathfrak{X}}^{(\bullet)})$. We say $\mathcal{E}^\bullet$ is $\mathcal{O}_{\mathfrak{X}}^{(\bullet)}$*-quasi-coherent* if the following conditions holds:

(a) The complex $\mathcal{O}_{X_0} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{(i)}$ has $\mathcal{O}_{X_0}$-quasi-coherent cohomology for any $i \in I$.

(b) The canonical morphism of $D^-(\mathcal{O}_{\mathfrak{X}}^{(\bullet)})$

$$\mathcal{E}^{(\bullet)} \to \mathbb{R}\underset{\leftarrow}{l}_{\mathfrak{X}(I)*}(\mathbb{L}\underset{\leftarrow}{l}_{\mathfrak{X}(I)}^{*}\mathcal{E}^{(\bullet)}), \tag{8.5.1.6.1}$$

where $\underset{\leftarrow}{l}_{\mathfrak{X}(I)}$ is the ringed topos morphism 8.5.1.2.3 in the case $\mathcal{D}^{(\bullet)} = \mathcal{O}_{\mathfrak{X}}^{(\bullet)}$, is an isomorphism.

Similarly to 7.3.1.10, we have the following quasi-coherent notion:

Definition 8.5.1.7. Let $\mathcal{O}_{X_n}^{(\bullet)} := \underset{\rightarrow}{l}_{X_n,I*}(\mathcal{O}_{X_n})$ for any $n \in \mathbb{N}$ (see notation 8.1.1.2.4). In the case where $\mathcal{D}^{(\bullet)} = \mathcal{O}_{\mathfrak{X}}^{(\bullet)}$, the sheaf of rings $\mathcal{D}_\bullet^{(\bullet)}$ is written $\mathcal{O}_{X_\bullet}^{(\bullet)}$. Let $\mathcal{E}_\bullet^{(\bullet)} \in D^-(\mathcal{O}_{X_\bullet}^{(\bullet)})$. We say $\mathcal{E}_\bullet^{(\bullet)}$ is $\mathcal{O}_{X_\bullet}^{(\bullet)}$*-quasi-coherent* if the following conditions hold:

(a) The complex $\mathcal{E}_0^{(i)}$ is in $D^-_{\mathrm{qc}}(\mathcal{O}_{X_0})$ for any $i \in I$.

(b) The canonical morphisms of $D^-(\mathcal{O}_{X_n}^{(\bullet)})$

$$\mathcal{O}_{X_n}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_{n+1}}^{(\bullet)}} \mathcal{E}_{n+1}^{(\bullet)} \to \mathcal{E}_n^{(\bullet)} \tag{8.5.1.7.1}$$

are isomorphisms for all $n \in \mathbb{N}$.

8.5.1.8. Let $* \in \{\mathrm{l}, \mathrm{r}\}$. We denote by $D^-_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)})$ (resp. $D^-_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}_\bullet)$) the full subcategory of $D^-({}^*\mathcal{D}^{(\bullet)})$ (resp. $D^-({}^*\mathcal{D}^{(\bullet)}_\bullet)$) consisting of complexes which are $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$-quasi-coherent (resp. $\mathcal{O}^{(\bullet)}_{X_\bullet}$-quasi-coherent modules) in the sense of 8.5.1.6 (resp. 8.5.1.7). The objects of $D^-_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)})$ (resp. $D^-_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}_\bullet)$) are called $\mathcal{D}^{(\bullet)}$-quasi-coherent complexes of $D^-({}^*\mathcal{D}^{(\bullet)})$ (resp. $\mathcal{D}^{(\bullet)}_\bullet$-quasi-coherent complexes of $D^-({}^*\mathcal{D}^{(\bullet)}_\bullet)$), or simply quasi-coherent complexes. Moreover, $D^-_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)})$ is a triangulated subcategory of $D^-(\mathcal{D}^{(\bullet)})$ and $D^-_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}_\bullet)$ is a triangulated subcategory of $D^-(\mathcal{D}^{(\bullet)}_\bullet)$. The full subcategory of $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)})$ (resp. $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}_\bullet)$) consisting of quasi-coherent complexes of finite tor dimension on $\mathcal{D}^{(\bullet)}_\bullet$ is denoted by $D_{\mathrm{qc,tdf}}({}^*\mathcal{D}^{(\bullet)})$ (resp. $D_{\mathrm{qc,tdf}}({}^*\mathcal{D}^{(\bullet)}_\bullet)$). Finally, the $\mathcal{D}^{(\bullet)}$-quasi-coherence is a local notion on $\mathfrak{X}$: 1) a complex $\mathcal{E}^{(\bullet)}$ of $D^-(\mathcal{D}^{(\bullet)})$ is quasi-coherent if and only if there exists an open covering $(\mathfrak{U}_\alpha)_\alpha$ of $\mathfrak{X}$ such that $\mathcal{E}^{(\bullet)}|_{\mathfrak{U}_\alpha}$ is quasi-coherent for any α ; 2) if $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ then $\mathcal{E}^{(\bullet)}|_{\mathfrak{U}} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}|_{\mathfrak{U}})$ for any open set $\mathfrak{U}$ of $\mathfrak{X}$. Similarly, the $\mathcal{D}^{(\bullet)}_\bullet$-quasi-coherence is a local notion on $\mathfrak{X}$.

8.5.1.9. Let $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$. It follows from the isomorphisms 8.5.1.4.1 and 8.5.1.5.1 that the property $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ is equivalent to saying $\mathcal{E}^{(i)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(i)})$ for any $i \in I$. Since this is the case when I has cardinal one, this yields this notion of quasi-coherence does not depend on the choice of the ideal of definition of $\mathfrak{X}$.

Let $\mathcal{E}^{(\bullet)}_\bullet \in D^-(\mathcal{D}^{(\bullet)}_\bullet)$. It follows from the isomorphism 8.5.1.5.1 that the property $\mathcal{E}^{(\bullet)}_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_\bullet)$ is equivalent to saying $\mathcal{E}^{(i)}_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}^{(i)}_\bullet)$ for any $i \in I$.

Let $\mathcal{E}^{(\bullet)}_\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}_\bullet)$. It follows from the flatness properties given at 7.1.3.6 that for any integers $a \le b$, $\mathcal{E}^{(\bullet)}_\bullet$ has tor amplitude in $[a, b]$ if and only if $\mathcal{E}^{(i)}_\bullet$ has tor amplitude in $[a, b]$ for any $i \in I$. We have similar properties replacing $\mathcal{D}^{(\bullet)}_\bullet$ by $\mathcal{D}^{(\bullet)}$.

Using again the flatness properties of 7.1.3.6, we notice that the property $\mathcal{D}^{(\bullet)}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}^{(\bullet)}$) has right finite tor dimension on $\mathcal{D}^{(\bullet)}$ (resp. $\mathcal{D}^{(\bullet)}_0$) is equivalent to saying that $\mathcal{D}^{(i)}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}^{(i)}$) has right finite tor dimension on $\mathcal{D}^{(i)}$ (resp. $\mathcal{D}^{(i)}_0$) for any $i \in I$.

Theorem 8.5.1.10. *With notation 8.5.1.2, we have the following properties.*

(a) The functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ induce canonically quasi-inverse equivalences of categories between $D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_\bullet)$ and $D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$

(b) Suppose that $\mathcal{D}^{(\bullet)}_0$ (resp. $gr^\bullet_{\mathcal{I}}\mathcal{D}^{(\bullet)}$) has right finite tor dimension on $\mathcal{D}^{(\bullet)}$ (resp. $\mathcal{D}^{(\bullet)}_0$). Then the functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ induce canonically quasi-inverse equivalences of categories between $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_\bullet)$ and $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$.

(c) The functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ induce canonically quasi-inverse equivalences of categories between $D_{\mathrm{qc,tdf}}(\mathcal{D}^{(\bullet)}_\bullet)$ and $D_{\mathrm{qc,tdf}}(\mathcal{D}^{(\bullet)})$.

More precisely let $\mathcal{E}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ (resp. $\mathcal{E}^{(\bullet)}_\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_\bullet)$) and $a \le b$ be two integers. Then $\mathcal{E}^{(\bullet)}$ (resp. $\mathcal{E}^{(\bullet)}_\bullet$) has tor amplitude in $[a, b]$ if and only if $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}(\mathcal{E}^{(\bullet)})$ (resp. $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}\mathcal{E}^{(\bullet)}_\bullet$) has tor amplitude in $[a, b]$.

Proof. By using the isomorphisms 8.5.1.4.1 and 8.5.1.5.1 and the remarks of 8.5.1.9, this is a consequence of Theorems 7.3.2.10, 7.3.2.15. □

8.5.2 Derived completed tensor products in $D(\mathcal{D}^{(\bullet)})$ or $D(\mathcal{D}^{(\bullet)}_\bullet)$

We would extend the definition of derived completed tensor products as in 7.3.4 to the case of inductive systems and we would like to give few properties.

Let $\mathfrak{X}$ be a locally noetherian formal scheme of Krull dimension d and $\mathcal{I}$ be an ideal of definition of $\mathfrak{X}$. Let I be a partially ordered set. Let $\mathcal{D}^{(\bullet)}$ (resp. $\mathcal{D}'^{(\bullet)}$, resp. $\mathcal{D}''^{(\bullet)}$, resp. $\mathcal{D}'''^{(\bullet)}$) be a sheaf of rings on the topos $\mathfrak{X}^{(I)}$. We suppose there exists a homomorphism $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}^{(\bullet)}$ (resp. $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}'^{(\bullet)}$, resp. $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}''^{(\bullet)}$, resp. $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}'''^{(\bullet)}$) such that $\mathcal{D}^{(i)}$ (resp. $\mathcal{D}'^{(i)}$, resp. $\mathcal{D}''^{(i)}$, resp. $\mathcal{D}'''^{(i)}$) equipped with the induced map $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}^{(i)}$ (resp. $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}'^{(i)}$, resp. $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}''^{(i)}$, resp. $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}'''^{(i)}$) is a sheaf of rings on $\mathfrak{X}$ satisfying the conditions of 7.3.2.

8.5.2.1 (Independence of $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}}$). We have the topoi morphisms $\underleftarrow{l}_{\mathfrak{X}^{(I)}} \colon (X^{(I)}_{\bullet}, \mathcal{D}^{(\bullet)}_{\bullet}) \to\colon (\mathfrak{X}^{(I)}, \mathcal{D}^{(\bullet)})$ and $\underleftarrow{l}_{\mathfrak{X}^{(I)}} \colon (X^{(I)}_{\bullet}, \mathbb{Z}_{X^{(I)}_{\bullet}}) \to\colon (\mathfrak{X}^{(I)}, \mathbb{Z}_{\mathfrak{X}^{(I)}})$.

(a) Both functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ can be computed by taking the same flasque resolution (see 7.1.3.14) and therefore we have the canonical commutative diagram

$$\begin{array}{ccc} D^{-}(\mathcal{D}^{(\bullet)}_{\bullet}) & \xrightarrow{\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}} & D^{-}(\mathcal{D}^{(\bullet)}) \\ \downarrow & & \downarrow \\ D^{-}(\mathbb{Z}_{X^{(I)}_{\bullet}}) & \xrightarrow{\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}} & D^{-}(\mathbb{Z}_{\mathfrak{X}^{(I)}}), \end{array}$$

where the vertical maps are the forgetful functors. We have obviously the same property for the exact functor $\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}$.

(b) Following 7.3.2.3.d, the canonical morphism

$$\mathcal{O}_{X^{(\bullet)}_{\bullet}} \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}} \mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}(\mathcal{E}^{(\bullet)}) \to \mathcal{D}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}} \mathcal{D}^{(\bullet)}} \underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}(\mathcal{E}^{(\bullet)}) =\colon \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}}(\mathcal{E}^{(\bullet)})$$

is an isomorphism for any $\mathcal{E}^{(\bullet)} \in D^{-}({}^{l}\mathcal{D}^{(\bullet)})$, and similarly for right modules. Hence, the functor $\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}}$ does not depend, up to canonical forgetful functor, to the choice of such $\mathcal{D}^{(\bullet)}$.

8.5.2.2. Let $\mathcal{E}^{(\bullet)} \in D^{-}({}^{l}\mathcal{D}^{(\bullet)})$, $\mathcal{M}^{(\bullet)} \in D^{-}({}^{r}\mathcal{D}^{(\bullet)})$ be two complexes of $\mathcal{D}^{(\bullet)}$-modules, respectively to the left, to the right. Similarly to 7.3.4.2.1, we define their *completed tensor product* by setting

$$\mathcal{M}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} := \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}} \mathcal{M}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}} \mathcal{E}^{(\bullet)}). \tag{8.5.2.2.1}$$

By adjunction (see 7.3.2.2), similarly to 7.3.4.3, we construct the canonical morphism in $D^{-}(\mathbb{Z}_{\mathfrak{X}^{(I)}})$ of the form

$$\mathcal{M}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \to \mathcal{M}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \tag{8.5.2.2.2}$$

which is an isomorphism when one of the two complexes belongs to $D^{-}_{\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and the other one to $D^{-}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$.

When $\mathcal{D}^{(\bullet)}$ is commutative, we have the isomorphism of $D^{-}({}^{r}\mathcal{D}^{(\bullet)})$:

$$\mathcal{M}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*} \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}} (\mathcal{M}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)}) \tag{8.5.2.2.3}$$

and the map 8.5.2.2.2 is (modulo the canonical isomorphism 8.5.2.2.3) the adjunction morphism.

8.5.2.3. Suppose there exists a homomorphism of sheaf of rings on $\mathfrak{X}$ of the form $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ such that the composition of $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}^{(\bullet)}$ with $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ gives $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}'^{(\bullet)}$.

(a) Let $* \in \{\mathrm{l}, \mathrm{r}\}$ and $\star \in \{-, \mathrm{b}\}$. By definition (see 8.5.1.8), the forgetful functors $D^{\star}({}^{*}\mathcal{D}'^{(\bullet)}) \to D^{\star}({}^{*}\mathcal{D}^{(\bullet)})$ and $D^{\star}({}^{*}\mathcal{D}'^{(\bullet)}_{\bullet}) \to D^{\star}({}^{*}\mathcal{D}^{(\bullet)}_{\bullet})$ preserve the quasi-coherence, i.e. induces the functor

$$forg_{\mathcal{D},\mathcal{D}'} \colon D^{\star}_{\mathrm{qc}}({}^{*}\mathcal{D}'^{(\bullet)}) \to D^{\star}_{\mathrm{qc}}({}^{*}\mathcal{D}^{(\bullet)}), \qquad forg_{\mathcal{D},\mathcal{D}'} \colon D^{\star}_{\mathrm{qc}}({}^{*}\mathcal{D}'^{(\bullet)}_{\bullet}) \to D^{\star}_{\mathrm{qc}}({}^{*}\mathcal{D}^{(\bullet)}_{\bullet}). \tag{8.5.2.3.1}$$

(b) Since $\mathcal{D}^{(i)}_{\bullet}$ satisfied the condition 7.3.2.d for the left structure, then it follows from 7.3.1.14 and 8.5.1.9 that the functor $\mathcal{D}'^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} -$ preserves the quasi-coherence, i.e., induces the functor

$$\mathcal{D}'^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} - \colon D^{-}_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet}) \to D^{-}_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet}). \tag{8.5.2.3.2}$$

(c) Let $\mathcal{E}^{(\bullet)} \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)})$. Since $\mathcal{D}'^{(\bullet)}_{\bullet} \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}} \mathcal{D}'^{(\bullet)}$ (see 7.3.2.3.b), then we get

$$\mathcal{D}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xleftarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{D}'^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}} \mathcal{E}^{(\bullet)}), \tag{8.5.2.3.3}$$

Using the preservation of the quasi-coherence under the functors $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ and $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ of 8.5.1.10, the property 8.5.2.3.2 and the isomorphism 8.5.2.3.3, we get that the functor $\mathcal{D}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}-$ preserves the quasi-coherence, i.e. induces the functor

$$\mathcal{D}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}-\colon D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}) \to D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}). \tag{8.5.2.3.4}$$

Similarly, we get the functor $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{D}'^{(\bullet)}\colon D^-_{\mathrm{qc}}({}^{r}\mathcal{D}^{(\bullet)}) \to D^-_{\mathrm{qc}}({}^{r}\mathcal{D}'^{(\bullet)})$.

In order to derive complexes of bimodules, we need further hypotheses on $\mathcal{D}^{(\bullet)}$.

Definition 8.5.2.4. Let us introduce the following definition (compare with 4.6.3.2.b). A pair $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$ consisting of a sheaf $\mathcal{R}^{(\bullet)}$ of commutative rings on the topos $\mathfrak{X}^{(I)}$ and an ideal $\mathcal{K}^{(\bullet)}$ of $\mathcal{R}^{(\bullet)}$ is said to be solving $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ if it satisfies the following conditions

(i) $\mathcal{O}_{\mathfrak{X}}$ is endowed with a structure of $\mathcal{R}^{(i)}$-algebra such that $\mathcal{K}^{(i)}\mathcal{O}_{\mathfrak{X}} = \mathcal{I}$ for any $i \in I$;

(ii) $\mathcal{R}^{(\bullet)}$ is sent to the center of $\mathcal{D}^{(\bullet)}$ and of $\mathcal{D}'^{(\bullet)}$;

(iii) $\mathcal{D}^{(\bullet)}$ and $\mathcal{D}'^{(\bullet)}$ are flat on $\mathcal{R}^{(\bullet)}$.

In that case, we say that $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ is solvable. We remark that if $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$ solves $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$, then $\mathcal{R}^{(\bullet)}_{\bullet}$ is a solving ring of $(\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{D}'^{(\bullet)}_{\bullet})$ (see Definition 4.6.3.2.b), where $\mathcal{R}^{(\bullet)}_{\bullet} := (\mathcal{R}^{(\bullet)}/(\mathcal{K}^{(\bullet)})^{i+1})_{i\in\mathbb{N}}$, $\mathcal{D}^{(\bullet)}_{\bullet} := (\mathcal{D}^{(\bullet)}/\mathcal{I}^{i+1}\mathcal{D}^{(\bullet)})_{i\in\mathbb{N}}$ and $\mathcal{D}'^{(\bullet)}_{\bullet} := (\mathcal{D}'^{(\bullet)}/\mathcal{I}^{i+1}\mathcal{D}'^{(\bullet)})_{i\in\mathbb{N}}$. Finally, $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$ solves $(\mathcal{D}^{(\bullet)}, \mathcal{I})$ means by definition that $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$ solves $(\mathcal{D}^{(\bullet)}, \mathcal{D}^{(\bullet)}, \mathcal{I})$.

A pair $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$ is said to be "strongly solving $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$" is it is solving $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ and if the following condition holds.

(iv) $\mathcal{R}^{(\bullet)}_0$ (resp. $gr^{\bullet}_{\mathcal{K}^{(\bullet)}}\mathcal{R}^{(\bullet)}$) has right finite tor dimension on $\mathcal{R}^{(\bullet)}$ (resp. $\mathcal{R}^{(\bullet)}_0$).

By flatness, remark that we have therefore $\mathcal{D}^{(\bullet)}_0$ (resp. $gr^{\bullet}_{\mathcal{I}}\mathcal{D}^{(\bullet)}$) has right finite tor dimension on $\mathcal{D}^{(\bullet)}$ (resp. $\mathcal{D}^{(\bullet)}_0$) and similarly $\mathcal{D}'^{(\bullet)}_0$ (resp. $gr^{\bullet}_{\mathcal{I}}\mathcal{D}'^{(\bullet)}$) has right finite tor dimension on $\mathcal{D}'^{(\bullet)}$ (resp. $\mathcal{D}'^{(\bullet)}_0$).

Example 8.5.2.5. We will use essentially in this book the following cases. Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^{\sharp} \to \mathfrak{S}^{\sharp}$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension and is p-torsion free (see 3.3.1.12 for some example). For any integer $i \geq 0$, set $X^{\sharp}_i := X^{\sharp} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$. Let Z be a divisor of X_0. Let $\lambda \in L(\mathbb{N})$. Then it follows from 8.7.4.2 that we can choose $I = \mathbb{N}$, $\mathcal{R}^{(\bullet)}$ is the constant inductive system with value $\mathcal{V}$, $\mathcal{K}^{(\bullet)}$ is the constant inductive system with value $\mathfrak{m}$, $\mathcal{I} = \mathfrak{m}\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{D}^{(\bullet)}$ is such that $\mathcal{D}^{(m)} = \varprojlim_i \mathcal{B}^{(\lambda(m))}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X^{\sharp}_i/S^{\sharp}_i}$ for any $m \in \mathbb{N}$ with the canonical transition maps $\mathcal{D}^{(m)} \to \mathcal{D}^{(m+1)}$.

8.5.2.6. Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ is solved by $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. Let $\mathcal{E}^{\bullet} \in K({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$. The following condition

(v) The structures of $\mathcal{R}^{(\bullet)}$-module induced on the $(\mathcal{D}'^{(\bullet)}, \mathcal{D}^{(\bullet)})$-bimodules $\mathcal{E}^n$ by the structure of left $\mathcal{D}'^{(\bullet)}$-module and of right $\mathcal{D}^{(\bullet)}$-module coincide for any $n \in \mathbb{Z}$;

is equivalent to saying that $\mathcal{E}^{\bullet} \in K({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{R}^{(\bullet)}} \mathcal{D}^{(\bullet)\mathrm{o}})$.

We denote by $D({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{\mathrm{r}}\mathcal{D}^{(\bullet)})$ (resp. $D({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}^{(\bullet)}_{\bullet}, {}^{\mathrm{r}}\mathcal{D}^{(\bullet)}_{\bullet})$) the strictly full subcategory of $D({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, {}^{\mathrm{r}}\mathcal{D}^{(\bullet)})$ (resp. $D({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet}, {}^{\mathrm{r}}\mathcal{D}^{(\bullet)}_{\bullet})$) consisting of complexes isomorphic to an object of $K({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{R}^{(\bullet)}} \mathcal{D}^{(\bullet)\mathrm{o}})$ (resp. $K({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet} \otimes_{\mathcal{R}^{(\bullet)}_{\bullet}} \mathcal{D}^{(\bullet)\mathrm{o}}_{\bullet})$).

8.5.2.7. Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ is solved by $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. The functor $\underleftarrow{l}^*_{\mathfrak{X}^{(I)}} = -\otimes_{\mathcal{R}^{(\bullet)}}\mathcal{R}^{(\bullet)}_{\bullet}\colon \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}}) \to \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{D}^{(\bullet)\mathrm{r}}_{\bullet})$ is canonically isomorphic (modulo forgetful functors) to $-\otimes_{\mathcal{D}^{(\bullet)}}\mathcal{D}^{(\bullet)}_{\bullet}\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)\mathrm{r}}) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)\mathrm{r}}_{\bullet})$ and $\mathcal{D}'^{(\bullet)}_{\bullet} \otimes_{\mathcal{D}'^{(\bullet)}} -\colon \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}) \to \mathrm{Mod}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet})$.

Let $\star \in \{\emptyset, -, \mathrm{b}\}$. When $\star = \mathrm{b}$, we always suppose that $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ is strongly solved by $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. Since $\mathcal{D}^{(\bullet)} \otimes_{\mathcal{R}^{(\bullet)}} (\mathcal{D}'^{(\bullet)})^o$ is flat over $\mathcal{D}^{(\bullet)}$, $(\mathcal{D}'^{(\bullet)})^o$ and $\mathcal{R}^{(\bullet)}$, then by using resolutions by K-flat complexes of left $\mathcal{D}'^{(\bullet)} \otimes_{\mathcal{R}^{(\bullet)}} (\mathcal{D}^{(\bullet)})^o$-modules we get the functor

$$\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}} = - \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}(\mathcal{R}^{(\bullet)})} \mathcal{R}^{(\bullet)}_\bullet \colon D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}}) \to D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet) \tag{8.5.2.7.1}$$

which is canonically isomorphic to the functors $- \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}(\mathcal{D}^{(\bullet)})} \mathcal{D}^{(\bullet)}_\bullet \colon D^\star(\mathcal{D}^{(\bullet)\mathrm{r}}) \to D^\star(\mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$ and $\mathcal{D}'^{(\bullet)}_\bullet \otimes^{\mathbb{L}}_{\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}(\mathcal{D}'^{(\bullet)})} - \colon D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}) \to D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet)$. Similarly, by using resolutions by K-injective complexes of left $\mathcal{D}'^{(\bullet)}_\bullet \otimes_{\underleftarrow{l}^{-1}_{\mathfrak{X}^{(I)}}(\mathcal{R}^{(\bullet)})} (\mathcal{D}^{(\bullet)}_\bullet)^o$-modules, we get the functor

$$\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*} \colon D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet) \to D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}\mathcal{D}^{(\bullet)\mathrm{r}}) \tag{8.5.2.7.2}$$

which is canonically isomorphic to the functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*} \colon D^\star(\mathcal{D}^{(\bullet)\mathrm{r}}_\bullet) \to D^\star(\mathcal{D}^{(\bullet)\mathrm{r}})$ and $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*} \colon \colon D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet) \to D^\star({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)})$.

Let $\mathcal{E}^{(\bullet)}_\bullet \in D^-({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}})$. The property $\mathcal{E}^{(\bullet)}_\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet)$ (resp. $\mathcal{E}^{(\bullet)}_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$) is satisfied if and only if both conditions hold:

(a) The image via the forgetful functor $D^-({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_0, \mathcal{R}^{(\bullet)}_0, \mathcal{D}^{(\bullet)\mathrm{r}}_0) \to D^-({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_0) \to D^-(\mathcal{O}^{(\bullet)}_{X_0})$ (resp. $D^-({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_0, \mathcal{R}^{(\bullet)}_0, \mathcal{D}^{(\bullet)\mathrm{r}}_0) \to D^-(\mathcal{D}^{(\bullet)\mathrm{r}}_0) \to D^-(\mathcal{O}^{(\bullet)}_{X_0})$)) of the complex $\mathcal{E}^{(\bullet)}_0$ is in $D^-_{\mathrm{qc}}(\mathcal{O}^{(\bullet)}_{X_0})$.

(b) The canonical map
$$\mathcal{R}^{(\bullet)}_i \overset{\mathbb{L}}{\otimes}_{\mathcal{R}^{(\bullet)}_{i+1}} \mathcal{E}^{(\bullet)}_{i+1} \to \mathcal{E}^{(\bullet)}_i \tag{8.5.2.7.3}$$
is an isomorphism.

We denote respectively by $D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$ and $D^-_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$ the full subcategory of $D^-({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$ consisting of complexes $\mathcal{E}^{(\bullet)}_\bullet$ which belongs to $\mathcal{E}^{(\bullet)}_\bullet \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet)$ (resp. $\mathcal{E}^{(\bullet)}_\bullet \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$). Beware that the property 8.5.2.6.(v) is not necessarily satisfied for $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ instead of $\mathcal{R}^{(\bullet)}$, so we do have to distinguish the categories $D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$ and $D^-_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$.

Similarly, we denote by $D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$ and $D^-_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$ the full subcategory of complexes $\mathcal{E}^{(\bullet)}$ which belong to $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)})$ (resp. $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)\mathrm{r}})$). It follows from Theorem 8.5.1.10 that the functors $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ of 8.5.2.7.1 and $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ of 8.5.2.7.2 induce quasi-inverse equivalences of categories between $D^-_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$ and $D^-_{.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$ and between $D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$ and $D^-_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}^{(\bullet)\mathrm{r}}_\bullet)$.

8.5.2.8. Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ and $(\mathcal{D}^{(\bullet)}, \mathcal{D}''^{(\bullet)}, \mathcal{I})$ are solved by $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. We have the bifunctors

$$\mathbb{R}\mathscr{H}om_{{}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_\bullet}(-,-) \colon D({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)\mathrm{r}}_\bullet) \times D({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}''^{(\bullet)\mathrm{r}}_\bullet) \to D({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}''^{(\bullet)\mathrm{r}}_\bullet), \tag{8.5.2.8.1}$$

$$- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_\bullet} - \colon D({}^{*}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}_\bullet{}^{\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{D}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, {}^{*}\mathcal{D}''^{(\bullet)}_\bullet) \to D({}^{*}\mathcal{D}'^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, {}^{*}\mathcal{D}''^{(\bullet)}_\bullet). \tag{8.5.2.8.2}$$

We have similar bifunctors by changing the indices l and r. Via the functors 8.5.2.7.1 and 8.5.2.7.2, we can check the bifunctor 8.5.2.2.1 induces

$$- \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon D({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times D({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{*}\mathcal{D}''^{(\bullet)}) \to D({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{*}\mathcal{D}''^{(\bullet)}). \tag{8.5.2.8.3}$$

Let $\mathcal{M}^{(\bullet)} \in D({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$, $\mathcal{E}^{(\bullet)} \in D({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{*}\mathcal{D}''^{(\bullet)})$ be two complexes. Since $\mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{M}^{(\bullet)}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_\bullet} \mathbb{L}\underleftarrow{l}^*_X(\mathcal{E}^{(\bullet)})$ (remark we have three different functors $\mathbb{L}\underleftarrow{l}^*_X$ such as 8.5.2.7.1), then we get by adjunction via the adjoint pair $(\mathbb{L}\underleftarrow{l}^*_X \dashv \mathbb{R}\underleftarrow{l}_{X*})$, (see the functors 8.5.2.7.1 and 8.5.2.7.2) the canonical morphism

$$\mathcal{M}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \to \mathcal{M}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \tag{8.5.2.8.4}$$

which is an isomorphism when either $\mathcal{M}^{(\bullet)} \in D^-_{\mathrm{coh}}(\mathcal{D}^{(\bullet)\mathrm{r}})$ and $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)})$, or $\mathcal{M}^{(\bullet)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)\mathrm{r}})$ and $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)})$. Indeed, for instance, suppose $\mathcal{M}^{(\bullet)} \in D^-_{\mathrm{coh}}(\mathcal{D}^{(\bullet)\mathrm{r}})$ and $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)})$. Then, since both functors $- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)}$ and $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{E}^{(\bullet)}$ are way-out left (use 7.3.1.2 for the second functor), then following (the way-out left version of) [Har66, I.7.1 (ii) and (iv)], since this is local, we reduce to the case where $\mathcal{M}^{(\bullet)} = \mathcal{D}^{(\bullet)}$, which is obvious.

Proposition 8.5.2.9. *Suppose* $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ *and* $(\mathcal{D}^{(\bullet)}, \mathcal{D}''^{(\bullet)}, \mathcal{I})$ *are solved by* $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. *With* $*, ** \in \{r, l\}$, *the functors 8.5.2.8.2 and 8.5.2.8.3 preserve the quasi-coherence for bounded above complexes, i.e. they factor through the functor*

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} - \colon D^{-}_{\mathrm{qc},.}({}^{*}\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}^{(\bullet)}_{\bullet}, \mathcal{D}^{(\bullet)\mathrm{r}}_{\bullet}) \times D^{-}_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}^{(\bullet)}_{\bullet}, {}^{*}\mathcal{D}''^{(\bullet)}_{\bullet}) \to D^{-}_{\mathrm{qc},.}({}^{*}\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}^{(\bullet)}_{\bullet}, {}^{**}\mathcal{D}''^{(\bullet)}_{\bullet}), \tag{8.5.2.9.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon D^{-}_{\mathrm{qc},.}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times D^{-}_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{*}\mathcal{D}''^{(\bullet)}) \to D^{-}_{\mathrm{qc},.}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{**}\mathcal{D}''^{(\bullet)}), \tag{8.5.2.9.2}$$

and similarly replacing the indexes "qc, ." *by* "., qc".

Proof. We can copy the proof of 7.3.4.10. □

Example 8.5.2.10. When $\mathcal{D}$ is commutative, we get the factorisations

$$- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} - \colon D_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet}) \times D_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet}) \to D_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet}), \tag{8.5.2.10.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon D^{-}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}) \times D^{-}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}) \to D^{-}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}). \tag{8.5.2.10.2}$$

Proposition 8.5.2.11. *Suppose* $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$, $(\mathcal{D}^{(\bullet)}, \mathcal{D}''^{(\bullet)}, \mathcal{I})$ *and* $(\mathcal{D}''^{(\bullet)}, \mathcal{D}'''^{(\bullet)}, \mathcal{I})$ *are solved by* $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. *Let* $*, ** \in \{r, l\}$. *Let* $\mathcal{E}^{(\bullet)} \in D^{-}_{\mathrm{qc},.}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, \mathcal{D}^{(\bullet)\mathrm{r}})$, $\mathcal{F}^{(\bullet)} \in D^{-}_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{\mathrm{r}}\mathcal{D}''^{(\bullet)})$, $\mathcal{G}^{(\bullet)} \in D^{-}_{\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}''^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{**}\mathcal{D}'''^{(\bullet)})$. *The associativity isomorphism of the derived complete tensor product of quasi-coherent complexes holds, i.e. we have the isomorphism in* $D^{-}_{\mathrm{qc},.}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}^{(\bullet)}, {}^{**}\mathcal{D}'''^{(\bullet)})$:

$$\left(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)}\right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''^{(\bullet)}} \mathcal{G}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \left(\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}''^{(\bullet)}} \mathcal{G}^{(\bullet)}\right). \tag{8.5.2.11.1}$$

Proof. By using Theorem 8.5.1.10 and Proposition 8.5.2.9, we can copy the proof of 7.3.4.12. □

In the case of extension by ring homomorphisms, contrary to 8.5.2.11, we get the following associativity without the use of solving pairs:

Proposition 8.5.2.12. *Suppose there exists a homomorphism of sheaf of rings on* $\mathfrak{X}$ *of the form* $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)} \to \mathcal{D}''^{(\bullet)}$ *such that the composition of* $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}^{(\bullet)}$ *with* $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ *gives* $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}'^{(\bullet)}$ *and the composition of* $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}'^{(\bullet)}$ *with* $\mathcal{D}'^{(\bullet)} \to \mathcal{D}''^{(\bullet)}$ *gives* $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}''^{(\bullet)}$.

(a) For any Let $\mathcal{E}^{(\bullet)}_{\bullet} \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet})$, *we have the associativity isomorphism in* $D^{-}({}^{\mathrm{l}}\mathcal{D}''^{(\bullet)}_{\bullet})$:

$$\mathcal{D}''^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{E}^{(\bullet)}_{\bullet} \xrightarrow{\sim} \left(\mathcal{D}''^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}'^{(\bullet)}_{\bullet}} \mathcal{D}'^{(\bullet)}\right) \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{E}^{(\bullet)}_{\bullet} \xrightarrow{\sim} \mathcal{D}''^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}'^{(\bullet)}_{\bullet}} \left(\mathcal{D}'^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{E}^{(\bullet)}_{\bullet}\right). \tag{8.5.2.12.1}$$

(b) For any $\mathcal{E}^{(\bullet)} \in D^{-}_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)})$, *we have the associativity isomorphism in* $D^{-}_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}''^{(\bullet)})$:

$$\mathcal{D}''^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \left(\mathcal{D}''^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}'^{(\bullet)}} \mathcal{D}'^{(\bullet)}\right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{D}''^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}'^{(\bullet)}} \left(\mathcal{D}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)}\right). \tag{8.5.2.12.2}$$

Proof. By using a flat resolution of we get the isomorphism 8.5.2.12.1. By using Theorem 8.5.1.10, we get 8.5.2.12.2 from 8.5.2.12.1. □

Proposition 8.5.2.13. *Suppose* $\mathfrak{X}$ *is quasi-compact. Suppose* $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ *is solved by* $(\mathcal{R}^{(\bullet)}, \mathcal{K}^{(\bullet)})$. *Let* $*, ** \in \{r, l\}$, $\star \in \{\mathrm{b}, -\}$. *Let* $\mathcal{E}^{(\bullet)}_{\bullet} \in D_{\mathrm{perf}}({}^{*}\mathcal{D}^{(\bullet)}_{\bullet})$, $\mathcal{F}^{(\bullet)}_{\bullet} \in D^{\star}_{.,\mathrm{qc}}({}^{*}\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}^{(\bullet)}_{\bullet}, {}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$, *Then* $\mathbb{R}\mathscr{H}om_{\mathcal{D}_{\bullet}}(\mathcal{E}^{(\bullet)}_{\bullet}, \mathcal{F}^{(\bullet)}_{\bullet}) \in D^{\star}_{\mathrm{qc}}({}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$. *If moreover* $\mathcal{F}^{(\bullet)}_{\bullet} \in D_{\mathrm{perf}}({}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$, *then* $\mathbb{R}\mathscr{H}om_{\mathcal{D}_{\bullet}}(\mathcal{E}^{(\bullet)}_{\bullet}, \mathcal{F}^{(\bullet)}_{\bullet}) \in D_{\mathrm{perf}}({}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$.

Proof. We can copy the proof of 7.3.4.17. □

8.5.3 $LD(\mathcal{D}^{(\bullet)}_\bullet)$

We keep notation 8.5.2.

8.5.3.1. Similarly to 8.1.2.2, we define the notion of ind-isogenies of a left $\mathcal{D}^{(\bullet)}_\bullet$-module as follows. Let $\mathcal{E}^{(\bullet)}_\bullet = (\mathcal{E}^{(i)}_n, \alpha^{j,i}_{m,n})_{i\le j, m\le n}$, be a left $\mathcal{D}^{(\bullet)}_\bullet$-module. For any map $\chi \in M(I)$ (see notation 8.1.2.1), for any $\mathcal{E}^{(\bullet)}_\bullet = (\mathcal{E}^{(i)}, \alpha^{j,i}_{m,n})$ we set

$$\chi^*(\mathcal{E}^{(\bullet)}_\bullet) := (\mathcal{E}^{(i)}_n, p^{\chi(j)-\chi(i)}\alpha^{j,i}_{m,n}). \tag{8.5.3.1.1}$$

We obtain the functor $\chi^*\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}_\bullet) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)}_\bullet)$ as follows: if $f^{(\bullet)}_\bullet\colon \mathcal{E}^{(\bullet)}_\bullet \to \mathcal{F}^{(\bullet)}_\bullet$ is a morphism of $\mathrm{Mod}(\mathcal{D}^{(\bullet)}_\bullet)$, then $\chi^*(f^{(\bullet)}_\bullet)\colon \chi^*(\mathcal{E}^{(\bullet)}_\bullet) \to \chi^*(\mathcal{F}^{(\bullet)}_\bullet)$ is the morphism of left $\mathcal{D}^{(\bullet)}_\bullet$-modules such that $(\chi^*(f^{(\bullet)}_\bullet))^{(i)} = f^{(i)}_\bullet$. Since the functor $\chi^*\colon \mathrm{Mod}(\mathcal{D}^{(\bullet)}_\bullet) \to \mathrm{Mod}(\mathcal{D}^{(\bullet)}_\bullet)$ is exact, then for any $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$, this induces the functor $\chi^*\colon D^\sharp(\mathcal{D}^{(\bullet)}_\bullet) \to D^\sharp(\mathcal{D}^{(\bullet)}_\bullet)$.

Notation 8.5.3.2 (Ind-isogenies)**.** (a) Let $\chi_1, \chi_2 \in M(I)$ such that $\chi_1 \le \chi_2$. For any $\mathcal{E}^{(\bullet)}_\bullet \in D(\mathcal{D}^{(\bullet)}_\bullet)$, let

$$\theta_{\mathcal{E},\chi_2,\chi_1}\colon \chi^*_{1\bullet}(\mathcal{E}^{(\bullet)}_\bullet) \to \chi^*_{2\bullet}(\mathcal{E}^{(\bullet)}_\bullet) \tag{8.5.3.2.1}$$

be the morphism defined by $p^{\chi_2(n)-\chi_1(n)}\colon \mathcal{E}^{(n)} \to \mathcal{E}^{(n)}$ for any $n \in \mathbb{N}$. When $\chi_1 = 0$, we simply write $\theta_{\mathcal{E},\chi_2}$. We get a functor $\theta_{\mathcal{E}}\colon M(I) \to D(\mathcal{D}^{(\bullet)}_\bullet)$ given by $\chi \mapsto \chi^*(\mathcal{E}^{(\bullet)}_\bullet)$, where $M(I)$ is the category associated with its structure of partially ordered set (see 7.1.2.1).

(b) A morphism $f^{(\bullet)}_\bullet\colon \mathcal{E}^{(\bullet)}_\bullet \to \mathcal{F}^{(\bullet)}_\bullet$ of $D(\mathcal{D}^{(\bullet)}_\bullet)$ is an "ind-isogeny" of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet)$ if there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)}_\bullet\colon \mathcal{F}^{(\bullet)}_\bullet \to \chi^*\mathcal{E}^{(\bullet)}_\bullet$ of $D(\mathcal{D}^{(\bullet)}_\bullet)$ such that $g^{(\bullet)}_\bullet \circ f^{(\bullet)}_\bullet = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}_\bullet) \circ g^{(\bullet)}_\bullet = \theta_{\mathcal{F},\chi}$. We denote by $\Xi(\mathcal{D}^{(\bullet)}_\bullet)$ the set of ind-isogenies of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet)$. For any $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$, we set $\Xi^\sharp(\mathcal{D}^{(\bullet)}_\bullet) := \Xi(\mathcal{D}^{(\bullet)}_\bullet) \cap D^\sharp(\mathcal{D}^{(\bullet)}_\bullet)$. If no confusion is possible with respect to $\mathcal{D}^{(\bullet)}_\bullet$, we simply write $\Xi^\sharp := \Xi^\sharp(\mathcal{D}^{(\bullet)}_\bullet)$.

8.5.3.3. With the notation 8.5.1.2.4, for any $\mathcal{E}^{(\bullet)}_\bullet \in \mathrm{Mod}(\mathcal{D}^{(\bullet)}_\bullet)$, we compute

$$\chi^* \circ \underleftarrow{l}_{\mathfrak{X}(I)*}(\mathcal{E}^{(\bullet)}_\bullet) = (\varprojlim_{n\in\mathbb{N}} \mathcal{E}^{(i)}_n, p^{\chi(j)-\chi(i)}\beta^{j,i}) = \underleftarrow{l}_{\mathfrak{X}(I)*}(\mathcal{E}^{(i)}_n, p^{\chi(j)-\chi(i)}\alpha^{j,i}_{m,n}) = \underleftarrow{l}_{\mathfrak{X}(I)*} \circ \chi^*(\mathcal{E}^{(\bullet)}_\bullet). \tag{8.5.3.3.1}$$

It follows from the last sentence of the paragraph 8.5.1.3 that the functor χ^* sends $\underleftarrow{l}_{\mathfrak{X}(I),*}$-acyclic modules to $\underleftarrow{l}_{\mathfrak{X}(I),*}$-acyclic modules. Hence, for any $\mathcal{E}^{(\bullet)\bullet}_\bullet \in D(\mathcal{D}^{(\bullet)}_\bullet)$, by using a resolution $\mathcal{F}^{(\bullet)\bullet}_\bullet$ of $\mathcal{E}^{(\bullet)\bullet}_\bullet$ by $\underleftarrow{l}_{\mathfrak{X}(I),*}$-acyclic modules (following 4.6.1.6.b, such resolutions exist because of the boundedness of the cohomological dimension of $\underleftarrow{l}_{\mathfrak{X}(I),*}$ of 8.5.1.4 and such resolutions compute derived functors), we get the functorial in χ isomorphism of $D(\mathcal{D}^{(\bullet)})$:

$$\chi^* \circ \mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*}(\mathcal{E}^{(\bullet)\bullet}_\bullet) \xrightarrow{\sim} \chi^* \circ \underleftarrow{l}_{\mathfrak{X}(I)*}(\mathcal{F}^{(\bullet)\bullet}_\bullet) \overset{8.5.3.3.1}{\xrightarrow{\sim}} \underleftarrow{l}_{\mathfrak{X}(I)*} \circ \chi^*(\mathcal{F}^{(\bullet)\bullet}_\bullet) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*} \circ \chi^*(\mathcal{E}^{(\bullet)\bullet}_\bullet). \tag{8.5.3.3.2}$$

8.5.3.4. Let $\mathcal{E}^{(\bullet)\bullet} \in D(\mathcal{D}^{(\bullet)})$. Choose $\mathcal{P}^{(\bullet)\bullet}$ a K-flat complex of $K(\mathcal{D}^{(\bullet)})$ together with a quasi-isomorphism $\mathcal{P}^{(\bullet)\bullet} \to \mathcal{E}^{(\bullet)\bullet}$ of $K(\mathcal{D}^{(\bullet)})$. Following 7.1.3.6, $\mathcal{P}^{(\bullet)\bullet}$ is a K-flat complex means $\mathcal{P}^{(i)\bullet}$ is a K-flat complex of $K(\mathcal{D}^{(i)})$. Hence, using again 7.1.3.6 and the exactness of χ^*, we get that $\chi^*(\mathcal{P}^{(\bullet)\bullet})$ is a K-flat complex of $K(\mathcal{D}^{(\bullet)})$ together with a quasi-isomorphism $\chi^*\mathcal{P}^{(\bullet)\bullet} \to \chi^*\mathcal{E}^{(\bullet)\bullet}$ of $K(\mathcal{D}^{(\bullet)})$. This yields the functorial in χ and $\mathcal{E}^{(\bullet)\bullet}$ isomorphism in $D(\mathcal{D}^{(\bullet)})$:

$$\chi^* \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}(\mathcal{E}^{(\bullet)\bullet}) \xleftarrow{\sim} \chi^* \circ \underleftarrow{l}^*_{\mathfrak{X}(I)}(\mathcal{P}^{(\bullet)\bullet}) \xrightarrow{\sim} \underleftarrow{l}^*_{\mathfrak{X}(I)} \circ \chi^*(\mathcal{P}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)} \circ \chi^*(\mathcal{E}^{(\bullet)\bullet}). \tag{8.5.3.4.1}$$

Notation 8.5.3.5. The subset of ind-isogenies is a saturated multiplicative system compatible with its triangulated structure (this follows from Proposition 7.4.1.7 and the fact that by copying its proof we can check Lemma 8.1.2.3 is still valid replacing $\mathcal{D}^{(\bullet)}$ by $\mathcal{D}^{(\bullet)}_\bullet$). For any $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$, the localisation of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet)$ with respect to ind-isogenies is denoted by $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet)$.

8.5.3.6. We have the following notations and definitions. For any map $\lambda \in L(I)$, for any left $\mathcal{D}_{\bullet}^{(\bullet)}$-module $\mathcal{E}_{\bullet}^{(\bullet)} = (\mathcal{E}_n^{(i)}, \alpha_{m,n}^{j,i})_{i\leq j, m\leq n}$, we set

$$\lambda^*(\mathcal{E}_{\bullet}^{(\bullet)}) := (\mathcal{E}_n^{(\lambda(i))}, \alpha_{m,n}^{\lambda(j),\lambda(i)})_{i\leq j, m\leq n}.$$

We obtain the functor $\lambda^* \colon \mathrm{Mod}(\mathcal{D}_{\bullet}^{(\bullet)}) \to \mathrm{Mod}(\lambda^*\mathcal{D}_{\bullet}^{(\bullet)})$ as follows: if $f_{\bullet}^{(\bullet)} \colon \mathcal{E}_{\bullet}^{(\bullet)} \to \mathcal{F}_{\bullet}^{(\bullet)}$ is a morphism of $\mathrm{Mod}(\mathcal{D}_{\bullet}^{(\bullet)})$, then $\lambda^*(f_{\bullet}^{(\bullet)}) \colon \lambda^*(\mathcal{E}_{\bullet}^{(\bullet)}) \to \lambda^*(\mathcal{F}_{\bullet}^{(\bullet)})$ is the morphism of left $\lambda^*\mathcal{D}_{\bullet}^{(\bullet)}$-modules such that $(\lambda^*(f_{\bullet}^{(\bullet)}))^{(i)} = f_{\bullet}^{(\lambda(i))}$. Since the functor $\lambda^* \colon \mathrm{Mod}(\mathcal{D}_{\bullet}^{(\bullet)}) \to \mathrm{Mod}(\lambda^*\mathcal{D}_{\bullet}^{(\bullet)})$ is exact, then this induces the functor $\lambda^* \colon D^\sharp(\mathcal{D}_{\bullet}^{(\bullet)}) \to D^\sharp(\lambda^*\mathcal{D}_{\bullet}^{(\bullet)})$.

Notation 8.5.3.7 (Lim-isomorphims)**.** (a) Let $\lambda_1, \lambda_2 \in L(I)$. When $\lambda_1 \leq \lambda_2$, for any left $\mathcal{D}_{\bullet}^{(\bullet)}$-module $\mathcal{E}_{\bullet}^{(\bullet)} = (\mathcal{E}_n^{(i)}, \alpha_{m,n}^{j,i})_{i\leq j, m\leq n}$ we have the canonical morphism $\lambda_1^*(\mathcal{E}_{\bullet}^{(\bullet)}) \to \lambda_2^*(\mathcal{E}_{\bullet}^{(\bullet)})$ defined by the morphism $\alpha_{\bullet}^{\lambda_2(i),\lambda_1(i)} \colon \mathcal{E}_{\bullet}^{(\lambda_1(i))} \to \mathcal{E}_{\bullet}^{(\lambda_2(i))}$. The morphism $\lambda_1^*\mathcal{D}_{\bullet}^{(\bullet)} \to \lambda_2^*\mathcal{D}_{\bullet}^{(\bullet)}$ is in fact a ring homomorphism and the morphism $\lambda_1^*(\mathcal{E}_{\bullet}^{(\bullet)}) \to \lambda_2^*(\mathcal{E}_{\bullet}^{(\bullet)})$ is $\lambda_1^*\mathcal{D}_{\bullet}^{(\bullet)}$-linear. This yields the morphisms of functors $\rho_{\lambda_1,\lambda_2} \colon D^\sharp(\mathcal{D}_{\bullet}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\lambda_1^*\mathcal{D}_{\bullet}^{(\bullet)})$ (resp. $\rho_{\lambda_2,\lambda_1} \colon \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)}) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\lambda_1^*\mathcal{D}_{\bullet}^{(\bullet)})$) of the form $\lambda_1^* \to \lambda_2^*$. For any $\mathcal{E}_{\bullet}^{(\bullet)} \in \mathrm{Ob}\, D^\sharp(\mathcal{D}_{\bullet}^{(\bullet)}) = \mathrm{Ob}\, \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$, we set $\rho_{\mathcal{E},\lambda_2,\lambda_1} := \rho_{\lambda_2,\lambda_1}(\mathcal{E}_{\bullet}^{(\bullet)}) \colon \lambda_1^*(\mathcal{E}_{\bullet}^{(\bullet)}) \to \lambda_2^*(\mathcal{E}_{\bullet}^{(\bullet)})$. When $\lambda_1 = \mathrm{id}$, we set $\rho_{\mathcal{E},\lambda_2} := \rho_{\mathcal{E},\lambda_2,\mathrm{id}}$. We get a functor $\rho_{\mathcal{E}} \colon L(I) \to \underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$ given by $\rho \mapsto \rho^*(\mathcal{E}_{\bullet}^{(\bullet)})$, where $L(I)$ is the category associated with its structure of partially ordered set (see 7.1.2.1).

(b) We denote by $\Lambda^\sharp(\mathcal{D}_{\bullet}^{(\bullet)})$ the set of morphisms $f_{\bullet}^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$ such that there exist $\lambda \in L(I)$ and a morphism $g_{\bullet}^{(\bullet)} \colon \mathcal{F}_{\bullet}^{(\bullet)} \to \lambda^*\mathcal{E}_{\bullet}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$ such that $g_{\bullet}^{(\bullet)} \circ f_{\bullet}^{(\bullet)} = \rho_{\mathcal{E},\lambda}$ and $\lambda^*(f_{\bullet}^{(\bullet)}) \circ g_{\bullet}^{(\bullet)} = \rho_{\mathcal{F},\lambda}$ in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$. If no confusion is possible with respect to $\mathcal{D}_{\bullet}^{(\bullet)}$, we simply write $\Lambda^\sharp := \Lambda^\sharp(\mathcal{D}_{\bullet}^{(\bullet)})$. The morphisms belonging to Λ are called "lim-isomorphisms".

Notation 8.5.3.8. It follows from the analogous Lemma 8.1.3.4 and Proposition 7.4.1.7 that $\Lambda^\sharp$ is a saturated multiplicative system of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$ compatible with its triangulated structure. By localizing $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$ with respect to lim-isomorphisms we get a category denoted by $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}_{\bullet}^{(\bullet)})$.

8.5.3.9. With the notation 8.5.1.2.4, for any $\lambda \in L(I)$, for any $\mathcal{E}_{\bullet}^{(\bullet)} \in \mathrm{Mod}(\mathcal{D}_{\bullet}^{(\bullet)})$, we compute

$$\lambda^* \circ \underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{E}_{\bullet}^{(\bullet)}) = (\varprojlim_{n\in\mathbb{N}} \mathcal{E}_n^{(\lambda(i))}, \beta_{m,n}^{\lambda(j),\lambda(i)}) = \underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{E}_n^{(\lambda(i))}, \alpha_{m,n}^{\lambda(j),\lambda(i)}) = \underleftarrow{l}_{\mathfrak{X}^{(I)}*} \circ \lambda^*(\mathcal{E}_{\bullet}^{(\bullet)}). \tag{8.5.3.9.1}$$

It follows from the last sentence of the paragraph 8.5.1.3 that the functor λ^* sends $\underleftarrow{l}_{\mathfrak{X}^{(I)},*}$-acyclic modules to $\underleftarrow{l}_{\mathfrak{X}^{(I)},*}$-acyclic modules. Hence, for any $\mathcal{E}_{\bullet}^{(\bullet)\bullet} \in D(\mathcal{D}_{\bullet}^{(\bullet)})$, by using a resolution $\mathcal{F}_{\bullet}^{(\bullet)\bullet}$ of $\mathcal{E}_{\bullet}^{(\bullet)\bullet}$ by $\underleftarrow{l}_{\mathfrak{X}^{(I)},*}$-acyclic modules (following 4.6.1.6.b, such resolutions exist because of the boundedness of the cohomological dimension of $\underleftarrow{l}_{\mathfrak{X}^{(I)},*}$ of 8.5.1.4 and such resolutions compute derived functors), we get the functorial in χ isomorphism of $D(\mathcal{D}^{(\bullet)})$:

$$\lambda^* \circ \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{E}_{\bullet}^{(\bullet)\bullet}) \xrightarrow{\sim} \lambda^* \circ \underleftarrow{l}_{\mathfrak{X}^{(I)}*}(\mathcal{F}_{\bullet}^{(\bullet)\bullet}) \overset{8.5.3.9.1}{\xrightarrow{\sim}} \underleftarrow{l}_{\mathfrak{X}^{(I)}*} \circ \lambda^*(\mathcal{F}_{\bullet}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*} \circ \lambda^*(\mathcal{E}_{\bullet}^{(\bullet)\bullet}). \tag{8.5.3.9.2}$$

8.5.3.10. Let $\lambda \in L(I)$. Let $\mathcal{E}^{(\bullet)\bullet} \in D(\mathcal{D}^{(\bullet)})$. Choose $\mathcal{P}^{(\bullet)\bullet}$ a K-flat complex of $K(\mathcal{D}^{(\bullet)})$ together with a quasi-isomorphism $\mathcal{P}^{(\bullet)\bullet} \to \mathcal{E}^{(\bullet)\bullet}$ of $K(\mathcal{D}^{(\bullet)})$. We get the functorial in λ and $\mathcal{E}^{(\bullet)\bullet}$ isomorphisms of $D(\lambda^*\mathcal{D}^{(\bullet)})$:

$$\lambda^* \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}(\mathcal{E}^{(\bullet)\bullet}) \xleftarrow{\sim} \lambda^* \circ \underleftarrow{l}^*_{\mathfrak{X}^{(I)}}(\mathcal{P}^{(\bullet)\bullet}) \xrightarrow{\sim} \underleftarrow{l}^*_{\mathfrak{X}^{(I)}} \circ \lambda^*(\mathcal{P}^{(\bullet)\bullet}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}} \circ \lambda^*(\mathcal{E}^{(\bullet)\bullet}). \tag{8.5.3.10.1}$$

Notation 8.5.3.11. Let $(\lambda_1, \chi_1) \leq (\lambda_2, \chi_2)$ in $L(I) \times M(I)$. Let $\mathcal{E}_{\bullet}^{(\bullet)} \in D^\sharp(\mathcal{D}_{\bullet}^{(\bullet)})$. We get the canonical morphism $\chi_1^*\lambda_1^*\mathcal{E}_{\bullet}^{(\bullet)} \to \chi_2^*\lambda_2^*\mathcal{E}_{\bullet}^{(\bullet)}$ which is given by

$$\sigma_{\mathcal{E},(\lambda_2,\chi_2),(\lambda_1,\chi_1)} := \theta_{\lambda_2^*\mathcal{E},(\chi_2,\chi_1)} \circ \chi_1^*(\rho_{\mathcal{E},(\lambda_2,\lambda_1)}) = \chi_2^*(\rho_{\mathcal{E},(\lambda_2,\lambda_1)}) \circ \theta_{\lambda_1^*\mathcal{E},(\chi_2,\chi_1)} \tag{8.5.3.11.1}$$

We set $\sigma_{\mathcal{E},(\lambda_2,\chi_2)} := \sigma_{\mathcal{E},(\lambda_2,\chi_2),(\mathrm{id},0)}$.

Definition 8.5.3.12. For any $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$, let $S^\sharp(\mathcal{D}^{(\bullet)})$ be the collection of morphisms $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^\sharp(\mathcal{D}^{(\bullet)})$ such that there exist $\chi \in M(I)$, $\lambda \in L(I)$ and a morphism $g^{(\bullet)} \colon \mathcal{F}^{(\bullet)} \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \sigma_{\mathcal{E},(\lambda,\chi)}$ and $\chi^*\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \sigma_{\mathcal{F},(\lambda,\chi)}$. If no confusion is possible with respect to $\mathcal{D}^{(\bullet)}$, we simply write $S^\sharp := S^\sharp(\mathcal{D}^{(\bullet)})$. The morphisms of $S^\sharp$ are called lim-ind-isogenies.

Similarly to 8.1.4.9, we have the canonical equivalence of categories

$$S^{\sharp-1} D^\sharp(\mathcal{D}^{(\bullet)}_\bullet) \cong \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet)$$

which is the identity over the objects. f $K(\mathcal{D}^{(\bullet)} \otimes_{\mathcal{R}^{(\bullet)}} \mathcal{D}'^{(\bullet)})$.

8.5.3.13. A morphism $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ is an "ind-isogeny of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$" if there exist $\chi \in M(I)$ and a morphism $g^{(\bullet)} \colon \mathcal{F}^{(\bullet)} \to \chi^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \theta_{\mathcal{E},\chi}$ and $\chi^*(f^{(\bullet)}) \circ g^{(\bullet)} = \theta_{\mathcal{F},\chi}$. We denote by $\Xi^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ the set of ind-isogenies of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$. The localisation of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ with respect to ind-isogenies is denoted by $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$.

We denote by $\Lambda^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ the set of morphisms $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ such that there exist $\lambda \in L(I)$ and a morphism $g^{(\bullet)} \colon \mathcal{F}^{(\bullet)} \to \lambda^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{D}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \rho_{\mathcal{E},\lambda}$ and $\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \rho_{\mathcal{F},\lambda}$ in $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$. The morphisms belonging to $\Lambda^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ are called "lim-isomorphisms". By localising $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ with respect to lim-isomorphisms, we get the category $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$.

Let $S^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ be the collection of morphisms $f^{(\bullet)} \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ of $D^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ such that there exist $\chi \in M(I)$, $\lambda \in L(I)$ and a morphism $g^{(\bullet)} \colon \mathcal{F}^{(\bullet)} \to \chi^*\lambda^*\mathcal{E}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ such that $g^{(\bullet)} \circ f^{(\bullet)} = \sigma_{\mathcal{E},(\lambda,\chi)}$ and $\chi^*\lambda^*(f^{(\bullet)}) \circ g^{(\bullet)} = \sigma_{\mathcal{F},(\lambda,\chi)}$. The morphisms of $S^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ are called "lim-ind-isogenies". Similarly to 8.1.4.9, we have the canonical equivalence of categories

$$S^{\sharp-1} D^\sharp(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet) \cong \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$$

which is the identity over the objects. We define also the abelian category $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$. Similarly to 8.1.5.3 we establish that the canonical functor $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet) \to \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ is fully faithful.

Suppose $(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ is left or right solved by $\mathcal{R}^{(\bullet)}_\bullet$ (see definition 4.6.3.2). Then we denote by $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ (resp. $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{R}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$) the strictly full subcategory of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$ (resp. $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_\bullet, \mathcal{D}'^{(\bullet)}_\bullet)$) consisting of complexes isomorphic to a complex of $K(\mathcal{D}^{(\bullet)}_\bullet \otimes_{\mathcal{R}^{(\bullet)}_\bullet} \mathcal{D}'^{(\bullet)}_\bullet)$.

8.5.3.14. The lemma 8.3.2.2 is still valid by adding some index $\bullet$, i.e. by replacing $X^{(I)}$ by $X^{(I)}_\bullet$ and $\mathcal{D}^{(\bullet)}$ by $\mathcal{D}^{(\bullet)}_\bullet$ etc. For instance, let $\sharp \in \{\emptyset, -\}$, let $\lambda \le \mu$ be two elements of $L(I)$, and $\mathcal{F}^{(\bullet)}_\bullet \in D^\sharp(\lambda^*\mathcal{D}^{(\bullet)}_\bullet)$. The canonical morphism of $\underrightarrow{D}^\sharp_{\mathbb{Q}}(\mu^*\mathcal{D}^{(\bullet)})$

$$\mu^*\mathcal{D}^{(\bullet)}_\bullet \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}_\bullet} \mathcal{F}^{(\bullet)}_\bullet \to \mu^*\mathcal{F}^{(\bullet)}_\bullet \tag{8.5.3.14.1}$$

belongs to $\Lambda(\mu^*\mathcal{D}^{(\bullet)}_\bullet)$.

8.5.4 Quasi-coherence, finite tor dimension, tensor products in $LD_{\mathbb{Q}}$

We keep notation 8.5.2 and we suppose I is a strictly filtered partially ordered set. We set $\mathcal{D}^\dagger = \varinjlim_{i\in I} \mathcal{D}^{(i)}$, $\mathcal{D}'^\dagger = \varinjlim_{i\in I} \mathcal{D}'^{(i)}$, $\mathcal{D}''^\dagger = \varinjlim_{i\in I} \mathcal{D}''^{(i)}$. Let $\sharp \in \{-, \mathrm{b}\}$.

Definition 8.5.4.1. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We say that $\mathcal{E}^{(\bullet)}$ is *quasi-coherent* (as an object of $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$) if it is isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ to a complex $\mathcal{F}^{(\bullet)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ (see notation 8.5.1.8), i.e., following 8.5.1.9 to a complex $\mathcal{F}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$ such that for all $i \in I$, we have $\mathcal{F}^{(i)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(i)})$. We denote by $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ the full subcategory of $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ consisting of quasi-coherent complexes.

Remark 8.5.4.2. With the notation of 8.5.1.1.b, let $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Following 8.5.1.9, the property $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ is equivalent to the property that, for any $m \in \mathbb{Z}$, $\mathcal{E}^{(m)} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Hence, the above definition of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ is equal to that of Berthelot's one formulated in [Ber02, 4.2.3] (without singularities along a divisor).

Proposition 8.5.4.3. *We have the following properties.*

(a) $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ *is a triangulated subcategory of* $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$.

(b) If $u\colon I' \to I$ *is an L-equivalence then the equivalence of 8.3.1.3.1 induces the equivalence of categories*

$$\underrightarrow{u}_X^{-1}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}). \tag{8.5.4.3.1}$$

(c) $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ *(see notation 8.4.1.1) is a full triangulated subcategory of* $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)})$.

Proof. a) Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$ such that for all $i \in I$, we have $\mathcal{E}^{(i)}, \mathcal{F}^{(i)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(i)})$. Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Hence, there exist $\lambda \in L(I)$, $\chi \in M(I)$ and a morphism $u^{(\bullet)}\colon \mathcal{E}^{(\bullet)} \to \chi^*\lambda^*\mathcal{F}^{(\bullet)}$ of $D(\mathcal{D}^{(\bullet)})$ representing f (see 8.1.4.12.1). For any $i \in I$, $u^{(i)}\colon \mathcal{E}^{(i)} \to \mathcal{F}^{(\lambda(i))}$ is a morphism of $D^-_{\mathrm{qc}}(\mathcal{D}^{(i)})$ which is a triangulated subcategory of $D^-(\mathcal{D}^{(i)})$. By using 8.5.1.9, this implies that the cone of $u^{(\bullet)}$ belongs to $D_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$. Hence, we are done.

b) By using 8.5.1.9, we can check that the functor $\underrightarrow{u}_X^{-1}$ factors through $D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}) \to D^-_{\mathrm{qc}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)})$. Hence, we get the functor 8.5.4.3.1. Let $\mathcal{E}^{\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. With the equivalence 8.3.1.3.1, it remains to check that if $\underrightarrow{u}_X^{-1}(\mathcal{E}^{(\bullet)})$ is quasi-coherent, then $\mathcal{E}^{(\bullet)}$ is quasi-coherence. We easily reduce to the case $I' = I$ and $\lambda := u \in L(I)$. In that case, since the canonical morphism $\mathcal{E}^{(\bullet)} \to \lambda^*(\mathcal{E}^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, then we are done.

c) Using theorem 8.4.1.7, we reduce to check that $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ (see notation 8.4.1.1) is a subcategory of $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)})$. Let $\mathcal{E}^{\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. We can suppose there exists $\lambda \in L(I)$ such that $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{coh}}(\lambda^*\mathcal{D}^{(\bullet)})$. Since $\mathcal{D}^{(\lambda(i))}$ is quasi-coherent, then we can apply 7.3.1.16, i.e. $D^-_{\mathrm{coh}}(\mathcal{D}^{(\lambda(i))})$ is a triangulated subcategory of $D^-_{\mathrm{qc}}(\mathcal{D}^{(\lambda(i))})$. Hence, $\mathcal{E}^{(i)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\lambda(i))})$ and therefore $\mathcal{E}^{(i)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(i)})$, i.e. $\mathcal{E}^{(\bullet)} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ (use 8.5.1.9). □

Definition 8.5.4.4. Let $\mathcal{E}^{(\bullet)}_{\bullet} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$. We say that $\mathcal{E}^{(\bullet}$ is *quasi-coherent* if it is isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$ to a complex $\mathcal{F}^{(\bullet)}_{\bullet} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet})$ (see notation 8.5.1.8), i.e., to a complex $\mathcal{F}^{(\bullet)}_{\bullet} \in D^-(\mathcal{D}^{(\bullet)}_{\bullet})$ such that for all $i \in I$, we have $\mathcal{F}^{(i)}_{\bullet} \in D^-_{\mathrm{qc}}(\mathcal{D}^{(i)}_{\bullet})$. We denote by $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet})$ the full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$ consisting of quasi-coherent complexes.

Theorem 8.5.4.5. *Let* $\sharp \in \{-, \mathrm{b}\}$. *In the case where* $\sharp = \mathrm{b}$ *we suppose moreover that* $\mathcal{D}^{(\bullet)}_0$ *(resp.* $gr^{\bullet}_{\mathcal{I}}\mathcal{D}^{(\bullet)}$*) has right finite tor dimension on* $\mathcal{D}^{(\bullet)}$ *(resp.* $\mathcal{D}^{(\bullet)}_0$*). Then the functors* $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ *and* $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ *induce canonically quasi-inverse equivalences of categories between* $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet})$ *and* $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)})$.

Proof. It follows from the theorem 8.5.1.10 that we reduce to check that the functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ of 8.5.1.10 preserve ind-isogenies (resp. lim-isomorphisms), which is a consequence of 8.5.3.3.2 and 8.5.3.9.2. □

Notation 8.5.4.6 (Bimodules case for quasi-coherence in LD categories)**.** We give a quasi-coherent context of the notation and hypotheses of 8.3.1.5 as follows: Let $\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}$ be two sheaves of rings on the topos $X^{(I)}$. Let $\mathcal{R}$ be a sheaf of commutative rings on $\mathfrak{X}$, $\mathcal{K}$ be an ideal of $\mathcal{R}$. We still denote by $\mathcal{R}$ (resp. $\mathcal{K}$) the constant inductive system of rings of X indexed by I with value $\mathcal{R}$ (resp. $\mathcal{K}$). Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ and $(\mathcal{D}^{(\bullet)}, \mathcal{D}''^{(\bullet)}, \mathcal{I})$ are solved by $(\mathcal{R}, \mathcal{K})$ (see definition 8.5.2.4). Set $\mathcal{R}_i := \mathcal{R} \otimes_{\mathcal{K}} \mathcal{K}^{i+1}$ for any $i \in \mathbb{N}$ and $\mathcal{R}_{\bullet} := (\mathcal{R}_i)_{i \in \mathbb{N}}$ the inductive system with the canonical transition map.

Definition 8.5.4.7. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$. We keep the notation and hypotheses of 8.5.4.6. Let $*, ** \in \{l, r\}$. We denote by $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},.,\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ the full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$

(see notation 8.1.4.10) consisting of complexes $\mathcal{E}^{(\bullet)}$ such that there exists $\mathcal{F}^{(\bullet)} \in D^{\sharp}_{\cdot,\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ (see notation 8.5.2.7) together with an isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$. We define similarly $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc},.}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$, $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc},.}({}^*\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$, $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},.,\mathrm{qc}}({}^*\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$.

Lemma 8.5.4.8. *We keep the notation and hypotheses of 8.5.4.6. Let* $*, ** \in \{l, r\}$*.*

(a) The functor $- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} -$ *factors through*

$$- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} - \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, \mathcal{D}^{(\bullet)\mathrm{r}}_{\bullet}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}''^{(\bullet)}_{\bullet}) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}''^{(\bullet)}_{\bullet}). \tag{8.5.4.8.1}$$

(b) The functor 8.5.2.8.3 induces the functor:

$$- \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}''^{(\bullet)}). \tag{8.5.4.8.2}$$

(c) Moreover, both above functors preserve the ($\mathcal{D}^{(\bullet)}$ or $\mathcal{D}'^{(\bullet)}$) quasi-coherence, e.g. we get the factorization

$$- \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc},.}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^*\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc},.}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^*\mathcal{D}''^{(\bullet)}). \tag{8.5.4.8.3}$$

Proof. 1) Let us prove the factorisation 8.5.4.8.2. We can suppose $* = l$ and $** = r$. We need to check that the functor 8.5.2.8.3 sends a lim-ind-isogeny to a lim-ind-isogeny. Let $\mathcal{E}^{(\bullet)} \in D^{-}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}})$, $\mathcal{F}^{(\bullet)} \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{r}\mathcal{D}''^{(\bullet)})$. Let $\chi \in M(I)$, $\lambda \in L(I)$. Set $\mathcal{E}^{(\bullet)}_{\bullet} := \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}(I)}(\mathcal{E}^{(\bullet)}) \in D^{-}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, \mathcal{D}^{(\bullet)\mathrm{r}}_{\bullet})$ and $\mathcal{F}^{(\bullet)}_{\bullet} := \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}(I)}(\mathcal{F}^{(\bullet)}) \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{r}\mathcal{D}''^{(\bullet)}_{\bullet})$.

By using a resolution of $\mathcal{E}^{(\bullet)}_{\bullet}$ by flat left $\mathcal{D}'^{(\bullet)}_{\bullet} \otimes_{\mathcal{R}_{\bullet}} (\mathcal{D}^{(\bullet)}_{\bullet})^{o}$-modules, we can check there exists a morphism $\mathcal{E}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \chi^*\lambda^*\mathcal{F}^{(\bullet)}_{\bullet} \to \chi^*\lambda^*\left(\mathcal{E}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{F}^{(\bullet)}_{\bullet}\right)$ making commutative the canonical diagram

$$\begin{array}{ccc} \mathcal{E}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{F}^{(\bullet)}_{\bullet} & \xrightarrow{\mathrm{id} \otimes \sigma_{\mathcal{F},(\lambda,\chi)}} & \mathcal{E}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \chi^*\lambda^*\mathcal{F}^{(\bullet)}_{\bullet} \\ \downarrow \sigma_{\mathcal{E}\otimes\mathcal{F},(\lambda,\chi)} & \swarrow & \downarrow \sigma_{\mathcal{E}\otimes\chi^*\lambda^*\mathcal{F},(\lambda,\chi)} \\ \chi^*\lambda^*\left(\mathcal{E}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{F}^{(\bullet)}_{\bullet}\right) & \xrightarrow[\chi^*\lambda^*(id\otimes\sigma_{\mathcal{F},(\lambda,\chi)})]{} & \chi^*\lambda^*\left(\mathcal{E}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \chi^*\lambda^*\mathcal{F}^{(\bullet)}_{\bullet}\right), \end{array} \tag{8.5.4.8.4}$$

where σ are the canonical morphisms (see notation 8.5.3.11.1) and where the square is commutative by functoriality. By applying the functor $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*}$ to the diagram 8.5.4.8.4 and by using the commutation isomorphisms 8.5.3.3.2 and 8.5.3.9.2, we get the morphism $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\chi^*\lambda^*\mathcal{F}^{(\bullet)} \to \chi^*\lambda^*\left(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)}\right)$ making commutative the diagram:

$$\begin{array}{ccc} \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)} & \xrightarrow{\mathrm{id} \otimes \sigma_{\mathcal{F},(\lambda,\chi)}} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\chi^*\lambda^*\mathcal{F}^{(\bullet)} \\ \downarrow \sigma_{\mathcal{E}\otimes\mathcal{F},(\lambda,\chi)} & \swarrow & \downarrow \\ \chi^*\lambda^*\left(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)}\right) & \longrightarrow & \chi^*\lambda^*\left(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\chi^*\lambda^*\mathcal{F}^{(\bullet)}\right). \end{array} \tag{8.5.4.8.5}$$

This yields the factorization $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{r}\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^{r}\mathcal{D}''^{(\bullet)})$ and then by functoriality,

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon D^{-}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{r}\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^{r}\mathcal{D}''^{(\bullet)}).$$

Similarly, we get the factorization with respect to the first factor.

2) The factorisation 8.5.4.8.1 is more or less contained in the proof of 1) : this is a consequence of the commutative diagram 8.5.4.8.4.

3) The preservation of quasi-coherence follows from 8.5.2.9. □

Remark 8.5.4.9. We keep the notation and hypotheses of 8.5.4.6. Similarly to 8.5.4.8.1 (more precisely, we use a similar to 8.5.4.8.4 diagram), we check that we have the factorisation:

$$- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} - \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}''^{(\bullet)}). \tag{8.5.4.9.1}$$

But contrary to the complete version, it does not preserves the quasi-coherence.

8.5.4.10. We keep the notation and hypotheses of 8.5.4.6. Let $\lambda \in L(I)$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}})$ (resp. either $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},.,\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}})$ or $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc},.}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}})$). Moreover, let $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{r}\mathcal{D}''^{(\bullet)})$. We have the morphisms (resp. isomorphisms) in $D({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}})$:

$$\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)} \circ \lambda^*(\mathcal{D}^{(\bullet)}) \overset{8.5.3.10.1}{\xrightarrow{\sim}} \lambda^* \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}(\mathcal{D}^{(\bullet)}) \xleftarrow{\sim} \lambda^*(\mathcal{D}^{(\bullet)}_{\bullet}) \tag{8.5.4.10.1}$$

This yields the isomorphisms of $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\lambda^*\mathcal{D}^{(\bullet)\mathrm{r}})$

$$\begin{aligned}\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\lambda^*\mathcal{D}^{(\bullet)} = \mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*}(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}\mathcal{E}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}\lambda^*\mathcal{D}^{(\bullet)}) &\overset{8.5.4.10.1}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*}(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}\mathcal{E}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \lambda^*\mathcal{D}^{(\bullet)}_{\bullet})\\ \underset{8.5.3.14.1}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*} \circ \lambda^* \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}(\mathcal{E}^{(\bullet)}) \overset{8.5.3.9.2}{\xrightarrow{\sim}} \lambda^* \circ \mathbb{R}\underleftarrow{l}_{\mathfrak{X}(I)*} \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(I)}(\mathcal{E}^{(\bullet)}) &\leftarrow \lambda^*\mathcal{E}^{(\bullet)}\end{aligned} \tag{8.5.4.10.2}$$

Then we have in $\underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^*\mathcal{D}''^{(\bullet)})$ the morphisms (resp. isomorphisms):

$$\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)} \overset{8.5.2.11.1}{\xrightarrow{\sim}} (\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\lambda^*\mathcal{D}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)} \underset{8.5.4.10.2}{\xleftarrow{\sim}} (\lambda^*\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)}. \tag{8.5.4.10.3}$$

Similarly, we have in $\underrightarrow{LD}^{-}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^*\mathcal{D}''^{(\bullet)})$ the isomorphisms:

$$\mathcal{E}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \overset{4.6.3.5.1}{\xrightarrow{\sim}} (\mathcal{E}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \lambda^*\mathcal{D}^{(\bullet)}) \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \underset{8.3.2.2.1}{\xrightarrow{\sim}} (\lambda^*\mathcal{E}^{(\bullet)}) \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)}. \tag{8.5.4.10.4}$$

Definition 8.5.4.11. Let $a \leq b$ be two integers. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We say that $\mathcal{E}^{(\bullet)}$ has *has finite tor dimension in* $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ (resp. *LD-tor amplitude in* $[a, b]$) if there exist $\lambda \in L(I)$ and a complex $\mathcal{F}^{(\bullet)} \in D_{\mathrm{tdf}}(\lambda^*\mathcal{D}^{(\bullet)})$ (resp. $\mathcal{F}^{(\bullet)} \in D(\lambda^*\mathcal{D}^{(\bullet)})$ of tor amplitude in $[a, b]$) together with an isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$. We denote by $\underrightarrow{LD}_{\mathbb{Q},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$ the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ consisting of complexes of finite tor dimension in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. We denote by $\underrightarrow{LD}_{\mathbb{Q},\mathrm{qc},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$ the full subcategory of $\underrightarrow{LD}_{\mathbb{Q},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$ consisting of quasi-coherent complexes.

Let $\mathcal{E}^{(\bullet)}_{\bullet} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$. We say that $\mathcal{E}^{(\bullet)}_{\bullet}$ has *finite tor dimension in* $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$ (resp. *LD-tor amplitude in* $[a, b]$) if there exist $\lambda \in L(I)$ and a complex $\mathcal{F}^{(\bullet)}_{\bullet} \in D_{\mathrm{tdf}}(\lambda^*\mathcal{D}^{(\bullet)}_{\bullet})$ (resp. $\mathcal{F}^{(\bullet)}_{\bullet} \in D(\lambda^*\mathcal{D}^{(\bullet)}_{\bullet})$ of tor amplitude in $[a, b]$) together with an isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$ of the form $\mathcal{E}^{(\bullet)}_{\bullet} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}_{\bullet}$. We denote by $\underrightarrow{LD}_{\mathbb{Q},\mathrm{tdf}}(\mathcal{D}^{(\bullet)}_{\bullet})$ the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$ consisting of complexes of finite tor dimension in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{\bullet})$. We denote by $\underrightarrow{LD}_{\mathbb{Q},\mathrm{qc},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$ the full subcategory of $\underrightarrow{LD}_{\mathbb{Q},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$ consisting of quasi-coherent complexes.

Remark 8.5.4.12. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},\mathrm{qc},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$. Then this is not clear there exist $\lambda \in L(I)$ and a complex $\mathcal{F}^{(\bullet)} \in D_{\mathrm{qc},\mathrm{tdf}}(\lambda^*\mathcal{D}^{(\bullet)})$ (see notation 8.5.1.8) together with an isomorphic in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$.

Notation 8.5.4.13. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$. Let $*, ** \in \{l, r\}$. We keep the notation and hypotheses of 8.5.4.6. We denote by $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},.,\mathrm{tdf}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ the full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ consisting of complexes $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ (see notation 8.1.4.10) such that there exist $\lambda \in L(I)$ and $\mathcal{F}^{(\bullet)} \in D^{\sharp}_{.,\mathrm{tdf}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\lambda^*\mathcal{D}'^{(\bullet)})$ (see notation 8.5.2.7) together with an isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$. We define similarly $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{tdf},.}({}^*\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}'^{(\bullet)})$, $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{tdf},.}({}^*\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$, $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},.,\mathrm{tdf}}({}^*\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}'^{(\bullet)}_{\bullet})$.

Lemma 8.5.4.14. *We keep the notation and hypotheses of 8.5.4.6.*

(a) The functor $- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} -$ *factors through*

$$- \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} - \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, \mathcal{D}^{(\bullet)\mathrm{r}}_{\bullet}) \times \underrightarrow{LD}_{\mathbb{Q},.,\mathrm{tdf}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}''^{(\bullet)}_{\bullet}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^*\mathcal{D}'^{(\bullet)}_{\bullet}, \mathcal{R}_{\bullet}, {}^{**}\mathcal{D}''^{(\bullet)}_{\bullet}). \tag{8.5.4.14.1}$$

(b) *The functor 8.5.2.8.3 induces the functor:*

$$- \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times \underrightarrow{LD}_{\mathbb{Q},\mathrm{tdf},.}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^{**}\mathcal{D}''^{(\bullet)}). \tag{8.5.4.14.2}$$

(c) *Moreover, both functors preserve the quasi-coherence, e.g. we get the factorization*

$$- \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}, \mathcal{D}^{(\bullet)\mathrm{r}}) \times \underrightarrow{LD}_{\mathbb{Q},\mathrm{qc},\mathrm{tdf},.}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}, \mathcal{R}, {}^{*}\mathcal{D}''^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{*}\mathcal{D}'^{(\bullet)}, \mathcal{R}, {}^{*}\mathcal{D}''^{(\bullet)}). \tag{8.5.4.14.3}$$

Proof. By using the isomorphisms 8.5.4.10.3, 8.5.4.10.4 and the Lemma 8.3.2.9 we get the boundedness of the functors of Lemma 8.5.4.8. □

Theorem 8.5.4.15. *The functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}}$ induce canonically quasi-inverse equivalences of categories between $D_{\mathrm{qc,tdf}}(\mathcal{D}^{(\bullet)}_{\bullet})$ and $D_{\mathrm{qc,tdf}}(\mathcal{D}^{(\bullet)})$.*

More precisely let $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)})$ (resp. $\mathcal{E}^{(\bullet)}_{\bullet} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{\bullet})$) and $a \leq b$ be two integers. Then $\mathcal{E}^{(\bullet)}$ (resp. $\mathcal{E}^{(\bullet)}_{\bullet}$) has LD-tor amplitude in $[a,b]$ if and only if $\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}}(\mathcal{E}^{(\bullet)})$ (resp. $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}\mathcal{E}^{(\bullet)}_{\bullet}$) has LD-tor amplitude in $[a,b]$.

Proof. It follows from the theorem 8.5.1.10 that we reduce to check that the functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)}*}$ and $\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(I)}}$ of 8.5.1.10 preserve ind-isogenies (resp. lim-isomorphisms), which is a consequence of the functoriality of the isomorphisms 8.5.3.3.2 and 8.5.3.9.2. □

8.5.4.16. Suppose there exists a homomorphism of sheaf of rings on $\mathfrak{X}$ of the form $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ such that the composition of $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}^{(\bullet)}$ with $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ gives $\mathcal{O}^{(\bullet)}_{\mathfrak{X}} \to \mathcal{D}'^{(\bullet)}$.

(a) Let $* \in \{\mathrm{l},\mathrm{r}\}$ and $\star \in \{-,\mathrm{b}\}$. Since the functors $\mathrm{forg}_{\mathcal{D},\mathcal{D}'}$ of 8.5.2.3.1 send a lim-ind-isogeny to a lim-ind-isogeny, then we obtain the factorization of the form:

$$forg_{\mathcal{D},\mathcal{D}'}\colon \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\mathcal{D}^{(\bullet)}), forg_{\mathcal{D},\mathcal{D}'}\colon \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\mathcal{D}'^{(\bullet)}_{\bullet}) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\mathcal{D}^{(\bullet)}_{\bullet}). \tag{8.5.4.16.1}$$

When $\mathcal{D}^{(\bullet)} \to \mathcal{D}'^{(\bullet)}$ is flat, we get the factorization

$$forg_{\mathcal{D},\mathcal{D}'}\colon \underrightarrow{LD}_{\mathbb{Q},\mathrm{qc,tdf}}({}^{*}\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}_{\mathbb{Q},\mathrm{qc,tdf}}({}^{*}\mathcal{D}^{(\bullet)}). \tag{8.5.4.16.2}$$

(b) Following 8.3.2.2.(a), the functor 8.5.2.3.2 sends a lim-ind-isogeny to a lim-ind-isogeny, then we obtain the factorization of the form:

$$\mathcal{D}'^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} -\colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet}) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}_{\bullet}). \tag{8.5.4.16.3}$$

If $\mathcal{E}^{(\bullet)}_{\bullet} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\bullet})$ has LD-tor amplitude in $[a,b]$, then so is $\mathcal{D}'^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\bullet}} \mathcal{E}^{(\bullet)}_{\bullet}$.

(c) Following 8.3.2.2.(a), the functor 8.5.2.3.4 sends a lim-ind-isogeny to a lim-ind-isogeny. We obtain therefore the factorization of the form:

$$\mathcal{D}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}). \tag{8.5.4.16.4}$$

If $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)})$ has LD-tor amplitude in $[a,b]$, then so is $\mathcal{D}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)}$ (use Theorem 8.5.4.15).

Similarly, we get the functor $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{D}'^{(\bullet)}\colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{r}\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{r}\mathcal{D}'^{(\bullet)})$.

Proposition 8.5.4.17. *With the notation 8.5.4.16, the functor 8.5.4.16.4 preserves the coherence and induces:*

$$\mathcal{D}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} -\colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}'^{(\bullet)}). \tag{8.5.4.17.1}$$

Moreover, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$, the canonical morphism of $D^{-}(\mathcal{D}^{(\bullet)})$ (see 8.5.2.2.2)

$$\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \to \mathcal{D}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \tag{8.5.4.17.2}$$

is an isomorphism in $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}'^{(\bullet)})$.

Proof. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. By hypothesis, there exist $\lambda \in L(I)$ and $\mathcal{F}^{(\bullet)} \in D^{-}_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$. This yields the isomorphism of $\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$

$$\mathcal{D}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{D}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)} \underset{8.5.4.10.3}{\xrightarrow{\sim}} \lambda^{*}\mathcal{D}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\lambda^{*}\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)}. \tag{8.5.4.17.3}$$

Since $\mathcal{F}^{(\bullet)} \in D^{-}_{\mathrm{coh}}(\lambda^{*}\mathcal{D}^{(\bullet)})$, then the canonical morphism of $D^{-}(\lambda^{*}\mathcal{D}'^{(\bullet)})$

$$\lambda^{*}\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^{*}\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \to \lambda^{*}\mathcal{D}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\lambda^{*}\mathcal{D}^{(\bullet)}}\mathcal{F}^{(\bullet)} \tag{8.5.4.17.4}$$

is an isomorphism (see 8.5.2.2.2) and therefore its target belongs to $D^{-}_{\mathrm{coh}}(\lambda^{*}\mathcal{D}'^{(\bullet)})$. Hence, it follows from the isomorphism 8.5.4.17.3 that we get the factorization 8.5.4.17.1. We have moreover the isomorphisms of $\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$

$$\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)} \underset{8.5.4.10.4}{\xrightarrow{\sim}} \lambda^{*}\mathcal{D}'^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^{*}\mathcal{D}^{(\bullet)}} \mathcal{F}^{(\bullet)}. \tag{8.5.4.17.5}$$

By composing 8.5.4.17.3, 8.5.4.17.4 and 8.5.4.17.5, we get the isomorphism 8.5.4.17.2.

Similarly, we get the functor $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}}\mathcal{D}'^{(\bullet)} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}({}^{r}\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{coh}}({}^{r}\mathcal{D}'^{(\bullet)})$.

□

8.5.4.18. Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0,p)$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10, $\mathfrak{X}^{\sharp}$ be a log smooth $\mathfrak{S}^{\sharp}$-log-formal scheme. For any $m \in \mathbb{N}$, let $\mathcal{B}^{(m)}_{\mathfrak{X}}$ be a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible structure of $\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$-module and satisfying the hypotheses of 7.3.2. We suppose that, for any $m \in \mathbb{N}$, we have a morphism of $\mathcal{O}_{\mathfrak{X}}$-algebras $\mathcal{B}^{(m)}_{\mathfrak{X}} \to \mathcal{B}^{(m+1)}_{\mathfrak{X}}$ which is moreover $\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$-linear. This yields the homomorphism $\mathcal{B}^{(m)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} \to \mathcal{B}^{(m+1)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m+1)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ of sheaves of rings on $\mathfrak{X}$ and we denote by $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} := (\mathcal{B}^{(m)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})_{m\in\mathbb{N}}$ the corresponding object of $\mathfrak{X}^{(\mathbb{N})}$. Let $* \in \{\mathrm{l},\mathrm{r}\}$. In the case where $\mathcal{D}^{(\bullet)} = \mathcal{B}^{(\bullet)}_{\mathfrak{X}}$ (resp. $\mathcal{D}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$), we set $\mathcal{B}^{(\bullet)}_{X_{\bullet}} := \mathcal{D}^{(\bullet)}_{\bullet}$ (resp. $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} := \mathcal{D}^{(\bullet)}_{\bullet}$). In both cases, we have the ringed topos

$$\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}} = (\underleftarrow{l}^{-1}_{\mathfrak{X}^{(\mathbb{N})}} \dashv \underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}) \colon (X^{(\mathbb{N})}_{\bullet}, \mathcal{D}^{(\bullet)}_{\bullet}) \to (\mathfrak{X}^{(\mathbb{N})}, \mathcal{D}^{(\bullet)}). \tag{8.5.4.18.1}$$

Since $\mathcal{B}^{(\bullet)}_{\mathfrak{X}} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ is flat, for any $\mathcal{E}^{(\bullet)} \in D(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$, it follows from 4.3.4.6.1 that by taking K-flat representation of $\mathcal{E}^{(\bullet)}$ we get the canonical morphism

$$\mathcal{B}^{(\bullet)}_{X_{\bullet}} \otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}} \mathcal{E}^{(\bullet)} \tag{8.5.4.18.2}$$

is an isomorphism.

8.5.4.19. We keep notation 8.5.4.18. Since a resolution of $\mathcal{F}^{(\bullet)}_{\bullet}$ by flat left $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}$-modules is also a resolution by flat $\mathcal{B}^{(\bullet)}_{X_{\bullet}}$-modules, then it follows from 4.2.3.1 that we have the tensor product

$$-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_{\bullet}}} - \colon D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \times D^{-}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \to D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}).$$

By copying the proof of 8.5.4.8, we can check it preserves lim-ind-isogenies and we get therefore the bifunctor

$$-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_{\bullet}}} - \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \tag{8.5.4.19.1}$$

making commutative the diagram

$$\begin{array}{ccc}
\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{X_{\bullet}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{X_{\bullet}}) & \xrightarrow[8.5.4.8.1]{-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_{\bullet}}}-} & \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{X_{\bullet}}) \\
\uparrow & & \uparrow \\
\underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_{\bullet}}}-} & \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}})
\end{array} \tag{8.5.4.19.2}$$

The bifunctor 8.5.4.19.1 preserves the quasi-coherence.

Lemma 8.5.4.20. *Let us keep notation 8.5.4.18. We have the bifunctor*

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}-\colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})\times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})\to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}) \tag{8.5.4.20.1}$$

making commutative the diagram

$$\begin{array}{ccc}
\underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}})\times \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}) & \xrightarrow[8.5.4.8.2]{-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}-} & \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}) \\
\uparrow & & \uparrow \\
\underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})\times \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}) & \xrightarrow{-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}-} & \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}).
\end{array} \tag{8.5.4.20.2}$$

The bifunctor 8.5.4.20.1 preserves the quasi-coherence and corresponds to 8.5.4.19.1 via the equivalence of categories of Theorem 8.5.1.10, i.e. we have the commutative (up to canonical isomorphism) diagram:

$$\begin{array}{ccc}
\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}})\times \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_{\bullet}}}-} & \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}) \\
\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}\times\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}\uparrow\ \downarrow\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}\times\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*} & & \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}\uparrow\ \downarrow\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*} \\
\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})\times \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}) & \xrightarrow{-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}-} & \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}).
\end{array} \tag{8.5.4.20.3}$$

Proof. 1) First, we check we have a commutative diagram like 8.5.4.20.2 but with $\underrightarrow{LD}^{-}_{\mathbb{Q}}$ replaced by D^{-}. Let $\mathcal{E}^{(\bullet)}\in D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$. It follows from 4.4.4.1.4, that the canonical morphism

$$\mathcal{B}^{(\bullet)}_{X_{\bullet}}\otimes_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)}\to \widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}\otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}}\mathcal{E}^{(\bullet)}=:\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)})$$

is an isomorphism in $D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}})$. We set $\mathcal{E}^{(\bullet)}_{\bullet}:=\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)})\in D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}})$, $\mathcal{F}^{(\bullet)}_{\bullet}:=\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{F}^{(\bullet)})\in D^{-}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}})$. Hence, by applying $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}$ and by definition (see 8.5.2.2.1), we get

$$\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}\mathcal{F}^{(\bullet)}\xrightarrow{\sim}\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}\left(\mathcal{E}^{(\bullet)}_{\bullet}\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_{\bullet}}}\mathcal{F}^{(\bullet)}_{\bullet}\right)\in D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}). \tag{8.5.4.20.4}$$

2) We can copy the proof of 8.5.4.8 to get the localized functor with respect to lim-ind-isogenies. Moreover, by construction we get the commutative diagram 8.5.4.20.3 up to canonical isomorphism. □

Lemma 8.5.4.21. *Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic $(0,p)$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10, $\mathfrak{X}^{\sharp}$ be a log smooth $\mathfrak{S}^{\sharp}$-log-formal scheme. For any $m\in\mathbb{N}$, let $\mathcal{B}^{(m)}_{\mathfrak{X}}$ and $\mathcal{C}^{(m)}_{\mathfrak{X}}$ be two commutative $\mathcal{O}_{\mathfrak{X}}$-algebras endowed with a compatible structure of $\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$-module and satisfying the hypotheses of 7.3.2. We suppose that, for any $m\in\mathbb{N}$, we have the commutative diagram*

$$\begin{array}{ccc}
\mathcal{B}^{(m)}_{\mathfrak{X}} & \longrightarrow & \mathcal{B}^{(m+1)}_{\mathfrak{X}} \\
\downarrow & & \downarrow \\
\mathcal{C}^{(m)}_{\mathfrak{X}} & \longrightarrow & \mathcal{C}^{(m+1)}_{\mathfrak{X}}
\end{array} \tag{8.5.4.21.1}$$

whose arrows are both morphisms of $\mathcal{O}_{\mathfrak{X}}$-algebras and morphisms of left $\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$-modules. With notation 8.5.4.20, this yields the homomorphism of sheaves of rings on $\mathfrak{X}^{(\mathbb{N})}$ of the form $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}\to\mathcal{C}^{(\bullet)}_{\mathfrak{X}}$ and $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}\to\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$.

(a) *For any* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$*, we have the canonical morphism of* $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$

$$\mathcal{F}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{C}^{(\bullet)}_{\mathfrak{X}}}\left(\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)}\right). \tag{8.5.4.21.2}$$

(b) *For any* $\mathcal{E}^{(\bullet)} \in D^{-}({}^{\mathrm{l}}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$ *the canonical morphism of* $D^{-}({}^{\mathrm{l}}\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$

$$\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)} \to (\mathcal{C}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}}\mathcal{E}^{(\bullet)} \tag{8.5.4.21.3}$$

is an isomorphism.

Proof. By using 4.3.4.12.1, we get the isomorphism of $D^{-}({}^{\mathrm{l}}\mathcal{C}^{(\bullet)}_{X_\bullet} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}})$

$$\mathcal{F}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_\bullet}} \mathcal{E}^{(\bullet)}_{\bullet} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}_{\bullet} \otimes^{\mathbb{L}}_{\mathcal{C}^{(\bullet)}_{X_\bullet}} \left(\mathcal{C}^{(\bullet)}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_\bullet}} \mathcal{E}^{(\bullet)}_{\bullet}\right). \tag{8.5.4.21.4}$$

Hence, via the quasi-inverse equivalences of categories of 8.5.4.5, we get 8.5.4.21.2 from 8.5.4.21.4. Similarly, the isomorphism 8.5.4.21.3 follows from 4.3.4.6.1 and from the quasi-inverse equivalences of categories of 8.5.4.5. □

Remark 8.5.4.22. Beware the isomorphism 8.5.4.21.2 might be wrong if complexes are not quasi-coherent.

8.5.5 Strongly quasi-flat relative I-ringed $\mathcal{V}$-log formal schemes

In order to give examples when we can apply for instance theorems 8.5.4.5 and 8.5.4.15 in the context of bounded quasi-coherent complexes, let us introduce the following notion of relative I-ringed $\mathcal{V}$-log formal scheme.

Definition 8.5.5.1. We fix I is a partially ordered set. For convenience, let us define the following categories.

(a) We define the category of "I-ringed $\mathcal{V}$-log formal scheme" as follows:

   (i) Its objects consist in pairs $(\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}})$ where $\mathfrak{X}^{\sharp}$ is a $\mathcal{V}$-log formal scheme, $\mathfrak{X}$ is the underlying topological space of $\mathfrak{X}^{\sharp}$, $\mathfrak{X}^{(\bullet)} = \mathfrak{X}^{(I)}$, and $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}$ is a commutative $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$-algebra. The underlying ringed I-topos of $(\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}})$ is $(\mathfrak{X}^{(\bullet)}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}})$.

   (ii) A morphism of I-ringed $\mathcal{V}$-log formal scheme $(\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}}) \to (\mathfrak{Y}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}})$ is the data a morphism of log schemes of the form $\alpha\colon \mathfrak{X}^{\sharp} \to \mathfrak{Y}^{\sharp}$ and of a morphism of $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$-algebras $\alpha^{*}\mathcal{B}^{(\bullet)}_{\mathfrak{Y}} \to \mathcal{B}^{(\bullet)}_{\mathfrak{X}}$.

(b) We define the category of "relative I-ringed $\mathcal{V}$-log formal scheme" as follows: a "relative I-ringed $\mathcal{V}$-log formal scheme" $(\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}})/(\mathfrak{Y}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}})$ is a morphism of I-ringed $\mathcal{V}$-log formal schemes of the form $(\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}}) \to (\mathfrak{Y}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}})$. A morphism of relative I-ringed $\mathcal{V}$-log formal schemes $(\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}})/(\mathfrak{Y}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}}) \to (\mathfrak{X}'^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}'})/(\mathfrak{Y}'^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}'})$ is a commutative square of the form

$$\begin{array}{ccc} (\mathfrak{X}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}}) & \xrightarrow{f} & (\mathfrak{X}'^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{X}'}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ (\mathfrak{Y}^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}}) & \xrightarrow{g} & (\mathfrak{Y}'^{\sharp}, \mathcal{B}^{(\bullet)}_{\mathfrak{Y}'}), \end{array} \tag{8.5.5.1.1}$$

such that the arrows are morphisms of I-ringed $\mathcal{V}$-log formal schemes.

Example 8.5.5.2. A (logarithmic) $\mathcal{V}$-formal scheme can be viewed as a I-ringed $\mathcal{V}$-log formal scheme via the faithfully flat functor given by $\mathfrak{X}^{\sharp} \mapsto (\mathfrak{X}^{\sharp}, \mathcal{O}^{(\bullet)}_{\mathfrak{X}})$. By abuse of notation, we can simply write $\mathfrak{X}^{\sharp}$ for $(\mathfrak{X}^{\sharp}, \mathcal{O}^{(\bullet)}_{\mathfrak{X}})$.

Definition 8.5.5.3. Let $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(\bullet)})/\mathfrak{S}^\sharp$ be a relative I-ringed $\mathcal{V}$-log formal scheme.

(a) We say that $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(\bullet)})/\mathfrak{S}^\sharp$ is quasi-flat, if there exists a morphism $\mathfrak{S} \to \mathfrak{T}$ of $\mathcal{V}$-formal schemes such that the induced morphism of ringed spaces $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(i)}) \to \mathfrak{T}$ is flat, for any $i \in I$.

(b) We say that $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(\bullet)})/\mathfrak{S}^\sharp$ is strongly quasi-flat if there exists a morphism $\mathfrak{S} \to \mathfrak{T}$ of $\mathcal{V}$-formal schemes such that the induced morphism of ringed spaces $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(i)}) \to \mathfrak{T}$ is flat for any $i \in I$ and such that, denoting by $\mathcal{I}_{\mathfrak{T}}$ an ideal of definition of $\mathfrak{T}$, the sheaf $\mathcal{O}_{T_0}$ (resp. $gr^{\bullet}_{\mathcal{I}_{\mathfrak{T}}} \mathcal{O}_{\mathfrak{T}}$) has finite tor dimension on $\mathcal{O}_{\mathfrak{T}}$ (resp. $\mathcal{O}_{T_0}$).

(c) Let $\widetilde{\alpha} \colon (\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(\bullet)})/\mathfrak{S}^\sharp \to (\mathfrak{X}'^\sharp, \mathcal{B}_{\mathfrak{X}'}^{(\bullet)})/\mathfrak{S}'^\sharp$ be a morphism of relative I-ringed $\mathcal{V}$-log formal schemes. We say that $\widetilde{\alpha}$ is "strongly quasi-flat" if there exists a morphism $\mathfrak{S} \to \mathfrak{T}$ of $\mathcal{V}$-formal schemes such that both induced morphisms of ringed spaces $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(i)}) \to \mathfrak{T}$ and $(\mathfrak{X}'^\sharp, \mathcal{B}_{\mathfrak{X}'}^{(i)}) \to \mathfrak{T}$ are flat for any $i \in I$ and such that denoting by $\mathcal{I}_{\mathfrak{T}}$ an ideal of definition of $\mathfrak{T}$ the sheaf $\mathcal{O}_{T_0}$ (resp. $gr^{\bullet}_{\mathcal{I}_{\mathfrak{T}}} \mathcal{O}_{\mathfrak{T}}$) has finite tor dimension on $\mathcal{O}_{\mathfrak{T}}$ (resp. $\mathcal{O}_{T_0}$). Remark that in both $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(\bullet)})/\mathfrak{S}^\sharp$ and $(\mathfrak{X}'^\sharp, \mathcal{B}_{\mathfrak{X}'}^{(\bullet)})/\mathfrak{S}^\sharp$ are strongly quasi-flat.

8.5.5.4. The notion of strong quasi-flatness is interesting because of the following: With notation 8.5.4.18, suppose that $(\mathfrak{X}^\sharp, \mathcal{B}_{\mathfrak{X}}^{(\bullet)})/\mathfrak{S}^\sharp$ is strongly quasi-flat in the sense of 8.5.5.3. Set $\mathcal{D}^{(\bullet)} := \mathcal{B}_{\mathfrak{X}}^{(\bullet)}$ (resp. $\mathcal{D}^{(\bullet)} := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$). By definition (resp. and by flatness of the homomorphism $\mathcal{B}_{\mathfrak{X}}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$), we get that $\mathcal{D}_0^{(\bullet)}$ (resp. $gr^{\bullet}_{\mathcal{I}} \mathcal{D}^{(\bullet)}$) has right and left finite tor dimension on $\mathcal{D}^{(\bullet)}$ (resp. $\mathcal{D}_0^{(\bullet)}$). Hence, we can apply for instance theorems 8.5.4.5 and 8.5.4.15.

The main example of such context will be given at 8.7.4.2.

8.6 Duality

Let I be a partially ordered set and let X be a topological space. Let $\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}$ be two sheaves of rings on the topos $X^{(I)}$. Let $\mathcal{R}$ be a sheaf of commutative rings on $\mathfrak{X}$, $\mathcal{K}$ be an ideal of $\mathcal{R}$. We still denote by $\mathcal{R}$ (resp. $\mathcal{K}$) the constant inductive system of rings of X indexed by I with value $\mathcal{R}$ (resp. $\mathcal{K}$). Suppose $(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)}, \mathcal{I})$ is solved by $(\mathcal{R}, \mathcal{K})$ (see definition 8.5.2.4).

8.6.1 Perfectness in $LD_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$, comparison with the $\mathcal{D}^{\dagger}_{\mathbb{Q}}$-perfectness

Definition 8.6.1.1. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. The complex $\mathcal{E}^{(\bullet)}$ is said to be perfect if there exist $\lambda \in L(I)$ and $\mathcal{F}^{(\bullet)} \in D_{\mathrm{perf}}(\lambda^* \mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$.

Remark 8.6.1.2. Beware that this is not clear that the perfectness in the sense of 8.6.1.1 is local on X or on $X^{(\bullet)}$.

8.6.1.3. Let $\lambda \in L(I)$. Following 7.1.3.13, $\mathcal{F}^{(\bullet)} \in D_{\mathrm{perf}}(\lambda^* \mathcal{D}^{(\bullet)})$ means that $\mathcal{F}^{(\bullet)}$ satisfies the following conditions:

(i) For any $i \in I$, $\mathcal{F}^{(i)} \in D_{\mathrm{perf}}(\mathcal{D}^{(\lambda(i))})$;

(ii) For any $i, j \in I$ such that $i \leq j$, the canonical homomorphism

$$\mathcal{D}^{(\lambda(j))} \otimes^{\mathbb{L}}_{\mathcal{D}^{(\lambda(i))}} \mathcal{F}^{(\lambda(i))} \to \mathcal{F}^{(\lambda(j))} \tag{8.6.1.3.1}$$

is an isomorphism.

Notation 8.6.1.4. Let $\sharp \in \{\emptyset, -, \mathrm{b}, 0\}$. We denote by $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})$ the strictly full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ consisting of perfect complexes. The category $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})$ is a strictly full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$ and of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{tdf}}(\mathcal{D}^{(\bullet)})$ (see notation 8.5.4.11).

Lemma 8.6.1.5. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$. Let $\lambda \leq \mu$ be two elements of $L(I)$. Consider the following two properties.*

(a) There exists $\mathcal{F}^{(\bullet)} \in D_{\mathrm{perf}}(\lambda^*\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$.

(b) There exists $\mathcal{G}^{(\bullet)} \in D_{\mathrm{perf}}(\mu^*\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$.

Then (a) $\Rightarrow$ (b).

Proof. Using the fact the functor $\mu^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} -$ preserves the perfectness, i.e. induces $\mu^*\mathcal{D}^{(\bullet)} \otimes^{\mathbb{L}}_{\lambda^*\mathcal{D}^{(\bullet)}} - \colon D_{\mathrm{perf}}(\lambda^*\mathcal{D}^{(\bullet)}) \to D_{\mathrm{perf}}(\mu^*\mathcal{D}^{(\bullet)})$, we can copy the proof of 8.4.1.4. □

Proposition 8.6.1.6. *Let $u\colon I' \to I$ be an L-equivalence between two partially ordered sets (see definition 8.1.3.8). Let $\sharp \in \{\emptyset, -, \mathrm{b}, 0\}$. The equivalence of categories $\underrightarrow{u}_X^{-1}$ of 8.3.1.3.1 preserves the perfectness, i.e. it induces the equivalence of categories*

$$\underrightarrow{u}_X^{-1}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}). \tag{8.6.1.6.1}$$

Suppose $I' = I$ and $u \in L(I)$. Denoting by $\lambda := u$, we have the forgetful functor $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}\,\mathrm{perf}}(\lambda^*\mathcal{D}^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q}\,\mathrm{perf}}(\mathcal{D}^{(\bullet)})$ which is a quasi-inverse equivalence of 8.6.1.6.1.

Proof. For any $\mu \in L(I)$, by using the functor $\underrightarrow{u}_X^{-1}$ preserves the perfectness, i.e. induces $\underrightarrow{u}_X^{-1}\colon D_{\mathrm{perf}}(\mu^*\mathcal{D}^{(\bullet)}) \to D_{\mathrm{perf}}(\underrightarrow{u}_X^{-1}\mu^*\mathcal{D}^{(\bullet)})$, we can copy the proof of 8.4.1.5. □

Proposition 8.6.1.7. *Suppose I has a smallest element or is strictly filtered (see 8.1.3.8). Let $\sharp \in \{\emptyset, -, \mathrm{b}\}$. Then the subcategory $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})$ of $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ is a triangulated subcategory.*

Proof. We can copy the proof of 8.4.1.7. □

Theorem 8.6.1.8. *Suppose that either I is filtered and has a smallest element or I is strictly filtered. Suppose moreover that X is coherent. Suppose the following conditions are satisfied*

(i) The rings $\mathcal{D}^{(i)}$ are coherent sheaves for any $i \in I$

(ii) The transition maps $\mathcal{D}^{(i)}_{\mathbb{Q}} \to \mathcal{D}^{(j)}_{\mathbb{Q}}$ are flat for any elements $i \leq j$ of I;

(iii) There exists an integer d such that, for any elements $i \leq j$ of I, the ring $\mathcal{D}^{(j)}$ is of tor-dimension $\leq d$ on $\mathcal{D}^{(i)}$;

(iv) For any $i \in I$, for any coherent $\mathcal{D}^{(i)}_{\mathbb{Q}}$-module $\mathcal{E}$, there exists a coherent $\mathcal{D}^{(i)}$-module $\mathcal{E}'$ together with an isomorphism $\mathcal{E}'_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ of $\mathcal{D}^{(i)}_{\mathbb{Q}}$-modules.

Under these hypotheses, the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ induces a fully faithful functor $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$.

Proof. The factorisation of $\underrightarrow{l}^*_{\mathbb{Q}}$ is obvious and the fully faithfulness follows from 8.4.1.15. □

Remark 8.6.1.9. Keep notation and hypotheses of 8.6.1.8. This is not clear that the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ induces a fully faithful functor $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)}) \to D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$. It follows from 7.1.3.12 that if the ring $\mathcal{D}^{(i)}$ has finite tor dimension for any $i \in I$ then by using 1.4.3.29, we get $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)}) = \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)})$. Moreover, in the case $\mathcal{D}^{\dagger}$ has also finite tor dimension, we get $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathbb{Q}}) = D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathbb{Q}})$ (use loc.cit.).

Let us finish the subsection by considering the bimodules case.

Definition 8.6.1.10. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$. We denote by $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},.,\mathrm{perf}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ the full subcategory of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that there exist $\lambda \in L(I)$ and $\mathcal{F}^{(\bullet)} \in D^{\sharp}_{.,\mathrm{perf}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$. We define similarly $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf},.}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$.

Remark 8.6.1.11. Recall the category $D^{\sharp}_{.,\mathrm{perf}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)})$ is the strictly full subcategory of $D^{\sharp}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)})$ consisting in complexes whose image in $D^{\sharp}(\mathcal{D}'^{(\bullet)})$ belongs to $D^{\sharp}_{\mathrm{perf}}(\mathcal{D}'^{(\bullet)})$. However, this is not clear that if $\mathcal{E}^{(\bullet)}$ is a complex of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ whose image in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$ belongs to $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}'^{(\bullet)})$, then $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},.,\mathrm{perf}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$.

Lemma 8.6.1.12. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$. Let $\lambda \leq \mu$ be two elements of $L(I)$. Consider the following two properties.*

(a) There exists $\mathcal{F}^{(\bullet)} \in D_{\mathrm{perf},.}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$.

(b) There exists $\mathcal{G}^{(\bullet)} \in D_{\mathrm{perf},.}(\mu^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ of the form $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$.

Then (a) ⇒ (b).

Proof. Using the fact the functor $\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda^*\mathcal{D}^{(\bullet)}} -$ preserves the perfectness, i.e. induces $\mu^*\mathcal{D}^{(\bullet)} \overset{\mathbb{L}}{\otimes}_{\lambda^*\mathcal{D}^{(\bullet)}} - \colon D_{\mathrm{perf}}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to D_{\mathrm{perf}}(\mu^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$, by using 8.3.2.9.(b), we can copy the proof of 8.4.1.4. □

Lemma 8.6.1.13. *Let λ, μ be two elements of $L(I)$. Let $\mathcal{E}^{(\bullet)} \in D_{\mathrm{perf},.}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$. Then $\mu^*\mathcal{E}^{(\bullet)} \in D_{\mathrm{perf},.}(\mu^*\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \mu^*\mathcal{D}'^{(\bullet)})$.*

Proof. This is obvious from 8.6.1.3. □

Proposition 8.6.1.14. *Let $u\colon \widetilde{I} \to I$ be an increasing map of partially ordered sets which is an L-equivalence (see definition 8.1.3.8). The equivalence of categories $\underrightarrow{u}_X^{-1}$ of 8.3.1.7.1 preserves the perfectness, i.e. it induces the equivalence of categories*

$$\underrightarrow{u}_X^{-1} \colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf},.}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf},.}(\underrightarrow{u}_X^{-1}\mathcal{D}^{(\bullet)}, \mathcal{R}, \underrightarrow{u}_X^{-1}\mathcal{D}'^{(\bullet)}). \tag{8.6.1.14.1}$$

Suppose $\widetilde{I} = I$ and $u \in L(I)$. Denoting by $\lambda := u$, the forgetful functor $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf},.}(\lambda^*\mathcal{D}^{(\bullet)}, \mathcal{R}, \lambda^*\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf},.}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ is a quasi-inverse equivalence of 8.6.1.14.1.

Proof. We proceed similarly to the proofs 8.3.1.3 and 8.4.1.5 (use also 8.6.1.13). □

Proposition 8.6.1.15. *Suppose I has a smallest element or is strictly filtered (see 8.1.3.8). Let $\sharp \in \{\emptyset, -, \mathrm{b}\}$. Then the subcategory $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ of $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$ is a triangulated subcategory.*

Proof. We can copy the proof of 8.4.1.7. □

8.6.2 Dual functor

Suppose that either I is filtered and has a smallest element or I is strictly filtered.

8.6.2.1. Since $L(I) \times M(I)$ is a filtered set, then we get the functor

$$\mathscr{F}(-,-) := \underset{(\lambda,\chi)\in L(I)\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-) \colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D^+(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \mathrm{Ind}\, \underrightarrow{LD}^+_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)}) \tag{8.6.2.1.1}$$

where $Q_{LD}\colon D^+({}^r\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^+_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)})$ is the localisation functor, where $\underset{(\lambda,\chi)\in L(I)\times M(I)}{\text{“}\varinjlim\text{”}}$ means the inductive limits in the category of ind-objects of $\underrightarrow{LD}^+_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)})$, and where

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\ \chi^*\lambda^*-) \colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times D^+(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to D^+({}^r\mathcal{D}'^{(\bullet)})$$

is the functor defined similarly to 8.5.2.8.1.

Lemma 8.6.2.2. *Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{E}'^{(\bullet)}$ be a morphism belonging to $S^{-}(\mathcal{D}^{(\bullet)})$. Let $g\colon \mathcal{F}^{(\bullet)} \to \mathcal{F}'^{(\bullet)}$ be a morphism belonging to $S^{+}(\mathcal{D}^{(\bullet)}, \mathcal{D}'^{(\bullet)})$ (see notation 8.1.4.10) between two objects of $D^{+}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$. Then the canonical morphism*

$$\mathscr{F}(\mathcal{E}'^{(\bullet)}, \mathcal{F}^{(\bullet)}) \to \mathscr{F}(\mathcal{E}^{(\bullet)}, \mathcal{F}'^{(\bullet)}) \tag{8.6.2.2.1}$$

is an isomorphism.

Proof. We easily reduce to the following two cases.

1) Suppose there exists $\chi_0 \in M(I)$ and $\lambda_0 \in L(I)$ such that $g = \sigma_{\mathcal{F},(\lambda_0,\chi_0)}$ and $f = \mathrm{id}$ (see notation 8.1.4.1). Then we can copy the part 1) of the proof the 8.2.4.21.

2) Suppose there exist $\chi_0 \in M(I)$ and $\lambda_0 \in L(I)$ such that $f = \sigma_{\mathcal{E},(\lambda_0,\chi_0)}$. We copy the part 2) of the proof the 8.2.4.21 as follows: We denote by $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$ (resp. $\mathcal{D}'^{(\bullet),(\bullet,\bullet)}$, resp. $\mathcal{R}$) the constant inductive system of rings of $X^{(I)}$ indexed by $L(I) \times M(I)$ with value $\mathcal{D}^{(\bullet)}$ (resp. $\mathcal{D}'^{(\bullet)}$, resp $\mathcal{R}$). Let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^{+}(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)} \otimes_{\mathcal{R}} (\mathcal{D}'^{(\bullet),(\bullet,\bullet)})^{o}$-modules representing $\mathfrak{c}(\mathcal{F}^{(\bullet)})$. Let $\mathcal{I}'^{(\bullet),(\bullet,\bullet)} \in K^{+}(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)} \otimes_{\mathcal{R}} (\mathcal{D}'^{(\bullet),(\bullet,\bullet)})^{o}$-modules endowed with a quasi-isomorphism $\chi_0^{*}\lambda_0^{*}\mathcal{I}^{(\bullet),(\bullet,\bullet)} \to \mathcal{I}'^{(\bullet),(\bullet,\bullet)}$ of $K^{+}(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$, where $\chi_0^{*}\lambda_0^{*}\mathcal{I}^{(\bullet),(\bullet,\bullet)} := (\chi_0^{*}\lambda_0^{*}\mathcal{I}^{(\bullet),(\lambda,\chi)})_{\lambda,\chi}$. Consider the following diagram of $K^{+}(\mathcal{D}'^{(\bullet)})$:

$$\begin{array}{ccc}
\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi_0^{*}\lambda_0^{*}\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(5)} & \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) \\
\downarrow{\scriptstyle (1)} & \swarrow{\scriptstyle (\star)} & \downarrow{\scriptstyle (3)} \\
\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi_0^{*}\lambda_0^{*}\mathcal{E}^{(\bullet)}, \chi_0^{*}\lambda_0^{*}\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(6)} & \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi_0^{*}\lambda_0^{*}\mathcal{I}^{(\bullet),(\lambda,\chi)}) \\
\downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (4)} \\
\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\chi_0^{*}\lambda_0^{*}\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(7)} & \mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)}),
\end{array} \tag{8.6.2.2.2}$$

where the numbered arrows are given by functoriality of the bifunctor $\mathscr{H}om^{\bullet}_{\mathcal{D}^{(\bullet)}}(-,-)$ and where the arrow $(\star)$ is defined by functoriality from the functor $\chi_0^{*}\lambda_0^{*}$ (i.e. is the composition of 8.2.4.18.3 with the first morphism of 8.2.4.19.1). It follows from the $\mathscr{H}om$ version of the commutative diagram 8.2.4.20.1 that we get the commutativity of both triangles of the diagram 8.6.2.2.2. Since its bottom square is commutative by functoriality, then the diagram 8.6.2.2.2 is commutative.

Let $Q^{\mathrm{qi}}_{LD}\colon K^{+}({}^{r}\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{+}_{\mathbb{Q}}({}^{r}\mathcal{D}'^{(\bullet)})$ be the localisation functor. By applying the functor "$\varinjlim_{(\lambda,\chi)}$" Q^{qi}_{LD} to the diagram 8.6.2.2.2 we get the commutative diagram

$$\begin{array}{ccc}
\text{“}\varinjlim_{(\lambda,\chi)}\text{”}\, Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\chi_0^{*}\lambda_0^{*}\mathcal{E}^{(\bullet)}, \chi^{*}\lambda^{*}\mathcal{D}^{(\bullet)}) & \xrightarrow{(5)} & \text{“}\varinjlim_{(\lambda,\chi)}\text{”}\, Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi^{*}\lambda^{*}\mathcal{D}^{(\bullet)}) \\
\downarrow{\scriptstyle (2)\circ(1)} & \swarrow{\scriptstyle (2)\circ(\star)} & \downarrow{\scriptstyle (4)\circ(3)} \\
\text{“}\varinjlim_{(\lambda,\chi)}\text{”}\, Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\chi_0^{*}\lambda_0^{*}\mathcal{E}^{(\bullet)}, \chi_0^{*}\lambda_0^{*}\chi^{*}\lambda^{*}\mathcal{D}^{(\bullet)}) & \xrightarrow{(7)} & \text{“}\varinjlim_{(\lambda,\chi)}\text{”}\, Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \chi_0^{*}\lambda_0^{*}\chi^{*}\lambda^{*}\mathcal{D}^{(\bullet)})
\end{array} \tag{8.6.2.2.3}$$

where in the notation concerning the arrows, we have omitted indicating "$\varinjlim_{(\lambda,\chi)}$" Q^{qi}_{LD}. Remark that the top morphism (5) is 8.6.2.2.1. By copying the part 2.iii) of the proof of 8.2.4.21, we get that both vertical arrows of 8.6.2.2.3 are isomorphisms. Hence, we are done. □

8.6.2.3. It follows from 8.6.2.2 that the functor $\mathscr{F}$ of 8.6.2.1.1 induces the functor

$$\mathscr{F}(-,-)\colon \underrightarrow{LD}^{-}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{+}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to \mathrm{Ind}\, \underrightarrow{LD}^{+}_{\mathbb{Q}}({}^{r}\mathcal{D}'^{(\bullet)}). \tag{8.6.2.3.1}$$

Lemma 8.6.2.4. *Let $\lambda_0 \in L(I)$, $\mathcal{E}^{(\bullet)} \in D^-(\mathcal{D}^{(\bullet)})$ and $\mathcal{F}^{(\bullet)} \in D^+(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$. The canonical morphism*

$$\underset{(\lambda,\chi)\in L(I)^{\lambda_0}\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \underset{(\lambda,\chi)\in L(I)\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)})$$

is an isomorphism.

Proof. We reduce to check

$$\underset{(\lambda,\chi)\in L(I)^{\lambda_0}\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \underset{(\lambda,\chi)\in L(I)^{\lambda_0}\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)})$$

is an isomorphism. Only in this proof, we denote by $\mathcal{D}^{(\bullet),(\bullet,\bullet)}$ (resp. $\mathcal{D}'^{(\bullet),(\bullet,\bullet)}$, resp. $\mathcal{R}$) the constant inductive system of sheaves of rings on $X^{(I)}$ indexed by $L(I)^{\lambda_0} \times M(I)$ with value $\mathcal{D}^{(\bullet)}$ (resp $\mathcal{D}'^{(\bullet)}$, resp. $\mathcal{R}$). We denote by $\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)}$ (resp. $\widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)}$, resp. $\widetilde{\mathcal{R}}$) the constant inductive system of rings of $X^{(I)}$ indexed by $L(I)^{\lambda_0} \times M(I)$ with value $\lambda_0^*\mathcal{D}^{(\bullet)}$ (resp. $\lambda_0^*\mathcal{D}'^{(\bullet)}$, resp $\mathcal{R}$). We define the functor $\widetilde{\mathfrak{c}}\colon K^+(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)}) \to K^+(\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})$ by setting, for any $\mathcal{G}^{(\bullet)} \in K^+(\mathcal{D}^{(\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet)})$, for any $\lambda \geq \lambda_0$ in $L(I)$ and any $\chi \in M(I)$, $\widetilde{\mathfrak{c}}(\mathcal{G}^{(\bullet)})^{(\lambda,\chi)} := \chi^*\lambda^*(\mathcal{G}^{(\bullet)})$ and where for any $(\lambda_1, \chi_1) \leq (\lambda_2, \chi_2)$ the transition maps $\chi_1^*\lambda_1^*\mathcal{G}^{(\bullet)} \to \chi_2^*\lambda_2^*\mathcal{G}^{(\bullet)}$ are the canonical ones i.e. are equal to $\sigma_{\mathcal{G},(\lambda_2,\chi_2),(\lambda_1,\chi_1)}$ (see 8.1.4.1.1).

Let $\widetilde{\mathcal{I}}^{(\bullet),(\bullet,\bullet)} \in K^+(\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)} \otimes_{\widetilde{\mathcal{R}}} (\widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})^o$-modules representing $\widetilde{\mathfrak{c}}(\mathcal{F}^{(\bullet)})$. Let $\mathcal{I}^{(\bullet),(\bullet,\bullet)} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)} \otimes_{\mathcal{R}} (\mathcal{D}'^{(\bullet),(\bullet,\bullet)})^o$-modules endowed with a quasi-isomorphism $\widetilde{\mathcal{I}}^{(\bullet),(\bullet,\bullet)} \to \mathcal{I}^{(\bullet),(\bullet,\bullet)}$ of the category $K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$.

Let $\widetilde{\mathcal{I}}'^{(\bullet),(\bullet,\bullet)} \in K^+(\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)} \otimes_{\widetilde{\mathcal{R}}} (\widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})^o$-modules endowed with a quasi-isomorphism $\lambda_0^*\mathcal{I}^{(\bullet),(\bullet,\bullet)} \to \widetilde{\mathcal{I}}'^{(\bullet),(\bullet,\bullet)}$ of $K^+(\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})$, where $\lambda_0^*\mathcal{I}^{(\bullet),(\bullet,\bullet)} = (\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)})_{\lambda,\chi}$. Let $\mathcal{I}'^{(\bullet),(\bullet,\bullet)} \in K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$ be a complex of injective left $\mathcal{D}^{(\bullet),(\bullet,\bullet)} \otimes_{\mathcal{R}} (\mathcal{D}'^{(\bullet),(\bullet,\bullet)})^o$-modules endowed with a quasi-isomorphism $\widetilde{\mathcal{I}}'^{(\bullet),(\bullet,\bullet)} \to \mathcal{I}'^{(\bullet),(\bullet,\bullet)}$ of $K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)}, \mathcal{R}, \mathcal{D}'^{(\bullet),(\bullet,\bullet)})$.

Consider the diagram of $K^+(\mathcal{D}'^{(\bullet)})$:

$$\begin{array}{ccccc}
\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \widetilde{\mathcal{I}}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(\alpha)} & \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \widetilde{\mathcal{I}}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(4)} & \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{I}^{(\bullet),(\lambda,\chi)}) \\
\downarrow{\scriptstyle (1)} & \swarrow{\scriptstyle (\star\star)} & & \swarrow{\scriptstyle (\star)} & \\
\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \lambda_0^*\widetilde{\mathcal{I}}^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(2)} & \mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) & \xrightarrow[(i)]{} & \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)}) \\
& \swarrow{\scriptstyle (3)} & & \swarrow{\scriptstyle (5)} & \\
\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \widetilde{\mathcal{I}}'^{(\bullet),(\lambda,\chi)}) & \xrightarrow[(ii)]{} & \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \widetilde{\mathcal{I}}'^{(\bullet),(\lambda,\chi)}) & \xrightarrow{(6)} & \mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)}, \mathcal{I}'^{(\bullet),(\lambda,\chi)})
\end{array}$$

(8.6.2.4.1)

where (i) and (ii) are the forgetful functors (see 8.2.4.19.2), where α is the composition of the forgetful functor 8.2.4.19.2 with the map induced by functoriality from the canonical map $\mathcal{E}^{(\bullet)} \to \lambda_0^*\mathcal{E}^{(\bullet)}$, where the numbered arrows are given by functoriality of the bifunctors $\mathscr{H}om^\bullet_{\mathcal{D}^{(\bullet)}}(-,-)$ or $\mathscr{H}om^\bullet_{\lambda_0^*\mathcal{D}^{(\bullet)}}(-,-)$ and where the arrows $(\star)$ and $(\star\star)$ are defined at 8.2.4.18.3. The triangle is commutative (see 8.2.4.20.3), the diamonds are commutative by functoriality. Hence, we get the commutativity of the diagram:

$$\begin{array}{ccc}
\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)}) & \xrightarrow{(4)\circ(\alpha)} & \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \\
{\scriptstyle (3)\circ(2)\circ(1)}\downarrow & \swarrow{\scriptstyle (3)\circ(\star)} & \downarrow{\scriptstyle (6)\circ(5)\circ(i)\circ(\star)} \\
\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)},\ \lambda_0^*\chi^*\lambda^*\mathcal{F}^{(\bullet)}) & \xrightarrow{(6)\circ(ii)} & \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)},\ \lambda_0^*\chi^*\lambda^*\mathcal{F}^{(\bullet)}).
\end{array} \tag{8.6.2.4.2}$$

Since both maps $\lambda_0^*\widetilde{\mathcal{I}}^{(\bullet),(\lambda,\chi)} \to \lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)} \to \widetilde{\mathcal{I}}'^{(\bullet),(\lambda,\chi)}$ are quasi-isomorphisms of $K^+(\widetilde{\mathcal{D}}^{(\bullet),(\bullet,\bullet)}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{D}}'^{(\bullet),(\bullet,\bullet)})$ then the left vertical map of 8.6.2.4.2 is equal to the canonical morphism. Since both maps $\lambda_0^*\mathcal{I}^{(\bullet),(\lambda,\chi)} \to$

$\widetilde{\mathcal{I}}^{(\bullet),(\lambda,\chi)} \to \mathcal{I}'^{(\bullet),(\lambda,\chi)}$ are quasi-isomorphisms of $K^+(\mathcal{D}^{(\bullet),(\bullet,\bullet)},\mathcal{R},\mathcal{D}'^{(\bullet),(\bullet,\bullet)})$ then the right vertical map of 8.6.2.4.2 is equal to the canonical morphism. Finally, it is tautological that the three other arrows of 8.6.2.4.2 are the canonical ones.

By applying the functor $\underset{(\lambda,\chi)\in L(I)^{\lambda_0}\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}$ to the diagram 8.6.2.4.2, we get a commutative diagram say $(\mathcal{D}iag)$. Since the right vertical morphism of $(\mathcal{D}iag)$ is equal to the map denoted by $(2)\circ(\star)$ of the diagram 8.6.2.2.3, then it is an isomorphism (see the proof of 8.6.2.2). Let $\widetilde{\mathcal{P}}^{(\bullet)} \in K^+(\lambda_0^*\mathcal{D}^{(\bullet)})$ be a complex of left $\lambda_0^*\mathcal{D}^{(\bullet)}$-modules belonging to $\mathscr{P}(\lambda_0^*\mathcal{D}^{(\bullet)})$ representing $\lambda_0^*\mathcal{E}^{(\bullet)}$. By copying the part 2.iii) of the proof of 8.2.4.21, we get that the left vertical arrows of the diagram $(\mathcal{D}iag)$ is an isomorphism. Hence, we are done. □

Lemma 8.6.2.5. *Suppose I is strictly filtered and X is quasi-compact. Let $\lambda_0 \in L(I)$, $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{perf}}(\lambda_0^*\mathcal{D}^{(\bullet)})$, let $g\colon \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ be a morphism belonging to $S^{\mathrm{b}}(\mathcal{D}^{(\bullet)},\mathcal{D}'^{(\bullet)})$ (see notation 8.1.4.10) between two objects of $D^{\mathrm{b}}(\mathcal{D}^{(\bullet)},\mathcal{R},\mathcal{D}'^{(\bullet)})$. Then the canonical morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)})$*

$$Q_{LD}\circ \mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)}) \to Q_{LD}\circ \mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet)}) \tag{8.6.2.5.1}$$

induced by g is an isomorphism.

Proof. Since $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{perf}}(\lambda_0^*\mathcal{D}^{(\bullet)})$, then both complexes of the map 8.6.2.5.1 do belong to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}'^{(\bullet)})$. We can therefore apply Corollary 8.1.5.11. Hence, with lemma 8.3.3.8, the property that 8.6.2.5.1 is an isomorphism is local on $X^{(\bullet)}$. This implies that we can suppose $\mathcal{E}^{(\bullet)}$ is strictly perfect. Hence, by devissage, we reduce to the case where $\mathcal{E}^{(\bullet)} = \lambda_0^*\mathcal{D}^{(\bullet)}$, which is obvious. □

Proposition 8.6.2.6. *Suppose I is strictly filtered and X is quasi-compact. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})$.*

(a) For any $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)},\mathcal{R},\mathcal{D}'^{(\bullet)})$, we have $\mathscr{F}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)})$ (see notation 8.6.2.3.1).

(b) For any $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},.,\mathrm{perf}}(\mathcal{D}^{(\bullet)},\mathcal{R},\mathcal{D}'^{(\bullet)})$, we have $\mathscr{F}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^r\mathcal{D}'^{(\bullet)})$.

Proof. We can suppose there exists $\lambda_0 \in L(I)$ such that $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{perf}}(\lambda_0^*\mathcal{D}^{(\bullet)})$.

a) Similarly to 8.6.2.2, we establish that the canonical morphism

$$\underset{(\lambda,\chi)\in L(I)^{\lambda_0}\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\lambda_0^*\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)}) \to \underset{(\lambda,\chi)\in L(I)^{\lambda_0}\times M(I)}{\text{“}\varinjlim\text{”}} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \chi^*\lambda^*\mathcal{F}^{(\bullet)})$$

is an isomorphism. It follows from Lemmas 8.6.2.4 and 8.6.2.5 that we have the canonical isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)})$:

$$Q_{LD}\circ \mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \lambda_0^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathscr{F}(\mathcal{E}^{(\bullet)},\lambda_0^*\mathcal{F}^{(\bullet)}).$$

Since $\mathscr{F}(\mathcal{E}^{(\bullet)},\lambda_0^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{F}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)})$, then we are done.

b) Suppose now $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},.,\mathrm{perf}}(\mathcal{D}^{(\bullet)},\mathcal{R},\mathcal{D}'^{(\bullet)})$. Following 8.6.1.14, we have $\lambda_0^*\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}_{\mathbb{Q},.,\mathrm{perf}}(\lambda_0^*\mathcal{D}^{(\bullet)},\mathcal{R},\lambda_0^*\mathcal{D}'^{(\bullet)})$. Hence, there exists $\mu_0 \in L(I)$, $\mathcal{G}^{(\bullet)} \in D^{\mathrm{b}}_{.,\mathrm{perf}}(\lambda_0^*\mathcal{D}^{(\bullet)},\mathcal{R},\mu_0^*\lambda_0^*\mathcal{D}^{(\bullet)})$ together with an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\lambda_0^*\mathcal{D}^{(\bullet)},\mathcal{R},\lambda_0^*\mathcal{D}'^{(\bullet)})$ of the form $\lambda_0^*\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}$. This yields the isomorphisms

$$\mathscr{F}(\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathscr{F}(\mathcal{E}^{(\bullet)},\lambda_0^*\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathscr{F}(\mathcal{E}^{(\bullet)},\mathcal{G}^{(\bullet)}) \xleftarrow{\sim} Q_{LD}\circ\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet)}),$$

where the last isomorphism comes from 8.6.2.5.1. Since $\mathbb{R}\mathscr{H}om_{\lambda_0^*\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)},\ \mathcal{G}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{perf}}(\mu_0^*\lambda_0^*\mathcal{D}'^{(\bullet)})$, then we are done. □

Notation 8.6.2.7 (Duality)**.** Suppose I is strictly filtered and X is quasi-compact. We denote by

$$\mathbb{R}_{\mathrm{LD}}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,-) := \mathscr{F}(-,-)\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)},\mathcal{R},\mathcal{D}'^{(\bullet)}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^r\mathcal{D}'^{(\bullet)}) \tag{8.6.2.7.1}$$

the functor induced by $\mathscr{F}$ (see 8.6.2.6). Following 8.6.2.3.1, we get also the functor:

$$\mathbb{R}_{\mathrm{LD}}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\mathcal{D}^{(\bullet)})\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^r\mathcal{D}^{(\bullet)}). \tag{8.6.2.7.2}$$

Following the proof of 8.6.2.6, for any $\lambda_0 \in L(I)$ and $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{perf}}(\lambda_0^* \mathcal{D}^{(\bullet)})$, we have the isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^r\mathcal{D}^{(\bullet)})$:

$$\mathbb{R}_{\mathrm{LD}} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{D}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\lambda_0^* \mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \lambda_0^* \mathcal{D}^{(\bullet)}). \tag{8.6.2.7.3}$$

A translation of 8.6.2.6 is that the functor $Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-, \mathcal{D}^{(\bullet)}) \colon D^-(\mathcal{D}^{(\bullet)})^{\mathrm{op}} \to \underrightarrow{LD}^+_{\mathbb{Q}}({}^r\mathcal{D}^{(\bullet)})$ is universally right localisable with respect to $S^{\mathrm{b}}(\mathcal{D}^{(\bullet)})$ at any object of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\mathcal{D}^{(\bullet)})$ (see definition 7.4.1.9) and we have

$$\mathbb{R}_{S^{\mathrm{b}}(\mathcal{D}^{(\bullet)})} Q_{LD} \circ \mathbb{R}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{D}^{(\bullet)}) = \mathbb{R}_{\mathrm{LD}} \mathscr{H}om_{\mathcal{D}^{(\bullet)}}(\mathcal{E}^{(\bullet)}, \mathcal{D}^{(\bullet)}). \tag{8.6.2.7.4}$$

8.7 Sheaf of differential operators with infinite order and finite level

8.7.1 The sheaf of differential operators $\mathcal{D}^\dagger$

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian.

8.7.1.1. Let $\mathcal{B}^{(\bullet)}$ be an inductive system of commutative $\mathcal{O}_{\mathfrak{X}}$-algebras indexed by $\mathbb{N}$ satisfying the following conditions.

(a) For any $m \in \mathbb{N}$, $\mathcal{B}^{(m)}$ is endowed with a structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra such that the transition map $\mathcal{B}^{(m)} \to \mathcal{B}^{(m+1)}$ are $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear ;

(b) For any $m \in \mathbb{N}$, for any affine open $\mathfrak{U}$ of $\mathfrak{X}$, the ring $\Gamma(U, \mathcal{B}^{(m)})$ is noetherian ;

(c) For any $m \in \mathbb{N}$, for any $i \geq 0$, $\mathcal{B}^{(m)}/\mathfrak{m}^{i+1}\mathcal{B}^{(m)}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module and the canonical homomorphism $\mathcal{B}^{(m)} \to \varprojlim_{i \in \mathbb{N}} \mathcal{B}^{(m)}/\mathfrak{m}^{i+1}\mathcal{B}^{(m)}$ is an isomorphism.

When the transition maps $\mathcal{B}^{(m)} \to \mathcal{B}^{(m+1)}$ are the identities, we say that $\mathcal{B}^{(\bullet)}$ is constant and we denote it by $\mathcal{B}$. We remark that the conditions b and c are equal to that of 7.2.3 in the case where $\mathcal{I} = \mathfrak{m}$. Hence, for instance we get following 7.2.3.3 that for any open immersion $\mathfrak{V} \subset \mathfrak{U}$ of affine opens, the homomorphism $\Gamma(\mathfrak{U}, \mathcal{B}^{(\bullet)}) \to \Gamma(\mathfrak{V}, \mathcal{B}^{(\bullet)})$ is flat.

Definition 8.7.1.2. We keep notation 8.7.1.1 and we suppose $\mathcal{B}^{(\bullet)}$ is constant.

(a) We define the *sheaf of differential operators of infinite order and finite level with coefficients in* $\mathcal{B}$ to be

$$\mathcal{B} \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \varinjlim_m \mathcal{B} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}.$$

When $\mathcal{B} = \mathcal{O}_{\mathfrak{X}}$, we simply write $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and call it the *sheaf of differential operators of infinite order and finite level.*

(b) We define the *sheaf of differential operators of infinite order and infinite level with coefficients in* $\mathcal{B}$ to be $\mathcal{B} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, the p-adic completion of $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

8.7.1.3. We keep notation 8.7.1.1. Since $\mathcal{B}^{(m)}$ is endowed with a structure of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module, then $\mathcal{B} := \varinjlim_m \mathcal{B}^{(m)}$ is endowed with a structure of left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. We set

$$\mathcal{B} \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \varinjlim_m \mathcal{B}^{(m)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp},$$

$$\mathcal{B} \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} := \left(\mathcal{B} \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\right)_{\mathbb{Q}}.$$

Let $\mathcal{B}^{(\bullet)}_t$ be the ideal of $\mathcal{B}^{(\bullet)}$ consisting of p-torsion sections and $\mathcal{B}'^{(\bullet)} := \mathcal{B}^{(\bullet)}/\mathcal{B}^{(\bullet)}_t$. Then $\mathcal{B}'^{(\bullet)}$ satisfies the condition (a) of 8.7.1.1. Moreover, $\mathcal{B}'^{(m)}$ is a coherent $\mathcal{B}^{(m)}$-module for any $m \in \mathbb{N}$ (see 7.4.5.1). With

7.2.3.16, this implies that $\mathcal{B}'^{(\bullet)}$ satisfies the properties (b) and (c) of 8.7.1.1. By setting $\mathcal{B}' := \varinjlim_m \mathcal{B}'^{(m)}$, it follows from 7.5.1.10 that the canonical homomorphism

$$\mathcal{B} \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \mathcal{B}' \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \tag{8.7.1.3.1}$$

is an isomorphism.

8.7.1.4. With notation 8.7.1.3, $\mathcal{B} \otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is right and left flat over $\mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ Indeed, this follows from 7.5.1.4 by taking inductive limits.

Proof. Following 4.1.2.17.(d), $\Gamma(\mathfrak{U}, \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ is right and left noetherian for any affine open $\mathfrak{U} \subset \mathfrak{X}$. According to 7.2.2.2, $\Gamma(\mathfrak{U}, \mathcal{B} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ is the p-adic completion of $\Gamma(\mathfrak{U}, \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Hence, we are done. □

8.7.1.5. We keep notation 8.7.1.1. For any $m' \geq m$, by p-adic completion of 3.2.3.5.1 and by using the transition maps, we get the maps

$$\mathcal{B}^{(m)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\rho_{m',m}} \mathcal{B}^{(m')} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\rho_{m'}} \mathcal{B} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{8.7.1.5.1}$$

It follows from 3.1.4.3.1, 3.2.3.1.1 and 3.2.2.14.3 that the maps of 3.2.3.5.1 are homomorphisms of rings. From 3.2.3.5.2, we get the formula

$$\rho_{m',m}(\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) = \frac{q^{(m)}_{\underline{k}}!}{q^{(m')}_{\underline{k}}!} \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m')}} \text{ and } \rho_m(\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}) = \underline{q}^{(m)}_{\underline{k}}! \underline{\partial}_\sharp^{[\underline{k}]}. \tag{8.7.1.5.2}$$

Hence, when $\mathcal{B}^{(\bullet)}$ is constant and $\mathcal{B}$ is p-torsion free, the homomorphisms 8.7.1.5.1 are injective.

8.7.1.6 (p-adic norm, v_p, v_π)**.** We denote by e the absolute ramification index of $\mathcal{V}$. We keep notation 8.7.1.1 and suppose $\mathcal{B}^{(m)}$ is p-torsion free for some $m \in \mathbb{N}$. Let $\mathfrak{U}^\sharp$ be an affine open of $\mathfrak{X}^\sharp$. We endow $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathbb{Q}}) \xrightarrow{\sim} \Gamma(\mathfrak{U}, \mathcal{B}^{(m)})_K$ with a norm induced by the p-adic topology on $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)})$ as follows. For any $b \in \Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathbb{Q}})$, we set $\|b\| = p^{-v_p(b)}$, where $E_b = \{n \in \mathbb{Z} \ ; \ b \in p^n \Gamma(\mathfrak{U}, \mathcal{B}^{(m)})\}$ and $v_p(b) := \max E_b$ if $b \neq 0$ and $v_p(0) = +\infty$. This norm is called the p-adic norm on $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathbb{Q}})$ given by $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)})$.

Remark moreover that $\|bb'\| \leq \|b\| \|b'\|$ and then $v_p \colon B_K \to \mathbb{Z} \cup \{+\infty\}$ is a quasi-valuation. We call v_p to be the p-adic quasi-valuation of $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathbb{Q}})$ induced by $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)})$. We also define $v_\pi(b) := \max\{n \in \mathbb{Z} \ ; \ b \in \pi^n \Gamma(\mathfrak{U}, \mathcal{B}^{(m)})\}$ if $b \neq 0$ and $v_\pi(b) = +\infty$ otherwise. We say that $v_\pi \colon B_K \to \mathbb{Z} \cup \{+\infty\}$ the π-adic quasi-valuation of $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathbb{Q}})$ induced by $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)})$. For any $b \in \Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathbb{Q}})$, we have

$$v_p(b) \leq \frac{v_\pi(b)}{e} < v_p(b) + 1. \tag{8.7.1.6.1}$$

We denote by $| - |$ the p-adic norm on $\mathbb{Q}$ given by $\mathbb{Z}_{(p)}$. Hence, by convention $|p| = p^{-1}$.

Lemma 8.7.1.7. *With notation 1.2.1.2 we have the following estimates.*

(a) For any $k, m \in \mathbb{N}$, we have the inequalities

$$\frac{k}{p^m(p-1)} - \log_p(k+1) - \frac{p}{(p-1)} < v_p(q^{(m)}_k!) \leq \frac{k}{p^m(p-1)}. \tag{8.7.1.7.1}$$

(b) For any $m \in \mathbb{N}$ there exists $\eta' < 1$, $c' \in \mathbb{R}$ such that $|q^{(m)}_k!| \leq c' \eta'^k$ for all $k \in \mathbb{N}$.

(c) For any $\eta < 1$ there exist $m \in \mathbb{N}$, $c \in \mathbb{R}$ such that $\eta^k \leq c |q^{(m)}_k!|$ for all $k \in \mathbb{N}$.

Proof. Let us write $q = q^{(m)}_k$. Then $\frac{k}{p^m(p-1)} - \frac{1}{(p-1)} < \frac{q}{(p-1)} \leq \frac{k}{p^m(p-1)}$. Using the estimates 1.2.1.1.2 of $\sigma(q)$, we get moreover

$$-\log_p(k+1) - 1 < \frac{-\sigma(q)}{(p-1)} \leq 0.$$

Since $v_p(q!) = \frac{(q-\sigma(q))}{(p-1)}$ (see 1.2.1.1.1), then we get 8.7.1.7.1 by addition.

The inequality $|q_k^{(m)}!| \leq c'\eta'^k$ is equivalent to saying that $-v_p(q_k^{(m)}!) \leq \log_p(c') + k\log_p(\eta')$. The inequality $\eta^k \leq c|q_k^{(m)}!|$ is equivalent to saying that $k\log_p(\eta) \leq \log_p(c) - v_p(q_k^{(m)}!)$. Hence, the estimates 8.7.1.7.1 imply easily (b) and (c). ☐

Proposition 8.7.1.8. We keep notation 8.7.1.1. Suppose $\mathcal{B}^{(\bullet)}$ is constant and $\mathcal{B}$ is p-torsion free. Let $\mathfrak{U}^\sharp$ be an affine open of $\mathfrak{X}^\sharp$ endowed with logarithmic coordinates. Let $P \in \Gamma(\mathfrak{U}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Following 7.5.1.5.2 (in the case where $m = +\infty$), we can uniquely write

$$P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}\underline{\partial}_\sharp^{[\underline{k}]},$$

with $b_{\underline{k}} \in \Gamma(\mathfrak{U}, \mathcal{B})$ a sequence converging to 0 for the p-adic topology when $|\underline{k}|$ goes to infinity. For any $i \in \mathbb{N}$, let $P_i \in \Gamma(U, \mathcal{B}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{D}_{X_i^\sharp/S_i^\sharp})$ be the image of P, where $\mathcal{B}_i := \mathcal{B}/\mathfrak{m}^{i+1}\mathcal{B}$. The following conditions are equivalent:

(a) $P \in \Gamma(\mathfrak{U}, \mathcal{B}\otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$;

(b) $\exists \alpha, \beta \in \mathbb{R}$ such that $\mathrm{ord}(P_i) \leq \alpha i + \beta$ for any $i \in \mathbb{N}$;

(c) $\exists c, \eta \in \mathbb{R}_+$ such that $\eta < 1$ and $\|b_{\underline{k}}\| \leq c\eta^{|\underline{k}|}$, for any $\underline{k} \in \mathbb{N}^d$.

Proof. (a) $\Rightarrow$ (b). Set $B := \Gamma(\mathfrak{U}, \mathcal{B})$. Set $B_i := \Gamma(\mathfrak{U}, \mathcal{B}_i)$. Suppose $P \in \Gamma(\mathfrak{U}, \mathcal{B}\otimes^\dagger_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Then there exists $m \in \mathbb{N}$ large enough such that $P \in \Gamma(\mathfrak{U}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Let us fix such an m. Following 7.5.1.5.1, there exists a unique sequence $(b_{\underline{k}}^{(m)})_{\underline{k}\in\mathbb{N}^d}$ of elements of B such that $b_{\underline{k}}^{(m)} \to 0$ for the p-adic topology when $|\underline{k}| \to \infty$ and $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}^{(m)}\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}}$. Since $\underline{\partial}_\sharp^{\langle\underline{k}\rangle_{(m)}} = \underline{q}_{\underline{k}}^{(m)}!\underline{\partial}_\sharp^{[\underline{k}]}$ (see 8.7.1.5.2), then we get $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}^{(m)}\underline{q}_{\underline{k}}^{(m)}!\underline{\partial}_\sharp^{[\underline{k}]}$, i.e. $b_{\underline{k}} = b_{\underline{k}}^{(m)}\underline{q}_{\underline{k}}^{(m)}!$. Using 8.7.1.7, $\exists \eta < 1$, $\exists c \in \mathbb{R}$ such that $|\underline{q}_{\underline{k}}^{(m)}!| \leq c\eta^{|\underline{k}|}$ for any $\underline{k} \in \mathbb{N}^d$. Setting $a := e\log_p(1/\eta)$ and $b := e\log_p(1/c)$, using the first inequality 8.7.1.6.1, this yields $v_\pi(b_{\underline{k}}) \geq ev_p(\underline{q}_{\underline{k}}^{(m)}!) \geq a|\underline{k}| + b$, where v_p is the p-adic valuation of $\mathbb{Z}_{(p)}$ and v_π is the π-adic quasi-valuation of B. This is equivalent to saying that $\mathrm{ord}(P_i) \leq \frac{1}{a}i - \frac{b}{a}$ for any $i \in \mathbb{N}$.

(b) $\Rightarrow$ (c). Suppose $\exists \alpha, \beta \in \mathbb{R}$ such that $\mathrm{ord}(P_i) \leq \alpha i + \beta$ for any $i \in \mathbb{N}$. Let $\underline{k} \in \mathbb{N}^d$. Set $i_{\underline{k}} := v_\pi(b_{\underline{k}})$. Since the image of $b_{\underline{k}}$ in $B_{i_{\underline{k}}}$ is not null, then $|\underline{k}| \leq \alpha i_{\underline{k}} + \beta$. Hence, using the second inequality 8.7.1.6.1, we get $\|b_{\underline{k}}\| = p^{-v_p(b_{\underline{k}})} < p^{1-\frac{i_{\underline{k}}}{e}} \leq p^{1-\frac{(|\underline{k}|-\beta)}{\alpha e}} = p^{1+\frac{\beta}{\alpha e}}\left(p^{-\frac{1}{\alpha e}}\right)^{|\underline{k}|}$. Hence, we can choose $c = p^{1+\frac{\beta}{\alpha e}}$ and $\eta = p^{-1/\alpha e}$.

(c) $\Rightarrow$ (a). Suppose $\exists c, \eta \in \mathbb{R}_{>0}$ such that $\eta < 1$ and $\|b_{\underline{k}}\| \leq c\eta^{|\underline{k}|}$, for any $\underline{k} \in \mathbb{N}^d$. We have to prove that for m large enough, $b_{\underline{k}}/\underline{q}_{\underline{k}}^{(m)}! \in B$. The inequality $\|b_{\underline{k}}\| \leq c\eta^{|\underline{k}|}$ is equivalent to $v_p(b_{\underline{k}}) \geq \lambda|\underline{k}| + \mu$, with $\mu = -\log_p(c)$ and $\lambda = -\log_p(\eta) > 0$. Using 8.7.1.7.1, this yields

$$v_p(b_{\underline{k}}/\underline{q}_{\underline{k}}^{(m)}!) \geq \lambda|\underline{k}| + \mu - |\underline{k}|/p^m(p-1) = (\lambda - 1/p^m(p-1))\,|\underline{k}| + \mu. \tag{8.7.1.8.1}$$

Suppose m large enough such that $\lambda - 1/p^m(p-1) > 0$. Hence, if $\mu \geq 0$, then $v_p(b_{\underline{k}}/\underline{q}_{\underline{k}}^{(m)}!) \geq 0$, i.e. $b_{\underline{k}}/\underline{q}_{\underline{k}}^{(m)}! \in B$ and we are done. Suppose now $\mu < 0$ and m large enough such that the inequalities hold

$$p^m \geq -\frac{\mu}{(\lambda - 1/p^m(p-1))} \Leftrightarrow p^m\lambda - 1/(p-1) \geq -\mu \Leftrightarrow p^m \geq (-\mu + 1/(p-1))/\lambda. \tag{8.7.1.8.2}$$

Let $\underline{k} \in \mathbb{N}^d$. If $|\underline{k}| \geq -\mu/(\lambda - 1/p^m(p-1))$, then $(\lambda - 1/p^m(p-1))\,|\underline{k}| + \mu \geq 0$ and we are done thanks to 8.7.1.8.1. On the other hand, if $|\underline{k}| \leq -\mu/(\lambda - 1/p^m(p-1))$, then from 8.7.1.8.2 we get $|\underline{k}| \leq p^m$. Hence, $\underline{q}_{\underline{k}}^{(m)}! = 1$ and then $b_{\underline{k}}/\underline{q}_{\underline{k}}^{(m)}! \in B$. ☐

The Spencer exact sequence (see 4.7.3.7.2) is wrong at the level m when $m \geq 1$. However, we get the exactness of the end of the sequence up to a multiplication by p^m (or $p^m!$ with logarithmic log structures):

Proposition 8.7.1.9. *Suppose $\mathfrak{X}/\mathfrak{S}$ is smooth equipped with coordinates $t_1, \dots, t_d$. We use notation 7.5.1.5.*

(a) Let $P = \sum_{\underline{k}} a_{\underline{k}} \partial^{\langle \underline{k} \rangle} \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$ *with* $a_{\underline{k}} = 0$ *for all* $\underline{k}$ *such that* $k_1 = 0$*. Then there exists a unique operator* $Q \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$ *such that* $p^m P = Q\partial_1$*.*

(b) Let $P = \sum_{\underline{k}} a_{\underline{k}} \partial^{[\underline{k}]} \in \Gamma(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$*. If* $a_{\underline{k}} = 0$ *for all* $\underline{k}$ *such that* $k_1 = 0$*, then there exists a unique operator* $Q \in \Gamma(\mathfrak{U}, \mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$ *such that* $P = Q\partial_1$*.*

Proof. The part (b) follows from (a) by taking limit. For (a) we now take $\underline{k}$ such that $k_1 \neq 0$. In $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$ we have the relation:

$$\left\langle \begin{matrix} k_1 \\ 1 \end{matrix} \right\rangle \partial_1^{\langle k_1 \rangle} = \partial_1^{\langle k_1 - 1 \rangle} \partial_1, \quad \text{with} \quad \left\langle \begin{matrix} k_1 \\ 1 \end{matrix} \right\rangle = \begin{cases} k_1 & \text{if } p^m \nmid k, \\ p^m & \text{if } p^m | k. \end{cases}$$

For any k_1, $\left\langle \begin{smallmatrix} k_1 \\ 1 \end{smallmatrix} \right\rangle$ divides p^m in $\mathbb{Z}_p$, thus we can take Q to be

$$Q = \sum_{k_1 \neq 0} (p^m \left\langle \begin{matrix} k_1 \\ 1 \end{matrix} \right\rangle^{-1}) a_{\underline{k}} \underline{\partial}^{\langle \underline{k} - \epsilon_1 \rangle}.$$

Finally, the uniqueness of Q follows easily from the fact that $\mathcal{O}_{\mathfrak{X}}$ is p-torsion free and from the unicity of the writing of Q of the form $Q = \sum_{\underline{k}} b_{\underline{k}} \partial^{\langle \underline{k} \rangle} \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$. □

Proposition 8.7.1.10. *Suppose* $\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}$ *is equipped with logarithmic coordinates* $u_1, \ldots, u_d$ *and* $\mathfrak{X}$ *is* p*-torsion free. We use notation 7.5.1.5.*

(a) Let $P = \sum_{\underline{k}} a_{\underline{k}} \partial_{\sharp}^{\langle \underline{k} \rangle} \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$ *with* $a_{\underline{k}} = 0$ *for all* $\underline{k}$ *such that* $k_1 = 0$*. Then there exists a unique operator* $Q \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$ *such that* $p^m! P = Q\partial_{\sharp i}$*.*

(b) Let $P = \sum_{\underline{k}} a_{\underline{k}} \partial_{\sharp}^{[\underline{k}]} \in \Gamma(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$*. If* $a_{\underline{k}} = 0$ *for all* $\underline{k}$ *such that* $k_1 = 0$*, then there exists a unique operator* $Q \in \Gamma(\mathfrak{U}, \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$ *such that* $P = Q\partial_{\sharp i}$*.*

Proof. The part (b) follows from (a) by taking limit. We can suppose P is of the form $\partial_{\sharp 1}^{\langle n \rangle_{(m)}}$ for some integer $n \geq 0$. Let us prove by induction on $n \geq 0$ that there exists a unique operator $Q \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$ such that $p^m! \partial_{\sharp 1}^{\langle n \rangle_{(m)}} = Q\partial_1$. Following 3.2.3.11.c, there exists a unique operator $Q \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$ such that $n! \partial_{\sharp 1}^{\langle n \rangle_{(m)}} = Q\partial_1$. Hence, when $1 \leq n \leq p^m$ we are done. Suppose now $n \geq p^m$ and let $q \geq 1$ and $0 \leq r < p^m$ be some integers such that $n = p^m q + r$. Following 3.2.3.13.1, we have the formula

$$\partial_{\sharp 1}^{\langle p^m(q-1)+r \rangle_{(m)}} \partial_{\sharp 1}^{\langle p^m \rangle_{(m)}} = \left\langle \begin{matrix} p^m q + r \\ p^m \end{matrix} \right\rangle \partial_{\sharp 1}^{\langle p^m q + r \rangle_{(m)}}$$
$$+ \sum_{k=\max\{p^m(q-1)+r, p^m\}}^{p^m q + r - 1} \frac{k!}{(p^m q + r - k)!(k - p^m)!(k - (p^m(q-1) + r))!} \frac{q^{(m)}_{p^m}! q^{(m)}_{p^m(q-1)+r}!}{q^{(m)}_{k}!} \partial_{\sharp 1}^{\langle k \rangle_{(m)}}$$

with $\frac{k!}{(p^m q + r - k)!(k - p^m)!(k - (p^m(q-1) + r))!} \frac{q^{(m)}_{p^m}! q^{(m)}_{p^m(q-1)+r}!}{q^{(m)}_{k}!} \in \mathbb{Z}_{(p)}$. Following 1.2.1.5.1, we have $\left\langle \begin{smallmatrix} p^m q + r \\ p^m \end{smallmatrix} \right\rangle \in \mathbb{Z}^*_{(p)}$. Hence, we conclude the induction. Finally, the uniqueness of Q follows from the fact that $\mathcal{O}_{\mathfrak{X}}$ is p-torsion free, from the property 3.2.3.14.1 and the unicity of the writing $P = \sum_{\underline{k}} a_{\underline{k}} \partial_{\sharp}^{\langle \underline{k} \rangle}$. □

8.7.2 Swapping left and right $\mathcal{D}^{\dagger}$-modules

Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^{\sharp} \to \mathfrak{S}^{\sharp}$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian.

Let $\mathcal{B}^{(\bullet)}$ be an inductive system of commutative $\mathcal{O}_{\mathfrak{X}}$-algebras indexed by $\mathbb{N}$ satisfying the conditions of 8.7.1.1. Set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} := \mathcal{B}^{(m)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$, $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$, i.e. $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} = \mathcal{B} \otimes^{\dagger}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ (see notation 8.7.1.3). We get a canonical left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure on $\mathcal{B} := \varinjlim_m \mathcal{B}^{(m)}$. This yield a structure of left $\widehat{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure on $\widehat{\mathcal{B}}$, where $\widehat{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is the p-adic completion of $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (or of $\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$). We suppose $\mathcal{B}^{(\bullet)}$ has no p-torsion (this is an harmless hypothesis: see 8.7.1.3.1). In that case $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ has no p-torsion and is included in its p-adic completion $\mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

8.7.2.1. Following 7.5.1.11, the sheaf $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}^{(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is endowed with a canonical right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure extending its structure of $\mathcal{B}^{(m)}$-module. By taking the inductive limits on the level, the sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is endowed with a canonical right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure extending its structure of $\mathcal{B}$-module.

Suppose $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d \in M_{\mathfrak{X}^\sharp}$. By p-adic completion of the logarithmic adjoint operator (see 3.4.1.2.3), we get the map $\Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ given by $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}} \mapsto \widetilde{P} := \sum_{\underline{k}\in\mathbb{N}^d} \widetilde{\underline{\partial}}_\sharp^{\langle \underline{k}\rangle_{(m)}} b_{\underline{k}}$, where $b_{\underline{k}}$ is a sequence of elements of $\Gamma(\mathfrak{X}, \mathcal{B}^{(m)})$ converging to 0 for the p-adic topology when $|\underline{k}|$ goes to infinity. With the local description 8.7.1.8, we can check the logarithmic adjoint operator (see 3.4.1.2.3) extends to a By taking inductive limits on the level, this yields the map $\Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. This map is also induced (via the inclusion $\Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \subset \Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$), by the p-adic completion of the logarithmic adjoint operator $\Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{X}, \mathcal{B}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ given by $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}} \underline{\partial}_\sharp^{[\underline{k}]} \mapsto \widetilde{P} := \sum_{\underline{k}\in\mathbb{N}^d} \widetilde{\underline{\partial}}_\sharp^{[\underline{k}]} b_{\underline{k}}$, where $b_{\underline{k}}$ is a sequence of elements of $\Gamma(\mathfrak{X}, \mathcal{B})$ converging to 0 for the p-adic topology when $|\underline{k}|$ goes to infinity. It follows from the formula 7.5.1.11.1 that the action of $P \in \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ on the section $b\, d\log t_1 \wedge \cdots \wedge d\log t_d$, where b is section of $\mathcal{B}$ is given by the formula

$$(b\ d\log t_1 \wedge \cdots \wedge d\log t_d) \cdot P = \widetilde{P}(b) d\log t_1 \wedge \cdots \wedge d\log t_d. \tag{8.7.2.1.1}$$

8.7.2.2. We have the following properties.

(a) By taking the inductive limits on the level, we get from 7.5.1.13.(a) a structure of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-bimodule on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

(b) Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Via the canonical isomorphism of $\mathcal{B}$-modules:

$$\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E} \xrightarrow{\sim} \left(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\right) \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E} \tag{8.7.2.2.1}$$

we get a structure of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E}$. Suppose $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d \in M_{\mathfrak{X}^\sharp}$. With notation 8.7.2.1, we compute the action of $P \in \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ on the section $d\log t_1 \wedge \cdots \wedge d\log t_d \otimes x$ of $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E}$, where x is a section of $\mathcal{E}$, is given by the formula

$$(d\log t_1 \wedge \cdots \wedge d\log t_d \otimes x) \cdot P = d\log t_1 \wedge \cdots \wedge d\log t_d \otimes \widetilde{P} \cdot x. \tag{8.7.2.2.2}$$

Hence, the structure of right $\mathcal{B}^{(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}^{(m)}} \mathcal{E}$ given by 7.5.1.12 is equal to the one induced (via the canonical map $\mathcal{B}^{(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$) by its structure of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module.

(c) Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Via the canonical isomorphism

$$\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M}) \xrightarrow{\sim} \mathscr{H}om_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M}), \tag{8.7.2.2.3}$$

we get a structure of left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M})$. Suppose $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ has logarithmic coordinates $u_1, \ldots, u_d \in M_{\mathfrak{X}^\sharp}$. With notation 8.7.2.1, we compute the action of $P \in \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ on the section $x \otimes (d\log t_1 \wedge \cdots \wedge d\log t_d)^*$ of $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M})$, where x is section of $\mathcal{M}$, is given by the formula

$$P \cdot (x \otimes (d\log t_1 \wedge \cdots \wedge d\log t_d)^*) = x \cdot \widetilde{P} \otimes (d\log t_1 \wedge \cdots \wedge d\log t_d)^*. \tag{8.7.2.2.4}$$

Hence, the induced structure of left $\mathcal{B}^{(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module corresponds to that of 7.5.1.12.

8.7.2.3. Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module.

(a) Using 8.7.2.2.4, we easily compute that the canonical $\mathcal{B}$-linear isomorphism

$$(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_r \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E} \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E} \tag{8.7.2.3.1}$$

is right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear.

(b) Since $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is locally free (of rank one), the canonical $\mathcal{B}$-linear morphism

$$\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{H}om_{\mathcal{B}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \mathcal{M}) \xrightarrow{\sim} \mathcal{M} \tag{8.7.2.3.2}$$

is an isomorphism. Using 8.7.2.2.4, we compute this isomorphism is moreover $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. Similarly, the canonical $\mathcal{B}$-linear isomorphism :

$$\mathcal{E} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \mathcal{E}). \tag{8.7.2.3.3}$$

is $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear.

(c) $\mathcal{H}om_{\mathcal{B}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -)$ and $\mathcal{H}om_{\mathcal{O}}(\omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -)$ (resp. $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} -$ and $\omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_X} -$) are canonically isomorphism on the category of right (resp. left) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules.

(d) The functors $- \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} = \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -)$ and $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} -$ are exact and induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules. Hence, for any $\star \in \{-, +, \mathrm{b}, \emptyset\}$, the functors $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} -$ and $\mathscr{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, -)$ induce quasi-inverse equivalences of categories between $D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $D^{\star}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Moreover, these equivalences preserve K-flat complexes and K-injective complexes.

8.7.2.4. By taking the projective limits, the inductive limits on the level, we get from 4.2.5.6.1 the transposition isomorphism of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules:

$$\widetilde{\delta}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \delta_{\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} : \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{8.7.2.4.1}$$

By applying to this isomorphism the functor $- \otimes_{\mathcal{B}_X} \widetilde{\omega}^{-1}_{X^\sharp/S^\sharp}$ to the right (resp. left) structure from the source (resp. target), we get the isomorphism of $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules:

$$\widetilde{\alpha}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} : \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{8.7.2.4.2}$$

By applying to this isomorphism the functor $- \otimes_{\mathcal{B}} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ to the left (resp. right) structure from the source (resp. target), this yields the isomorphism

$$\widetilde{\beta}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} : \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{8.7.2.4.3}$$

8.7.2.5. Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. By copying the proof of 8.7.2.5, we get the following isomorphism of $\mathcal{O}_S$-modules:

$$\mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} (\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}). \tag{8.7.2.5.1}$$

8.7.2.6. The sheaf $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} := \mathcal{B}^{(m)}_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is endowed with a canonical right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module structure extending its structure of $\mathcal{B}^{(m)}_{\mathbb{Q}}$-module. By taking the inductive limits on the level, the sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} := \mathcal{B}_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is endowed with a canonical right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module structure extending its structure of $\mathcal{B}_{\mathbb{Q}}$-module. The results of this subsection are still valid by replacing respectively $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ by $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$.

8.7.3 Overconvergent singularities

Proposition 8.7.3.1. We keep notation 3.4. We suppose with S is endowed with a quasi-coherent (resp. coherent of the formal case) m-PD-ideal $(\mathfrak{a}_S, \mathfrak{b}_S, \alpha_S)$ such that $p \in \mathfrak{a}_S$. Let $m, r \in \mathbb{N}$ be two integers such that p^{m+1} divides r. Fix $f \in \Gamma(X, \mathcal{O}_X)$, put $\mathcal{B}'_X(f,r) := \mathcal{O}_X[T]/(f^r T - p)$, Y be the open of X complementary to $V(f)$ and $j\colon Y \subset X$ be the inclusion.

(a) There exists on $\mathcal{B}'_X(f,r)$ a canonical structure of left $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module compatible with its structure of $\mathcal{O}_X$-algebra such that $\mathcal{B}'_X(f,r) \to j_* \mathcal{O}_Y$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

(b) If $g \in \Gamma(X, \mathcal{O}_X)$, and $f' = gf$, the $\mathcal{O}_X$-algebra homomorphism

$$\rho_g \colon \mathcal{B}'_X(f,r) = \mathcal{O}_X[T]/(f^r T - p) \to \mathcal{B}'_X(f',r) = \mathcal{O}_X[T']/(f'^r T' - p)$$

such that $\rho_g(T) = g^r T'$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear.

(c) If r is divisible by $p^{m'+1}$ with $m' \geq m$, then the structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{B}'_X(f,r)$ is equal to that induced by its structure of $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$-module.

Proof. a) Since the formal case is checked identically, let us focus on the algebraic one. Consider the following commutative diagram

$$\begin{array}{ccc} X^\sharp & \xrightarrow{\phi_f} & \mathbb{A}^1_{\mathbb{Z}_{(p)}} = \operatorname{Spec} \mathbb{Z}_{(p)}[t] \\ \downarrow & & \downarrow \\ (S^\sharp, \mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) & \longrightarrow & (\operatorname{Spec} \mathbb{Z}_{(p)}, (p), (p), \alpha_{(p)}), \end{array} \tag{8.7.3.1.1}$$

where ϕ_f is the morphism given by f and where $((p), \alpha_{(p)})$ is the canonical m-PD-structure of (p) (see 1.2.4.2.a). Hence, by using the preservation of $\mathcal{D}$-module structures under pullbacks (see 4.4.2.4), since $\mathcal{B}'_X(f,r) = \phi_f^*(\mathcal{B}'_{\mathbb{A}^1_{\mathbb{Z}_{(p)}}}(t,r))$, we reduce to the case where $S^\sharp = \operatorname{Spec} \mathbb{Z}_{(p)}$ and $X^\sharp = \operatorname{Spec} \mathbb{Z}_{(p)}[t]$, $f = t$. Since the ideal of the closed immersion $\Delta_{X/S}$ is the ideal of $\mathbb{Z}_{(p)}[t_1, t_2] = (\mathbb{Z}_{(p)}[t_1])[t_2 - t_1]$ generated by $t_2 - t_1$, then following 1.3.2.6, we have $(|X|, \mathcal{P}^n_{X/S,(m)}) = \operatorname{Spec}(\mathbb{Z}_{(p)}[t_1]\langle t_2 - t_1\rangle_{(m)}/I^{\{n+1\}})$, where and $I = \overline{(t_2 - t_1)}$ is the m-PD-ideal generated by $t_2 - t_1$. We have $r = p^{m+1} q$. Since $\mathbb{Z}_{(p)}[t_1]\langle t_2 - t_1\rangle_{(m)}/I^{\{n+1\}}$ is p-torsion free, then we compute there exists a of degree r homogeneous m-PD-polynomial $\phi^{(m)}_r(t_1, t_2) \in I$ such that

$$t_2^r - t_1^r = p \phi^{(m)}_r(t_1, t_2). \tag{8.7.3.1.2}$$

Set $T_1 := T \otimes 1 \in \mathcal{O}_X[T]/(f^r T - p) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)}$ and $T_2 := 1 \otimes T \in \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X[T]/(f^r T - p)$. We are looking for an $\mathcal{O}_X$-algebra isomorphism ε_n making commutative the diagram below:

$$\begin{array}{ccccc} \mathcal{P}^n_{X/S,(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_X[T]/(f^r T - p) & & \xrightarrow[\sim]{\varepsilon_n} & & \mathcal{O}_X[T]/(f^r T - p) \otimes_{\mathcal{O}_X} \mathcal{P}^n_{X/S,(m)} \\ \downarrow & & & & \downarrow \\ j_*(\mathcal{P}^n_{Y/S,(m)} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) & \xrightarrow{\sim} & j_* \mathcal{P}^n_{Y/S,(m)} & \xleftarrow[\sim]{} & j_*(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{P}^n_{Y/S,(m)}) \end{array} \tag{8.7.3.1.3}$$

where the bottom isomorphisms are the canonical isomorphisms and the composite bottom isomorphism is $\varepsilon_n^{\mathcal{O}_Y}$ (see 3.4.2.4.1). The image of T_2 (resp. T_1) in $\mathcal{P}^n_{Y/S,(m)}$ is $p t_2^{-r}$ (resp. $p t_1^{-r}$). To do so, we define an $\mathcal{O}_X$-algebra map $\varepsilon_n(T_2)$ by setting

$$\varepsilon_n(T_2) = T_1(1 + T_1 \phi^{(m)}_r(t_1, t_2))^{-1},$$

which has a meaning because $\phi^{(m)}_r(t_1, t_2)$ is nilpotent in $\mathcal{P}^n_{X/S,(m)}$. Its inverse is given by $T_1 \mapsto T_2(1 - T_2 \phi^{(m)}_r(t_1, t_2))^{-1}$. Since $\phi^{(m)}_r(t_1, t_2) \in I$, then $\varepsilon_0 = \mathrm{id}$. Moreover, the cocycle condition is a consequence of the formula

$$\phi^{(m)}_r(t_1, t_2) + \phi^{(m)}_r(t_2, t_3) = \phi^{(m)}_r(t_1, t_3) \tag{8.7.3.1.4}$$

in $\mathbb{Z}_{(p)}[t_1]\langle t_2 - t_1, t_3 - t_2\rangle_{(m)}$.

b) and c) We leave the proof to the reader. □

Remark 8.7.3.2. Contrary to Berthelot's notation of [Ber96c, 4.2], we have added a prime, i.e. the ring $\mathcal{B}'_X(f,r)$ was written $\mathcal{B}_X(f,r)$. The reason is that in order to use Huyghe sheaves of 8.7.3.16, we prefer to work by default with the slight modification of 8.7.3.8 introduced by Huyghe where p is replaced by π.

Proposition 8.7.3.3. We consider the algebraic case of 3.4 (beware the nilpotence condition is too strong in the formal case). Let $m, r \in \mathbb{N}$ be two integers such that p^{m+1} divides r. Let $\mathcal{I} \subset \mathcal{O}_S$ be an m-PD-nilpotent quasi-coherent ideal which extends to X, $f, g \in \Gamma(X, \mathcal{O}_X)$, $h \in \Gamma(X, \mathcal{I}\mathcal{O}_X)$, and $f' = gf + h$. There exists the canonical $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear homomorphism of $\mathcal{O}_X$-algebras

$$\eta_{g,h} \colon \mathcal{B}'_X(f,r) \to \mathcal{B}'_X(f',r),$$

satisfying the following properties:

(a) If $g' \in \Gamma(X, \mathcal{O}_X)$, $h' \in \Gamma(X, \mathcal{I}\mathcal{O}_X)$, and $f'' = g'f' + h'$, $g'' = g'g$, $h'' = g'h + h'$, then $\eta_{g'',h''} = \eta_{g',h'} \circ \eta_{g,h}$.

(b) $\eta_{g,0} = \rho_g$, $\eta_{1,0} = \mathrm{id}$.

(c) If f is not a divisor of 0 in $\mathcal{O}_X/\mathcal{I}\mathcal{O}_X$, $\eta_{g,h}$ only depends on f, f', r and will be denoted $\eta_{f',f,r}$.

(d) If r is divisible by $p^{m'+1}$, with $m' \geq m$, $\eta_{g,h}$ is independent from $m \leq m'$.

Proof. This is checked similarly to [Ber96c, 4.2.2]. For the reader, we will only recall below the construction of $\eta_{g,h}$.

1) Suppose $g = 1$. Let $u \colon Z^\sharp \hookrightarrow X^\sharp$ be the exact closed immersion given by $\mathcal{I}\mathcal{O}_X$. Put $S_0 = \operatorname{Spec} \mathbb{Z}_{(p)}$ and $X_0 = \operatorname{Spec} \mathbb{Z}_{(p)}[t]$. Let $\overline{f}$ and $\overline{f'}$ be the image of f and f' via the morphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(Z, \mathcal{O}_Z)$ induced by u. Since $\overline{f} = \overline{f'}$, then f an f' induce the same morphism $\phi_{\overline{f}} \colon Z^\sharp \to X_0$. Since $\mathcal{I}$ is an m-PD-nilpotent ideal, then by using the universal property of the m-PD-envelope, for any integer n large enough, we get a unique factorization $\theta \colon X \to \Delta^n_{X_0/S_0^\sharp,(m)}$ making commutative the following diagram

$$\begin{array}{ccccc} Z^\sharp & \hookrightarrow & & \longrightarrow & X^\sharp \\ \downarrow{\scriptstyle \phi_{\overline{f}}} & & \swarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi_{f'} \times \phi_f} \\ X_0 & \longrightarrow & \Delta^n_{X_0/S_0,(m)} & \longrightarrow & X_0 \times_{S_0} X_0. \end{array} \qquad (8.7.3.3.1)$$

Let $\epsilon_n \colon \mathcal{P}^n_{X_0/S_0,(m)} \otimes_{\mathcal{O}_{X_0}} \mathcal{B}'_{X_0}(t,r) \xrightarrow{\sim} \mathcal{B}'_{X_0}(t,r) \otimes_{\mathcal{O}_{X_0}} \mathcal{P}^n_{X_0/S_0,(m)}$ be the isomorphism given by the $\mathcal{D}^{(m)}_{X_0/S_0}$-module structure of $\mathcal{B}'_{X_0}(t,r)$. Taking the inverse image by θ we get the isomorphism $\epsilon_h \colon \mathcal{B}'_X(f,r) \xrightarrow{\sim} \mathcal{B}'_X(f',r)$ which is independent on the choice of n.

2) In general, $\eta_{g,h} := \epsilon_h \circ \rho_g$. □

Notation 8.7.3.4. We consider the algebraic case of 3.4. Let $m, r \in \mathbb{N}$ be two integers such that p^{m+1} divides r. Let $\mathcal{I} \subset \mathcal{O}_S$ be an m-PD-nilpotent quasi-coherent ideal which extends to X, let $X_0^\sharp = V(\mathcal{I}\mathcal{O}_X) \hookrightarrow X^\sharp$ be the exact closed immersion induced $\mathcal{I}$. We suppose X_0 regular.

(a) Let Z be a divisor of X_0. Let $U^\sharp$ be an open set of $X^\sharp$, $f \in \Gamma(U, \mathcal{O}_X)$ such that the closed immersion $Z^\sharp \cap U_0^\sharp \hookrightarrow U_0^\sharp$ is given by $\overline{f} \in \Gamma(U_0, \mathcal{O}_{X_0})$ the image of f via $\Gamma(U, \mathcal{O}_X) \to \Gamma(U_0, \mathcal{O}_{X_0})$. Following 8.7.3.3, $\mathcal{B}'_U(f,r)$ only depends on Z up to canonical isomorphism. More precisely, let $f' \in \Gamma(U, \mathcal{O}_X)$ be such that the closed immersion $Z^\sharp \cap U_0^\sharp \hookrightarrow U_0^\sharp$ is given by $\overline{f'} \in \Gamma(U_0, \mathcal{O}_{X_0})$ the image of f' via $\Gamma(U, \mathcal{O}_X) \to \Gamma(U_0, \mathcal{O}_{X_0})$. There exists $g \in \Gamma(U, \mathcal{O}_X)$ such that $\overline{g}$ is a unit of $\Gamma(U, \mathcal{O}_{X_0})$ and $\overline{f'} = \overline{g}\overline{f}$. Hence, there exists $h \in \Gamma(U, \mathcal{I}\mathcal{O}_X)$ such that $f' = gf + h$. Since $\mathcal{I}$ is m-PD-nilpotent then $\mathcal{I}$ is nilpotent. Hence, since $\overline{g}$ is a unit of $\Gamma(U, \mathcal{O}_{X_0})$ then g is a unit of $\Gamma(U, \mathcal{O}_X)$. We get $f = g^{-1}f' - h$. Hence $\eta_{f',f,r} \colon \mathcal{B}'_U(f',r)$ is an isomorphism with $\eta_{f,f',r}$ as an inverse (indeed, following 8.7.3.3.a, we have $\eta_{f,f',r} \circ \eta_{f',f,r} = \eta_{f,f,r}$ and $\eta_{f',f,r} \circ \eta_{f,f',r} = \eta_{f',f',r}$). Hence, glueing $\mathcal{B}'_U(f,r)$ we get the $\mathcal{O}_X$-algebra $\mathcal{B}'_X(Z \hookrightarrow X_0, r)$ endowed with a compatible $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module structure. Put $\mathcal{B}'^{(m)}_X(Z \hookrightarrow X_0) := \mathcal{B}'_X(Z \hookrightarrow X_0, p^{m+1})$. If there is no ambiguity with $\mathcal{I}$ then we can simply write $\mathcal{B}'_X(Z,r)$ and $\mathcal{B}'^{(m)}_X(Z)$.

(b) Let $Z' \supset Z$ be two divisors of X_0. If f and f' are two liftings of a local equation of respectively Z and Z' in X_0, then the homomorphism $\eta_{f',f,r} \colon \mathcal{B}'_X(f,r) \to \mathcal{B}'_X(f',r)$ of 8.7.3.3.c are compatible with the glueing isomorphisms which yields

$$\eta_{Z',Z,r} \colon \mathcal{B}'_X(Z,r) \to \mathcal{B}'_X(Z',r). \tag{8.7.3.4.1}$$

If $Z'' \subset Z'$ is a third divisor of X_0 then we get

$$\eta_{Z'',Z,r} = \eta_{Z'',Z',r} \circ \eta_{Z',Z,r}. \tag{8.7.3.4.2}$$

(c) Let $a \geq 1$ be an integer. Then we can check that the canonical isomorphism $\mathcal{B}'(f^a, r) \xrightarrow{\sim} \mathcal{B}'(f, ar)$ is $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-linear and that this is compatible with the glueing isomorphism. The yields the equality

$$\mathcal{B}'_X(aZ, r) = \mathcal{B}'_X(Z, ar). \tag{8.7.3.4.3}$$

We get

$$\eta_{Z,ar,r} := \eta_{aZ,Z,r} \colon \mathcal{B}'_X(Z,r) \to \mathcal{B}'_X(Z,ar). \tag{8.7.3.4.4}$$

If $a' \geq 1$ is another integer, we get

$$\eta_{Z,a'ar,r} = \eta_{Z,a'ar,ar} \circ \eta_{Z,ar,r}. \tag{8.7.3.4.5}$$

(d) When r' is a multiple of $p^{m'+1}$, with $m' \geq m$, the structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{B}'_X(Z,r)$ is induced by its structure of $\mathcal{D}^{(m')}_{X^\sharp/S^\sharp}$-module, and the homomorphisms $\eta_{Z',Z,r}$ and $\eta_{Z,ar,r}$ are independent of $m \geq m'$.

Notation 8.7.3.5. Let $m \in \mathbb{N}$ be an integer such that $p^m \geq e/(p-1)$. Let $r \in \mathbb{N}$ be an integer such that p^{m+1} divides r. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian. Let $X^\sharp_i := \mathfrak{X}^\sharp \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\mathfrak{m}^{i+1})$, $X^\sharp := X^\sharp_0$. We suppose X regular. Let Z be a divisor of X.

Following 1.2.4.2.(a) $\pi\mathcal{V}$ has an m-PD-structure. Hence, we can use 8.7.3.4 and we get the $\mathcal{O}_{X_i}$-algebra $\mathcal{B}'_{X_i}(Z,r)$ endowed with a compatible $\mathcal{D}^{(m)}_{X^\sharp_i/S^\sharp_i}$-module structure for any $i \in \mathbb{N}$. We observe that the canonical isomorphism

$$\mathcal{B}'_{X_i}(Z,r)/\mathfrak{m}^i \mathcal{B}'_{X_i}(Z,r) \xrightarrow{\sim} \mathcal{B}'_{X_{i-1}}(Z,r)$$

is $\mathcal{D}^{(m)}_{X^\sharp_{i-1}/S^\sharp_{i-1}}$-linear (the structure of the left term is induced by base change). This yields a structure of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on

$$\mathcal{B}'_{\mathfrak{X}}(Z,r) := \varprojlim_i \mathcal{B}'_{X_i}(Z,r), \quad \mathcal{B}'^{(m)}_{\mathfrak{X}}(Z) = \mathcal{B}'_{\mathfrak{X}}(Z, p^{m+1}). \tag{8.7.3.5.1}$$

Moreover, the induced structure of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{B}'_{\mathfrak{X}}(Z,r)$ is compatible with its $\mathcal{O}_{\mathfrak{X}}$-algebra structure (we can use the remark 4.1.1.2). Beware, contrary to what might appear in the literature, to lighten the notation we have chosen to write $\mathcal{B}'_{\mathfrak{X}}(Z,r)$ instead of $\widehat{\mathcal{B}'}_{\mathfrak{X}}(Z,r)$ because we can only define a priori $\mathcal{B}'_{\mathfrak{X}}(Z,r)$ by p-adic completion. However, when the divisor Z lift to an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$, we can get $\mathcal{B}'_{\mathfrak{X}}(Z,r)$ as p-adic completion of a sheaf of rings on $\mathfrak{P}$ (see just below 8.7.3.9.1).

Finally, we put

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z) := \varinjlim_m \mathcal{B}'^{(m)}_{\mathfrak{X}}(Z) \tag{8.7.3.5.2}$$

and call it the sheaf of *functions* on $\mathfrak{X}$ *with overconvergent singularities* along the divisor Z. An important result is: $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module. The proof will be given below at Theorem 12.2.7.1.

If $Z \subset T$ are two divisors of X, we get from 8.7.3.3 the canonical morphisms $\mathcal{B}'^{(m)}_{X_i}(Z) \to \mathcal{B}'^{(m)}_{X_i}(T)$, $\mathcal{B}'^{(m)}_{\mathfrak{X}}(Z) \to \mathcal{B}'^{(m)}_{\mathfrak{X}}(T)$, and $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z) \to \mathcal{O}_{\mathfrak{X}}({}^\dagger T)$.

Suppose $\mathfrak{X}$ is noetherian. Then for any divisor Z of X, there exists $a \in \mathbb{N}$ such that $Z_{\mathrm{red}} \subset Z \subset p^a Z_{\mathrm{red}}$. Since $\mathcal{B}'^{(m)}_{\mathfrak{X}}(p^a Z) = \mathcal{B}'^{(m+a)}_{\mathfrak{X}}(Z)$ (see 8.7.3.4.3), this yields that the canonical morphism

$$\mathcal{B}'^{(\bullet)}_{\mathfrak{X}}(Z_{\mathrm{red}}) \to \mathcal{B}'^{(\bullet)}_{\mathfrak{X}}(Z) \tag{8.7.3.5.3}$$

is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{O}^{(\bullet)}_{\mathfrak{X}})$. By applying the inductive limit on the level (without tensoring with $\mathbb{Q}$), we get the isomorphism

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z_{\mathrm{red}}) \to \mathcal{O}_{\mathfrak{X}}({}^\dagger Z). \tag{8.7.3.5.4}$$

Proposition 8.7.3.6. *We keep notation 8.7.3.5. Let $r \in \mathbb{N}$ be an integer such that p^{m+1} divides r. Fix $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and put $\mathcal{B}_{\mathfrak{X}}(f,r) := \mathcal{O}_X\{T\}/(f^r T - \pi)$. Let $\mathfrak{Y}$ be the open of $\mathfrak{X}$ complementary to $V(f)$ and $j\colon \mathfrak{Y} \subset \mathfrak{X}$ be the inclusion. There exists on $\mathcal{B}_{\mathfrak{X}}(f,r)$ a canonical structure of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra such that the inclusion $\mathcal{B}_{\mathfrak{X}}(f,r) \to j_*\mathcal{O}_{\mathfrak{Y}}$ is $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear.*

Proof. Since $\mathcal{B}_{\mathfrak{X}}(f,r)$ is p-adically complete, then we reduce to check that there exists on $\mathcal{B}_{\mathfrak{X}}(f,r)$ a canonical structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module compatible with its structure of $\mathcal{O}_{\mathfrak{X}}$-algebra such that the inclusion $\mathcal{B}_{\mathfrak{X}}(f,r) \to j_*\mathcal{O}_{\mathfrak{Y}}$ is $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear. Consider the following commutative diagram

$$\begin{array}{ccc} \mathfrak{X}^\sharp & \xrightarrow{\phi_f} & \widehat{\mathbb{A}}^1_{\mathcal{V}} = \operatorname{Spf} \mathcal{V}\{t\} \\ \downarrow & & \downarrow \\ (S^\sharp, \mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) & \longrightarrow & (\operatorname{Spf} \mathcal{V}, (\pi), (\pi^{e+1}), \alpha), \end{array} \tag{8.7.3.6.1}$$

where ϕ_f is the morphism given by f and where $((\pi^{e+1}), \alpha))$ is the canonical m-PD-structure of (π) (see 1.2.4.2.a). Hence, by using the preservation of $\mathcal{D}$-module structures under pullbacks (see 4.4.2.4), since $\mathcal{B}_{\mathfrak{X}}(f,r) = \phi_f^*(\mathcal{B}_{\widehat{\mathbb{A}}^1_{\mathcal{V}}}(t,r))$, we reduce to the case where $\mathfrak{S}^\sharp = \operatorname{Spf} \mathcal{V}$ and $\mathfrak{X}^\sharp = \operatorname{Spf} \mathcal{V}\{t\}$, $f = t$. Since the ideal of the closed immersion $\Delta_{\mathfrak{X}/\mathfrak{S}}$ is the ideal of $\mathcal{V}\{t_1, t_2\} = (\mathcal{V}\{t_1\})\{t_2 - t_1\}$ generated by $t_2 - t_1$, since $\mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)} = \varprojlim_i \mathcal{P}^n_{X_i/\mathfrak{S}_i,(m)}$, then following 1.3.2.6, we have $(|\mathfrak{X}|, \mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)}) = \operatorname{Spf}(\mathcal{V}\{t_1\}\langle t_2 - t_1\rangle_{(m)}/I^{\{n+1\}})$, where and $I = \overline{(t_2 - t_1)}$ is the m-PD-ideal generated by $t_2 - t_1$. We have $r = p^{m+1}q$. Following 8.7.3.1.2, there exists a of degree r homogeneous m-PD-polynomial $\phi^{(m)}_r(t_1, t_2), \psi^{(m)}_r(t_1, t_2) \in I$ (and even with coefficient in $\mathbb{Z}_{(p)}$) such that $t_2^r - t_1^r = p\phi^{(m)}_r(t_1, t_2) = \pi\psi^{(m)}_r(t_1, t_2)$ Set $T_1 := T \otimes 1 \in \mathcal{O}_{\mathfrak{X}}\{T\}/(f^r T - \pi) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)}$ and $T_2 := 1 \otimes T \in \mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}\{T\}/(f^r T - \pi)$. We are looking for an $\mathcal{O}_{\mathfrak{X}}$-algebra isomorphism ε_n making commutative the diagram below:

$$\begin{array}{ccccc} \mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}\{T\}/(f^r T - \pi) & & \xrightarrow[\sim]{\varepsilon_n} & & \mathcal{O}_{\mathfrak{X}}\{T\}/(f^r T - \pi) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)} \\ \downarrow & & & & \downarrow \\ j_*(\mathcal{P}^n_{\mathfrak{Y}/\mathfrak{S},(m)} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{Y}}) & \xrightarrow{\sim} & j_*\mathcal{P}^n_{\mathfrak{Y}/\mathfrak{S},(m)} & \xleftarrow[\sim]{} & j_*(\mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{P}^n_{\mathfrak{Y}/\mathfrak{S},(m)}) \end{array} \tag{8.7.3.6.2}$$

where the bottom isomorphisms are the canonical isomorphisms and the composite bottom isomorphism is $\varepsilon_n^{\mathcal{O}_{\mathfrak{Y}}}$ (see 3.4.2.4.1). The image of T_2 (resp. T_1) in $\mathcal{P}^n_{\mathfrak{Y}/\mathfrak{S},(m)}$ is πt_2^{-r} (resp. πt_1^{-r}). To do so, we define an $\mathcal{O}_{\mathfrak{X}}$-algebra map $\varepsilon_n(T_2)$ by setting

$$\varepsilon_n(T_2) = T_1(1 + T_1\psi^{(m)}_r(t_1, t_2))^{-1},$$

which has a meaning because $\psi^{(m)}_r(t_1, t_2)$ is nilpotent in $\mathcal{P}^n_{\mathfrak{X}/\mathfrak{S},(m)}$. Its inverse is given by $T_1 \mapsto T_2(1 - T_2\psi^{(m)}_r(t_1, t_2))^{-1}$. Since $\psi^{(m)}_r(t_1, t_2) \in I$, then $\varepsilon_0 = \mathrm{id}$. Moreover, the cocycle condition is a consequence of the formula

$$\psi^{(m)}_r(t_1, t_2) + \psi^{(m)}_r(t_2, t_3) = \psi^{(m)}_r(t_1, t_3) \tag{8.7.3.6.3}$$

in $\mathcal{V}\{t_1\}\langle t_2 - t_1, t_3 - t_2\rangle_{(m)}$ □

Proposition 8.7.3.7. *We keep notation 8.7.3.5. Let $r \in \mathbb{N}$ be an integer such that p^{m+1} divides r. Let $f, g, h \in \Gamma(X, \mathcal{O}_{\mathfrak{X}})$ such that $f' = gf + \pi h$. There exists the canonical $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear homomorphism of $\mathcal{O}_{\mathfrak{X}}$-algebras*

$$\eta_{g,h} \colon \mathcal{B}_{\mathfrak{X}}(f, r) \to \mathcal{B}_{\mathfrak{X}}(f', r),$$

satisfying the following properties:

(a) If $g' \in \Gamma(X, \mathcal{O}_{\mathfrak{X}})$, $h' \in \Gamma(X, \pi\mathcal{O}_{\mathfrak{X}})$, and $f'' = g'f' + \pi h'$, $g'' = g'g$, $h'' = g'h + h'$, then $\eta_{g'',h''} = \eta_{g',h'} \circ \eta_{g,h}$.

(b) $\eta_{1,0} = \mathrm{id}$.

(c) If f is not a divisor of 0 in $\mathcal{O}_{\mathfrak{X}}/\pi\mathcal{O}_{\mathfrak{X}}$, $\eta_{g,h}$ only depends on f, f', r and will be denoted $\eta_{f',f,r}$.

(d) If r is divisible by $p^{m'+1}$, with $m' \geq m$, $\eta_{g,h}$ is independent from $m \leq m'$.

Proof. 1) Suppose $g = 1$. Then $f = f' - \pi h = f'u$, where $u := (\mathrm{id} - \pi f'^{-1} h) \in \mathcal{B}^{\times}_{\mathfrak{X}}(f', r)$. we get the isomorphism $\eta_{1,h}$ defined by setting $\eta_{1,h}(T) = u^{-1}T'$, i.e. $\eta_{1,h}(\pi f^{-r}) = u^{-1}\pi f'^{-r}$. 2) Suppose $h = 0$. Then we get the morphism $\eta_{g,0}$ defined by setting $\eta_{g,0}(T) = g^r T'$, i.e. $\eta_{g,0}(\pi f^{-r}) = g^r \pi f'^{-r}$. 3) In general, we get the map $\eta_{g,h} := \eta_{1,h} \circ \eta_{g,0}$. The $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linearity follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{\mathfrak{X}}(f, r) & \xrightarrow{\eta_{g,h}} & \mathcal{B}_{\mathfrak{X}}(f', r) \\ \downarrow & & \downarrow \\ j_* \mathcal{O}_{\mathfrak{Y}} & = & j_* \mathcal{O}_{\mathfrak{Y}} \end{array} \tag{8.7.3.7.1}$$

where the vertical map are $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear (see 8.7.3.6).

□

Notation 8.7.3.8. We keep notation 8.7.3.5. Let $\mathfrak{U}^\sharp$ be an open set of $\mathfrak{X}^\sharp$, $f \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ such that the closed immersion $Z^\sharp \cap U^\sharp \hookrightarrow U^\sharp$ is given by $\overline{f} \in \Gamma(U, \mathcal{O}_X)$ the image of f via $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \to \Gamma(U, \mathcal{O}_X)$. Following 8.7.3.7, $\mathcal{B}_{\mathfrak{U}}(f, r)$ only depends on Z up to canonical isomorphism. Hence, glueing $\mathcal{B}_{\mathfrak{U}}(f, r)$ we get the $\mathcal{O}_X$-algebra $\mathcal{B}_{\mathfrak{X}}(Z, r)$ endowed with a compatible $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module structure. Put $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) := \mathcal{B}_{\mathfrak{X}}(Z, p^{m+1})$.

We have the isomorphism:

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z) \xrightarrow{\sim} \varinjlim_{m} \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \tag{8.7.3.8.1}$$

If $Z \subset T$ are two divisors of X, we get from 8.7.3.7 the canonical morphism $\eta_{T,Z,r} \colon \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m)}_{\mathfrak{X}}(T)$, and $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z) \to \mathcal{O}_{\mathfrak{X}}({}^\dagger T)$. These algebras satisfy the same properties than in 8.7.3.4 and we keep the corresponding notation, e.g.

.

Suppose $\mathfrak{X}$ is noetherian. Then for any divisor Z of X, there exists $a \in \mathbb{N}$ such that $Z_{\mathrm{red}} \subset Z \subset p^a Z_{\mathrm{red}}$. Since $\mathcal{B}^{(m)}_{\mathfrak{X}}(p^a Z) = \mathcal{B}^{(m+a)}_{\mathfrak{X}}(Z)$ (similarly to 8.7.3.4.3), this yields that the canonical morphism

$$\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z_{\mathrm{red}}) \to \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z) \tag{8.7.3.8.2}$$

is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{O}^{(\bullet)}_{\mathfrak{X}})$. By applying the inductive limit on the level (without tensoring with $\mathbb{Q}$), we get the isomorphism

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z_{\mathrm{red}}) \to \mathcal{O}_{\mathfrak{X}}({}^\dagger Z). \tag{8.7.3.8.3}$$

Notation 8.7.3.9. With notation 8.7.3.5, suppose there exists an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$ (see definition 4.5.2.1) which is a lifting of Z. Let $m, r \in \mathbb{N}$ be two integers such that p^{m+1} divides r. Then, similarly to 8.7.3.4, by glueing $\mathcal{B}_{\mathfrak{X}}(f, r)$ when f is an equation of $\mathfrak{Z} \subset \mathfrak{X}$ we can define $\mathcal{B}_{\mathfrak{X}}(\mathfrak{Z}, r)$ (indeed we can get a formal version of 8.7.3.3 in the case where $\mathcal{I}$ is null). We get the isomorphism:

$$\widehat{\mathcal{B}}_{\mathfrak{X}}(\mathfrak{Z}, r) \xrightarrow{\sim} \mathcal{B}_{\mathfrak{X}}(Z, r), \tag{8.7.3.9.1}$$

where the LHS is the p-adic completion of $\mathcal{B}_{\mathfrak{X}}(\mathfrak{Z}, r)$. We also set $\mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}) := \mathcal{B}_{\mathfrak{X}}(\mathfrak{Z}, p^{m+1})$.

8.7.3.10 (p-adic completion, p-adic weak completion)**.** Let A be a noetherian p-adic ring. We denote by $A\{T\}$ the p-adic completion of the polynomial ring $A[T]$ with one variable. This is called the ring of restricted power series (see [Gro60, 0.7.5]). We get

$$A\{T\} := \widehat{A[T]} = \left\{ \sum_{n=0}^{\infty} a_n T^n : a_n \in A, \lim_{n\to+\infty} a_n = 0 \right\},$$

where A is endowed with the p-adic topology. Following [Gro60, 0.7.6.15], for any $f \in A$, we write

$$\begin{aligned} A_{\{f\}} &:= \widehat{A_f} = A\{T\}/(fT-1) \\ &= \left\{ \sum_{n=0}^{\infty} \frac{a_n}{f^n} : a_n \in A, \lim_{n\to+\infty} a_n = 0 \right\} \end{aligned}$$

where A is endowed with the p-adic topology.

The *p-adic weak completion of $A[T]$ as A-algebra* is

$$A[T]^{\dagger} = \left\{ \sum_{n=0}^{\infty} a_n T^n : \exists\, c, \eta \in \mathbb{R}, \eta < 1 \text{ such that } \|a_n\| \leq c\eta^n \right\},$$

where $\| - \|$ is the p-adic norm of A (see 8.7.1.6). The p-adic weak completion of A_f as A-algebra is

$$A_f^{\dagger} := A[T]^{\dagger}/(fT-1) = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{f^n} : a_n \in A, \exists\, c, \eta \in \mathbb{R}, \eta < 1 \text{ such that } \|a_n\| \leq c\eta^n \text{ for all } n \right\}.$$

Using the computation of 8.7.1.8, we can check

$$A_f^{\dagger} = \left\{ \sum_{n=0}^{\infty} a_n f^n : a_n \in A, \exists\, \lambda > 0 \text{ such that } v_p(a_n) \geq \frac{n}{\lambda} - 1 \right\}.$$

8.7.3.11 (Local description)**.** Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^{\sharp} \to \mathfrak{S}^{\sharp}$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian and X_0 is regular. Let Z be a divisor of X, $\mathfrak{Y}^{\sharp}$ be the open of $\mathfrak{X}^{\sharp}$ complementary to the support of Z and $j \colon \mathfrak{Y}^{\sharp} \to \mathfrak{X}^{\sharp}$ be the canonical morphism. Let $\mathfrak{U} = \operatorname{Spf} A$ be an affine open of $\mathfrak{X}$, $f \in A$ such that $Z = V(\bar{f}) \subset X_0$ where $\bar{f}$ is the image of f in $A/\mathfrak{m}A$.

(a) We have

$$\Gamma(\mathfrak{U}, j_* \mathcal{O}_{\mathfrak{Y}}) = A_{\{f\}} = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{f^n} : a_n \in A, \lim_{n\to+\infty} a_n = 0 \right\}.$$

(b) Let r' and r be two any positive integer such that r divides r'. Then $\Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}(Z,r)) = A\{T\}/(f^r T - p)$ and $\Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}(Z,r')) = A\{T'\}/(f^{r'} T' - p)$. Moreover, by taking the inverse limits over $i \in \mathbb{N}$ of the morphisms $\eta_{Z,r',r} \colon \mathcal{B}_{X_i}(Z,r) \to \mathcal{B}_{X_i}(Z,r')$ of 8.7.3.4.4 and taking the sections over $\mathfrak{U}$ functor, we get the morphism $\Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}(Z,r)) \to \Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}(Z,r'))$ which is given by $T \mapsto f^{r'-r} T'$.

(c) After tensorising by $\mathbb{Q}$ the sheaf $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)$ we have the following nice description from the computation of (b):

$$\begin{aligned} \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}) &\xrightarrow{\sim} \varinjlim_{m} \Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}^{(m)}(Z)_{\mathbb{Q}}) \xrightarrow{\sim} \varinjlim_{m} \left(A\{t\}/(f^{p^{m+1}} t - p) \right)_K \xrightarrow{\sim} (A_f^{\dagger})_K \\ &\xrightarrow{\sim} \left\{ \sum_{n=0}^{\infty} \frac{a_n}{f^n} : a_n \in A_K, \exists\, c, \eta \in \mathbb{R}, \eta < 1 \text{ such that } \|a_n\| \leq c\eta^n \right\}, \end{aligned} \tag{8.7.3.11.1}$$

where A_K is endowed with the topology induced by the p-adic topology of A which induces a p-adic norm on A_K (see 8.7.1.6).

8.7.3.12. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose X is regular. Let Z be a divisor of X. We define the sheaf of "differential operators of finite level on $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ with overconvergent singularities along of Z" by setting

$$\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z) = \varinjlim_m \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}.$$

We have the ring homomorphism $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)$.

Remark 8.7.3.13. With notation 8.7.3.12, let $\mathfrak{Y}^\sharp$ be the open of $\mathfrak{X}^\sharp$ complementary to Z and let $j\colon \mathfrak{Y}^\sharp \subset \mathfrak{X}^\sharp$ be the inclusion of the complementary open. We will see via 8.7.6.8 that the sheaf $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ (resp. $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$) plays the role of an "overconvergent direct image" of $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$ (resp. $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$). Bearing in mind the algebraic case, where j_* induces an equivalence between the category of $\mathcal{O}_Y$-quasi-coherent modules and that of $j_*\mathcal{O}_Y$-quasi-coherent modules, we are led to replace the category of coherent $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules by that of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-modules, and the functor j_* (resp. j^*) by the restriction of the scalars of $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$ to $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ (resp. the extension of the scalars from $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ to $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$). The main reason is to get finiteness results. For instance, when $\mathfrak{X}$ is a proper and smooth formal scheme over $\mathcal{V}$, then the sheaf $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ is coherent as $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module (see 12.2.7.1) and has in particular finite de Rham cohomology, which is not the case for $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$ (for instance when $\mathfrak{Y}$ is the affine space).

Proposition 8.7.3.14. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension and that X is regular. Let Z be a divisor of X, $\mathfrak{Y}^\sharp$ be the open of $\mathfrak{X}^\sharp$ complementary to the support of Z and $j\colon \mathfrak{Y}^\sharp \to \mathfrak{X}^\sharp$ be the canonical morphism.

(a) For any affine open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$, $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z))$, and $\Gamma(\mathfrak{U}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}})$ are noetherian.

(b) The extensions $\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$ and $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} \to \mathcal{B}^{(m+1)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$ are flat.

(c) The sheaves $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$, $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$, and $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ are coherent. Moreover, coherent modules over these sheaves satisfy theorems A and B.

Proof. We can copy the proof of [Ber96c, 4.3.2]. □

In the rest of the section, we keep notation and hypotheses of 8.7.3.14. We give below few constructions of Huyghe (see [Huy03, 2.2]).

Lemma 8.7.3.15. *Suppose there exist two sections f and g of $\mathcal{O}_{\mathfrak{X}}$ lifting an equation of de Z in X. Then the modules $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)[1/f]$ and $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)[1/g]$ are canonically isomorphic.*

Proof. There exists $v \in \mathcal{O}_{\mathfrak{X}}$, $u \in \mathcal{O}_{\mathfrak{X}}^\times$, such that $g = u(f - \pi v)$. We have $h := u(1 - \pi v f^{-1}) \in (\mathcal{B}^{(m)}_{\mathfrak{X}}(Z))^\times$ (its inverse is given by $u^{-1} \sum_{n\in\mathbb{N}} \pi^n v^n f^{-n}$). Since $g = fh$, then we are done. □

Notation 8.7.3.16. It follows from Lemma 8.7.3.15 that we can glue the rings $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)[1/f]$ which gives a sheaf of rings on $\mathfrak{X}$ denoted by $\mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)$ We denote by $(*Z)\colon \mathrm{Mod}(\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)) \to \mathrm{Mod}(\mathcal{B}^{(m)}_{\mathfrak{X}}(*Z))$ the functor defined by setting for any $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$-module $\mathcal{E}$:

$$\mathcal{E}(*Z) = \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z) \otimes_{\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)} \mathcal{E}.$$

Notation 8.7.3.17. Suppose there exists an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z (see definition 4.5.2.1). If there exist two sections f and g of $\mathcal{O}_{\mathfrak{X}}$ giving an equation of $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$, then Then the modules $\mathcal{O}_{\mathfrak{X}}[1/f]$ and $\mathcal{O}_{\mathfrak{X}}[1/g]$ are canonically isomorphic. By glueing, we get the sheaf of rings $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})$. We define the localisation functor $(\mathfrak{Z})\colon \mathrm{Mod}(\mathcal{O}_{\mathfrak{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}))$ by setting for any $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{E}$:

$$\mathcal{E}(\mathfrak{Z}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}.$$

.

Lemma 8.7.3.18. *The canonical inclusion $\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \cap j_*\mathcal{O}_{\mathfrak{Y}}$ is an isomorphism.*

Proof. Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine and integral, and there exists $f \in A$ such that $\mathfrak{Y} = \mathrm{Spf}\, A\{1/f\}$. Since $A/\pi A$ is integral then we get the injective map $\alpha\colon A/\pi A \to A\{1/f\}/\pi A\{1/f\}$. Let $h \in A_{\mathbb{Q}} \cap A\{1/f\}$. Let $s \in \mathbb{Z}$ be minimal such that $\pi^s h \in A$. Set $k := \pi^s h$. By the absurd, suppose $s \geq 1$. Denoting by $\overline{k}$ the image of k in $A/\pi A$, since $k \in \pi A\{1/f\}$ then $\alpha(\overline{k}) = 0$ and therefore $k \in \pi A$. Since A is p-torsion free, this yields $\pi^{s-1}h \in A$, which is in contradiction with respect to the minimality of s. Hence, we are done. □

Lemma 8.7.3.19. *Suppose there exists an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z (see definition 4.5.2.1). Denote by $(\mathfrak{Z})$ the localisation functor $(\mathfrak{Z})\colon \mathrm{Mod}(\mathcal{O}_{\mathfrak{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}))$.*

(a) For any $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$-module $\mathcal{E}$, we have the canonical isomorphism $\mathcal{E}(*Z) \xrightarrow{\sim} \mathcal{E}(\mathfrak{Z})$.

(b) We have the equality $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} \cap j_*\mathcal{O}_{\mathfrak{Y}}$.

(c) We have moreover the equality:

$$\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) + \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}) = \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z). \tag{8.7.3.19.1}$$

(d) We have the equalities $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}}$ (see notation of 8.7.3.9):

$$\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) + \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)_{\mathbb{Q}}. \tag{8.7.3.19.2}$$

Proof. a) The first statement is tautological.

b) Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine and integral, and there exists $f \in A$ such that $\mathfrak{Z} = V(fA)$. Then $B := \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})) = A[1/f]$, $B_{\mathbb{Q}} = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}})$ and $\Gamma(\mathfrak{X}, j_*\mathcal{O}_{\mathfrak{Y}}) = A\{1/f\}$. The inclusion $B \subset A\{1/f\}$ induces the isomorphism $\iota\colon B/\pi B \xrightarrow{\sim} A\{1/f\}/\pi A\{1/f\}$. Let $h \in B_{\mathbb{Q}} \cap A\{1/f\}$. Let $s \in \mathbb{Z}$ be minimal such that $\pi^s h \in B$. Set $k := \pi^s h$. By the absurd, suppose $s \geq 1$. Denoting by $\overline{k}$ the image of k in $B/\pi B$, since $k \in \pi A\{1/f\}$ then $\iota(\overline{k}) = 0$ and therefore $k \in \pi B$. Since B is p-torsion free, this yields $\pi^{s-1}h \in B$, which is in contradiction with respect to the minimality of s. Hence, we are done.

c) We have to check that the inclusion $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) + \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}) \subset \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)$ is an equality. Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine and there exists $f \in A$ such that $\mathfrak{Z} = V(f)$. Let $P \in \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z))$. Then we get $P = f^{-n}Q$, for some $n \in \mathbb{N}$ and $Q \in \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z))$. We can write $Q = Q_0 + p^n R$, with $Q_0 \in A[1/f]$ and $R \in \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z))$. Since $p^n f^{-n} \in \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z))$, then we are done.

d) Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine and there exists $f \in A$ such that $\mathfrak{Z} = V(f)$. Let us check the first equality. Since $1/f \in \mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}}$, then we get the inclusion $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} \subset \mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}}$. Conversely, the fact that the inclusion $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} \subset \mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}}$ is an equality follows from the fact that $p/f^{p^{m+1}} \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}}$.

Since $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) + \mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$, then we get the first equality of 8.7.3.19.2. Finally if follows from 8.7.3.19.1 the last equality: $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} + \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)_{\mathbb{Q}}$. Finally if follows from 8.7.3.19.1 the last equality: $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} + \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})_{\mathbb{Q}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)_{\mathbb{Q}}$. □

Remark 8.7.3.20. Since the canonical inclusion $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} \to \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)_{\mathbb{Q}}$ is an isomorphism, then for any $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$-module $\mathcal{E}$, we have the canonical morphism $\mathcal{E}_{\mathbb{Q}} \to \mathcal{E}(*Z)_{\mathbb{Q}}$ is an isomorphism.

Lemma 8.7.3.21. *We have the equality $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} \cap j_*\mathcal{O}_{\mathfrak{Y}} = \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)$.*

Proof. Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine and integral, and there exists $f \in A$ such that $\mathfrak{Z} := V(f)$ is an effective Cartier divisor in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z. Let $B = \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}))$, $\widehat{B}$ its p-adic completion. Let $u \in \widehat{B}_{\mathbb{Q}} \cap A\{1/f\}$. Since $u \in \widehat{B}_{\mathbb{Q}}$ then following 8.7.3.19.2, we can write $u = h + r$, with $r \in \widehat{B}$ and $h \in A[1/f]_{\mathbb{Q}}$. Since $u \in A\{1/f\}$ and $\widehat{B} \subset A\{1/f\}$, then $h \in A[1/f]_{\mathbb{Q}} \cap A\{1/f\}$. Following 8.7.3.19.(b) we get the equality $A[1/f]_{\mathbb{Q}} \cap A\{1/f\} = A[1/f] \subset \widehat{B}[1/f]$ and we are done. □

Notation 8.7.3.22. We introduce the following sheaf of differential operators by adding some coefficients.

(a) By using Leibnitz formula, since $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$ is a sub-$\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module of $j_*\mathcal{O}_{\mathfrak{Y}}$, then so is $\mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)$. Hence, $\mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)$ is an $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. We get the sheaf of rings

$$\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z) := \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \qquad \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z) := \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z) \otimes_{\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z).$$

The sheaf of rings $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ is endowed with two structure of $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$-algebras (the left and the right one). This yields two sheaves:

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z) := \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z) \otimes_{\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z), \qquad \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\natural Z) := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) \otimes_{\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)} \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z),$$

where we choose the left (resp. right) structure for the left (resp. right) notation. Since $j^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z) = \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ and $j^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\natural Z) = \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$, then we can identify both sheaves as subsheaves of $j_*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$. We will see they are identical (see 8.7.3.23.1).

(b) Suppose there exists an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z (see definition 4.5.2.1). By using Leibnitz formula, we compute $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})$ is a sub-$\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module of $j_*\mathcal{O}_{\mathfrak{Y}}$. Hence, $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})$ is an $\mathcal{O}_{\mathfrak{X}}$-algebra endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module and we get the sheaf of rings

$$\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\mathfrak{Z}) := \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}.$$

Lemma 8.7.3.23. *Suppose there exists an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z (see definition 4.5.2.1). We have the equality*

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z) = \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) + \mathcal{D}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}) = \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\natural Z). \tag{8.7.3.23.1}$$

Proof. i) Let us check that the inclusion $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) + \mathcal{D}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}) \subset \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z)$ is an equality. Since this is local, we can suppose $\mathfrak{X} = \operatorname{Spf} A$ affine and integral, and there exists $f \in A$ such that $\mathfrak{Z} := V(f)$ is an effective Cartier divisor in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z. Let $P \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z))$. Then we get $P = f^{-n}Q$, for some $n \in \mathbb{N}$ and $Q \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. We can write $Q = Q_0 + p^n R$, with $Q_0 \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}))$ and $R \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Since $p^n f^{-n} \in \Gamma(\mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, hence we are done.

ii) Similarly we check the second inclusion is the identity: let $P \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\natural Z))$. Then we get $P = Qf^{-n}$, for some $n \in \mathbb{N}$ and $Q \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. We can write $Q = Q_0 + p^n R$, with $Q_0 \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}))$ and $R \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Since $p^n f^{-n} \in \Gamma(\mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, hence we are done. □

Proposition 8.7.3.24. *The sheaf $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z)$ is a subring of $j_*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$.*

Proof. Since this is local, then we can suppose there exists an effective Cartier divisor $\mathfrak{Z}$ in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z. Hence, this follows from the equality 8.7.3.23.1. Let $P \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ $Q \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}}(\mathfrak{Z}))$. Then there exists $n \in \mathbb{N}$ such that $f^n Q \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}})$ and $Qf^n \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}})$. Since $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ is a ring, this implies that $(PQ)f^n = P(Qf^n) \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ and $f^n(QP) = (f^nQ)P \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Hence, $PQ, QP \in \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z))$. □

Notation 8.7.3.25. We set $\mathcal{O}_{\mathfrak{X}}(^\dagger *Z) := \varinjlim_m \mathcal{B}^{(m)}_{\mathfrak{X}}(*Z)$. Moreover, for all integer m, we write $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z) := \mathcal{O}_{\mathfrak{X}}(^\dagger * Z) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z) := \varinjlim_m \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(*Z) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(^\dagger * Z) \otimes_{\mathcal{O}_{\mathfrak{X}}(^\dagger Z)} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)$.

Lemma 8.7.3.26. *Suppose $\mathfrak{X} = \operatorname{Spf} A$ is affine and there exists $f \in A$ such that $\mathfrak{Z} := V(f)$ is an effective Cartier divisor in $\mathfrak{X}/\mathfrak{S}$ which is a lifting of Z. Then $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(^\dagger * Z)) = A[\frac{1}{f}]^\dagger$, where $A[\frac{1}{f}]^\dagger$ the p-adic weak completion of $A[\frac{1}{f}]$ as A-algebra (see 8.7.3.10).*

Proof. Set $B_{\mathfrak{X}}(Z,r) := \Gamma(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}(Z,r))$ and $B_{\mathfrak{X}}(*Z,r) := \Gamma(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}}(*Z,r))$. We have:

$$B_{\mathfrak{X}}(Z,r) = \left\{ \sum_{n=0}^{\infty} a_n f^{-n} \ \mid a_n \in A,\ v_\pi(a_n) \geq \frac{n}{r} \text{ for all } n \right\}.$$

Hence, $B_{\mathfrak{X}}(*Z,r) = \left\{ \sum_{n=0}^{\infty} a_n f^{-n} \mid a_n \in A,\ \exists s \in \mathbb{N} \text{ satisfying } v_\pi(a_n) \geq \frac{n}{r} - s \text{ for all } n \right\}$. This yields the equality $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(^\dagger * Z)) = \left\{ \sum_{n=0}^{\infty} a_n f^{-n} : a_n \in A, \exists\, \lambda > 0 \text{ such that } v_\pi(a_n) \geq \frac{n}{\lambda} - 1 \text{ for all } n \right\}$ and we are done. □

Proposition 8.7.3.27. *The sheaves of rings $\mathcal{O}_{\mathfrak{X}}(^\dagger * Z)$ and $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z)$ are coherent.*

Proof. This follows from 8.7.3.26 and 4.1.2.17.(e). □

8.7.3.28. Suppose $\mathfrak{X}$ is noetherian.

(a) Let $\mathcal{E}$ a left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)$-module. Suppose that there exists a section $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ lifting a local equation of Z. Then, similarly to 7.4.3.1.1, we can check that the canonical map

$$\mathbb{R}\Gamma(\mathfrak{X}, \mathcal{E})[1/f] \to \mathbb{R}\Gamma(\mathfrak{X}, \mathcal{E}(*Z)). \qquad (8.7.3.28.1)$$

is an isomorphism.

(b) Moreover, if $\mathcal{F}$ is a $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z)$-module of finite presentation, then there exists a $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)$-module of finite presentation $\mathcal{E}$ such that $\mathcal{E}(*Z) \xrightarrow{\sim} \mathcal{F}$. Indeed, this is checked similarly to 8.4.1.11.b.ii).

Theorem 8.7.3.29. *Suppose $\mathfrak{X}$ is affine. We obtain then theorems A and B:*

A) The functors $\Gamma(\mathfrak{X}, -)$ and $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z) \otimes_{\Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z))} -$ are quasi-inverse equivalences between the category of $\Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z))$-modules of finite presentation and that of $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger * Z)$-modules of finite presentation.

B) For all $\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger * Z)$-module of finite presentation $\mathcal{F}$, for all integer $q \geq 1$, $H^q(\mathfrak{X}, \mathcal{F}) = 0$.

Proof. By 8.4.1.14.(a), we know this without the symbol "$*$". We can conclude by using 8.7.3.28. □

Remark 8.7.3.30. After tensoring with $\mathbb{Q}$, we get a coherent version of theorem A and B (see 8.7.5.5).

Let us first state a general theorem of C. Huyghe, which shows that, when Z is a large divisor in a diagram projective, the coherent $\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$-modules behave well from the cohomological point of view as coherent sheaves on an affine scheme; we can see this theorem as an analogue of theorems A and B, with overconvergence at infinity:

Theorem 8.7.3.31 ([Huy04, 5.3.3])**.** *Let $\mathfrak{X}$ be a projective and smooth $\mathcal{V}$-formal scheme, $Z \subset X$ be an ample divisor of its special fiber, and let $D^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}} := \Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}})$.*

(i) The ring $D^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$ is coherent, and the functor $\Gamma(\mathfrak{X}, -)$ induces an equivalence of categories between the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$-modules and that of $D^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$-coherent modules.

(ii) For all $n \geq 1$, and all coherent $\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$-module, we have $H^n(\mathfrak{X}, \mathcal{E}) = 0$.

C. Huyghe further establishes an invariance theorem which shows that the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$-modules does not depend on the smooth compactification X of $\mathfrak{Y}$:

Theorem 8.7.3.32 ([Huy04, 7.3.3])**.** *Let $f \colon \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of smooth $\mathcal{V}$-formal schemes, $Z \subset X$, $Z' \subset X'$ two divisors of the special fibers of X and X', $\mathfrak{Y} = \mathfrak{X} \setminus Z$, $\mathfrak{Y}' = \mathfrak{X}' \setminus Z'$. We assume that f induces an isomorphism $\mathfrak{Y}' \xrightarrow{\sim} f^{-1}(\mathfrak{Y})$. Then the functors $f^!_{Z',Z}$ and $f_{Z,Z',+}$ induce quasi-inverse category equivalences between the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$-modules, and that coherent $\mathcal{D}^\dagger_{\mathfrak{X}'}(^\dagger Z')_{\mathbb{Q}}$-modules.*

8.7.4 Increasing the level: finiteness of the tor-dimension with overconvergent coefficients

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension.

Lemma 8.7.4.1. *Let $r \in \mathbb{N}$, A be a separated complete $\mathcal{V}$-algebra (for the p-adic topology), noetherian (of finite Krull dimension) and p-torsion free, $f \in A$ which is not a zero divisor modulo $\mathfrak{m}$, $A_{\{f\}}$ be the p-adic completion of A_f, $B = A[T]/(f^rT - p)$.*

(a) The ring B is p-torsion free and is p-adically separated.

(b) The homomorphism $\widehat{B} \to A_{\{f\}}$ sending T to p/f^r is injective.

(c) If $A \otimes_{\mathcal{V}} k$ is integral then so is $\widehat{B}$.

Proof. This is [Ber96c, 4.3.3]. □

Corollary 8.7.4.2. *Suppose $\mathfrak{X}^\sharp$ is p-torsion free (see 3.3.1.12 for some example). Let Z be a divisor of X_0. Let $m, r \in \mathbb{N}$ be two integers such that p^{m+1} divides r.*

(a) With the notation 8.5.4.18, 8.7.3.4 and 8.7.3.5.1, the sheaf $\mathcal{B}_{\mathfrak{X}}(Z,r)$ is p-torsion free and we have the canonical isomorphism

$$\mathbb{L}\underset{\leftarrow X}{l}^*(\mathcal{B}_{\mathfrak{X}}(Z,r)) := \mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{B}_{\mathfrak{X}}(Z,r)) \xrightarrow{\sim} \mathcal{B}_{X_\bullet}(Z,r), \tag{8.7.4.2.1}$$

$$\mathbb{L}\underset{\leftarrow X}{l}^*(\mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \xrightarrow{\sim} \mathcal{B}_{X_\bullet}(Z,r) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}. \tag{8.7.4.2.2}$$

(b) We have $\mathcal{B}_{\mathfrak{X}}(Z,r) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$, $\mathcal{B}_{X_\bullet}(Z,r) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ and $\mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ and $\mathcal{B}_{X_\bullet}(Z,r) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$.

(c) Moreover, $(\mathfrak{X}^\sharp, \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))/\mathfrak{S}^\sharp$ is strongly quasi-flat (see definition 8.5.5.3).

Proof. This is a consequence of 8.7.4.1 and of 7.3.1.7 (for the last property, more precisely, in the definition 8.5.5.3, we can choose $\mathfrak{T} = \mathrm{Spf}\,\mathcal{V}$). □

8.7.4.3. It follows from 7.3.2.10 that the functors $\mathbb{R}\underset{\leftarrow X *}{l}$ and $\mathbb{L}\underset{\leftarrow X}{l}^*$ induce canonically quasi-inverse equivalences of categories between $D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ and $D^-_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$.

Since $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{O}_{X_i}$ have tor-dimension ≤ 1 for any $i \in \mathbb{N}$, then the functor $\mathbb{L}\underset{\leftarrow X}{l}^* = \mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} -$ preserves the boundedness. Hence, the functors $\mathbb{R}\underset{\leftarrow X *}{l}$ and $\mathbb{L}\underset{\leftarrow X}{l}^*$ induce canonically quasi-inverse equivalences of categories between $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$ and $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{X_\bullet})$.

Lemma 8.7.4.4. *Let $m' \geq m \geq 0$ be two integers, $D \subset T$ be two divisors of X. We suppose $\mathfrak{X}$ is p-torsion free.*

(a) The kernel of the canonical epimorphism $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ is a quasi-coherent $\mathcal{O}_X$-module.

(b) The canonical morphism $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T) \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ is an isomorphism, where the complete tensor product is defined at 7.3.4.2.1.

(c) The $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$-module $\mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 2.

(d) The $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 1.

Proof. Since the lemma is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\,A$ affine, and $T \subset X$ (resp $D \subset X$) has a local equation $\phi \in A/\pi A$ (resp. $\psi \in A/\pi A$). Since $D \subset T$, then ϕ is a multiple of ψ. Choose $g \in A$ a lifting of ψ and then f a lifting of ϕ which is a multiple of g. For any integer $r \geq 0$, put $\mathcal{B}(g,r) := \mathcal{O}_{\mathfrak{X}}[X]/(g^rX - p)$ and $\mathcal{B}^{(m)}(g) := \mathcal{B}^{(m)}(g, p^{m+1})$ where $\mathcal{O}_{\mathfrak{X}}[X]$ is a polynomial ring in the variable X with coefficients in

$\mathcal{O}_{\mathfrak{X}}$. For any integer $s \geq 0$, put $\mathcal{B}(f,s) := \mathcal{O}_{\mathfrak{X}}[Y]/(f^s Y - p)$, $\mathcal{B}^{(m')}(f) := \mathcal{B}(f, p^{m'+1})$ where $\mathcal{O}_{\mathfrak{X}}[Y]$ is a polynomial ring in the variable Y with coefficients in $\mathcal{O}_{\mathfrak{X}}$ (we use two distinct letters for the variable to avoid confusion later). Recall $\mathcal{B}^{(m')}_{\mathfrak{X}}(T) \xrightarrow{\sim} \mathcal{B}^{(m')}(f)$ (see 8.7.3.5.1) and do not depend (up to a canonical isomorphism) on the choice of f such that f modulo πA is a local equation of $T \subset X$. Since $\mathcal{B}^{(m')}(f) = \mathcal{B}(f^{p^{m'-m}}, p^{m+1})$ and since f is a multiple of g then we get from 8.7.3.1 the canonical morphism $\mathcal{B}^{(m)}(g) \to \mathcal{B}^{(m')}(f)$. It follows from 8.7.4.1 that we have the commutative diagram of injective ring morphisms

$$\begin{array}{ccc} B^{(m)}(g) & \hookrightarrow & A_{\{g\}} \\ \downarrow & & \downarrow \\ B^{(m')}(f) & \hookrightarrow & A_{\{f\}}. \end{array} \tag{8.7.4.4.1}$$

Since the image in $A_{\{g\}}$ of the class of X in $B^{(m)}(g)$ is $\frac{p}{g^{p^{m+1}}}$, since $B^{(m)}(g) \to A_{\{g\}}$ is injective then we still denote by $\frac{p}{g^{p^{m+1}}}$ the class of X in $B^{(m)}(g)$ or its image in $B^{(m')}(f)$ (following 8.7.4.4.1 this is harmless).

1) Let us prove first that, for any integer $j \leq -1$, $\mathcal{H}^j(\mathcal{B}^{(m)}(g) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f)) = 0$.

We have the short exact sequence $0 \to \mathcal{O}_{\mathfrak{X}}[X] \underset{\alpha}{\longrightarrow} \mathcal{O}_{\mathfrak{X}}[X] \underset{\beta}{\longrightarrow} \mathcal{B}^{(m)}(g) \to 0$, where α is the multiplication by $g^{p^{m+1}}X - p$ and β is the morphism of $\mathcal{O}_{\mathfrak{X}}$-algebras given by $X \mapsto \frac{p}{g^{p^{m+1}}}$. Since this exact sequence gives a canonical resolution of $\mathcal{B}^{(m)}(g)$ by some flat $\mathcal{O}_{\mathfrak{X}}$-modules, by applying the functor $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f)$ to this exact sequence, we see that it is a question of proving that p (or $g^{p^{m+1}}$) is not a zero divisor of $B^{(m')}(f)$. Since $B^{(m')}(f)$ is p-torsion free (see 8.7.4.1), then we are done.

2) Consider the diagram of $B^{(m')}(f)$-modules:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{\mathfrak{X}}[X] \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f) & \xrightarrow{\alpha\otimes \mathrm{id}} & \mathcal{O}_{\mathfrak{X}}[X] \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f) & \xrightarrow{\beta\otimes \mathrm{id}} & \mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f) & \to & 0 \\ & & \downarrow a & & \| & & \downarrow c & & \\ 0 & \to & \mathcal{B}^{(m')}(f)[X] & \xrightarrow{\alpha'} & \mathcal{B}^{(m')}(f)[X] & \xrightarrow{\beta'} & \mathcal{B}^{(m')}(f) & \to & 0, \end{array} \tag{8.7.4.4.2}$$

where α' is the multiplication by $X - \frac{p}{g^{p^{m+1}}}$, β' is the morphism the morphism of $\mathcal{B}^{(m')}(f)$-algebras given by $X \mapsto \frac{p}{g^{p^{m+1}}}$, a is the multiplication by $g^{p^{m+1}}$, c is induced by the ring morphism $\mathcal{B}^{(m)}(g) \to \mathcal{B}^{(m')}(f)$ and the identity of $\mathcal{B}^{(m')}(f)$. We compute easily that the diagram 8.7.4.4.2 is commutative. Moreover, the horizontal sequences are exact. Hence by using the snake lemma, we get the isomorphism $\mathcal{N}_0 := \operatorname{Ker} c \xrightarrow{\sim} \operatorname{Coker} a$. This implies that $\mathcal{N}_0$, is killed by $1 \otimes g^{p^{m+1}}$ and therefore by p. Since $\mathcal{B}^{(m')}(f)$ is p-torsion free, since $\mathcal{B}^{(m)}_{X_i}(D) \otimes_{\mathcal{O}_{X_i}} \mathcal{B}^{(m')}_{X_i}(T) \xrightarrow{\sim} \mathcal{V}/\pi^{i+1}\mathcal{V} \otimes_{\mathcal{V}} \mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f)$, this implies the short exact sequence $0 \to \mathcal{N}_0 \to \mathcal{B}^{(m)}_{X_i}(D) \otimes_{\mathcal{O}_{X_i}} \mathcal{B}^{(m')}_{X_i}(T) \to \mathcal{B}^{(m')}_{X_i}(T) \to 0$. For $i = 0$, this implies que $\mathcal{N}_0$ is a quasi-coherent $\mathcal{O}_X$-module. Following 7.3.1.9, this yields $\mathcal{N}_0 \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$. By taking the projective limits, we get moreover the exact sequence: $0 \to \mathcal{N}_0 \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(T) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(T) \to 0$. Since $\mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ is p-torsion free and separated complete, then $\mathcal{B}^{(m')}_{\mathfrak{X}}(T) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ (see 7.3.2.15). Since the notion of quasi-coherence of 7.3.1.5 is closed under devissage, i.e. quasi-coherent complexes are a triangulated subcategory of that all complexes, then $\mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(T) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$.

3) On the other hand, since $\mathcal{B}^{(m)}_{\mathfrak{X}}(D), \mathcal{B}^{(m')}_{\mathfrak{X}}(T) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ (see 8.7.4.2), then following 7.3.4.10, $\mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(T) \in D^{-}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$. Moreover, we get the isomorphism

$$\mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \left(\mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(T)\right) \xrightarrow{\sim} \mathcal{B}^{(m)}_{X_\bullet}(D) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_\bullet}} \mathcal{B}^{(m)}_{X_\bullet}(T).$$

4) Hence, with the last result of 2) and with 3), in order to prove the isomorphism of (b), it remains to check the canonical isomorphism $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \left(\mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(T)\right) \xrightarrow{\sim} \mathcal{B}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(T)$. This

is checked as follows: By applying the functor $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} -$ to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}_0 & \longrightarrow & \mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f) & \longrightarrow & \mathcal{B}^{(m')}(f) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N}_0 & \longrightarrow & \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T) & \longrightarrow & \mathcal{B}^{(m')}_{\mathfrak{X}}(T) & \longrightarrow & 0, \end{array} \tag{8.7.4.4.3}$$

we get $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \left(\mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f)\right) \xrightarrow{\sim} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \left(\mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T)\right)$. Finally following the step 1), we get $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \left(\mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f)\right) \xrightarrow{\sim} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m)}(g)) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f)) \xrightarrow{\sim} \mathcal{B}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{B}^{(m')}_{X_i}(T)$.

5) It remains to check the property (c) via the following steps 5,6,7). We have the short exact sequence $0 \to \mathcal{O}_{\mathfrak{X}}[Y] \underset{\alpha''}{\longrightarrow} \mathcal{O}_{\mathfrak{X}}[Y] \underset{\beta''}{\longrightarrow} \mathcal{B}^{(m')}(f) \to 0$, where α'' is the multiplication by $f^{p^{m'+1}}Y - p$ and β'' is the morphism of $\mathcal{O}_{\mathfrak{X}}$-algebras given by $Y \mapsto \frac{p}{f^{p^{m'+1}}}$. Since $\mathcal{B}^{(m)}(g)$ is p-torsion free, by applying the functor $\mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} -$, similarly to the step 1) we get the exact sequence

$$0 \longrightarrow \mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}[Y] \xrightarrow{\mathrm{id}\otimes\alpha''} \mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}[Y] \xrightarrow{\mathrm{id}\otimes\beta''} \mathcal{B}^{(m)}(g) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}(f) \longrightarrow 0. \tag{8.7.4.4.4}$$

We set $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)\{Y\} := \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}[Y]$. By taking the p-adic completion of 8.7.4.4.4, we get the sequence

$$0 \longrightarrow \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\{Y\} \xrightarrow{\widehat{\alpha}} \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\{Y\} \xrightarrow{\widehat{\beta}} \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T) \longrightarrow 0 \tag{8.7.4.4.5}$$

where $\widehat{\alpha}$ is the multiplication by $f^{p^{m'+1}}Y - p$ and $\widehat{\beta}$ is the morphism of complete $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$-algebras given by $Y \mapsto \frac{p}{f^{p^{m'+1}}}$. Since $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$ is p-torsion free, then the sequence is exact. Hence, $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 1 as $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$-module.

6) With notation of 2), $\mathcal{N}_0 \xrightarrow{\sim} \mathrm{Coker}\, a \xrightarrow{\sim} (\mathcal{B}^{(m')}(f)/g^{p^{m+1}}\mathcal{B}^{(m')}(f))[X]$. Since $\mathcal{B}^{(m')}(f) := \mathcal{O}_{\mathfrak{X}}[Y]/(f^{p^{m'+1}}Y - p)$, then we get $\mathcal{B}^{(m')}(f)/g^{p^{m+1}}\mathcal{B}^{(m')} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}[Y]/(g^{p^{m+1}}, f^{p^{m'+1}}Y - p)$, where $(g^{p^{m+1}}, f^{p^{m'+1}}Y - p)$ is the ideal of $\mathcal{O}_{\mathfrak{X}}[Y]$ generated by the elements $g^{p^{m+1}}$ and $f^{p^{m'+1}}Y - p$. Using the same notation Y instead of X to define $\mathcal{B}^{(m)}(g)$, since $(g^{p^{m+1}}, f^{p^{m'+1}}Y - p) = (g^{p^{m+1}}, p) = (g^{p^{m+1}}, g^{p^{m+1}}Y - p)$, then we get

$$\mathcal{B}^{(m')}(f)/g^{p^{m+1}}\mathcal{B}^{(m')}(f) \xrightarrow{\sim} \mathcal{O}_X[Y]/(\overline{g}^{p^{m+1}}) \xrightarrow{\sim} \mathcal{B}^{(m)}(g)/g^{p^{m+1}}\mathcal{B}^{(m)}(g),$$

where $\overline{g}$ is the image of g via the projection $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_X$. Similarly, since $\mathcal{B}^{(m)}(g) \xrightarrow{\sim} \mathcal{B}^{(m)}_{\mathfrak{X}}(D)$ taking the p-adic completion, we get

$$\mathcal{B}^{(m)}_{\mathfrak{X}}(D)/g^{p^{m+1}}\mathcal{B}^{(m)}_{\mathfrak{X}}(D) \xrightarrow{\sim} \mathcal{O}_X[Y]/(\overline{g}^{p^{m+1}}).$$

This yields

$$\mathcal{N}_0 \xrightarrow{\sim} (\mathcal{B}^{(m)}_{\mathfrak{X}}(D)/g^{p^{m+1}}\mathcal{B}^{(m)}_{\mathfrak{X}}(D))[Y].$$

Since $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$ is p-torsion free, since $g^{p^{m+1}}Y = p$ in $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$, then $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$ is $g^{p^{m+1}}$-torsion free. Hence, $\mathcal{N}_0$ has tor-dimension ≤ 1 as $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$-module.

7) It follows from the parts 5) and 6) of the proof and from the bottom exact sequence of the diagram 8.7.4.4.3 that $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$-module $\mathcal{B}^{(m')}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 2.

□

Remark 8.7.4.5. With notation 8.7.4.4, it seems false in general that for any $i \in \mathbb{N}$, the canonical morphism $\mathcal{B}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(T) \to \mathcal{B}^{(m)}_{X_i}(D) \otimes_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(T)$ is an isomorphism. But, when the divisors are reduced and irreducible components are two by two distinct, then this becomes true (see 9.1.3.1).

8.7.4.6. We keep notation 8.7.3.12. Following 8.1, we denote by $M(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$ the category of $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$-modules. Let $\sharp \in \{\emptyset, +, -, \mathrm{b}\}$. We get $D^{\sharp}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$ the derived category of complexes of $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$-modules with the corresponding boundedness property. Moreover, we denote similarly to 8.1 by $\underrightarrow{D}^{\sharp}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$, $\underrightarrow{M}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$, $\underrightarrow{LD}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$, $\underrightarrow{LM}^{\sharp}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$ the localized categories of $D^{\sharp}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$. We have also the notion of coherent up to lim-ind-isogeny $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$-modules and we denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$ of coherent up to lim-ind-isogeny $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$-modules. By using the flatness property of 8.7.3.14, the finiteness tor-dimension of 8.7.4.4.c, we can applying 8.4.1.15.b. Hence, we get the canonical fully faithful functor $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))$ and of the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ induces the equivalence of categories

$$\underrightarrow{l}^{*}_{\mathbb{Q}} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}). \tag{8.7.4.6.1}$$

Corollary 8.7.4.7. *Let $\mu \in L(\mathbb{N})$, $m' \geq m \geq 0$ be two integers and $n = \mu(m)$, $n' = \mu(m')$. Let $D \subset T$ be two divisors of P. We suppose $\mathfrak{X}$ is p-torsion free. We have the following tor finiteness results.*

(a) The $\mathcal{O}_{X_\bullet}$-module $\mathcal{B}^{(n')}_{X_\bullet}(T)$ (resp. $\mathcal{B}^{(n)}_{X_\bullet}(D)$-module $\mathcal{B}^{(n')}_{X_\bullet}(T)$) has tor-dimension ≤ 1 (resp. has tor-dimension ≤ 2).

(b) The (left or right) $\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module $\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ has tor-dimension ≤ 2.

(c) The (left or right) $\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module $\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}$ has tor-dimension $\leq d+2$.

(d) The $\mathcal{B}^{(n)}_{\mathfrak{X}}(D)$-module $\mathcal{B}^{(n')}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 2. The (left or right) $\mathcal{B}^{(n)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{B}^{(n')}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ has tor-dimension ≤ 2. The (left or right) $\mathcal{B}^{(n)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{B}^{(n')}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ has tor-dimension $\leq d+2$.

Proof. a) The first statement follows from 8.7.4.4.c.

b) Let us check b). Since the proof is the same, we reduce to check the left case. Let $\mathcal{M}_\bullet$ be a right $\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module. Let $\mathcal{P}^\bullet_\bullet$ be a resolution of $\mathcal{M}_\bullet$ by flat right $\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module. Since $\mathcal{P}^\bullet_\bullet$ is also a resolution of $\mathcal{M}_\bullet$ by flat $\mathcal{B}^{(n)}_{X_\bullet}(D)$-modules, then we get:

$$\begin{aligned} \mathcal{M}_\bullet \otimes^{\mathbb{L}}_{\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \left(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) &\xrightarrow{\sim} \mathcal{P}^\bullet_\bullet \otimes_{\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \left(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \\ \underset{4.3.4.8.3}{\xleftarrow{\sim}} \mathcal{P}^\bullet_\bullet \otimes_{\mathcal{B}^{(n)}_{X_\bullet}(D)} \mathcal{B}^{(n')}_{X_\bullet}(T) &\xleftarrow{\sim} \mathcal{M}_\bullet \otimes^{\mathbb{L}}_{\mathcal{B}^{(n)}_{X_\bullet}(D)} \mathcal{B}^{(n')}_{X_\bullet}(T). \end{aligned} \tag{8.7.4.7.1}$$

Hence, it follows from the part (a) that the object $\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$ of $D^{\mathrm{b}}_{\mathrm{qc}}({}^{l}\mathcal{B}^{(n)}_{X_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ has tor-amplitude in $[-2, 0]$.

c) Let us check c). Let $\mathcal{N}_\bullet$ be a right $\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module. Let $\mathcal{Q}^\bullet_\bullet$ be a resolution of $\mathcal{N}_\bullet$ by flat $\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module. We have the isomorphisms of $(\mathcal{B}^{(n')}_{X_\bullet}(T), \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodules:

$$\begin{aligned} \mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet} &\xleftarrow{\sim} \mathcal{B}^{(n')}_{X_\bullet}(T) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet} \xrightarrow{\sim} \mathcal{B}^{(n')}_{X_\bullet}(T) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_\bullet}} \left(\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \\ \underset{4.6.3.5}{\xrightarrow{\sim}} \left(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet} &\xrightarrow{\sim} \left(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}. \end{aligned}$$

This yields the canonical morphism of complexes of $(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}, \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodules

$$\left(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet} \to \left(\mathcal{B}^{(n')}_{X_\bullet}(T) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \otimes_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}} \mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}$$

is an isomorphism. By using 4.3.1.2.1, we get the isomorphism of complexes of $(\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet},\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet})$-bimodules

$$\left(\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right)\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}\xrightarrow{\sim}\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}. \tag{8.7.4.7.2}$$

This yields the first isomorphism:

$$\mathcal{N}_\bullet\otimes^{\mathbb{L}}_{\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\left(\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}\right)\underset{8.7.4.7.2}{\xrightarrow{\sim}}$$

$$\mathcal{N}_\bullet\otimes^{\mathbb{L}}_{\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\left(\left(\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right)\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}\right) \tag{8.7.4.7.3}$$

$$\underset{4.6.3.5}{\xrightarrow{\sim}}\left(\mathcal{N}_\bullet\otimes^{\mathbb{L}}_{\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\left(\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}\right)\right)\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet} \tag{8.7.4.7.4}$$

$$\xrightarrow{\sim}\mathcal{N}_\bullet\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet} \tag{8.7.4.7.5}$$

Hence, it follows from 3.2.4.3 that the object $\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet}$ of $D^{\mathrm{b}}_{\mathrm{qc}}({}^l\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet})$ has tor amplitude in $[-d,0]$. Hence, c) follows from b).

d) Let us check the non-respective case (the respective ones are checked similarly). Since $\mathcal{B}^{(n')}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\xrightarrow{\sim}\mathbb{R}\underleftarrow{l}_{X*}(\mathcal{B}^{(n')}_{X_\bullet}(T)\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{D}^{(m')}_{X^\sharp_\bullet/S^\sharp_\bullet})$ (for any $n'\geq m'$ and T), then the part c) is a consequence of the part c) and of the Theorem 7.3.2.15.

□

8.7.5 Flatness by adding overconvergent singularities

Lemma 8.7.5.1. *We keep notation 3.4 in the algebraic case. Let $m\in\mathbb{N}$, r be a multiple of p^{m+1}, r' be a multiple of r. Fix $f\in\Gamma(X,\mathcal{O}_X)$ and put $\mathcal{B}_X(f,r):=\mathcal{O}_X[T]/(f^rT-p)$. Let $\rho_{f^{\frac{r'}{r}-1}}\colon\mathcal{B}_X(f,r)=\mathcal{O}_X[T]/(f^rT-p)\to\mathcal{B}_X(f^{\frac{r'}{r}},r)=\mathcal{B}_X(f,r')=\mathcal{O}_X[T']/(f^{r'}T'-p)$ be the homomorphism of $\mathcal{O}_X$ algebras of 8.7.3.1 given by $T\mapsto f^{r'-r}T'$. Then for any $k\in\mathbb{N}$, the sub-$\mathcal{B}_X(f,r)$-module of $\mathcal{B}_X(f,r')$ generated by T'^i for $i\leq k$ is a left sub-$\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module of $\mathcal{B}_X(f,r')$.*

Proof. First, suppose $S^\sharp=\operatorname{Spec}\mathbb{Z}_{(p)}$ and $X^\sharp=\operatorname{Spec}\mathbb{Z}_{(p)}[t]$, $f=t$. This case has already been proved in [Ber96c, 4.3.4]. More generally, using the diagram of the form 8.7.3.1.1, we get by construction $\rho_{f^{\frac{r'}{r}-1}}=\phi_f^*(\rho_{t^{\frac{r'}{r}-1}})$. Hence, we are done. Since the structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module on $\mathcal{B}_X(f,r')$ is induced by pullback via 4.4.2.4, then we are done. □

Theorem 8.7.5.2. *Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp\to\mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is locally noetherian of finite Krull dimension and p-torsion free. Let Z be a divisor of X, $m'\geq m$, r (resp. r') a multiple of p^{m+1} (resp. $p^{m'+1}$). Then with notation 8.7.3.5 the extension*

$$\mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}\to\mathcal{B}_{\mathfrak{X}}(Z,r')\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$$

is left and right flat.

Proof. It follows from 7.5.3.1 that we only need to show

$$\mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}\to\mathcal{B}_{\mathfrak{X}}(Z,r')\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$$

is flat. This check is similar to that of 7.5.3.1. Since this is local, we can suppose $\mathfrak{X}^\sharp\to\mathfrak{S}^\sharp$ is endowed with logarithmic coordinates $u_1,\dots,u_d$, $\mathcal{X}$ is connected affine and there exists $f\in\Gamma(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})=:A$ such that its image in $A/\pi A$ is a local equation of the divisor Z of X.

Let $B = A[T]/(f^r T - p)$, $B' = A[T']/(f^{r'} T' - p)$, $D = \Gamma(\mathfrak{U}, \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$; $\widehat{B}, \widehat{B'}, \widehat{D}$ be their respective p-adic completion. Let $\eta \colon B \to B'$ be the A-algebra homomorphism given by $T \mapsto f^{r'-r} T'$. Let $\eta' \colon B \otimes D \to B' \otimes D$ be the homomorphism induced by η (we simply write in the proof $\otimes$ for $\otimes_A$). It suffices to prove $B' \widehat{\otimes} D_{\mathbb{Q}}$ is flat over $B \widehat{\otimes} D_{\mathbb{Q}}$. Observe that since f is not a zero divisor modulo π, the rings B and B' are p-torsion free (see 8.7.4.1). Hence, since B and B' are noetherian then $\widehat{B}$ and $\widehat{B}'$ are p-torsion free by flatness. Again, by flatness $B \otimes D$, $B' \otimes D$, $B \widehat{\otimes} D$, $\widehat{B'} \widehat{\otimes} \widehat{D}$ are p-torsion free. It follows from 8.7.4.1 that $B \to \widehat{B}$ and $\widehat{B} \to \widehat{B}'$ are injective. Hence, since $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ is endowed with logarithmic coordinates then $B \widehat{\otimes} D \to B' \widehat{\otimes} D$ and $B' \otimes D \to B' \widehat{\otimes} D$ are injective (use the description 7.5.1.5.2). Let us introduce the set

$$D' = B \widehat{\otimes} D + B' \otimes D \subset B' \widehat{\otimes} D.$$

In fact D' is a subring: let $P \in B \widehat{\otimes} D$ and $Q \in B' \otimes D$. It is enough to check that $PQ, QP \in D'$. There exists $a \in \mathbb{N}$ such that $f^a Q \in B \otimes D$. Since $p = f^r T$in B, this yields $p^a Q \in B \otimes D$. We can write $P = P_1 + p^a P_2$ with $P_1 \in B \otimes D$ and $P_2 \in B \widehat{\otimes} D$. Since $PQ = P_1 Q + P_2 (p^a Q)$ and $QP = QP_1 + (p^a Q) P_2$, then we are done.

Now let's prove that the canonical morphism $\widehat{D'} \to B' \widehat{\otimes} D$ is an isomorphism. We have to check the homomorphisms $D'/p^i D' \to B' \widehat{\otimes} D / p^i (B' \widehat{\otimes} D)$ are isomorphisms for all integer $i \geq 1$. Since $B' \otimes D / p^i B' \otimes D) \xrightarrow{\sim} B' \widehat{\otimes} D / p^i (B' \widehat{\otimes} D)$ and $B' \otimes D \subseteq D'$ then we get the surjectivity. For the injectivity, an operator of $D' \cap p^i (B' \widehat{\otimes} D)$ can be written of the form $P + Q = p^i P'$ with $P \in B \widehat{\otimes} D$, $Q \in B' \otimes D$ and $P' \in B' \widehat{\otimes} D$. We can write $P = P_1 + p^i P_2$ with $P_1 \in B \otimes D$ and $P_2 \in B \widehat{\otimes} D$. Then $P_1 + Q \in (B' \otimes D) \cap p^i (B' \widehat{\otimes} D) = p^i B' \otimes D$. Hence, $P + Q = (P_1 + Q) + p^i P_2 \in p^i D'$ and we are done.

Since $B_{\mathbb{Q}} \to B'_{\mathbb{Q}}$ is an isomorphism, then so is $(B \otimes D)_{\mathbb{Q}} \to (B' \otimes D)_{\mathbb{Q}}$. This yields the equality $B \widehat{\otimes} D_{\mathbb{Q}} = D'_{\mathbb{Q}}$. Hence, it is sufficient to prove that D' is noetherian because this implies the flatness of $D' \to \widehat{D'}$. We prove this noetherianity as follows.

For any $k \in \mathbb{N}$, let B'_k be the sub-B-module of B' generated by T'^j for $0 \leq j \leq k$. Following 8.7.5.1, B'_k is a left sub-D-module of B'. Hence, by using the formula 4.1.2.2.3, we can check that $E'_k := B'_k \otimes D$ is a left sub-$B \otimes D$-module of $B' \otimes D$.

The element P' of E'_k are the element of $B' \otimes D$ which can be written of the form

$$P' = \sum_{\underline{i} \in \mathbb{N}^d} b'_{\underline{i}} \underline{\partial}_\sharp^{\langle \underline{i} \rangle} = \sum_{\underline{i} \in \mathbb{N}^d} \left(\sum_{j=0}^k T'^j b_{\underline{i},j} \right) \underline{\partial}_\sharp^{\langle \underline{i} \rangle} = \sum_{j=0}^k T'^j \left(\sum_{\underline{i} \in \mathbb{N}^d} b_{\underline{i},j} \underline{\partial}_\sharp^{\langle \underline{i} \rangle} \right), \tag{8.7.5.2.1}$$

where the sums are finite, $b_{\underline{i},j} \in B$ and $b'_{\underline{i}} = \sum_{j=0}^k T'^j b_{\underline{i},j} \in B'_k$. In the left equality of 8.7.5.2.1, the elements $b'_{\underline{i}} \in B'$ are uniquely determined by P' but be aware that this is not clear in the right equality for $b_{\underline{i},j}$.

By p-adic completion, for any integer $k \geq 0$, we get the homomorphism of $B \widehat{\otimes} D$-modules $B'_k \widehat{\otimes} D \to B' \widehat{\otimes} D$ whose image is denoted by D'_k. Set $D'_k = 0$ for any integer $k < 0$. Since E'_k is generated as sub-$B \otimes D$-module of $B' \otimes D$ by T'^j for $j = 0, \ldots, k$, then D'_k is the sub-$B \widehat{\otimes} D$-module of $B' \widehat{\otimes} D$ generated by T'^j for $j = 0, \ldots, k$. We prove now that $\operatorname{gr} D' := \oplus_{k \in \mathbb{N}} D'_k / D'_{k-1}$ is noetherian, which will imply that so is D' (recall 1.4.2.9).

The element P' of D'_k are the element of $B' \otimes D$ which can be written of the form

$$P' = \sum_{j=0}^k T'^j \left(\sum_{\underline{i} \in \mathbb{N}^d} b_{\underline{i},j} \underline{\partial}_\sharp^{\langle \underline{i} \rangle} \right) = \sum_{\underline{i} \in \mathbb{N}^d} \left(\sum_{j=0}^k T'^j b_{\underline{i},j} \right) \underline{\partial}_\sharp^{\langle \underline{i} \rangle} = \sum_{\underline{i} \in \mathbb{N}^d} b'_{\underline{i}} \underline{\partial}_\sharp^{\langle \underline{i} \rangle}, \tag{8.7.5.2.2}$$

where $b_{\underline{i},j} \in \widehat{B}$ converges to 0 when $|\underline{i}|$ goes to infinity, where $b'_{\underline{i}} := \sum_{j=0}^k T'^j b_{\underline{i},j} \in \widehat{B'}$.

For $k \geq 1$ we have $f^{r'} T'^k = p T'^{k-1} \in D'_{k-1}$ because $f^{r'} T' = p$ in $B' \widehat{\otimes} D$; thus $f^{r'} \operatorname{gr}_k D' = 0$. Write $r' = ar$. Since $p = f^r T$, then $f^{r'}$ divides p^a in $B \widehat{\otimes} D$ and thus $p^a \operatorname{gr}_k D' = 0$ for $k \geq 1$. Since $\operatorname{gr}_0 D' = D'_0 = B \widehat{\otimes} D$ then we get the ring homomorphism $B \widehat{\otimes} D \to \operatorname{gr} D'$. This yields a structure of $B \widehat{\otimes} D$-module on $p^a \operatorname{gr} D'$ such that $p^a \operatorname{gr} D' = p^a \operatorname{gr}_0 D' = p^a B \widehat{\otimes} D$ is $B \widehat{\otimes} D$-linear. Since $B \widehat{\otimes} D$ is noetherian, then so is $p^a \operatorname{gr} D'$.

Now let us check that $\operatorname{gr} D' / p^a \operatorname{gr} D'$ is noetherian. We have

$$\operatorname{gr} D' / p^a \operatorname{gr} D' = (B \widehat{\otimes} D) / p^a (B \widehat{\otimes} D) \oplus_{k \geq 1} \operatorname{gr}_k D'. \tag{8.7.5.2.3}$$

For any $n \in \mathbb{N}$, we set $D_n := \Gamma(\mathfrak{U}, \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,n})$ which yields the order filtration $(D_n)_{n\in\mathbb{N}}$ on D. For any $k, n \in \mathbb{N}$, we put $C'_{I,k} := D'_k \cap (B' \otimes D)$, $C'_{II,n} := B' \otimes D_n$.

Let $k \geq 1$ be an integer. Since $p^a D'_k \subset D'_{k-1}$, since $\widehat{B} = B + p^a \widehat{B}$, then we get from 8.7.5.2.1 and 8.7.5.2.2 $D'_k \subset E'_k + D'_{k-1}$. Since we have also $E'_k \subset D'_k \cap (B' \otimes D) = C'_{I,k}$, then the monomorphism $\mathrm{gr}^I_k(B' \otimes D) = C'_{I,k}/C'_{I,k-1} \hookrightarrow D'_k/D'_{k-1}$ is in fact an isomorphism. Since $C'_{I,0} = (B\widehat{\otimes}D) \cap (B' \otimes D) \to (B\widehat{\otimes}D)/p^a(B\widehat{\otimes}D)$ is surjective, then it follows from 8.7.5.2.3 that the canonical morphism $\mathrm{gr}^I(B' \otimes D) \to \mathrm{gr}\, D'/p^a \mathrm{gr}\, D'$ is an epimorphism. Hence, we reduce to prove $\mathrm{gr}^I(B' \otimes D)$ is noetherian. Since $\mathrm{gr}^{II}(B' \otimes D) \xrightarrow{\sim} B' \otimes \mathrm{gr}\, D$ is noetherian, then $B' \otimes D$ is noetherian and so is $\mathrm{gr}^I(B' \otimes D)$. Hence $\mathrm{gr}\, D'/p^a \mathrm{gr}\, D'$ is noetherian. Finally the exact sequence

$$0 \to p^a \mathrm{gr}\, D' \to \mathrm{gr}\, D' \to \mathrm{gr}\, D'/p^a \mathrm{gr}\, D' \to 0$$

implies that $\mathrm{gr}\, D'$ is noetherian. This completes the proof of theorem 8.7.5.2. □

Corollary 8.7.5.3. *With notation and hypotheses of 8.7.5.5, the sheaf $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)_\mathbb{Q}$ is right and left flat over $\mathcal{B}_\mathfrak{X}(Z,r)\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$.*

8.7.5.4. It follows from Corollary 7.2.3.16 (see also the example 7.2.3.2 which can be applied in the case where $\mathcal{B} = \mathcal{B}^{(m)}_\mathfrak{X}(Z)$ thanks to 8.7.3.14), Proposition 7.4.5.2, the corollary 8.7.4.7 and Theorem 8.7.5.2 that the condition of Theorem 8.4.1.15.b are fully satisfied in the case where $I = \mathbb{N}$ and for any $i \in I$, $\mathcal{D}^{(i)} := \mathcal{B}^{(\lambda(i))}_\mathfrak{X}(Z)\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}}\mathcal{D}^{(i)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, where $\lambda \in L(\mathbb{N})$. Hence, we have the equivalence of categories:

$$\underrightarrow{l}^*_\mathbb{Q} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)_\mathbb{Q}). \tag{8.7.5.4.1}$$

Theorem 8.7.5.5. *With notation and hypotheses of 8.7.5.2, the sheaf $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)_\mathbb{Q}$ (see 8.7.3.12) is coherent. We have theorems of type A and B for right or left coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)_\mathbb{Q}$-modules (in the sense of 1.4.3.14).*

Proof. Following 8.4.1.14, this is a consequence of 8.7.5.2. □

Remark 8.7.5.6. With notation and hypotheses of 8.7.5.5, we do not know if $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ (and a fortiori $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(^\dagger Z)$) is coherent.

8.7.6 Restriction outside the overconvergent singularities: full faithfulness

Lemma 8.7.6.1. *Let $S^\sharp$ be a nice fine log scheme over* $\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z})$, *(see definition 3.1.1.1). Let $X^\sharp \to S^\sharp$ be a log smooth morphism of log schemes, $f \in \Gamma(X, \mathcal{O}_X)$, r be a multiple of p^{m+2} and consider the $\mathcal{O}_X$-algebra $\mathcal{B}_X(f,r) := \mathcal{O}_X[T]/(f^r T)$ endowed with its canonical structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module compatible with its structure of $\mathcal{O}_X$-algebra (see 8.7.3.1). Then $T \otimes 1$ belongs to the center of the sheaf of rings $\mathcal{B}_X(f,r) \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$.*

Proof. Since this is local, we can suppose $X^\sharp/S^\sharp$ has logarithmic coordinates. By using 4.1.2.2.3, we reduce to check that $\underline{\partial}^{\langle k\rangle}_\sharp(T) = 0$. Following Taylor formula (see 4.2.1.5.4), this is equivalent to saying that for any $n \in \mathbb{N}$ we have $\varepsilon_n^{\mathcal{B}_{X(f,r)}}(1 \otimes T) = T \otimes 1$. Consider the following commutative diagram

$$\begin{array}{ccc} X^\sharp & \xrightarrow{\phi_f} & \mathbb{A}^1_{\mathbb{Z}_{(p)}} = \mathrm{Spec}\,\mathbb{Z}_{(p)}[t] \\ \downarrow & & \downarrow \\ (S^\sharp, \mathfrak{a}_S, \mathfrak{b}_S, \alpha_S) & \longrightarrow & (\mathrm{Spec}\,\mathbb{Z}_{(p)}, (p), (p), \alpha_{(p)}), \end{array} \tag{8.7.6.1.1}$$

where ϕ_f is the morphism given by f, $\alpha_{(p)}$ is the canonical m-PD (see 1.2.4.2.a). By construction, the canonical structure of $\mathcal{D}^{(m)}_{X^\sharp/S^\sharp}$-module compatible with its structure of $\mathcal{O}_X$-algebra on $\mathcal{B}_X(f,r)$ is given by pullback (see 4.4.2.4) via $\mathcal{B}_X(f,r) = \phi^*_f(\mathcal{B}_{\mathbb{A}^1_{\mathbb{Z}_{(p)}}}(t,r))$. By construction the m-PD-stratification

$\varepsilon_n^{\mathcal{B}_{X(f,r)}}$ is therefore given by pullback from the m-PD-stratification $\varepsilon_n^{\mathcal{B}_{\mathbb{A}^1}(t,r)}$. Hence, we reduce to the case where $S^\sharp = \operatorname{Spec} \mathbb{Z}_{(p)}$ and $X^\sharp = \operatorname{Spec} \mathbb{Z}_{(p)}[t]$, $f = t$. This latter case has already been proved in [Ber96c, 4.3.9]. □

Lemma 8.7.6.2. *Let $A \to B$ be a ring homomorphism. The following property are equivalent.*

(a) B is right flat on A and for any monogeneous of finite presentation left A-module M, we have the implication $B \otimes_A M = 0 \Rightarrow M = 0$;

(b) B is faithfully flat as A-module.

Proof. The implication $(b) \Rightarrow (a)$ is straightforward. Let us prove $(a) \Rightarrow (b)$. Let M be a left A-module such that $B \otimes_A M = 0$. Let $x \in M$. If we set $M' = A \cdot x \subset M$, then by flatness $B \otimes_A M' = 0$. Hence, to check that $M = 0$ we reduce to the case where M is monogeneous. We get $M = A/J$ with J a left ideal of A. Let Λ be the set of finite subsets of J. For any $\lambda \in \Lambda$, let J_λ be the left ideal of A generated by the element of λ. Then $J = \cup_{\lambda \in \Lambda} J_\lambda$. Let $M_\lambda := A/J_\lambda$. Since filtered inductive limits are exact, then we have $\varinjlim_{\lambda \in \Lambda} M_\lambda = M$ and therefore $\varinjlim_{\lambda \in \Lambda} B \otimes_A M_\lambda = 0$. Since Λ is filtered, then for λ large enough, $1 \otimes \overline{1} = 0$ in $B \otimes_A M_\lambda$, where $\overline{1}$ is the image of 1 in M_λ. Since $B \otimes_A M_\lambda = B/BJ_\lambda$ is generated by $1 \otimes \overline{1}$, this yields $B \otimes_A M_\lambda = 0$. Hence, $M_\lambda = 0$. This yields $M = 0$. □

Definition 8.7.6.3. Let X be a topological space, $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. We say that $\mathcal{A} \to \mathcal{B}$ is right (resp. left) faithfully flat if for any $x \in X$, $\mathcal{A}_x \to \mathcal{B}_x$ is right (resp. left) faithfully flat.

Lemma 8.7.6.4. *Let X be a topological space, $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. The following properties are equivalent:*

(a) $\mathcal{A} \to \mathcal{B}$ is right faithfully flat ;

(b) $\mathcal{A} \to \mathcal{B}$ is right flat and that for any left $\mathcal{A}$-module $\mathcal{M}$, we have the implication $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M} = 0 \Rightarrow \mathcal{M} = 0$.

Proof. Since $\mathcal{M} = 0$ if and only if $\mathcal{M}_x = 0$ for any $x \in X$, then we get $(a) \Rightarrow (b)$ (for instance use Lemma 8.7.6.2). Suppose now $\mathcal{A} \to \mathcal{B}$ satisfies (b) and let us prove (a). Let $x \in X$ and let M be a monogeneous left $\mathcal{A}_x$-module of finite presentation such that $\mathcal{B}_x \otimes_{\mathcal{A}_x} M = 0$. Following 8.7.6.2, it is enough to check that $M = 0$. Let $\mathfrak{B}$ be the collection of open subsets of X containing x. Following 8.4.1.11.b, since $\mathfrak{B}$ is filtered, then there exist $U \in \mathfrak{B}$ and M_U a left $\Gamma(U, \mathcal{A})$-module of finite presentation endowed with the isomorphism $M \xrightarrow{\sim} \mathcal{A}_x \otimes_{\Gamma(U,\mathcal{A})} M_U$. Using again 8.4.1.11.b, shrinking U if necessary, we can suppose that M_U is monogeneous. Since $\mathcal{B}_x \otimes_{\Gamma(U,\mathcal{A})} M_U \xrightarrow{\sim} \mathcal{B}_x \otimes_{\mathcal{A}_x} M = 0$, since $\mathfrak{B}$ is filtered, since M_U is monogeneous, then there exists $U' \in \mathfrak{B}$ such that $U' \subset U$ and $\Gamma(U', \mathcal{B}) \otimes_{\Gamma(U,\mathcal{A})} M_U = 0$. Set $\mathcal{M}_{U'} := \mathcal{A}|_{U'} \otimes_{\Gamma(U,\mathcal{A})} M_U$. Then $\mathcal{B}|_{U'} \otimes_{\mathcal{A}|_{U'}} \mathcal{M}_{U'} = 0$. Let $j \colon U' \subset X$ be the inclusion and $j_!(\mathcal{M}_{U'})$ be the extension by zero outside U' of $\mathcal{M}_{U'}$ (recall $j_!(\mathcal{M}_{U'})$ is the sheaf associated with the presheaf $V \mapsto \Gamma(V, \mathcal{M}_{U'})$ is $V \subset U'$ and $V \mapsto 0$ if $V \not\subset U'$). Then $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M} = 0$. Hence by hypothesis $\mathcal{M} = 0$ and therefore $M = \mathcal{M}_x = 0$.

□

Lemma 8.7.6.5. *Let X be a coherent topological space, $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Suppose there exists a basis $\mathfrak{B}$ of opens of X such that, for any $U \in \mathfrak{B}$, $\Gamma(U, \mathcal{A}) \to \Gamma(U, \mathcal{B})$ is right faithfully flat. Then, $\mathcal{A} \to \mathcal{B}$ is right faithfully flat.*

Proof. Let $x \in X$ and let M be a monogeneous left $\mathcal{A}_x$-module of finite presentation such that $\mathcal{B}_x \otimes_{\mathcal{A}_x} M = 0$. Following 8.7.6.2, it is enough to check that $M = 0$. Using the same arguments as the beginning of the proof of $(b) \Rightarrow (a)$ of 8.7.6.4, we can check there exist $U' \subset U$ two elements of $\mathfrak{B}$, M_U a monogeneous left $\Gamma(U, \mathcal{A})$-module of finite presentation endowed with the isomorphism $M \xrightarrow{\sim} \mathcal{A}_x \otimes_{\Gamma(U,\mathcal{A})} M_U$ such that $\Gamma(U', \mathcal{B}) \otimes_{\Gamma(U,\mathcal{A})} M_U = 0$. Hence, $\Gamma(U', \mathcal{B}) \otimes_{\Gamma(U',\mathcal{A})} (\Gamma(U', \mathcal{A}) \otimes_{\Gamma(U,\mathcal{A})} M_U) = 0$. By hypothesis, this implies $\Gamma(U', \mathcal{A}) \otimes_{\Gamma(U,\mathcal{A})} M_U = 0$ and therefore $M = 0$. □

8.7.6.6. Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is p-torsion free. Let Z be a divisor of X, $\mathfrak{Y}^\sharp$ be the open of $\mathfrak{X}^\sharp$ complementary to the support of Z and $j \colon \mathfrak{Y}^\sharp \to \mathfrak{X}^\sharp$ be the canonical morphism.

For any multiple r of p^{m+1}, the canonical $\mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$-linear ring homomorphism $\mathcal{O}_{X_i} \to \mathcal{B}_{X_i}(Z,r)$ is an isomorphism outside Z, i.e. it becomes an isomorphism on the open Y_i. This yields the ring homomorphism $\mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp} \to \mathcal{B}_{X_i}(Z,r) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$ is an isomorphism outside Z. By completion, this yields the ring homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}}(Z,r)$, $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ are isomomorphisms outside Z. Going through the limit on the level this implies the ring homomorphisms $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}({}^\dagger Z)$, $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)$ are isomorphisms outside Z. By adjointness, this yields the ring homomorphisms

$$\mathcal{B}_{\mathfrak{X}}(Z,r) \to j_*\mathcal{O}_{\mathfrak{Y}}, \quad \mathcal{O}_{\mathfrak{X}}({}^\dagger Z) \to j_*\mathcal{O}_{\mathfrak{Y}}, \tag{8.7.6.6.1}$$

$$\mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to j_*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}, \qquad \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z) \to j_*\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}. \tag{8.7.6.6.2}$$

Lemma 8.7.6.7. *With notation 8.7.6.6, suppose $\mathfrak{X}^\sharp$ is affine and there exists $f \in \Gamma(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ lifting a local equation of Z in X. Let $m' \geq m$. Set $D^{(m)}_{m'} := \Gamma(\mathfrak{X}, \mathcal{B}^{(m'')}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ for any $m'' \geq m$. Let E be an f-torsion free monogeneous left $D^{(m)}_{m'}$-module such that $(E/pE)_f = 0$. Then for $m'' \geq m'$ large enough we have $D^{(m)}_{m''} \otimes_{D^{(m)}_{m'}} E = 0$.*

Proof. Choose a generator e of E. Then there exists $s \in \mathbb{N}$, $R \in D^{(m)}_{m'}$ such that $(f^s - pR)e = 0$. Set $B_{m'} := A\{T\}/(f^{p^{m'+1}}T - p) = \Gamma(\mathfrak{U}, \mathcal{B}^{(m')}_{\mathfrak{X}}(Z))$. Increasing m' if necessary (i.e. replacing E by $D^{(m)}_{m''} \otimes_{D^{(m)}_{m'}} E$ with m'' large enough such that $s \leq p^{m''+1}$), we get $f^s(1-TR')e = 0$ with $R' = f^{p^{m'+1}-s}R$. Since E is f-torsion free, then $(1-TR')e = 0$ and therefore $(1-TR')^p e = 0$. Let $S \in D^{(m)}_{m'}$ such that $(1-TR')^p = 1+pS+(TR')^p$. We can suppose $m' \geq m+1$ and then following 8.7.6.1 we get that T belongs to the center of $D^{(m)}_{m'}/pD^{(m)}_{m'}$. Hence, we can write $(1-TR')^p = 1+pS'+T^pR''$ with $R'', S' \in D^{(m)}_{m'}$. Set $B_{m'+1} := A\{T'\}/(f^{p^{m'+2}}T' - p) = \Gamma(\mathfrak{U}, \mathcal{B}^{(m'+1)}_{\mathfrak{X}}(Z))$. Using the computation of 8.7.3.11.(b), we get that the homomorphism $B_{m'} \to B_{m'+1}$ (and therefore $D^{(m)}_{m'} \to D^{(m)}_{m'+1}$) sends T to $f^{(p-1)p^{m'+1}}T'$ and then sends T^p to $f^{(p-1)p^{m'+2}}(T')^p = pf^{(p-2)p^{m'+2}}(T')^{p-1}$. This yields that we can write $(1-TR')^p = 1+pQ$ with $Q \in D^{(m)}_{m'+1}$. Since $D^{(m)}_{m'+1}$ is p-adically complete, this implies that $(1-TR')^p$ is invertible in $D^{(m)}_{m'+1}$. Hence, $1 \otimes e = 0$ in $D^{(m)}_{m'+1} \otimes_{D^{(m)}_{m'}} E$ and then $D^{(m)}_{m'+1} \otimes_{D^{(m)}_{m'}} E = 0$. Hence, we are done. □

Theorem 8.7.6.8. *We keep notation 8.7.6.6*

(a) The ring homomorphisms 8.7.6.6.1 and 8.7.6.6.2 are injective, left and right flat.

(b) The induced ring homomorphisms

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \to j_*\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}, \tag{8.7.6.8.1}$$

$$\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \to j_*\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \tag{8.7.6.8.2}$$

are left and right faithfully flat.

Proof. (a) The injectivity of 8.7.6.6.1 is a consequence of 8.7.4.1.b. By using the local description 7.5.1.5.2, we get the injectivity of 8.7.6.6.2 from that of 8.7.6.6.1.

(i) Let us now check the flatness. Since this is local, we can suppose $\mathcal{X}$ is affine and there exists $f \in \Gamma(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) =: A$ giving a local equation of the divisor Z of X. Let $B = A[T]/(f^rT - p)$. Since the canonical morphism $B \to A_f$ is induced by adjointness from the isomorphism $B_f \xrightarrow{\sim} A_f$, this morphism $B \to A_f$ is flat. Since these rings are noetherian, then so is the p-adic completion $\widehat{B} \to A_{\{f\}}$, which corresponds to the canonical morphism $\Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}(f,r)) \to \Gamma(\mathfrak{U}, j_*\mathcal{O}_{\mathfrak{Y}})$. Going through the limits, we get that the the canonical morphism $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}({}^\dagger Z)) \to \Gamma(\mathfrak{U}, j_*\mathcal{O}_{\mathfrak{Y}})$ is flat. Hence, we have checked that the homomorphisms of 8.7.6.6.1 are flat.

(ii) Using the same arguments as in (i), for any affine open $\mathfrak{U}$ of $\mathfrak{X}$ such that there exists $f \in \Gamma(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) =: A$ giving a local equation of the divisor Z of X, we get the flatness of $\Gamma(\mathfrak{U}, \mathcal{B}_{\mathfrak{X}}(Z,r)\widehat{\otimes}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{U}, j_*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ and $\Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)) \to \Gamma(\mathfrak{U}, j_*\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ and therefore that of the morphisms of 8.7.6.6.2.

(b) Let us now check the full faithfulness of 8.7.6.8.2. Let $\mathfrak{U}^\sharp = (\operatorname{Spf} A, M_{\mathfrak{U}^\sharp})$ be an affine open of $\mathfrak{X}^\sharp$ such that there exists $f \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ lifting a local equation of Z in X. Set $D = \Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z))$, $D^{(m)}_{m'} := \Gamma(\mathfrak{U}, \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ for $m' \geq m$, $D' := \Gamma(\mathfrak{U} \cap \mathfrak{Y}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $D'^{(m)} = \Gamma(\mathfrak{U} \cap \mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Let M be a monogeneous left $D_{\mathbb{Q}}$-module of finite presentation such that $D'_{\mathbb{Q}} \otimes_{D_{\mathbb{Q}}} M = 0$. With 8.7.6.2 and 8.7.6.5, we reduce to check that $M = 0$.

Since $\mathfrak{U}$ is coherent, then filtered inductive limits commute with the section functor $\Gamma(\mathfrak{U}, -)$ (see [SGA4.2, VI.5.1-2]) and therefore $\Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \xrightarrow{\sim} D_{\mathbb{Q}} \xrightarrow{\sim} \varinjlim_{m' \geq m} D^{(m)}_{m',\mathbb{Q}}$. Following 8.4.1.11.b, there exists a monogeneous left $D^{(m)}_{m',\mathbb{Q}}$-module N for $m' \geq m$ large enough together with an isomorphism of left $D_{\mathbb{Q}}$-module $M \xrightarrow{\sim} D_{\mathbb{Q}} \otimes_{D^{(m)}_{m',\mathbb{Q}}} N$. Hence $D'_{\mathbb{Q}} \otimes_{D^{(m)}_{m',\mathbb{Q}}} N = 0$. Since $D'_{\mathbb{Q}} \xrightarrow{\sim} \varinjlim D'^{(m)}_{\mathbb{Q}}$, then increasing $m' \geq m$ is necessary we can suppose $D'^{(m)}_{\mathbb{Q}} \otimes_{D^{(m)}_{m',\mathbb{Q}}} N = 0$.

The left $D^{(m)}_{m',\mathbb{Q}}$-module N is the cokernel of some morphism of the form $(D^{(m)}_{m',\mathbb{Q}})^n \to D^{(m)}_{m',\mathbb{Q}}$ for some integer n. Since such morphism is the image via $- \otimes_{\mathbb{Z}} \mathbb{Q}$ of some morphism of the form $(D^{(m)}_{m'})^n \to D_{m'}$, then there exists a monogeneous p-torsion free left $D^{(m)}_{m'}$-module E endowed with an isomorphism of left $D^{(m)}_{m',\mathbb{Q}}$-module $N \xrightarrow{\sim} E_{\mathbb{Q}}$. Following the proof of the part (a), the extension $D^{(m)}_{m'} \to D'^{(m)}$ is flat. Hence, $D'^{(m)} \otimes_{D^{(m)}_{m'}} E$ is p-torsion free and therefore is null. Since $D'^{(m)}/pD'^{(m)} \xrightarrow{\sim} (D^{(m)}_{m'}/pD^{(m)}_{m'})_f$, this implies that $(E/pE)_f = 0$. Moreover, since E is p-torsion free then *a fortiori* E is f-torsion free. Hence, it follows from 8.7.6.7 that $D_{\mathbb{Q}} \otimes_{D^{(m)}_{m'}} E = 0$. Since $M \xrightarrow{\sim} D_{\mathbb{Q}} \otimes_{D^{(m)}_{m'}} E$ then we are done.

(b') Using similar arguments, we can check the full faithfulness of 8.7.6.8.1. □

Remark 8.7.6.9. Let $\mathcal{E}$ be a coherent $\mathcal{B}^{(m_0)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$-module. If $\mathcal{E}|\mathfrak{Y}$ is a locally projective $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-module of finite type, then for $m \geq m_0$ large enough, $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(m_0)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}} \mathcal{E}$ is a locally projective $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$-module of finite type. Indeed, since $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \to j_* \mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$ is faithfully flat (see 8.7.6.8), then $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(m_0)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}} \mathcal{E}$ is a projective $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module of finite type. We conclude using Proposition 8.4.1.11.

Corollary 8.7.6.10. *With the notation 8.7.6.6, the homomorphism $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$ is left and right flat.*

Proof. It follows from 7.2.3.3.a that $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to j_* \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ is flat. Passing to the limits and tensoring by $\mathbb{Q}$ this yields that $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to j_* \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ is flat. We conclude from the full faithfulness of 8.7.6.8.2. □

Proposition 8.7.6.11. *We keep the notations 8.7.6.8.*

(a) For any coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-module $\mathcal{E}$, the canonical homomorphism

$$j_* \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} \mathcal{E} \to j_* j^* \mathcal{E} \tag{8.7.6.11.1}$$

is an isomorphism.

(b) A coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-module $\mathcal{E}$ is null if and only if $j^*\mathcal{E}$ is null.

Proof. The first assertion is local and then we can suppose $\mathcal{X}$ is affine, there exists $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) =: A$ giving a lifting of a local equation of the divisor Z of X and that $\mathcal{M}$ has a global finite presentation. In that case j is an open immersion of affine formal log schemes. Hence the functor j_* (and therefore j_*j^*) is exact (this follows from theorem B of 8.7.5.5). By flatness (see 8.7.6.8) the functor $j_* \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} -$ is exact. Hence, using the five lemma we reduce to check that the morphism 8.7.6.11.1 is an isomorphism when $\mathcal{M}$ is a free $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-module of finite type. This latter case is obvious.

By using the full faithfulness of 8.7.6.8.2, we get the statement (b) from (a). □

Theorem 8.7.6.12. *With the notation 8.7.6.8, the canonical homomorphisms $\mathcal{O}_{\mathfrak{X}}(\dagger * Z) \to j_* \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\dagger * Z) \to j_* \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$ are left and right faithfully flat.*

Proof. We already know that these morphisms without the symbol "$*$" are flat (see 8.7.6.8). This yields that the morphisms of 8.7.6.12 are flat. Let $\mathfrak{X}'$ be an affine open of $\mathfrak{X}$ on which there exists $f \in \Gamma(\mathfrak{X}', \mathcal{O}_{\mathfrak{X}})$ lifting a local equation of Z in X. Let M be a monogeneous left $\Gamma(\mathfrak{X}', \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z))$-module of finite presentation such that $\Gamma(\mathfrak{X}' \cap \mathfrak{Y}, \mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}}) \otimes_{\Gamma(\mathfrak{X}', \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger}*Z))} M = 0$. By using 8.7.6.2 and 8.7.6.5, it suffices to prove that $M = 0$.

Let $m' \geq m$ be large enough such that there exists a monogeneous $\Gamma(\mathfrak{X}', \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \widehat{\otimes} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(*Z))$-module M' of finite presentation inducing M by scalar extension and such that

$$\Gamma(\mathfrak{X}' \cap \mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}) \otimes_{\Gamma(\mathfrak{X}', \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \widehat{\otimes} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(*Z))} M' = 0.$$

On $\mathfrak{X}'$, the functor $(*Z)$ is isomorphic to the localisation functor of localisation $(\mathfrak{Z})$. Let e be a generator of M'. Write E' for the sub-$\Gamma(\mathfrak{X}', \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \widehat{\otimes} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$-module of M' generated by e. Then E' is a f-torsion free monogeneous $\Gamma(\mathfrak{X}', \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \widehat{\otimes} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})$-module such that $E'[1/f] \xrightarrow{\sim} M'$. As $M'/pM' = 0$, we obtain the equality $(E'/pE')_f = 0$, Hence, it follows from Lemma 8.7.6.7 that $E := \Gamma(\mathfrak{X}', \mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger}Z)) \otimes_{\Gamma(\mathfrak{X}', \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \widehat{\otimes} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})} E' = 0$. Since $M \xrightarrow{\sim} E_f$, then we are done. □

Corollary 8.7.6.13. *With the notation 8.7.6.8, the homomorphism $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} \to \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$ is left and right flat.*

Proposition 8.7.6.14. For all $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$-module of finite presentation $\mathcal{E}$, the canonical homomorphism

$$j_* \mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger}*Z)} \mathcal{E} \to j_* j^* \mathcal{E} \tag{8.7.6.14.1}$$

is an isomorphism. Moreover, a morphism of $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$-modules of finite presentation is injective (resp. surjective, resp. bijective) if and only if so is its restriction to $\mathfrak{Y}$.

We have analogous results when we replace "$\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$" by "$\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$" and "$\mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}}$" by "$\mathcal{D}^{(m)}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}}$".

Proof. We argue as in 8.7.6.11: the right exactness of the functors in $\mathcal{E}$ (because of the theorem B for coherent $\mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}}$-modules or coherent $\mathcal{D}^{(m)}_{\mathfrak{Y}^{\sharp}/\mathfrak{S}^{\sharp}}$-modules implies the exactness of j_* in our case) we end up to establish the isomorphism 8.7.6.14.1, in the immediate case where $\mathcal{E} = \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$ or $\mathcal{E} = \mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(^{\dagger} * Z)$. Proposition 8.7.6.12 then implies the required criteria. □

8.7.7 Homological global dimension

8.7.7.1. With its notation, it follows from the equivalence between left and right $\mathcal{D}$-modules of the subsection 8.7.2 that when $\mathfrak{X}$ is affine we have

$$\mathrm{l.gl.dim}\, \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) = \mathrm{r.gl.dim}\, \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}).$$

Hence, we simply denote by $\mathrm{gl.dim}\, \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$ this number.

Proposition 8.7.7.2. *Let $\mathfrak{S}^{\sharp}$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^{\sharp} \to \mathfrak{S}^{\sharp}$ be a log smooth morphism of log formal schemes. We suppose $\mathfrak{X}$ is locally noetherian. Let $\mathcal{B}^{(\bullet)}$ be an inductive system of commutative $\mathcal{O}_{\mathfrak{X}}$-algebras indexed by $\mathbb{N}$ satisfying the conditions of 8.7.1.1. We keep notation 8.7.2.1 and 8.7.2. The following conditions hold.*

(a) We have the exact sequence of coherent left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$-modules:

$$0 \to \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \wedge^d \mathcal{T}_{\mathfrak{X}^{\sharp}/\mathfrak{Y}^{\sharp}} \cdots \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}_{\mathfrak{X}^{\sharp}/\mathfrak{Y}^{\sharp}} \xrightarrow[\delta]{} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \to \mathcal{B}_{\mathbb{Q}} \to 0, \tag{8.7.7.2.1}$$

i.e. the canonical complex morphism

$$\widetilde{\mathrm{Sp}}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \to \mathcal{B}_{\mathbb{Q}}. \tag{8.7.7.2.2}$$

is a quasi-isomorphism. In particular,

$$\mathcal{B}_{\mathbb{Q}} \in D^{\mathrm{b}}_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}). \tag{8.7.7.2.3}$$

(b) *The map* $\widetilde{\omega}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \otimes_{\mathcal{B}} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \xrightarrow{\beta} \widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$ *given by the structure of a right* $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$*-module on* $\widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$ *induces a* $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$*-linear resolution* $\mathrm{DR}(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})[d_{\mathfrak{X}/\mathfrak{S}}] \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$ *of* $\widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$.

Proof. We get the rest of the proposition by taking the inductive limit on the level of 7.5.10.4. □

8.7.7.3 (Biduality)**.** The $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$-linear functor denoted by $\mathbb{D}_{\widetilde{\mathfrak{X}}^{\sharp}} \colon D(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) \to D(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$ is defined by setting for any $\mathcal{E} \in D(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$

$$\mathbb{D}_{\widetilde{\mathfrak{X}}^{\sharp}}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}}(\mathcal{E}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}))[d_{X/S}]. \tag{8.7.7.3.1}$$

Similarly to 5.1.4.4, we can check that for any $\mathcal{E} \in D(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$, we have a canonical morphism $\mathcal{E} \to \mathbb{D}_{\widetilde{\mathfrak{X}}^{\sharp}} \circ \mathbb{D}_{\widetilde{\mathfrak{X}}^{\sharp}}(\mathcal{E})$, which is an isomorphism when $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$.

Proposition 8.7.7.4. *We keep notation 8.7.7.2 and we suppose the rank of* $\Omega^1_{X^{\sharp}/S^{\sharp}}$ *is constant and equal to d.*

(a) *We have* $\mathcal{E}xt^{i}_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}}(\mathcal{B}_{\mathbb{Q}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) = 0$ *for* $i \neq d$.

(b) *There are the canonical isomorphisms of right (left)* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$*-modules*

$$\mathcal{E}xt^{d}_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}}(\mathcal{B}_{\mathbb{Q}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}},$$
$$\mathcal{E}xt^{d}_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}}(\widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{B}_{\mathbb{Q}}.$$

(c) *Suppose* $\mathfrak{X}$ *is affine. Then we have* $d \leq \mathrm{tor}.\dim \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}})$.

Proof. We get the first statement and the first isomorphism of the proposition by taking the inductive limit on the level of 7.5.10.5. The second isomorphism is a consequence of the biduality isomorphism (see 8.7.7.3). When $\mathfrak{X}$ is affine, by using theorems A and B for coherent $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$-modules, by applying the exact functor $\Gamma(\mathfrak{X}, -)$ on the isomorphisms of (b) we get

$$\mathrm{Ext}^{d}_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}}(\mathcal{B}_{\mathbb{Q}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}, \mathrm{Ext}^{d}_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}}(\widetilde{\omega}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{B}_{\mathbb{Q}}.$$

The third assertion is therefore a consequence of 1.4.3.30. □

Example 8.7.7.5. We denote by $\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}$ one of the sheaves of rings $\mathcal{D}^{(0)}_{\mathfrak{X}^{\sharp}}$, $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}}$ (resp. $\mathcal{D}_{\mathfrak{X}^{\sharp},\mathbb{Q}}$, $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}$, $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp},\mathbb{Q}}$, resp. $\mathcal{D}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$, $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$). Let us denote by $\omega_{\mathfrak{X}^{\sharp}}({}^{\dagger}T) := \omega_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)$. With 4.7.3.7 and 4.7.3.14 (level 0 case), with 4.7.3.15 (algebraic infinite level case), with 7.5.10.5 (level m case), with 8.7.7.2.3 and 8.7.7.4, the following assertions hold.

(a) The canonical morphism $\omega_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}} \to \omega_{\mathfrak{X}^{\sharp}}$ (resp. $\omega_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}} \to \omega_{\mathfrak{X}^{\sharp},\mathbb{Q}}$, resp. $\omega_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}} \to \omega_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$) induces a quasi-isomorphism $\Omega^{\bullet}_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}[d] \xrightarrow{\sim} \omega_{\mathfrak{X}^{\sharp}}$ (resp. $\Omega^{\bullet}_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}[d] \xrightarrow{\sim} \omega_{\mathfrak{X}^{\sharp},\mathbb{Q}}$, resp. $\Omega^{\bullet}_{\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}[d] \xrightarrow{\sim} \omega_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$).

(b) $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}T) \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$ and $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}T) \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$.

(c) We have a canonical isomorphism of $D_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$:

$$\mathbb{D}^{\mathrm{alg}}_{T}(\mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}, \qquad \mathbb{D}_{T}(\mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}. \tag{8.7.7.5.1}$$

Beware that this isomorphism is not compatible with Frobenius: we need a Tate twist (see 11.3.5.2).

Proposition 8.7.7.6 (Berthelot)**.** *Let $\mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of $\mathcal{V}$-formal schemes. We suppose $\mathfrak{X}$ is affine and $\mathcal{V}$-flat, S is regular and the rank of $\Omega^1_{X/S}$ is constant and equal to d. Let $r := \sup_{s \in f(X)} \dim \mathcal{O}_{S,s}$. Set $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}} := \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$, $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} := \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$, $D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} := \Gamma(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$.*

(a) The ring $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$ has homological global dimension equal to $2d + r + 1$.

(b) Let $E^{(m)}$ be a p-torsion free left $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-module of finite type. Then $E^{(m)}$ admits a resolution by projective of finite type left $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-modules of length $\leq 2d + r$.

(c) Let $E^{(m)}$ be a $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-module of finite type. Then $E^{(m)}$ admits a resolution by projective of finite type left $\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-modules of length $\leq 2d + r$.

(d) We have the inequalities $d \leq \mathrm{gl}.\dim \widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} \leq 2d + r$.

(e) We have $d \leq \mathrm{tor}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} \leq 2d + r$ and coherent left $D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-module admits a resolution by projective of finite type left $D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-modules of length $\leq 2d + r$.

(f) We have the inequalities $d \leq \mathrm{tor}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} \leq \mathrm{gl}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} \leq 2d + r + 1$.

Proof. The parts a), b), c), d) of the proposition follow from 6.1.4.1, 7.5.10.5 and from 1.4.3.31. The part (e) follows from 1.4.3.32. It remains to check (f). The inequality $d \leq \mathrm{gl}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$ is already known in a wider context (see 8.7.7.4). The inequality $\mathrm{gl}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}} \leq 2d + r + 1$ follows from (see 2.3.4.3).

□

Proposition 8.7.7.7 (Montagnon)**.** *Let $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$. Let $\mathfrak{X} \to \mathfrak{S}$ be a smooth adic morphism of formal schemes. We suppose is locally noetherian and the rank of $\Omega^1_{X/S}$ is constant and equal to d. Let $\mathfrak{D}$ be a strict normal crossing divisor relative to $\mathfrak{X}/\mathfrak{S}$. Let $M_{\mathfrak{D}}$ be the log structure of $\mathfrak{X}$ given by $\mathfrak{D}$ and let $\mathfrak{X}^{\sharp} := (\mathfrak{X}, M_{\mathfrak{D}})$.*

(a) If $\mathfrak{X}$ is affine then the rings $\Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}})$, $\Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}})$, $\Gamma(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}})$ have finite global dimensions.

(b) The rings $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}$, $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}$, $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}$ have finite tor dimensions.

Proof. By copying the proof of 8.7.7.6, this is a consequence 6.1.4.2. □

Remark 8.7.7.8. With the notation of 8.7.7.6, since the rings are noetherian, then we have $\mathrm{tor}.\dim(\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}) = \mathrm{gl}.\dim(\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$, $\mathrm{tor}.\dim(\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}) = \mathrm{gl}.\dim(\widehat{D}^{(m)}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$ (see 2.3.4.2). Beware that $\mathrm{tor}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$ is equal to $\mathrm{gl}.\dim D^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$. We have a similar remark for the log version 8.7.7.7.

Theorem 8.7.7.9 (Noot-Huyghe)**.** *Let $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$. Let $\mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of $\mathcal{V}$-formal schemes. We suppose $\mathfrak{X}$ is locally noetherian and the rank of $\Omega^1_{X/S}$ is constant and equal to d. Let Z be a divisor of X, $\mathfrak{U}$ be an affine open subscheme of $\mathfrak{X}$. We have the inequality:* $\mathrm{gl}.\dim(\Gamma(\mathfrak{U}, \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})) \leq 2d + 2$.

Proof. This has been proved by C. Noot-Huyghe in [Huy07, Th. 3.1.3.2 and 3.2.1.2]. □

8.7.7.10. Let $\mathfrak{X} \to \mathfrak{S}$ be a noetherian smooth morphism of $\mathcal{V}$-formal schemes and Z be a divisor of X. Following 8.4.1.15, for any $* \in \{\mathrm{r}, \mathrm{l}\}$, the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ induces the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z)) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$. It follows from 8.7.7.6 and 7.1.3.13 that for any $\lambda \in L(\mathbb{N})$ we have the equality $D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\lambda^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) = D^{\mathrm{b}}_{\mathrm{coh}}({}^{*}\lambda^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. With notation 8.6.1.4, this yields $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. Then the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ induces an equivalence $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \cong D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$.

We get from 8.7.7.9 the equality $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}) = D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$. Howerver, when Z is not empty, it is not clear that the inclusion $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z)) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z))$ is an equality. Hence, we only get a priori a fully faithful functor $\underrightarrow{l}^{*}_{\mathbb{Q}} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z)) \to D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$.

8.7.8 Arithmetic differential operators of finite congruence, link with Ardakov-Wadsley's theory of $\mathcal{D}$-modules

8.7.8.1. Let $\mathfrak{X}_0$ be a smooth formal scheme over $\mathcal{V}$ and let $\mathcal{D}^{(m)}_{\mathfrak{X}_0}$ be Berthelot's sheaf of arithmetic differential operators of level m on $\mathfrak{X}_0$. For any number $k \geq 0$, let $\mathcal{D}^{(k,m)}_{\mathfrak{X}_0}$ be the $\mathcal{V}$-subalgebra of $\mathcal{D}^{(m)}_{\mathfrak{X}_0}$ consisting of those differential operators which are generated, locally where we have coordinates $x_1, ..., x_M$ and corresponding derivations $\partial_1, ..., \partial_M$, by operators of the form

$$\pi^{k|\underline{\nu}|}\underline{\partial}^{\langle\underline{\nu}\rangle_{(m)}} = \pi^{k(\nu_1+\ldots+\nu_M)}\prod_{l=1}^{M}\partial_l^{\langle\nu_l\rangle_{(m)}}.$$

Given an admissible blow-up $\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}_0$, we let $k_{\mathfrak{X}}$ be the minimal k such that $\pi^k \mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$ for any coherent ideal sheaf $\mathcal{I}$ on $\mathfrak{X}_0$ whose blow-up is $\mathfrak{X}$. In [HSS21, 2.1.12], C. Huyghe, T. Schmidt, and M. Strauch checked that

$$\mathcal{D}^{(k,m)}_{\mathfrak{X}} := \mathrm{pr}^* \mathcal{D}^{(k,m)}_{\mathfrak{X}_0} = \mathcal{O}_{\mathfrak{X}} \otimes_{\mathrm{pr}^{-1}\mathcal{O}_{\mathfrak{X}_0}} \mathrm{pr}^{-1}\mathcal{D}^{(k,m)}_{\mathfrak{X}_0}$$

is naturally a sheaf of rings on $\mathfrak{X}$ whenever $k \geq k_{\mathfrak{X}}$. Similarly to Berthelot's sheaves of arithmetic $\mathcal{D}$-module, we write

$$\widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}} = \varprojlim_i \mathcal{D}^{(k,m)}_{\mathfrak{X}}/\pi^i \quad \text{and} \quad \mathcal{D}^{\dagger}_{\mathfrak{X},k} = \varinjlim_m \widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}} \to \mathbb{Q}\ ,$$

and call these sheaves *arithmetic differential operators of congruence level*[1] k on $\mathfrak{X}$. The structure theory of these differential operators goes largely parallel to the classical smooth setting (when $\mathfrak{X} = \mathfrak{X}_0$ and $k = 0$), as developed in this book. In particular, the sheaves $\mathcal{D}^{(k,m)}_{\mathfrak{X}}, \widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}}$ and $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$ are sheaves of coherent rings on $\mathfrak{X}$. Cartan's theorems A and B hold for the sheaf $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$, when restricted to an affine open subscheme $\mathfrak{U}$ of $\mathfrak{X}$ (see [HSS21, 3.1.12 and 3.1.15]). A key result (the 'invariance theorem') shows that in case of a morphism $\mathfrak{X}' \to \mathfrak{X}$ between admissible blow-ups of $\mathfrak{X}_0$, the categories of coherent modules over $\mathcal{D}^{\dagger}_{\mathfrak{X}',k}$ and over $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$, respectively, are naturally equivalent. As a consequence, this yields theorems A and B on the whole blow-up $\mathfrak{X}$ provided the base $\mathfrak{X}_0$ is affine (see [HSS21, 2.3.12]).

Passing to the projective limit

$$\mathcal{D}_{\mathfrak{X},\infty} = \varprojlim_k \mathcal{D}^{\dagger}_{\mathfrak{X},k},$$

we get the category $\mathcal{C}_{\mathfrak{X}}$ of *coadmissible* $\mathcal{D}_{\mathfrak{X},\infty}$*.modules.* This is a full abelian subcategory of the category of all $\mathcal{D}_{\mathfrak{X},\infty}$-modules. Its construction relies on the fact that the ring of local sections $\Gamma(\mathfrak{V}, \mathcal{D}_{\mathfrak{X},\infty})$ over an open affine $\mathfrak{V} \subseteq \mathfrak{X}$ is a Fréchet-Stein algebra. This concept and study of coadmissible modules over Fréchet-Stein algebra were introduced by P. Schneider and J. Teitelbaum in [ST03]. For such coadmissible $\mathcal{D}_{\mathfrak{X},\infty}$-module, C. Huyghe, T. Schmidt, and M. Strauch established Cartan's theorems A and B hold (see [HSS21, 3.1.12 an 3.1.15]).

Consider the Zariski-Riemann space of $\mathfrak{X}_0$, i.e., the projective limit $\langle\mathfrak{X}_0\rangle = \varprojlim \mathfrak{X}$ of all admissible formal blow-ups $\mathfrak{X} \to \mathfrak{X}_0$, cf. [Bos14]. One can then form the inductive limit

$$\mathcal{D}_{\langle\mathfrak{X}_0\rangle} = \varinjlim_{\mathfrak{X}} \mathrm{sp}_{\mathfrak{X}}^{-1}\mathcal{D}_{\mathfrak{X},\infty}\ ,$$

where $\mathrm{sp}_{\mathfrak{X}} : \langle\mathfrak{X}_0\rangle \to \mathfrak{X}$ is the projection map. This is a sheaf of rings on $\langle\mathfrak{X}_0\rangle$ and we get the abelian category of coadmissible $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-modules. C. Huyghe, T. Schmidt, and M. Strauch proved analogues of Theorems A and B in this setting (see [HSS21, 3.2.6])

This work of C. Huyghe, T. Schmidt, M. Strauch of [HSS21], [HSS22] presented above is a generalisation of the construction of D. Patel, T. Schmidt, and M. Strauch in [PSS14, PSS17] and is related to distribution algebras of arithmetic $\mathcal{D}$-modules (see [HS18]). Moreover, A. Shiho introduced in [Shi15] sheaves of p-adic differential operators of negative level $-m$, as they are called there. These are closely related to the sheaves considered here, where the congruence level k corresponds to Shiho's level $-m$.

Some application of this generalization is the localization theorem of [HPSS]: in this context $\mathfrak{X}_0$ is the smooth model of the flag variety of a connected split reductive group $\mathbb{G}$ over K, and the main result of [HPSS] establishes then an anti-equivalence between the category of admissible locally analytic $\mathbb{G}(L)$-representations (with trivial character) [ST03] and the category of so-called coadmissible equivariant

[1] The terminology is motivated by the relation to congruence subgroups in reductive groups in the case of formal models of flag varieties, cf. [HPSS].

arithmetic $\mathcal{D}$-modules on the system of all formal models of the rigid analytic flag variety of $\mathbb{G}$. We also have a twisted version by A. Sarrazola in [SA23]: when $\mathcal{V}$ is the ring of integers of a finite extension $\mathbb{Q}_p$, A. Sarrazola defined and studied the category of coadmissible $\mathbb{G}$-equivariant twisted (by an algebraic character of a split maximal torus of $\mathbb{G}$) arithmetic $\mathcal{D}$-modules over the induced rigid flag variety. He generalized the main results from a paper by Huyghe, Patel, Schmidt and Strauch in [HPSS19] for algebraic characters. A. Sarrazola also proved in [SA24] a twisted version of a Beilinson-Bernstein theorem for twisted arithmetic differential operators on the formal flag variety. This extends for instance the Beilinson-Bernstein's theorem for arithmetic $\mathcal{D}$-modules proved by C. Huyghe in [Huy09].

8.7.8.2 (Comparison with K. Ardakov and S. Wadsley theory)**.** K. Ardakov and S. Wadsley are developing a theory of $\mathcal{D}$-modules on general smooth rigid-analytic spaces, cf. [Ard14, AW18, AW19]. In their work they consider deformations of crystalline differential operators (as in [AW13]), whereas we take as a starting point deformations of Berthelot's arithmetic differential operators. The category of coadmissible $\mathcal{D}_{\langle \mathfrak{X}_0 \rangle}$-modules as defined by C. Huyghe, T. Schmidt, M. Strauch, when pulled back to the site of the rigid-analytic space of classical points, coincides with the corresponding category studied in [Ard14, AW18, AW19].

8.8 Frobenius structures

8.8.1 Frobenius descente of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-modules

Let $m \in \mathbb{N}$ be an integer. Let $\mathfrak{a} := \pi^h \mathcal{V}$ and $\mathfrak{b} := \pi^k \mathcal{V}$. Following 1.2.4.2.(a) $(\mathfrak{a}, \mathfrak{b}, ^{[\,]})$ is an m-ideal if and only if $e/(p-1) \leq k$, $e+h \geq k$, $hp^m \geq k$ and $k \geq h$. In particular, $\mathfrak{a}$ has an m-PD-structure if and only if $hp^m \geq e/(p-1)$. From now, fix such a m-PD-ideal $(\mathfrak{a}, \mathfrak{b}, ^{[\,]})$ of $\mathcal{V}$.

To avoid being limited in certain applications by hypotheses on the branching index, we will thus be led to consider not only liftings of the relative Frobenius modulo $\mathfrak{m}$ of a $\mathcal{V}$-formal scheme $\mathfrak{X}$, but, more generally, raisings of the Frobenius relative to the reduction of $\mathfrak{X}$ modulo $\mathfrak{a}$. For any integer $i \in \mathbb{N}$, for any $\mathcal{V}$-formal scheme $\mathfrak{S}$, we denote here S_i the reduction of $\mathfrak{S}$ modulo $\mathfrak{a}^{i+1}$ (and not $\mathfrak{m}^{i+1}$). The ideal $\mathfrak{b}\mathcal{O}_{S_i}$ endows $\mathfrak{a}\mathcal{O}_{S_i}$ with an m-PD nilpotent m-PD-structure. Remark that if $\mathfrak{a}$ is topologically m-PD-nilpotent (i.e. if $hp^m > e/(p-1)$ following 1.3.1.3), then $\mathfrak{a}\mathcal{O}_{S_i}$ is m-PD nilpotent.

Proposition 8.8.1.1. *Let $s \in \mathbb{N}$, $\mathfrak{S}$ be a flat $\mathcal{V}$-formal scheme, $\mathfrak{X}$ be an $\mathfrak{S}$-smooth formal scheme, $X_0 \to S_0$ be its reduction modulo $\mathfrak{a}$, $F\colon \mathfrak{X} \to \mathfrak{X}'$ a morphism of smooth $\mathfrak{S}$-schemes lifting $F^s_{X_0/S_0}$. Let $\mathcal{B}_{\mathfrak{X}'}$ be an $\mathcal{O}_{\mathfrak{X}'}$-algebra endowed with a compatible structure of left $\mathcal{D}_{\mathfrak{X}'/\mathfrak{S}}$-module. Set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} := \mathcal{B}_{\mathfrak{X}'} \widehat{\otimes} \mathcal{D}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$, $\mathcal{B}_{\mathfrak{X}} := F^* \mathcal{B}_{\mathfrak{X}'}$ and $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}} := \mathcal{B}_{\mathfrak{X}} \widehat{\otimes} \mathcal{D}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$. So:*

(a) $F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$, $F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$ and $F^* F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$ are respectively provided with canonical structures of $(\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$-bimodule , $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}, \widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}})$-bimodule, and of $(\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}, \widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}})$- bimodule; moreover, the canonical homomorphism

$$F^* F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \to F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}, \qquad (\text{resp. } F^* F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \to F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$$

is $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-linear on the left (resp. on the right), and locally identifies $F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$ (resp. $F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$) to a direct factor of $F^* F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$ on $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$.

(b) There exists a canonical isomorphism of $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-bimodules

$$\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} F^* F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}. \tag{8.8.1.1.1}$$

(c) There exists a canonical isomorphism of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-bimodules

$$F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}} F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}. \tag{8.8.1.1.2}$$

Proof. By p-adic completion, this is a consequence of such results over schemes (see 6.1.3). For more details, see [Ber00, 4.1.2]. □

Remark 8.8.1.2. Suppose $\mathfrak{X}/\mathfrak{S}$ has coordinates $t_1, \ldots, t_d$. If $m \geq 1$, then the sheaf $\Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$ contains zero divisors. Indeed, let $F\colon \mathfrak{X} \to \mathfrak{X}'$ is a morphism of $\mathcal{V}$-formal schemes lifting the m-th power of Frobenius of X. Then $(\underline{t}^{\underline{\alpha}})$ with $0 \leq \alpha_j < p^M$ is a basis of $\mathcal{O}_{\mathfrak{X}}$ over $\mathcal{O}_{\mathfrak{X}'}$. Let $(\theta_{\underline{\alpha}})$ be the dual basis of $F^\flat \mathcal{O}_{\mathfrak{X}'} = \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}'}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}'})$. Then the isomorphism 8.8.1.1.1 becomes an isomorphism of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-modules of the form

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} \oplus_{\underline{\alpha}} F^* \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}/\mathfrak{S}} \cdot \theta_{\underline{\alpha}}.$$

Let $1 = \sum_{\underline{\alpha}} e_{\underline{\alpha}}$ be the decomposition of 1. Since the map $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}} \to F^* \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}/\mathfrak{S}} \cdot \theta_{\underline{\alpha}}$ is a surjective morphism of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-modules which sends 1 to $e_{\underline{\alpha}}$, we get $e_{\underline{\alpha}} \neq 0$ for any $\underline{\alpha}$. Since $e_{\underline{\beta}} = e_{\underline{\beta}} \cdot 1 = \sum_{\underline{\alpha}} e_{\underline{\beta}} e_{\underline{\alpha}}$, then we obtain $e_{\underline{\beta}} e_{\underline{\alpha}} = 0$ if $\underline{\beta} \neq \underline{\alpha}$, $e_{\underline{\beta}}^2 = e_{\underline{\beta}}$.

Theorem 8.8.1.3. *We keep notations and hypotheses of 8.8.1.1.*

(a) For any left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-module $\mathcal{E}'$ (resp. right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-modules $\mathcal{M}'$), $F^*\mathcal{E}'$ (resp. $F^\flat \mathcal{M}'$) is endowed with a functorial structure of left (resp. right) $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-module.

(b) The functor F^* (resp. $F^\flat$) induces an equivalence between the category of left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-modules and left (resp. right) $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-modules.

(c) The functor which associates from a left (resp. right) $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-module $\mathcal{E}$ the left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-module $F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}} \mathcal{E}$ (resp. $\mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}} F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$) is quasi-inverse to F^* (resp. $F^\flat$).

(d) A left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-module $\mathcal{E}'$ (resp. $\mathcal{M}'$) is coherent if and only if $F^*(\mathcal{E}')$ (resp. $F^\flat \mathcal{M}'$) is $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-coherent.

(e) For any left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-module $\mathcal{E}'$, we have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-modules of the form

$$\omega_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}}} F^*(\mathcal{E}') \xrightarrow{\sim} F^\flat(\omega_{\mathfrak{X}'/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{E}'). \tag{8.8.1.3.1}$$

Proof. By p-adic completion, this is a consequence of such results over schemes (see 6.1.3). For more details, see [Ber00, 4.1.3]. □

8.8.1.4. Suppose $\mathfrak{a}$ is topologically m-PD-nilpotent. If F and F' are two liftings of the relative Frobenius $F^s_{X_0/S_0}$, then by taking the p-adic completion of the glueing isomorphisms of the form (see 6.1.5.1), we get the canonical isomorphism $\tau_{F,F'}\colon F'^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \xrightarrow{\sim} F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$. Hence, by glueing these bimodules, we get the $(\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$-bimodule $(F^s_{X_0/S_0})^*(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$. This yields the functor from the category of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-modules to that of left $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-modules by setting

$$(F^s_{X_0/S_0})^*(\mathcal{E}') := (F^s_{X_0/S_0})^*(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}} \mathcal{E}'. \tag{8.8.1.4.1}$$

Similarly, by p-adic completion, we get from 6.1.5.1, the canonical isomorphism $\sigma_{F,F'}\colon F'^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} \xrightarrow{\sim} F^\flat \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$. By glueing, this yields the $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}, \widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}})$-bimodule $(F^s_{X_0/S_0})^\flat(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$. This yields the functor from the category of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-modules to that of right $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-modules by setting

$$(F^s_{X_0/S_0})^\flat(\mathcal{M}') := \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}} (F^s_{X_0/S_0})^\flat(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}).$$

The theorem 8.8.1.3 (in particular) is still valid in this context, i.e. we do not need to suppose that $F^s_{X_0/S_0}$ has a lifting.

Theorem 8.8.1.5. *We keep notations and hypotheses of 8.8.1.1. Let $m' \geq m$, $\mathcal{C}_{\mathfrak{X}'}$ be an $\mathcal{O}_{\mathfrak{X}'}$-algebra endowed with a compatible structure of left left $\mathcal{D}^{(m')}_{\mathfrak{X}'/\mathfrak{S}}$-module, $\mathcal{B}_{\mathfrak{X}'} \to \mathcal{C}_{\mathfrak{X}'}$ a homomorphism of $\mathcal{O}_{\mathfrak{X}'}$-algebras which is $\mathcal{D}^{(m')}_{\mathfrak{X}'/\mathfrak{S}}$-linear. Set*

$$\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}'/\mathfrak{S}} := \mathcal{C}_{\mathfrak{X}'} \widehat{\otimes} \mathcal{D}^{(m')}_{\mathfrak{X}'/\mathfrak{S}}, \qquad \mathcal{C}_{\mathfrak{X}} := F^* \mathcal{C}_{\mathfrak{X}'}, \qquad \widetilde{\mathcal{D}}^{(m'+s)}_{\mathfrak{X}/\mathfrak{S}} := \mathcal{C}_{\mathfrak{X}} \widehat{\otimes} \mathcal{D}^{(m'+s)}_{\mathfrak{X}/\mathfrak{S}}.$$

For any $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$, there exists in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}})$ the canonical isomorphism:

$$\widetilde{\mathcal{D}}^{(m'+s)}_{\mathfrak{X}/\mathfrak{S}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}} F^* \mathcal{E}' \xrightarrow{\sim} \left(F^* (\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}'/\mathfrak{S}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}} \mathcal{E}' \right). \tag{8.8.1.5.1}$$

Proof. By associativity of the tensor products, we reduce to check

$$\widetilde{\mathcal{D}}^{(m'+s)}_{\mathfrak{X}} \overset{\mathbb{L}}{\underset{\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}{\otimes}} F^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \xrightarrow{\sim} F^* (\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}'}).$$

Since $F^*(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'})$ is a locally projective of finite type left $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-module, then the tensor product is p-adically complete and this therefore follows from 6.2.3.5. □

8.8.2 Frobenius descente of $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-modules

8.8.2.1. Let $\mathfrak{a} \subset \mathfrak{m}$ be an ideal containing p. Fix a PD-ideal $\mathfrak{b} \subset \mathfrak{a}$ (e.g. $\mathfrak{b} = (p)$). Let $m_0 \in \mathbb{N}$ be large enough such that $\mathfrak{b}$ endows $\mathfrak{a}$ with an m_0-PD-structure. Replacing m_0 by $m_0 + 1$ if necessary, we can suppose that this structure is topologically m_0-PD-nilpotent. Let $\mathfrak{S}$ be a flat $\mathcal{V}$-formal scheme. For any integer $i \in \mathbb{N}$, we denote here S_i the reduction of $\mathfrak{S}$ modulo $\mathfrak{a}^{i+1}$ (and not $\mathfrak{m}^{i+1}$).

Let $\mathfrak{X}$ be a smooth formal $\mathfrak{S}$-scheme. Let X_0 be its special fiber, $s \in \mathbb{N}$ and integer and $X_0^{(s)}$ be the base change of X_0 by the s-th power of the absolute Frobenius of S_0. Suppose there exists $F \colon \mathfrak{X} \to \mathfrak{X}'$ a morphism of smooth formal $\mathfrak{S}$-schemes which is a lifting of the relative Frobenius $F^s_{X_0/S_0} \colon X_0 \to X_0^{(s)}$.

Let $(\mathcal{B}^{(m)}_{\mathfrak{X}'})_{m \geq m_0}$ be an inductive system of commutative $\mathcal{O}_{\mathfrak{X}'}$-algebras indexed by integers great or equal to m_0 and satisfying the conditions of 8.7.1.1. For any $m \geq m_0$, set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} := \mathcal{B}^{(m)}_{\mathfrak{X}'} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}'}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$, $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$. We set $\mathcal{B}^{(m+s)}_{\mathfrak{X}'} := F^* \mathcal{B}^{(m)}_{\mathfrak{X}'}$, $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}} := \mathcal{B}^{(m+s)}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$, $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}} := \varinjlim_m \widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$.

Proposition 8.8.2.2. *We keep notations and hypotheses of 8.8.2.1.*

(a) $F^* \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$, $F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$ and $F^* F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$ are respectively provided with canonical structures of $(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}})$-bimodule , $(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}})$-bimodule , and of $(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}})$- bimodule; moreover, the canonical homomorphism

$$F^* F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \to F^* \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}, \qquad (\textit{resp. } F^* F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \to F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}})$$

is $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-linear on the left (resp. on the right), and locally identifies $F^* \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$ (resp. $F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$) to a direct factor of $F^* F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$ on $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$.

(b) There exists a canonical isomorphism of $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-bimodules

$$\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} F^* F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}. \tag{8.8.2.2.1}$$

(c) There exists a canonical isomorphism of $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-bimodules

$$F^\flat \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}} F^* \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}. \tag{8.8.2.2.2}$$

Proof. By taking inductive limits on the level, this is a consequence of 8.8.1.1. For more details, see [Ber00, 4.2.2]. □

Theorem 8.8.2.3. *We keep notations and hypotheses of 8.8.2.1.*

(a) *For any left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-module $\mathcal{E}'$ (resp. right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-modules $\mathcal{M}'$), $F^*\mathcal{E}'$ (resp. $F^\flat\mathcal{M}'$) is endowed with a functorial structure of left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module.*

(b) *The functor F^* (resp. $F^\flat$) induces an equivalence between the category of left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-modules and left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-modules.*

(c) *The functor F_+ which associates from a left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module $\mathcal{E}$ the left (resp. right) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-module $F_+(\mathcal{E}) := F^\flat\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}} \mathcal{E}$ (resp. $F_+(\mathcal{M}) := \mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}} F^*\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$) is quasi-inverse to F^* (resp. $F^\flat$).*

(d) *For any left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-module $\mathcal{E}'$ (resp. right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-modules $\mathcal{M}'$), $F^*\mathcal{E}'$ (resp. $F^\flat\mathcal{M}'$) is a flat (coherent) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module if and only if $\mathcal{E}'$ (resp. $\mathcal{M}'$) is a flat (coherent) $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-module.*

(e) *For any left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-module $\mathcal{E}'$, we have the canonical isomorphism of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module-modules of the form*
$$\omega_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}}} F^*(\mathcal{E}') \xrightarrow{\sim} F^\flat(\omega_{\mathfrak{X}'/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{E}'). \tag{8.8.2.3.1}$$

Proof. By taking inductive limits on the level, this is a consequence of 8.8.1.3. For more details, see [Ber00, 4.2.4]. □

8.8.2.4. If F and F' are two liftings of the relative Frobenius $F^s_{X_0/S_0}$, then by taking the inductive limit on the level of the glueing isomorphisms of 8.8.1.4 we get the canonical isomorphism $\tau_{F,F'}\colon F'^*\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \xrightarrow{\sim} F^*\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$. Hence, by glueing these bimodules, we get the $(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}})$-bimodule $(F^s_{X_0/S_0})^*(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}})$. This yields the functor from the category of left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-modules to that of left $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-modules by setting
$$(F^s_{X_0/S_0})^*(\mathcal{E}') := (F^s_{X_0/S_0})^*(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}) \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}} \mathcal{E}'. \tag{8.8.2.4.1}$$

Similarly, by taking the inductive limit on the level of the glueing isomorphisms of 8.8.1.4 we get the canonical isomorphism $\sigma_{F,F'}\colon F'^\flat\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \xrightarrow{\sim} F^\flat\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$. By glueing, this yields the $(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}, \widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}})$-bimodule $(F^s_{X_0/S_0})^\flat(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}})$. This yields the functor from the category of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}$-modules to that of right $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-modules by setting
$$(F^s_{X_0/S_0})^\flat(\mathcal{M}') := \mathcal{M}' \otimes_{\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}} (F^s_{X_0/S_0})^\flat(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}).$$

The theorem 8.8.2.3 is still valid in this context, i.e. we do not need to suppose that $F^s_{X_0/S_0}$ has a lifting.

Theorem 8.8.2.5. *We keep notations and hypotheses of 8.8.2.1. Let $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}})$. There exists in $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}})$ the canonical isomorphism:*
$$\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}} F^*\mathcal{E}' \xrightarrow{\sim} \left(F^*(\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}} \mathcal{E}'\right). \tag{8.8.2.5.1}$$

Proof. By taking inductive limits on the level, this is a consequence of 8.8.1.5. □

8.8.3 F-complexes

8.8.3.1. Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. Suppose there exists an automorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining. Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme. We denote by $\mathfrak{X}' := \mathfrak{X}^\sigma$ the $\mathcal{V}$-formal scheme deduced from $\mathfrak{X}$ by the base change defined by σ. Following 8.8.2.3 and the remark 8.8.2.4, the functor $(F^s_{X_0/S_0})^*$ (resp. $(F^s_{X_0/S_0})^\flat$) induces an equivalence between the category of left (resp. right) $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}'/\mathfrak{S}}$-modules and left (resp. right) $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-modules. For simplicity, we simply denote by F^* and $F^\flat$ these functors, which is compatible with such a functor when $F\colon \mathfrak{X} \to \mathfrak{X}'$ is a lifting of $F^s_{X_0/S_0}$. If $\mathcal{E}$ is a left $\mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-module (or a left $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module etc.), we denote by $\mathcal{E}^\sigma$ is the left $\mathcal{D}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-module (or left $\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}$-module etc.) induced from $\mathcal{E}$ by base change via σ.

Let us further assume that we have the following data

(a) an inductive system of $\mathcal{O}_{\mathfrak{X}}$-algebra $(\mathcal{B}^{(m)}_{\mathfrak{X}})_{m\geq m_0}$, such that each $\mathcal{B}^{(m)}_{\mathfrak{X}}$ is p-adically complete, and endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-module so that $\mathcal{B}^{(m)}_{\mathfrak{X}} \to \mathcal{B}^{(m+1)}_{\mathfrak{X}}$ is $\mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-linear ; by setting $\mathcal{B}^{(m)}_{\mathfrak{X}'} := (\mathcal{B}^{(m)}_{\mathfrak{X}})^\sigma$ this yields inductive system of $\mathcal{O}_{\mathfrak{X}}$-algebra $(\mathcal{B}^{(m)}_{\mathfrak{X}'})_{m\geq m_0}$, such that each $\mathcal{B}^{(m)}_{\mathfrak{X}'}$ is p-adically complete, and endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-module so that $\mathcal{B}^{(m)}_{\mathfrak{X}'} \to \mathcal{B}^{(m+1)}_{\mathfrak{X}'}$ is $\mathcal{D}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$-linear ;

(b) a family of $\mathcal{D}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-linear isomorphism $F^*(\mathcal{B}^{(m)}_{\mathfrak{X}'}) \xrightarrow{\sim} \mathcal{B}^{(m+s)}_{\mathfrak{X}}$ which commutes with the transition maps, i.e., making commutative the diagram

$$\begin{array}{ccc} F^*(\mathcal{B}^{(m)}_{\mathfrak{X}'}) & \longrightarrow & \mathcal{B}^{(m+s)}_{\mathfrak{X}} \\ \downarrow & & \downarrow \\ F^*(\mathcal{B}^{(m+1)}_{\mathfrak{X}'}) & \longrightarrow & \mathcal{B}^{(m+1+s)}_{\mathfrak{X}}. \end{array} \tag{8.8.3.1.1}$$

For any $m \geq m_0$, set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}} := \mathcal{B}^{(m)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$, $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}} := \mathcal{B}^{(m)}_{\mathfrak{X}'}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}'}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$, $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}'/\mathfrak{S}} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}$. The isomorphisms $F^*(\mathcal{B}^{(m)}_{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{B}^{(m+s)}_{\mathfrak{X}}$ induce the ring isomorphisms

$$F^*(\mathcal{B}^{(m)}_{\mathfrak{X}'})\widehat{\otimes}\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$$

which are compatible when m varies. This yields the ring isomorphism

$$\varinjlim_m F^*(\mathcal{B}^{(m)}_{\mathfrak{X}'})\widehat{\otimes}\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}. \tag{8.8.3.1.2}$$

Let $\mathcal{E}$ be a left $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module and let $\mathcal{E}^\sigma$ be the left $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}'/\mathfrak{S}}$-module induced from $\mathcal{E}$ by base change via σ. It follows from 8.8.2.3 that $F^*(\mathcal{E}^\sigma)$ is endowed with a structure of left $\varinjlim_m F^*(\mathcal{B}^{(m)}_{\mathfrak{X}'})\widehat{\otimes}\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-module. Via 8.8.3.1.2, we get a structure of left $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module on $F^*\mathcal{E} := F^*(\mathcal{E}^\sigma)$ (we keep this abuse of notation). Similarly, for any integer $n \in \mathbb{N}$, we get the left $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module by setting $F^{n*}\mathcal{E} := (F^{ns}_{X_0/S_0})^*(\mathcal{E}^{\sigma^n})$.

Definition 8.8.3.2. With notation and hypotheses of 8.8.3.1, a "left F-$\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module" (or a left F^s-$\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module if we would like to clarify s) is the data of a left $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module $\mathcal{E}$ together with a $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-linear isomorphism $\Phi\colon \mathcal{E} \xrightarrow{\sim} F^*\mathcal{E}$. A morphism of left F-$\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-modules $u\colon (\mathcal{E},\Phi) \to (\mathcal{F},\Psi)$ is a morphism of $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-modules $u\colon \mathcal{E} \to \mathcal{F}$ such that $\Psi \circ u = F^*(u) \circ \Phi$.

A "coherent (of finite presentation) left F-$\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module" is a left F-$\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module $(\mathcal{E},\Phi)$ such that $\mathcal{E}$ is a coherent (of finite presentation) $\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}}$-module.

Similarly, for $\star \in \{\emptyset, \mathrm{coh}, \mathrm{tdf}, \mathrm{perf}\}$, $* \in \{\mathrm{b}, -, +, \emptyset\}$, we define an F-complex of $D^*_\star({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$ (resp. $D({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$) is the data of a complex $\mathcal{E}$ of $D^*_\star({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$ together with an isomorphism of $D^*_\star({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$ of the form $\Phi\colon \mathcal{E} \xrightarrow{\sim} F^*\mathcal{E}$. A morphism $u\colon (\mathcal{E},\Phi) \to (\mathcal{F},\Psi)$ of F-complexes of $D^*_\star({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$ is a morphism $u\colon \mathcal{E} \to \mathcal{F}$ of $D({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$ such that $\Psi \circ u = F^*(u) \circ \Phi$. We denote by F-$D^*_\star({}^1\widetilde{\mathcal{D}}^\dagger_{\mathfrak{X}/\mathfrak{S}})$ the corresponding category of F-complexes.

Definition 8.8.3.3 (Tate twist)**.** Let $(\mathcal{E}, \Phi)$ be an F-$\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module (resp. complex). For any integer d, we define an F-$\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module (resp. complex) $(\mathcal{E}, \Phi)(d) := (\mathcal{E}, \Phi(d))$ as follows: the underlying $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-module (resp. complex) is the same as $\mathcal{E}$ and $\Phi(d) := q^{-d}\Phi$, where $q = p^s$ (beware s is hidden in the notation "F"). The object $(\mathcal{E}, \Phi)(d)$ is called the Tate twist of $(\mathcal{E}, \Phi)$.

Theorem 8.8.3.4. *There exists a canonical equivalence between the category of left F-$\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module of finite presentation and the category of couples $(\mathcal{E}^{(m)}, \Phi^{(m+s)})$, where $\mathcal{E}^{(m)}$ is left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$-module of finite presentation, and*

$$\Phi^{(m+s)} \colon \widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}} \mathcal{E}^{(m)} \xrightarrow{\sim} F^*((\mathcal{E}^{(m)})^{\sigma})$$

is a $\widetilde{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}$-linear isomorphism.

Proof. We have the functor $(\mathcal{E}^{(m)}, \Phi^{(m+s)}) \mapsto (\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}} \mathcal{E}^{(m)}, \Phi)$, where Φ is induced from $\Phi^{(m+s)}$ thanks to the commutation isomorphism 8.8.2.5.1. For the details, see [Ber00, 4.5.4]. □

Corollary 8.8.3.5. *Suppose $\mathfrak{X}$ is quasi-compact, the transition maps $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}} \to \widetilde{\mathcal{D}}^{(m+1)}_{\mathfrak{X}/\mathfrak{S}}$ are flat, the algebras $\mathcal{B}_{\mathfrak{X}}/\mathfrak{m}^{i+1}\mathcal{B}_{\mathfrak{X}}$ are quasi-coherent with noetherian sections on affine opens. The category of coherent left F-$\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}$-module is a noetherian.*

Proof. See [Ber00, 4.5.5]. □

Chapter 9

Cohomological operations for coherent $\mathcal{D}^\dagger$-modules or quasi-coherent inductive systems of arithmetic $\mathcal{D}$-modules

9.1 Localization functor outside a divisor

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is p-torsion free, noetherian of finite Krull dimension. For any integer $i \geq 0$, set $X_i^\sharp := \mathfrak{X}^\sharp \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$. We suppose X_0 is regular.

Let $\lambda_0 \leq \mu_0$ be two maps of $L(\mathbb{N})$. Let $D \subset Z$ be two divisors of X_0. For any $m \in \mathbb{N}$, for any $i \in \mathbb{N}$, we set $\widetilde{\mathcal{B}}^{(m)}_{X_i}(D) := \mathcal{B}^{(\lambda_0(m))}_{X_i}(D)$, $\widetilde{\mathcal{B}}^{(m)}_{X_i}(Z) := \mathcal{B}^{(\mu_0(m))}_{X_i}(Z)$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D) := \widetilde{\mathcal{B}}^{(m)}_{X_i}(D) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S^\sharp}(Z) := \widetilde{\mathcal{B}}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$. For any $m \in \mathbb{N}$, we set $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}}(D)$, $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z) := \mathcal{B}^{(\mu_0(m))}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. When $\lambda_0 = \mathrm{id}$ or $\mu_0 = \mathrm{id}$, we write $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$ and $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ or $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$ and $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$.

Following 8.7.4.2 the sheaves $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ satisfy 7.3.2 ; similarly with D replaced by Z. Via the canonical homomorphisms of sheaves of rings $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \widetilde{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ (resp. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \to \widetilde{\mathcal{B}}^{(m+1)}_{\mathfrak{X}}(D)$) we get a sheaf of rings denoted by $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ (resp. $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)$) on $\mathfrak{X}^{(\mathbb{N})}$; similarly with D replaced by Z. We get also the sheaf of rings $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(D)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet^\sharp/S_\bullet^\sharp}(D)$ on $X_\bullet^{(\mathbb{N})}$; similarly with D replaced by Z. When D is empty, $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)$ (resp. $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(D)$) is denoted by $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ (resp. $\mathcal{O}^{(\bullet)}_{X_\bullet}$, i.e. $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ is the subring of $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ whose transition morphisms are the identity of $\mathcal{O}_{\mathfrak{X}}$. We have the canonical ring homomorphisms $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$.

Following 8.7.4.2, $(\mathfrak{X}^\sharp, \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z))/\mathfrak{S}^\sharp$ is strongly quasi-flat (see definition 8.5.5.3). Hence, we can apply 8.5.4.5 and 8.5.4.15 in this context where $\mathcal{D}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ or $\mathcal{D}^{(\bullet)} = \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$ (see 8.5.5.4).

9.1.1 Internal tensor products, localization functor outside a divisor

9.1.1.1. It follows from 8.7.4.7, we have the following tor finiteness results.

(a) The $\mathcal{O}^{(\bullet)}_{X_\bullet}$-module $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(T)$ has tor-dimension ≤ 1. The $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(D)$-module $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(T)$ has tor-dimension ≤ 2.

(b) The (left or right) $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet^\sharp/S_\bullet^\sharp}(D)$-module $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet^\sharp/S_\bullet^\sharp}(Z)$ has tor-dimension $\leq d+2$.

(c) Let $\nu \in L(\mathbb{N})$. Denoting the constant sheaf $\underrightarrow{l}_{\mathfrak{X}^{(\mathbb{N})},\mathbb{N}*}(\widetilde{\mathcal{D}}^{(\nu(0))}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$ on $X^{(\mathbb{N})}_{\mathbb{N}}$ simply by $\widetilde{\mathcal{D}}^{(\nu(0))}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)$, the (left or right) $\widetilde{\mathcal{D}}^{(\nu(0))}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)$-module $\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ has tor-dimension $\leq d+2$. Moreover, the (left or right) $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)$-module $\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ has tor-dimension $\leq d+2$.

(d) The $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$-module $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 1. The $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)$-module $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)$ has tor-dimension ≤ 2.

(e) The (left or right) $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ has tor-dimension $\leq d+2$.

(f) Let $\nu \in L(\mathbb{N})$. Denoting the constant sheaf $\underrightarrow{l}_{X,\mathbb{N}*}(\widetilde{\mathcal{D}}^{(\nu(0))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ on $\mathcal{X}^{(\mathbb{N})}$ simply by $\widetilde{\mathcal{D}}^{(\nu(0))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$, the (left or right) $\widetilde{\mathcal{D}}^{(\nu(0))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module $\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ has tor-dimension $\leq d+2$. Moreover, the (left or right) $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module $\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ has tor-dimension $\leq d+2$.

9.1.1.2. We have the morphism of ringed topoi

$$\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}} = (\underleftarrow{l}^{-1}_{\mathfrak{X}^{(\mathbb{N})}} \dashv \underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}) \colon (X^{(\mathbb{N})}_\bullet, \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to (\mathfrak{X}^{(\mathbb{N})}, \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)). \tag{9.1.1.2.1}$$

We keep notation 8.5.4.18, e.g. for any $\mathcal{E}^{(\bullet)} \in D^-({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ and $\mathcal{M}^{(\bullet)} \in D^-({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, we set:

$$\mathcal{M}^{(\bullet)}_\bullet := \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{M}^{(\bullet)}) = \mathcal{M}^{(\bullet)} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z), \qquad \mathcal{E}^{(\bullet)}_\bullet := \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)}) = \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \mathcal{E}^{(\bullet)},$$

9.1.1.3. Let $* = r$ or $* = l$. Following respectively 8.5.4.19.1 and 8.5.4.20 (see also the last example of 7.3.2.1), we have the following tensor product bifunctors

$$- \otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_\bullet}(Z)} - \colon \underrightarrow{LD}^-_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \times \underrightarrow{LD}^-_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to \underrightarrow{LD}^-_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)), \tag{9.1.1.3.1}$$

$$- \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} - \colon \underrightarrow{LD}^-_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \times \underrightarrow{LD}^-_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^-_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)). \tag{9.1.1.3.2}$$

Notation 9.1.1.4 (Quasi-coherence and partial forgetful functor of the divisor)**.** Let $\star \in \{-, \mathrm{b}\}$.

(a) The *partial forgetful functors of the divisor* $D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$ and $D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to D^\star_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ (thanks to the example 8.5.1.1.b, this is a special case of 8.5.2.3.1) will be denoted by $\mathrm{forg}_{D,Z}$.

(b) Following 8.5.4.16.1, the forgetful functors $\mathrm{forg}_{D,Z}$ induces

$$\mathrm{forg}_{D,Z} \colon \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)), \qquad \mathrm{forg}_{D,Z} \colon \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)). \tag{9.1.1.4.1}$$

(c) We still denote by $\mathrm{forg}_{D,Z} \colon D^\star(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D^\star(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}})$ the partial forgetful functor of the divisor.

9.1.1.5. Let $\star \in \{-, \mathrm{b}\}$.

(a) It follows from 4.3.4.6.1 (and from 9.1.1.1.a-b in the case where $\star = \mathrm{b}$) that for any $\mathcal{E}^{(\bullet)}_\bullet \in D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$, the morphism of $D^\star({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$:

$$\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(D)} \mathcal{E}^{(\bullet)}_\bullet \to \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)} \mathcal{E}^{(\bullet)}_\bullet =: ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}_\bullet) \tag{9.1.1.5.1}$$

is an isomorphism.

(b) For any $\mathcal{E}^{(\bullet)} \in D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, it follows from 8.5.4.21.3 (and from 9.1.1.1.d-e) that the morphism of $D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$:

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\mathcal{E}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)}\mathcal{E}^{(\bullet)} =: ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}) \tag{9.1.1.5.2}$$

is an isomorphism.

(c) Following 8.5.4.16.4, we get from 9.1.1.5.2 the localization outside Z functor:

$$({}^\dagger Z,\, D) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)} -\colon \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)). \tag{9.1.1.5.3}$$

We also write $\mathcal{E}^{(\bullet)}({}^\dagger D,\, Z) := ({}^\dagger Z,\, D)(\mathcal{E}^{(\bullet)})$. This functor $({}^\dagger Z,\, D)$ is *the localization outside Z functor*. When $D = \emptyset$, we omit writing it.

9.1.1.6. Following 7.5.1.13, the $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$-module $\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$, where $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ for any $m \in \mathbb{N}$ with the canonical transition maps, is endowed with a canonical structure of right $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$-module. Moreover, $\widetilde{\omega}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ satisfies the conditions of 7.3.2 for any $m \in \mathbb{N}$, i.e. $\widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) := \widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ is quasi-coherent in the sense of 8.5.1.7.

9.1.1.7. Let $\star \in \{-, \mathrm{b}\}$. We have a right version of 9.1.1.5.

(a) It follows from 4.3.4.8.3 (and from 9.1.1.1.a-b in the case where $\star = \mathrm{b}$) that for any $\mathcal{M}^{(\bullet)}_\bullet \in D^{\star}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$, the morphism of $D^{\star}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$:

$$\mathcal{M}^{(\bullet)}_\bullet \otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(D)} \widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z) \to \mathcal{M}^{(\bullet)}_\bullet \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)} \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) =: ({}^\dagger Z, D)(\mathcal{M}^{(\bullet)}_\bullet) \tag{9.1.1.7.1}$$

is an isomorphism. For any $\mathcal{E}^{(\bullet)}_\bullet \in D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$, with notation 9.1.1.6, it follows from 9.1.1.5.1 and 9.1.1.7.1 that we have the canonical isomorphism

$$({}^\dagger Z, D)(\widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D) \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{E}^{(\bullet)}_\bullet) \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes_{\mathcal{O}_{X_\bullet}} ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}_\bullet). \tag{9.1.1.7.2}$$

(b) For any $\mathcal{M}^{(\bullet)} \in D^{\star}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, it follows from 8.5.4.21.3 (and from 9.1.1.1.d-e) that the morphism of $D^{\star}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$:

$$\mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \to \mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) =: ({}^\dagger Z, D)(\mathcal{M}^{(\bullet)}) \tag{9.1.1.7.3}$$

is an isomorphism. For any $\mathcal{E}^{(\bullet)} \in D^{\star}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, it follows from 9.1.1.5.2 and 9.1.1.7.3 that we have the canonical isomorphism

$$({}^\dagger Z, D)(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) \otimes_{\mathcal{O}_{\mathfrak{X}}} ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}). \tag{9.1.1.7.4}$$

(c) Following 8.5.4.16.4, we get from 9.1.1.7.3 the localization outside Z functor:

$$({}^\dagger Z,\, D) := -\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)\colon \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)). \tag{9.1.1.7.5}$$

We also write $\mathcal{M}^{(\bullet)}({}^\dagger D,\, Z) := ({}^\dagger Z,\, D)(\mathcal{M}^{(\bullet)})$. This functor $({}^\dagger Z,\, D)$ is *the localization outside Z functor*. When $D = \emptyset$, we omit writing it.

9.1.1.8. For any $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$ and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, it follows from 8.5.4.5 that we have the functorial isomorphisms

$$({}^\dagger Z, D) \circ \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}(\mathcal{E}^{(\bullet)}_\bullet) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*} \circ ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}_\bullet), \qquad ({}^\dagger Z, D) \circ \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}} \circ ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}). \tag{9.1.1.8.1}$$

For any $\mathcal{E}^{(\bullet)}_{\bullet} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z))$ and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$, it follows from 8.5.4.5 that we have the functorial isomorphisms

$$\mathrm{forg}_{D,Z} \circ \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}(\mathcal{E}^{(\bullet)}_{\bullet}) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*} \circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}_{\bullet}), \qquad \mathrm{forg}_{D,Z} \circ \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}} \circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}). \tag{9.1.1.8.2}$$

Remark 9.1.1.9. Let $\star \in \{-, \mathrm{b}\}$. Let $\mathfrak{X}^{\sharp} \to \mathfrak{X}'^{\sharp}$ be a morphism of log formal schemes which are log smooth over $\mathfrak{S}^{\sharp}$ whose underlying morphism of formal schemes is the identity. We keep similar to 9.1 notation replacing $\mathfrak{X}$ by $\mathfrak{X}'$. Since a flat $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(D)$-module or a flat $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(D)$-module is also a flat $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)$-module, then it follows that the functor $({}^{\dagger}Z, D)$ of 9.1.1.5.1 that we get the commutative (up to canonical isomorphism) square:

$$\begin{array}{ccc}
D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(D)) & \xrightarrow[9.1.1.5.1]{({}^{\dagger}Z,D)} & D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)) \\
\downarrow & & \downarrow \\
D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(D)) & \xrightarrow[9.1.1.5.1]{({}^{\dagger}Z,D)} & D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z))
\end{array}$$

where the vertical maps are the forgetful functors. Since the functors $\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}$ and $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}$ commute with the forgetful functors (induced by the homomorphisms of the form $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z)$), then we get the commutative (up to canonical isomorphism) square:

$$\begin{array}{ccc}
D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(D)) & \xrightarrow[9.1.1.5.2]{({}^{\dagger}Z,D)} & D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z)) \\
\downarrow & & \downarrow \\
D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D)) & \xrightarrow[9.1.1.5.2]{({}^{\dagger}Z,D)} & D^{\star}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))
\end{array}$$

where the vertical maps are the forgetful functors.

9.1.1.10. If follows from 8.7.3.8.2 that the canonical morphism $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z_{\mathrm{red}}) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$ is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z_{\mathrm{red}}))$ and that the canonical morphism $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z_{\mathrm{red}}) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)$ is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z_{\mathrm{red}}))$. This yields that the canonical morphism $\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z_{\mathrm{red}}) \to \widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)$ is an isomorphism of $\underrightarrow{LM}(\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z_{\mathrm{red}}))$, and that canonical morphism $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z_{\mathrm{red}}) \to \widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)$ is an isomorphism of $\underrightarrow{LM}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z_{\mathrm{red}}))$, Hence, the functors $({}^{\dagger}Z, D)$ and $({}^{\dagger}Z_{\mathrm{red}}, D_{\mathrm{red}})$ are canonically isomorphic.

9.1.1.11 (Preservation of the coherence and perfection)**.** Let $\star \in \{-, \mathrm{b}\}$ and $? \in \{\mathrm{coh}, \mathrm{perf}\}$. The functors $({}^{\dagger}Z,\, D)$ of 9.1.1.5.3 and 9.1.1.7.5 preserve coherence and perfectness, i.e. they induce the functors

$$({}^{\dagger}Z,\, D)\colon \underrightarrow{LD}^{\star}_{\mathbb{Q},?}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D)) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},?}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)). \tag{9.1.1.11.1}$$

9.1.1.12 (Coherent complexes)**.** Let $\sharp \in \{-, \mathrm{b}\}$. We write in the same way the associated functor for coherent complexes:

$$({}^{\dagger}Z, D) := \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}}} -\colon D^{\sharp}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}}) \to D^{\sharp}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}). \tag{9.1.1.12.1}$$

The functor 9.1.1.12.1 is exact, which justifies the absence of the symbol $\mathbb{L}$. Following 8.4.1.9.c, we have the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T)) \to D^{\sharp}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$ in the case where $T = D$ or $T = Z$. The functors 9.1.1.5.3 and 9.1.1.12.1 are compatible with these functors $\underrightarrow{l}^{*}_{\mathbb{Q}}$ i.e. for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D))$ we have the canonical isomorphism of $D^{\sharp}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$

$$({}^{\dagger}Z, D) \circ \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \underrightarrow{l}^{*}_{\mathbb{Q}} \circ ({}^{\dagger}Z, D)(\mathcal{E}^{(\bullet)}). \tag{9.1.1.12.2}$$

Hence, both notation of $({}^{\dagger}Z, D)$ are compatible.

9.1.1.13. Let $D' \subset D$ be a third divisor of X and let $\lambda'_0 \leq \lambda_0$ be a third map of $L(\mathbb{N})$. Let $\sharp \in \{-, \mathrm{b}\}$. For any $m \in \mathbb{N}$, we set $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D') := \mathcal{B}^{(\lambda'_0(m))}_{\mathfrak{X}}(D)$, and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D') := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D')\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. We get similarly to 9.1.1.5 the functors $({}^\dagger Z, D')$ and $({}^\dagger D, D')$. It follows from 8.5.2.12.1 (resp. 8.5.2.12.2) that we have the isomorphism

$$({}^\dagger Z, D) \circ ({}^\dagger D, D') \xrightarrow{\sim} ({}^\dagger Z, D') \tag{9.1.1.13.1}$$

of functors $D^\sharp({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D')) \to D^\sharp({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$ (resp. $D^\sharp_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D')) \to D^\sharp_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$).

Proposition 9.1.1.14. *Let $\nu \in L(\mathbb{N})$. Let $\star \in \{-, \mathrm{b}\}$. We have the functors:*

$$\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\nu(0))}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} -\colon D^\star(\widetilde{\mathcal{D}}^{(\nu(0))}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to D^\star(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)), \tag{9.1.1.14.1}$$

$$\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\nu(0))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} -\colon D^\star(\widetilde{\mathcal{D}}^{(\nu(0))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to D^\star(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)), \tag{9.1.1.14.2}$$

and similarly replacing $\widetilde{\mathcal{D}}^{(\nu(0))}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ (resp. $\widetilde{\mathcal{D}}^{(\nu(0))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$) by $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ (resp. $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$).

Proof. It follows from 9.1.1.1.(c) that these functors preserve the boundedness of the cohomology. □

Remark 9.1.1.15. Let $\star \in \{-, \mathrm{b}\}$. Let $\nu \in L(\mathbb{N})$ and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^\star_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$.

(a) Following 8.3.2.2.(bi) (and 9.1.1.1.(f)), we have the canonical isomorphism of $\underrightarrow{LD}^\star_{\mathbb{Q}}(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$

$$\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \nu^*\mathcal{E}^{(\bullet)}. \tag{9.1.1.15.1}$$

(b) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. By using $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) \xrightarrow{\sim} \nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$, we get the first of the canonical isomorphisms of $\underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$:

$$\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}\mathcal{E}^{(\bullet)})$$
$$\xrightarrow[8.5.3.14.1]{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}(\nu^*\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}\mathcal{E}^{(\bullet)}) \xrightarrow[8.5.3.9.2]{\sim} \nu^*\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \nu^*\mathcal{E}^{(\bullet)}.$$

With 9.1.1.15.1, this yields the canonical morphisms

$$\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \mathcal{E}^{(\bullet)} \to \nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)}\mathcal{E}^{(\bullet)} \to \nu^*\mathcal{E}^{(\bullet)}$$

are isomorphisms in $\underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Moreover, it follows from 8.3.1.3 that we have the equivalence of categories

$$\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} -\colon \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \cong \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \tag{9.1.1.15.2}$$

whose forgetful functor $\underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\nu^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^\star_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ is canonically a quasi-inverse equivalence.

Proposition 9.1.1.16. *Let $\sharp \in \{-, \mathrm{b}\}$.*

(a) The forgetful functor and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -$ induce quasi-inverse equivalences of categories between $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ and $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

(b) The forgetful functor and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -$ (or $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -$) induce quasi-inverse equivalences of categories between $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ and $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$, between $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$) and $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

(c) The forgetful functor and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -$ *are quasi-inverse equivalences of categories between* $\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ *and* $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$*, between* $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ *and* $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$*.*

Proof. 1) We have the ring extensions $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \to \mu_0^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)$. Hence, $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)$ is an isomorphism in $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. By using the functor 8.5.4.9.1 (which preserves here the boundedness thanks to 9.1.1.1.(e)), this yields for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ the isomorphisms of $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$:

$$\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(\bullet)}, \tag{9.1.1.16.1}$$

which implies the first assertion.

2) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. We have the isomorphisms of $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$

$$\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(\bullet)}. \tag{9.1.1.16.2}$$

This yields the second assertion with respect to the functor $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -$. Hence, with 9.1.1.16.1 and 9.1.1.16.2, the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(\bullet)} \tag{9.1.1.16.3}$$

is an isomorphism, which yields the quasi-coherent case of the second statement.

3) Let $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. Since $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \to \mu_0^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)$ is an isomorphism in $\underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$, this yields the isomorphisms of $\underrightarrow{LD}^\sharp_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$:

$$\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mu_0^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{F}^{(\bullet)}. \tag{9.1.1.16.4}$$

Since $\mu_0^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{coh}}(\mu_0^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ (use 8.5.4.17), then $\mu_0^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^\sharp_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ (use 8.4.1.5). Hence, with Proposition 8.5.4.17, we have checked the coherent case of the second statement.

4) We check the third one similarly. □

9.1.2 Preservation of bounded quasi-coherence by localization functor outside a divisor, internal tensor products

Proposition 9.1.2.1. *Let $D' \subset D$ be a third divisors of X. Let $m' \geq m \geq 0$ be two integers. The canonical homomorphisms of* $D^-(\widetilde{\mathcal{B}}^{(m')}_{\mathfrak{X}}(Z))$

$$\mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(m)}_{\mathfrak{X}}(D')} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(Z), \tag{9.1.2.1.1}$$

$$\mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}_{\mathcal{B}^{(m)}_{\mathfrak{X}}(D')} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \tag{9.1.2.1.2}$$

are p^3-isogenies (see definition 7.4.2.1.(c)).

Proof. 1) Following the lemma 8.7.4.4.a, the kernel of the canonical epimorphism $a\colon \mathcal{B}^{(m)}_{\mathfrak{X}}(D') \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(Z)$ is a quasi-coherent $\mathcal{O}_X$-module. Hence, it follows from 7.4.2.6.b that a is a p-isogeny. Moreover, following lemma 8.7.4.4.b, the canonical morphism $\mathcal{B}^{(m)}_{\mathfrak{X}}(D') \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D') \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z)$ is an isomorphism. Hence, we get by composition the p-isogeny $\alpha\colon \mathcal{B}^{(m)}_{\mathfrak{X}}(D') \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(Z)$.

2) Since the functor $\mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(m)}_{\mathfrak{X}}(D')} -$ sends a p-isogeny to a p-isogeny, then we get the p-isogeny

$$\beta\colon \mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \xrightarrow{\sim} \mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(m)}_{\mathfrak{X}}(D')} (\mathcal{B}^{(m)}_{\mathfrak{X}}(D') \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z)) \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(m)}_{\mathfrak{X}}(D')} \mathcal{B}^{(m')}_{\mathfrak{X}}(Z).$$

Let $\gamma \colon \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(m)}_{\mathfrak{X}}(D')}\mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(Z)$ be the canonical morphism (the last one of 9.1.2.1.1). Following the part 1) of the proof (replace D' by D), the composition $\gamma\circ\beta \colon \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m')}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(m')}_{\mathfrak{X}}(Z)$ is a p-isogeny. Since so is β, then γ is a p^3-isogeny (see **??.??**). Since the composition of the following two canonical morphisms of 9.1.2.1.1 is the identity, by using **??.??**, this yields that the first morphism is also a p^3-isogeny.

3) Likewise (we remove the symbols $\mathbb{L}$) we can check that the canonical morphisms of 9.1.2.1.2 are p^3-isogenies.

□

Corollary 9.1.2.2. *The canonical morphisms of $D^{\mathrm{b}}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$*

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \tag{9.1.2.2.1}$$

are lim-ind-isogenies.

Proof. This is a consequence of 9.1.2.1. □

Proposition 9.1.2.3. *Let $\sharp \in \{-, \mathrm{b}\}$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ (resp. $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$).*

(a) The functorial in $\mathcal{E}^{(\bullet)}$ canonical morphism:

$$({}^\dagger Z,\, D)\circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \tag{9.1.2.3.1}$$

is an isomorphism of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ (resp. $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$).

(b) The functorial in $\mathcal{E}^{(\bullet)}$ canonical morphism:

$$\mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \to \mathrm{forg}_{D,Z}\circ({}^\dagger Z, D)\circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \tag{9.1.2.3.2}$$

is an isomorphism of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ (resp. $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$).

(c) The functor $\mathrm{forg}_{D,Z}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ (resp. $\mathrm{forg}_{D,Z}\colon \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$) is fully faithful.

Proof. 1) Since the functors $({}^\dagger Z,\, D)$ and $\mathrm{forg}_{D,Z}$ commute with $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}$ (see 9.1.1.8.1 and 9.1.1.8.2), then via the equivalence of categories of Theorem 8.5.1.10 we reduce to check the non-respective case.

2) Let us check the non-respective case. With the notations 9.1.1.5.2, we have canonical isomorphisms of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$:

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \left(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\right)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)} \underset{9.1.2.2}{\xrightarrow{\sim}} \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)} \xleftarrow{\sim} \mathcal{E}^{(\bullet)}.$$

This composition is the canonical morphism $({}^\dagger Z,\, D)\circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ which is therefore an isomorphism. This yields the canonical isomorphism $\mathrm{forg}_{D,Z}\circ({}^\dagger Z, D)\circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)})$ of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. Since the composition $\mathrm{forg}_{T,D}(\mathcal{E}^{(\bullet)}) \to \mathrm{forg}_{D,Z}\circ({}^\dagger Z, D)\circ \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)})$ is the identity, this implies that the canonical morphism 9.1.2.3.2 is an isomorphism. The fully faithfulness of $\mathrm{forg}_{D,Z}$ is a consequence of both previous assertions. □

Corollary 9.1.2.4. *Let $D' \subset D$ be a third divisor of X and let $\lambda'_0 \le \lambda_0$ be a third map of $L(\mathbb{N})$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ (resp. $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D))$). With notation 9.1.1.13, we get the functorial in $\mathcal{E}^{(\bullet)}$ canonical morphism*

$$({}^\dagger Z,\, D')\circ \mathrm{forg}_{D',D}(\mathcal{E}^{(\bullet)}) \to ({}^\dagger Z,\ D)(\mathcal{E}^{(\bullet)}) \tag{9.1.2.4.1}$$

is an isomorphism of $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ (resp. $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$).

Proof. Following 9.1.2.3.1, we benefit from the canonical isomorphism $({}^\dagger D,\ D')\circ \mathrm{forg}_{D',D}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. By applying the functor $({}^\dagger Z,\ D)$ to this latter isomorphism, the isomorphism $({}^\dagger Z,\ D') \xrightarrow{\sim} ({}^\dagger Z,\ D)\circ ({}^\dagger D,\ D')$ of 9.1.1.13.1 allows us then to conclude. □

Notation 9.1.2.5. Let $D' \subset D \subset Z$ be some divisors of X. Following 9.1.2.4, by forgetting to write some forgetful functors, the functors $({}^\dagger Z,\ D')$ and $({}^\dagger Z,\ D)$ are canonically isomorphic over categories of the form $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}$. Hence, we can simply write $({}^\dagger Z)$ in both case.

Proposition 9.1.2.6. *Suppose the log structure of $\mathfrak{S}^\sharp$ is trivial and that S is regular. Suppose $\mathfrak{X}/\mathfrak{S}$ is smooth and that $M_{\mathfrak{X}^\sharp}$ is the log structure given by a strict normal crossing divisor of $\mathfrak{X}/\mathfrak{S}$.*

(a) We have the equalities $D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}) = D_{\mathrm{qc,tdf}}(\widehat{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp})$, $D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}) = D_{\mathrm{qc,tdf}}(\widehat{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, $D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) = D_{\mathrm{qc,tdf}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

(b) With notation 8.5.4.11, we have $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc,tdf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc,tdf}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z))$ and similarly for $\mathfrak{X}$ and $\mathfrak{S}$ replaced respectively by $X_\bullet$ and $S_\bullet$.

(c) The bifunctors 9.1.1.3.2 and 9.1.1.3.1 factor respectively through the bifunctors

$$-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_\bullet}(Z)} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)), \tag{9.1.2.6.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)). \tag{9.1.2.6.2}$$

Proof. a) i) Let us check $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_{X^\sharp/S}) = D^{\mathrm{b}}_{\mathrm{qc,tdf}}(\mathcal{D}^{(m)}_{X^\sharp/S})$. Since this is local, we can suppose X affine. With the theorems of type A for the quasi-coherent sheaves (see 4.6.1.7.c), we reduce then to check $D^{\mathrm{b}}(D^{(m)}_{X^\sharp/S}) = D^{\mathrm{b}}_{\mathrm{tdf}}(D^{(m)}_{X^\sharp/S})$. This last equality is a consequence of the fact that $D^{(m)}_{X^\sharp/S}$ is a ring of global dimension $\leq N$ for some integer N independent from m (see 6.1.4.2 and [Sta22, 066P]).

ii) By using Theorem 8.5.1.10, proposition 7.3.1.13 and the paragraph 8.5.1.9, since $D^{(m)}_{X^\sharp/S}$ is a ring of finite global dimension

b) The inclusion is a consequence of the flatness of $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$. Let us now prove the equality. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}}(Z))$. It follows from 9.1.2.3.1 (in the case where D empty), that we have the isomorphism $({}^\dagger Z)\circ \mathrm{forg}_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}}(Z))$. It follows from a) that $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) = \underrightarrow{LD}_{\mathbb{Q},\mathrm{qc,tdf}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$. Hence, we get $\mathrm{forg}_Z(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc,tdf}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}})$. Since the functor $({}^\dagger Z)$ preserves the LD-tor amplitude, we conclude the proof.

c) Similarly to 8.5.4.14, we get c) from the b). □

Corollary 9.1.2.7. *Let $\sharp \in \{-, \mathrm{b}\}$. When $\sharp = \mathrm{b}$, we suppose we are in the context of Proposition 9.1.2.6. Let $\mathcal{M}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$. We have the canonical isomorphism in $\underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ of the form*

$$({}^\dagger Z,\, D)(\mathcal{M}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}({}^\dagger Z,\, D)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z,\, D)\left(\mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\mathcal{E}^{(\bullet)}\right). \tag{9.1.2.7.1}$$

Proof. Following notation 9.1.1.2, we set $\mathcal{E}^{(\bullet)}_\bullet = \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)})$ and $\mathcal{M}^{(\bullet)}_\bullet = \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{M}^{(\bullet)})$. By using 9.1.1.5.1, we get the isomorphism:

$$({}^\dagger Z,\, D)(\mathcal{M}^{(\bullet)}_\bullet)\otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)}({}^\dagger Z,\, D)(\mathcal{E}^{(\bullet)}_\bullet) \xrightarrow{\sim} ({}^\dagger Z,\, D)\left(\mathcal{M}^{(\bullet)}_\bullet\otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(D)}\mathcal{E}^{(\bullet)}_\bullet\right). \tag{9.1.2.7.2}$$

Hence, with the commutativity isomorphisms 9.1.1.8.1 and using the commutativity of the diagram 8.5.4.20.3, we get 9.1.2.7.1 from 9.1.2.7.2. □

Corollary 9.1.2.8. *Let $\sharp \in \{-, \mathrm{b}\}$. When $\sharp = \mathrm{b}$, we suppose we are in the context of Proposition 9.1.2.6. Let $\mathcal{M}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$, and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\sharp}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$. We have the isomorphism*

$$\mathrm{forg}_{D,Z}(\mathcal{M}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{forg}_{D,Z}\left(\mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}\right). \tag{9.1.2.8.1}$$

Proof. We construct the isomorphism 9.1.2.8.1 by composing the following isomorphisms:

$$\mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)} \overset{9.1.2.3.1}{\xrightarrow{\sim}} \mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\left(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}(\mathrm{forg}_{D,Z}(\mathcal{E}^{(\bullet)}))\right) \overset{8.5.4.21.4}{\xrightarrow{\sim}} \mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)}\mathcal{E}^{(\bullet)}.$$

□

9.1.3 Composition of localisation functors

Let Z' be a second divisor of X. Let $\mu'_0 \in L(\mathbb{N})$. For any $m \in \mathbb{N}$, for any $i \in \mathbb{N}$, we set $\widetilde{\mathcal{B}}^{(m)}_{X_i}(Z') := \mathcal{B}^{(\mu'_0(m))}_{X_i}(Z')$, and $\widetilde{\mathcal{D}}^{(m)}_{X_i^{\sharp}/S_i^{\sharp}}(Z') := \widetilde{\mathcal{B}}^{(m)}_{X_i}(Z')\otimes_{\mathcal{O}_{X_i}}\mathcal{D}^{(m)}_{X_i^{\sharp}/S_i^{\sharp}}$. For any $m \in \mathbb{N}$, we set $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z') := \mathcal{B}^{(\mu'_0(m))}_{\mathfrak{X}}(Z')$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z') := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z')\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$. When $Z \subset Z'$, we can suppose $\mu_0 \leq \mu'_0$ (see 9.1.1.16).

The following Lemma improves the part (a) and (b) of Lemma 8.7.4.4 by adding hypotheses on the divisors.

Lemma 9.1.3.1. *Suppose Z, Z' are two reduced divisors of X whose irreducible components are two by two distinct. Let $\mathfrak{U}''$ the open set of $\mathfrak{X}$ complementary to $Z \cup Z'$.*

(a) For any $i \in \mathbb{N}$, the canonical morphism $\mathcal{B}^{(m)}_{X_i}(Z) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(Z') \to \mathcal{B}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(Z')$ is an isomorphism.

(b) The canonical morphism $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m)}_{\mathfrak{X}}(Z') \to \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m)}_{\mathfrak{X}}(Z')$ is an isomorphism and the $\mathcal{O}_{\mathfrak{X}}$-algebra $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m)}_{\mathfrak{X}}(Z')$ is p-torsion free.

(c) The canonical morphism of $\mathcal{O}_{\mathfrak{X}}$-algebras $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(m)}_{\mathfrak{X}}(Z') \to j_*\mathcal{O}_{\mathfrak{U}''}$, where $j : \mathfrak{U}'' \hookrightarrow \mathfrak{X}$ is the inclusion, is a monomorphism.

(d) Let χ, $\lambda : \mathbb{N} \to \mathbb{N}$ defined respectively by setting for any integer $m \in \mathbb{N}$, $\chi(m) := p^{p-1}$ and $\lambda(m) := m+1$. We have two canonical ring monomorphisms $\alpha^{(\bullet)} : \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z') \to \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z \cup Z')$ and $\beta^{(\bullet)} : \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z \cup Z') \to \chi^*\lambda^*(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z'))$ such that $\chi^*\lambda^*(\alpha^{(\bullet)}) \circ \beta^{(\bullet)} = \sigma_{\mathcal{F},(\lambda,\chi)}$ and $\beta^{(\bullet)} \circ \alpha^{(\bullet)} = \sigma_{\mathcal{E},(\lambda,\chi)}$ (see notation 8.1.4.1.1).

Proof. 0) Since the assertions a), b) and c) are local on $\mathfrak{X}$, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine, integral and there exist $f_1, \ldots, f_r \in \mathcal{O}_{\mathfrak{X}}$ (resp. $f'_1, \ldots, f'_{r'} \in \mathcal{O}_{\mathfrak{X}}$) lifting a local equation of the irreducible components of $Z \subset X$ (resp $Z' \subset X$). Denote by $f := \prod_{s=1}^{r} f_s$ and $f' := \prod_{s=1}^{r'} f'_s$. We have therefore the isomorphisms $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}\{T\}/(f^{p^{m+1}}T - p)$ and $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z') \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}\{T'\}/((f')^{p^{m+1}}T' - p)$, where we took care to distinguish the variables T and T'.

a) We have to prove that for any $j \leq -1$, $\mathcal{H}^j(\mathcal{B}^{(m)}_{X_i}(Z) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(Z')) = 0$.

Since $\mathfrak{X}$ is p-torsion free, we have the short exact sequence $0 \to \mathcal{O}_{\mathfrak{X}}\{T\} \underset{f^{p^{m+1}}T-p}{\longrightarrow} \mathcal{O}_{\mathfrak{X}}\{T\} \to \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \to 0$. Since $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$ is p-torsion free (see 8.7.4.1), this yields the short exact sequence $0 \to \mathcal{O}_{X_i}[T] \underset{\overline{f}^{p^{m+1}}T-p}{\longrightarrow} \mathcal{O}_{X_i}[T] \to \mathcal{B}^{(m)}_{X_i}(Z) \to 0$ where $\overline{f}$ is the reduction of f modulo π^{i+1}. Since this exact sequence gives a canonical resolution of $\mathcal{B}^{(m)}_{X_i}(Z)$ by some flat $\mathcal{O}_{X_i}$-modules, by applying the functor $- \otimes_{\mathcal{O}_{X_i}} \mathcal{B}^{(m)}_{X_i}(Z')$ to this exact sequence, we see that it is a question of proving that the image de $\overline{f}$ in $B^{(m)}_{X_i}(Z')$ is nonzero and is nonzero divisor. Since $\widehat{B}^{(m)}_{\mathfrak{X}}(Z')$ is p-torsion free (see 8.7.4.1), we reduce to the case $i = 0$. Set $\overline{A} = \Gamma(X, \mathcal{O}_X)$. In that case, we get $\mathcal{B}^{(m)}_X(Z') \xrightarrow{\sim} \mathcal{O}_X[T']/(\overline{f'}^{p^{m+1}}T')$ where $\overline{f'}$ is the reduction of f' modulo π, i.e. is the image of f' on $\overline{A}$. Let $P(T'), Q(T') \in \overline{A}[T']$ satisfying the equality $\overline{f} \cdot P(T') = (\overline{f'}^{p^{m+1}} T') \cdot Q(T')$ in $\overline{A}[T']$. We have to check that $\overline{f'}^{p^{m+1}} T'$ divides $P(T')$. Since $\overline{A}/\overline{f'_s}\,\overline{A}$ is integral for any $s = 1, \ldots, r'$ (because Z' is reduced), since the image of $\overline{f}$ on $\overline{A}/\overline{f'_s}\,\overline{A}$ is not null (because the irreducible components of Z and Z' are two by two distinct), since $\overline{A}$ is integral, then from an equality of the form $\overline{f}\overline{a} = \overline{f'}^{p^{m+1}} \overline{a'}$ in $\overline{A}$ with $a, a' \in A$, we get $\overline{f'}^{p^{m+1}}$ divides $\overline{a}$. Hence we are done.

b) By applying the functor $\mathbb{R}\varprojlim_i$ to the canonical isomorphisms of (a), via Mittag-Leffler, we get the desired isomorphism of (b) (or we can invoke 8.7.4.4.b). Moreover, following 8.7.4.2, $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m)}_{\mathfrak{X}}(Z') \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{\mathfrak{X}})$ and $k \otimes^{\mathbb{L}}_{\mathcal{V}} \left(\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m)}_{\mathfrak{X}}(Z') \right) \xrightarrow{\sim} \mathcal{B}^{(m)}_{X}(Z) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{B}^{(m)}_{X}(Z')$. This yields following a) and the isomorphism of b) that $\mathcal{H}^{-1}(k \otimes^{\mathbb{L}}_{\mathcal{V}} (\mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m)}_{\mathfrak{X}}(Z'))) \xrightarrow{\sim} 0$.

c) It is sufficient to copy the proof of 8.7.4.1.(b): denote $B^{(m)}_{\mathfrak{X}}(Z) := \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$ and $B^{(m)}_{\mathfrak{X}}(Z') := \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z')$. We get $\widehat{B} := \Gamma(\mathfrak{X}, \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m)}_{\mathfrak{X}}(Z')) \xrightarrow{\sim} B^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}_A B^{(m)}_{\mathfrak{X}}(Z')$. Since $\widehat{B}$ is p-torsion free, then it is f-torsion free and we get therefore the monomorphism $\widehat{B} \hookrightarrow \widehat{B}_f$. Denote by $\mathfrak{U} = D(f)$ the open set of $\mathfrak{X}$ complementary to Z. Since $(B^{(m)}_{\mathfrak{X}}(Z)_f)^{\widehat{}} \xrightarrow{\sim} A_{\{f\}}$, then we get the first isomorphism: $(\widehat{B}_f)^{\widehat{}} \xrightarrow{\sim} A_{\{f\}} \widehat{\otimes}_A B^{(m)}_{\mathfrak{X}}(Z') \xrightarrow{\sim} A_{\{f\}}\{T'\}/((f')^{p^{m+1}} T' - p)$. Since f' modulo π is not a zero divisor of $A_{\{f\}}$, then following 8.7.4.1.b, the canonical morphism $A_{\{f\}}\{T'\}/((f')^{p^{m+1}} T' - p) \to A_{\{ff'\}} = \Gamma(\mathfrak{U}'', \mathcal{O}_{\mathfrak{X}})$ is injective. Hence, we reduce to check $\widehat{B}_f$ is p-adically separated. We still denote by T, the image of T in $\widehat{B}$. Since T is a divisor of p in $B^{(m)}_{\mathfrak{X}}(Z)$ and therefore in $\widehat{B}_f$, then we reduce to check that $\widehat{B}_f$ is T-adically separated. Let $b, x \in \widehat{B}_f$ such that $(1 - bT)x = 0$. With Krull theorem (see [Mat89, Theorem 8.9]), it is enough to prove $x = 0$. By multiplying by a power of f large enough if necessary, we can suppose $b, x \in \widehat{B}$ and there exists $s \in \mathbb{N}$ large enough such that $(f^s - bT)x = 0$ in $\widehat{B}$. However, since $B^{(m)}_{\mathfrak{X}}(Z)/TB^{(m)}_{\mathfrak{X}}(Z) \xrightarrow{\sim} A\{T\}/(f^{p^{m+1}} T - p, T) = A\{T\}/(p, T) \xrightarrow{\sim} A/pA$, then we get the first isomorphism: $\widehat{B}/T\widehat{B} \xrightarrow{\sim} A/pA \otimes_A A\{T'\}/((f')^{p^{m+1}} T' - p) \xrightarrow{\sim} (A/pA)[T']/(\overline{f'}^{p^{m+1}} T') \xrightarrow{\sim} B^{(m)}_{X}(Z')$. Hence, since the image of f^s in $B^{(m)}_{X}(Z')$ is nonzero and is not a zero divisor (see the step a) of the proof), then the equality $(f^s - bT)x = 0$ implies that $x \in T\widehat{B}$. Since $\widehat{B}$ is p-torsion free, then it is T-torsion free. Hence, by iterating the reasoning, we get $x \in \cap_{n \in \mathbb{N}} T^n \widehat{B}$. With Krull theorem (see [Mat89, Theorem 8.9]), this yields there exists $c \in \widehat{B}$ such that $(1 - cT)x = 0$.

The relation $(1 - cT)x = 0$ in $\widehat{B}$ induces in $\widehat{B}/p\widehat{B}$ the equality $(1 - \overline{c}T)\overline{x} = 0$. On the other hand, $\widehat{B}/p\widehat{B} \xrightarrow{\sim} B^{(m)}_{X}(Z) \otimes_A B^{(m)}_{X}(Z') \xrightarrow{\sim} B^{(m)}_{X}(Z')[T]/(\overline{f}^{p^{m+1}} T)$. We can choose some elements $\widetilde{c}(T)$, $\widetilde{X}(T)$ of $B^{(m)}_{X}(Z')[T]$ inducing modulo $\overline{f}^{p^{m+1}} T$ the elements $\overline{c}$, $\overline{x}$. Hence, there exists $Q(T) \in B^{(m)}_{X}(Z')[T]$ such that $(1 - \widetilde{c}(T)T)\widetilde{X}(T) = \overline{f}^{p^{m+1}} TQ(T)$ in $B^{(m)}_{X}(Z')[T]$. This yields there exists $R(T) \in B^{(m)}_{X}(Z')[T]$ such that $\widetilde{X}(T) = TR(T)$. Hence we get in $B^{(m)}_{X}(Z')[T]$ the equality $(1 - \widetilde{c}(T)T)R(T) = \overline{f}^{p^{m+1}} Q(T)$. This implies that $\overline{f}^{p^{m+1}}$ divides $R(T)$ and therefore $\overline{x} = 0$, i.e. $x \in p\widehat{B}$. Since $\widehat{B}$ is p-torsion free, repeating the process we get $x \in \cap_{n \in \mathbb{N}} p^n \widehat{B} = \{0\}$ and we are done.

d) Since $Z, Z' \subset Z \cup Z'$, then we get the canonical morphism $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z') \to \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z \cup Z')$ and $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z) \to \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z \cup Z')$. This yields by extension $\alpha^{(\bullet)}$. The fact that this is a monomorphism is a consequence of $d)$. This yields that the construction Moreover, since $\sigma_{\mathcal{F},(\lambda,\chi)}$ is also a monomorphism (use 8.7.4.1), then the existence of such morphism $\beta^{(\bullet)}$ is local on $\mathfrak{X}$. Hence, we reduce to local situation of 0) and we keep its notations. Let $\gamma^{(m)} \colon \mathcal{O}_{\mathfrak{X}}[T] \to (\mathcal{B}^{(m+1)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m+1)}_{\mathfrak{X}}(Z'))_{\mathbb{Q}}$ be the morphism of $\mathcal{O}_{\mathfrak{X}}$-algebras defined by setting $\gamma^{(m)}(T) := \frac{1}{p}(\frac{p}{f^{p^{m+1}}} \widehat{\otimes} \frac{p}{(f')^{p^{m+1}}})$. Let $n \in \mathbb{N}$, $q \in \mathbb{N}$, $r \in \mathbb{N}$ such that $0 \leq r < p$ and $n = pq + r$. We compute $\gamma^{(m)}(T^n) = \frac{1}{p^r}(\frac{p}{f^{p^{m+1}}} \widehat{\otimes} \frac{p}{(f')^{p^{m+1}}})^r \left(p^{p-2}(\frac{p}{f^{p^{m+2}}} \widehat{\otimes} \frac{p}{(f')^{p^{m+2}}}) \right)^q$. Hence, we get the factorisation $p^{p-1}\gamma^{(m)} \colon \mathcal{O}_{\mathfrak{X}}\{T\} \to (\mathcal{B}^{(m+1)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{B}^{(m+1)}_{\mathfrak{X}}(Z'))$ of $\mathcal{O}_{\mathfrak{X}}\{T\}$-modules. Since $\gamma^{(m)}((ff')^{p^{m+1}} T - p) = 0$, then $p^{p-1}\gamma^{(m)}$ induces the desired morphism $\beta^{(m)}$. □

Proposition 9.1.3.2. *For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, the canonical functorial in Z, Z', $\mathcal{E}^{(\bullet)}$ morphism $({}^\dagger Z') \circ ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \to (Z' \cup Z)(\mathcal{E}^{(\bullet)})$ is an isomorphism.*

Proof. By using 9.1.1.10, we can suppose Z and Z' are reduced. Let Z_1, Z_2, Z'' be some divisors whose irreducible components are two by two distinct and such that $Z = Z_1 \cup Z''$ et $Z' = Z_2 \cup Z''$. It follows from 9.1.3.1 the isomorphisms $({}^\dagger Z') \circ ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z_2) \circ ({}^\dagger Z'') \circ ({}^\dagger Z'') \circ ({}^\dagger Z_1)(\mathcal{E}^{(\bullet)})$. It follows from 9.1.2.3 that we get $({}^\dagger Z'') \circ ({}^\dagger Z'') \xrightarrow{\sim} ({}^\dagger Z'')$. Using again 9.1.3.1, since $Z \cup Z' = Z_1 \cup Z'' \cup Z_2$, then we are done. □

Let us five the following corollary of 9.1.3.2.

Lemma 9.1.3.3. *Let* $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. *The canonical morphism* $\mathcal{G}^{(\bullet)}({}^\dagger Z') \to \mathcal{G}^{(\bullet)}({}^\dagger Z \cup Z')$ *is an isomorphism of* $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

Proof. Following 9.1.2.4, by omitting to indicate the forgetful functor, we have the isomorphism $\mathcal{G}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}({}^\dagger Z)$. Moreover, by 9.1.3.2, $({}^\dagger Z') \circ ({}^\dagger Z) \xrightarrow{\sim} ({}^\dagger Z \cup Z')$. We deduce the desired isomorphism $\mathcal{G}^{(\bullet)}({}^\dagger Z') \xrightarrow{\sim} \mathcal{G}^{(\bullet)}({}^\dagger Z \cup Z')$. □

Proposition 9.1.3.4. *Let* $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z'))$. *The canonical morphism:*

$$\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)}({}^\dagger Z \cup Z') \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z\cup Z')} \mathcal{E}^{(\bullet)}({}^\dagger Z \cup Z') \tag{9.1.3.4.1}$$

is an isomorphism of $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

Proof. Following 9.1.1.5.2, the morphism $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z \cup Z') \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z \cup Z') \widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{G}^{(\bullet)}$ is an isomorphism. In the same way, for $\mathcal{E}^{(\bullet)}$. Via the associativity properties of the derived complete tensor product for quasi-coherent complexes, this yields the canonical isomorphism: $({}^\dagger Z \cup Z') \left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}\right) \xrightarrow{\sim} \mathcal{G}^{(\bullet)}({}^\dagger Z \cup Z') \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z\cup Z')} \mathcal{E}^{(\bullet)}({}^\dagger Z \cup Z')$. Since we have the canonical morphisms $({}^\dagger Z) \left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}\right) \xrightarrow{\sim} ({}^\dagger Z)(\mathcal{G}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}$ and $({}^\dagger Z') \left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}\right) \to \mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} ({}^\dagger Z')(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}$, then by 9.1.3.2 the canonical morphism $\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)} \to ({}^\dagger Z \cup Z') \left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}\right)$ is an isomorphism. □

9.1.4 The case of pseudo quasi-coherent modules

We keep notation 9.1.3.

Notation 9.1.4.1. Let $\mathcal{E}^{(\bullet)} \in M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. We define the divisor extension functor $({}^\dagger Z, D)^0 \colon M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \to M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ by putting $({}^\dagger Z, D)^0(\mathcal{E}^{(\bullet)}) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)}$. As for 9.1.1.3.2, we verify that the functor $({}^\dagger Z, D)^0$ factorizes into

$$({}^\dagger Z, D)^0 \colon \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)). \tag{9.1.4.1.1}$$

The canonical forgetful functor $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ factorizes into the functor which we denote by

$$\mathrm{forg}_{D,Z} \colon \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)). \tag{9.1.4.1.2}$$

It seems false that the functor 9.1.4.1.2 is fully faithful without finiteness hypothesis. To get satisfying properties we need to work with pseudo-coherent modules whose notion is recalled just below.

Definition 9.1.4.2. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ and $\mathcal{E}^{(\bullet)}_{\bullet} \in \underrightarrow{LM}_{\mathbb{Q}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$.

1. We say that $\mathcal{E}^{(\bullet)}$ is *pseudo-quasi-coherent* (as an object of $\underrightarrow{LM}_{\mathbb{Q}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$) if it is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)})$ to a complex $\mathcal{F}^{(\bullet)} \in \mathrm{Mod}_{\mathrm{pqc}}(\mathcal{D}^{(\bullet)})$ (see notation 7.2.3.5).

   We denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{pqc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ consisting of quasi-coherent complexes.

2. We say that $\mathcal{E}^{(\bullet)}$ is *of local presentation* (as an object of $\underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$) if it is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ to a complex $\mathcal{F}^{(\bullet)} \in \mathrm{Mod}_{\mathrm{lp}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ (see notation 7.1.3.9).

   We denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{lp}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ consisting of local presentation.

9.1.4.3. The functors $\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}\colon \mathrm{Mod}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z)) \to \mathrm{Mod}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ and $\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}\colon \mathrm{Mod}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \mathrm{Mod}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ send lim-ind-isogenies to lim-ind-isogenies. Hence, we get the functors $\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}\colon \underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ and $\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}\colon \underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$, which yields the adjoint functors ($\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}} \dashv \underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}$). It follows from 7.2.3.7 that for any $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{lp}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ the adjoint morphism $\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}} \circ \underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}(\mathcal{E}^{(\bullet)}_\bullet) \to \mathcal{E}^{(\bullet)}_\bullet$ is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{lp}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$. By using 7.2.1.5 and 7.2.2.1, this implies that the functor $\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}\colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{lp}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ is fully faithful and its essential image is $\underrightarrow{LM}_{\mathbb{Q},\mathrm{pqc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. Moreover, the functors $\underleftarrow{l}_{\mathfrak{X}^{(\mathbb{N})},*}$ and $\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}$ induce canonically quasi-inverse equivalences of categories between $\underrightarrow{LM}_{\mathbb{Q},\mathrm{lp}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(Z))$ and $\underrightarrow{LM}_{\mathbb{Q},\mathrm{pqc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

9.1.4.4. As pseudo quasi-coherence is a notion which is independant of the choice of divisors, the forgetful functor factors into a functor of the form $\mathrm{forg}_{D,Z}\colon M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. The factorization $({}^\dagger Z, D)^0\colon M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \to M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ is even more obvious. As for 9.1.2.3 (i.e., it suffices to remove the "$\mathbb{L}$" in the proof), we deduce from 9.1.2.2 that, for all $\mathcal{E}'^{(\bullet)} \in M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$, the functorial in $\mathcal{E}'^{(\bullet)}$ canonical morphisms:

$$\mathrm{forg}_{D,Z}(\mathcal{E}'^{(\bullet)}) \to \mathrm{forg}_{D,Z} \circ ({}^\dagger Z, D)^0 \circ \mathrm{forg}_{D,Z}(\mathcal{E}'^{(\bullet)}),$$
$$({}^\dagger Z, D)^0 \circ \mathrm{forg}_{D,Z}(\mathcal{E}'^{(\bullet)}) \to \mathcal{E}'^{(\bullet)} \qquad (9.1.4.4.1)$$

are isomorphisms. The functor $\mathrm{forg}_{D,Z}\colon M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to M_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ is therefore fully faithful.

9.1.4.5. Similarly to 9.1.3.2, for any $\mathcal{E}^{(\bullet)} \in \mathrm{Mod}_{\mathrm{pqc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, the canonical functorial in Z, Z', $\mathcal{E}^{(\bullet)}$ morphism $({}^\dagger Z') \circ ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \to (Z' \cup Z)(\mathcal{E}^{(\bullet)})$ is an isomorphism.

9.1.5 Theorem of type A in $LM_{\mathbb{Q},\mathrm{coh}}$

9.1.5.1. Let $\lambda \in L(\mathbb{N})$ and $m_0 = \lambda(0)$. We denote by $\mu_{m_0} \in L(\mathbb{N})$ so that $\mu_{m_0}(m) = m + m_0$, by $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z) := \mu^*_{m_0}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)|_{(m_0,X)}$. Then it follows from 8.4.2.1 that we have an equivalence of categories between the category of $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z)$-module of global finite presentation and that of $\lambda^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)$-module of global finite presentation.

Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. Hence, it follows from 8.4.5.10 that the property $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ is equivalent to saying that there exists $m_0 \in \mathbb{N}$ such that for any affine open $\mathfrak{U}^\sharp$ of $\mathfrak{X}^\sharp$, the object $\mathcal{E}^{(\bullet)}|\mathfrak{U}$ is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}^\sharp}(Z))$ to a $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{U}^\sharp}(Z \cap U)$-module $\mathcal{F}^{(\bullet)}$ having a global finite presentation.

9.1.5.2. Suppose that $\mathfrak{X}$ is affine.

(a) We have the global section functor $\Gamma(\mathfrak{X},-)\colon M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to M(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ defined by putting $\Gamma(\mathfrak{X},\mathcal{E}^{(\bullet)}) := E^{(\bullet)}$, where $E^{(m)} \to E^{(m+1)}$ is the image by the functor $\Gamma(\mathfrak{X},-)$ of the arrow $\mathcal{E}^{(m)} \to \mathcal{E}^{(m+1)}$. As the functor $\Gamma(\mathfrak{X},-)$ commutes canonically with the functor χ^*, the functor $\Gamma(\mathfrak{X},-)$ transforms the ind-isogenies of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ into ind-isogenies of $M(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. It induces therefore the functor $\Gamma(\mathfrak{X},-)\colon \underrightarrow{M}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{M}_{\mathbb{Q}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. Likewise, $\Gamma(\mathfrak{X},-)$ send a lim-ind-isogeny to a lim-ind-isogeny and we obtain the factorization $\Gamma(\mathfrak{X},-)\colon \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

(b) We have the functor $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -\colon M(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. We check in the same way that we obtains the factorization $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -\colon \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

Lemma 9.1.5.3. *Let* $\lambda \in L(\mathbb{N})$.

(a) *For all* $\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)$*-module* $\mathcal{E}^{(\bullet)}$*, the canonical morphisms* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \to \lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)}$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \to \lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)}$ *are isomorphisms of* $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

(b) *For all* $\lambda^* \widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)$*-module* $F^{(\bullet)}$*, the canonical morphisms* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)} \to \lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\lambda^* \widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)}$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)} \to \lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\lambda^* \widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)}$ *are isomorphisms of* $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

Proof. We have the factorization $\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \to \lambda^* \left(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \right)$ inscribed in the canonical commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} & \longrightarrow & \lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \\ \downarrow & \swarrow & \downarrow \\ \lambda^* \left(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \right) & \longrightarrow & \lambda^* \left(\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\lambda^* \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \right), \end{array} \tag{9.1.5.3.1}$$

and the same in replacing "$\otimes$" by "$\widehat{\otimes}$". From this the first assertion follows. For the second assertion we proceed in the same manner. □

Lemma 9.1.5.4. *We have the following properties.*

(a) *The functor* $({}^\dagger Z, D)^0$ *factors through*

$$({}^\dagger Z, D)^0 \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)). \tag{9.1.5.4.1}$$

(b) *For all* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$*, the canonical morphism* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(\bullet)}$ *is an isomorphism of* $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

(c) *Suppose that* $\mathfrak{X}$ *is affine. For all* $F^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$*, the canonical morphism* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)}$ *is an isomorphism of* $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

Proof. It follows from 8.4.5.2 that the statement (a) and (b) are local on $\mathfrak{X}$. Hence, we can suppose $\mathfrak{X}$ is affine. According to lemma 8.4.5.10, to verify the preservation of coherence, we can suppose that there exists $m_0, n_0 \in \mathbb{N}$ such that $\mathcal{E}^{(\bullet)}$ (resp. $F^{(\bullet)}$) is a $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z)$-module having a global finite presentation (resp. a $\widetilde{D}^{(\bullet+n_0)}_{\mathfrak{X}^\sharp}(Z)$-module having a global finite presentation). Following Lemma 8.4.2.1, we thus obtain the equalities (up to canonical isomorphisms): $\mathcal{E}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp}(Z)} \mathcal{E}^{(m_0)}$ and $F^{(\bullet)} = \widetilde{D}^{(\bullet+n_0)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(n_0)}_{\mathfrak{X}^\sharp}(Z)} F^{(n_0)}$. We conclude by using the lemma 9.1.5.3 in the case where $\lambda \in L(\mathbb{N})$ is defined by setting $\lambda(m) = m + m_0$. □

Proposition 9.1.5.5. *Suppose that* $\mathfrak{X}$ *is affine. Then the functor* $\Gamma(\mathfrak{X}, -)$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} -$ *factor trough the quasi-inverse equivalences of the categories* $\Gamma(\mathfrak{X}, -) \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)} - \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

Proof. One proceeds in an analogous manner as in the proof of the lemma 9.1.5.4: by using 8.4.5.10 and 9.1.5.3 we reduce to the theorem of type A of the remark 8.4.2.2.b. □

9.1.5.6. We have the canonical commutative (up to canonical isomorphism) diagram

$$\begin{array}{ccccc} \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) & \xrightarrow{\cong} & \underrightarrow{LD}^0_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) & \xrightarrow[H^0]{\cong} & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \\ \downarrow {\scriptstyle ({}^\dagger Z,D)^0} & & \downarrow {\scriptstyle ({}^\dagger Z,D)} & & \downarrow {\scriptstyle ({}^\dagger Z,D)^0} \\ \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) & \xrightarrow{\cong} & \underrightarrow{LD}^0_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) & \xrightarrow[H^0]{\cong} & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \\ \downarrow {\scriptstyle \mathrm{forg}_{D,Z}} & & \downarrow {\scriptstyle \mathrm{forg}_{D,Z}} & & \downarrow {\scriptstyle \mathrm{forg}_{D,Z}} \\ \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) & \xrightarrow{\cong} & \underrightarrow{LD}^0_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) & \xrightarrow[H^0]{\cong} & \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \end{array} \tag{9.1.5.6.1}$$

whose horizontal functors are the equivalence of categories of Lemma 8.4.5.5 or of Lemma 8.1.5.10. As the middle forgetful functor $\mathrm{forg}_{D,Z}$ is fully faithful (see 9.1.2.3), we deduce that it is the same for $\mathrm{forg}_{D,Z}\colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$.

The functor 9.1.1.12.1 is exact and induces the following functor for coherent modules:

$$({}^\dagger Z, D) := \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}} -\colon \mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \to \mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}). \tag{9.1.5.6.2}$$

The functors 9.1.5.4.1 and 9.1.5.6.2 are compatible with the equivalence of categories of 8.4.5.6.3 i.e. for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ we have the isomorphism of $\mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$

$$({}^\dagger Z, D) \circ \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}} \circ ({}^\dagger Z, D)^0(\mathcal{E}^{(\bullet)}) \tag{9.1.5.6.3}$$

where $\underrightarrow{l}^*_{\mathbb{Q}}$ is the equivalence of categories of 8.4.5.6.3 of the form $\underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T)) \to \mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$ in the case where $T = Z$ or $T = D$.

9.1.6 A coherence stability criterion by localisation outside a divisor

Notation 9.1.6.1. Following 8.4.1.15 in this context where $\mathcal{D}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$, the functor $\underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LD}^-_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to D^-(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ induces we have the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$. When we would like to clarify the divisor (specially when we consider a quasi-inverse functor), this latter equivalence of categories will be denoted by $\underrightarrow{l}^*_{\mathbb{Q},Z}$.

The functors $\underrightarrow{l}^*_{\mathbb{Q}}$ commute with the forgetful of the divisor functor, i.e. we have the commutative (up to canonical isomorphism) square

$$\begin{array}{ccc} \underrightarrow{LD}^-_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) & \xrightarrow{\underrightarrow{l}^*_{\mathbb{Q}}} & D^-(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \\ {\scriptstyle 9.1.1.4}\downarrow {\scriptstyle \mathrm{forg}_{D,Z}} & & {\scriptstyle 9.1.1.4}\downarrow {\scriptstyle \mathrm{forg}_{D,Z}} \\ \underrightarrow{LD}^-_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) & \xrightarrow{\underrightarrow{l}^*_{\mathbb{Q}}} & D^-(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}), \end{array} \tag{9.1.6.1.1}$$

which justifies why this harmless to write $\underrightarrow{l}^*_{\mathbb{Q}}$ instead of $\underrightarrow{l}^*_{\mathbb{Q},Z}$.

Theorem 9.1.6.2. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ and $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q},D}(\mathcal{E}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}})$. We suppose that the morphism $\mathcal{E} \to ({}^\dagger Z, D)(\mathcal{E})$ is an isomorphism of $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}})$. Then, the canonical morphism $\mathcal{E}^{(\bullet)} \to ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)})$ is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$.*

Proof. 0) *We reduce to the case where $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$.*

For any integer $n \in \mathbb{Z}$, following the lemma 8.4.5.4, we have $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. Moreover, following the corollary 8.1.5.11, the canonical morphism $\phi\colon \mathcal{E}^{(\bullet)} \to ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$, if and only if, for any integer $n \in \mathbb{Z}$, the morphism $\mathcal{H}^n(\phi)\colon \mathcal{H}^n(\mathcal{E}^{(\bullet)}) \to \mathcal{H}^n(({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}))$ is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. Let $\psi_n\colon ({}^\dagger Z, D)^0(\mathcal{H}^n(\mathcal{E}^{(\bullet)})) \to \mathcal{H}^n(({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}))$ be the morphism of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ induced by adjunction from the morphism $\mathcal{H}^n(\phi)$.

a) Let us check that ψ_n is an isomorphism. Since the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ is fully faithful on $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$, we reduce to check that $\underrightarrow{l}^*_{\mathbb{Q}}(\psi_n)$ is an isomorphism. Moreover, since the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ commutes canonically to $\mathcal{H}^n$ up to canonical isomorphism (see the diagram 8.4.1.10.1), via the isomorphisms 9.1.1.12.2 and 9.1.5.6.3, we get that $\underrightarrow{l}^*_{\mathbb{Q}}(\psi_n)$ is canonically isomorphic to the canonical morphism $({}^\dagger Z, D)(\mathcal{H}^n\mathcal{E}) \to \mathcal{H}^n(({}^\dagger Z, D)(\mathcal{E}))$. Since the functor $({}^\dagger Z, D)\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ is exact, the morphism $({}^\dagger Z, D)(\mathcal{H}^n\mathcal{E}) \to \mathcal{H}^n(({}^\dagger Z, D)(\mathcal{E}))$ is therefore an isomorphism. Hence we are done.

b) If the theorem holds when $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$, then this impies in general context that the canonical morphism $\mathcal{H}^n(\mathcal{E}^{(\bullet)}) \to ({}^\dagger Z, D)^0(\mathcal{H}^n(\mathcal{E}^{(\bullet)}))$ is an isomorphism. By composing the latter isomorphism with ψ_n we get $\mathcal{H}^n(\phi)$ which is therefore an isomorphism. Hence, we are done.

1) It follows from 8.4.5.2 that the statement (a) and (b) are local on $\mathfrak{X}$. Hence, we can suppose $\mathfrak{X}$ is affine. According to lemma 8.4.5.10, since this theorem is closed under isomorphism of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$, we can suppose that there exists $m_0 \in \mathbb{N}$ such that $\mathcal{E}^{(\bullet)}$ is a $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)$-module having a global finite presentation, where we set $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)|_{(m_0,X)}$. Following Lemma 8.4.2.1, we thus obtain the equalities (up to canonical isomorphisms): $\mathcal{E}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) \otimes_{\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp}(D)} \mathcal{E}^{(m_0)}$. Put $E := \Gamma(\mathfrak{X}, \mathcal{E})$, $F := \Gamma(\mathfrak{X}, \mathcal{F})$, $E^{(\bullet)} = \Gamma(\mathfrak{X}, \mathcal{E}^{(\bullet)})$, $\mathcal{F}^{(\bullet)} := ({}^\dagger Z, D)^0(\mathcal{E}^{(\bullet)})$, $F^{(\bullet)} = \Gamma(\mathfrak{X}, \mathcal{F}^{(\bullet)})$. Let us check in this step 1) that the canonical morphism $E^{(\bullet)} \to F^{(\bullet)}$ of $M(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$ is a lim-ind-isogeny, with $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp}(D) := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(D))$ for any $m \in \mathbb{N}$. (Recall that from 8.3.1.3.1, this means that the image of the morphism $E^{(\bullet)} \to F^{(\bullet)}$ via the composition of functor $M(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)) \to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ is an isomorphism.)

a) Let $E := \varinjlim_m E^{(m)}$ and $N^{(m)}$ the kernel of the morphism canonical $E^{(m)} \to E$. It follows from the theorem of type A (see the second point of the remark 8.4.2.2) that, for any integer $m \in \mathbb{N}$, $E^{(m)}$ is a $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(D)$-module of finite type. By noetherianity of $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(D)$, the $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(D)$-module $N^{(m)}$ is therefore of finite type. This implies there exists $\lambda(m) \geq m$ such that $E^{(m)} \to E^{(\lambda(m))}$ factors through $E^{(m)}/N^{(m)} \to E^{(\lambda(m))}$. We can choose $\lambda\colon \mathbb{N} \to \mathbb{N}$ such that $\lambda \in L(\mathbb{N})$. This implies that $E^{(\bullet)} \to E^{(\bullet)}/N^{(\bullet)}$ is a lim-ind-isogeny. Replacing $E^{(\bullet)}$ by $E^{(\bullet)}/N^{(\bullet)}$ if necessary, we can therefore assume that the transition maps $E^{(m)} \to E^{(m+1)}$ are injective. Let $E^{(\bullet)}_t$ be the sub-$\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)$-module of $E^{(\bullet)}$ of p-torsion sections, since $E^{(m)}_t$ is a $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(D)$-module of finite type, then we can check that the morphism $E^{(\bullet)} \to E^{(\bullet)}/E^{(\bullet)}_t$ is an ind-isogeny. Hence, replacing $E^{(\bullet)}$ by $E^{(\bullet)}/E^{(\bullet)}_t$ if necessary, we can therefore assume that $E^{(m)}$ is p-torsion free for any $m \in \mathbb{N}$.

Put $F := \varinjlim_m F^{(m)}$ and, for any integer $m \geq 0$, let $G^{(m)}$ be the quotient of $F^{(m)}$ by the kernel of the canonical morphism $F^{(m)} \to F_{\mathbb{Q}}$. Hence, the transition maps $G^{(m)} \to G^{(m+1)}$ are injective and the $G^{(m)}$ are p-torsion free $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(Z)$-modules of finite type. As above, we see that the canonical arrow $F^{(\bullet)} \to G^{(\bullet)}$ is a lim-ind-isogeny of $M(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z))$ and therefore of $M(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$. Hence, we reduce to check that the canonical morphism $E^{(\bullet)} \to G^{(\bullet)}$ is a lim-ind-isogeny of $M(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$.

b) Since the functor $\Gamma(\mathfrak{X}, -)$ commutes with filtrant inductive limites and with the functor $- \otimes_{\mathcal{D}} \mathbb{Q}$, then it commutes with $\underrightarrow{l}^*_{\mathbb{Q}}$. Moreover, it follows from 9.1.1.12.2 that the canonical morphism $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \to \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})$ is an isomorphism of $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules. Hence, the canonical morphism $E_{\mathbb{Q}} \to F_{\mathbb{Q}}$ is an isomorphism of $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules. Since $F^{(\bullet)} \to G^{(\bullet)}$ is a lim-ind-isogeny of $M(\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$ (see part a), then by applying $\underrightarrow{l}^*_{\mathbb{Q}}$ we get that the canonical morphism $F_{\mathbb{Q}} \to G_{\mathbb{Q}}$ is an isomorphism of $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-modules, where $G := \varinjlim_m G^{(m)}$. By composition, the canonical morphism $E_{\mathbb{Q}} \to G_{\mathbb{Q}}$ is thus an isomorphism of $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules.

c) The K-spaces $E^{(m)}_{\mathbb{Q}}$ and $G^{(m)}_{\mathbb{Q}}$ are endowed with canonical structure of Banach K-spaces induced respectively by their structure of $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(D)_{\mathbb{Q}}$-module of finite type and of $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(Z)_{\mathbb{Q}}$-module of finite type. For theses topologies, the canonical morphisms $E^{(m)}_{\mathbb{Q}} \to G^{(m)}_{\mathbb{Q}}$ are continuous morphisms of Banach K-spaces. Denote by $W := \varinjlim_m G^{(m)}_{\mathbb{Q}}$ endowed with the inductive limit topology of locally convex K-space. Denote by $i_m\colon E^{(m)}_{\mathbb{Q}} \to W$ the composite of continuous maps $E^{(m)}_{\mathbb{Q}} \to G^{(m)}_{\mathbb{Q}} \to W$. Moreover, following the part *b*), the canonical morphism $E_{\mathbb{Q}} \to W$ is an isomorphism of $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-

modules and is in particular bijective. Since $E^{(m)}_{\mathbb{Q}} \to E_{\mathbb{Q}}$ is injective, then so is i_m. We have also the equality of sets $W = \cup_{m\in\mathbb{N}} i_m(E^{(m)}_{\mathbb{Q}})$. Following [Sch02, 8.9], for any integer m, since $E^{(m)}_{\mathbb{Q}}$ and $G^{(m)}_{\mathbb{Q}}$ are both Banach K-spaces and since the canonical morphism $\alpha_m \colon G^{(m)}_{\mathbb{Q}} \to W$ is continuous, this yields there exists $\lambda(m) \geq m$ and a unique continuous morphism $\beta^{(m)} \colon G^{(m)}_{\mathbb{Q}} \to E^{(\lambda(m))}_{\mathbb{Q}}$ such that $\alpha_m = i_{\lambda(m)} \circ \beta^{(m)}$. We choose such a $\lambda(m) \geq m$ as small as possible. Since $i_{\lambda(m)}$ is injective, since $i_{\lambda(m)}$ and α_m are both $\widetilde{D}^{(m+m_0)}_{\mathfrak{X}^\sharp}(D)_{\mathbb{Q}}$-linear, then so is $\beta^{(m)}$. Moreover, by unicity of such factorisations, the morphisms $\beta^{(m)}$ are compatible with transition maps. Hence, we get the morphism of $\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)_{\mathbb{Q}}$-modules $\beta^{(\bullet)} \colon G^{(\bullet)}_{\mathbb{Q}} \to \lambda^* E^{(\bullet)}_{\mathbb{Q}}$.

Since the family $(p^n G^{(m)})_{n\in\mathbb{N}}$ forms a basis of neighborhood of 0, since $E^{(\lambda(m))}$ is an open of $E^{(\lambda(m))}_{\mathbb{Q}}$, (recall that $E^{(\lambda(m))}$ and $G^{(m)}$ are both p-torsion free), the fact that $\beta^{(m)}$ is continuous implies then there exists $\chi(m) \in \mathbb{N}$ large enough such that $\beta^{(m)}(p^{\chi(m)} G^{(m)}) \subset E^{(\lambda(m))}$ (or we can invoke the $\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)_{\mathbb{Q}}$-linearity of $\beta^{(m)}$ and the type finiteness of $G^{(m)}$). We can choose the $\chi(m)$ such that the induced map $\chi \colon \mathbb{N} \to \mathbb{N}$ is increasing. Denote by $\gamma^{(\bullet)}$ the composition of $\beta^{(\bullet)} \colon G^{(\bullet)}_{\mathbb{Q}} \to \lambda^* E^{(\bullet)}_{\mathbb{Q}}$ with the canonical morphism $\lambda^* E^{(\bullet)}_{\mathbb{Q}} \to \chi^* \lambda^* E^{(\bullet)}_{\mathbb{Q}}$. From what we have just seen, $\gamma^{(\bullet)}$ factors through (in a unique way) the morphism of the form $g^{(\bullet)} \colon G^{(\bullet)} \to \chi^* \lambda^* E^{(\bullet)}$. Denote by $f^{(\bullet)} \colon E^{(\bullet)} \to G^{(\bullet)}$ the canonical morphism. Since for any $m \in \mathbb{N}$, the canonical morphisms $E^{(m)} \to W$ and $G^{(m)} \to W$ are injective, we can check that $g^{(\bullet)} \circ f^{(\bullet)}$ and $\chi^* \lambda^* (f^{(\bullet)}) \circ g^{(\bullet)}$ are the canonical morphisms. Hence, we are done.

2) We deduce from the step 1) that the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) \widehat{\otimes}_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)} E^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) \widehat{\otimes}_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)} F^{(\bullet)}$$

is a lim-ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$.

3) By quasi-coherence of both sheaves $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}(D)$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}(Z)$, we get that the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}(D) \otimes_{\widetilde{D}^{(m)}_{X_i}(D)} \widetilde{D}^{(m)}_{X_i}(Z) \to \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}(Z)$ is an isomorphism. This implies that the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}(D) \otimes_{\widetilde{D}^{(m)}_{X_i}(D)} F_i^{(m)} \to \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp}(Z) \otimes_{\widetilde{D}^{(m)}_{X_i}(Z)} F_i^{(m)}$ is an isomorphism. Passing to the projective limit, this implies the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) \widehat{\otimes}_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)} F^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)}$$

is an isomorphism of $M(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$.

4) Il follows from the steps 2) and 3) that the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) \widehat{\otimes}_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)} E^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z) \widehat{\otimes}_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)}$$

is a lim-ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$. Hence, it follows from Lemma 9.1.5.4.b (and also from the Lemma 9.1.5.3) that the top morphism of the canonical commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D) \otimes_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D)} E^{(\bullet)} & \longrightarrow & \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z) \otimes_{\widetilde{D}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(Z)} F^{(\bullet)} \\ \downarrow & & \downarrow \\ \mathcal{E}^{(\bullet)} & \longrightarrow & \mathcal{F}^{(\bullet)} \end{array}$$

is a lim-ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$. Moreover, we deduce from the theorem of type A of the remark 8.4.2.2.b that the vertical morphisms are isomorphisms of $M(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}(D))$. Hence we are done. □

Corollary 9.1.6.3. *Let* $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ *and* $\mathcal{E}' := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}'^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$. *If* $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}})$, *then* $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$.

Proof. Assume $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}})$. Then it follows from Theorem 8.4.1.15 there exists $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ and a $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-linear isomorphism of the form $\mathcal{E}' \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$. Moreover, since the canonical morphism $\mathcal{E}' \to ({}^\dagger Z, D)(\mathcal{E}')$ is a morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ which an isomorphism outside Z, this yields following 8.7.6.11 that it is an isomorphism. This implies that the canonical morphism $\mathcal{E}' \to ({}^\dagger Z, D)(\mathcal{E}')$ of $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}})$ is an isomorphism. By applying the theorem 9.1.6.2 to $\mathcal{E}^{(\bullet)}$, this implies that the canonical morphism $\mathcal{E}^{(\bullet)} \to ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)})$ is therefore an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. Since $({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}),\ \mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ and since

$$\underrightarrow{l}^*_{\mathbb{Q}} \circ ({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}) \overset{9.1.1.12.2}{\underset{\sim}{\longrightarrow}} ({}^\dagger Z, D) \circ \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z, D)(\mathcal{E}') \xrightarrow{\sim} \mathcal{E}' \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}'^{(\bullet)},$$

by fully faithfulness of the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ we get that $({}^\dagger Z, D)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$. □

Corollary 9.1.6.4. *The equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q},Z}$ factors through the equivalence of categories*

$$\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \cong D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}}). \qquad (9.1.6.4.1)$$

Proof. This is a straightforward consequence of 9.1.6.3 and of the full faithfulness of the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ (see Theorem 8.4.1.15). □

Proposition 9.1.6.5. *Suppose $D \subset Z \subset Z'$.*

(a) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D)) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z'))$. Then $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$.

(b) Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z')_{\mathbb{Q}})$. Then $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$.

Proof. Using 9.1.2.3.1, we check that the canonical morphism $({}^\dagger Z,\, D) \circ \mathrm{forg}_{D,Z}(\mathrm{forg}_{Z,Z'}(\mathcal{E}^{(\bullet)})) \to \mathrm{forg}_{Z,Z'}(\mathcal{E}^{(\bullet)})$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ is an isomorphism. Hence, we get the first assertion. Using 9.1.6.4, we get (b) from (a). □

In order to define the bifunctor 9.1.6.8.1, we need to introduce the following notation.

Notation 9.1.6.6. Let $\mathfrak{X}^\sharp_1 \to \mathfrak{S}^\sharp$ and $\mathfrak{X}^\sharp_2 \to \mathfrak{S}^\sharp$ be two log smooth morphisms of log formal schemes. We suppose the underlying formal schemes $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are p-torsion free, noetherian of finite Krull dimension. For $i = 1, 2$, for $T_i = D_i$, choose $(\underrightarrow{l}^*_{\mathbb{Q},T_1})^{-1} \colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_i}({}^\dagger T_i)_{\mathbb{Q}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_i}(T_i))$ a functor quasi-inverse functor of the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q},T_1}$ (see notation 9.1.6.1).

Let $D_1 \subset Z_1$ be two divisors of X_1, $D_2 \subset Z_2$ be two divisors of X_2. Let $\phi^{(\bullet)} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_1}(Z_1)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ be a functor and

$$\psi^{(\bullet)} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_1}(Z_1)) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_2}(Z_2)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$$

be a bifunctor. This yields a functor $\mathrm{Coh}_{Z_1}(\phi^{(\bullet)}) \colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}}) \to D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ by setting $\mathrm{Coh}_{Z_1}(\phi^{(\bullet)}) := \underrightarrow{l}^*_{\mathbb{Q}} \circ \phi^{(\bullet)} \circ (\underrightarrow{l}^*_{\mathbb{Q},Z_1})^{-1}$. In the same way, we obtain the bifunctor

$$\mathrm{Coh}_{Z_1,Z_2}(\psi^{(\bullet)}) \colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}}) \times D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp}({}^\dagger Z_2)_{\mathbb{Q}}) \to D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$$

by setting $\mathrm{Coh}_{Z_1,Z_2}(\psi^{(\bullet)}) := \underrightarrow{l}^*_{\mathbb{Q}} \circ \psi^{(\bullet)} \circ ((\underrightarrow{l}^*_{\mathbb{Q},Z_1})^{-1} \times (\underrightarrow{l}^*_{\mathbb{Q},Z_2})^{-1})$. Beware that these functors is only well defined up to non-canonical isomorphism. Hence, it is more convenient to work with categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$ than with $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$.

Remark 9.1.6.7. We keep notation 9.1.6.6. It follows from 9.1.6.4.1 that the functors $\mathrm{Coh}_{Z_1}(\phi^{(\bullet)})$ and $\mathrm{Coh}_{D_1}(\phi^{(\bullet)})$ are (non-canonically) isomorphic on $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_1}({}^\dagger D_1)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_1}({}^\dagger Z_1)_{\mathbb{Q}})$. Similarly, the bifunctors $\mathrm{Coh}_{Z_1,Z_2}(\psi^{(\bullet)})$ and $\mathrm{Coh}_{D_1,D_2}(\psi^{(\bullet)})$ are isomorphic on $(D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_1}({}^\dagger D_1)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_1}({}^\dagger Z_1)_{\mathbb{Q}})) \times (D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_2}({}^\dagger D_2)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_2}({}^\dagger Z_2)_{\mathbb{Q}}))$.

Notation 9.1.6.8. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, $\mathcal{M}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Let us denote by $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ and $\mathcal{M} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{M}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{*}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$. With the notations of 9.1.6.6, we get the bifunctor $-\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} - := \mathrm{Coh}_{T,}(-\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} -)$ of the form:

$$-\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} - \colon D^{\mathrm{b}}_{\mathrm{coh}}({}^{*}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \times D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D^{\mathrm{b}}({}^{*}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}). \tag{9.1.6.8.1}$$

Recall this functor is well defined up to non-canonical isomorphism. Via the equivalence 8.4.1.15, then we have the (non-canonical) functorial isomorphisms

$$\mathcal{M}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}\mathcal{E} \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}\left(\mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}\right). \tag{9.1.6.8.2}$$

Proposition 9.1.6.9. *Let $\mathcal{E}, \mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^\dagger D)_{\mathbb{Q}})$. We have the canonical isomorphism*

$$\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger D)_{\mathbb{Q}}}\mathcal{F} \xrightarrow{\sim} \mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}\mathcal{F}. \tag{9.1.6.9.1}$$

Proof. Following 9.1.6.4.1, there exist $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D)) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z))$ such that $\mathcal{E} \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$, $\mathcal{F} \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})$. Then, this is a consequence of 9.1.2.8.1. □

Remark 9.1.6.10. (a) For any divisors $D \subset Z$, following 9.1.1.12.2, we have the isomorphism of functors $\mathrm{Coh}_D(({}^\dagger Z',\ D)) \xrightarrow{\sim} ({}^\dagger Z',\ D)$.

(b) Let Z and $D \subset D'$ be some divisors of X. Suppose X is regular. Following 9.1.3.3, the functors $({}^\dagger Z)$ and $({}^\dagger Z \cup D)$ are canonically isomorphic on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ and similarly replacing D by D'. We obtain the functor $({}^\dagger Z \cup D, D)\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp}({}^\dagger Z \cup D)_{\mathbb{Q}})$ is isomorphic to $\mathrm{Coh}_D(({}^\dagger Z))$, and similarly replacing D by D'. With the remark 9.1.6.7, this yields the functors $({}^\dagger Z \cup D, D)$ and $({}^\dagger Z \cup D', D')$ are (canonically) isomorphic over $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp}({}^\dagger D')_{\mathbb{Q}})$. Hence, it is harmless to remove D in the notation.

9.2 Extraordinary inverse image, direct image, duality, base change, exterior tensor product

9.2.1 Extraordinary inverse images

Let

$$\begin{array}{ccc} \mathfrak{X}'^\sharp & \xrightarrow{f} & \mathfrak{X}^\sharp \\ \downarrow{\scriptstyle p_{\mathfrak{X}'^\sharp}} & & \downarrow{\scriptstyle p_{\mathfrak{X}^\sharp}} \\ \mathfrak{S}'^\sharp & \xrightarrow{\phi} & \mathfrak{S}^\sharp, \end{array} \tag{9.2.1.0.1}$$

be a commutative diagram of very nice (see definition 3.3.1.10.(b)) fine $\mathcal{V}$-log formal schemes, where $p_{\mathfrak{X}^\sharp}$ and $p_{\mathfrak{X}'^\sharp}$ are log smooth morphisms. We suppose S, $S^\sharp$, S', $S'^\sharp$, X and X' are regular (to have an idea about theses hypotheses, see [GR, 12.5.19]) and noetherian. Let Z and Z' be some divisors of respectively X and X' such that $f(X' \setminus Z') \subset X \setminus Z$.

We suppose the underlying formal schemes of $\mathfrak{S}$, $\mathfrak{S}'$, $\mathfrak{X}$ and $\mathfrak{X}'$ are p-torsion free. Let $\mathfrak{a} := \pi^h \mathcal{O}_{\mathfrak{S}}$ for some integer h. By flatness of the structural morphism $\mathfrak{S} \to \mathrm{Spf}\,\mathcal{V}$ we get from 1.2.4.2.a that $\mathfrak{a}$ has a canonical m-PD-structure if $hp^m \geq e/(p-1)$, where e is the absolute ramification of $\mathcal{V}$. Choosing h large enough, we can endowing $\mathfrak{S}$ and $\mathfrak{S}'$ with these canonical m-PD-structures, the bottom arrow ϕ can be viewed as an m-PD-morphism.

We define in this section the extraordinary inverse image by f with overconvergent singularities along Z and Z'. We fix $\lambda_0 \in L(\mathbb{N})$. We set $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) := \lambda_0^* \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Finally, we set $\mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}(Z) := \mathcal{V}/\pi^{i+1} \otimes_{\mathcal{V}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) = \mathcal{B}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(Z) := \widetilde{\mathcal{B}}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}$ for any $m \in \mathbb{N}$. We use similar notation by adding some primes, e.g. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(Z') :=$

$\mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}'}(Z')$. Recall also, following 7.3.2.1, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z)$ satisfy the conditions of 7.3.2, i.e. in particular $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$ is $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$-quasi-coherent in the sense of 8.5.1.6 and $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S'^\sharp_\bullet}(Z)$ is $\mathcal{O}^{(\bullet)}_{X_\bullet}$-quasi-coherent in the sense of 8.5.1.7.

We denote by $\widetilde{\mathfrak{X}}^{(\bullet)} := (\mathfrak{X}^\sharp, \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z))$ and by $\widetilde{\mathfrak{X}}'^{\sharp(\bullet)} := (\mathfrak{X}'^\sharp, \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z'))$ the associated $\mathbb{N}$-ringed $\mathcal{V}$-log formal schemes (see 8.5.5.1). With notation 8.5.5.2, let $\tilde{f}^{(\bullet)} \colon \widetilde{\mathfrak{X}}'^{(\bullet)}/\mathfrak{S}'^\sharp \to \widetilde{\mathfrak{X}}^{(\bullet)}/\mathfrak{S}^\sharp$ be the morphism of relative $\mathbb{N}$-ringed $\mathcal{V}$-log formal schemes induced by the diagram 9.2.1.0.1 and by $f^*\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')$ (see definition 8.5.5.1). When $\phi\colon \mathfrak{S}'^\sharp \to \mathfrak{S}^\sharp$ in understood, by abuse of notation, we also denote by $\tilde{f}^{(\bullet)}$ the induced morphism $\widetilde{\mathfrak{X}}'^{(\bullet)}/\mathfrak{S}'^\sharp \to \widetilde{\mathfrak{X}}^{(\bullet)}/\mathfrak{S}^\sharp$ of $\mathbb{N}$-ringed $\mathcal{V}$-log formal schemes.

We denote by $\widetilde{X}^{(\bullet)}_\bullet := (X^\sharp_\bullet, \widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z))$ and $\widetilde{X}'^{(\bullet)}_\bullet := (X^\sharp_\bullet, \widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z))$ the induced ringed topoi. Let $\tilde{f}^{(\bullet)}_\bullet \colon \widetilde{X}'^{(\bullet)}_\bullet/S'^\sharp_\bullet \to \widetilde{\mathfrak{X}}^{(\bullet)}/S^\sharp_\bullet$ be the induced morphism of relative ringed topoi.

Following 8.7.4.2.(b), $\widetilde{\mathfrak{X}}^{(\bullet)}/\mathfrak{S}^\sharp$ and $\widetilde{\mathfrak{X}}'^{(\bullet)}/\mathfrak{S}^\sharp$ are strongly quasi-flat (see definition 8.5.5.3). In fact, using the proof of 8.7.4.2.(b) we can check that $\tilde{f}^{(\bullet)}$ is strongly quasi-flat (more precisely, in the definition 8.5.5.3, we can choose $\mathfrak{S}^\sharp = \operatorname{Spf}\mathcal{V}$). In particular, we can apply 8.5.4.5 and 8.5.4.15 in this context where $\mathcal{D}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, or also similarly $\mathcal{D}'^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}$.

Let $\mathfrak{U} := \mathfrak{X}^{\sharp *}$ be the open of $\mathfrak{X}$ where $M_{\mathfrak{X}^\sharp}$ is trivial and $j_{\mathfrak{U}} \colon \mathfrak{U} \hookrightarrow \mathfrak{X}^\sharp$ be the canonical open immersion. Let $\mathfrak{U}' := \mathfrak{X}'^{\sharp *}$ be the open of $\mathfrak{X}'$ where $M_{\mathfrak{X}'^\sharp}$ is trivial and $j_{\mathfrak{U}'} \colon \mathfrak{U}' \hookrightarrow \mathfrak{X}'^\sharp$ be the canonical open immersion.

9.2.1.1 (Decomposition into a regular exact closed immersion and a log smooth morphism)**.** Suppose the morphism ϕ of the diagram 9.2.1.0.1 is the identity. Etale locally on $\mathfrak{X}$, there exists an exact closed immersion $\mathfrak{X}'^\sharp \hookrightarrow \mathfrak{X}''^\sharp$ and a log etale morphism $\mathfrak{X}''^\sharp \to \mathfrak{X}'^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{X}^\sharp$ whose composite map gives $\mathfrak{X}'^\sharp \hookrightarrow \mathfrak{X}'^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{X}^\sharp$, the graph of f (see 3.3.3.2). Since $\mathfrak{X}''^\sharp \to \mathfrak{S}^\sharp$ is log flat and since $\mathfrak{S}^\sharp$ is very nice, then so is $\mathfrak{X}''^\sharp$ is very nice. Since $X''^\sharp \to S^\sharp$ is log smooth, since $S^\sharp$ is regular, then it follows from [Ogu18, IV.3.5.3] (or [GR, 12.5.28]) that $X''^\sharp$ is regular. Hence, it follows from [GR, 12.5.14] that the underlying morphism of schemes of the exact closed immersion $X'^\sharp \hookrightarrow X''^\sharp$ is a regular closed immersion. Hence, following [Gro67, 19.1.2], the underlying scheme of $X''^\sharp$ is regular. Denoting by g the composite morphism $\mathfrak{X}''^\sharp \to \mathfrak{X}'^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{X}^\sharp \to \mathfrak{X}^\sharp$, since $f^{-1}(Z)$ is a divisor of X'' then $Z'' := g^{-1}(Z)$ is a divisor of X''. In other words, we can decompose f by a regular exact closed immersion and a log smooth morphism, both morphisms satisfying the hypotheses of 9.2.1.

Following [Liu02, 6.3.15], we get a similar decomposition $X'^\sharp_i \hookrightarrow X''^\sharp_i \to X'^\sharp_i \times_{S^\sharp_i} X^\sharp_i \to X^\sharp_i$ such that the underlying morphism of schemes of $X'^\sharp_i \hookrightarrow X''^\sharp_i$ is a regular closed immersion.

9.2.1.2 (Finite tor-dimension)**.** It follows from [Sta22, 066P], that since X is regular, then $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ has finite tor dimension, i.e. the functor $\mathbb{L}f^* = \mathcal{O}_{X'} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_X} f^{-1}-$ from the category of $\mathcal{O}_X$-modules to that of $\mathcal{O}_{X'}$-modules has bounded cohomological dimension (see definition 4.6.1.4). Since $\mathfrak{X}$ is noetherian and is flat over $\operatorname{Spf}\mathcal{V}$, since X is regular then so is $\mathfrak{X}$ and $f^{-1}\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}'}$ has finite tor dimension. Since the functor $\mathbb{L}f^* = \mathcal{O}_{\mathfrak{X}'} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_{\mathfrak{X}}} f^{-1}-$ has finite cohomological dimension, then following 7.5.5.1 so is the functor $\widetilde{f}^*_\bullet = \mathcal{O}^{(\bullet)}_{X'_\bullet} \otimes_{f^{-1}\mathcal{O}^{(\bullet)}_{X_\bullet}} f^{-1}-$.

9.2.1.3. We have the functors

$$\mathbb{L}\widetilde{f}^{(\bullet)*}_{\mathrm{alg}} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z') \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} f^{-1}(-) \colon D^-(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)) \to D^-(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')), \tag{9.2.1.3.1}$$

$$\mathbb{L}\widetilde{f}^{(\bullet)*}_\bullet = \widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z') \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} f^{-1}(-) \colon D^-(\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)) \to D^-(\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')). \tag{9.2.1.3.2}$$

Let $\mathcal{F}^{(\bullet)}_\bullet \in D^-(\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z))$. It follows from 7.1.3.6.1 that we have the isomorphism:

$$\big(\mathbb{L}\widetilde{f}^{(\bullet)*}_\bullet(\mathcal{F}^{(\bullet)}_\bullet)\big)^{(m)}_i \xrightarrow{\sim} \mathbb{L}\widetilde{f}^{(m)*}_i(\mathcal{F}^{(m)}_i). \tag{9.2.1.3.3}$$

Since $\widetilde{\mathcal{B}}^{(\bullet)}_{X'_i}(Z') \otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_{i+1}}(Z')} \mathbb{L}\widetilde{f}^{(\bullet)*}_{i+1}(\mathcal{F}^{(\bullet)}_{i+1}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^{(\bullet)*}_i(\widetilde{\mathcal{B}}^{(\bullet)}_{X_i}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{i+1}}(Z)} \mathcal{F}^{(\bullet)}_{i+1})$, then it follows from 7.5.4.4 that the bottom functor of 9.2.1.3.1 preserves the quasi-coherence (in the sense of 8.5.1.7), i.e. induces the functor

$$\mathbb{L}\widetilde{f}^{(\bullet)*}_\bullet \colon D^-_{\mathrm{qc}}(\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)) \to D^-_{\mathrm{qc}}(\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')). \tag{9.2.1.3.4}$$

Since $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ is flat, then a K-flat complex of left $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$-modules is a K-flat complex of $\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)$-modules. Hence, we get the functor $\mathbb{L}\widetilde{f}^{(\bullet)*}_\bullet \colon D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$ making commutative the diagram:

$$\begin{array}{ccc} D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) & \xrightarrow{\mathbb{L}\widetilde{f}^{(\bullet)*}_\bullet} & D^-_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')) \\ \downarrow & & \downarrow \\ D^-_{\mathrm{qc}}(\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)) & \xrightarrow{\mathbb{L}\widetilde{f}^{(\bullet)*}_\bullet} & D^-_{\mathrm{qc}}(\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')) \end{array} \quad (9.2.1.3.5)$$

where the vertical functors are the forgetful ones.

Notation 9.2.1.4. We deduce by functoriality from 7.5.5.2 that we get a structure of $(\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'), f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$-bimodule on $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z) := \widetilde{f}^{(\bullet)*}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) = \widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z') \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$. When $\mathfrak{S}'^\sharp \to \mathfrak{S}^\sharp$ is the identity, we may simply write $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet\to X^\sharp_\bullet/S'^\sharp_\bullet}(Z',Z)$ and when moreover there is no doubt about $\mathfrak{S}^\sharp$ we write $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet\to X^\sharp_\bullet}(Z',Z)$. By functoriality from 7.5.5.2 and 7.5.4.6, with notation 9.1.1.6, we get a structure of $(f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z), \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$-bimodule on

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet\leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z') := \widetilde{\omega}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z') \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')} \widetilde{f}^{(\bullet)*}_{\bullet r}\left(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)^{-1}\right),$$

where the index "r" means that we have chosen the right (i.e. the twisted) structure of left $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$-module on $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)^{-1}$ to compute the structure of left $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')$-module via the functor $\widetilde{f}^{(\bullet)*}_\bullet$.

When $\mathfrak{S}'^\sharp \to \mathfrak{S}^\sharp$ is the identity, we can simply write $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet\leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z')$ and when moreover there is no doubt about $\mathfrak{S}^\sharp$ we write $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet\leftarrow X'^\sharp_\bullet}(Z,Z')$.

We have the isomorphism of left $(f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z), \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$-bimodules

$$\widetilde{f}^{(\bullet)*}_{\bullet r}\left(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)^{-1}\right) \underset{\widetilde{f}^{(\bullet)*}_\bullet(4.2.5.6.3)}{\xrightarrow{\sim}} \widetilde{f}^{(\bullet)*}_{\bullet l}\left(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)^{-1}\right),$$

where the index "l" (resp. "r ") means that we have chosen the left (resp. right) structure to compute $\widetilde{f}^{(\bullet)*}_\bullet$. By tensoring this latter isomorphism with $\widetilde{\omega}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z') \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')} -$, we get the isomorphism of $(f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z), \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$-bimodules

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet\leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z') \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z') \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')} \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z) \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} f^{-1}\widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)^{-1}. \quad (9.2.1.4.1)$$

Definition 9.2.1.5. We keep notation 9.2.1.4. Set $d_f := \delta_{X'^\sharp/S'^\sharp} - \delta_{X^\sharp/S^\sharp}\circ f$, $\delta_{X'^\sharp/S'^\sharp}$, $\delta_{X^\sharp/S^\sharp}$ are respectively the rank (as a locally constant function on X' or X respectively) of the locally free modules $\Omega_{X'^\sharp/S'^\sharp}$ and $\Omega_{X^\sharp/S^\sharp}$.

(a) The (left version of the) extraordinary inverse image functor by $\widetilde{f}^{(\bullet)}_\bullet$ is the functor $\widetilde{f}^{(\bullet)!}_\bullet \colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$ which is defined by setting

$$\widetilde{f}^{(\bullet)!}_\bullet(\mathcal{F}^{(\bullet)}_\bullet) := \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z) \otimes^{\mathbb{L}}_{f^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} f^{-1}_\bullet\mathcal{F}^{(\bullet)}_\bullet\,[d_f],$$

where $\mathcal{F}^{(\bullet)}_\bullet \in D(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$.

(b) The (right version of the) extraordinary inverse image functor by $\widetilde{f}^{(\bullet)}$ is the functor $\widetilde{f}_{\bullet}^{(\bullet)!}\colon D({}^r\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to D({}^r\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$ which is defined by setting

$$\widetilde{f}_{\bullet}^{(\bullet)!}(\mathcal{M}_\bullet) := f_\bullet^{-1}\mathcal{M}_\bullet \otimes^{\mathbb{L}}_{f_\bullet^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet \leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z')\ [d_f],$$

where $\mathcal{M}_\bullet \in D({}^r\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$.

(c) For any $* \in \{r,l\}$, the extraordinary inverse image functor $\widetilde{f}^{(\bullet)!}\colon D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$ by $\widetilde{f}$ is defined by setting

$$\widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)}) := \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*} \circ \widetilde{f}_\bullet^{(\bullet)!} \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}(\mathcal{F}^{(\bullet)})$$

where $\mathcal{F}^{(\bullet)} \in D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$.

9.2.1.6 (Left to right)**.** For any $\mathcal{M}_\bullet \in D({}^r\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$, by copying the proof of 5.1.1.5.1 (replace the use of 5.1.1.2.1 by that of 9.2.1.4.1), we get the canonical isomorphism:

$$\widetilde{f}_{\bullet}^{(\bullet)!}(\mathcal{M}_\bullet \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)^{-1}) \xrightarrow{\sim} \widetilde{f}_{\bullet}^{(\bullet)!}(\mathcal{M}_\bullet) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')} \widetilde{\omega}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')^{-1}. \tag{9.2.1.6.1}$$

For any $\mathcal{E}_\bullet \in D({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$, this yields the isomorphism

$$\widetilde{f}_{\bullet}^{(\bullet)!}(\mathcal{E}_\bullet \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \xrightarrow{\sim} \widetilde{f}_{\bullet}^{(\bullet)!}(\mathcal{E}_\bullet) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')} \widetilde{\omega}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'). \tag{9.2.1.6.2}$$

Hence, for any $\mathcal{E} \in D^-({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ we get the isomorphisms

$$\begin{aligned}\widetilde{f}^{(\bullet)!}(\mathcal{E} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) &\underset{7.5.4.13.1}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*} \circ \widetilde{f}_\bullet^{(\bullet)!}(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}(\mathcal{E}) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \widetilde{\omega}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \underset{9.2.1.6.2}{\xrightarrow{\sim}} \\ \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*}(\widetilde{f}_\bullet^{(\bullet)!} \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}(\mathcal{E}) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')} \widetilde{\omega}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')) &\underset{7.5.4.13.2}{\xrightarrow{\sim}} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*} \circ \widetilde{f}_\bullet^{(\bullet)!} \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}(\mathcal{E}) \otimes_{\mathcal{B}_{\mathfrak{X}'}} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z') \\ &= \widetilde{f}^{(\bullet)!}(\mathcal{E}) \otimes_{\mathcal{B}_{\mathfrak{X}'}} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z').\end{aligned} \tag{9.2.1.6.3}$$

Notation 9.2.1.7. We have a structure of $(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'), f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$-bimodule on

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z',Z) := \underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet \to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z) \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet \to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z).$$

We get a structure of $(f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z), \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$-bimodule on

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \leftarrow \mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z,Z') := \underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet \leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z') \xrightarrow{\sim} \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet \leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z').$$

Proposition 9.2.1.8. *We have the following properties.*

(a) For any $\mathcal{F}^{(\bullet)} \in D({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, we have the canonical morphism

$$\widetilde{f}^{(\bullet)!}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z',Z) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}^{(\bullet)}\ [d_f] \to \widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)}). \tag{9.2.1.8.1}$$

(b) For any $\mathcal{F}^{(\bullet)} \in D({}^r\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, we have the canonical morphism

$$\widetilde{f}^{(\bullet)!}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) := f^{-1}\mathcal{F}^{(\bullet)} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \leftarrow \mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z,Z')\ [d_f] \to \widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)}). \tag{9.2.1.8.2}$$

(c) *For any* $* \in \{r, l\}$, *if* $\mathcal{F}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, *then the morphism 9.2.1.8.1 or 9.2.1.8.2 is an isomorphism.*

Proof. Let us construct 9.2.1.8.1. Denoting by $\mathcal{G}$ the left term of 9.2.1.8.1, we have the adjoint morphism $\mathcal{G} \to \mathbb{R}\underleftarrow{l}_{\mathfrak{X}'(\mathbb{N}),*} \circ \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}'(\mathbb{N})}(\mathcal{G})$. Since $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}'(\mathbb{N})}(\mathcal{G})$ is isomorphic to $\widetilde{f}^{(\bullet)!}_{\bullet}(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}\mathcal{F}^{(\bullet)})$, we are done. Similarly, we construct 9.2.1.8.2. Finally, to check the last statement, we reduce to the case where $\mathcal{F}^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)$, which is obvious. □

Proposition 9.2.1.9. *For any* $\mathcal{F}^{(\bullet)} \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, *we have* $\widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)}) \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$. *For any* $\mathcal{F}^{(\bullet)}_{\bullet} \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$, *we have* $\widetilde{f}^{(\bullet)!}_{\bullet}(\mathcal{F}^{(\bullet)}_{\bullet}) \in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$.

Proof. By using the equivalence of categories 8.5.1.10 and the isomorphims 9.2.1.6.2, we reduce to check that for any $\mathcal{F}^{(\bullet)}_{\bullet} \in D^-_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$, we have $\widetilde{f}^{(\bullet)!}_{\bullet}(\mathcal{F}^{(\bullet)}_{\bullet}) \in D^-_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$. Since the canonical morphism

$$\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z') \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet \to X^\sharp_\bullet/S^\sharp_\bullet}(Z', Z)$$

is an isomorphism, then so is the canonical morphism

$$\mathbb{L}\widetilde{f}^{(\bullet)*}_{\bullet}(\mathcal{F}^{(\bullet)}_{\bullet}) = \widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z') \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} f^{-1}\mathcal{F}^{(\bullet)}_{\bullet} \to \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet \to X^\sharp_\bullet/S^\sharp_\bullet}(Z', Z) \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} f^{-1}\mathcal{F}^{(\bullet)}_{\bullet}. \tag{9.2.1.9.1}$$

Since the functor $\mathbb{L}\widetilde{f}^{(\bullet)*}_{\bullet}$ induces $\mathbb{L}\widetilde{f}^{(\bullet)*}_{\bullet} \colon D^-_{\mathrm{qc}}(\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)) \to D^-_{\mathrm{qc}}(\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z'))$ (see 9.2.1.3.4), then we are done. □

Lemma 9.2.1.10. *Let* $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. *With notation 9.2.1.15.3, the canonical morphism*

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z') \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} f^{-1}\mathcal{F}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z', Z) \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} f^{-1}\mathcal{F}^{(\bullet)} = \widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)})[-d_f]$$

is an isomorphism.

Proof. This follows from 9.2.1.9.1. □

9.2.1.11. It follows from 7.1.3.6.1 that we have the isomorphism for any $\mathcal{F}^{(\bullet)}_{\bullet} \in D^-({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$:

$$\left(\widetilde{f}^{(\bullet)!}_{\bullet}(\mathcal{F}^{(\bullet)}_{\bullet})\right)^{(m)}_i \xrightarrow{\sim} \widetilde{f}^{(m)!}_i(\mathcal{F}^{(m)}_i). \tag{9.2.1.11.1}$$

By using the equivalences of categories 7.3.3.3, it follows from 9.2.1.9 that for any $\mathcal{F}^{(\bullet)} \in D^-_{\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}'(\mathbb{N})}\widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \widetilde{f}^{(\bullet)!}_{\bullet}(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}\mathcal{F}^{(\bullet)})$. Since $(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}(\mathbb{N})}\mathcal{F}^{(\bullet)})_i \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \mathcal{F}^{(\bullet)}$ and since we have the isomorphism $\left(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}'(\mathbb{N})}\widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)})\right)_i \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_i/S'^\sharp_i}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z)} \widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)})$, this yields

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \widetilde{f}^{(\bullet)!}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \widetilde{f}^{(\bullet)!}_i(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \mathcal{F}^{(\bullet)}), \tag{9.2.1.11.2}$$

and similarly for right modules.

Notation 9.2.1.12. Let

$$\begin{array}{ccc} \mathfrak{X}''^\sharp & \xrightarrow{g} & \mathfrak{X}'^\sharp \\ \downarrow{\scriptstyle p_{\mathfrak{X}''^\sharp}} & & \downarrow{\scriptstyle p_{\mathfrak{X}'^\sharp}} \\ \mathfrak{S}''^\sharp & \xrightarrow{\phi'} & \mathfrak{S}'^\sharp, \end{array} \tag{9.2.1.12.1}$$

be a commutative diagram of nice fine $\mathcal{V}$-log formal schemes where $p_{\mathfrak{X}''^\sharp}$ is log smooth morphism. We suppose the underlying formal schemes of $\mathfrak{S}''$, $\mathfrak{X}'$ are p-torsion free. We suppose $S''^\sharp$, S'' are regular and that S'' is noetherian. We suppose X'' is regular. Let Z'' be some divisor of X'' such that $g(X'' \setminus Z'') \subset X' \setminus Z'$. We set $\widetilde{\mathfrak{X}}''^{\sharp(\bullet)} := (\mathfrak{X}''^\sharp, \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z''))$. We denote by $\widetilde{g}^{(\bullet)}$ the induced morphism $\widetilde{\mathfrak{X}}''^{(\bullet)}/\mathfrak{S}''^\sharp \to \widetilde{\mathfrak{X}}'^{(\bullet)}/\mathfrak{S}^\sharp$ of $\mathbb{N}$-ringed $\mathcal{V}$-log formal schemes.

Lemma 9.2.1.13. *We keep notation 9.2.1.12.*

(a) We have the canonical isomorphism of $D(\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet}(Z''), (f\circ g)^{-1}_\bullet \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$:

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'',Z')\otimes^{\mathbb{L}}_{g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}} g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z)\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z'',Z). \tag{9.2.1.13.1}$$

(b) We have the canonical isomorphism of $D((f\circ g)^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z),\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet}(Z''))$:

$$g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet\leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z')\otimes^{\mathbb{L}}_{g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\leftarrow X''^\sharp_\bullet/S''^\sharp_\bullet}(Z',Z'')\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet\leftarrow X''^\sharp_\bullet/S''^\sharp_\bullet}(Z,Z''). \tag{9.2.1.13.2}$$

Proof. By quasi-flatness of $\widetilde{f}$, it follows from 5.1.1.3 that the morphisms are well defined. a) Let us check 9.2.1.13.1. i) We have the isomorphism of left $\mathcal{D}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet}(Z'')$-modules $\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'',Z')\otimes_{g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')}$ $g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z)\xrightarrow{\sim}\widetilde{g}^{(\bullet)*}_\bullet\widetilde{f}^*_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'',Z')\xrightarrow{\sim}(\widetilde{f}\circ\widetilde{g})^*_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$. Hence, it follows from 4.4.5.6 that we get the canonical isomorphism of left $\mathcal{D}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet}(Z'')$-modules

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'',Z')\otimes_{g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')} g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z)\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z'',Z).$$

We obtain by functoriality the fact that this isomorphism is an isomorphism of $(\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet}(Z''),(f\circ g)^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$-bimodules.

ii) Moreover, since $\widetilde{f}^*_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)$ is $\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')$-flat then $\mathbb{L}\widetilde{g}^{(\bullet)*}_\bullet\left(\widetilde{f}^*_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)\right)\xrightarrow{\sim}\widetilde{g}^{(\bullet)*}_\bullet\left(\widetilde{f}^*_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)\right)$. Hence, it follows from 9.2.1.9.1 the isomorphism

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'',Z')\otimes^{\mathbb{L}}_{g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')} f^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z)$$
$$\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(\bullet)}_{X''^\sharp_\bullet/S''^\sharp_\bullet\to X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'',Z')\otimes_{g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')} g^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet\to X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z).$$

b) Finally, we get the isomorphism 9.2.1.13.2 from 9.2.1.13.1 by twisting (use 9.2.1.4.1 and 4.3.5.6.1). □

Proposition 9.2.1.14. *With notation 9.2.1.12, let* $*\in\{r,l\}$.

(a) For any $\mathcal{E}_\bullet\in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$*, we have the canonical isomorphism*

$$\widetilde{g}^{(\bullet)!}_\bullet\circ\widetilde{f}^{(\bullet)!}_\bullet(\mathcal{E}_\bullet)\xrightarrow{\sim}(\widetilde{f\circ g})^{(\bullet)!}_\bullet(\mathcal{E}_\bullet). \tag{9.2.1.14.1}$$

(b) For any $\mathcal{E}\in D^-_{\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$*, we have the canonical isomorphism*

$$\widetilde{g}^{(\bullet)!}\circ\widetilde{f}^{(\bullet)!}(\mathcal{E})\xrightarrow{\sim}(\widetilde{f\circ g})^{(\bullet)!}(\mathcal{E}). \tag{9.2.1.14.2}$$

Proof. By using the equivalence of categories 8.5.1.10, we reduce to check the isomorphism 9.2.1.14.1. This latter one can be checked similarly to 5.1.1.13 by using 9.2.1.13. □

9.2.1.15. For any $*\in\{r,l\}$, similarly to 8.5.4.8.1, the extraordinary inverse image functors $\widetilde{f}^{(\bullet)!}_\bullet$ and $\widetilde{f}^{(\bullet)!}$ of 9.2.1.5 send lim-ind-isogenies to lim-ind-isogenies and preserve the quasi-coherence. This yields the factorizations denoted by:

$$(f,\phi)^{(\bullet)!}_{\bullet Z',Z}\colon \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))\to\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')), \tag{9.2.1.15.1}$$

$$(f,\phi)^{(\bullet)!}_{Z',Z}\colon \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))\to\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')). \tag{9.2.1.15.2}$$

They are called the *extraordinary inverse image* by (f,ϕ) (or simply f when ϕ is understood) with overconvergent singularities along Z and Z'. Remark that $(\mathrm{id},\phi)^{(\bullet)!}_{Z',Z}=({}^{\dagger}Z',Z)$. When $\phi=\mathrm{id}_{\mathfrak{S}^\sharp}$, we replace $\mathrm{id}_{\mathfrak{S}^\sharp}$ by $_{/\mathfrak{S}^\sharp}$ and $\mathrm{id}_{S^\sharp_\bullet}$ by $_{/S^\sharp_\bullet}$, e.g. $f^{(\bullet)!}_{\bullet/S^\sharp_\bullet,Z',Z}:=(f,\mathrm{id}_{S^\sharp_\bullet})^{(\bullet)!}_{\bullet Z',Z}$ and $f^{(\bullet)!}_{/\mathfrak{S}^\sharp,Z',Z}:=(f,\mathrm{id}_{\mathfrak{S}^\sharp})^{(\bullet)!}_{Z',Z}$. To lighten the notations, when ϕ is understood, we simply write f instead of (f,ϕ), e.g. $f^{(\bullet)!}_{Z',Z}$ instead of $(f,\phi)^{(\bullet)!}_{Z',Z}$. We also denote by

$$\mathbb{L}(f,\phi)^{*(\bullet)}_{\bullet Z',Z}:=(f,\phi)^{(\bullet)!}_{\bullet Z',Z}[-d_f]\qquad \mathbb{L}(f,\phi)^{*(\bullet)}_{Z',Z}:=(f,\phi)^{(\bullet)!}_{Z',Z}[-d_f], \tag{9.2.1.15.3}$$

or simply $\mathbb{L}f^{*(\bullet)}_{\bullet Z',Z}$ and $\mathbb{L}f^{*(\bullet)}_{Z',Z}$; when f is smooth, we remove $\mathbb{L}$. When $Z'=f^{-1}(Z)$, we remove Z' in the notation, e.g. $(f,\phi)^{(\bullet)!}_{Z}:=(f,\phi)^{(\bullet)!}_{f^{-1}(Z),Z}$. If moreover Z is empty, we remove Z in the notation.

For any $\mathcal{E}^{(\bullet)}\in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, $\mathcal{M}^{(\bullet)}\in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)))$ we have the isomorphisms of $D^{-}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$:

$$(f,\phi)^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z',Z)\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)}f^{-1}\mathcal{E}^{(\bullet)}[d_{\mathfrak{X}'/\mathfrak{X}}],$$

$$(f,\phi)^{(\bullet)!}_{Z',Z}(\mathcal{M}^{(\bullet)})\xrightarrow{\sim}\mathcal{M}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z,Z')[d_{\mathfrak{X}'/\mathfrak{X}}] \tag{9.2.1.15.4}$$

where the tensor product is defined at 8.5.4.8.3.

9.2.1.16. For any $*\in\{r,l\}$, the extraordinary inverse image functors $\widetilde{f}^{(\bullet)!}_{\mathrm{alg}}$ of 9.2.1.8 send lim-ind-isogenies to lim-ind-isogenies but beware they do not preserve the quasi-coherence. This yields the factorizations denoted by:

$$(f,\phi)^{(\bullet)!}_{\mathrm{alg}\,Z',Z}\colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))\to\underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')). \tag{9.2.1.16.1}$$

We can slightly improve 9.2.1.8.(c) with the following lemma. When ϕ is understood, we simply write f instead of (f,ϕ), etc.

Lemma 9.2.1.17. *Let $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. The canonical morphism $f^{(\bullet)!}_{\mathrm{alg}\,Z',Z}(\mathcal{E}^{(\bullet)})\to f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$.*

Proof. By definition, there exist $\lambda\in L$ and an isomorphism $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ of the form $\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathcal{F}^{(\bullet)}$ such that $\mathcal{F}^{(m)}\in D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\lambda(m))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ (plus another condition that we do not need here). We reduce to check the lemma for $\mathcal{F}^{(\bullet)}$ in the place of $\mathcal{E}^{(\bullet)}$. Denote by $f^{!\lambda(\bullet)}_{\mathrm{alg}\,Z',Z}(\mathcal{F}^{(\bullet)}):=(f^{!(\lambda(m))}_{\mathrm{alg}\,Z',Z}(\mathcal{F}^{(m)}))_{m\in\mathbb{N}}\in D^{\mathrm{b}}(\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ and $f^{!\lambda(\bullet)}_{Z',Z}(\mathcal{F}^{(\bullet)}):=(f^{!(\lambda(m))}_{Z',Z}(\mathcal{F}^{(m)}))_{m\in\mathbb{N}}\in D^{\mathrm{b}}(\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Moreover, we have canonical morphism $f^{!\lambda(\bullet)}_{Z',Z}(\mathcal{F}^{(\bullet)})\to\lambda^{*}\left(f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)})\right)$ inducing the canonical commutative diagram:

$$\begin{array}{ccc}
f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)}) & \longrightarrow & f^{!\lambda(\bullet)}_{Z',Z}(\mathcal{F}^{(\bullet)}) \\
\downarrow & \swarrow & \downarrow \\
\lambda^{*}\left(f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)})\right) & \longrightarrow & \lambda^{*}\left(f^{!\lambda(\bullet)}_{Z',Z}(\mathcal{F}^{(\bullet)})\right).
\end{array}$$

This yields that $f^{!\lambda(\bullet)}_{Z',Z}(\mathcal{F}^{(\bullet)})$ is canonically isomorphic to $f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)})$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Similarly, we get that $f^{!\lambda(\bullet)}_{\mathrm{alg}\,Z',Z}(\mathcal{F}^{(\bullet)})$ is canonically isomorphic to $f^{(\bullet)!}_{\mathrm{alg}\,Z',Z}(\mathcal{F}^{(\bullet)})$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Hence, we reduce to check that the canonical morphism

$$f^{!\lambda(\bullet)}_{\mathrm{alg}\,Z',Z}(\mathcal{F}^{(\bullet)})\to f^{!\lambda(\bullet)}_{Z',Z}(\mathcal{F}^{(\bullet)}) \tag{9.2.1.17.1}$$

is an isomorphism of $D^{\mathrm{b}}(\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, which is a consequence of the fact that $\mathcal{F}^{(m)}\in D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\lambda(m))}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. □

Proposition 9.2.1.18. *Let D and D' be some divisors of respectively X and X' such that $f(X'\setminus D')\subset X\setminus D$, $D\subset Z$, and $D'\subset Z'$. Let $\mathcal{E}^{(\bullet)}\in D^{-}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$. We have the isomorphism in $D^{-}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$ of the form*

$$({}^\dagger Z',D')\circ f^{(\bullet)!}_{D',D}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim} f^{(\bullet)!}_{Z',D}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim} f^{(\bullet)!}_{Z',Z}\circ({}^\dagger Z,D)(\mathcal{E}^{(\bullet)}).$$

Proof. This is a consequence of the transitivity of the extraordinary inverse image (see 9.2.1.14). □

Proposition 9.2.1.19. *The functors 9.2.1.15.1 and 9.2.1.15.2 preserves bounded complexes, i.e. it induces*

$$f^{(\bullet)!}_{\bullet Z',Z}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))\to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')), \tag{9.2.1.19.1}$$

$$f^{(\bullet)!}_{Z',Z}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))\to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')). \tag{9.2.1.19.2}$$

Proof. Since the functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}'^{(\mathbb{N})},*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}$ preserve the boundedness, then we reduce to only check 9.2.1.19.1. Let $\mathcal{E}^{(\bullet)}_\bullet\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$. We deduce from 9.2.1.18 the isomorphism $({}^\dagger Z')\circ f^{(\bullet)!}_\bullet(\mathrm{forg}_Z(\mathcal{E}^{(\bullet)}_\bullet))\xrightarrow{\sim} f^{(\bullet)!}_{\bullet Z',Z}\circ({}^\dagger Z)(\mathrm{forg}_Z(\mathcal{E}^{(\bullet)}_\bullet))$. Following 9.1.2.3.1, we have the isomorphism $({}^\dagger Z)(\mathrm{forg}_Z(\mathcal{E}^{(\bullet)}_\bullet)$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. Since the boundedness is preserved by the functors $({}^\dagger Z')$ (see 9.1.1.5) and $f^{(\bullet)!}_\bullet$ (see 9.2.1.2 and 9.2.1.9.1), then we are done. □

Notation 9.2.1.20. By passing to the limit on the level, we get the following bimodules.

(a) We get a $(\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}}, f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$-bimodule by setting

$$\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z',Z)_{\mathbb{Q}}:=\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z',Z)_{\mathbb{Q}}.$$

(b) We get the $(f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}},\, \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}})$-bimodule by setting

$$\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z,Z')_{\mathbb{Q}}:=\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z,Z')_{\mathbb{Q}}.$$

(c) By taking the projective limits and then inductive limits on the level, it follows from 5.1.1.2.1, that we have the isomorphism:

$$\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z,Z')_{\mathbb{Q}}\xrightarrow{\sim}\omega_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}\otimes_{\mathcal{O}_{\mathfrak{X}'}}\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z',Z)_{\mathbb{Q}}\otimes_{f^{-1}\mathcal{O}_{\mathfrak{X}}} f^{-1}\omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}. \tag{9.2.1.20.1}$$

Notation 9.2.1.21 (Extraordinary inverse image)**.** Let $*\in\{\mathrm{l},\mathrm{r}\}$.

(a) The extraordinary inverse image by (f,ϕ) (or simply f when ϕ is understood) with overconvergent singularities along Z and Z' is also a functor of the form $f^!_{Z',Z}\colon D^-({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})\to D^-({}^*\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}})$ which is defined for any $\mathcal{E}\in D^-({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ and $\mathcal{M}\in D^-({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ by setting:

$$f^!_{Z',Z}(\mathcal{E}):=\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z',Z)_{\mathbb{Q}}\otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} f^{-1}\mathcal{E}[d_f], \tag{9.2.1.21.1}$$

$$f^!_{Z',Z}(\mathcal{M}):=f^{-1}\mathcal{M}\otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z,Z')_{\mathbb{Q}}[d_f]. \tag{9.2.1.21.2}$$

(b) We can also consider the functors $\mathbb{L}f^*_{Z',Z}:=f^!_{Z',Z}[-d_f]$, which is the left derived functor of the functor denoted by $f^*_{Z',Z}$ which corresponds to the above tensor product without $\mathbb{L}$ and shift.

(c) When $Z'=f^{-1}(Z)$, we simply write respectively $f^!_Z$ and $\mathbb{L}f^*_Z$. If moreover Z is empty, we write $f^!$ and $\mathbb{L}f^*$.

9.2.1.22 (Log-étale case)**.** Suppose the bottom square of 9.2.1.0.1 is the identity and f is log-étale. Then, following 5.1.3.6, the canonical morphism of left $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')$-modules $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z') \to \widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}\to X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}$ is an isomorphism and the composite induced map $f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} \to \widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}\to X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')$ is a ring homomorphism. Taking the projective and inductive limits, tensoring with $\mathbb{Q}$, this yields that the canonical morphism $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}\to\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z',Z)_{\mathbb{Q}}$ is an isomorphism and the induced map $f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}}$ is a ring homomorphism. For any $\mathcal{E} \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$, we have therefore the isomorphism:

$$f^{!}_{Z',Z}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}} f^{-1}\mathcal{E}[d_f]. \tag{9.2.1.22.1}$$

Similarly, the canonical morphism of right $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')$-modules $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z') \to \widetilde{\mathcal{D}}^{(\bullet)}_{Y^{\sharp}_{\bullet}\leftarrow X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}$ is an isomorphism and the composite induced map $f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} \to \widetilde{\mathcal{D}}^{(\bullet)}_{Y^{\sharp}_{\bullet}\leftarrow X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')$ is the ring homomorphism. For any $\mathcal{M} \in D^{-}({}^{\mathrm{r}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$, this yields the isomorphism:

$$f^{!}_{Z',Z}(\mathcal{M}) \xrightarrow{\sim} f^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}}[d_f]. \tag{9.2.1.22.2}$$

9.2.1.23. Similarly to 5.1.1.5.1, it follows from 9.2.1.20.1 that for any $\mathcal{E} \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$, we check the isomorphism

$$\omega_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} f^{!}_{Z',Z}(\mathcal{E}) \xrightarrow{\sim} f^{!}_{Z',Z}(\omega_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{9.2.1.23.1}$$

Proposition 9.2.1.24. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$. The functors 9.2.1.15.2 and 9.2.1.21.1 are compatible with the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ (see notation 9.1.6.1), i.e. we have the functorial isomorphism:*

$$f^{!}_{Z',Z} \circ \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \underrightarrow{l}^{*}_{\mathbb{Q}} \circ f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}).$$

Proof. This follows from 9.2.1.17. □

9.2.1.25. With notation 9.1.6.6, the proposition 9.2.1.24 means that we have the isomorphism of functors $\mathrm{Coh}_{Z}(f^{(\bullet)!}_{Z',Z}) \xrightarrow{\sim} f^{!}_{Z',Z}$.

Lemma 9.2.1.26 (Tor independence)**.** *Suppose $Z' := f^{-1}(Z)$. We have the canonical isomorphism*

$$\mathcal{O}_{X'_i} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_{X_i}} f^{-1}\mathcal{B}^{(m)}_{X_i}(Z) \xrightarrow{\sim} \mathcal{B}^{(m)}_{X'_i}(Z').$$

We have also the canonical isomorphism $f^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)) \xrightarrow{\sim} \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')[d_f]$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$.

Proof. 1) To check the first isomorphism, since the canonical morphism $\mathcal{O}_{X'_i} \otimes_{f^{-1}\mathcal{O}_{X_i}} f^{-1}\mathcal{B}^{(m)}_{X_i}(Z) \xrightarrow{\sim} \mathcal{B}^{(m)}_{X'_i}(Z')$ is an isomorphism, we reduce to check that the canonical morphism $\mathcal{O}_{X'_i} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_{X_i}} f^{-1}\mathcal{B}^{(m)}_{X_i}(Z) \to \mathcal{O}_{X'_i} \otimes_{f^{-1}\mathcal{O}_{X_i}} f^{-1}\mathcal{B}^{(m)}_{X_i}(Z)$ is an isomorphism. Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ affine, integral and there exist $a \in \mathcal{O}_{\mathfrak{X}}$ lifting a local equation of $Z \subset X$. We get $a' := f^{*}(a) \in \mathcal{O}_{\mathfrak{X}'}$ a lifting of a local equation of $Z' \subset X'$. Since $\mathfrak{X}$ is p-torsion free, we have the short exact sequence $0 \to \mathcal{O}_{\mathfrak{X}}\{T\} \underset{a^{p^{m+1}}}{\longrightarrow} \mathcal{O}_{\mathfrak{X}}\{T\} \to \mathcal{B}^{(m)}_{\mathfrak{X}}(Z) \to 0$. Since $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)$ is p-torsion free (see 8.7.4.1), this yields the short exact sequence $0 \to \mathcal{O}_{X_i}[T] \underset{\overline{a}^{p^{m+1}}T-p}{\longrightarrow} \mathcal{O}_{X_i}[T] \to \mathcal{B}^{(m)}_{X_i}(Z) \to 0$ where $\overline{a}$ is the reduction of a modulo π^{i+1}, which is a resolution of $\mathcal{B}^{(m)}_{X_i}(Z)$ by free $\mathcal{O}_{X_i}$-modules. By applying $\mathcal{O}_{X'_i} \otimes_{f^{-1}\mathcal{O}_{X_i}} f^{-1}-$ to this resolution, we get the sequence $0 \to \mathcal{O}_{X'_i}[T] \underset{\overline{a'}^{p^{m+1}}T-p}{\longrightarrow} \mathcal{O}_{X'_i}[T] \to \mathcal{B}^{(m)}_{X'_i}(Z') \to 0$ where $\overline{a'}$ is the reduction of a' modulo π^{i+1}, which is exact (for the same reason as above) and we are done.

2) Since, for any nonnegative integer m, the $\mathcal{V}$-modules $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(Z')$ and $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z)$ are p-torsion free, we get the second isomorphism from the first one. □

Proposition 9.2.1.27. *Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$.*

(a) *We have the canonical isomorphism in* $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$*:*

$$f^{(\bullet)!}_{Z',Z}\left(\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}\right)[d_f] \xrightarrow{\sim} f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')}f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.1.27.1}$$

(b) *With notation 9.2.1.12, the isomorphisms of the form 9.2.1.27.1 are transitive, i.e., the following diagram*

$$\begin{array}{ccc} g^{(\bullet)!}_{Z'',Z'}\circ f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}\mathcal{F}^{(\bullet)})[d_{f\circ g}] & \xrightarrow{\sim} & (f\circ g)^{(\bullet)!}_{Z'',Z}(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}\mathcal{F}^{(\bullet)})[d_{f\circ g}] \\ \downarrow{\scriptstyle 9.2.1.27.1} & & \\ g^{(\bullet)!}_{Z'',Z'}(f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)}))[d_f] & & \downarrow{\scriptstyle 9.2.1.27.1} \\ \downarrow{\scriptstyle 9.2.1.27.1} & & \\ g^{(\bullet)!}_{Z'',Z'}\circ f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}g^{(\bullet)!}_{Z'',Z'}\circ f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & (f\circ g)^{(\bullet)!}_{Z'',Z}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}(f\circ g)^{(\bullet)!}_{Z'',Z}(\mathcal{F}^{(\bullet)}) \end{array} \tag{9.2.1.27.2}$$

is commutative.

Proof. Since pullback commutes with tensor products (see 7.5.5.9), we get

$$f^{(\bullet)!}_{\bullet Z',Z}\left(\mathcal{F}^{(\bullet)}_{\bullet}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}_{\bullet}\right)[d_f] \xrightarrow{\sim} f^{(\bullet)!}_{\bullet Z',Z}(\mathcal{F}^{(\bullet)}_{\bullet})\otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}(Z')}f^{(\bullet)!}_{\bullet Z',Z}(\mathcal{E}^{(\bullet)}_{\bullet}). \tag{9.2.1.27.3}$$

This yields the isomorphism 9.2.1.27.1 by construction.

Following 8.5.4.5, the functors $\mathbb{R}\underleftrightarrow{l}_{\mathfrak{X}^{(\mathbb{N})}*}$ and $\mathbb{L}\underleftrightarrow{l}^{*}_{\mathfrak{X}^{(\mathbb{N})}}$ induce canonically quasi-inverse equivalences of categories between $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ and $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$. Hence, to get the commutative diagram 9.2.1.27.2, we reduce to check that the isomorphisms of the form 9.2.1.27.3 is transitive, which is obvious. □

By using Lemma 9.2.1.26, we get a derived functor version of 4.4.5.2:

Proposition 9.2.1.28. *Suppose* $Z' = f^{-1}(Z)$*.*

(a) *Let* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$*. We have the canonical isomorphism* $f^{(\bullet)!}\circ \mathrm{forg}_Z \circ ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{forg}_{Z'}\circ({}^\dagger Z')\circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$*, which we can simply write* $f^{(\bullet)!}\circ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z')\circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$*.*

(b) *Let* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$*. We have the canonical isomorphism* $\mathrm{forg}_{Z'}\circ f^{(\bullet)!}_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} f^{(\bullet)!}\circ \mathrm{forg}_Z(\mathcal{E}^{(\bullet)})$*. Hence, it is harmless to write by abuse of notation* $f^{(\bullet)!}$ *instead of* $f^{(\bullet)!}_Z$*.*

Proof. Using 9.2.1.27.1,9.2.1.26, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, we get the isomorphism

$$\begin{aligned} f^{(\bullet)!}\circ \mathrm{forg}_Z\circ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) &= f^{(\bullet)!}\left(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)}\right) \\ &\xrightarrow{\sim} \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}'}}f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) = \mathrm{forg}_{Z'}\circ({}^\dagger Z')\circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}). \end{aligned}$$

By using 9.1.2.3.1 and 9.2.1.18, we check the second part from the first one. □

Remark 9.2.1.29. With notation 9.2.1.28, using the remark 9.1.6.7 we check that the functors $\mathrm{Coh}_Z(f^{(\bullet)!}_Z)$ and $\mathrm{Coh}(f^{(\bullet)!})$ are isomorphic over $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})\cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$. Since we have the canonical isomorphisms of functors $\mathrm{Coh}_Z(f^{(\bullet)!}_Z)\xrightarrow{\sim} f^!_Z$ and $\mathrm{Coh}_Z(f^{(\bullet)!})\xrightarrow{\sim} f^!$ (9.2.1.25), then it is harmless to write $f^!$ instead of $f^!_Z$.

9.2.2 Glueing pullbacks

We keep notation 9.2.1 and 9.2.1.12. Let $f', f''\colon \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ be two other $\mathfrak{S}^\sharp$ morphisms such that $f_0 = f'_0 = f''_0$. Let $g', g''\colon \mathfrak{X}''^\sharp \to \mathfrak{X}'^\sharp$ be two other $\mathfrak{S}^\sharp$ morphisms such that $g_0 = g'_0 = g''_0$.

9.2.2.1. It follows from 4.4.5.3 that we have the canonical glueing isomorphism of functors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z'))$ of the form

$$\tau^{(\bullet)}_{f,f'} \colon f'^{(\bullet)!}_{Z',Z} \xrightarrow{\sim} f^{(\bullet)!}_{Z',Z},$$

where to lighten notation we have avoid indicating Z and Z'. It follows from 4.4.5.9 that these isomorphisms satisfy the following formulas $\tau^{(\bullet)}_{f,f} = \mathrm{id}$, $\tau^{(\bullet)}_{f,f''} = \tau^{(\bullet)}_{f,f'} \circ \tau^{(\bullet)}_{f',f''}$, $g^{(\bullet)!}_{Z'',Z'} \circ \tau^{(\bullet)}_{f,f'} = \tau^{(\bullet)}_{f\circ g,f'\circ g}$ and $\tau^{(\bullet)}_{g,g'} \circ f^{(\bullet)!}_{Z',Z} = \tau^{(\bullet)}_{f\circ g,f\circ g'}$.

Proposition 9.2.2.2. *Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. We have the commutative diagram :*

$$\begin{array}{ccc} f^{(\bullet)!}_{Z',Z}\left(\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}\right)[d_f] & \xrightarrow[\sim]{9.2.1.27.1} & f^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')} f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}) \\ \sim\downarrow \tau^{(\bullet)}_{f',f} & & \sim\downarrow \tau^{(\bullet)}_{f',f}\otimes\tau^{(\bullet)}_{f',f} \\ f'^{(\bullet)!}_{Z',Z}\left(\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}\right)[d_f] & \xrightarrow[\sim]{9.2.1.27.1} & f'^{(\bullet)!}_{Z',Z}(\mathcal{F}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')} f'^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}), \end{array} \tag{9.2.2.2.1}$$

the isomorphisms of glueing $\tau^{(\bullet)}_{f',f}$ having been defined at 9.2.2.1.

Proof. This is a consequence of 4.4.5.8.2. □

Proposition 9.2.2.3. *We have the following properties.*

(a) There exists a canonical glueing isomorphism of functors $D(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D(\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}})$ of the form

$$\tau_{f,f'} \colon f'^{!}_{Z',Z} \xrightarrow{\sim} f^{!}_{Z',Z}, \tag{9.2.2.3.1}$$

such that $\tau_{f,f} = \mathrm{id}$, $\tau_{f,f''} = \tau_{f,f'}\circ\tau_{f',f''}$. For any $\mathcal{E} \in D(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$, we write $\tau_{f,f'}(\mathcal{E}) \colon f'^{!}_{Z',Z}(\mathcal{E}) \xrightarrow{\sim} f^{!}_{Z',Z}(\mathcal{E})$ or $\tau^{\mathcal{E}}_{f,f'}$ this corresponding canonical isomorphism.

(b) The diagram of functors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)) \to D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}})$

$$\begin{array}{ccc} \underrightarrow{l}^*_{\mathbb{Q}} \circ f'^{(\bullet)!}_{Z',Z} & \xrightarrow[\underrightarrow{l}^*_{\mathbb{Q}}\circ\tau^{(\bullet)}_{f,f'}]{\sim} & \underrightarrow{l}^*_{\mathbb{Q}} \circ f^{(\bullet)!}_{Z',Z} \\ \downarrow\sim & & \downarrow\sim \\ f'^{!}_{Z',Z} \circ \underrightarrow{l}^*_{\mathbb{Q}} & \xrightarrow{\tau_{f,f'}\circ\underrightarrow{l}^*_{\mathbb{Q}}} & f^{!}_{Z',Z} \circ \underrightarrow{l}^*_{\mathbb{Q}} \end{array}$$

is commutative up to canonical isomorphism.

(c) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z))$ such that $f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z))$. Set $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E})$. Then $g^{!}_{Z'',Z'}(\tau_{f,f'}(\mathcal{E})) = \tau_{f\circ g,f'\circ g}(\mathcal{E})$ and $\tau_{g,g'}(f^{!}_{Z',Z}(\mathcal{E})) = \tau_{f\circ g,f\circ g'}(\mathcal{E})$.

Proof. Let us prove a). Let $\mathcal{E} \in D(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$. Tensoring by $\mathbb{Q}$ the inductive limit on the level of the inverse limits on i of the glueing isomorphisms $\mathcal{D}^{(m)}_{X'^\sharp_i/S'^\sharp_i \xrightarrow{g_i} X^\sharp_i/S^\sharp_i}({}^\dagger Z) \xrightarrow{\sim} \mathcal{D}^{(m)}_{X'^\sharp_i/S'^\sharp_i \xrightarrow{f_i} X^\sharp_i/S^\sharp_i}({}^\dagger Z)$ of 4.4.5.3, we get the isomorphism $\tau_{f,g} \colon \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \xrightarrow{g} \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \xrightarrow{f} \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$. Finally, we construct $\tau_{f,g}(\mathcal{E})$ to be the composition $f'^{!}_{Z',Z}\mathcal{E} = \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \xrightarrow{g} \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f_0^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} f_0^{-1}\mathcal{F}[d_f] \xrightarrow[\tau_{f,g}\otimes^{\mathbb{L}}\mathrm{id}]{\sim} \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \xrightarrow{f} \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f_0^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} f_0^{-1}\mathcal{F}[d_f] = f^{!}_{Z',Z}\mathcal{E}$. It follows from 4.4.5.3 the desired properties of the part a).

By construction of both $\tau_{f,g}$ and $\tau^{(\bullet)}_{f,g}$, we can easily check b). The part c) follows from the part b) and from 9.2.1.24 and 9.2.2.1. □

9.2.2.4. To define pullbacks (or the pushforwards) with notation 9.2.1, we do not need that the morphism f_0 has a lifting. More precisely, let be the commutative diagram

$$\begin{array}{ccccc} X'^{\sharp} & \hookrightarrow & \mathfrak{X}'^{\sharp} & \xrightarrow{p_{\mathfrak{X}'^{\sharp}}} & \mathfrak{S}'^{\sharp} \\ \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle \phi} \\ X^{\sharp} & \hookrightarrow & \mathfrak{X}^{\sharp} & \xrightarrow{p_{\mathfrak{X}^{\sharp}}} & \mathfrak{S}^{\sharp}, \end{array} \tag{9.2.2.4.1}$$

where $p_{\mathfrak{X}^{\sharp}}$ and $p_{\mathfrak{X}'^{\sharp}}$ are log smooth morphisms of very nice (see definition 3.3.1.10.(b)) fine $\mathcal{V}$-log formal schemes. We suppose S, $S^{\sharp}$, S', $S'^{\sharp}$, X and X' are regular (to have an idea about theses hypotheses, see [GR, 12.5.19]) and noetherian. Let Z and Z' be some divisors of respectively X and X' such that $f(X' \setminus Z') \subset X \setminus Z$. Then similarly to 9.2.1.15.1 and 9.2.1.15.2, it follows from 4.4.5.11 that we get the functors

$$(f_0,\phi)^{(\bullet)!}_{\bullet Z',Z} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')), \tag{9.2.2.4.2}$$

$$(f_0,\phi)^{(\bullet)!}_{Z',Z} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')). \tag{9.2.2.4.3}$$

In the case where f_0 is the relative Frobenius, we will study more precisely later at 9.5.1.2.

9.2.2.5. Suppose f is finite and $Z' = f_0^{-1}(Z)$. Then using 7.2.1.4, we can check that the canonical morphism

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(Z') \otimes^{\mathbb{L}}_{f_0^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z)} f_0^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z) \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp} \xrightarrow{f} \mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)$$

is an isomorphism. Tensoring by $\mathbb{Q}$ and taking the inductive limit over the level, this yields the canonical isomorphism

$$\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f_0^{-1}\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}} f_0^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp} \xrightarrow{f} \mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}.$$

Let $\mathcal{E} \in D^{-}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$. Hence, by associativity of the tensor product, so is the canonical morphism

$$\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f_0^{-1}\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}} f_0^{-1}\mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp} \xrightarrow{f} \mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f_0^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}} f_0^{-1}\mathcal{F} = \mathbb{L}f^{*}_{Z}\mathcal{E} = f^{!}_{Z}\mathcal{E}[-d_f]$$

is an isomorphism. Hence, if $\mathcal{P}$ is a complex of left $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}$-modules which are $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$-flat, if $\mathcal{P} \to \mathcal{E}$ is a quasi-isomorphism of $K^{-}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$ then we get the isomorphism $f^{*}_{Z}(\mathcal{P}) \xrightarrow{\sim} \mathbb{L}f^{*}_{Z}\mathcal{E}$ of $D^{-}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$.

Remark 9.2.2.6. Let $\mathcal{E} \in D^{-}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$.

(a) Let $\mathcal{P}$ be a complex of flat left $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}$-modules endowed with a quasi-isomorphism $\mathcal{P} \to \mathcal{E}$. The isomorphism $\tau_{f,f'} \colon f'^{!}_{Z',Z}\mathcal{E} \xrightarrow{\sim} f^{!}_{Z',Z}\mathcal{E}$ is represented (up to the shift $[d_{f_0}]$) by the isomorphism of $\tau_{f,f'} \colon f'^{*}_{Z',Z}(\mathcal{P}) \xrightarrow{\sim} f^{*}_{Z',Z}(\mathcal{P})$ of $C(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$ which is computed term by term.

(b) Suppose f and g are finite morphisms. Suppose $Z' = f_0^{-1}(Z)$ and $Z'' = g_0^{-1}(Z')$. Let $\mathcal{P}$ be a complex of flat left $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}$-modules which are $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$-flat together with a quasi-isomorphism $\mathcal{P} \to \mathcal{E}$. It follows from 9.2.2.5 that the isomorphism $\tau_{f,f'} \colon f'^{!}_{Z}\mathcal{E} \xrightarrow{\sim} f^{!}_{Z}\mathcal{E}$ is represented (up to the shift $[d_{f_0}]$) by the isomorphism of $\tau_{f,f'} \colon f'^{*}_{Z}(\mathcal{P}) \xrightarrow{\sim} f^{*}_{Z}(\mathcal{P})$ of $C(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$ which is computed term by term.

9.2.3 Projection formula given by a ringed topoi morphism

We keep notation 9.2.1. In order to check later that the pushforwards are compatible with the quasi-inverse functors $- \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)} \widetilde{\omega}^{(\bullet)-1}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}$ and $\widetilde{\omega}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)} -$ exchanging left and right $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)$-module structures, we will need the following projection formula (see 9.2.4.8).

Proposition 9.2.3.1. Let $\mathcal{F}_\bullet^{(\bullet)} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$ and $\mathcal{G}_\bullet^{(\bullet)} \in D({}^{\mathrm{l}}f_\bullet^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$.

(i) We have the canonical morphism in $D(\mathbb{Z}_{X_\bullet})$:

$$\mathcal{F}_\bullet^{(\bullet)} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} \mathbb{R}f_{\bullet *}(\mathcal{G}_\bullet^{(\bullet)}) \to \mathbb{R}f_{\bullet *}\left(f_\bullet^{-1}\mathcal{F}_\bullet^{(\bullet)} \otimes^{\mathbb{L}}_{f_\bullet^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} \mathcal{G}_\bullet^{(\bullet)}\right). \tag{9.2.3.1.1}$$

Let $\mathcal{D}_\bullet$ be a sheaf of rings on $X_\bullet$ such that $(\mathcal{D}_\bullet, \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$ is solvable by $\mathcal{R}_\bullet$ and $\mathcal{F}_\bullet^{(\bullet)} \in D(\mathcal{D}_\bullet, \mathcal{R}_\bullet, \widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$ (see definition and notation 4.6.3.2). Then the morphism 9.2.3.1.1 can also be viewed as a morphism of $D(\mathcal{D}_\bullet)$.

(ii) Suppose f is quasi-compact and quasi-separated. Suppose moreover for any $i \in \mathbb{Z}$ one of the following conditions hold:

(a) either $\mathcal{F}_i^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$, and $\mathcal{G}_i^{(\bullet)} \in D({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$,

(b) or S_i is a noetherian scheme of finite Krull dimension, and $\mathcal{F}_i^{(\bullet)} \in D^{-}_{\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$, and $\mathcal{G}_i^{(\bullet)} \in D^{-}({}^{\mathrm{l}}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$.

Then the morphism 9.2.3.1.1 is an isomorphism.

Proof. We construct 9.2.3.1.1 as 5.1.2.5.1. The second statement is a consequence of 5.1.2.5.1. □

Remark 9.2.3.2. Inverting r and l, we get the morphism

$$\mathbb{R}f_{\bullet *}(\mathcal{G}_\bullet) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} \mathcal{F}_\bullet^{(\bullet)} \to \mathbb{R}f_{\bullet *}\left(\mathcal{G}_\bullet \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} f^{-1}\mathcal{F}_\bullet^{(\bullet)}\right), \tag{9.2.3.2.1}$$

which is an isomorphism when we invert the corresponding hypotheses.

Corollary 9.2.3.3. *Let $*, ** \in \{\mathrm{l}, \mathrm{r}\}$ such that both are not equal to* r. *Suppose f is quasi-compact and quasi-separated.*

(a) either $\mathcal{F}_i^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$, and $\mathcal{G}_i \in D({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$,

(b) or S_i is a noetherian scheme of finite Krull dimension, and $\mathcal{F}_i^{(\bullet)} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$, and $\mathcal{G}_i \in D^{-}({}^{**}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$.

Then we have the following isomorphism of $D^{-}({}^{**}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_i/S^\sharp_i}(Z))$:

$$\mathcal{F}_\bullet^{(\bullet)} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \mathbb{R}f_{\bullet *}(\mathcal{G}_\bullet) \xrightarrow{\sim} \mathbb{R}f_{\bullet *}\left(f^{-1}\mathcal{F}_\bullet^{(\bullet)} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{X_\bullet}(Z)} \mathcal{G}_\bullet\right). \tag{9.2.3.3.1}$$

Proof. As 5.1.2.8, this is a consequence of 9.2.3.1. □

Proposition 9.2.3.4. Let $\mathcal{F} \in D({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ and $\mathcal{G} \in D({}^{\mathrm{l}}f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$.

(i) We have the canonical morphism in $D(\mathbb{Z}_{\mathfrak{X}})$:

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} \mathbb{R}f_*(\mathcal{G}) \to \mathbb{R}f_*\left(f^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}} \mathcal{G}\right). \tag{9.2.3.4.1}$$

Let $\mathcal{D}$ be a sheaf of rings on $\mathfrak{X}$ such that $(\mathcal{D}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ is solvable by $\mathcal{R}$ and $\mathcal{F} \in D(\mathcal{D}, \mathcal{R}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$. Then the morphism 9.2.3.4.1 can also be viewed as a morphism of $D(\mathcal{D})$.

(ii) Suppose f is quasi-compact and quasi-separated. Suppose moreover $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$. Then the morphism 9.2.3.4.1 is an isomorphism.

Proof. We construct 9.2.3.4.1 as 5.1.2.5.1. The second statement is a consequence of 5.1.2.5.1. To check that this is an isomorphism, using the remark 5.1.2.1 and using [Har66, I.7.1 (ii), (iii) and (iv)] and 5.1.2.4, we reduce to the case where $\mathcal{F} = \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$, which is obvious. □

9.2.4 Direct image and duality

We keep notation 9.2.1. Assume that f is quasi-compact and quasi-separated, S, S', X, X' have finite Krull dimension,

Definition 9.2.4.1. We keep notation 9.2.1.4.

(a) The (left version of the) direct image functor of level m by $\widetilde{f}_\bullet$ is the functor $\widetilde{f}^{(\bullet)}_{\bullet +}\colon D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')) \to D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$ which is defined by setting

$$\widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{E}'_\bullet) := \mathbb{R}f_{\bullet *}\left(\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet \leftarrow X'^\sharp_\bullet/S'^\sharp_\bullet}(Z,Z') \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')} \mathcal{E}'_\bullet\right),$$

where $\mathcal{E}'_\bullet \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$.

(b) The (right version of the) direct image functor of level m by $\widetilde{f}_\bullet$ is the functor $\widetilde{f}^{(\bullet)}_{\bullet +}\colon D({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')) \to D({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$ which is defined by setting

$$\widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{M}'_\bullet) := \mathbb{R}f_{\bullet *}\left(\mathcal{M}_\bullet \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')} \widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet \rightarrow X^\sharp_\bullet/S^\sharp_\bullet}(Z',Z)\right),$$

where $\mathcal{M}'_\bullet \in D({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z'))$.

(c) For any $* \in \{r,l\}$, the direct image functor of level m by $\widetilde{f}$ of the form $\widetilde{f}^{(\bullet)}_{+}\colon D({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')) \to D({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ is defined by setting

$$\widetilde{f}^{(\bullet)}_{+}(\mathcal{E}') := \mathbb{R}\underleftarrow{l}_{Y *}(\widetilde{f}^{(\bullet)}_{\bullet +}(\mathbb{L}\underleftarrow{l}^{*}_{X}\mathcal{E}'))$$

where $\mathcal{E}' \in D({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$.

9.2.4.2. For any $\mathcal{E}'^{(\bullet)} \in D^{-}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$, we have the isomorphism of $D^{-}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$:

$$\widetilde{f}^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}f_{*}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}'^\sharp \leftarrow \mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z,Z') \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')} \mathcal{E}'^{(\bullet)}). \tag{9.2.4.2.1}$$

where the tensor product is defined at 8.5.4.8.3, and similarly for complexes of right modules.

Proposition 9.2.4.3. *We have the following properties.*

(a) For any $\mathcal{M}' \in D({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$, we have the canonical morphism

$$\widetilde{f}^{(\bullet)}_{\mathrm{alg}\,+}(\mathcal{M}') := \mathbb{R}f_{*}\left(\mathcal{M}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \rightarrow \mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}\right) \to \widetilde{f}^{(\bullet)}_{+}(\mathcal{M}'). \tag{9.2.4.3.1}$$

(b) For any $\mathcal{E}' \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$, we have the canonical morphism

$$\widetilde{f}^{(\bullet)}_{\mathrm{alg}\,+}(\mathcal{E}') := \mathbb{R}f_{*}\left(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp \leftarrow \mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z,Z') \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')} \mathcal{E}'\right) \to \widetilde{f}^{(\bullet)}_{+}(\mathcal{E}'). \tag{9.2.4.3.2}$$

(c) For any $* \in \{r,l\}$, if $\mathcal{E}'$ and respectively $\mathcal{M}'$ belong to $D^{-}_{\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$, then the morphisms 9.2.4.3.1 and 9.2.4.3.2 are isomorphisms.

Proof. We can copy 7.5.8.2. □

9.2.4.4. We have the following boundedness preservation results.

(a) Since S is a noetherian scheme of finite Krull dimension, then it follows from 5.1.2.4.i that we get the factorization $\widetilde{f}^{(\bullet)}_{\bullet +}\colon D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')) \to D^{-}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z))$.

(b) Suppose log-structures are trivial. Then, by copying the proof of 5.3.2.4, we can check the right (resp. left) $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)$-module $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}\leftarrow X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z,Z')$ (resp. $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}\to X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z',Z)$) has finite tor-dimension. Hence, we get the induced functor

$$\widetilde{f}^{(\bullet)}_{\bullet +}\colon D^{\mathrm{b}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')) \to D^{\mathrm{b}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)). \tag{9.2.4.4.1}$$

Proposition 9.2.4.5. *Let $* \in \{r,l\}$. Then, we have the following properties.*

(a) For any $\mathcal{E}_{\bullet} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z'))$, $\widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{E}_{\bullet}) \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z))$.

(b) For any $\mathcal{E} \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$, $\widetilde{f}^{(\bullet)}_{+}(\mathcal{E}) \in D^{-}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$.

Proof. This is checked similarly to 7.5.8.7. □

9.2.4.6. For any $* \in \{r,l\}$, for any $\mathcal{E}^{(\bullet)}_{\bullet} \in D({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z'))$, it follows from 7.1.3.15.2 and 7.1.3.6.1 that we have for any $i, m \in \mathbb{N}$:

$$\big(\widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{E}^{(\bullet)}_{\bullet})\big)^{(m)}_{i} \xrightarrow{\sim} \widetilde{f}^{(m)}_{i+}(\mathcal{E}^{(m)}_{i}). \tag{9.2.4.6.1}$$

9.2.4.7. Suppose the bottom square of 9.2.1.0.1 is the identity and f is log-étale. Then, it follows from 9.2.1.22 that for any $\mathcal{M}'_{\bullet} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z'))$ and for any $\mathcal{E}'_{\bullet} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z'))$ we have the isomorphisms:

$$\widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{M}_{\bullet}) \xrightarrow{\sim} \mathbb{R}f_{\bullet *}(\mathcal{M}_{\bullet}), \qquad \widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{E}_{\bullet}) \xrightarrow{\sim} \mathbb{R}f_{\bullet *}(\mathcal{M}_{\bullet}).$$

9.2.4.8. The functors of 9.2.4.1 are compatible with the quasi-inverse functors $-\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)} \widetilde{\omega}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)^{-1}$ and $\widetilde{\omega}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)} -$ exchanging left and right $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)$-module structures. More precisely, similarly to 7.5.8.10.1, for any $\mathcal{E}'_{\bullet} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z'))$ we construct the canonical isomorphism

$$\widetilde{\omega}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)} \widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{E}'_{\bullet}) \xrightarrow{\sim} \widetilde{f}^{(\bullet)}_{\bullet +}(\widetilde{\omega}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_{\bullet}}(Z')} \mathcal{E}'_{\bullet}). \tag{9.2.4.8.1}$$

Let $\star \in \{-,\mathrm{b}\}$, $\mathcal{E}' \in D^{\star}_{\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$. Since $\mathbb{L}\underleftarrow{l}^{*}_{X'}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')} \mathcal{E}') \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_{\bullet}}(Z')} \mathcal{E}'_{\bullet}$ and since $\mathbb{L}\underleftarrow{l}^{*}_{X}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} \widetilde{f}^{(\bullet)}_{+}(\mathcal{E}')) \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X_{\bullet}}(Z)} \widetilde{f}^{(\bullet)}_{\bullet +}(\mathcal{E}'_{\bullet})$, then we deduce from 9.2.4.8.1 the canonical isomorphism

$$\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} \widetilde{f}^{(\bullet)}_{+}(\mathcal{E}') \xrightarrow{\sim} \widetilde{f}^{(\bullet)}_{+}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')\otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')} \mathcal{E}'). \tag{9.2.4.8.2}$$

Proposition 9.2.4.9. *With notation 9.2.1.12, we suppose X'' has finite Krull dimension and g is quasi-compact and quasi-separated. Let $* \in \{r,l\}$, and $\star \in \{-,\mathrm{b}\}$. We have the following properties.*

1. *For any $\mathcal{E}'_{\bullet} \in D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}})$, we have the canonical isomorphism of $D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X''^{\sharp}_{\bullet}/S''^{\sharp}_{\bullet}}(Z''))$:*

$$\widetilde{f}^{(\bullet)}_{\bullet +}\circ \widetilde{g}^{(\bullet)}_{\bullet +}(\mathcal{E}'_{\bullet}) \xrightarrow{\sim} \widetilde{(f\circ g)}^{(\bullet)}_{\bullet +}(\mathcal{E}'_{\bullet}). \tag{9.2.4.9.1}$$

2. *For any $\mathcal{E}' \in D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$, we have the canonical isomorphism of $D^{\star}_{\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}''^{\sharp}/\mathfrak{S}''^{\sharp}}(Z''))$:*

$$\widetilde{f}^{(\bullet)}_{+}\circ \widetilde{g}^{(\bullet)}_{+}(\mathcal{E}') \xrightarrow{\sim} \widetilde{(f\circ g)}^{(\bullet)}_{+}(\mathcal{E}'). \tag{9.2.4.9.2}$$

Proof. By using 9.2.3.1 and 9.2.1.13.2, we can check 9.2.4.9.1 by copying the proof of 5.1.3.8. By using the equivalence of categories 8.5.1.10 (see also example 8.5.1.1), this yields 9.2.4.9.2. ☐

9.2.4.10. For any $* \in \{r, l\}$, similarly to 8.5.4.8.1, the pushforward functors $\widetilde{f}^{(\bullet)}_{\bullet +}$ and $\widetilde{f}^{(\bullet)}_{+}$ of 9.2.4.1 send lim-ind-isogenies to lim-ind-isogenies and preserve the quasi-coherence. This yields the factorizations denoted by:

$$(f,\phi)^{(\bullet)}_{Z,Z',\bullet +} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S'^{\sharp}_{\bullet}}(Z')) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z)), \tag{9.2.4.10.1}$$

$$(f,\phi)^{(\bullet)}_{Z,Z'+} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)). \tag{9.2.4.10.2}$$

They are called the *push forward* by (f,ϕ) (or simply f when ϕ is understood) with overconvergent singularities along Z and Z'. When $\phi = \mathrm{id}$, we will see that the functor $f^{(\bullet)}_{Z,Z+}$ preserves bounded complexes (see 9.4.2.3), even when log structures are not trivial.

9.2.4.11. For any $* \in \{r, l\}$, the pushforward functors $\widetilde{f}^{(\bullet)}_{\mathrm{alg}\,+}$ of 9.2.4.3 send lim-ind-isogenies to lim-ind-isogenies (but do not preserve the quasi-coherence). Hence, they induces the functors:

$$(f,\phi)^{(\bullet)}_{\mathrm{alg}\,Z,Z'+} \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)). \tag{9.2.4.11.1}$$

The following lemma improves 9.2.4.3.(c).

Lemma 9.2.4.12. *Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{alg}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$. The canonical morphism $f^{(\bullet)}_{\mathrm{alg}\,Z,Z'+}(\mathcal{E}'^{(\bullet)}) \to f^{(\bullet)}_{Z,Z'+}(\mathcal{E}'^{(\bullet)})$ is an isomorphism in $\underrightarrow{LD}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$.*

Proof. We proceed similarly to 9.2.1.17 (and use 7.5.8.2). ☐

Notation 9.2.4.13 (Pushforwards)**.** Let $* \in \{\mathrm{l}, \mathrm{r}\}$. The direct image by (f,ϕ) (or simply f when ϕ is understood) with overconvergent singularities along Z and Z' is also a functor of the form $f_{Z,Z',+} \colon D^{-}({}^{*}\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}}) \to D^{-}({}^{*}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$ which is defined for any $\mathcal{E}' \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$ and $\mathcal{M}' \in D^{-}({}^{\mathrm{r}}\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$ by setting:

$$(f,\phi)_{Z,Z'+}(\mathcal{E}') := \mathbb{R}f_{*}\left(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}\to\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z,Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}}} \mathcal{E}'\right), \tag{9.2.4.13.1}$$

$$(f,\phi)_{Z,Z'+}(\mathcal{M}') := \mathbb{R}f_{*}\left(\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}\to\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z',Z)_{\mathbb{Q}}\right). \tag{9.2.4.13.2}$$

Notation 9.2.4.14. In the notation 9.2.4.13 and 9.2.4.10 of the pushforwards, when $Z' = f^{-1}(Z)$, we simply write respectively $f^{(\bullet)}_{\bullet Z,+}$, $f^{(\bullet)}_{Z,+}$ and $f_{Z,+}$. If moreover Z is empty, we write $f^{(\bullet)}_{\bullet +}$, $f^{(\bullet)}_{+}$ and f_{+}.

When $Z' = f^{-1}(Z)$, we remove Z' in the notation, e.g. $(f,\phi)^{(\bullet)}_{Z,\bullet +} := (f,\phi)^{(\bullet)}_{Z,f^{-1}(Z),+}$. If moreover Z is empty, we remove Z in the notation.

When $\phi = \mathrm{id}_{\mathfrak{S}^{\sharp}}$, we replace $\mathrm{id}_{\mathfrak{S}^{\sharp}}$ by ${}_{/\mathfrak{S}^{\sharp}}$ and $\mathrm{id}_{S^{\sharp}_{\bullet}}$ by ${}_{/S^{\sharp}_{\bullet}}$, e.g. $f^{(\bullet)}_{\bullet/S^{\sharp}_{\bullet},Z,Z'+} := (f,\mathrm{id}_{S^{\sharp}_{\bullet}})^{(\bullet)}_{\bullet Z,Z'+}$ and $f^{(\bullet)}_{/\mathfrak{S}^{\sharp},Z,Z'+} := (f,\mathrm{id}_{\mathfrak{S}^{\sharp}})^{(\bullet)!}_{Z,Z'+}$. To lighten the notations, when ϕ is understood, by abuse of notation we simply write f instead of (f,ϕ), e.g. $f^{(\bullet)}_{Z,Z',\bullet +}$ instead of $(f,\phi)^{(\bullet)}_{Z,Z',\bullet +}$.

9.2.4.15 (Log-étale case)**.** Suppose the bottom square of 9.2.1.0.1 is the identity and f is log-étale. Then, using 9.2.1.22 we get the following properties:

(a) For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$, we have the isomorphism:

$$f^{(\bullet)}_{Z,Z',+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}f_{*}\left(\mathcal{E}'^{(\bullet)}\right); \tag{9.2.4.15.1}$$

If f is moreover finite then the functor $f^{(\bullet)}_{Z,Z',+}$ is t-exact.

(b) For any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$, we have the isomorphism:

$$f_{Z,Z',+}(\mathcal{E}') \xrightarrow{\sim} \mathbb{R}f_*\mathcal{E}'. \tag{9.2.4.15.2}$$

If f is moreover finite then the functor $f_{Z,Z',+}$ is exact.

Remark 9.2.4.16. This is not clear (except when complexes are coherent thanks to 9.2.4.8.2) that the functors of 9.2.4.13 commute with the quasi-inverse functors $-\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}}$ and $\omega_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}-$ exchanging left and right $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}}$-module structures. However, when f is an exact closed immersion, this will be checked later (see 9.3.2.2.1).

Proposition 9.2.4.17. *Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$. The functors 9.2.4.10.2 and 9.2.1.21.1 are compatible with the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ (see notation 9.1.6.1), i.e. we have the functorial isomorphism:*

$$f_{Z,Z',+}\circ \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \underrightarrow{l}^{*}_{\mathbb{Q}}\circ f^{(\bullet)}_{Z,Z'+}(\mathcal{E}'^{(\bullet)}).$$

Proof. This follows from 9.2.4.12. □

Remark 9.2.4.18. With notation 9.1.6.6, the proposition 9.2.4.17 means that we have the isomorphism of functors $\mathrm{Coh}_{Z'}(f^{(\bullet)}_{Z,Z',+}) \xrightarrow{\sim} f_{Z,Z',+}$.

Proposition 9.2.4.19. *Let $D \subset Z$ and $D' \subset Z'$ be some divisors of respectively X and X' such that $f(X'\setminus D') \subset X\setminus D$.*

(a) Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z'))$. We have the canonical isomorphism

$$\mathrm{forg}_{D,Z}\circ f^{(\bullet)}_{Z,Z',+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f^{(\bullet)}_{D,D',+}\circ \mathrm{forg}_{D',Z'}(\mathcal{E}'^{(\bullet)}).$$

Hence, it is harmless to write by abuse of notation $f^{(\bullet)}_{+}$ instead of $f^{(\bullet)}_{Z,Z',+}$.

(b) Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp},\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$. We have the canonical isomorphism

$$\mathrm{forg}_{D,Z}\circ f_{Z,Z',+}(\mathcal{E}') \xrightarrow{\sim} f_{D,D',+}\circ \mathrm{forg}_{D',Z'}(\mathcal{E}').$$

Hence, it is harmless to write by abuse of notation f_{+} instead of $f_{Z,Z',+}$ for coherent complexes.

Proof. It follows from 5.1.3.3 that we have the canonical isomorphism $\mathrm{forg}_{Z}\circ f^{(\bullet)}_{Z,Z',+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f^{(\bullet)}_{+}\circ \mathrm{forg}_{Z'}(\mathcal{E}'^{(\bullet)})$. Hence, we conclude the first part from 9.1.2.3. Using the remark 9.1.6.7 this yields that the functors $\mathrm{forg}_{D,Z}\circ \mathrm{Coh}_{Z'}(f^{(\bullet)}_{Z,Z',+})$ and $\mathrm{Coh}_{D'}(f^{(\bullet)}_{D,D',+})$ are isomorphic over $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'}({}^{\dagger}D')_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}})$. Since we have the canonical isomorphisms of functors $\mathrm{Coh}_{Z'}(f^{(\bullet)}_{Z,Z',+}) \xrightarrow{\sim} f_{Z,Z',+}$ and $\mathrm{Coh}_{D'}(f^{(\bullet)}_{D,D',+}) \xrightarrow{\sim} f_{D,D',+}$ (9.2.4.18), then we are done. □

Notation 9.2.4.20 (Duality in LD)**.** Suppose X is quasi-compact. With notation 8.6.2.7, we get the functor:

$$\mathbb{D}^{(\bullet)}_{Z} := \mathbb{R}_{\mathrm{LD}}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{(\bullet)-1}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})[\delta_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}]\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)), \tag{9.2.4.20.1}$$

$$\mathbb{D}^{(\bullet)}_{Z} := \mathbb{R}_{\mathrm{LD}}\mathscr{H}om_{\mathcal{D}^{(\bullet)}}(-,\omega^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))[\delta_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}]\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{r}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)), \tag{9.2.4.20.2}$$

that we call the dual functor. It follows from 5.1.4.3.1, that for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$ we get the isomorphism:

$$\mathbb{D}^{(\bullet)}_{Z}(\omega^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \omega^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathbb{D}^{(\bullet)}_{Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.4.20.3}$$

By using 4.6.4.7.1, we can check that dual functors commute with extensions, i.e. for any divisor Z' containing Z, we have the isomorphism

$$\mathbb{D}^{(\bullet)}_{Z'}\circ({}^{\dagger}Z') \xrightarrow{\sim} ({}^{\dagger}Z')\circ\mathbb{D}^{(\bullet)}_{Z} \tag{9.2.4.20.4}$$

of functors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$.

9.2.4.21 (Empty divisor)**.** If follow from 8.7.7.10 that the dual functors of 9.2.4.20 are for any $* \in \{\mathrm{l}, \mathrm{r}\}$, we get the functor:

$$\mathbb{D}^{(\bullet)} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}), \tag{9.2.4.21.1}$$

Notation 9.2.4.22 (Duality)**.** Suppose X is quasi-compact.

(a) We have the $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-linear functor denoted by $\mathbb{D}^{\mathrm{alg}}_{\mathfrak{X}^\sharp,Z} \colon D_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ defined by setting for any $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$

$$\mathbb{D}^{\mathrm{alg}}_{\mathfrak{X}^\sharp,Z}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}))[d_{X/S}]. \tag{9.2.4.22.1}$$

If there is no risk of confusion, we simply write $\mathbb{D}^{\mathrm{alg}}_Z := \mathbb{D}^{\mathrm{alg}}_{\mathfrak{X}^\sharp,Z}$.

(b) We have the $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-linear functor denoted by $\mathbb{D}_{\mathfrak{X}^\sharp,Z} \colon D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ defined by setting for any $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$

$$\mathbb{D}_{\mathfrak{X}^\sharp,Z}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}))[d_{X/S}]. \tag{9.2.4.22.2}$$

If there is no risk of confusion, we simply write $\mathbb{D}_Z := \mathbb{D}_{\mathfrak{X}^\sharp,Z}$.

(c) Similarly to 8.7.7.3, we can check we have the biduality isomorphism $\mathbb{D}_{\mathfrak{X}^\sharp,Z} \circ \mathbb{D}_{\mathfrak{X}^\sharp,Z}(\mathcal{G}) \xrightarrow{\sim} \mathcal{G}$ for any $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$.

(d) We have the functor $({}^\dagger Z) \colon D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp\mathbb{Q}}) \to D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ defined by setting $({}^\dagger Z)(\mathcal{G}) := \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp\mathbb{Q}}} \mathcal{G}$ for any $\mathcal{G} \in D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp\mathbb{Q}})$. Dual functors commute with extensions (see 4.6.4.7.1), i.e. for any divisor Z' containing Z, we have the isomorphism

$$\mathbb{D}_{Z'} \circ ({}^\dagger Z', Z) \xrightarrow{\sim} ({}^\dagger Z', Z) \circ \mathbb{D}_Z \tag{9.2.4.22.3}$$

of functors $D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z')_{\mathbb{Q}})$.

(e) Following 8.7.7.9, when log structure are trivial, we get $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) = D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) = D_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ (recall X is quasi-compact).

(f) For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, it follows from 8.6.2.7.3 that we have the isomorphism

$$\underrightarrow{l}^*_{\mathbb{Q}} \circ \mathbb{D}^{(\bullet)}_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{D}_Z \circ \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}).$$

9.2.5 External tensor products

Let $\mathfrak{S}^\sharp$ be a nice (see definition 3.3.1.10) fine $\mathcal{V}$-log formal scheme. Assume that $\mathfrak{S}$ has finite Krull dimension. Let $\mathfrak{X}^\sharp$ and $\mathfrak{Q}^\sharp$ be two log smooth, quasi-compact and quasi-separated formal log schemes over $\mathfrak{S}^\sharp$, $p \colon \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{X}^\sharp$, $q \colon \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{Q}^\sharp$ be the structural maps. Let $* \in \{\mathrm{r}, \mathrm{l}\}$.

We suppose S, $S^\sharp$, X and Q are regular. Let Z_1 be a divisor of X, Z_2 be a divisor of Q and $Z := p^{-1}(Z_1) \cup q^{-1}(Z_2)$. Set $\mathfrak{R}^\sharp := \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp$ and $r \colon \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{S}^\sharp$ be the structural map.

9.2.5.1. We get the bifunctor

$$- \boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2} - \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z_1)) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{Q^\sharp_\bullet/S^\sharp_\bullet}(Z_2)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{R^\sharp_\bullet}(Z)), \tag{9.2.5.1.1}$$

for any $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z_1))$, $\mathcal{F}^{(\bullet)}_\bullet \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{Q^\sharp_\bullet/S^\sharp_\bullet}(Z_2))$, we set

$$\mathcal{E}^{(\bullet)}_\bullet \boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2} \mathcal{F}^{(\bullet)}_\bullet := p^{(\bullet)!}_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)}_\bullet) \otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)} q^{(\bullet)!}_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)}_\bullet)[d_R].$$

When Z_1 and Z_2 are empty, we simply write $-\boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}-$.

Using the tensor product defined in 9.1.2.6.2, we get the bifunctor

$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))\times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))\to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp}(Z)), \tag{9.2.5.1.2}$$

defined as follows: for any $\mathcal{E}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$, we set

$$\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}:=\mathbb{R}\underleftarrow{l}_{\mathfrak{R}^{(\mathbb{N})}*}\left(\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathbb{L}\underleftarrow{l}^*_{\mathfrak{Q}^{(\mathbb{N})}}(\mathcal{F}^{(\bullet)})\right)\xrightarrow{\sim} p^{(\bullet)!}_{Z,Z_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{R}}(Z)}q^{(\bullet)!}_{Z,Z_2}(\mathcal{F}^{(\bullet)})[d_R]. \tag{9.2.5.1.3}$$

When Z_1 and Z_2 are empty, we simply write $-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}-$.

Similarly we define the functor:

$$-\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}-\colon \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))\times \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))\to \underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}(Z)), \tag{9.2.5.1.4}$$

defined as follows: for any $\mathcal{E}^{(\bullet)}\in \underrightarrow{LM}_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LM}_{\mathbb{Q}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$, we set

$$\mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}:=p^{*(\bullet)}_{Z,Z_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}_{\mathcal{B}^{(\bullet)}_{\mathfrak{R}}(Z)}q^{*(\bullet)}_{Z,Z_2}(\mathcal{F}^{(\bullet)}). \tag{9.2.5.1.5}$$

9.2.5.2. It follows from 7.5.9.4 (or from 9.1.2.7.1 and 9.2.1.18) that for any $\mathcal{E}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp})$, we have the isomorphism

$$({}^\dagger Z)\left(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)}\right)\xrightarrow{\sim}({}^\dagger Z_1)(\mathcal{E}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}({}^\dagger Z_2)(\mathcal{F}^{(\bullet)}). \tag{9.2.5.2.1}$$

9.2.5.3. Modulo the forgetful functors of divisors forg_{Z_1}, forg_{Z_2}, forg_{Z} which are fully faithful, the external tensor products do not depend on the divisors Z_1 or Z_2. Indeed, it follows from 9.1.2.8.1 and 9.2.1.28 that for any $\mathcal{E}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp})$, by omitting the indication of the functor forg_{Z_1}, forg_{Z_2} and forg_{Z} we get the isomorphisms

$$\begin{aligned}
\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)} &\overset{9.2.5.1.5}{\xrightarrow{\sim}} p^{(\bullet)!}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{R}}}q^{(\bullet)!}(\mathcal{F}^{(\bullet)})[d_R]\\
&\underset{9.2.1.28}{\xrightarrow{\sim}}\left(({}^\dagger p^{-1}Z_1)(p^{(\bullet)!}(\mathcal{E}^{(\bullet)}))\right)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{R}}}\left(({}^\dagger q^{-1}Z_2)(q^{(\bullet)!}(\mathcal{F}^{(\bullet)}))\right)[d_R]\\
&\underset{9.1.3.2}{\xrightarrow{\sim}}({}^\dagger Z)\left(p^{(\bullet)!}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{R}}}q^{(\bullet)!}(\mathcal{F}^{(\bullet)})\right)[d_R]\underset{9.2.5.1.5}{\xrightarrow{\sim}}({}^\dagger Z)(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)})\\
&\underset{9.2.5.2.1}{\xrightarrow{\sim}}({}^\dagger Z_1)(\mathcal{E}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}({}^\dagger Z_2)\mathcal{F}^{(\bullet)}\underset{9.1.2.3}{\xrightarrow{\sim}}\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}.
\end{aligned}\tag{9.2.5.3.1}$$

By abuse of notation, it is harmless to avoid indicating the divisors Z_1 and Z_2 in the notation $-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}-$.

9.2.5.4. Set $\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2):=\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)\underset{\mathrm{top}}{\boxtimes}\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2):=p^{-1}\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)\otimes_{\mathcal{O}_{S_\bullet}}q^{-1}\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)$. Set $\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2):=\mathcal{D}^{(\bullet)}_{X^\sharp_\bullet}(Z_1)\underset{\mathrm{top}}{\boxtimes}\mathcal{D}^{(\bullet)}_{Q^\sharp_\bullet}(Z_2):=p^{-1}\mathcal{D}^{(\bullet)}_{X^\sharp_\bullet}(Z_1)\otimes_{\mathcal{O}_{S_\bullet}}q^{-1}\mathcal{D}^{(\bullet)}_{Q^\sharp_\bullet}(Z_2)$.

(a) Since $\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)$ and $\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)$ are flat over $\mathcal{O}_{S_\bullet}$, then similarly to 5.1.5.4.2, for any $\mathcal{E}^{(\bullet)}_\bullet\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z_1))$, $\mathcal{F}^{(\bullet)}_\bullet\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{Q^\sharp_\bullet/S^\sharp_\bullet}(Z_2))$, we get the isomorphisms:

$$\begin{aligned}
\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}^{\mathbb{L}}\mathcal{F}^{(\bullet)}_\bullet &:=\left(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\otimes^{\mathbb{L}}_{p^{-1}\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\right)\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}\left(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\otimes^{\mathbb{L}}_{q^{-1}\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)}q^{-1}\mathcal{F}^{(\bullet)}_\bullet\right)\\
&\xrightarrow{\sim}\left(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\otimes^{\mathbb{L}}_{p^{-1}\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\right)\otimes^{\mathbb{L}}_{q^{-1}\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)}q^{-1}\mathcal{F}^{(\bullet)}_\bullet\xrightarrow{\sim}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\otimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}q^{-1}\mathcal{F}^{(\bullet)}_\bullet.
\end{aligned}\tag{9.2.5.4.1}$$

When $*=\mathrm{l}$, viewing $\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)$ as a $({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2),{}^{\mathrm{r}}p^{-1}\mathcal{D}^{(\bullet)}_{X^\sharp_\bullet}(Z_1),{}^{\mathrm{r}}q^{-1}\mathcal{D}^{(\bullet)}_{Q^\sharp_\bullet}(Z_2))$-trimodule, we get the $\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)$-linear isomorphism:

$$p^{-1}\mathcal{E}^{(\bullet)}_\bullet\otimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}q^{-1}\mathcal{F}^{(\bullet)}_\bullet\xrightarrow{\sim}\left(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)\otimes^{\mathbb{L}}_{p^{-1}\mathcal{D}^{(\bullet)}_{X^\sharp_\bullet}(Z_1)}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\right)\otimes^{\mathbb{L}}_{q^{-1}\mathcal{D}^{(\bullet)}_{Q^\sharp_\bullet}(Z_2)}q^{-1}\mathcal{F}^{(\bullet)}_\bullet. \tag{9.2.5.4.2}$$

(b) Similarly to 5.1.5.2.2, for any $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z_1))$, $\mathcal{F}^{(\bullet)}_\bullet \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{Q^\sharp_\bullet/S^\sharp_\bullet}(Z_2))$, we get the isomorphism:

$$\mathcal{E}^{(\bullet)}_\bullet \underset{\mathrm{top}}{\boxtimes} \mathcal{F}^{(\bullet)}_\bullet := \left(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\otimes_{p^{-1}\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)} p^{-1}\mathcal{E}^{(\bullet)}_\bullet\right)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}\left(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\otimes_{q^{-1}\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)} q^{-1}\mathcal{F}^{(\bullet)}_\bullet\right)$$
$$\xrightarrow{\sim} p^{-1}\mathcal{E}^{(\bullet)}_\bullet \otimes_{\mathcal{O}_{S_\bullet}} q^{-1}\mathcal{F}^{(\bullet)}_\bullet. \tag{9.2.5.4.3}$$

9.2.5.5. The functor $\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}-\colon \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2))\to\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{B}^{(\bullet)}_{R_\bullet}(Z))$ is exact. Moreover the functor $\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2))\to\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\mathcal{B}^{(\bullet)}_{R_\bullet}(Z))$ is well defined and is isomorphic to $\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}-$. Indeed, setting $\mathcal{O}^{(\bullet)}_{R_\bullet,\mathrm{top}} := p^{-1}\mathcal{O}^{(\bullet)}_{X_\bullet}\otimes_{\mathcal{O}_{S^{(\bullet)}_\bullet}} q^{-1}\mathcal{O}^{(\bullet)}_{Q_\bullet}$, similarly to 5.1.5.2.2, we check the isomorphism

$$\mathcal{O}^{(\bullet)}_{R_\bullet}\otimes_{\mathcal{O}^{(\bullet)}_{R_\bullet,\mathrm{top}}}\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\xrightarrow{\sim} p^*_\bullet\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)\otimes_{\mathcal{O}_{R_\bullet}} q^*_\bullet\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2) =: \mathcal{B}^{(\bullet)}_{R_\bullet}(Z_1,Z_2).$$

Since the extension $\mathcal{O}^{(\bullet)}_{R_\bullet,\mathrm{top}}\to\mathcal{O}^{(\bullet)}_{R_\bullet}$ is flat, then so is $\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)\to\mathcal{B}^{(\bullet)}_{R_\bullet}(Z_1,Z_2)$. Moreover, following 9.1.3.(d), the canonical ring homomorphism

$$\mathcal{B}^{(\bullet)}_{\mathfrak{R}}(Z_1,Z_2) := p^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z_1)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{R}}} q^*\mathcal{B}^{(\bullet)}_{\mathfrak{Q}}(Z_2)\xrightarrow{\sim}\mathcal{B}^{(\bullet)}_{\mathfrak{R}}(p^{-1}Z_1)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{R}}}\mathcal{B}^{(\bullet)}_{\mathfrak{R}}(q^{-1}Z_2)\to\mathcal{B}^{(\bullet)}_{\mathfrak{R}}(Z) \tag{9.2.5.5.1}$$

is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{qc}}(\mathcal{O}^{(\bullet)}_{\mathfrak{R}^\sharp})$. This yields that the canonical ring homomorphism

$$\mathcal{B}^{(\bullet)}_{R_\bullet}(Z_1,Z_2) := p^*_\bullet\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)\otimes_{\mathcal{O}_{R_\bullet}} q^*_\bullet\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)\to\mathcal{B}^{(\bullet)}_{R_\bullet}(Z) \tag{9.2.5.5.2}$$

is an isomorphism of $\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{O}}^{(\bullet)}_{R^\sharp_\bullet})$. Hence, we are done.

9.2.5.6. Similarly to 9.2.5.5, the functor $\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes_{\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)}-\colon \underrightarrow{LM}_{\mathbb{Q}}(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2))\to\underrightarrow{LM}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z))$ is exact. Moreover the functor $\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes_{\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2))\to\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z))$ is well defined and is isomorphic to $\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)}-$. Moreover, for any $\mathcal{G}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2))$, the canonical morphism

$$\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}\mathcal{G}^{(\bullet)}\to\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes_{\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)}\mathcal{G}^{(\bullet)}$$

is an isomorphism.

9.2.5.7. Let $\mathcal{E}^{(\bullet)}_\bullet\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z_1))$, $\mathcal{F}^{(\bullet)}_\bullet\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{Q^\sharp_\bullet/S^\sharp_\bullet}(Z_2))$. We get the isomorphisms

$$\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes_{\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)}\left(\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}{}^{\mathbb{L}}\mathcal{F}^{(\bullet)}_\bullet\right)\xrightarrow{\sim}\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet,\mathrm{top}}(Z_1,Z_2)}\left(\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}{}^{\mathbb{L}}\mathcal{F}^{(\bullet)}_\bullet\right)$$
$$\xrightarrow{\sim}\left(\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes^{\mathbb{L}}_{p^{-1}\mathcal{B}^{(\bullet)}_{X_\bullet}(Z_1)}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\right)\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}\left(\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes^{\mathbb{L}}_{q^{-1}\mathcal{B}^{(\bullet)}_{Q_\bullet}(Z_2)}q^{-1}\mathcal{F}^{(\bullet)}_\bullet\right)$$
$$\xrightarrow{\sim} p^{(\bullet)!}_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)}_\bullet)\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}q^{(\bullet)!}_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)}_\bullet)[d_R]=\mathcal{E}^{(\bullet)}_\bullet\boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathcal{F}^{(\bullet)}_\bullet. \tag{9.2.5.7.1}$$

Similarly, for any $\mathcal{E}^{(\bullet)}_\bullet\in\underrightarrow{LM}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z_1))$, $\mathcal{F}^{(\bullet)}_\bullet\in\underrightarrow{LM}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{Q^\sharp_\bullet/S^\sharp_\bullet}(Z_2))$, we get the isomorphism

$$\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z_1,Z_2)}(\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}\mathcal{F}^{(\bullet)}_\bullet)\xrightarrow{\sim}p^*_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)}_\bullet)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}q^*_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)}_\bullet)=:\mathcal{E}^{(\bullet)}_\bullet\widehat{\boxtimes}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathcal{F}^{(\bullet)}_\bullet.$$

By using the composition of 9.2.5.4.1 and 9.2.5.4.2, we get the last isomorphism

$$\mathcal{E}^{(\bullet)}_\bullet\boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathcal{F}^{(\bullet)}_\bullet\overset{9.2.5.7.1}{\xrightarrow{\sim}}\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes_{\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet,\mathrm{top}}(Z_1,Z_2)}\left(\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}{}^{\mathbb{L}}\mathcal{F}^{(\bullet)}_\bullet\right)$$
$$\xrightarrow{\sim}\left(\mathcal{D}^{(\bullet)}_{R^\sharp_\bullet}(Z)\otimes^{\mathbb{L}}_{p^{-1}\mathcal{D}^{(\bullet)}_{X^\sharp_\bullet}(Z_1)}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\right)\otimes^{\mathbb{L}}_{q^{-1}\mathcal{D}^{(\bullet)}_{Q^\sharp_\bullet}(Z_2)}q^{-1}\mathcal{F}^{(\bullet)}_\bullet. \tag{9.2.5.7.2}$$

9.2.5.8 (Preservation of the coherence)**.** It follows from 7.5.9.3.1 that the bifunctor exterior tensor product of 9.2.5.1.2 induces the bifunctor

$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))\times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))\to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp}(Z)). \qquad (9.2.5.8.1)$$

Lemma 9.2.5.9. *Suppose $\mathfrak{S}^\sharp=\operatorname{Spf}\mathcal{V}$. The bifunctor 9.2.5.1.4 induces therefore the exact bifunctor*

$$-\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}-\colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))\times \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))\to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}(Z)).$$

Moreover, it is compatible with the functor 9.2.5.8.1, i.e., for any $\mathcal{E}^{(\bullet)}\in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$, we have the canonical isomorphism

$$\mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}\xrightarrow{\sim}\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}.$$

Proof. 1) Let $\mathcal{E}^{(\bullet)}\in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$. It follows from 8.4.5.12(a) that we can suppose $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ are p-torsion free. If no confusion is possible, the sheaf $r^{-1}\mathcal{O}_{S_\bullet}$ will be denoted by $\mathcal{O}_{S_\bullet}$. Since $\mathcal{E}^{(m)}$ and $\mathcal{F}^{(m)}$ are p-torsion free for any $m\geq 0$, then $\mathcal{E}^{(\bullet)}_\bullet:=\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)})$ and $\mathcal{F}^{(\bullet)}_\bullet:=\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{F}^{(\bullet)})\xrightarrow{\sim}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{F}^{(\bullet)})$. Using again the p-torsion freeness, we get the middle isomorphism

$$\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}{}^{\mathbb{L}}\mathcal{F}^{(\bullet)}_\bullet\overset{9.2.5.4.1}{\xrightarrow{\sim}}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\otimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet}}q^{-1}\mathcal{F}^{(\bullet)}_\bullet\xrightarrow{\sim}p^{-1}\mathcal{E}^{(\bullet)}_\bullet\otimes_{\mathcal{O}_{S_\bullet}}q^{-1}\mathcal{F}^{(\bullet)}_\bullet\overset{9.2.5.4.3}{\xrightarrow{\sim}}\mathcal{E}^{(\bullet)}_\bullet\underset{\mathrm{top}}{\boxtimes}\mathcal{F}^{(\bullet)}_\bullet. \quad (9.2.5.9.1)$$

By using 9.2.5.9.1 and 9.2.5.7, we get

$$\mathcal{E}^{(\bullet)}_\bullet\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathcal{F}^{(\bullet)}_\bullet\xrightarrow{\sim}\mathcal{E}^{(\bullet)}_\bullet\widehat{\boxtimes}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathcal{F}^{(\bullet)}_\bullet. \qquad (9.2.5.9.2)$$

This yields the first isomorphism

$$\begin{aligned}\mathbb{R}\underleftarrow{l}_{\mathfrak{R}^{(\mathbb{N})}*}(\mathcal{E}^{(\bullet)}_\bullet\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}\mathcal{F}^{(\bullet)}_\bullet)&\overset{9.2.5.9.2}{\xrightarrow{\sim}}\mathbb{R}\underleftarrow{l}_{\mathfrak{R}^{(\mathbb{N})}*}(p^*_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)}_\bullet)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}q^*_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)}_\bullet))\\ &\xrightarrow{\sim}\underleftarrow{l}_{\mathfrak{R}^{(\mathbb{N})}*}(p^*_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)}_\bullet)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}q^*_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)}_\bullet))\xrightarrow{\sim}\mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}\end{aligned} \quad (9.2.5.9.3)$$

where the second one is checked by using Mittag-Leffler. □

Corollary 9.2.5.10. *Suppose $\mathfrak{S}^\sharp=\operatorname{Spf}\mathcal{V}$. We get the t-exact bifunctor*

$$\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\colon D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1)))\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2)))\to D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}(Z))). \quad (9.2.5.10.1)$$

Proposition 9.2.5.11. Suppose $\mathfrak{S}^\sharp=\operatorname{Spf}\mathcal{V}$ and $\mathfrak{X}$ affine. Let $\mathcal{E}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$. Choose $\mathscr{E}^{(\bullet)}\in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1)))$, $\mathscr{F}^{(\bullet)}\in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2)))$ such that $\mathfrak{e}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim}\mathscr{E}^{(\bullet)}$ and $\mathfrak{e}(\mathcal{F}^{(\bullet)})\xrightarrow{\sim}\mathscr{F}^{(\bullet)}$ (see 8.4.5.11). Then we have the isomorphism of $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}(Z)))$:

$$\mathscr{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathscr{F}^{(\bullet)}\xrightarrow{\sim}\mathfrak{e}(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}). \qquad (9.2.5.11.1)$$

Proof. Following 8.4.5.13 there exist a representant $\mathcal{E}^{(\bullet)\bullet}\in K^{\mathrm{b}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$ of $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)n}$ is p-torsion free and $\mathfrak{e}(\mathcal{E}^{(\bullet)n})\in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$ for any $n\in\mathbb{Z}$. We can choose $\mathscr{E}^{(\bullet)\bullet}:=\mathfrak{e}(\mathcal{E}^{(\bullet)\bullet})\in K^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1)))$ as a representant of $\mathscr{E}^{(\bullet)}$; and similarly we choose such an $\mathcal{F}^{(\bullet)\bullet}$ and $\mathscr{F}^{(\bullet)\bullet}:=\mathfrak{e}(\mathcal{F}^{(\bullet)\bullet})$. Set $\mathcal{E}^{(\bullet)\bullet}_\bullet:=\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)\bullet})\xrightarrow{\sim}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{E}^{(\bullet)\bullet})$ and $\mathcal{F}^{(\bullet)\bullet}_\bullet:=\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{F}^{(\bullet)\bullet})\xrightarrow{\sim}\underleftarrow{l}^*_{\mathfrak{X}^{(\mathbb{N})}}(\mathcal{F}^{(\bullet)\bullet})$. We have the canonical morphism of $D^-(\widetilde{\mathcal{D}}^{(\bullet)}_{R^\sharp_\bullet/S^\sharp_\bullet}(Z))$:

$$p^{(\bullet)!}_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)\bullet}_\bullet)\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}q^{(\bullet)!}_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)\bullet}_\bullet)[d_R]\to p^{*(\bullet)}_{\bullet Z,Z_1}(\mathcal{E}^{(\bullet)\bullet}_\bullet)\otimes_{\mathcal{B}^{(\bullet)}_{R_\bullet}(Z)}q^{*(\bullet)}_{\bullet Z,Z_2}(\mathcal{F}^{(\bullet)\bullet}_\bullet). \qquad (9.2.5.11.2)$$

To check that 9.2.5.11.2 is an isomorphism, by induction using exact triangles, we reduce to the case where $\mathcal{E}^{(\bullet)\bullet}$ and $\mathcal{F}^{(\bullet)\bullet}$ are modules, which was already checked (see 9.2.5.9.2). Using Mittag-Leffler, we conclude by applying the functor $\mathbb{R}\underleftarrow{l}_{\mathfrak{R}^{(\mathbb{N})}*}$. □

Proposition 9.2.5.12. (a) Let $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)))$, $\mathcal{F}^{(\bullet)} \in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2)))$. We get the spectral sequence in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp})$ of the form

$$H^r(\mathcal{E}^{(\bullet)})\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2} H^s(\mathcal{F}^{(\bullet)}) =: E_2^{r,s} \Rightarrow E^n := H^n\left(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}\right).$$

In particular, when $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, this yields $H^n\left(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}\right) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2} H^n(\mathcal{F}^{(\bullet)})$.

(b) Suppose $\mathfrak{S}^\sharp = \operatorname{Spf}\mathcal{V}$ and $\mathfrak{X}$ affine. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$. We have $H^n\left(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}\right) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2} H^n(\mathcal{F}^{(\bullet)})$.

Proof. The first statement is a consequence of the t-exactness of the functor 9.2.5.10.1. Moreover, when $\mathfrak{Q}^\sharp$ is affine, this follows from Proposition 9.2.5.11 and the commutativity of the diagram 8.4.5.7.1. □

We end this subsection with exterior tensor products for coherent complexes.

9.2.5.13 (Exterior tensor products of complexes coherent)**.** With the notations of 9.1.6.6, it follows from 9.2.5.8.1 that we get the bifunctor $-\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}- := \mathrm{Coh}_{Z_1,Z_2}(-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}-)$ of the form:

$$-\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}-\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_1)_{\mathbb{Q}}) \times D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_2)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z)_{\mathbb{Q}}). \tag{9.2.5.13.1}$$

For instance

$$\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_1)_{\mathbb{Q}}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_2)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z)_{\mathbb{Q}}.$$

Remark 9.2.5.14. Let $D_1 \subset Z_1$ be a second divisor of X, $D_2 \subset Z_2$ be a second divisor of Q. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}D_1)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_1)_{\mathbb{Q}})$ and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp}({}^{\dagger}D_2)_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp}({}^{\dagger}Z_2)_{\mathbb{Q}})$. Following 9.1.6.6 and 9.2.5.3, the canonical morphism

$$\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},D_1,D_2}\mathcal{F} \to \mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F} \tag{9.2.5.14.1}$$

is an isomorphism.

Remark 9.2.5.15. Let us suppose $\mathfrak{X}^\sharp = \mathfrak{Q}^\sharp$, $Z_1 = Z_2$. Denoting by $\delta\colon \mathfrak{X}^\sharp \hookrightarrow \mathfrak{X}^\sharp \times \mathfrak{X}^\sharp$ the diagonal immersion, for any $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, then we have the canonical isomorphism

$$\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z_1)}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \delta_Z^{(\bullet)!}(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_1}\mathcal{F}^{(\bullet)})[-d_X]. \tag{9.2.5.15.1}$$

This yields, for any $\mathcal{E},\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_1)_{\mathbb{Q}})$, the canonical isomorphism

$$\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z_1)_{\mathbb{Q}}}\mathcal{F} \xrightarrow{\sim} \delta_Z^{!}(\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_1}\mathcal{F})[-d_X]. \tag{9.2.5.15.2}$$

Lemma 9.2.5.16. *We have, for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^{\dagger}Z_1)_{\mathbb{Q}})$, $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp}({}^{\dagger}Z_2)_{\mathbb{Q}})$, of the canonical isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp}({}^{\dagger}Z)_{\mathbb{Q}})$:*

$$\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F} \xrightarrow{\sim} f^*_{Z,Z_1}(\mathcal{E})\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{R}}({}^{\dagger}Z)_{\mathbb{Q}}} g^*_{Z,Z_2}(\mathcal{F}). \tag{9.2.5.16.1}$$

Proof. For any complexes $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$ such that $\mathcal{E} = \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)}$ and $\mathcal{F} = \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{F}^{(\bullet)}$, as $({}^{\dagger}Z)\circ p^{*(\bullet)}(\mathcal{E}^{(\bullet)})$ and $({}^{\dagger}Z)\circ q^{*(\bullet)}(\mathcal{F}^{(\bullet)})$ are objects of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp}(Z))$, by applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ to 9.2.5.1.5, we get 9.2.5.16.1 from 9.2.5.8.1. □

Lemma 9.2.5.17. *Let $Z_1' \supset Z_1$ be a second divisor of X, $Z_2' \supset Z_2$ be a second divisor of Q and $Z' := p^{-1}(Z_1') \cup q^{-1}(Z_2')$.*

(a) *For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$, we have the canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp}(Z'))$:*

$$({}^{\dagger}Z')\left(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)}\right) \xrightarrow{\sim} ({}^{\dagger}Z_1')(\mathcal{E}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}({}^{\dagger}Z_2')(\mathcal{F}^{(\bullet)}). \tag{9.2.5.17.1}$$

(b) *We have, for any* $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}})$, $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp}({}^\dagger Z_2)_{\mathbb{Q}})$, *the canonical isomorphism in* $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp}({}^\dagger Z')_{\mathbb{Q}})$*:*

$$({}^\dagger Z')\left(\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}\right) \xrightarrow{\sim} ({}^\dagger Z'_1)(\mathcal{E})\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z'_2}({}^\dagger Z'_2)(\mathcal{F}). \tag{9.2.5.17.2}$$

Proof. To check 9.2.5.17.1, we proceed similarly to 9.2.5.3. By applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ to 9.2.5.17.1, we obtain therefore 9.2.5.17.2. □

Proposition 9.2.5.18. *For any* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$, *we have the canonical isomorphism in* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp}(Z))$*:*

$$\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \left(\mathcal{D}^{(\bullet)}_{\mathfrak{R}^\sharp}(Z)\widehat{\otimes}^{\mathbb{L}}_{p^{-1}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z_1)}p^{-1}\mathcal{E}^{(\bullet)}\right)\widehat{\otimes}^{\mathbb{L}}_{q^{-1}\mathcal{D}^{(\bullet)}_{\mathfrak{Q}^\sharp}(Z_2)}q^{-1}\mathcal{F}^{(\bullet)}. \tag{9.2.5.18.1}$$

Proof. This follows from 9.2.5.7.2. □

Corollary 9.2.5.19. *For any* $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}})$, $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_2)_{\mathbb{Q}})$, *we have the canonical isomorphism in* $D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$*:*

$$\left(\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}\otimes_{q^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_2)_{\mathbb{Q}}} q^{-1}\mathcal{F}\right)\otimes^{\mathbb{L}}_{p^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}}} p^{-1}\mathcal{E} \xrightarrow{\sim} \mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}. \tag{9.2.5.19.1}$$

Moreover, by taking $\mathcal{E} = \mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}}$, *the isomorphism*

$$\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}\otimes_{q^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_2)_{\mathbb{Q}}} q^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F} \tag{9.2.5.19.2}$$

is $(\mathcal{D}^{\dagger}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}, p^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}})$*-bilinear.*

9.2.6 Base change and their commutation with cohomological operations

We keep notation 9.2.1 and we suppose $\phi = \mathrm{id}$. Moreover, let

$$\begin{array}{ccc} \mathfrak{Y}^\sharp & \xrightarrow{\varpi} & \mathfrak{X}^\sharp \\ \downarrow{\scriptstyle p_{\mathfrak{Y}^\sharp}} & \square & \downarrow{\scriptstyle p_{\mathfrak{X}^\sharp}} \\ \mathfrak{T}^\sharp & \xrightarrow{\psi} & \mathfrak{S}^\sharp, \end{array} \tag{9.2.6.0.1}$$

be a commutative diagram of very nice fine $\mathcal{V}$-log formal schemes. We denote by $\mathfrak{Y}'^\sharp := \mathfrak{Y}^\sharp \times_{\mathfrak{X}^\sharp} \mathfrak{X}'^\sharp$ and by $\varpi' \colon \mathfrak{Y}'^\sharp \to \mathfrak{X}'^\sharp$, $g \colon \mathfrak{Y}'^\sharp \to \mathfrak{Y}^\sharp$ the projection. We suppose T, $T^\sharp$ are regular, $D := \varpi^{-1}(Z)$ is a divisor of Y and $D' := \varpi^{-1}(Z')$ is a divisor of Y'.

9.2.6.1. Let $\star \in \{-, \mathrm{b}\}$. With notation 9.2.1.15, it follows from 9.2.1.19 that we get the functors

$$\varpi^{(\bullet)!}_{\bullet Z} := (\varpi,\psi)^{(\bullet)!}_{\bullet Z} \colon \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{Y^\sharp_\bullet/T^\sharp_\bullet}(D)), \tag{9.2.6.1.1}$$

$$\varpi^{(\bullet)!}_{Z} := (\varpi,\psi)^{(\bullet)!}_{Z} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D)). \tag{9.2.6.1.2}$$

These functors are called the *base change by* $\psi \colon \mathfrak{S}^\sharp \to \mathfrak{T}^\sharp$ *with overconvergent singularities along* Z.

9.2.6.2. The morphisms appearing in the propositions 4.4.4.1 or 7.5.6.1 are compatible with the level change. In particular, we get the composite map

$$\varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D,Z) \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D). \tag{9.2.6.2.1}$$

is a homomorphism of sheaves of rings which fits into the commutative diagram

$$\begin{array}{ccc} \varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z) & \xrightarrow{9.2.6.2.1} & \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D) \\ \uparrow & & \uparrow \\ \varpi^{-1}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z) & \longrightarrow & \mathcal{B}^{(\bullet)}_{\mathfrak{Y}}(D); \end{array} \tag{9.2.6.2.2}$$

and similarly by replacing gothic letters by their corresponding roman letter with a bullet as an index. For any $\mathcal{E}^{(\bullet)}_{\bullet} \in \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z))$ (resp. $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$), similarly to 4.4.4.1.4 (resp. 7.5.6.4.1) we get the first (resp. second) isomorphism of $D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{Y^\sharp_\bullet/T^\sharp_\bullet}(D))$ (resp. $D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D))$:

$$\widetilde{\mathcal{D}}^{(\bullet)}_{Y^\sharp_\bullet/T^\sharp_\bullet}(D) \otimes^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)} \varpi^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}_{\bullet}), \tag{9.2.6.2.3}$$

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D) \widehat{\otimes}^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \varpi^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.6.2.4}$$

Remark this do have a meaning since the functors 9.2.6.1.1 and 9.2.6.1.2 are well defined before localising by lim-ind-isogenies. It follows from 5.1.1.15.3 and 7.5.6.4.2 the isomorphisms:

$$\mathcal{O}^{(\bullet)}_{T_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{S_\bullet}} \varpi^{-1}\mathcal{E}^{(\bullet)} := p^{-1}_{\mathfrak{Y}^\sharp}\mathcal{O}^{(\bullet)}_{T_\bullet} \otimes^{\mathbb{L}}_{\varpi^{-1}p^{-1}_{\mathfrak{X}^\sharp}\mathcal{O}^{(\bullet)}_{S_\bullet}} \varpi^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}_{\bullet}), \tag{9.2.6.2.5}$$

$$\mathcal{O}^{(\bullet)}_{\mathfrak{T}} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{S}}} \varpi^{-1}\mathcal{E}^{(\bullet)} := p^{-1}_{\mathfrak{Y}^\sharp}\mathcal{O}^{(\bullet)}_{\mathfrak{T}} \widehat{\otimes}^{\mathbb{L}}_{\varpi^{-1}p^{-1}_{\mathfrak{X}^\sharp}\mathcal{O}^{(\bullet)}_{\mathfrak{S}}} \varpi^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.6.2.6}$$

9.2.6.3 (Preservation of the coherence)**.** For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, similarly to 9.2.1.17, we can check that the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D) \otimes^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \varpi^{-1}\mathcal{E}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D) \widehat{\otimes}^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \varpi^{-1}\mathcal{E}^{(\bullet)} \tag{9.2.6.3.1}$$

is an isomorphism of $\underrightarrow{LD}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D))$. With 9.2.6.2.4, this yields the isomorphism

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D) \otimes^{\mathbb{L}}_{\varpi^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)} \varpi^{-1}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.6.3.2}$$

We deduce from 9.2.6.3.2 that the base change preserves the coherence, i.e. the functor 9.2.6.1.2 induces:

$$\varpi^{(\bullet)!}_{Z} : \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(D)). \tag{9.2.6.3.3}$$

Via 9.2.1.24 and via the equivalence of categories 8.4.1.15, this yields

$$\varpi^{!}_{Z} : D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \tag{9.2.6.3.4}$$

9.2.6.4. Taking inductive limits, we deduce from 7.5.6.1 the following assertions : the canonical morphism

$$\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_{\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D, Z)_{\mathbb{Q}} \tag{9.2.6.4.1}$$

is an isomorphism. Moreover, the composite map

$$\varpi^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D, Z)_{\mathbb{Q}} \xleftarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_{\mathbb{Q}}. \tag{9.2.6.4.2}$$

is a homomorphism of sheaves of rings which fits into the commutative diagram

$$\begin{array}{ccc} \varpi^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} & \xrightarrow{9.2.6.4.2} & \mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_{\mathbb{Q}} \\ \uparrow & & \uparrow \\ \varpi^{-1}\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} & \longrightarrow & \mathcal{O}_{\mathfrak{Y}}({}^\dagger D)_{\mathbb{Q}}. \end{array} \tag{9.2.6.4.3}$$

The canonical morphism of left $\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules 9.2.6.4.1 is more precisely an isomorphism of $(\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_{\mathbb{Q}}, \varpi^{-1}\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})$-bimodules, where the structure of right $\varpi^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}$-module on $\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_{\mathbb{Q}}$ is given via 9.2.6.4.2.

9.2.6.5. For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$, we get from 9.2.6.3.2 the isomorphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}})$:

$$\varpi^{!}_{Z}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\varpi^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}} \varpi^{-1}\mathcal{E}. \tag{9.2.6.5.1}$$

For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)_{\mathbb{Q}})$, via 7.5.6.6.1 and 9.2.6.5.1 we get the isomorphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}})$:

$$\mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}',\mathbb{Q}}} \varpi^{(m)!}_{Z}(\mathcal{E}^{(m)}) \xrightarrow{\sim} \varpi^{!}_{Z}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)_{\mathbb{Q}}} \mathcal{E}^{(m)}). \tag{9.2.6.5.2}$$

Proposition 9.2.6.6. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$. With notation 9.2.1.15, there exists a canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^{\sharp}/\mathfrak{T}^{\sharp}})$ of the form:

$$\varpi'^{(\bullet)!}_{Z'} \circ f^{(\bullet)!}_{/\mathfrak{S}^{\sharp},Z',Z}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} g^{(\bullet)!}_{/\mathfrak{T}^{\sharp},D',D} \circ \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.6.6.1}$$

Proof. This follows from 9.2.1.14 and 9.2.1.19. □

Proposition 9.2.6.7. For any $\mathcal{M}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$ and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$, we have the canonical isomorphism in $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}(D))$:

$$\varpi^{(\bullet)!}_{Z}(\mathcal{M}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)} \mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z}(\mathcal{M}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{Y}}(D)} \varpi^{(\bullet)!}_{Z}(\mathcal{E}^{(\bullet)}). \tag{9.2.6.7.1}$$

Proof. This is a particular case of 9.2.1.27.1. □

Corollary 9.2.6.8. *Let $Z_1 \hookrightarrow Z_2$ be two divisor of X. Let $D_i := \varpi^{-1}(Z_i)$ for $i = 1, 2$. For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z_1))$, we have the isomorphism*

$$({}^{\dagger}D_2, D_1) \circ \varpi^{(\bullet)!}_{Z_1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \varpi^{(\bullet)!}_{Z_2} \circ ({}^{\dagger}Z_2, Z_1)(\mathcal{E}^{(\bullet)}) \tag{9.2.6.8.1}$$

Proof. This is contained in 9.2.1.18. □

Theorem 9.2.6.9. *Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$. With notation 9.2.4.14, there exists a canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}(D))$ of the form:*

$$\varpi^{(\bullet)!}_{Z} \circ f^{(\bullet)}_{/\mathfrak{S}^{\sharp},Z,Z',+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} g^{(\bullet)}_{/\mathfrak{T}^{\sharp},D,D',+} \circ \varpi'^{(\bullet)!}_{Z'}(\mathcal{E}'^{(\bullet)}). \tag{9.2.6.9.1}$$

Proof. By using the equivalence of categories 8.5.4.5,

$$\varpi^{(\bullet)!}_{\bullet Z} \circ f^{(\bullet)}_{/S^{\sharp}_{\bullet},Z,Z',+}(\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}'^{(\mathbb{N})}}(\mathcal{E}'^{(\bullet)})) \xrightarrow{\sim} g^{(\bullet)}_{/T^{\sharp}_{\bullet},D,D',+} \circ \varpi'^{(\bullet)!}_{\bullet Z'}(\mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}'^{(\mathbb{N})}}(\mathcal{E}'^{(\bullet)})) \tag{9.2.6.9.2}$$

which follows from 5.3.3.3 (more precisely, the isomorphism is even valid in $D({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{Y^{\sharp}_{\bullet}/T^{\sharp}_{\bullet}}(D))$). □

9.2.6.10. It follows from 7.5.6.5.4 by going through to the limit on the level and tensorisation with $\mathbb{Q}$ the isomorphism:

$$\mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\varpi^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}}} f^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}})_{\mathrm{r}} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \omega^{-1}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}} \tag{9.2.6.10.1}$$

where the index r means that we take the right structure of left $\mathcal{D}$-module.

Proposition 9.2.6.11. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$. We have the canonical isomorphism

$$\varpi^{!}_{Z}(\mathbb{D}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(\mathcal{E})) \xrightarrow{\sim} \mathbb{D}_{\mathfrak{Y}^{\sharp}/\mathfrak{T}^{\sharp}}(\varpi^{!}_{Z}(\mathcal{E})). \tag{9.2.6.11.1}$$

Proof. Since $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}) = D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q})$, then we have the last canonical isomorphism

$$\varpi^!_Z(\mathbb{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(\mathcal{E})) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_\mathbb{Q} \otimes^\mathbb{L}_{\varpi^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}} \varpi^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})[d_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}]$$

$$\xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_\mathbb{Q} \otimes^\mathbb{L}_{\varpi^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}} \mathbb{R}\mathcal{H}om_{\varpi^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}}(\varpi^{-1}\mathcal{E}, \varpi^{-1}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}))[d_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}]$$

$$\underset{4.6.3.6.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\varpi^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}}(\varpi^{-1}\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_\mathbb{Q} \otimes^\mathbb{L}_{\varpi^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}} (\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \omega^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathrm{r}})[d_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}]$$

$$\underset{9.2.6.10.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\varpi^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}}(\varpi^{-1}\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{Y}} \omega^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp})[d_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}]$$

$$\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_\mathbb{Q}}(\varpi^!_Z\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}({}^\dagger D)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{Y}} \omega^{-1}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}))[d_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}] = \mathbb{D}_{\mathfrak{Y}^\sharp/\mathfrak{T}^\sharp}(\varpi^!_Z(\mathcal{E})).$$

□

Definition 9.2.6.12. We define the category $\mathrm{DVR}(\mathcal{V})$ as follows: an object is the data of a complete discrete valued ring $\mathcal{W}$ of mixed characteristic $(0,p)$ together with a morphism of local algebras $\mathcal{V} \to \mathcal{W}$. A morphism $\mathcal{W} \to \mathcal{W}'$ is the data of a morphism of local $\mathcal{V}$-algebras $\mathcal{W} \to \mathcal{W}'$.

Notation 9.2.6.13 (Base change of DVR)**.** Let $\alpha\colon \mathcal{W} \to \mathcal{W}'$ be a morphism of $\mathrm{DVR}(\mathcal{V})$ (see notation 9.2.6.12), $\mathfrak{T} := \mathrm{Spf}\,\mathcal{W}$, $\mathfrak{T}' := \mathrm{Spf}\,\mathcal{W}'$. let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{T}})$, $\mathfrak{X}' := \mathfrak{X} \times_\mathfrak{T} \mathrm{Spf}\,\mathfrak{T}'$, and $\pi\colon \mathfrak{X}'/\mathfrak{T}' \to \mathfrak{X}/\mathfrak{T}$ be the morphism induced by the projection. The base change of $\mathcal{E}^{(\bullet)}$ by α is the object $\pi^{(\bullet)!}(\mathcal{E}^{(\bullet)})$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{T}'})$ According to notation 9.2.6.2.6, we can simply write

$$\mathcal{W}'\widehat{\otimes}^\mathbb{L}_\mathcal{W}\mathcal{E}^{(\bullet)} := \pi^{(\bullet)!}(\mathcal{E}^{(\bullet)}).$$

9.2.7 Coherence descent by base change of a finite morphism of complete DVR

Let $\mathcal{V}'$ be an object of $\mathrm{DVR}(\mathcal{V})$ and $\rho\colon \mathcal{V} \to \mathcal{V}'$ be its structural morphism. We set $\mathfrak{S}' := \mathrm{Spf}\,\mathcal{V}'$. Let $\mathfrak{X}^\sharp/\mathfrak{S}$ be a flat, log smooth very nice fine $\mathcal{V}$-log formal scheme, $\mathfrak{X}'^\sharp := \mathfrak{X} \times_\mathfrak{S} \mathfrak{S}'$, and $f\colon \mathfrak{X}' \to \mathfrak{X}$ be the canonical projection. We suppose X and X' are regular. Let Z be a divisor of X and $Z' := f^{-1}(Z)$. We denote by $f^!_Z$ the extraordinary inverse image of $\mathfrak{X}' \to \mathfrak{X}$ above $\mathfrak{S}' \to \mathfrak{S}$ with overconvergent singularities along Z, i.e. $f^!_Z$ is the base change functor via ρ (see 9.2.6.1).

Lemma 9.2.7.1. *We have the following properties.*

(a) For any open $\mathfrak{U}$ of $\mathfrak{X}$, setting $\mathfrak{U}' := f^{-1}(\mathfrak{U})$, the ring homomorphisms $\Gamma(\mathfrak{U}, \mathcal{D}^{(\bullet)}_{X^\sharp_\bullet/S_\bullet}(Z)) \to \Gamma(\mathfrak{U}', \mathcal{D}^{(\bullet)}_{X'^\sharp_\bullet/S'_\bullet}(Z'))$, $\Gamma(\mathfrak{U}, \mathcal{D}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}}(Z)) \to \Gamma(\mathfrak{U}', \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'}(Z'))$ and $\Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}}({}^\dagger Z)_\mathbb{Q}) \to \Gamma(\mathfrak{U}', \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'}({}^\dagger Z')_\mathbb{Q})$ are flat.

(b) For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}}({}^\dagger Z)_\mathbb{Q})$, we have the isomorphism:

$$f^!_Z(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'}({}^\dagger Z')_\mathbb{Q} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}}({}^\dagger Z)_\mathbb{Q}} f^{-1}\mathcal{E}.$$

Proof. a) Since $\mathcal{V}'$ is ideally Hausdorff for the p-adic topology, then if follows from [Bou61a, Theorem 1 of III.§5.2] that $\mathcal{V}'$ is a flat $\mathcal{V}$-algebra. Hence $S'_\bullet \to S_\bullet$ is flat. It follows from 5.1.1.15.3 the isomorphism:

$$\mathcal{O}^{(\bullet)}_{S'_\bullet} \otimes^\mathbb{L}_{\mathcal{O}^{(\bullet)}_{S_\bullet}} \varpi^{-1}\mathcal{D}^{(\bullet)}_{X^\sharp_\bullet/S_\bullet}(Z) \xrightarrow{\sim} \mathcal{D}^{(\bullet)}_{X'^\sharp_\bullet/S'_\bullet}(Z'). \tag{9.2.7.1.1}$$

This yields the flatness of $\Gamma(\mathfrak{U}, \mathcal{D}^{(\bullet)}_{X^\sharp_\bullet/S_\bullet}(Z)) \to \Gamma(\mathfrak{U}', \mathcal{D}^{(\bullet)}_{X'^\sharp_\bullet/S'_\bullet}(Z'))$. Tensoring with $\mathbb{Q}$ and passing to the limits, we get the two other ones.

b) It follows from the part (a) that the ring homomorphism $f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}}({}^\dagger Z)_\mathbb{Q} \to \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'}({}^\dagger Z')_\mathbb{Q}$ is flat. Using 9.2.6.5.1, we are done. □

Lemma 9.2.7.2. *Suppose $\mathcal{V} \to \mathcal{V}'$ is finite.*

(a) *Suppose $\mathfrak{X}$ is affine. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-module. Then the canonical morphisms*

$$\mathcal{V}' \otimes_{\mathcal{V}} \Gamma(\mathfrak{X},\mathcal{E}) \to D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} \Gamma(\mathfrak{X},\mathcal{E}) \to \Gamma(\mathfrak{X}', f^{!}_{Z}(\mathcal{E})) \qquad (9.2.7.2.1)$$

are isomorphisms. Moreover, $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}}$ is a faithfully flat $D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-module for both left or right structure.

(b) *For any $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-module $\mathcal{E}$, the canonical morphisms*

$$f^{*}(\mathcal{E}) := \mathcal{O}_{\mathfrak{X}'} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{X}}} f^{-1}\mathcal{E} \to \mathcal{O}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}} f^{-1}\mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} f^{-1}\mathcal{E}$$

are isomorphisms.

(c) *Let $\phi\colon \mathcal{E}' \to \mathcal{E}$ be a morphism of $\mathcal{O}_{\mathfrak{X}}$-modules. Then ϕ is an isomorphism if and only if $f^{*}(\phi)$ is an isomorphism.*

Proof. a) i) Since $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-module and $f^{!}_{Z}(\mathcal{E})$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}$, then via the corresponding theorems of type A the canonical morphisms $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} \Gamma(\mathfrak{X},\mathcal{E}) \to \mathcal{E}$, and $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}} \Gamma(\mathfrak{X}', f^{!}_{Z}(\mathcal{E})) \to f^{!}_{Z}(\mathcal{E})$ are isomorphisms. This yields by associativity of tensor products that the canonical morphism $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} \Gamma(\mathfrak{X},\mathcal{E}) \to \Gamma(\mathfrak{X}', f^{!}_{Z}(\mathcal{E}))$ is an isomorphism.

ii) By associativity of tensor products, to check the first morphism of 9.2.7.2.1 is an isomorphism, we reduce to the case where $\mathcal{E} = \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$, which easily follows from the fact that the morphism $\mathcal{V} \to \mathcal{V}'$ is finite. Since the homomorphism $\mathcal{V} \to \mathcal{V}'$ is faithfully flat (e.g. use [Bou61a, I.3.5, Proposition 9 and III.5.2, Theorem 1]), then $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}}$ is a faithfully flat $D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-module for both left or right structure.

b) By associativity of tensor products, to check the third statement we reduce to the case where $\mathcal{E} = \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$, which is easy.

c) Let us prove the forth assertion. Since $\mathcal{V}'$ is a finite faithfully flat $\mathcal{V}$-algebra, since $\mathcal{V}$ and $\mathcal{V}'$ are complete for the p-adic topology then $\mathcal{V}'$ is a free $\mathcal{V}$-module of finite type (e.g. use [Bou61a, II.3.2, Proposition 5]). Let $\mathfrak{U}$ be an affine open of $\mathfrak{X}$ and $\mathfrak{U}' := f^{-1}(\mathfrak{U})$. Then, $\Gamma(\mathfrak{U}',\mathcal{O}_{\mathfrak{X}'})$ is also a free $\Gamma(\mathfrak{U},\mathcal{O}_{\mathfrak{X}})$-module of finite type, and we can conclude.

□

Remark 9.2.7.3. Suppose $\mathfrak{X}$ is affine. It is likely that we do not need the finiteness of $\mathcal{V} \to \mathcal{V}'$ to get that $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}}$ is a faithfully flat $D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-module for both left or right structure.

Proposition 9.2.7.4. *Suppose $\mathcal{V} \to \mathcal{V}'$ finite. Let $\mathcal{E}$ be a $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}$-coherent module. Then $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}$-module if and only if $f^{!}_{Z}(\mathcal{E})$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-module.*

Proof. 1) Using Lemma 9.2.7.2.b (for both cases Z and the empty set), we check that the canonical morphism

$$\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}} f^{-1}\mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} f^{-1}\mathcal{E} = f^{!}_{Z}(\mathcal{E})$$

is an isomorphism. If $\mathcal{E}$ is also a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}$-module, this implies that $f^{!}_{Z}(\mathcal{E})$ is $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-coherent.

2) Conversely, suppose $f^{!}_{Z}(\mathcal{E})$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-module.

a) Since the $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}$-coherence of $\mathcal{E}$ is local on $\mathfrak{X}$, we can suppose $\mathfrak{X}$ is affine. Using theorem of type A, this yields that $\Gamma(\mathfrak{X}', f^{!}_{Z}(\mathcal{E}))$ is a $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-module of finite presentation. Set $E := \Gamma(\mathfrak{X},\mathcal{E})$. Following 9.2.7.2.a, this implies that $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} E$ is a $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-module of finite presentation. Using again 9.2.7.2.a, we get both isomorphism $\mathcal{V}' \otimes_{\mathcal{V}} D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}} \xrightarrow{\sim} D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}}$, and

$\mathcal{V}'\otimes_{\mathcal{V}} D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}} \xrightarrow{\sim} D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$. Hence, the canonical morphisms $\mathcal{V}'\otimes_{\mathcal{V}} E \to D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}} E \to D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}}} E$ are isomorphisms. This yields that $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}} E$ is a $D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-module of finite presentation. By full faithfulness of $D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}} \to D^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$, then E is a $D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}$-module of finite presentation.

c) Let $E^{\Delta} := \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}} \otimes_{D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S},\mathbb{Q}}} E$. By applying the functor $f^* = \mathcal{O}_{\mathfrak{X}'} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{X}}} -$ to the morphism $E^{\Delta} \to \mathcal{E}$, we get (up to canonical isomorphisms) the homomorphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-modules $f^!(E^{\Delta}) \to f^!_Z(\mathcal{E})$. Since $\Gamma(\mathfrak{X}', f^!(E^{\Delta})) \xrightarrow{\sim} \mathcal{V}'\otimes_{\mathcal{V}} \Gamma(\mathfrak{X}, E^{\Delta}) \xrightarrow{\sim} \mathcal{V}'\otimes_{\mathcal{V}} E$ and $\Gamma(\mathfrak{X}', f^!_Z(\mathcal{E})) \xrightarrow{\sim} \mathcal{V}'\otimes_{\mathcal{V}} E$, then by applying the functor $\Gamma(\mathfrak{X}', -)$ to $f^!(E^{\Delta}) \to f^!_Z(\mathcal{E})$, we get an isomorphism. Since $\mathfrak{X}'^{\sharp}$ is affine, using the theorem of type A satisfied by coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-modules, this yields that the morphism $f^!(E^{\Delta}) \to f^!_Z(\mathcal{E})$ of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}',\mathbb{Q}}$-modules is an isomorphism. Using 9.2.7.2.c, this implies that the morphism $E^{\Delta} \to \mathcal{E}$ is an isomorphism. □

9.3 Exact closed immersions

Let $\mathfrak{S}^{\sharp}$ be a p-torsion free noetherian nice fine $\mathcal{V}$-log formal scheme as defined in 3.3.1.10). Moreover, let $u\colon \mathfrak{Z}^{\sharp} \hookrightarrow \mathfrak{X}^{\sharp}$ be an exact closed immersion of p-torsion free log smooth $\mathfrak{S}^{\sharp}$-log formal schemes such that the underlying closed immersion of schemes of u_0 is regular. Let $\mathfrak{Y}^{\sharp}$ be the open of $\mathfrak{X}^{\sharp}$ complementary to the underlying topological space of $\mathfrak{Z}^{\sharp}$. Let $\mathcal{I}$ be the ideal defining u.

Suppose S_0, X_0, Z_0 are regular. Let D be some divisor of respectively X_0 such that $E := D\cap Z_0$ is a divisor of Z_0. We fix $\lambda_0 \in L(\mathbb{N})$. We set $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D) := \lambda_0^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(D)$, $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{Z}}(E) := \lambda_0^*\mathcal{B}^{(\bullet)}_{\mathfrak{Z}}(E)$, $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$, $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}(E) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{Z}}(E)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{Z}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}$. We denote by respectively $\widetilde{\mathfrak{X}}^{\sharp(m)}$ and $\widetilde{\mathfrak{Z}}^{\sharp(m)}$ the ringed $\mathcal{V}$-log formal scheme $(\mathfrak{X}^{\sharp}, \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D))$ and $(\mathfrak{Z}^{\sharp}, \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E))$, and by $u^{(m)}_D = \tilde{u}^{(m)}\colon \widetilde{\mathfrak{Z}}^{\sharp}/\mathfrak{S}^{\sharp(m)} \to \widetilde{\mathfrak{X}}^{\sharp}/\mathfrak{S}^{\sharp(m)}$ the induced morphism of relative ringed $\mathcal{V}$-log formal schemes. In this subsection, by $\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}$ we mean $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D)$, resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D)_{\mathbb{Q}}$, resp. $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(D)_{\mathbb{Q}}$) ; and by $\widetilde{\mathcal{D}}_{\mathfrak{Z}^{\sharp}}$ we mean $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)\otimes_{\mathcal{O}_{\mathfrak{Z}}}\mathcal{D}^{(m)}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}(E)$, resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}(E)_{\mathbb{Q}}$, resp. $\mathcal{D}^{\dagger}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}(E)_{\mathbb{Q}}$). By $\mathcal{B}_{\mathfrak{X}}$ we means $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$ (resp. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$, resp. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)_{\mathbb{Q}}$, resp. $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$) and by $\mathcal{B}_{\mathfrak{Z}}$ we means $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)$ (resp. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)$, resp. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)_{\mathbb{Q}}$, resp. $\mathcal{O}_{\mathfrak{Z}}({}^{\dagger}E)_{\mathbb{Q}}$).

We get the $(\widetilde{\mathcal{D}}_{\mathfrak{Z}^{\sharp}}, u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}})$-bimodule $\widetilde{\mathcal{D}}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp}} := u^{-1}(\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}})$ (e.g. see 9.2.1.20 for the last respective case). We have the $(u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^{\sharp}})$-bimodule $\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}$ (resp. $\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp},\mathbb{Q}}$, resp. $\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}} := \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}(D)_{\mathbb{Q}}$).

9.3.1 The fundamental isomorphism for formal schemes

The following lemma implies that we can apply 5.2 in the case where $\widetilde{\mathcal{D}}_{\mathfrak{X}^{\sharp}}$ (resp. $\widetilde{\mathcal{D}}_{\mathfrak{Z}^{\sharp}}$) is equal to $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$ (resp. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)\otimes_{\mathcal{O}_{\mathfrak{Z}}}\mathcal{D}^{(m)}_{\mathfrak{Z}^{\sharp}/\mathfrak{S}^{\sharp}}$)).

Lemma 9.3.1.1. *For any $m\in\mathbb{N}$, $\mathcal{O}_{\mathfrak{Z}}$ and $u^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$ are tor independent over $u^{-1}\mathcal{O}_{\mathfrak{X}}$. The sheaves $\mathcal{O}_{\mathfrak{Z}}$ and $u^{-1}\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$ are tor independent over $u^{-1}\mathcal{O}_{\mathfrak{X}}$. For any $m,i\in\mathbb{N}$, $\mathcal{O}_{Z_i}$ and $u^{-1}\widetilde{\mathcal{B}}^{(m)}_{X_i}(D)$ are tor independent over $u^{-1}\mathcal{O}_{X_i}$.*

Proof. Since this is local on $\mathfrak{X}$, we can suppose $\mathfrak{X} = \operatorname{Spf} A$ affine, integral and there exist $f\in\mathcal{O}_{\mathfrak{X}}$ lifting a local equation of D in X_0. Since $\mathfrak{X}$ is p-torsion free, we have the short exact sequence $0\to\mathcal{O}_{\mathfrak{X}}\{T\} \underset{f^{p^{m+1}}T-p}{\longrightarrow} \mathcal{O}_{\mathfrak{X}}\{T\} \to \mathcal{B}^{(m)}_{\mathfrak{X}}(D)\to 0$, which gives a flat resolution $\mathcal{B}^{(m)}_{\mathfrak{X}}(D)$. By applying $\mathcal{O}_{\mathfrak{Z}}\otimes_{u^{-1}\mathcal{O}_{\mathfrak{X}}} u^{-1}(-)$, this yields the short exact sequence $0\to\mathcal{O}_{\mathfrak{Z}}\{T\} \underset{[f^{p^{m+1}}T-p]_{\mathfrak{Z}}}{\longrightarrow} \mathcal{O}_{\mathfrak{Z}}\{T\} \to \mathcal{B}^{(m)}_{\mathfrak{Z}}(E)\to 0$ where $[-]_{\mathfrak{Z}}$ is the reduction modulo $\mathcal{I}$ and we get the first tor independence. By flatness of the extension $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\to\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$, we get the second one. Finally, the last part was already checked (see 9.2.1.26). □

9.3.1.2 (Local description)**.** Suppose $\mathfrak{X}^\sharp$ is affine and there exist some integers $0 \leq r \leq d$ and a cartesian diagram of morphisms of p-torsion free nice fine log smooth $\mathfrak{S}^\sharp$-log-formal schemes) of the form:

$$\begin{array}{ccc} \mathfrak{X}^\sharp & \xrightarrow{\alpha} & A^{d,r}_{\mathfrak{S}^\sharp} \\ {\scriptstyle u}\uparrow & \square & \uparrow \\ \mathfrak{Z}^\sharp & \longrightarrow & A^{r,r}_{\mathfrak{S}^\sharp} \end{array}$$

such that the horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion whose ideal is generated by $t_{r+1}, \ldots, t_d$. Recall following 5.2.1.1, this is Zariski locally possible. We denote by straight letter, the global section of a sheaf on $\mathfrak{X}$ or on $\mathfrak{Z}$, e.g. $\widetilde{D}_{\mathfrak{X}^\sharp} := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$ and $\widetilde{D}_{\mathfrak{Z}^\sharp} := \Gamma(\mathfrak{Z}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$.

Let $\mathfrak{Y} := \alpha^{-1}(B^{n,r}_{\mathfrak{S}^\sharp})$ be the open $\mathfrak{S}^\sharp$-(formal) subscheme of $\mathfrak{X}^\sharp$ with trivial log-structure (see [Ogu18, III.1.2.8]). Let $t_1, \ldots, t_r \in M_{\mathfrak{X}^\sharp}$ and $t_{r+1}, \ldots, t_d \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be the element given by α. Then $t_{r+1}, \ldots, t_d$ generate $I := \Gamma(\mathfrak{X}, \mathcal{I})$, $\bar{t}_1, \ldots, \bar{t}_r$ are semi-logarithmic coordinates of $\mathfrak{Z}^\sharp$ over $\mathfrak{S}^\sharp$, and $\bar{t}_{r+1}, \ldots, \bar{t}_d$ is a basis of $\mathcal{I}/\mathcal{I}^2$, where $\bar{t}_1, \ldots, \bar{t}_r \in \Gamma(\mathfrak{Z}, M_{\mathfrak{Z}^\sharp})$ (resp. $\bar{t}_{r+1}, \ldots, \bar{t}_d \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{Z}})$) are the images of $t_1, \ldots, t_r$ (resp. $t_{r+1}, \ldots, t_d$) via $\Gamma(\mathfrak{X}, M_{\mathfrak{X}^\sharp}) \to \Gamma(\mathfrak{Z}, M_{\mathfrak{Z}^\sharp})$ (resp. $\Gamma(\mathfrak{X}, \mathcal{I}) \to \Gamma(\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}})$). Remark that since the closed immersion $\underline{u}$ is regular then it follows from [Gro67, 16.9.3] that $t_{r+1}, \ldots, t_d$ is a quasi-regular sequence of $I := \Gamma(\mathfrak{X}, \mathcal{I})$ and then by noetherianity (see [Gro67, 16.9.10]) is a regular sequence of I.

We simply write $\tau_{i\sharp} := \mu^n_{(m),\gamma}(t_i) - 1 \in \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}$ for $i = 1, \ldots, r$ (see notation 3.2.2.4), $\tau_j := 1 \otimes t_j - t_j \otimes 1 \in \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}$ for $j = r+1, \ldots, d$. We write $\overline{\tau}_{i\sharp} := \mu^n_{(m),\gamma}(\bar{t}_i) - 1 \in \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E) \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{P}^n_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,(m)}$ for $i = 1, \ldots, r$. The sheaf of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-algebras $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^n_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,(m)}$ is a free $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-module with the basis $\{\underline{\tau}_{(r)}{}^{\{(\underline{i},\underline{0})\}_{(m)}} = \underline{\tau}_{\sharp}{}^{\{(\underline{i},\underline{0})\}_{(m)}} \mid \underline{i} \in \mathbb{N}^r,$ such that $|\underline{i}| \leq n\}$, and $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^n_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,(m)}$ is a free $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)$-module with the basis $\{\underline{\overline{\tau}}_{\sharp}^{\{\underline{i}\}_{(m)}} \mid \underline{i} \in \mathbb{N}^r$ such that $|\underline{i}|| \leq n\}$. According to 4.5.1.1, the corresponding dual basis of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,n}$ is denoted $\{\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} \mid \underline{k} \in \mathbb{N}^d,\ |\underline{k}| \leq n\}$ and the corresponding dual basis of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E) \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,n}$ is denoted by $\{\underline{\partial}_{\sharp}^{\langle \underline{i} \rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^r,\ |\underline{i}| \leq n\}$ The sheaf $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a free $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-module with the basis $\{\underline{\partial}_{(r)}^{\langle \underline{k} \rangle_{(m)}} \mid \underline{k} \in \mathbb{N}^d\}$, and $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E) \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$ is a free $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(E)$-module with the basis $\{\underline{\partial}_{\sharp}^{\langle \underline{i} \rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^r\}$.

9.3.1.3. Let $\mathcal{I}$ (resp. $\mathcal{I}_i$) be the ideal of $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{X_i}$) induced by the exact closed immersion $u \colon \mathfrak{Z}^\sharp \hookrightarrow \mathfrak{X}^\sharp$ (resp. $u_i \colon Z_i^\sharp \hookrightarrow X_i^\sharp$).

(a) Since $\mathcal{O}_{Z_i} = u^{-1}\mathcal{O}_{X_i}/u^{-1}\mathcal{I}_i$, then we check

$$\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp \to X_i^\sharp/S_i^\sharp}(D) = \widetilde{u}_i^* \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D) = u^{-1}(\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D)/\mathcal{I}_i \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D)).$$

Since $\mathcal{I}$ is $\mathcal{O}_{\mathfrak{X}}$-coherent, since $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ is a coherent ring, then $\mathcal{I} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ is a coherent right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module. Since $\mathcal{I}$ is $\mathcal{O}_{\mathfrak{X}}$-flat (recall X and Z are regular and use [Gro66, 15.4.2]), this yields that $\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \xrightarrow{\sim} \mathcal{I} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ are coherent right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-modules. Moreover, since $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \otimes_{\mathcal{V}} (\mathcal{V}/\pi^{i+1}\mathcal{V}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D)$, then we get the isomorphisms

$$\begin{aligned} \mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\pi^{i+1}\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) &\xrightarrow{\sim} (\mathcal{I} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\pi^{i+1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \\ &\xrightarrow{\sim} \mathcal{I} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D) \xrightarrow{\sim} \mathcal{I}_i \otimes_{\mathcal{O}_{X_i}} \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D) \xrightarrow{\sim} \mathcal{I}_i\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D). \end{aligned}$$

Via 7.2.3.16.(i), this yields that the canonical morphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \varprojlim_i \widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D)/\mathcal{I}_i\widetilde{\mathcal{D}}^{(m)}_{X_i^\sharp/S_i^\sharp}(D) \tag{9.3.1.3.1}$$

is an isomorphism.

Recall $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) := \varprojlim_i \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_i\to X^\sharp_i/S^\sharp_i}(D)$ (see 7.5.5.10). Since u_* commutes with projective limits, we deduced from 9.3.1.3.1 the isomorphism

$$u_*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D). \tag{9.3.1.3.2}$$

Recall $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)_{\mathbb{Q}}$ (see 9.2.1.20). Taking the tensor product by $\mathbb{Q}$ on $\mathbb{Z}$ and next to the inductive limit on the level, we get therefore:

$$\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}} \xrightarrow{\sim} u^{-1}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}/\mathcal{I}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \tag{9.3.1.3.3}$$

(b) Suppose $\mathfrak{X}$ is affine and put $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$. We prove similarly that $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ is a coherent right $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module. Since $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \otimes_{\mathcal{V}} (\mathcal{V}/\pi^{i+1}\mathcal{V}) \xrightarrow{\sim} \widetilde{D}^{(m)}_{X^\sharp_i/S^\sharp_i}(D)$, with 7.2.1.3, this implies that the canonical morphism

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \varprojlim_i \widetilde{D}^{(m)}_{X^\sharp_i/S^\sharp_i}(D)/I_i\widetilde{D}^{(m)}_{X^\sharp_i/S^\sharp_i}(D) \tag{9.3.1.3.4}$$

is an isomorphism. Since the functor $\Gamma(\mathfrak{X}, -)$ commutes with projective limits, the isomorphisms 9.3.1.3.1 and 9.3.1.3.4 induce $\Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$. This yields:

$$\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D). \tag{9.3.1.3.5}$$

Taking the tensor product by $\mathbb{Q}$ on $\mathbb{Z}$ and next to the inductive limit on the level, we get

$$D^\dagger_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}} \xrightarrow{\sim} D^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}/ID^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}. \tag{9.3.1.3.6}$$

9.3.1.4. Suppose we are in the local situation of 9.3.1.2.

(a) According to 5.2.2.3, we denote by $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}/\mathfrak{S}^\sharp}$ the free $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-module with the basis $\{\underline{\partial}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}} \mid \underline{i} \in \mathbb{N}^r\}$, where $\underline{0} := (0, \ldots, 0) \in \mathbb{N}^{d-r}$. The sheaf $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}/\mathfrak{S}^\sharp}$ is equal to the sub-$\mathcal{O}_{\mathfrak{S}}$-algebra of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ which is generated by $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$, by $\partial_{\sharp,i}^{\langle p^h\rangle_{(m)}}$ and $\partial_j^{\langle p^h\rangle_{(m)}}$ for any $1 \le i \le r$, $r+1 \le j \le d$ and $0 \le h \le p^m$.

(b) Let $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$ be the p-adic completion of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}/\mathfrak{S}^\sharp}$. A section of the sheaf $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ can uniquely be written of the form $\sum_{\underline{k}\in\mathbb{N}^d} a_{\underline{k}}\underline{\partial}^{\langle\underline{k}\rangle_{(m)}}$ such that $a_{\underline{k}} \in \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$ converges to 0 when $|\underline{k}| \to \infty$. A section of the sheaf $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$ can uniquely be written of the form $\sum_{\underline{l}\in\mathbb{N}^r} b_{\underline{l}}\underline{\partial}^{\langle\underline{l}\rangle_{(m)}}$ such that $b_{\underline{l}} \in \mathcal{O}_{\mathfrak{Z}}$ converges to 0 when $|\underline{l}| \to \infty$. Then $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$ is a subring of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ whose elements can uniquely be written of the form $\sum_{\underline{l}\in\mathbb{N}^r} a_{\underline{l}}\underline{\partial}^{\langle(\underline{l},\underline{0})\rangle_{(m)}}$ (recall $\underline{0} := (0, \ldots, 0) \in \mathbb{N}^{d-r}$) where $a_{\underline{l}} \in \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$ converges to 0 when $|\underline{l}| \to \infty$. Taking p-adic completion of the diagram 5.2.2.5.7, with notation 7.5.5.10 we get the canonical diagram

$$\begin{array}{ccc} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D) & \hookrightarrow & \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \\ \sim\uparrow \vartheta & & \sim\uparrow 9.3.1.3 \\ u_*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E) & \xrightarrow{\theta} & u_*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \end{array} \tag{9.3.1.4.1}$$

where $\vartheta\colon u_*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$ is an isomorphism of $\mathcal{V}$-algebras.

(c) We set $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D) := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$. We get a similar diagram than 9.3.1.4.1 by replacing $\widetilde{\mathcal{D}}^{(m)}$ with $\mathcal{D}^\dagger$ and by adding applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$.

(d) By $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ we mean $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}/\mathfrak{S}^\sharp}$ (resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$, resp. $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)_{\mathbb{Q}}$, resp. $\mathcal{D}^{\dagger}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)_{\mathbb{Q}}$).

9.3.1.5. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module. We set $\widetilde{u}^{\flat 0}(\mathcal{M}) := u^{-1}\mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(u_*\mathcal{B}_{\mathfrak{Z}}, \mathcal{M})$.

(a) We endow $\widetilde{u}^{\flat 0}(\mathcal{M})$ with a structure of right $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-module as follows. Suppose $\mathfrak{X}$ affine. Let $x \in \Gamma(\mathfrak{Z}, \widetilde{u}^{\flat 0}(\mathcal{M}))$ and $Q \in \widetilde{D}_{\mathfrak{Z}}$. Choose $Q_{\mathfrak{X}} \in \widetilde{D}_{\mathfrak{X}}$ such that $\vartheta(Q) = \overline{Q_{\mathfrak{X}}}$ (see 9.3.1.4.1). We define $x \cdot Q$ so that we get the equality

$$\mathrm{ev}_1(x \cdot Q) := \mathrm{ev}_1(x) \cdot Q_{\mathfrak{X}}, \tag{9.3.1.5.1}$$

where $\mathrm{ev}_1 \colon \Gamma(\mathfrak{Z}, \widetilde{u}^{\flat 0}(\mathcal{M})) \hookrightarrow \Gamma(\mathfrak{X}, \mathcal{M})$ is the evaluation at 1 homomorphism (which is injective). Since I annihilates $\mathrm{ev}_1(x)$, we remark that this is well defined.

(b) When $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} = \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, we have a canonical way to endow $\widetilde{u}^{\flat 0}(\mathcal{M})$ with a canonical structure of right $\mathcal{D}^{(m)}_{\mathfrak{Z}}$-module (see 5.2.5.1). It follows from the computation 5.2.5.2.1 that both structures are identical.

9.3.1.6 (Local description of the right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module structure of $\widetilde{u}^{\flat 0}(\mathcal{M})$)**.** Suppose we are in the local situation of 9.3.1.2. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module. Since $\vartheta \colon u_*\mathcal{B}_{\mathfrak{Z}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$, we have the isomorphism

$$\widetilde{u}^{\flat 0}(\mathcal{M}) \xrightarrow{\sim} u^{-1}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}, \mathcal{M}). \tag{9.3.1.6.1}$$

Following 9.3.1.4.1, we have the isomorphism of $\mathcal{O}_{\mathfrak{S}}$-algebras $\vartheta \colon u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ (see notation 9.3.1.4.1). Hence, we get from 9.3.1.6.1 the isomorphism

$$\widetilde{u}^{\flat 0}(\mathcal{M}) \xrightarrow{\sim} u^{-1}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, \mathcal{M}) \tag{9.3.1.6.2}$$

By using 9.3.1.5.1 formula, we compute that this isomorphism 9.3.1.6.2 is an isomorphism of right $u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-modules, and 9.3.1.6.1 is therefore an isomorphism of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-modules. If there is no ambiguity, we can avoid writing u^{-1} or u_*, e.g. we can simply write $\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, \mathcal{M})$ instead of $u^{-1}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, \mathcal{M})$.

9.3.1.7. Suppose we are in the local situation of 9.3.1.2. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}_{\mathfrak{X}/\mathfrak{S}}$-module.

(a) By derivating 9.3.1.6.2, we get the isomorphism of $D^{\mathrm{b}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}})$ of the form

$$\widetilde{u}^{\flat}(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}, \mathcal{M}). \tag{9.3.1.7.1}$$

Let $s := d - r$, and $f_1 = t_{r+1}, \ldots, f_s := t_d$. Let $K_\bullet(\underline{f})$ be the Koszul complex of $\underline{f} = (f_1, \ldots, f_s)$ (see notation 5.2.5.4). We set $\widetilde{K}_\bullet(\underline{f}) := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} K_\bullet(\underline{f})$.

Since $f_1, \ldots, f_s$ is a regular sequence of $\mathcal{I}$, since $\mathcal{O}_{\mathfrak{Z}}$ and $u^{-1}\mathcal{B}_{\mathfrak{X}}$ are tor independent over $u^{-1}\mathcal{O}_{\mathfrak{X}}$ (see 9.3.1.1), then the canonical morphism $\widetilde{K}_\bullet(\underline{f}) \to \mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}}$ (given by the canonical map $\widetilde{K}_0(\underline{f}) = \mathcal{B}_{\mathfrak{X}} \to \mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}}$) is a quasi-isomorphism. Hence, we get the isomorphism of $D^{\mathrm{b}}(\mathcal{B}_{\mathfrak{X}})$

$$\phi_{\underline{f}} \colon \widetilde{u}^{\flat}(\mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{K}_\bullet(\underline{f}), \mathcal{M}).$$

Since $f_1, \ldots f_s$ are in the center of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ and since $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ is a flat $\mathcal{B}_{\mathfrak{X}}$-algebra, then the quasi-isomorphism $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{K}_\bullet(\underline{f}) \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ in the category of complexes of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-bimodules. We get the commutativity of diagram

$$\begin{array}{ccc} \widetilde{u}^{\flat}(\mathcal{M}) & \xrightarrow[\sim]{\phi_{\underline{f}}} & \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\widetilde{K}_\bullet(\underline{f}), \mathcal{M}) \\ {\scriptstyle 9.3.1.7.1}\downarrow\wr & \searrow^{\phi_{\underline{t}}} & \downarrow\wr \\ \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}, \mathcal{M}) & \xleftarrow[\sim]{} & \mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{K}_\bullet(\underline{f}), \mathcal{M}), \end{array} \tag{9.3.1.7.2}$$

where $\phi_{\underline{t}}$ is the homomorphism making commutative the upper triangle. By commutativity of 5.2.5.4.2, $\phi_{\underline{t}}$ is an isomorphism of $D^{\mathrm{b}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}})$. Hence, we get the isomorphism of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-modules

$$\phi^s_{\underline{t}} := \mathcal{H}^s(\phi_{\underline{t}})\colon R^s\widetilde{u}^{\flat 0}(\mathcal{M}) \xrightarrow{\sim} \mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{K}_\bullet(\underline{f}), \mathcal{M}). \tag{9.3.1.7.3}$$

We have the homomorphism of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-modules $\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{K}_s(\underline{f}), \mathcal{M}) \to \mathcal{M}$ (the structure of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-module on $\mathcal{M}$ comes from its structure of $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$ via ϑ) given by $\phi \mapsto \phi(e_1 \wedge \cdots \wedge e_s)$. Since $\mathcal{M} \to \mathcal{M}/\mathcal{I}\mathcal{M}$ is a morphism of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-modules, then this induces the morphism of complex of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-modules of the form $\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{K}_\bullet(\underline{f}), \mathcal{M}) \to \mathcal{M}/\mathcal{I}\mathcal{M}$. This yields the isomorphism of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$-modules

$$\mathcal{H}^s\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{K}_\bullet(\underline{f}), \mathcal{M}) \xrightarrow{\sim} \mathcal{M}/\mathcal{I}\mathcal{M}. \tag{9.3.1.7.4}$$

Notation 9.3.1.8. Let $\mathcal{E} \in D({}^l\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$. We set:

$$\widetilde{u}^!(\mathcal{E}) := \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} u^{-1}\mathcal{E}[d_u]. \tag{9.3.1.8.1}$$

Beware, in the first respective case, this notation has not to be confused with its completed version denoted by $\widetilde{u}^{(m)!}$ of 7.5.5.6. We will work with the completed version functor in the subsection 9.3.3. Using 9.3.1.3.2 and 9.3.1.3.3 for the less obvious cases, we can check the canonical morphism

$$\begin{aligned}\mathbb{L}\widetilde{u}^*(\mathcal{E}) &:= \mathcal{B}_{\mathfrak{Z}} \otimes^{\mathbb{L}}_{u^{-1}\mathcal{B}_{\mathfrak{X}}} u^{-1}\mathcal{E} \xrightarrow{\sim} u^{-1}(\mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}) \\ &\xrightarrow{\sim} u^{-1}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{E}) \to \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} u^{-1}\mathcal{E}\end{aligned} \tag{9.3.1.8.2}$$

is an isomorphism, which yields $\mathbb{L}\widetilde{u}^*(\mathcal{E})[d_u] \xrightarrow{\sim} \widetilde{u}^!(\mathcal{E})$.

Suppose now we are in the local situation of 9.3.1.2. Let $Q \in \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$. Choose $Q_{\mathfrak{X}} \in \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ such that $[Q_{\mathfrak{X}}]_{\mathfrak{Z}} = \vartheta(Q)$ (use 9.3.1.4.1). For any section x of $\mathcal{E}$, we have the formula in $\widetilde{u}^*(\mathcal{E})$:

$$Q(\widetilde{u}^*(x)) = \widetilde{u}^*(Q_{\mathfrak{X}} \cdot x). \tag{9.3.1.8.3}$$

Indeed, by fonctoriality, we reduce to the case where $\mathcal{E} =$, which easily follows from 5.2.2.6.1 by p-adic completion (and inductive limits on the level). Via the monomorphism of rings $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \hookrightarrow \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, we check the canonical homorphism

$$\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}} u^{-1}\mathcal{E} \to \mathbb{L}\widetilde{u}^*(\mathcal{E})$$

is an isomorphism of $D({}^l\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$. Via the isomorphisms $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_X} \widetilde{K}_\bullet(\underline{f}) \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \overset{\vartheta}{\underset{\sim}{\longleftarrow}} \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$, this yields the isomorphism of $D(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}})$:

$$(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \otimes_{\mathcal{B}_X} \widetilde{K}_\bullet(\underline{f})) \otimes_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}} u^{-1}\mathcal{E} \xrightarrow{\sim} \mathbb{L}\widetilde{u}^*(\mathcal{E}). \tag{9.3.1.8.4}$$

9.3.1.9. Suppose we are in the local situation of 9.3.1.2. Since $u_*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} = \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, then $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$ is a $(u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$-bimodule. Using the formula 9.3.1.8.3 applied to the case $\mathcal{E} = \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, we compute that $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ is also a left $u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-submodule of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, and then a $(u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}})$-sub-bimodule of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$.

Since $u_*\theta\colon u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp} \to u_*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ is a homomorphism of left $u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-modules, then via the commutativity of 9.3.1.4.1, this implies that the bijection $\vartheta\colon u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ is an isomorphism of left $u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-modules.

9.3.1.10 (Semi-logarithmic adjoint operator)**.** Suppose we are in the local situation of 9.3.1.2. Let $\underline{k} \in \mathbb{N}^d$. We still denote by ${}^\tau\underline{\partial}_{(r)}^{\langle \underline{k}\rangle_{(m)}}$ the image of the operator ${}^\tau\underline{\partial}_{(r)}^{\langle \underline{k}\rangle_{(m)}}$ defined at 4.5.1.1.2 via the ring homomorphism $D^{(m)}_{\mathfrak{A}^{d,r}_{\mathfrak{S}^\sharp}/\mathfrak{S}^\sharp} \to \widetilde{D}_{\mathfrak{X}^\sharp} \to \widetilde{D}_{\mathfrak{X}^\sharp}$.

Let $P = \sum_{\underline{k}\in\mathbb{N}^d} b_{\underline{k}}\underline{\partial}_{(r)}^{\langle \underline{k}\rangle_{(m)}} \in \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$, with $b_{\underline{k}} \in B_X$ converging to 0 for the p-adic topology. We define the "semi-logarithmic adjoint operator" of P by setting ${}^\tau P := \sum_{\underline{k}} {}^\tau\underline{\partial}_{(r)}^{\langle \underline{k}\rangle_{(m)}} b_{\underline{k}}$, which is a mixed version of the logarithmic and non-logarithmic adjoint operator. By tensoring with or by taking the inductive limits on the level, we define the "semi-logarithmic adjoint operator" ${}^\tau\colon \widetilde{D}_{\mathfrak{X}^\sharp} \to \widetilde{D}_{\mathfrak{X}^\sharp}$.

Let P, Q be two differential operators of $\widetilde{D}_{\mathfrak{X}^\sharp}$. It follows from 9.3.1.10 (e.g. by p-adic completion, tensorisation by $\mathbb{Q}$ and inductive limits on the level if necessary) that the following properties hold:

(a) We have ${}^\tau({}^\tau P) = P$ and ${}^\tau(PQ) = {}^\tau Q {}^\tau P$

(b) We have the equality
$$\rho({}^\tau P) = \rho\left(\underline{t}_{(r)}{}^{\mathrm{t}}P\frac{1}{\underline{t}_{(r)}}\right), \tag{9.3.1.10.1}$$
where $\underline{t}_{(r)} = t_1 \cdots t_r$ and where ρ is the canonical map $\widetilde{D}_{\mathfrak{X}^\sharp} \to \widetilde{D}^{(m)}_{\mathfrak{Y}}$ (which is an inclusion when $\mathcal{B}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$).

Hence, ${}^\tau\colon (\widetilde{D}_{\mathfrak{X}^\sharp})^o \to \widetilde{D}_{\mathfrak{X}^\sharp}$ is an involution of $\mathcal{O}_T$-algebras, which is called the semi-logarithmic adjoint automorphism. Beware that this depends on the choice of the semi-logarithmic coordinates $t_1, \ldots, t_d$. This induces ${}^\tau\colon \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \to \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ such that ${}^\tau(I\widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}) = I\widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$. This yields the automorphism ${}^\tau\colon \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/I\widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \to \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/I\widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$. On the other hand, via the local logarithmic coordinates $\bar{t}_1, \ldots, \bar{t}_{r+s}$ of $Z^\sharp$ over $T^\sharp$, we get the logarithmic adjoint operator automorphism ${}^\tau\colon \widetilde{D}_{\mathfrak{Z}^\sharp} \to \widetilde{D}_{\mathfrak{Z}^\sharp}$ given by $Q = \sum_{\underline{i}\in\mathbb{N}^{r+s}} b_{\underline{i}}\underline{\partial}_{(r)}^{\langle \underline{i}\rangle_{(m)}} \mapsto {}^\tau(Q) := \sum_{\underline{i}\in\mathbb{N}^{r+s}} {}^\tau\underline{\partial}_{(r)}^{\langle \underline{i}\rangle_{(m)}} b_{\underline{i}}$.

Since $\vartheta(\underline{\partial}_{(r)}^{\langle \underline{i}\rangle_{(m)}}) = \underline{\partial}_{(r)}^{\langle (\underline{i},\underline{0})\rangle_{(m)}}$, $\vartheta({}^\tau\underline{\partial}_{(r)}^{\langle \underline{i}\rangle_{(m)}}) = {}^\tau\underline{\partial}_{(r)}^{\langle (\underline{i},\underline{0})\rangle_{(m)}}$ (use the formula 3.4.1.2.2) for any $\underline{i} \in \mathbb{N}^{r+s}$, then the following diagram
$$\begin{array}{ccc} \widetilde{D}_{\mathfrak{Z}^\sharp} & \xrightarrow[\vartheta]{\sim} & \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/I\widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \\ \sim\downarrow{\scriptstyle\tau} & & \sim\downarrow{\scriptstyle\tau} \\ \widetilde{D}_{\mathfrak{Z}^\sharp}, & \xrightarrow[\vartheta]{\sim} & \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}/I\widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}} \end{array} \tag{9.3.1.10.2}$$
is commutative.

Lemma 9.3.1.11. *The sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a free $\mathcal{B}_{\mathfrak{X}}$-module of rank one with the basis $\widetilde{e}_0 := d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge t_d$ and $\widetilde{\omega}_{\mathfrak{Y}/\mathfrak{S}^\sharp}$ is a free $\mathcal{B}_Y$-module of rank one with the basis $dt_1 \wedge \cdots \wedge dt_d$.*

The sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-submodule of $j_*\widetilde{\omega}_{\mathfrak{Y}/\mathfrak{S}^\sharp}$. More precisely, the action of $P \in \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$ on the section $b\,\widetilde{e}_0$, where b is section of $\mathcal{B}_{\mathfrak{X}}$ is given by the formula
$$(b\ \widetilde{e}_0) \cdot P = {}^\tau P(b)\ \widetilde{e}_0. \tag{9.3.1.11.1}$$

Proof. This is a consequence of 4.5.1.6 (e.g. by p-adic completion, tensorisation by $\mathbb{Q}$ and inductive limits on the level if necessary). □

9.3.1.12. It follows from 4.2.5.5.1 (e.g. by p-adic completion, tensorisation by $\mathbb{Q}$ and inductive limits on the level if necessary, we get a structure of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-modules on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$. It follows from 4.2.5.5.1 (e.g. by p-adic completion, tensorisation by $\mathbb{Q}$ and inductive limits on the level if necessary) that there exists a unique involution of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-bimodules
$$\delta_{\widetilde{\omega}}\colon \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \xrightarrow{\sim} \widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \tag{9.3.1.12.1}$$
exchanging the two structures of right $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-modules and such that, for each section m of $\mathcal{M}$, $\delta_{\widetilde{\omega}}(m \otimes 1) = m \otimes 1$. In semi-logarithmic coordinates, we have the following formula
$$\delta_{\mathcal{M}}(m \otimes {}^\tau\underline{\partial}_{(r)}^{\langle \underline{k}\rangle}) = \sum_{\underline{h}\leq\underline{k}} m\,{}^\tau\underline{\partial}_{(r)}^{\langle \underline{k}-\underline{h}\rangle} \otimes \underline{\partial}_{(r)}^{\langle \underline{h}\rangle}. \tag{9.3.1.12.2}$$

9.3.1.13 (Left to right $\mathcal{D}$-module)**.** Suppose now we are in the local situation of 9.3.1.2. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module. We denote by $\widetilde{e}_0 := d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge dt_d$ a basis of the free $\mathcal{B}_{\mathfrak{X}}$-module $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ has the basis, and by $\widetilde{e}_0^\vee$ its corresponding dual basis of the free $\mathcal{B}_{\mathfrak{X}}$-module $\widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$.

(a) We compute that the right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module structure on $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}$ (see its definition 7.5.1.13.(f) or 8.7.2) is given by the formula

$$(\widetilde{e}_0 \otimes x)P = \widetilde{e}_0 \otimes {}^\tau P x, \tag{9.3.1.13.1}$$

for any local section x of $\mathcal{E}$ and P of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$.

(b) On the other hand, the left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module structure on $\mathcal{M} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\omega}^{-1}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is given by the formula

$$P(y \otimes \widetilde{e}_0^\vee) = y^\tau P \otimes \widetilde{e}_0^\vee, \tag{9.3.1.13.2}$$

for any local section y of $\mathcal{M}$ and P of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$.

9.3.1.14. Suppose now we are in the local situation of 9.3.1.2. For any left (resp. right) $\widetilde{D}_{\mathfrak{X}^\sharp}$-module E (resp. M), we denote by $[-]_{\mathfrak{Z}} \colon E \to E/IE$ (resp. $[-]'_{\mathfrak{Z}} \colon M \to M/MI$) the canonical surjections. Remark we add a prime to avoid some confusion in the case of a bimodule.

It follows from 5.2.2.14.1 (e.g. by p-adic completion, tensorisation by $\mathbb{Q}$ and inductive limits on the level if necessary) that we get the isomorphism of abelian sheaves:

$$\iota_{\widetilde{t}} \colon u_*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\mathcal{I}. \tag{9.3.1.14.1}$$

Via this map, we get a structure of $(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}, u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$-bimodule on $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\mathcal{I}$. By functoriality of the construction of the structure of $(u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$-bimodule of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}$, we check that the underlying structure of left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module on $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\mathcal{I}$ is equal to its natural structure. The structure of right $u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-module of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\mathcal{I}$ is characterized by the following formula:

$$[y]'_{\mathfrak{Z}} \cdot Q = [y \cdot Q_{\mathfrak{X}}]'_{\mathfrak{Z}}, \tag{9.3.1.14.2}$$

where $y \in M$, $Q \in \widetilde{D}_{\mathfrak{Z}^\sharp}$ and $Q_{\mathfrak{X}} \in \widetilde{D}_{\mathfrak{X}^\sharp, \mathfrak{Z}^\sharp, \underline{t}}$ is an element such that $[Q_{\mathfrak{X}}]_{\mathfrak{Z}} = \vartheta(Q)$ (use 9.3.1.4.1). Moreover, we compute that $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp, \mathfrak{Z}^\sharp, \underline{t}}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp, \mathfrak{Z}^\sharp, \underline{t}}$ is also a left $u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-submodule of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, and therefore a $(u_*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp, \mathfrak{Z}^\sharp, \underline{t}})$-sub-bimodule of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$.

Proposition 9.3.1.15. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module (resp. a $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-bimodule). Set $n := -d_u \in \mathbb{N}$. We have the canonical isomorphism of right $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-modules (resp. of right $(\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$-bimodules):*

$$R^n\widetilde{u}^{\flat 0}(\omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \xrightarrow{\sim} \omega_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{u}^*(\mathcal{E}). \tag{9.3.1.15.1}$$

Proof. Using 9.3.1.7 and 9.3.1.8, thanks to respectively 7.5.1.13 or 8.7.2.2, we can proceed as in 5.2.5.6. □

Remark 9.3.1.16. Since complexes are not coherent, Proposition 9.3.1.15 is not a straightforward consequence of Theorem 5.2.5.6 (but the check is similar).

Theorem 9.3.1.17. *Let $* \in \{\mathrm{l}, \mathrm{r}\}$ and let $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$ (resp. $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}, {}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$). With notation 9.3.1.8, we have the canonical isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ (resp. $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, {}^*u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$) of the form*

$$\omega_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^\flat(\omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{9.3.1.17.1}$$

Proof. Using [Har66, I.7.4], this is a consequence of 9.3.1.15 and of the isomorphism $\mathbb{L}\widetilde{u}^*(\mathcal{E})[d_u] \xrightarrow{\sim} \widetilde{u}^!(\mathcal{E})$ (see 9.3.1.8). □

9.3.1.18. By using the same arguments as in 9.3.1.3, it follows from 5.2.2.14.1 that we have the canonical isomorphism

$$\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{\sim} u^{-1}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\mathcal{I}). \tag{9.3.1.18.1}$$

We get the functor $\widetilde{u}^!\colon D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ by setting for any $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$,

$$\widetilde{u}^!(\mathcal{M}) := u^{-1}\mathcal{E} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}[d_u]. \tag{9.3.1.18.2}$$

With notation 9.3.1.17, following 5.1.1.5.1 or 9.2.1.23.1, the functor $\widetilde{u}^!\colon D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ (resp. $\widetilde{u}^!\colon D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}, {}^{*}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, {}^{*}u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$) satisfies the isomorphism

$$\omega_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^!(\omega_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{9.3.1.18.3}$$

Hence, with 9.3.1.17.1, we get the isomorphism

$$\widetilde{u}^\flat \xrightarrow{\sim} \widetilde{u}^! \tag{9.3.1.18.4}$$

of functors $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ (resp. $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}, {}^{*}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \to D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, {}^{*}u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$).

Corollary 9.3.1.19. *(a) We have the canonical isomorphism of right $(\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}, u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$-bimodules of the form*

$$\omega_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp} \xrightarrow{\sim} \widetilde{u}^\flat_{\mathrm{l}}(\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})[-d_u], \tag{9.3.1.19.1}$$

where "l" means that in we have chosen the left structure of right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module of the right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-bimodule $\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$.

(b) We have the canonical isomorphism of $(u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$-bimodules of the form

$$\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \xrightarrow{\sim} \widetilde{u}^\flat(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})[-d_u]. \tag{9.3.1.19.2}$$

Proof. By applying Theorem 9.3.1.17 in the case $\mathcal{E} = \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, we get the isomorphism 9.3.1.19.1. By applying Theorem 9.3.1.17 in the case $\mathcal{E} = \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1}$, and by using the transposition isomorphism $\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1}) \xrightarrow{\sim} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$, we get the isomorphism 9.3.1.19.2. □

9.3.1.20. Suppose we are in the local situation of 9.3.1.2. Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module. Following 5.2.5.7.1 we have the isomorphism of right $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-modules:

$$\widetilde{\omega}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Z}}} H^0\widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^{\flat 0}(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}). \tag{9.3.1.20.1}$$

The sheaf $\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is a free $\mathcal{B}_{\mathfrak{X}}$-module of rank one with the basis $d\log t_1 \wedge \cdots \wedge d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge t_d$ and $\widetilde{\omega}_{Y/T^\sharp}$ is a free $\mathcal{B}_Y$-module of rank one with the basis $dt_1 \wedge \cdots \wedge dt_d$ (see 4.5.1.6). Using these bases, we get the isomorphism of $\mathcal{B}_{\mathfrak{Z}}$-modules:

$$u_* H^0\widetilde{u}^!(\mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}}, \mathcal{E}) = \cap_{s=r+1}^{d} \ker(\mathcal{E} \xrightarrow{t_s} \mathcal{E}). \tag{9.3.1.20.2}$$

This yields on $\mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}}, \mathcal{E})$ a structure of left $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp/T^\sharp}$-module extending its structure of $\mathcal{B}_{\mathfrak{X}}$ (beware this structure depends a priori on the choice of the semi-logarithmic coordinates). Let us denote by $\mathrm{ev}_1\colon \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}}, \mathcal{E}) \hookrightarrow \mathcal{E}$ the canonical inclusion (this is the evaluation at 1) and by $\mathrm{ev}_1\colon H^0\widetilde{u}^!(\mathcal{E}) \hookrightarrow \mathcal{E}$ its composition with 9.3.1.20.2. Let $x \in \Gamma(Z, H^0 u^!(\mathcal{E}))$, $y \in \mathcal{H}om_{\mathcal{B}_{\mathfrak{X}}}(\mathcal{B}_{\mathfrak{X}}/\mathcal{I}\mathcal{B}_{\mathfrak{X}}, \mathcal{E})$ and $Q \in \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$. Choose any $Q_{\mathfrak{X}} \in \widetilde{D}_{\mathfrak{X}^\sharp, \mathfrak{Z}^\sharp, \underline{t}}$ such that $\vartheta(Q) = [Q_{\mathfrak{X}}]_Z$ (see 5.2.2.5.7 or 9.3.1.4.1). It follows from 5.2.5.2.1 or 9.3.1.5.1 that we have the formula

$$\mathrm{ev}_1(Q \cdot x) = Q_{\mathfrak{X}} \cdot \mathrm{ev}_1(x), \qquad \mathrm{ev}_1(Q \cdot y) = Q_{\mathfrak{X}} \cdot \mathrm{ev}_1(y). \tag{9.3.1.20.3}$$

9.3.2 Adjunction, relative duality isomorphism

Notation 9.3.2.1. For any $* \in \{\mathrm{r}, \mathrm{l}\}$, we get the acyclic functor $\widetilde{u}_+ : D({}^*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}) \to D({}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$ by setting for any $\mathcal{F} \in D({}^l\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ and $\mathcal{N} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$,

$$\widetilde{u}_+(\mathcal{F}) := u_*(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \otimes_{\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}} \mathcal{F}), \tag{9.3.2.1.1}$$

$$\widetilde{u}_+(\mathcal{N}) := u_*(\mathcal{M} \otimes_{\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}} \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}) \xrightarrow{\sim} u_*(\mathcal{M}) \otimes_{\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}} (\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \tag{9.3.2.1.2}$$

These functors preserve the coherence. Beware, in the first respective case, this notation has not to be confused with its completed version denoted by $\widetilde{u}^{(m)!}$ (see 7.5.8.1). We will work with the completed version functor in the subsection 9.3.3.

9.3.2.2. Let $\mathcal{G} \in D({}^{\mathrm{l}}u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$. Via the isomorphism

$$u^{-1}\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{u^{-1}\mathcal{B}_{\mathfrak{X}}} \mathcal{G} \xrightarrow{\sim} \left(u^{-1}\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{u^{-1}\mathcal{B}_{\mathfrak{X}}} u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\right) \otimes_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{G} \xrightarrow{\sim} u^{-1}\left(\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}\right) \otimes_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{G},$$

we get $u^{-1}\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{u^{-1}\mathcal{B}_{\mathfrak{X}}} \mathcal{G} \in D({}^{\mathrm{r}}u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$. Similarly to 5.1.2.8.1, we have the isomorphism of $D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$

$$\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} u_*(\mathcal{G}) \xrightarrow{\sim} u_*\left(u^{-1}\widetilde{\omega}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{G}\right). \tag{9.3.2.2.1}$$

By using 8.7.2.5.1 (resp. 9.3.2.2.1) instead of 4.3.5.6.1 (resp. 5.1.2.8.1), by copying the proof of 5.1.3.2.1, for any $\mathcal{F} \in D({}^l\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ we get the canonical isomorphism

$$\widetilde{\omega}_{\mathfrak{X}} \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{u}_+(\mathcal{F}) \xrightarrow{\sim} \widetilde{u}_+(\widetilde{\omega}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{Z}}} \mathcal{F}). \tag{9.3.2.2.2}$$

Proposition 9.3.2.3. *Let $\mathcal{M}$ be a right (resp. left) $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module, $\mathcal{N}$ be a right (resp. left) $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-module.*

(a) We have the canonical adjunction morphisms $\mathrm{adj}\colon \widetilde{u}_+ H^0\widetilde{u}^!(\mathcal{M}) \to \mathcal{M}$ *and* $\mathrm{adj}\colon \mathcal{N} \to H^0\widetilde{u}^!\widetilde{u}_+(\mathcal{N})$. *Moreover, the compositions* $H^0\widetilde{u}^!(\mathcal{M}) \xrightarrow{\mathrm{adj}} H^0\widetilde{u}^!\widetilde{u}_+H^0\widetilde{u}^!(\mathcal{M}) \xrightarrow{\mathrm{adj}} H^0\widetilde{u}^!(\mathcal{M})$ *and* $\widetilde{u}_+(\mathcal{N}) \xrightarrow{\mathrm{adj}} \widetilde{u}_+H^0\widetilde{u}^!\widetilde{u}_+(\mathcal{N}) \xrightarrow{\mathrm{adj}} \widetilde{u}_+(\mathcal{N})$ *are the identity.*

(b) Using the above adjunction morphisms, we construct maps

$$\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}(\widetilde{u}_+(\mathcal{N}), \mathcal{M}) \to u_*\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}}(\mathcal{N}, H^0\widetilde{u}^!(\mathcal{M})),\ u_*\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}}(\mathcal{N}, H^0\widetilde{u}^!(\mathcal{M})) \to \mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}(\widetilde{u}_+(\mathcal{N}), \mathcal{M}),$$

which are inverse of each other.

(c) If $\mathcal{M}$ is an injective right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module, then $H^0\widetilde{u}^!(\mathcal{M})$ is an injective right $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}$-module.

Proof. We can copy the proof of 5.2.6.1. □

Remark 9.3.2.4. The local computations of the adjunction morphisms in the left case detailed at 5.2.6.2 still hold in the context 9.3.2.3.

Corollary 9.3.2.5. *Let $* \in \{\mathrm{r}, \mathrm{l}\}$. Let $\mathcal{M} \in D^+({}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$, $\mathcal{N} \in D({}^*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$. We have the isomorphisms:*

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}(\widetilde{u}_+(\mathcal{N}), \mathcal{M}) \xrightarrow{\sim} u_*\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp}}(\mathcal{N}, \widetilde{u}^!(\mathcal{M})). \tag{9.3.2.5.1}$$

Proof. Taking a K-injective object representing $\mathcal{M}$, this isomorphism is a consequence of 9.3.2.3.b–c. □

Notation 9.3.2.6. Moreover, we get the functor $\mathbb{D}\colon D({}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}) \to D({}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$ by setting for any $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$, $\mathcal{E} \in D({}^l\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$

$$\mathbb{D}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}(\mathcal{M}, \omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})[d_X], \mathbb{D}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}(\mathcal{E}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1})[d_X], \tag{9.3.2.6.1}$$

which are computed respectively by taking an injective resolution of $\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$ and $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1}$. These functors preserve the perfectness and commute with the quasi-inverse functors $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1}$ and $\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} -$ exchanging left and right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}$-module structures, i.e. similarly to 9.2.4.20.3, we get the isomorphisms:

$$\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathbb{D}(\mathcal{E}) \xrightarrow{\sim} \mathbb{D}(\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{9.3.2.6.2}$$

Lemma 9.3.2.7. *Let $\mathcal{N} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ with $* = r$ or $* = l$. We have $\widetilde{u}_+(\mathcal{N}) \in D^{\mathrm{b}}_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$.*

Proof. We can copy 5.2.6.4. □

Corollary 9.3.2.8. *Let $\mathcal{N} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp})$ with $* = r$ or $* = l$. We have the isomorphism of $D^{\mathrm{b}}_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp})$:*

$$\mathbb{D} \circ \widetilde{u}_+(\mathcal{N}) \xrightarrow{\sim} \widetilde{u}_+ \circ \mathbb{D}(\mathcal{N}). \tag{9.3.2.8.1}$$

Proof. Using 9.3.1.19.1 and 9.3.2.5, we can copy the proof of 5.2.6.6. □

We can complete the subsection with the case of quasi-coherent complexes as follows.

Proposition 9.3.2.9. *Let $\mathcal{M}^{(\bullet)}_\bullet$ be a right (resp. left) $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module, $\mathcal{N}^{(\bullet)}_\bullet$ be a right (resp. left) $\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet}$-module.*

(a) There exists a canonical functorial $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-linear morphism $\mathrm{adj}\colon \widetilde{u}^{(\bullet)}_{\bullet+} H^0 \widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet) \to \mathcal{M}^{(\bullet)}_\bullet$ and a canonical functorial $\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet}$-linear morphism $\mathrm{adj}\colon \mathcal{N}^{(\bullet)}_\bullet \to H^0\widetilde{u}^{(\bullet)!}_\bullet \widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet)$ so that the compositions $H^0\widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet) \xrightarrow{\mathrm{adj}} H^0\widetilde{u}^{(\bullet)!}_\bullet \widetilde{u}^{(\bullet)}_{\bullet+} H^0\widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet) \xrightarrow{\mathrm{adj}} H^0\widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet)$ and $\widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet) \xrightarrow{\mathrm{adj}} \widetilde{u}^{(\bullet)}_{\bullet+} H^0\widetilde{u}^{(\bullet)!}_\bullet \widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet) \xrightarrow{\mathrm{adj}} \widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet)$ are the identity.

(b) Using the above adjoint morphisms, we construct maps

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}}(\widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet), \mathcal{M}^{(\bullet)}_\bullet) \to \mathrm{Hom}_{\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet}}(\mathcal{N}^{(\bullet)}_\bullet, H^0\widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet)),$$
$$\mathrm{Hom}_{\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet}}(\mathcal{N}^{(\bullet)}_\bullet, H^0\widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet)) \to \mathrm{Hom}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}}(\widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet), \mathcal{M}^{(\bullet)}_\bullet),$$

which are inverse of each other.

(c) If $\mathcal{M}^{(\bullet)}_\bullet$ is an injective right (resp. left) $\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}$-module, then $H^0\widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet)$ is an injective right (resp. left) $\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet}$-module.

Proof. It follows from the fact that the adjunction morphisms of 5.2.6.1 are compatible with base changes and level changes. □

Corollary 9.3.2.10. *Let $* \in \{\mathrm{r}, \mathrm{l}\}$, $\mathcal{M}^{(\bullet)}_\bullet \in D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet})$, $\mathcal{N}^{(\bullet)}_\bullet \in D({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet})$. We have the isomorphisms*

$$\mathbb{R}\mathrm{Hom}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}}(\widetilde{u}^{(\bullet)}_{\bullet+}(\mathcal{N}^{(\bullet)}_\bullet), \mathcal{M}^{(\bullet)}_\bullet) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet/S^\sharp_\bullet}}(\mathcal{N}^{(\bullet)}_\bullet, \widetilde{u}^{(\bullet)!}_\bullet(\mathcal{M}^{(\bullet)}_\bullet)).$$

Proof. Taking an injective resolution of $\mathcal{M}$, this is a consequence of 9.3.2.9.b–c. □

Corollary 9.3.2.11. *Let $* \in \{\mathrm{r}, \mathrm{l}\}$, $\mathcal{N}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$, $\mathcal{M}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. We have the isomorphisms*

$$\mathrm{Hom}_{\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))}(\widetilde{u}^{(\bullet)}_+(\mathcal{N}^{(\bullet)}), \mathcal{M}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Hom}_{\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))}(\mathcal{N}^{(\bullet)}, \widetilde{u}^{(\bullet)!}(\mathcal{M}^{(\bullet)})).$$

Proof. It follows by localisation from 9.3.2.10 that we have the adjoint paire $((\widetilde{u}^{(\bullet)}_{\bullet+}, \widetilde{u}^{(\bullet)!}_\bullet)$ with respect to categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet}(D))$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{Z^\sharp_\bullet}(E))$. We conclude with the equivalence of categories of 8.5.4.5. □

9.3.3 pushforwards and extraordinary pullbacks of p-torsion free separated complete modules, adjunction

We suppose in this subsection that we are in the local context of 9.3.1.2. When we deal with complexes of $\widetilde{\mathcal{D}}^{(m)}$-modules which are only quasi-coherent and not necessarily coherent, the extraordinary pullbacks and the pushforwards we prefer to work with are that defined respectively at 7.5.5.6 and 7.5.8.1 instead of 9.3.1.8.1 and 9.3.2.1. To consider the case of quasi-coherent complexes of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-modules which are in fact a $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-modules, the notion of p-torsion free, separated and complete (for the topology p-adic topology) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module appears naturally. Unfortunately, this category is a priori not stable by the functor $H^0 u_D^{(m)!}$ (see 9.3.3.2). To overcome this problem, in this subsection we focus on the categories of p-torsion free, separated and complete (for the topology p-adic) $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-modules (or $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-modules), where $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$. We define on these categories the notion of direct image (resp. of extraordinary inverse image) of level m with overconvergent singularities along D by u denoted by $u^{(m)}_{D+}$ and (resp. $H^0 u_D^{(m)!}$). We also check the adjunction morphisms are still valid in this context, which extends 9.3.2. The results of this subsections will be useful in order to prove the key lemma 15.3.5.3.

9.3.3.1. We explain in this paragraph why the functor $u^{(m)}_{D,+}$ preserves p-torsion free quasi-coherent left $\widetilde{\mathcal{D}}^{(m)}$-modules.

Let $\mathcal{F}$ be a p-torsion free quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module (see 7.2.3.5 and 7.3.1.7). We have the ringed topoi morphism $\underleftarrow{l}_{\mathfrak{X}} \colon (X_\bullet, \widetilde{\mathcal{D}}^{(m)}_{X^\sharp_\bullet/S^\sharp_\bullet}(D)) \to (\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, and $\underleftarrow{l}_{\mathfrak{Z}} \colon (Z_\bullet, \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/S^\sharp_\bullet}(E)) \to (\mathfrak{Z}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E))$. We get the quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/S^\sharp_\bullet}(E)$-module by setting

$$\mathcal{F}_\bullet := \mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}}(\mathcal{F}) = \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/S^\sharp_\bullet}(E) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} \mathcal{F} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z^\sharp_\bullet/S^\sharp_\bullet}(E) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} \mathcal{F}.$$

By definition (see 7.5.8.1, here $u_D = \widetilde{u}$), the direct image of level m with overconvergent singularities along D of $\mathcal{F}$ by u is defined by setting

$$u^{(m)}_{D,+}(\mathcal{F}) := \mathbb{R}\underleftarrow{l}_{\mathfrak{Z}*}(u^{(m)}_{\bullet D,+}(\mathcal{F}_\bullet)) \tag{9.3.3.1.1}$$

Now, applying the base change isomorphisms (5.2.5.11), for any integer $i' \geq i$, we get an isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i}(D)$-modules $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i}(D) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_{i'}/S^\sharp_{i'}}(D)} u_{i'D+}(\mathcal{F}_{i'}{}^{(m)}) \xrightarrow{\sim} u_{iD+}(\mathcal{F}_i{}^{(m)})$ (recall the functor u_{iD+} is exact following 5.2.5.11). By Mittag-Leffler of 7.3.1.3, we get the first isomorphism below

$$u^{(m)}_{D,+}(\mathcal{F}) \xleftarrow{\sim} \underleftarrow{l}_{\mathfrak{Z}*}(u^{(m)}_{\bullet D,+}(\mathcal{F}_\bullet)) = \varprojlim u_{iD,+}\mathcal{F}_i{}^{(m)} \xleftarrow{\sim} u_*\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D) \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} \mathcal{F}\right), \tag{9.3.3.1.2}$$

whilst the second isomorphism follows from the fact that u_* commutes with projective limit. According to the stability of quasi-coherence under direct image by u (resp. via the local description of 9.3.3), we deduce that $u^{(m)}_{D,+}(\mathcal{F})$ is a quasi-coherent (resp. p-torsion free) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module, and we are done.

9.3.3.2. We have defined the extraordinary inverse image functor of level m with overconvergent singularities along D by u (see 7.5.5.6) that we denote by $u_D^{(m)!} \colon D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)) \to D({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E))$ for any $* \in \{r, l\}$, which preserves the boundedness and quasi-coherence (see 9.2.1.9). Let $\mathcal{E}$ be a p-torsion free quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module.

(a) As $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, by stability of quasi-coherence (see 9.2.1.9), we get $u_D^{(m)!}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E))$.

(b) The canonical morphism

$$H^0 u_D^{(m)!}(\mathcal{E}) \to \varprojlim_i H^0 u_{iD}^{(m)!}(\mathcal{E}_i) \tag{9.3.3.2.1}$$

is an isomorphism. Indeed, this follows from the isomorphism $u_D^{(m)!}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\varprojlim_i u_{iD}^{(m)!}(\mathcal{E}_i)$, to which we apply the functor H^0. As $u_{iD}^{(m)!}(\mathcal{E}_i) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp/S_i^\sharp}(E) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} u_D^{(m)!}(\mathcal{E})$, we obtain the isomorphism $H^0 u_{iD}^{(m)!}(\mathcal{E}_i) \xrightarrow{\sim} H^0(\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp/S_i^\sharp}(E) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} u_D^{(m)!}(\mathcal{E}))$. In particular, the $\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp/S_i^\sharp}(E)$-modules appearing in the projective limit 9.3.3.2.1 are quasi-coherent. We recall that $u_* H^0 u_{iD}^{(m)!}(\mathcal{E}_i) = \cap_{s=r+1}^d \ker(\mathcal{E}_i \xrightarrow{t_s} \mathcal{E}_i)$ (see 9.3.1.20.2). Hence, we get the inclusion:

$$H^0 u_D^{(m)!}(\mathcal{E}) \xrightarrow{\sim} \varprojlim_i H^0 u_{iD}^!(\mathcal{E}_i) \subset \varprojlim_i \mathcal{E}_i = \mathcal{E}. \tag{9.3.3.2.2}$$

This yields that $H^0 u_D^{(m)!}(\mathcal{E})$ is p-torsion free and we compute $u_* H^0 u_D^{(m)!}(\mathcal{E}) = \cap_{s=r+1}^d \ker(\mathcal{E} \xrightarrow{t_s} \mathcal{E})$. Notice this is not clear that the isomorphism 9.3.3.2.1 is still valid by replacing $H^0 u^{(m)!}$ by u^*. But, we further finiteness condition hypothesis, we have such an equality after applying the global section functor (see 9.3.4.9).

(c) Since $H^0 u_D^{(m)!}(\mathcal{E})$ is p-torsion free then we get

$$(H^0 u_D^{(m)!}(\mathcal{E}))_i := \widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp/S_i^\sharp}(E) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} H^0 u_D^{(m)!}(\mathcal{E}) \xleftarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp/S_i^\sharp}(E) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} H^0 u_D^{(m)!}(\mathcal{E}).$$

It seems false that $(H^0 u_D^{(m)!}(\mathcal{E}))_i$ is a quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp/S_i^\sharp}(E)$-module (beware this has not to be confused with $H^0 u_{iD}^!(\mathcal{E}_i)$ which is quasi-coherent: see 9.3.3.11). Hence, we have a priori $H^0 u_D^{(m)!}(\mathcal{E}) \notin D^{\mathrm{b}}_{\mathrm{qc}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E))$. This means that the category of p-torsion free quasi-coherent $\mathcal{D}$-modules is not stable under the functor $H^0 u^{(m)!}$. It is for this reason why we will work in this section with global sections and not with the whole sheaves. In the next section (see 9.3.3.10), we will have locally some stability of the separated completeness p-torsion free for global sections (see also 7.2.3.13.(ii) to recall the link with pseudo quasi-coherence).

Notation 9.3.3.3. We denote by $\mathcal{V}\{\partial_{r+1}, \ldots, \partial_d\}^{(m)}$ the sub-$\mathcal{V}$-algebra of $\Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \subset \Gamma(\mathfrak{X}, \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ generated by the elements $\{\partial_{r+1}^{\langle j_1\rangle_{(m)}}, \partial_{r+2}^{\langle j_2\rangle_{(m)}}, \ldots, \partial_d^{\langle j_{d-r}\rangle_{(m)}} \mid j_1, \ldots, j_{d-r} \in \mathbb{N}\}$. It is equal to the free $\mathcal{V}$-module whose basis is given by $\{\underline{\partial}_{(r)}^{\langle(\underline{0},\underline{j})\rangle_{(m)}} \mid \underline{j} \in \mathbb{N}^{d-r}\}$, where $\underline{0} := (0, \ldots, 0) \in \mathbb{N}^r$.

9.3.3.4 (p-adic completed maps)**.** By taking global section of the isomorphism ϑ of 5.2.2.1, we get the isomorphism of $\mathcal{V}$-algebras

$$\widehat{\vartheta} \colon \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D) \tag{9.3.3.4.1}$$

where $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$ are the global sections of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$. Remark that for unified notation reason $\widehat{\vartheta}$ was simply denoted by ϑ in 9.3.1.4 ; we hope this will not be too much confusing. We denote by $[-]_{\mathfrak{Z}} \colon \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ and by $[-]'_{\mathfrak{Z}} \colon \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)I$ the canonical projections.

By taking the global section of the p-adic completion of 5.2.2.5.3, we get the homomorphism of $\widetilde{B}^{(m)}_{\mathfrak{X}}(D)$ and of $\mathcal{V}$-algebras (see 5.2.2.7):

$$\widehat{\sigma}^{(m)}_{\mathfrak{Z}^\sharp,\mathfrak{X}^\sharp,\widetilde{t}} \colon \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}/\mathfrak{S}^\sharp}(D) \to \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E), \tag{9.3.3.4.2}$$

which is given by the formula $\widehat{\sigma}^{(m)}_{\mathfrak{Z}^\sharp,\mathfrak{X}^\sharp,\widetilde{t}}(\sum_{\underline{i}\in\mathbb{N}^r} b_{\underline{i}} \underline{\partial}_{(r)}^{\langle(\underline{i},\underline{0})\rangle_{(m)}}) = \sum_{\underline{i}\in\mathbb{N}^r} [b_{\underline{i}}]_{\mathfrak{Z}} \underline{\partial}_{(r)}^{\langle\underline{i}\rangle_{(m)}}$, where $b_{\underline{i}} \in \widetilde{B}^{(m)}_{\mathfrak{X}}(D)$. The map $\widehat{\sigma}^{(m)}_{\mathfrak{Z}^\sharp,\mathfrak{X}^\sharp,\widetilde{t}}$ is surjective with kernel equal to $I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$; the induced factorisation is an inverse of $\widehat{\vartheta}$.

By taking the global section of the bijection 9.3.1.18.1, we get

$$\widehat{\iota}_{\widetilde{t}} \colon \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)I. \tag{9.3.3.4.3}$$

Via this bijection, we get a structure of $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D), \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D))$-bimodule on $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)I$ such that the underlying structure of left $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module is equal to its natural structure.

By taking the global section of the p-adic completion of the morphism 5.2.2.16.1, we get the $\widetilde{B}^{(m)}_{\mathfrak{X}}(D)$-linear homomorphism

$$(\varsigma^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}})^\frown\colon \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D) \to \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E). \tag{9.3.3.4.4}$$

Moreover, $(\overline{\varsigma}^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}})^\frown$ is surjective with kernel equal to $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)I$ and we denote by

$$(\overline{\varsigma}^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}})^\frown\colon \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)I \xrightarrow{\sim} \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E) \tag{9.3.3.4.5}$$

the induced $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-linear isomorphism (use 5.2.2.18).

The elements of $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ can be written uniquely of the form $\sum_{\underline{j}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}P_{\underline{j}}$, with $P_{\underline{j}}\in\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}/\mathfrak{S}^\sharp}(D)$, the sequence $P_{\underline{j}}$ converges to 0 for the p-adic topology. It follows from 5.2.2.16.3 that we get the formula

$$(\varsigma^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}})^\frown\left(\sum_{\underline{j}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}P_{\underline{j}}\right) = (\varsigma^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}})^\frown\left(\left[\sum_{\underline{j}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}P_{\underline{j}}\right]'_{\mathfrak{Z}}\right) = \sum_{\underline{j}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}\otimes\widehat{\sigma}^{(m)}_{\mathfrak{Z}^\sharp,\mathfrak{X}^\sharp,\underline{\widetilde{t}}}(P_{\underline{j}}). \tag{9.3.3.4.6}$$

In particular, we get the formula $\varsigma^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}}(\underline{\partial}^{\langle(\underline{i},\underline{j})\rangle_{(m)}}_{(r)}) = \overline{\varsigma}^{(m)}_{\mathfrak{X}^\sharp,\underline{\widetilde{t}}}(\underline{\xi}^{\langle(\underline{i},\underline{j})\rangle_{(m)}}_{(r)}) = \underline{\partial}^{\langle(\underline{0},\underline{j})\rangle_{(m)}}\otimes\underline{\partial}^{\langle\underline{i}\rangle_{(m)}}_{(r)}$, for any $\underline{i}\in\mathbb{N}^r$ and $\underline{j}\in\mathbb{N}^{d-r}$.

9.3.3.5 (Local computation of the pushforward)**.** By composing 9.3.3.4.3 with 9.3.3.4.5, we get the isomorphism:

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D) \xrightarrow{\sim} \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E). \tag{9.3.3.5.1}$$

This yields the isomorphism

$$\Gamma(\mathfrak{X}, u^{(m)}_+(\mathcal{F})) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{X}\leftarrow\mathfrak{Z}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)}F \underset{9.3.3.5.1}{\xrightarrow{\sim}} \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}F. \tag{9.3.3.5.2}$$

Notation 9.3.3.6. Let F be a p-torsion free $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module which is separated and complete for the p-adic topology. We define the direct image of F by u of level m as

$$u^{(m)}_{D,+}(F) := \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D)\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)}F \underset{9.3.3.5.1}{\xrightarrow{\sim}} \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}F. \tag{9.3.3.6.1}$$

The formula 9.3.3.5.2 justifies the notation. The set $u^{(m)}_{D,+}(F)$ is endowed with a p-torsion free left $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module structure which is separated and complete for the p-adic topology. Since F is p-torsion free, separated and complete, then it follows from 9.3.3.6.1 that the elements of $u^{(m)}_{D,+}(F)$ can uniquely be written as $\sum_{\underline{k}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\otimes x_{\underline{k}}$, with $x_{\underline{k}}\in F$ and $\lim_{|\underline{k}|\to\infty}x_{\underline{k}}=0$ (for the p-adic topology of F). The elements of $\left(u^{(m)}_{D,+}(F)\right)_{\mathbb{Q}}$ can be uniquely written as $\sum_{\underline{k}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\otimes x_{\underline{k}}$, with $x_{\underline{k}}\in F_{\mathbb{Q}}$ and $\lim_{|\underline{k}|\to\infty}x_{\underline{k}}=0$ (for the topology of $F_{\mathbb{Q}}$ induced by the p-adic topology of F).

Lemma 9.3.3.7. *Let $F\hookrightarrow G$ be a monomorphism of p-torsion free separated and complete (for the p-adic topology) $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-modules.*

(a) The morphism $u^{(m)}_{D,+}(F)\to u^{(m)}_{D,+}(G)$ (resp. $(u^{(m)}_{D,+}(F))_{\mathbb{Q}}\to(u^{(m)}_{D,+}(G))_{\mathbb{Q}}$) is a monomorphism.

(b) The canonical morphism $u^{(m)}_{D,+}(F)\to u^{(m)}_{D,+}(G)$ (resp. $(u^{(m)}_{D,+}(F))_{\mathbb{Q}}\to(u^{(m)}_{D,+}(G))_{\mathbb{Q}}$) is an isomorphism if and only if the canonical morphism $F\hookrightarrow G$ (resp. $F_{\mathbb{Q}}\hookrightarrow G_{\mathbb{Q}}$) is an isomorphism.

Proof. It follows from the description of the writing of the elements of $u^{(m)}_{D,+}(F)$ given at 9.3.3.6. □

Lemma 9.3.3.8. *Let F be a p-torsion free $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module which is separated and complete for the p-adic topology. Then F is a coherent $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module if and only if $u^{(m)}_{D,+}(F)$ is a coherent $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module.*

Proof. Since the functor $u^{(m)}_{D,+}$ defined at 9.3.3.1.1 for the sheaves preserve the coherence, since we have the theorems of type A (see 7.2.3.10), the isomorphism 9.3.3.5.2 allows us to conclude the necessity of the assertion of the lemma. Conversely, suppose that F is not a coherent $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module. Then there exists a strictly increasing sequence $(F_n)_{n\in\mathbb{N}}$ of coherent $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-submodule of F. Then it comes from the lemma 9.3.3.7.(b) that we get the strictly increasing sequence $(u^{(m)}_{D,+}(F_n))_{n\in\mathbb{N}}$ of coherent $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-submodule of $u^{(m)}_{D,+}(F)$. Since $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$ is noetherian, this last one is not a coherent $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module. □

Remark 9.3.3.9. Let $H^{(m)}$ be a $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)_{\mathbb{Q}}$-module such that there exists a p-torsion free , separated and complete (for the topology p-adic) $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module $\mathring{H}^{(m)}$, endowed with an isomorphism of $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)_{\mathbb{Q}}$-modules of the form $\mathring{H}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} H^{(m)}$. We remark that the topology on $H^{(m)}$ induced by the basis of neighborhood of $p^n\mathring{H}^{(m)}$ makes $H^{(m)}$ a Banach K-vector space.

Moreover, if $H^{(m)}$ is a coherent $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module, then this topology is independent of the choice of such a $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module $\mathring{H}^{(m)}$ and this latter is necessarily $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-coherent. Indeed, this comes from the theorem of Banach (e.g. see [BGR84, 2.8.1]) and from the noetherianity of $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$.

This implies that if $H^{(m)}$ is a coherent $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module then $\left(u^{(m)}_{D,+}(\mathring{H}^{(m)})\right)_{\mathbb{Q}}$ is a coherent $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module and does not depend of the $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module separated complete $\mathring{H}^{(m)}$ p-torsion free , such that $\mathring{H}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} H^{(m)}$. Then we write $u^{(m)}_{D,+}(H^{(m)})$.

Notation 9.3.3.10. Let E be a p-torsion free, separated and complete for the p-adic topology $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module Put $E_i := D^{(m)}_{X^\sharp_i/S^\sharp_i}(D) \otimes_{\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} E \xleftarrow{\sim} D^{(m)}_{X^\sharp_i/S^\sharp_i}(D) \otimes^{\mathbb{L}}_{\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)} E$. We define the extraordinary inverse image of E by u by setting

$$H^0u^{(m)!}_D(E) := \varprojlim_i H^0u^!_{iD}(E_i) \subset \varprojlim_i E_i = E. \tag{9.3.3.10.1}$$

It follows from 9.3.3.2.1 that for any p-torsion free quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module $\mathcal{E}$, by applying the functor $\Gamma(\mathfrak{Z}, -)$ to 9.3.3.2.2, we get the isomorphism

$$\Gamma(\mathfrak{Z}, H^0u^{(m)!}_D(\mathcal{E})) \xrightarrow{\sim} H^0u^{(m)!}_D(\Gamma(\mathfrak{Z}, \mathcal{E})), \tag{9.3.3.10.2}$$

which justifies the notation 9.3.3.10.1.

We calculate $H^0u^{(m)!}_D(E) \xrightarrow{\sim} \cap^d_{s=r+1} \ker(E \xrightarrow{t_s} E)$. By composing with the canonical inclusion in E, this yields the map denoted by $\mathrm{ev}_1\colon H^0\widetilde{u}^{(m)!}(E) \hookrightarrow E$. We equip $H^0u^{(m)!}_D(E)$ with a canonical structure of left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module as follows: for any $Q \in \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$, for any $x \in H^0u^{(m)!}_D(E)$, put

$$\mathrm{ev}_1(Q \cdot x) := Q_{\mathfrak{X}} \cdot \mathrm{ev}_1(x) \tag{9.3.3.10.3}$$

where $Q_{\mathfrak{X}} \in \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$ is such that $[Q_{\mathfrak{X}}]_{\mathfrak{Z}} = \widehat{\vartheta}(Q)$ where $[-]_{\mathfrak{Z}}$ is the canonical projection $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D) \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)/I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}(D)$ and where $\widehat{\vartheta}$ is the isomorphism 9.3.3.4.1. Indeed, since I is generated by $t_{r+1}, \dots, t_d$, this definition do not depend on the choice of such $Q_{\mathfrak{X}}$. By using 9.3.1.20.3, we remark that the isomorphism 9.3.3.10.2 becomes an isomorphism of left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-modules, which explains the definition.

Since for $s = r+1, \dots, d$ the maps $t_s\colon E \to E$ are continuous (for the p-adic topology), since $H^0u^{(m)!}_D(E) \xrightarrow{\sim} \cap^d_{s=r+1} \ker(E \xrightarrow{t_s} E)$, then $H^0u^{(m)!}_D(E)$ is separated and complete for the topology induced by the p-adic topology of E. Since E is p-torsion free, then we deduce from $H^0u_D^{(m)!}(E) = \cap_{s=r+1}^d \ker(E \xrightarrow{t_s} E)$ the equality $H^0u_D^{(m)!}(E) \cap p^{i+1}E = p^{i+1}H^0u_D^{(m)!}(E)$. In particular, the p-adic topology of $H^0u_D^{(m)!}(E)$ and the topology of $H^0u_D^{(m)!}(E)$ induced by the p-adic topology of E are identical. This yields that $H^0u_D^{(m)!}(E)$ is a p-torsion free, separated and complete left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module.

9.3.3.11. Let E be a p-torsion free, separated and complete for the p-adic topology $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module. Put $\left(H^0u_D^{(m)!}(E)\right)_i := D^{(m)}_{Z_i^\sharp/S_i^\sharp}(E) \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)} H^0u_D^{(m)!}(E)$, as $H^0u_D^{(m)!}(E) \cap p^{i+1}E = p^{i+1}H^0u_D^{(m)!}(E)$, we get the inclusion

$$\left(H^0u_D^{(m)!}(E)\right)_i \subset H^0u_{iD}^{(m)!}(E_i). \tag{9.3.3.11.1}$$

When we take the structure of $D^{(m)}_{Z_i^\sharp/S_i^\sharp}(E)$-module on $\left(H^0u_D^{(m)!}(E)\right)_i$ induced by the structure of $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module of $H^0u_D^{(m)!}(E)$ defined by the formula (9.3.3.10.3) and the structure of $D^{(m)}_{Z_i^\sharp/S_i^\sharp}(E)$-module of $H^0u_{iD}^{(m)!}(E_i)$ given by (9.3.1.20.3), we check that the inclusion 9.3.3.11.1 is $D^{(m)}_{Z_i^\sharp/S_i^\sharp}(E)$-linear.

Beware, it seems false that the map 9.3.3.11.1 is bijective in general. However, this becomes a bijection by taking projective limits (see 9.3.3.12).

Remark 9.3.3.12. With the notations of 9.3.3.11, the canonical morphism

$$u_+H^0u_D^{(m)!}(E) = \varprojlim_i u_{iD,+}\left(H^0u_D^{(m)!}(E)\right)_i \to \varprojlim_i u_{iD,+}H^0u_{iD}^{(m)!}(E_i)$$

is an isomorphism, i.e. that the canonical morphism

$$\mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}H^0u_D^{(m)!}(E) \to \varprojlim_i(O_{S_i}\{\partial_{r+1},\dots,\partial_d\}^{(m)} \otimes_{O_{S_i}} H^0u_i^{(m)!}(E_i))$$

is an isomorphism. Indeed, the elements $x \in \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}H^0u_D^{(m)!}(E)$ can be uniquely written of the form $x = \sum_{\underline{l}\in\mathbb{N}^{d-r}} \underline{\partial}^{\langle(\underline{0},\underline{l})\rangle} \otimes x_{\underline{l}}$, with $x_{\underline{l}} \in H^0u_D^{(m)!}(E)$ (and converging to 0). Let $(x_i)_i \in \varprojlim_i(O_{S_i}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\otimes_{O_{S_i}} H^0u_i^{(m)!}(E_i))$. We can write uniquely x_i of the form $x_i = \sum_{\underline{l}\in\mathbb{N}^{d-r}} \underline{\partial}^{\langle(\underline{0},\underline{l})\rangle}\otimes x_{\underline{l},i}$, with $x_{\underline{l},i} \in H^0u_D^{(m)!}(E_i)$ and the sum being this time finite. By unicity, we get that $(x_{\underline{l},i})_i \in \varprojlim_i H^0u_{iD}^{(m)!}(E_i)$. The equality $H^0u^{(m)!}(E) = \varprojlim_i H^0u_{iD}^{(m)!}(E_i)$ allows us then to conclude.

Proposition 9.3.3.13 (Adjunction morphism I)**.** *Let $F^{(m)}$ be a p-torsion free, separated and complete left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module. By identifying $H^0u_D^{(m)!}\left(u_{D+}^{(m)}(F^{(m)})\right)$ with*

$$\cap_{s=r+1}^d \ker\left(\mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}F \xrightarrow{t_s} \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}F\right) \subset \mathcal{V}\{\partial_{r+1},\dots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}}F$$

(see 9.3.3.6.1 and 9.3.3.10), we define the canonical adjunction morphism

$$\mathrm{adj}\colon F^{(m)} \to H^0u_D^{(m)!}\left(u_{D+}^{(m)}(F^{(m)})\right), \tag{9.3.3.13.1}$$

by putting $\mathrm{adj}(x) = 1\otimes x$ *for any* $x \in F^{(m)}$*. The morphism* adj *is an isomorphism of* $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$*-modules.*

Proof. a) Let us prove that adj is $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-linear. For any left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module G, we write $[-]_i$ for the canonical morphism $G \to G/\pi^{i+1}G$. We have the commutative diagram:

$$\begin{array}{ccccc} F^{(m)} & \xrightarrow{\mathrm{adj}} & H^0u_D^{(m)!}\circ u_{D+}^{(m)}(F^{(m)}) & & \\ \downarrow{\scriptstyle [-]_i} & & \downarrow{\scriptstyle [-]_i} & & \\ F_i{}^{(m)} & \longrightarrow & \left(H^0u_D^{(m)!}\circ u_{D+}^{(m)}(F^{(m)})\right)_i & \hookrightarrow & H^0u_{iD}^{(m)!}\circ u_{iD+}^{(m)}(F_i{}^{(m)}), \end{array} \tag{9.3.3.13.2}$$

Here the bottom right arrow is derived from (9.3.3.11.1).- We compute that the composition of the bottom two morphisms is the morphism adj of (5.2.6.1) (see 5.2.6.2.4) which is $D^{(m)}_{Z^\sharp_i/S^\sharp_i}(E)$-linear. As the inclusion (9.3.3.11.1) is $D^{(m)}_{Z^\sharp_i/S^\sharp_i}(E)$-linear, it follows that the bottom left arrow of 9.3.3.13.2 is also $D^{(m)}_{Z^\sharp_i/S^\sharp_i}(E)$-linear. By passing to projective limit, we get that the top morphism of 9.3.3.13.2 is $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-linear.

b) Let us check that adj is bijective. Since the injectivity is clear, we reduce to prove the surjectivity as follows. Modulo the isomorphism 9.3.3.6, an element of $u^{(m)}_{D,+}(F^{(m)})$ can uniquely be written in form $\sum_{\underline{k}\in\mathbb{N}^{d-r}} \underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\otimes x_{\underline{k}}$ with $x_{\underline{k}}\in F^{(m)}$ and $\lim_{|\underline{k}|\to\infty} x_{\underline{k}}=0$. For $r+1\leq i\leq d$, we compute in $u^{(m)}_{D,+}(F^{(m)})$ the relation

$$t_i\cdot \sum_{\underline{k}\in\mathbb{N}^{d-r}} (\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\otimes x_{\underline{k}}) = -\sum_{\underline{k}\in\mathbb{N}^{d-r}|k_{i-r}\geq 1} \left\{ {k_{i-r} \atop 1} \right\} \underline{\partial}^{\langle(\underline{0},\underline{k})-1_i\rangle}\otimes x_{\underline{k}},$$

where $1_i=(0,\ldots,0,1,0,\ldots,0)$ with 1 at the i-th place (in fact, il suffices to verify the equality modulo π^{i+1}, which follows from the formulas 1.4.2.7 and 9.3.3.4.6. Since $F^{(m)}$ is p-torsion free, then this yields that any element of $H^0u^{(m)!}_D\circ u^{(m)}_{D+}(F^{(m)})$ must be of the form $1\otimes x$ with $x\in F^{(m)}$. □

Proposition 9.3.3.14 (Adjunction morphism II). *Let $E^{(m)}$ be a p-torsion free $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module which is separated and complete for the p-adic topology. By identifying $u^{(m)}_{D+}\circ H^0u^{(m)!}_D(E^{(m)})$ with*

$$\mathcal{V}\{\partial_{r+1},\ldots,\partial_d\}^{(m)}\widehat{\otimes}_{\mathcal{V}} \cap_{s=r+1}^d \ker\left(E\xrightarrow{t_s}\mathcal{V}E\right),$$

we define the map

$$\mathrm{adj}\colon u^{(m)}_{D+}\circ H^0u^{(m)!}_D(E^{(m)})\to E^{(m)} \tag{9.3.3.14.1}$$

by setting $\mathrm{adj}(\sum_{\underline{k}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\otimes x_{\underline{k}})=\sum_{\underline{k}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\cdot x_{\underline{k}}$, *with* $x_{\underline{k}}\in H^0u^{(m)!}_D(E^{(m)})$ *and* $\lim_{|\underline{k}|\to\infty}x_{\underline{k}}=0$. *The map* adj *is a morphism of* $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$*-modules.*

Proof. Consider the following canonical diagram

$$\begin{array}{ccccc} u^{(m)}_{D+}\circ H^0u^{(m)!}_D(E^{(m)}) & & \xrightarrow[9.3.3.14.1]{\mathrm{adj}} & & E^{(m)} \\ \downarrow{\scriptstyle [-]_i} & & & & \downarrow{\scriptstyle [-]_i} \\ u^{(m)}_{iD,+}\left(H^0u^{(m)!}_D(E^{(m)})\right)_i & \xrightarrow{9.3.3.11.1} & u^{(m)}_{iD,+}\circ H^0u^!_{iD}(E_i{}^{(m)}) & \xrightarrow[5.2.6.1]{\mathrm{adj}} & E_i{}^{(m)}. \end{array} \tag{9.3.3.14.2}$$

We already know both morphisms of the bottom of the diagram (9.3.3.14.2) are $D^{(m)}_{X^\sharp_i/S^\sharp_i}(D)$-linear. Moreover, by using the description of 5.2.6.2.3, we get the commutativity of (9.3.3.14.2) by calculating the map carrying an element of the form $\sum_{\underline{k}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\otimes x_{\underline{k}}$ to $\sum_{\underline{k}\in\mathbb{N}^{d-r}}\underline{\partial}^{\langle(\underline{0},\underline{k})\rangle}\cdot[x_{\underline{k}}]_i$ along the two possible paths of the diagram. The proposition now follows by passage to projective limit. □

9.3.3.15. Let E (resp. F) be a p-torsion free ,separated and complete $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module (resp. $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-module). Via the adjunction morphism 9.3.3.13.1, the functor $H^0u^{(m)!}_D$ induces the application

$$\mathrm{Hom}_{\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)}(u^{(m)}_{D+}(F),E)\to\mathrm{Hom}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)}(F,H^0u^{(m)!}_DE). \tag{9.3.3.15.1}$$

Via the adjunction morphism 9.3.3.14.1, the functor $u^{(m)}_{D+}$ induces the application

$$\mathrm{Hom}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)}(F,H^0u^{(m)!}_DE)\to\mathrm{Hom}_{\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)}(u^{(m)}_{D+}(F),E). \tag{9.3.3.15.2}$$

We compute that the composition morphism

$$u^{(m)}_{D+}(F)\overset{9.3.3.13.1}{\longrightarrow}u^{(m)}_{D+}\circ H^0u^{(m)!}_D\circ u^{(m)}_{D+}(F)\overset{9.3.3.14.1}{\longrightarrow}u^{(m)}_{D+}(F), \tag{9.3.3.15.3}$$

whose first arrow is the image by $u_{D+}^{(m)}$ of 9.3.3.13.1 and whose second one is 9.3.3.14.1 used for $E = u_{D+}^{(m)}(F)$, is the identity. Since the first arrow is an isomorphism, so is the second one. By a computation, we check moreover that the composition morphism

$$H^0 u_D^{(m)!}(E) \overset{9.3.3.14.1}{\longrightarrow} H^0 u_D^{(m)!} \circ u_{D+}^{(m)} \circ H^0 u_D^{(m)!}(E) \overset{9.3.3.13.1}{\longrightarrow} H^0 u_D^{(m)!}(E)$$

is the identity. This implies that the applications 9.3.3.15.1 and 9.3.3.15.2 above are converse one from the other.

Remark 9.3.3.16. By lack of stability of the category of quasi-coherent sheaves highlighted in the section 9.3, we do not have a priori an analogue in this context.

9.3.4 Inverse image by an exact closed immersion of affine $\mathcal{V}$-formal schemes

We suppose $\mathfrak{X}$ is affine. We denote by straight letter, the global section of a sheaf on $\mathfrak{X}$, e.g. $\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

9.3.4.1. For any left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{G}$, for any left $\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module G, we define a complex of left $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$-modules or respectively of left $\widetilde{D}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$-modules by setting

$$\mathbb{L}\widetilde{u}^*(\mathcal{G}) := \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} u^{-1}\mathcal{G}, \tag{9.3.4.1.1}$$

$$\mathbb{L}\widetilde{u}^*(G) := \widetilde{D}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} G. \tag{9.3.4.1.2}$$

We define in the same way the functor $\mathbb{L}\widetilde{u}_i^*$ on the category of left $\mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}(D)$-modules (resp. of left $D^{(m)}_{X_i^\sharp/S_i^\sharp}(D)$-modules).

9.3.4.2. Let $\mathcal{G}$ be a left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Let G be a left $\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. In accordance with notation 9.2.1.21 (in the $\mathcal{D}^\dagger$ case), we have the isomorphisms:

$$u^!_D(\mathcal{G})[d_u] = \mathbb{L}\widetilde{u}^*(\mathcal{G}) \overset{9.3.1.3.(a)}{\underset{\sim}{\longrightarrow}} u^{-1}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{G}), \tag{9.3.4.2.1}$$

$$\mathbb{L}\widetilde{u}^*(G) \overset{9.3.1.3.(b)}{\underset{\sim}{\longrightarrow}} \widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/I\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} G. \tag{9.3.4.2.2}$$

9.3.4.3. It follows from 9.3.1.3.6 that the canonical morphism

$$u^* D^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}} \to \Gamma(\mathfrak{X}, u^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}) \tag{9.3.4.3.1}$$

is an isomorphism. We extend later this commutation isomorphism (see 9.3.4.9.1).

9.3.4.4. For any right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module $\mathcal{M}$, for any right $\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module M, we define a complex of right $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-modules or respectively of right $\widetilde{D}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)$-modules by setting

$$\mathbb{L}\widetilde{u}^*_{\mathrm{r}}(\mathcal{M}) := u^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D), \tag{9.3.4.4.1}$$

$$\mathbb{L}\widetilde{u}^*_{\mathrm{r}}(M) := M \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)} \widetilde{D}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(D). \tag{9.3.4.4.2}$$

We define in the same way the functor $\mathbb{L}\widetilde{u}^*_{i,\mathrm{r}}$ on the category of right $\mathcal{D}^{(m)}_{X_i^\sharp/S_i^\sharp}(D)$-modules (resp. of right $D^{(m)}_{X_i^\sharp/S_i^\sharp}(D)$-modules).

9.3.4.5. Let $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. Let M be a right $\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module. In accordance with notation 9.2.1.21, we have the isomorphisms:

$$u^!_D(\mathcal{M})[d_u] = \mathbb{L}\widetilde{u}^*_{\mathrm{r}}(\mathcal{M}) \overset{9.3.1.18.1}{\underset{\sim}{\longrightarrow}} u^{-1}(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\mathcal{I}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{M}), \tag{9.3.4.5.1}$$

$$\mathbb{L}\widetilde{u}^*_{\mathrm{r}}(M) \overset{9.3.3.4.3}{\underset{\sim}{\longrightarrow}} \widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/I\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} M. \tag{9.3.4.5.2}$$

9.3.4.6 (Inverse image for bimodules)**.** By functoriality, the functor $\mathbb{L}u^*$ defined at 9.3.4.1 induces a functor $\mathbb{L}u^*\colon D(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D(\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ etc. To avoid confusion with the functor $\mathbb{L}u^*_{\mathrm{r}}\colon D(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to D(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp})$ we prefer to denote $\mathbb{L}u^*$ by $\mathbb{L}u^*_{\mathrm{l}}$.

Lemma 9.3.4.7. *For any integer $q \geq 1$, we have $H^q(\mathfrak{Z}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) = 0$.*

Proof. Let $q \geq 1$. Recall $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} = u^{-1}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)/\mathcal{I}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$. Since the functor $H^q(\mathfrak{Z}, -)$ commutes with projective limits, then it follows from Theorem B for quasi-coherent modules that $H^q(\mathfrak{Z}, \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) = 0$. As the functor $H^q(\mathfrak{Z}, -)$ and the tensor product commute with the tensorization by $\mathbb{Q}$ (because $\mathfrak{Z}$ is noetherian) and with filtrant inductive limits of sheaves on $\mathfrak{Z}$, this yields the two other vanishing formulas. □

Lemma 9.3.4.8. *The functors $\mathbb{R}\Gamma(\mathfrak{Z}, -)$ and $\mathbb{L}u^*$ commute : For any left $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{G}$, we have the canonical isomorphism*

$$\mathbb{L}u^*(G) \xrightarrow{\sim} \mathbb{R}\Gamma(\mathfrak{Z}, \mathbb{L}u^*(\mathcal{G})). \tag{9.3.4.8.1}$$

Proof. Let $P^\bullet$ be a left resolution of G by free $\widetilde{D}_{\mathfrak{X}^\sharp}$-modules of finite type (see 4.6.1.7). The complex $\mathcal{P}^\bullet := \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{D}_{\mathfrak{X}^\sharp}} P^\bullet$ is then a left resolution of $\mathcal{G}$ by free finite type $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules. Hence we get the canonical isomorphisms:

$$\widetilde{D}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{D}_{\mathfrak{X}^\sharp}} G \xleftarrow{\sim} \widetilde{D}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{D}_{\mathfrak{X}^\sharp}} P^\bullet$$
$$\xrightarrow{\sim} \Gamma(\mathfrak{Z}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{D}_{\mathfrak{X}^\sharp}} P^\bullet) \xrightarrow{\sim} \Gamma(\mathfrak{Z}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} u^{-1}\mathcal{P}^\bullet) \xrightarrow{\sim}$$
$$\mathbb{R}\Gamma(\mathfrak{Z}, \widetilde{\mathcal{D}}^\dagger_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} u^{-1}\mathcal{P}^\bullet) \xrightarrow{\sim} \mathbb{R}\Gamma(\mathfrak{Z}, \widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{u^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} u^{-1}\mathcal{G}),$$

the penultimate isomorphism resulting from the fact that $\Gamma(Z, -)$ has bounded cohomological dimension and that $(\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})^n$ is $\Gamma(\mathfrak{Z}, -)$-acyclic for any nonnegative integer n (see the lemma 9.3.4.7) which allows us to use 4.6.1.6. □

The following Lemma will be useful when we deal with overcoherent module (see 15.3.4.17.1).

Lemma 9.3.4.9. *Suppose that the ideal of $\mathcal{O}_{\mathfrak{X}}$ given by u is principal. Let $\mathcal{G}$ be a coherent $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module such that $H^0u^!_D(\mathcal{G})$ is a coherent $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$-module. Then we get the canonical isomorphism:*

$$\widetilde{u}^*(G) \xrightarrow{\sim} \Gamma(\mathfrak{Z}, \widetilde{u}^*(\mathcal{G})). \tag{9.3.4.9.1}$$

Proof. Let t be an element generating the ideal of definition of a. Via the flat resolution

$$0 \to \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \xrightarrow{t\cdot} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/t\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \to 0, \tag{9.3.4.9.2}$$

we deduced from 9.3.4.2.1 the two equalities $u_*\widetilde{u}^*(\mathcal{G}) = \mathcal{G}/t\mathcal{G}$ and $u_*H^0u^!_D(\mathcal{G}) = \ker(t\colon \mathcal{G} \to \mathcal{G})$. Likewise, since the sequence 9.3.4.9.2 is a sequence of coherent right $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules, then it remains exact after applying the functor $\Gamma(\mathfrak{X}, -)$. Hence, via 9.3.4.2.2, we get $\widetilde{u}^*(G) = G/tG$. Since $H^0u^!_D(\mathcal{G})$ is a coherent $\widetilde{\mathcal{D}}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$-module, it satisfies the theorems of type A and B. Since the functor u_* is exact and preserve injective objects (since its left adjoint u^{-1} is exact), this yields that, for any $i \geq 1$, $H^i(\mathfrak{X}, u_*H^0u^!_D(\mathcal{G})) = 0$. Since $\mathcal{G}$ satisfies also the theorem of type B, by applying the functor $\mathbb{R}\Gamma(\mathfrak{X}, -)$ to the exact sequence $0 \to u_*H^0u^!_D(\mathcal{G}) \to \mathcal{G} \to \mathcal{G}/u_*H^0u^!_D(\mathcal{G}) \to 0$, we obtain that, for any $i \geq 1$, $H^i(\mathfrak{X}, \mathcal{G}/u_*H^0u^!_D(\mathcal{G})) = 0$ and $\Gamma(\mathfrak{X}, \mathcal{G}/u_*H^0u^!_D(\mathcal{G})) = G/\Gamma(\mathfrak{Z}, H^0u^!_D(\mathcal{G}))$. This implies that by applying the functor $\Gamma(\mathfrak{X}, -)$ to the exact sequence $0 \to \mathcal{G}/u_*H^0u^!_D(\mathcal{G}) \xrightarrow{t} \mathcal{G} \to \mathcal{G}/t\mathcal{G} \to 0$, we get the isomorphism $\Gamma(\mathfrak{X}, \mathcal{G}/t\mathcal{G}) \xrightarrow{\sim} G/tG$. □

Lemma 9.3.4.10. *Let $\mathring{\mathcal{G}}^{(m)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module such that for any $i \in \mathbb{N}$ the canonical morphism $\mathbb{L}\widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}) \to \widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)})$ is an isomorphism. Set $\mathring{G}^{(m)} := \Gamma(\mathfrak{X}, \mathring{\mathcal{G}}^{(m)})$, $\mathring{\mathcal{G}}^{(m)}_i :=$*

$\mathring{\mathcal{G}}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{V}} \mathcal{V}/\pi^{i+1}\mathcal{V} \xleftarrow{\sim} \mathring{\mathcal{G}}^{(m)} \otimes_{\mathcal{V}} \mathcal{V}/\pi^{i+1}\mathcal{V}$, $\mathring{G}^{(m)}_i := \Gamma(\mathfrak{X}, \mathring{\mathcal{G}}^{(m)}_i)$. *Then we have the canonical isomorphisms:*

$$\mathbb{L}\widetilde{u}^*(\mathring{\mathcal{G}}^{(m)}) \xrightarrow{\sim} \widetilde{u}^*(\mathring{\mathcal{G}}^{(m)}) \xrightarrow{\sim} \varprojlim_i \widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}), \tag{9.3.4.10.1}$$

$$\mathbb{L}\widetilde{u}^*(\mathring{G}^{(m)}) \xrightarrow{\sim} \widetilde{u}^*(\mathring{G}^{(m)}) \xrightarrow{\sim} \varprojlim_i \widetilde{u}^*_i(\mathring{G}_i{}^{(m)}), \tag{9.3.4.10.2}$$

$$\widetilde{u}^*(\mathring{G}^{(m)}) \xrightarrow{\sim} \Gamma(\mathfrak{Z}, \widetilde{u}^*(\mathring{\mathcal{G}}^{(m)})). \tag{9.3.4.10.3}$$

Proof. Following 7.5.5.13.(c), since $\mathring{\mathcal{G}}^{(m)}$ is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module then we have the isomorphism $\mathbb{L}\widetilde{u}^*(\mathring{\mathcal{G}}^{(m)}) \xrightarrow{\sim} \mathbb{R}\varprojlim_i \mathbb{L}\widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)})$. By using the two other properties satisfied by $\mathring{\mathcal{G}}^{(m)}$ and by using Mittag-Leffler, we obtain therefore the canonical isomorphisms

$$\widetilde{u}^*(\mathring{\mathcal{G}}^{(m)}) \xleftarrow{\sim} \mathbb{L}\widetilde{u}^*(\mathring{\mathcal{G}}^{(m)}) \xrightarrow{\sim} \mathbb{R}\varprojlim_i \mathbb{L}\widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}) \xrightarrow{\sim} \mathbb{R}\varprojlim_i \widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}) \xleftarrow{\sim} \varprojlim_i \widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}). \tag{9.3.4.10.4}$$

Hence, we get the isomorphisms 9.3.4.10.1.

Now let us deal with the second isomorphism. Via the theorem of type A for quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp_i/S^\sharp_i}(D)$-modules, we get by associativity of the tensor product the canonical isomorphism $\mathbb{L}\widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}) \xrightarrow{\sim} \mathcal{D}^{(m)}_{Z^\sharp_i/S^\sharp_i}(E) \otimes_{D^{(m)}_{Z^\sharp_i/S^\sharp_i}(E)} \mathbb{L}\widetilde{u}^*_i(\mathring{G}_i{}^{(m)})$. Via the theorem of type A for quasi-coherent complexes of $\mathcal{D}^{(m)}_{Z^\sharp_i/S^\sharp_i}(E)$-modules (see 4.6.1.7), since $\mathbb{L}\widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)}) \xrightarrow{\sim} \widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)})$, this yields the isomorphism $\mathbb{L}\widetilde{u}^*_i(\mathring{G}_i{}^{(m)}) \xrightarrow{\sim} \widetilde{u}^*_i(\mathring{G}_i{}^{(m)})$. Moreover, we have following 7.2.3.13.1 the canonical isomorphism $\mathring{G}^{(m)}_i \xrightarrow{\sim} \mathring{G}^{(m)} \otimes_{\mathcal{V}} \mathcal{V}/\pi^{i+1}\mathcal{V}$. Then we obtain analogously (we use the appendice B of [BO78] instead of 7.5.5.13.(c)) the isomorphisms 9.3.4.10.4 with some straight letters. In particular, we have checked 9.3.4.10.2. Since the functor $\Gamma(\mathfrak{Z}, -)$ commutes with projective limits, since $\Gamma(\mathfrak{Z}, \widetilde{u}^*_i(\mathring{\mathcal{G}}_i{}^{(m)})) \xrightarrow{\sim} \widetilde{u}^*_i(\mathring{G}_i{}^{(m)})$, we can deduce the isomorphism 9.3.4.10.3 from 9.3.4.10.1 and 9.3.4.10.2. □

9.3.5 Berthelot-Kashiwara theorem

In this subsection, we give a proof of Berthelot-Kashiwara theorem (see 9.3.5.9). Suppose $\mathfrak{X}$ is noetherian of finite Krull dimension.

9.3.5.1. Let $(\mathcal{B}^{(m)})_{m\in\mathbb{N}}$ be an inductive system of commutative p-torsion free $\mathcal{O}_{\mathfrak{X}}$-algebras satisfying the conditions of 7.5.1.1. We suppose that $\mathcal{B}^{(m)}$ is endowed with a compatible structure of left $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module such that the homomorphism of $\mathcal{O}_{\mathfrak{X}}$-algebra $\mathcal{B}^{(m)} \to \mathcal{B}^{(m+1)}$ is flat and is a monomorphism of $\mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules. We set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}^{(m)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} := (u^*\mathcal{B}^{(m)}) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$. We suppose that for any affine open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}$, the extensions $\Gamma(\mathfrak{U}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{U}, \widetilde{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and $\Gamma(\mathfrak{U}\cap\mathfrak{Z}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}) \to \Gamma(\mathfrak{U}\cap\mathfrak{Z}, \widetilde{\mathcal{D}}^{(m+1)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp})$ are flat for any $m \in \mathbb{N}$. Recall, following 7.2.3.3 and 7.2.3.16, for any $m \in \mathbb{N}$, the ring $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \mathcal{B}^{(m)} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ is coherent and satisfies the theorems of type A and B.

Example 9.3.5.2. We can choose $\mathcal{B}^{(m)} = \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(D)$ (e.g. see 8.7.5.2).

Notation 9.3.5.3. Suppose the local condition of 9.3.1.2 are satisfied in the case where $r = d - 1$. We simply denoting by $t := t_d$ and $\partial := \partial_d$. We write $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} := \Gamma(\mathfrak{X}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ and similarly we use straight letter to mean the global sections.

Lemma 9.3.5.4 (Berthelot's key lemma)**.** *With the notation 9.3.5.3, let* $m \in \mathbb{N}$, $s \geq 1$ *be an integer,* $R \in M_s(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ *be a matrix. There exists a large enough integer* $m' \geq m$ *and a matrix* $P \in M_s(\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ *such that the following properties hold:*

(a) $P \equiv I_s \mod \pi M_s(\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$*, where* I_s *is the identity matrix ;*

(b) $t^{p^m} P = P(t^{p^m} I_s - \pi R)$*, where* R *is viewed as an element of* $M_s(\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ *via the inclusion* $M_s(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}) \subset M_s(\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

Proof. Since the proof for $s > 1$ is identical, we shall suppose that $s = 1$.

0) i) *Notations.* Let $Q \in \widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$ (beware that $\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$ is a $(\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, B^{(m)})$-bimodule but not a $B^{(m)}$-algebra). The element Q can uniquely be written of the form $Q = \sum_{\underline{i}\in\mathbb{N}^d} \underline{\partial}_{(r)}^{[\underline{i}]} b_{\underline{i}}$, with $b_{\underline{i}} \in B^{(m)}$ converging π-adically to 0 when $|\underline{i}|$ goes to infinity and where the term $\underline{\partial}_{(r)}^{[\underline{i}]}$ is defined at the end of 4.5.1.1. Put $c_{\underline{i}}(Q) := b_{\underline{i}}$. When $Q \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$, the sum is finite and we can therefore define $\mathrm{ord}(Q)$ to be the maximal of the elements $|\underline{i}|$ such that $c_{\underline{i}}(Q) \neq 0$. Put $[Q]_l := \pi^{-l}\sum_{\underline{i}\in E_l} \underline{\partial}_{(r)}^{[\underline{i}]} c_{\underline{i}}(Q)$, where E_l is the (finite) sub-set of $\mathbb{N}^d$ consisting of the elements $\underline{i}$ such that $v_\pi(c_{\underline{i}}(Q)) = l$, where v_π means the π-adic valuation on $B^{(m)}$. Put $\sigma_l(Q) := \sum_{l'\leq l} \pi^{l'}[Q]_{l'}$.

ii) *a-boundedness.* Let $Q \in \widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$. Similarly to 8.7.1.8, we can check that the following properties:

(i) there exists an integer $m' \geq m$ such that $Q \in \widehat{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$;

(ii) there exists a real number $a > 0$ such that $\mathrm{ord}(\sigma_l(Q)) \leq a(l+1)$ for any $l \in \mathbb{N}$;

(iii) there exists a real number $a > 0$ such that $\mathrm{ord}([Q]_l) \leq a(l+1)$ for any $l \in \mathbb{N}$

are equivalent. If $a > 0$ is a real number, we say then that Q is a-bounded if, for any $\underline{i} \in \mathbb{N}^d$, the following unequality is satisfied: $v_\pi(c_{\underline{i}}(Q)) \geq \frac{|\underline{i}|}{a} - 1$. We notice that Q is a-bounded if and only if we have $\mathrm{ord}([Q]_l) \leq a(l+1)$ for any integer l.

1) By using the left unequality of 8.7.1.7.1 and the right formula of 8.7.1.5.2, we compute that for any $a > p^m(p-1)$, there exists $b \geq a$ large enough such that for any $P \in \widehat{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$ we have $v_\pi(c_{\underline{i}}(P)) \geq \frac{|\underline{i}|}{a} - \frac{b}{a}$ for any $\underline{i} \in \mathbb{N}^d$ (which is equivalent to saying that $\mathrm{ord}([Q]_l) \leq al + b$ for any $l \in \mathbb{N}$). This implies there exists a large enough real number α (e.g. $a+b$) such that for any $P \in \widehat{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$, P is α-bounded. We fix now such an α and we set $\beta := \alpha + p^m$.

2) Let P be a β-bounded element of $D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)} \subset \widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$. It is straightforward that Pb is β-bounded for any $b \in B^{(m)}$. Moreover, by using the formula 4.2.5.7.3 (in the case where $m = \infty$ with the constant coefficient $\mathcal{B}_X = \mathcal{O}_X$), we get that for any $a \in O_{\mathfrak{X}}$ the element aP is still β-bounded.

3) Let $P \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$.

i) We have $PR \in \widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$. Indeed since $\widehat{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$ is a ring, then for any $b \in B^{(m)}$, $bR \in \widehat{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)} \subset \widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$. Since $D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$ is a left $D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module generated by the elements of $B^{(m)}$, then we are done.

ii) Suppose that P is β-bounded. The operator P is therefore a finite sum of elements of the form $\pi^{l_2}\underline{\partial}_{(r)}^{[\underline{i}_2]} b$, with $b \in B^{(m)}$ and $|\underline{i}_2| \leq \beta(l_2+1)$. Since $bR \in \widehat{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$ (because $\widehat{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$ is a ring), then following the part 2) the element bR is α-bounded. We can therefore write bR as a sum of elements of the form $\pi^{l_1}\underline{\partial}_{(r)}^{[\underline{i}_1]} b'$ with $b' \in B^{(m)}$ and $|\underline{i}_1| \leq \alpha(l_1+1)$. Hence, PR is equal to a sum of elements of the form $\pi^{l_1+l_2}\underline{\partial}_{(r)}^{[\underline{i}_1]}\underline{\partial}_{(r)}^{[\underline{i}_2]} b'$ with $b' \in B^{(m)}$, $|\underline{i}_1| \leq \alpha(l_1+1)$ and $|\underline{i}_2| \leq \beta(l_2+1)$. Since $\alpha \leq \beta$, by using the formula 1.1.4.4.1 and 3.2.3.13.1 (this latter is still available for $m = +\infty$), PR is therefore equal to a sum of elements of the form $\pi^l\underline{\partial}_{(r)}^{[\underline{i}]} b''$ with $b'' \in B^{(m)}$ and $|\underline{i}| \leq \alpha + \beta(l+1)$. Remark PR is only a priori "almost" β-bounded.

4) We can easily check the formula for any $N \in \mathbb{N}$:

$$t^{p^m}\partial^{[N+p^m]} - \partial^{[N+p^m]}t^{p^m} \equiv -\partial^{[N]} \mod \pi D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}.$$

This implies that for any operator $U \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$ there exists an operator $Q \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$ such that

$$t^{p^m}Q - Qt^{p^m} \equiv U \mod \pi\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_{\mathfrak{X}}} B^{(m)}$$

and $\mathrm{ord}(Q) \leq \mathrm{ord}(U) + p^m$.

5) In this step, let us build by induction on $l \geq 0$, some operators $P_l \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$ satisfying for any $l \geq 0$ the following conditions:

(i) $P_0 = 1$ and for $l \geq 1$ on a $P_l \equiv P_{l-1} \mod \pi^1 D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_{\mathfrak{X}}} B^{(m)}$,

(ii) P_l is β-bounded,

(iii) $t^{p^m} P_l \equiv P_l(t^{p^m} - \pi R) \mod \pi^{l+1}\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$.

Remark that following the step 3), the elements of the congruence of the property (iii) belong to $\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$. We have necessarily $P_0 = 1$. Let $l \in \mathbb{N}$ and suppose built $P_0, \ldots, P_l$ satisfying the conditions (i), (ii) and (iii). Put $U := -[t^{p^m} P_l - P_l(t^{p^m} - \pi R)]_{l+1} \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_\mathfrak{X}} B^{(m)}$. Following the property (iii) satisfied by P_l, we have $t^{p^m} P_l - P_l(t^{p^m} - \pi R) \equiv 0 \mod \pi^{l+1}\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$. This implies the congruence:

$$-\pi^{l+1}U \equiv t^{p^m} P_l - P_l(t^{p^m} - \pi R) \mod \pi^{l+2}\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}. \tag{9.3.5.4.1}$$

Following the step 3.ii), since P_l is β-bounded then $\sigma_{l+1}(\pi P_l R) = \pi\sigma_l(P_l R)$ is a differential operator of order bounded by $\alpha + \beta(l+1)$. It follows from that the step 2), that $\mathrm{ord}(\sigma_l(t^{p^m} P_l - P_l(t^{p^m}) \leq \beta(l+1)$. This yields $\mathrm{ord}(U) \leq \beta(l+1) + \alpha$. Following the part 4) of the proof, there exists therefore $Q \in D_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp} \otimes_{O_\mathfrak{X}} B^{(m)}$ such that $t^{p^m} Q - Q t^{p^m} \equiv U \mod \pi\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$ and $\mathrm{ord}(Q) \leq \mathrm{ord}(U) + p^m$. This implies $\mathrm{ord}(Q) \leq \beta(l+1) + \alpha + p^m = \beta(l+2)$ (recall that $\beta := \alpha + p^m$). Hence, $\pi^{l+1}Q$ is β-bounded.

Following the step 3.i), $QR \in \widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$. By using 9.3.5.4.1, we get the congruences

$$\begin{aligned} t^{p^m}(P_l + \pi^{l+1}Q) - (P_l + \pi^{l+1}Q)(t^{p^m} - \pi R) &= t^{p^m} P_l - P_l(t^{p^m} - \pi R) + \pi^{l+1}(t^{p^m} Q - Q t^{p^m}) + \pi^{l+2} QR \\ &\equiv -\pi^{l+1}U + \pi^{l+1}(t^{p^m} Q - Q t^{p^m}) \mod \pi^{l+2}\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)} \\ &\equiv 0 \mod \pi^{l+2}\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}. \end{aligned}$$

Hence, $P_{l+1} := P_l + \pi^{l+1}Q$ is β-bounded and the equality $t^{p^m} P_{l+1} \equiv P_{l+1}(t^{p^m} - \pi R) \mod \pi^{l+2}\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$ holds.

6) Finally, following the step 5), we get the element $P := \lim_{l\to\infty} P_l$ of $\widehat{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)}$ which is β-bounded and $t^{p^m} P = P(t^{p^m} - \pi R)$ holds. Following the step 0.ii), since P is β-bounded, there exists therefore $m' \geq m$ such that $P \in \widehat{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{O_\mathfrak{X}} B^{(m)} \subset \widehat{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. Such operator P verifies the required properties. □

Notation 9.3.5.5. According to the notation 7.5.5.14 (this is also a respective case of 9.3.1.8), for any $\mathcal{E} \in ({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$ we set

$$\widetilde{u}^{(m)!}(\mathcal{E}) := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} u^{-1}(\mathcal{E})[d_u].$$

According to 7.5.8.3 (or a respective case of 9.3.2.1) for any $\mathcal{F} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$, we set

$$\widetilde{u}^{(m)}_{+}(\mathcal{F}) := u_*\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{F}\right).$$

For coherent complexes, both functors are compatible with that of the subsection 9.3.3. More precisely, on one hand for any $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$, following 7.5.5.16.(b) we have the isomorphism

$$\widetilde{u}^{(m)!}(\mathcal{G}_\mathbb{Q}) \xrightarrow{\sim} (\widetilde{u}^{(m)!}(\mathcal{G}))_\mathbb{Q}, \tag{9.3.5.5.1}$$

where $\widetilde{u}^{(m)!}(\mathcal{G})$ is the usual pullbacks (see definition 7.5.5.6.(c). On the other hand for any $\mathcal{H} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp})$ following 7.5.8.4.(b)

$$\widetilde{u}^{(m)}_{+}(\mathcal{H}_\mathbb{Q}) \xrightarrow{\sim} (\widetilde{u}^{(m)}_{+}(\mathcal{H}))_\mathbb{Q}, \tag{9.3.5.5.2}$$

where $\widetilde{u}^{(m)}_{+}(\mathcal{H})$ is the usual pushforward (see definition 7.5.8.1.(c)).

Notation 9.3.5.6. Suppose the local condition of 9.3.1.2 are satisfied. Following 9.3.1.14.1, we have the isomorphism

$$\widetilde{u}_{+}^{(m)}(\mathcal{F}) \xrightarrow[\iota_t \otimes \mathrm{id}]{\sim} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}/\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}\mathcal{I}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} u_*\mathcal{F}.$$

By taking the global section of the bijection 9.3.1.18.1, we get $\Gamma(\mathfrak{Z}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} I$, which yields a structure of $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}, \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp})$-bimodule on this latter term. Hence, for any left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$-module G, we get a left $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module by setting:

$$\widetilde{u}_{+}^{(m)}(G) := \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} I \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} G, \tag{9.3.5.6.1}$$

i.e. this is a global version of the notation 9.3.2.1 (do not confuse with 9.3.3.6.1). By setting $\mathcal{G} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}} G$, we get the isomorphism

$$\Gamma(\mathfrak{X}, \widetilde{u}_{+}^{(m)}(\mathcal{G})) \xrightarrow{\sim} \widetilde{u}_{+}^{(m)}(G). \tag{9.3.5.6.2}$$

Indeed, by associativity of the tensor product, we reduce to the case where $G = \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$, which is easy.

Following 9.3.1.20.2, for any left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module $\mathcal{E}$, we have the isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules:

$$H^0\widetilde{u}^{(m)!}(\mathcal{E}) \xrightarrow{\sim} \cap_{s=r+1}^{d} \ker(\mathcal{E} \xrightarrow{t_s} \mathcal{E}). \tag{9.3.5.6.3}$$

For any left $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module E, we set

$$H^0\widetilde{u}^{(m)!}(E) := \mathrm{Hom}_{B_{\mathfrak{X}}}(B_{\mathfrak{X}}/IB_{\mathfrak{X}}, E) \cap_{s=r+1}^{d} \ker(E \xrightarrow{t_s} E). \tag{9.3.5.6.4}$$

Then $H^0\widetilde{u}^{(m)!}(E)$ is endowed with a structure of left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is (well) defined by the formula

$$Q \cdot x := Q_{\mathfrak{X}} \cdot x. \tag{9.3.5.6.5}$$

for any $x \in H^0u^{(m)!}(E)$, $Q \in \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}$ and any choice $Q_{\mathfrak{X}} \in \widetilde{D}_{\mathfrak{X}^\sharp,\mathfrak{Z}^\sharp,\underline{t}}$ such that $\vartheta(Q) = [Q_{\mathfrak{X}}]_Z$ (see 5.2.2.5.7). Comparing 9.3.1.20.3 and 9.3.5.6.5, for any left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module $\mathcal{E}$, we have the isomorphism of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules:

$$\Gamma(\mathfrak{X}, H^0\widetilde{u}^{(m)!}(\mathcal{E})) \xrightarrow{\sim} \Gamma(\mathfrak{X}, \cap_{s=r+1}^{d} \ker(\mathcal{E} \xrightarrow{t_s} \mathcal{E})) \xrightarrow{\sim} H^0\widetilde{u}^{(m)!}(\Gamma(\mathfrak{X}, \mathcal{E})). \tag{9.3.5.6.6}$$

Lemma 9.3.5.7. *Let $\mathcal{F}$ be a coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module with support in $\mathfrak{Z}$. Then $\widetilde{u}_{+}^{(m)}(\mathcal{F})$ is a coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module with support in $\mathfrak{Z}$. Moreover, we have a canonical isomorphism*

$$H^0\widetilde{u}^{(m)!}\widetilde{u}_{+}^{(m)}(\mathcal{F}) \xrightarrow{\sim} (\mathcal{F}). \tag{9.3.5.7.1}$$

Proof. The stability of the coherence comes from the fact that u is proper (see 7.5.11.4). The morphism 9.3.5.7.1 is constructed in 9.3.2.3. The fact that this is an isomorphism is local and we can therefore suppose the local condition of 9.3.1.2 are satisfied. Hence, via 7.5.8.4.(b), this is a consequence of 9.3.3.13 (see also 9.3.5.6 and use theorem of type A). □

Theorem 9.3.5.8 (Berthelot)**.** *Let $\mathcal{E}$ be a coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module with support in $\mathfrak{Z}$ (i.e. such that $\mathcal{E}|\mathfrak{Y} = 0$).*

Then there exist a large enough integer $m' \geq m$, a coherent $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module $\mathcal{F}$ and an isomorphism of $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules of the form

$$\widetilde{u}_{+}^{(m')}(\mathcal{F}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}.$$

Proof. 0) For any coherent $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module $\mathcal{G}$, we have the canonical isomorphism $H^0\widetilde{u}^{!(m')}\widetilde{u}^{(m')}_+(\mathcal{G}) \xrightarrow{\sim} \mathcal{G}$ (see 9.3.5.7). This implies that the functor $\widetilde{u}^{(m')}_+$ is fully faithful. The theorem is therefore local, and thanks to 5.2.1.1 we can suppose $\mathfrak{X}$ is affine and there exist some integers $d \geq r$ and a cartesian diagram of formal log $\mathfrak{S}$-schemes of the form:

$$\begin{array}{ccc} \mathfrak{X}^\sharp & \longrightarrow & \mathfrak{A}^{d,r}_{\mathfrak{S}^\sharp} \\ {\scriptstyle u}\uparrow & \square & \uparrow \\ \mathfrak{Z}^\sharp & \longrightarrow & \mathfrak{A}^{r,r}_{\mathfrak{S}^\sharp} \end{array}$$

such that horizontal morphisms are log-étale and the right morphism is the canonical exact closed immersion. By proceeding by induction on d, we can suppose $d = r+1$, i.e. $\mathfrak{Z}^\sharp = V(t)$ where t is the section of $\mathcal{O}_{\mathfrak{X}}$ image of t_n via $\mathfrak{X}^\sharp \to \mathfrak{A}^{d,d-1}_{\mathfrak{S}^\sharp}$. In other words, we reduce to the context 9.3.5.3 and we will use its notation.

1) Let $\overset{\circ}{\mathcal{E}}$ be a π-torsion free coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module endowed with a $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-linear isomorphism of the form $\overset{\circ}{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ (see 7.4.5.2). Since $\mathfrak{X}$ is affine, we can choose some generators $e_1, \ldots, e_s$ of $\overset{\circ}{\mathcal{E}}$ as $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module (use theorem of type A of 7.2.3.10). Put $\underline{e} := \begin{pmatrix} e_1 \\ \vdots \\ e_s \end{pmatrix}$. Since $\overset{\circ}{\mathcal{E}}/\pi\overset{\circ}{\mathcal{E}}$ is a quasi-coherent $\mathcal{O}_X$-module with support in Z (i.e. $(\overset{\circ}{\mathcal{E}}/\pi\overset{\circ}{\mathcal{E}})|Y = 0$), then $t^{p^m}e_i \equiv 0 \mod \pi\overset{\circ}{\mathcal{E}}$ (by increasing m if necessary). There exists therefore a matrix $R \in M_s(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ such that $t^{p^m}\underline{e} = \pi R\underline{e}$. Let m' and $P \in M_s(\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$ satisfying the conditions of the Lemma 9.3.5.4. Put $E^{(m')} := \widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} E$. We denote by $1\otimes e_1, \ldots, 1 \otimes e_s$ the images of $e_1, \ldots, e_s$ in $E^{(m')}$. Put $\underline{e}' := P(1\otimes \underline{e})$. We compute $t^{p^m}\underline{e}' = t^{p^m}P(1\otimes\underline{e}) = P(t^{p^m}I_s - \pi R)(1\otimes\underline{e}) = P(1\otimes(t^{p^m}\underline{e} - \pi R\underline{e})) = 0$.

2) Recall $H^0\widetilde{u}^{!(m')}(E^{(m')})$ is the set some elements of $E^{(m')}$ which are killed by t. Denote by $K^{(m')}$ the sub-$\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module of $E^{(m')}$ generated by $H^0\widetilde{u}^{!(m')}(E^{(m')})$. Let us check by induction on $i \geq 1$ that for any $x \in E^{(m')}$ if $t^i x = 0$ then $x \in K^{(m')}$. When $i = 1$, this is straightforward. Suppose $i \geq 2$ and the property holds for $i-1$. Let $x \in E^{(m')}$ such that $t^i x = 0$. From the formula $\partial t^{i-1} = t^{i-1}\partial + (i-1)t^{i-2}$, we get by right multiplication by t the following equality $\partial t^i = t^{i-1}\partial t + (i-1)t^{i-1} = t^{i-1}(\partial t + (i-1))$. Since $\partial t^i \cdot x = 0$, by induction hypothesis, we get $(\partial t + (i-1)) \cdot x \in K^{(m')}$ (resp. $t \cdot x \in K^{(m')}$ and therefore $\partial t \cdot x \in K^{(m')}$). This yields $(i-1)x \in K^{(m')}$ and therefore $x \in K^{(m')}$, which concludes the induction. Since $E^{(m')}$ is generated as $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module by some elements killed by t^{p^m} (see the step 1), then this implies that $K^{(m')} = E^{(m')}$, i.e. that $E^{(m')}$ is generated as $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module by some elements killed by t.

3) Following 9.3.2.3 (still valid in our coefficient $\mathcal{B}^{(m')}_{\mathfrak{X}}$) and more explicitly following 9.3.2.4 the description given at 5.2.6.2.a) is still valid in our context, i.e., we get the morphism of $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules given by

$$\mathrm{adj}\colon \widetilde{u}^{(m')}_+ \circ H^0\widetilde{u}^{!(m')}(E^{(m')}) \to E^{(m')} \tag{9.3.5.8.1}$$

given by $\sum_k [P_k] \otimes x_k \mapsto \sum_k P \cdot x_k$, with $x_k \in H^0\widetilde{u}^{!(m')}(E^{(m')})$, such that $\lim_{|k|\to\infty} x_k = 0$ (where $[\partial^{\langle k\rangle}]'$ is the image of $\partial^{\langle k\rangle}$ in the bimodule $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \xrightarrow{\sim} \widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}/\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}I$). Since $E^{(m')}$ is generated as $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module by a finite number of elements of $H^0\widetilde{u}^{!(m')}(E^{(m')})$, this yields a surjective morphism of (coherent) $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules of the form $\widetilde{u}^{(m')}_+(L) \to E^{(m')}$ where L is a free $\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module of finite rank and which splits of the form $\widetilde{u}^{(m')}_+(L) \to \widetilde{u}^{(m')}_+ \circ H^0\widetilde{u}^{!(m')}(E^{(m')}) \to E^{(m')}$, where the last map is 9.3.5.8.1. This yields the map $\widetilde{u}^{(m')}_+ \circ H^0\widetilde{u}^{!(m')}(E^{(m')}) \to E^{(m')}$ is surjective.

4) Denote by N the kernel of the surjection $\widetilde{u}^{(m')}_+(L) \to E^{(m')}$. Since N is the global section of the kernel of a morphism of coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules with support in $\mathfrak{Z}$ (see theorem of type A and B), then N is the global section of a coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules with support in $\mathfrak{Z}$. Hence, by proceeding

as above (see the step 3) and increasing m' if necessary (use 7.5.8.15 and the fact that the extension $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m'')}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ is flat for any $m' \leq m''$), the adjunction morphism $\widetilde{u}^{(m')}_{+} H^0 \widetilde{u}^{!(m')}(N) \to N$ is surjective. Since $H^0 \widetilde{u}^{!(m')}(N)$ is a sub-module of L, then $H^0 \widetilde{u}^{!(m')}(N)$ is a coherent $\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module and the composite $\widetilde{u}^{(m')}_{+} H^0 \widetilde{u}^{!(m')}(N) \to N \to \widetilde{u}^{(m')}_{+} L$ is injective (because $\widetilde{u}^{(m')}_{+}$ is exact). In particular, $\widetilde{u}^{(m')}_{+} H^0 \widetilde{u}^{!(m')}(N) \to N$ is also injective. Since this latter map is also surjective, then the canonical arrow $\widetilde{u}^{(m')}_{+} H^0 \widetilde{u}^{!(m')}(N) \to N$ is an isomorphism. Hence, by exactness of $\widetilde{u}^{(m')}_{+}$, we get $E^{(m')} \xrightarrow{\sim} \widetilde{u}^{(m')}_{+}(L)/N \xrightarrow{\sim} \widetilde{u}^{(m')}_{+}(L)/\widetilde{u}^{(m')}_{+} H^0 \widetilde{u}^{!(m')}(N) \xrightarrow{\sim} \widetilde{u}^{(m')}_{+}(L/H^0 \widetilde{u}^{!(m')}(N))$. □

Theorem 9.3.5.8 is false if you fix m, i.e. do not allow m to increase; for example $\mathcal{D}^{(0)}_{\mathfrak{X}}/\mathcal{D}^{(0)}_{\mathfrak{X}}(t^p - p)$. This theorem is the first step in the proof of the Berthelot-Kashiwara theorem, it shows that essential surjectivity is true asymptotically.

Theorem 9.3.5.9 (Berthelot-Kashiwara)**.** *The extraordinary inverse image $u^!_D$ and the direct image u_{D+} functors (see 9.2.1.21.1 and 9.2.4.13) induce quasi-inverse equivalences between the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules with support in Z and that of coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}({}^\dagger E)_{\mathbb{Q}}$-modules. Theses functors $u^!_D$ and u_{D+} are exact on theses categories.*

Proof. We reduce to the geometical situation of the step 1) of the proof of 9.3.5.8. For any coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-module $\mathcal{F}$ with support in Z, it follows from 9.3.5.7 that we have the canonical isomorphism $H^0 u^!_D u_{D+}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. By using theorem 9.3.5.8, this yields that the functors $H^0 u^!_D$ and u_{D+} induce some equivalences quasi-inverse between the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules with support in Z and that of coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}({}^\dagger E)_{\mathbb{Q}}$-modules. The acyclicity of u_{D+} follows from a computation in local coordinates. It remains to check the acyclicity of $\widetilde{u}^!$. Following 1.4.2.5.2, the morphism $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to \widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ sends $\sum_{k\in\mathbb{N}} b_k \partial^{\langle k\rangle_{(m)}}$ to $\sum_{k\in\mathbb{N}} b_k \frac{q^{(m)}_k!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}}$. Via the formula 1.4.2.7.1 (still valid for formal schemes), we compute $t\partial^{\langle k+1\rangle_{(m+1)}} - \partial^{\langle k+1\rangle_{(m+1)}} t = \left\{ {k+1 \atop 1} \right\}_{(m+1)} \partial^{\langle k\rangle_{(m+1)}} = \frac{q^{(m+1)}_{k+1}!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}}$ By using 8.7.1.7.1, $v_p(\frac{q^{(m+1)}_{k+1}!}{q^{(m+1)}_k!}) < \log_p(k+1) + \frac{1}{p^{m+1}(p-1)} + \frac{p}{(p-1)}$ and $v_p(\frac{q^{(m)}_k!}{q^{(m+1)}_k!}) > \frac{k}{p^m(p-1)} - \log_p(k+1) - \frac{p}{(p-1)} - \frac{k}{p^{m+1}(p-1)} = \frac{k}{p^{m+1}(p-1)} - \log_p(k+1) - \frac{p}{(p-1)}$. Hence, there exists $N \in \mathbb{N}$ large enough such that for any $k \geq N$ we get $b_k \frac{q^{(m)}_k!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}} = b'_k \frac{q^{(m+1)}_{k+1}!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}}$, with $b'_k \in \mathcal{B}^{(m+1)}_{\mathfrak{X}}$ converging to zero for the p-adic topology. Hence, $\sum_{k\geq N} b_k \frac{q^{(m)}_k!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}} = \sum_{k\geq N} b'_k \frac{q^{(m+1)}_{k+1}!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}} = t\left(\sum_{k\geq N} b'_k \partial^{\langle k+1\rangle_{(m+1)}}\right) - \left(\sum_{k\geq N} b'_k \partial^{\langle k+1\rangle_{(m+1)}}\right) t \in I\widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} + \widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} I$. Similarly, since the sum is finite, we get $\sum_{k\geq N} b_k \frac{q^{(m)}_k!}{q^{(m+1)}_k!} \partial^{\langle k\rangle_{(m+1)}} \in I\widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} + \widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} I$. Hence, for any coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module $\mathcal{F}^{(m)}$, denoting by $\mathcal{F}^{(m+1)} := \widetilde{\mathcal{D}}^{(m+1)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{F}^{(m)}$, the canonical arrow $\mathcal{H}^1 \widetilde{u}^{(m)!} \circ \widetilde{u}^{(m)}_{+}(\mathcal{F}^{(m)}) \to \mathcal{H}^1 \widetilde{u}^{(m+1)!} \circ \widetilde{u}^{(m+1)}_{+}(\mathcal{F}^{(m+1)})$ is the zero morphism. Indeed, by using $\mathcal{H}^1 \widetilde{u}^{(m)!} = \mathcal{H}^1 \widetilde{u}^{(m+1)!} = \widetilde{u}^*$ for coherent modules, this latter morphism corresponds to the map

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/(I\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} + \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} I) \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} F^{(m)} \to \widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}/(I\widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} + \widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} I) \otimes_{\widetilde{D}^{(m+1)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} F^{(m+1)}.$$

This implies that $\mathcal{H}^1 u^!_D \circ u_{D+}(\mathcal{F}) = 0$. Since we have $\mathcal{H}^i u^!_D \circ u_{D+}(\mathcal{F}) = 0$ for $i \notin \{0,1\}$, then this implies the acyclicity of $u^!_D$. □

Remark 9.3.5.10. C. Huyghe and T. Schmidt establish Kashiwara's theorem for twisted arithmetic differential operators, which is a variation of 9.3.5.9 (see [HS24]).

Corollary 9.3.5.11. *With notation 9.3.5.9, let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-modules with support in Z. Then we have the isomorphism of coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}({}^\dagger E)_{\mathbb{Q}}$-modules:*

$$\mathbb{D}_{\mathfrak{Z}^\sharp,E} \circ u^!_D(\mathcal{E}) \xrightarrow{\sim} u^!_D \circ \mathbb{D}_{\mathfrak{X}^\sharp,D}(\mathcal{E}). \qquad (9.3.5.11.1)$$

Proof. We have the isomorphisms of coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}({}^\dagger E)_\mathbb{Q}$-modules:

$$u_{D+}\circ \mathbb{D}_{\mathfrak{Z}^\sharp,E}\circ u^!_D(\mathcal{E}) \underset{9.3.2.8}{\xrightarrow{\sim}} \mathbb{D}_{\mathfrak{Z}^\sharp,E}\circ u_{D+}\circ u^!_D(\mathcal{E}) \underset{9.3.5.9}{\xrightarrow{\sim}} \mathbb{D}_{\mathfrak{Z}^\sharp,E}(\mathcal{E}) \underset{9.3.5.9}{\xrightarrow{\sim}} u_{D+}\circ u^!_D\circ \mathbb{D}_{\mathfrak{Z}^\sharp,E}(\mathcal{E}).$$

Since the functor u_{D+} is fully faithful on the category of coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}({}^\dagger E)_\mathbb{Q}$-modules, then we are done. □

9.3.5.12. For any $\mathcal{E}^{(\bullet)}\in D^-(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, we have the isomorphism of $D^-(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E)))$

$$\mathcal{B}_{\mathfrak{Z}}(E)^{(\bullet)}\otimes^{\mathbb{L}}_{u^{-1}\mathcal{B}_{\mathfrak{X}}(D)^{(\bullet)}} u^{-1}\mathcal{E}^{(\bullet)}[d_u] \xrightarrow{\sim} u^{(\bullet)!}_{D\,\mathrm{alg}}(\mathcal{E}^{(\bullet)}) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)\otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)} f^{-1}\mathcal{F}^{(\bullet)}\ [d_u]. \tag{9.3.5.12.1}$$

Moreover, for any left $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D)$-module $\mathcal{E}^{(\bullet)}$, it follows from the local computation 9.3.1.8.4 that $H^i(u^{(\bullet)!}_{D\,\mathrm{alg}}(\mathcal{E}^{(\bullet)}))=0$ for any integer $i\notin[0,-d_u]$. Hence, by using 9.2.1.17, this yields that for any object of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, we have $H^i(u^{(\bullet)!}_D(\mathcal{E}^{(\bullet)}))=0$ in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}(E))$ for any integer $i\notin[0,-d_u]$. defined by setting,

Beware that this functor do not preserve the quasi-coherence. We benefit for any $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$ from a canonical morphism $u^{(\bullet)!}_{D,\mathrm{alg}}(\mathcal{E}^{(\bullet)})\to u^{(\bullet)!}_D(\mathcal{E}^{(\bullet)})$ (this morphism is even build in $D(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(D))$, i.e. before the localisation).

Theorem 9.3.5.13 (LD-version of Berthelot-Kashiwara Theorem). *Let $\mathcal{F}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$, $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ such that $\mathcal{E}^{(\bullet)}|\mathfrak{U}\xrightarrow{\sim}0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}^\sharp}(D))$.*

(a) We have the canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$:

$$\mathcal{F}^{(\bullet)}\xrightarrow{\sim}u^{(\bullet)!}_D\circ u^{(\bullet)}_{D+}(\mathcal{F}^{(\bullet)}). \tag{9.3.5.13.1}$$

(b) We have $u^{(\bullet)!}_D(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$ and we benefit from the canonical isomorphism

$$u^{(\bullet)}_{D+}\circ u^{(\bullet)!}_D(\mathcal{E}^{(\bullet)})\xrightarrow{\sim}\mathcal{E}^{(\bullet)}. \tag{9.3.5.13.2}$$

(c) The functors $u^{(\bullet)}_{D+}$ and $u^{(\bullet)!}_D$ induce t-exact quasi-inverse equivalences between the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$ and the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}|\mathfrak{U}\xrightarrow{\sim}0$.

Proof. 1) Let us check the isomorphism 9.3.5.13.1. It follows from 5.2.6.3 that we have the canonical adjunction morphism $\mathcal{F}^{(\bullet)}\to u^{(\bullet)!}_D\circ u^{(\bullet)}_{D+}(\mathcal{F}^{(\bullet)})$ of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$. It is a question of checking that this arrow is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$. Following the proposition 8.3.3.5, this is local. By transitivity of the adjunction morphisms, we reduce to the case where $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ is endowed with logarithmic coordinates $t_1,\dots,t_d$ such that $\mathfrak{Z}^\sharp=V(t_1)$. In this case, it follows from 9.3.5.12 that, for any integer $l\notin\{0,1\}$, we have $H^l u^{(\bullet)!}_D=0$ on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$. Since the functor $u^{(\bullet)}_{D+}$ preserves the coherence and is exact, the functor $u^{(\bullet)!}_D\circ u^{(\bullet)}_{D+}$ is therefore left way-out (this has a meaning by using the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))\cong D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E)))$ of 8.4.5.6). By copying the beginning of the proof of 8.4.5.3, we reduce therefore to the case where $\mathcal{F}^{(\bullet)}=\lambda^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E)$, for some $\lambda\in L$. Hence, following 9.3.3.13, we get the canonical isomorphism $\mathcal{F}^{(\bullet)}\xrightarrow{\sim}H^0u^{(\bullet)!}_D\circ u^{(\bullet)}_{D+}(\mathcal{F}^{(\bullet)})$ is an isomorphism of $D(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$. Moreover, following the proof of 9.3.5.8, the canonical arrow $H^1u^{(m)!}_D\circ u^{(m)}_{D+}(\mathcal{F}^{(m)})\to H^1u^{(m+1)!}\circ u^{(m+1)}_+(\mathcal{F}^{(m+1)})$ is the zero morphism. This implies $H^1u^{(\bullet)!}_D\circ u^{(\bullet)}_{D+}(\mathcal{F}^{(\bullet)})=0$ in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$. Hence, we are done.

2) Now, let us check 9.3.5.13.2. Put $\mathcal{E}:=\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$. Following the theorem of Berthelot-Kashiwara of 9.3.5.9, since $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger D)_\mathbb{Q})$ and is with support in Z, then $u^!_D(\mathcal{E})\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger E)_\mathbb{Q})$. Let $\mathcal{H}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$ such that $u^!_D(\mathcal{E})\xrightarrow{\sim}\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{H}^{(\bullet)})$. Via 9.2.4.17, this implies $u_{D+}(u^!_D(\mathcal{E}))\xrightarrow{\sim}$

$\underrightarrow{l}^*_{\mathbb{Q}} u^{(\bullet)}_{D+}\mathcal{H}^{(\bullet)}$. Since following the theorem of Berthelot-Kashiwara $\mathcal{E} \xrightarrow{\sim} u_{D+}(u^!_D(\mathcal{E}))$, since $\mathcal{E}^{(\bullet)}$, $u^{(\bullet)}_{D+}\mathcal{H}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$ and since the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ is fully faithful on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(D))$, this implies $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} u^{(\bullet)}_{D+}\mathcal{H}^{(\bullet)}$. This yields $u^{(\bullet)!}_D(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)!}_D \circ u^{(\bullet)}_{D+}\mathcal{H}^{(\bullet)} \underset{9.3.5.13.1}{\xrightarrow{\sim}} \mathcal{H}^{(\bullet)}$. By applying the functor $u^{(\bullet)}_{D+}$ to this composite morphism, we get the desired isomorphism 9.3.5.13.2.

3) The third point is straightforward from the first two. □

Remark 9.3.5.14. Let $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$. It is not clear that the property $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}({}^\dagger E)_{\mathbb{Q}})$ implies $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp}(E))$. Hence, the version 9.3.5.13 is slightly more precise than 9.3.5.9.

Remark 9.3.5.15. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}(D))$. Then $\mathcal{E}^{(\bullet)}|\mathfrak{U} \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}^\sharp}(D\cap U))$ if and only if the cohomology spaces of $\underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)}$ are $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp}({}^\dagger D)_{\mathbb{Q}}$-coherent with support in X, i.e. $\underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)}|\mathfrak{U} = 0$.

9.3.6 Adjunction morphism associated to the base change of an exact closed immersion by log smooth morphisms, transitivity, compatibility with glueing isomorphisms

Proposition 9.3.6.1. *Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Consider the following diagram of fine log formal schemes which are log smooth over $\mathfrak{S}^\sharp$:*

$$\begin{array}{ccccc} \mathfrak{P}''^\sharp & \xrightarrow{g} & \mathfrak{P}'^\sharp & \xrightarrow{f} & \mathfrak{P}^\sharp \\ {\scriptstyle u''}\uparrow & & {\scriptstyle u'}\uparrow & & {\scriptstyle u}\uparrow \\ \mathfrak{X}''^\sharp & \xrightarrow{\mathrm{b}} & \mathfrak{X}'^\sharp & \xrightarrow{a} & \mathfrak{X}^\sharp, \end{array} \tag{9.3.6.1.1}$$

where f, g, a and b are log-smooth, where u, u' and u'' are exact closed immersions. We suppose that the diagram 9.3.6.1.1 is commutative modulo π. Moreover, let T be a divisor of P such that $T' := f^{-1}(T)$ (resp. $T'' := g^{-1}(T')$, $Z := u^{-1}(T)$, $Z' := u'^{-1}(T')$ and $Z'' := u''^{-1}(T'')$) is a divisor of P' (resp. P'', X, X' and X'').

(a) We have the canonical adjunction morphisms

$$u'_+ \circ a^! \to f^! \circ u_+, \qquad u'^{(\bullet)}_+ \circ a^{(\bullet)!} \to f^{(\bullet)!} \circ u^{(\bullet)}_+ \tag{9.3.6.1.2}$$

of functors of respectively of the form $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'^\sharp/\mathfrak{S}^\sharp}({}^\dagger T')_{\mathbb{Q}})$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}^\sharp}(T'))$). If the right square of 9.3.6.1.1 is cartesian modulo π then both morphisms 9.3.6.1.2 are isomorphisms.

(b) Denoting by $\phi\colon u'_+ \circ a^! \to f^! \circ u_+$, (resp. $\phi'\colon u''_+ \circ b^! \to g^! \circ u'_+$, resp. $\phi''\colon u''_+ \circ (a\circ b)^! \to (f\circ g)^! \circ u_+$) the morphism of adjunction of the right square 9.3.6.1.1 (resp. the left square, resp. the outer of 9.3.6.1.1), then the following diagram

$$\begin{array}{ccc} u''_+ \circ (a\circ b)^! & \underset{\sim}{\longrightarrow} & u''_+ \circ b^! \circ a^! \\ \downarrow{\scriptstyle \phi''} & & \downarrow{\scriptstyle (g^!\circ\phi)\circ(\phi'\circ a^!)} \\ (f\circ g)^! \circ u_+ & \underset{\sim}{\longrightarrow} & g^! \circ f^! \circ u_+, \end{array}$$

is commutative ; and similarly the $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$ is valid. By abuse of notation, we get the transitivity equality $\phi'' = (g^! \circ \phi) \circ (\phi' \circ a^!)$.

(c) Let $a'\colon \mathfrak{X}' \to \mathfrak{X}$ (resp. $f'\colon \mathfrak{X}' \to \mathfrak{X}$) be a morphism whose reduction $X' \to \mathfrak{X}$ (resp. $P' \to \mathfrak{X}$) is equal to that of a (resp. f). Then the following diagram

$$\begin{array}{ccc} u'_+ a^! & \xrightarrow{\phi} & f^! \circ u_+ \\ {\scriptstyle u'_+(\tau_{a,a'})}\uparrow\wr & & {\scriptstyle \tau_{f,f'}u_+}\uparrow\wr \\ u'_+ a'^! & \xrightarrow{\psi} & f'^! \circ u_+, \end{array}$$

where ψ means the morphism of adjunction of the right square of 9.3.6.1.1 whose a and f have been replaced respectively by a' and f', is commutative. Similarly the $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$ is valid

Proof. a) Since we have the equivalence of categories $\underset{\rightarrow\mathbb{Q}}{l^*}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z)) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ (and the other similar ones), it follows from the compatibility of the pullbacks and pushforwards of 9.2.1.24 and 9.2.4.17 and from their coherence stability properties of 9.4.1.7 and 9.4.2.4 that we reduce to check the respective case.

Let $\mathcal{E}^{(\bullet)} \in \underset{\rightarrow\mathbb{Q}}{l^*}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(Z))$. Let'sconstruct the morphism $\phi(\mathcal{E}^{(\bullet)})\colon u'^{(\bullet)}_+ \circ a^{(\bullet)!} \to f^{(\bullet)!} \circ u^{(\bullet)}_+$. By applying the functor $u'^{(\bullet)}_+ \circ a^{(\bullet)!}$ to the adjunction morphism of u applied to $\mathcal{E}^{(\bullet)}$ (see 9.3.2.11), we obtain: $u'_+ a^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \to u'^{(\bullet)}_+ a^{(\bullet)!} u^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$. Now, as $(f\circ u')^{(\bullet)!} \xrightarrow{\sim} u'^{(\bullet)!} f^{(\bullet)!}$ and $(u\circ a)^{(\bullet)!} \xrightarrow{\sim} a^{(\bullet)!} u^{(\bullet)!}$, we have isomorphism $u'^{(\bullet)}_+ \tau^{(\bullet)}_{f\circ u', u\circ a} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})\colon u'^{(\bullet)}_+ a^{(\bullet)!} u^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u'^{(\bullet)}_+ u'^{(\bullet)!} f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$ (notations of 9.2.2.1). Following 9.3.2.11, we have the adjunction morphism of u' applied to $f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$: $u'^{(\bullet)}_+ u'^{(\bullet)!} f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)}) \to f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$. By composing these three morphisms, it comes: $\phi(\mathcal{E}^{(\bullet)}) := u'^{(\bullet)}_+ a^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \to f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$.

Now let us establish that the morphism $\phi(\mathcal{E}^{(\bullet)})$ is an isomorphism when the right diagram of 9.3.6.1.1 is Cartesian. First, since u is an exact closed immersion, the first arrow in the construction of $\phi(\mathcal{E}^{(\bullet)})$ is an isomorphism. Furthermore, the cartesian hypothesis implies than the complex $f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$ has support in X' and therefore the adjunction morphism of u' in $f^{(\bullet)!} u^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$ is an isomorphism (see 9.3.5.13).

b) Let us now prove the transitivity formula of (b). For this, since the respective case is checked similarly, to simplify notation we consider the non-respective one. We have the following diagram:

$$\begin{array}{ccccccccc}
u''_+(a\circ b)^! & \longrightarrow & u''_+ b^! a^! & \xrightarrow{\mathrm{adj}_{u'}} & u''_+ b^! u'^! u'_+ a^! & \xrightarrow{\tau} & u''_+ u''^! g^! u'_+ a^! & \xrightarrow{\mathrm{adj}_{u''}} & g^! u'_+ a^! \\
\downarrow \mathrm{adj}_u & & \downarrow \mathrm{adj}_u & & \downarrow \mathrm{adj}_u & & \downarrow \mathrm{adj}_u & & \downarrow \mathrm{adj}_u \\
u''_+(a\circ b)^! u^! u_+ & \longrightarrow & u''_+ b^! a^! u^! u_+ & \xrightarrow{\mathrm{adj}_{u'}} & u''_+ b^! u'^! u'_+ a^! u^! u_+ & \xrightarrow{\tau} & u''_+ u''^! g^! u'_+ a^! u^! u_+ & \xrightarrow{\mathrm{adj}_{u''}} & g^! u'_+ a^! u^! u_+ \\
\Big| & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
\Big| & & u''_+ b^! u'^! f^! u_+ & \xrightarrow{\mathrm{adj}_{u'}} & u''_+ b^! u'^! u'_+ u'^! f^! u_+ & \xrightarrow{\tau} & u''_+ u''^! g^! u'_+ u'^! f^! u_+ & \xrightarrow{\mathrm{adj}_{u''}} & g^! u'_+ u'^! f^! u_+ \\
\Big| \tau & & \| & & \downarrow \mathrm{adj}_{u'} & & \downarrow \mathrm{adj}_{u'} & & \downarrow \mathrm{adj}_{u'} \\
\Big| & & u''_+ b^! u'^! f^! u_+ & = & u''_+ b^! u'^! f^! u_+ & \xrightarrow{\tau} & u''_+ u''^! g^! f^! u_+ & \xrightarrow{\mathrm{adj}_{u''}} & g^! f^! u_+ \\
\downarrow & & \downarrow \tau & & \downarrow \tau & & \| & & \| \\
u''_+ u''^!(f\circ g)^! u_+ & \longrightarrow & u''_+ u''^! g^! f^! u_+ & = & u''_+ u''^! g^! f^! u_+ & = & u''_+ u''^! g^! f^! u_+ & \xrightarrow{\mathrm{adj}_{u''}} & g^! f^! u_+ \\
\downarrow \mathrm{adj}_{u''} & & \downarrow \mathrm{adj}_{u''} & & \downarrow \mathrm{adj}_{u''} & & \downarrow \mathrm{adj}_{u''} & & \| \\
(f\circ g)^! u_+ & \longrightarrow & g^! f^! u_+ & = & g^! f^! u_+ & = & g^! f^! u_+ & = & g^! f^! u_+ .
\end{array} \tag{9.3.6.1.3}$$

The commutativity of the rectangle (the only rectangle: on the left and in the middle) from 9.3.6.1.3 is deduced of all the properties, given in 9.2.2.3, isomorphisms of the form τ, as well as from the following diagram

$$\begin{array}{ccccccc}
\mathfrak{X}''^\sharp & \xrightarrow{b} & \mathfrak{X}'^\sharp & \xrightarrow{a} & \mathfrak{X}^\sharp & \xrightarrow{u} & \mathfrak{P}^\sharp \\
\wr\downarrow \tau & & \wr\downarrow \tau & & & & \| \\
\mathfrak{X}''^\sharp & \xrightarrow{b} & \mathfrak{X}'^\sharp & \xrightarrow{u'} & \mathfrak{P}'^\sharp & \xrightarrow{f} & \mathfrak{P} \\
\wr\downarrow \tau & & & & \| & & \| \\
\mathfrak{X}''^\sharp & \xrightarrow{u''} & \mathfrak{P}''^\sharp & \xrightarrow{g} & \mathfrak{P}'^\sharp & \xrightarrow{f} & \mathfrak{P}^\sharp .
\end{array}$$

Moreover, we notice that the composite morphism $u'^! \xrightarrow{\mathrm{adj}_{u'}} u'^! u'_+ u'^! \xrightarrow{\mathrm{adj}_{u'}} u'^!$ is the identity. Indeed, this results from the bifunctor adjunction isomorphism θ: $\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T_{P'})}(u'_+(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger T_{X'})}(-, u'^!(-))$ (we will only use the right functoriality), and because the identity morphism $u'_+ u'^! \xrightarrow{Id} u'_+ u'^!$ is sent via θ on $u'^! \xrightarrow{\mathrm{adj}_{u'}} u'^! u'_+ u'^!$ while $u'_+ u'^! \xrightarrow{\mathrm{adj}_{u'}}$ id is sent over $u'^! \xrightarrow{Id} u'^!$. This results in the commutativity of the left square of the third row of 9.3.6.1.3.

We then check, by definition or by functoriality, the commutativity of the other squares of 9.3.6.1.3. This diagram is therefore commutative.

However, we note that the left composite morphism of 9.3.6.1.3, $u''_+ \circ (a\circ b)^! \xrightarrow{\sim} (f\circ g)^! \circ u_+$, is none other than ϕ'', while the one constructing taking the path that goes from the top then to the right of the contour of 9.3.6.1.3, $u''_+ \circ b^! \circ a^! \xrightarrow{\sim} g^! \circ f^! \circ u_+$, matches $(g^! \circ \phi) \circ (\phi' \circ a^!)$. Hence (i).

Let us now prove (c). It follows from 9.2.2.3 that the diagram below

$$\begin{array}{ccccccc}
u'_+ a^!(\mathcal{E}) & \xrightarrow{\mathrm{adj}_u} & u'_+ a^! u^! u_+(\mathcal{E}) & \xrightarrow{u'_+ \tau_{f\circ u', u\circ a} u_+} & u'_+ u'^! f^! u_+(\mathcal{E}) & \xrightarrow{\mathrm{adj}_{u'}} & f^! u_+(\mathcal{E}) \\
{\scriptstyle u'_+(\tau_{a,a'})}\uparrow & & {\scriptstyle u'_+\tau_{u\circ a, u\circ a'} u_+}\uparrow & & \uparrow{\scriptstyle u'_+\tau_{f\circ u', f'\circ u'} u_+} & & {\scriptstyle \tau_{f,f'} u_+}\uparrow \\
u'_+ a'^{!(\mathcal{E})} & \xrightarrow[\mathrm{adj}_u]{} & u'_+ a'^! u^! u_+(\mathcal{E}) & \xrightarrow[u'_+ \tau_{f'\circ u', u\circ a'} u_+]{} & u'_+ u'^! f'^! u_+(\mathcal{E}) & \xrightarrow[\mathrm{adj}_{u'}]{} & f'^! u_+(\mathcal{E}).
\end{array}$$

is commutative. We conclude by noting that its contour corresponds to the diagram of (ii). □

9.3.7 Coherent arithmetic $\mathcal{D}$-modules over a realizable log smooth scheme

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Let $\mathfrak{P}^\sharp$ be a log smooth separated formal scheme over $\mathfrak{S}^\sharp$. Let $u_0 \colon X^\sharp \to P^\sharp$ be an exact closed immersion of log smooth schemes over $S^\sharp$. Let T be a divisor of P such that $Z := T \cap X$ is a divisor of X. We set $Y^\sharp := X^\sharp \setminus Z$. Let $(\mathfrak{P}^\sharp_\alpha)_{\alpha\in\Lambda}$ be an open covering of $\mathfrak{P}^\sharp$. We set $\mathfrak{P}^\sharp_{\alpha\beta} := \mathfrak{P}^\sharp_\alpha \cap \mathfrak{P}^\sharp_\beta$, $\mathfrak{P}^\sharp_{\alpha\beta\gamma} := \mathfrak{P}^\sharp_\alpha \cap \mathfrak{P}^\sharp_\beta \cap \mathfrak{P}^\sharp_\gamma$, $X^\sharp_\alpha := X^\sharp \cap P_\alpha$, $X^\sharp_{\alpha\beta} := X^\sharp_\alpha \cap X^\sharp_\beta$ and $X^\sharp_{\alpha\beta\gamma} := X^\sharp_\alpha \cap X^\sharp_\beta \cap X^\sharp_\gamma$. We denote by $Y^\sharp_\alpha := X^\sharp_\alpha \cap Y^\sharp$, $Y^\sharp_{\alpha\beta} := Y^\sharp_\alpha \cap Y^\sharp_\beta$, $Y^\sharp_{\alpha\beta\gamma} := Y^\sharp_\alpha \cap Y^\sharp_\beta \cap Y^\sharp_\gamma$, $Z_\alpha := X_\alpha \cap Z$, $Z_{\alpha\beta} := Z_\alpha \cap Z_\beta$, $Z_{\alpha\beta\gamma} := Z_\alpha \cap Z_\beta \cap Z_\gamma$, $j_\alpha \colon Y^\sharp_\alpha \hookrightarrow X^\sharp_\alpha$, $j_{\alpha\beta} \colon Y^\sharp_{\alpha\beta} \hookrightarrow X^\sharp_{\alpha\beta}$ and $j_{\alpha\beta\gamma} \colon Y^\sharp_{\alpha\beta\gamma} \hookrightarrow X^\sharp_{\alpha\beta\gamma}$ the canonical open immersions. We suppose that for every $\alpha \in \Lambda$, $X^\sharp_\alpha$ is affine, (for instance when the covering $(\mathfrak{P}^\sharp_\alpha)_{\alpha\in\Lambda}$ is affine). Since P is separated, for any $\alpha, \beta, \gamma \in \Lambda$, $X^\sharp_{\alpha\beta}$ and $X^\sharp_{\alpha\beta\gamma}$ are also affine.

For any 3uple $(\alpha, \beta, \gamma) \in \Lambda^3$, fix $\mathfrak{X}^\sharp_\alpha$ (resp. $\mathfrak{X}^\sharp_{\alpha\beta}$, $\mathfrak{X}^\sharp_{\alpha\beta\gamma}$) some log smooth formal $\mathfrak{S}^\sharp$-schemes lifting $X^\sharp_\alpha$ (resp. $X^\sharp_{\alpha\beta}$, $X^\sharp_{\alpha\beta\gamma}$), $p_1^{\alpha\beta} \colon \mathfrak{X}^\sharp_{\alpha\beta} \to \mathfrak{X}^\sharp_\alpha$ (resp. $p_2^{\alpha\beta} \colon \mathfrak{X}^\sharp_{\alpha\beta} \to \mathfrak{X}^\sharp_\beta$) some flat lifting of $X^\sharp_{\alpha\beta} \to X^\sharp_\alpha$ (resp. $X^\sharp_{\alpha\beta} \to X^\sharp_\beta$).

Similarly, for any $(\alpha, \beta, \gamma) \in \Lambda^3$, fix some lifting $p_{12}^{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \to \mathfrak{X}^\sharp_{\alpha\beta}$, $p_{23}^{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \to \mathfrak{X}^\sharp_{\beta\gamma}$, $p_{13}^{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \to \mathfrak{X}^\sharp_{\alpha\gamma}$, $p_1^{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \to \mathfrak{X}^\sharp_\alpha$, $p_2^{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \to \mathfrak{X}^\sharp_\beta$, $p_3^{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \to \mathfrak{X}^\sharp_\gamma$, $u_\alpha \colon \mathfrak{X}^\sharp_\alpha \hookrightarrow \mathfrak{P}^\sharp_\alpha$, $u_{\alpha\beta} \colon \mathfrak{X}^\sharp_{\alpha\beta} \hookrightarrow \mathfrak{P}^\sharp_{\alpha\beta}$ and $u_{\alpha\beta\gamma} \colon \mathfrak{X}^\sharp_{\alpha\beta\gamma} \hookrightarrow \mathfrak{P}^\sharp_{\alpha\beta\gamma}$.

Definition 9.3.7.1. For any $\alpha \in \Lambda$, let $\mathcal{E}_\alpha$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_\alpha/\mathfrak{S}^\sharp}({}^\dagger Z_\alpha)_{\mathbb{Q}}$-module. A *glueing data* on $(\mathcal{E}_\alpha)_{\alpha\in\Lambda}$ is the data for any $\alpha, \beta \in \Lambda$ of a $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_{\alpha\beta}/\mathfrak{S}^\sharp}({}^\dagger Z_{\alpha\beta})_{\mathbb{Q}}$-linear isomorphism

$$\theta_{\alpha\beta} \colon p_2^{\alpha\beta!}(\mathcal{E}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta!}(\mathcal{E}_\alpha),$$

satisfying the cocycle condition: $\theta_{13}^{\alpha\beta\gamma} = \theta_{12}^{\alpha\beta\gamma} \circ \theta_{23}^{\alpha\beta\gamma}$, where $\theta_{12}^{\alpha\beta\gamma}$, $\theta_{23}^{\alpha\beta\gamma}$ and $\theta_{13}^{\alpha\beta\gamma}$ are the isomorphisms making commutative the following diagram

$$\begin{array}{ccccccccc}
p_{12}^{\alpha\beta\gamma!} p_2^{\alpha\beta!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\tau} & p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta) & p_{23}^{\alpha\beta\gamma!} p_2^{\beta\gamma!}(\mathcal{E}_\gamma) & \xrightarrow[\sim]{\tau} & p_3^{\alpha\beta\gamma!}(\mathcal{E}_\gamma) & p_{13}^{\alpha\beta\gamma!} p_2^{\alpha\gamma!}(\mathcal{E}_\gamma) & \xrightarrow[\sim]{\tau} & p_3^{\alpha\beta\gamma!}(\mathcal{E}_\gamma) \\
\sim\downarrow{\scriptstyle p_{12}^{\alpha\beta\gamma!}(\theta_{\alpha\beta})} & & \downarrow{\scriptstyle \theta_{12}^{\alpha\beta\gamma}} & \sim\downarrow{\scriptstyle p_{23}^{\alpha\beta\gamma!}(\theta_{\beta\gamma})} & & \downarrow{\scriptstyle \theta_{23}^{\alpha\beta\gamma}} & \sim\downarrow{\scriptstyle p_{13}^{\alpha\beta\gamma!}(\theta_{\alpha\gamma})} & & \downarrow{\scriptstyle \theta_{13}^{\alpha\beta\gamma}} \\
p_{12}^{\alpha\beta\gamma!} p_1^{\alpha\beta!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\tau} & p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha), & p_{23}^{\alpha\beta\gamma!} p_1^{\beta\gamma!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\tau} & p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta), & p_{13}^{\alpha\beta\gamma!} p_1^{\alpha\gamma!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\tau} & p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha),
\end{array} \tag{9.3.7.1.1}$$

where τ are the glueing isomorphisms defined in 9.2.2.3.1.

Definition 9.3.7.2. We define the category $\mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$ as follows:

(a) an object is a family $(\mathcal{E}_\alpha)_{\alpha\in\Lambda}$ of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_\alpha/\mathfrak{S}^\sharp}({}^\dagger Z_\alpha)_{\mathbb{Q}}$-modules together with a glueing data $(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}$,

(b) a morphism $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \to ((\mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ is a family of morphisms $f_\alpha \colon \mathcal{E}_\alpha \to \mathcal{E}'_\alpha$ of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_\alpha/\mathfrak{S}^\sharp}({}^\dagger Z_\alpha)_{\mathbb{Q}}$-modules commuting with glueing data, i.e., such that the following diagrams are commutative:

$$\begin{array}{ccc}
p_2^{\alpha\beta!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\theta_{\alpha\beta}} & p_1^{\alpha\beta!}(\mathcal{E}_\alpha) \\
{\scriptstyle p_2^{\alpha\beta!}(f_\beta)}\downarrow & & \downarrow{\scriptstyle p_1^{\alpha\beta!}(f_\alpha)} \\
p_2^{\alpha\beta!}(\mathcal{E}'_\beta) & \xrightarrow[\sim]{\theta'_{\alpha\beta}} & p_1^{\alpha\beta!}(\mathcal{E}'_\alpha).
\end{array} \tag{9.3.7.2.1}$$

Remark 9.3.7.3. For all $\alpha, \beta \in \Lambda$, let $f_\alpha\colon \mathcal{E}_\alpha \to \mathcal{E}'_\alpha$ a morphism of $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_\alpha}({}^\dagger Z_\alpha)_\mathbb{Q}$-coherent modules, $\theta_{\alpha\beta}\colon p_2^{\alpha\beta !}(\mathcal{E}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta !}(\mathcal{E}_\alpha)$ and $\theta'_{\alpha\beta}\colon p_2^{\alpha\beta !}(\mathcal{E}'_\beta) \xrightarrow{\sim} p_1^{\alpha\beta !}(\mathcal{E}'_\alpha)$ isomorphisms $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_{\alpha\beta}}({}^\dagger Z_{\alpha\beta})_\mathbb{Q}$-linear. It is further assumed that the morphisms f_α and the isomorphisms $\theta_{\alpha\beta}$ and $\theta'_{\alpha\beta}$ induce the commutative diagram 9.3.7.2.1.

Then, the isomorphisms $\theta_{\alpha\beta}$ satisfy the cocycle condition if and only if the same is true of the isomorphisms $\theta'_{\alpha\beta}$. Indeed, by transforming, via 9.3.7.2.1, the squares 9.3.7.1.1 in three commutative cubes, we get the following commutative squares for any $1 \le i < j \le 3$:

$$\begin{array}{ccc} p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta) & \xrightarrow{p_2^{\alpha\beta\gamma!}(f_\beta)} & p_2^{\alpha\beta\gamma!}(\mathcal{E}'_\beta) \\ \sim\downarrow\theta_{12}^{\alpha\beta\gamma} & & \sim\downarrow\theta_{12}'^{\alpha\beta\gamma} \\ p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha) & \xrightarrow{p_1^{\alpha\beta\gamma!}(f_\alpha)} & p_1^{\alpha\beta\gamma!}(\mathcal{E}'_\alpha), \end{array} \quad \begin{array}{ccc} p_3^{\alpha\beta\gamma!}(\mathcal{E}_\gamma) & \xrightarrow{p_3^{\alpha\beta\gamma!}(f_\gamma)} & p_3^{\alpha\beta\gamma!}(\mathcal{E}'_\gamma) \\ \sim\downarrow\theta_{23}^{\alpha\beta\gamma} & & \sim\downarrow\theta_{23}'^{\alpha\beta\gamma} \\ p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta) & \xrightarrow{p_2^{\alpha\beta\gamma!}(f_\beta)} & p_2^{\alpha\beta\gamma!}(\mathcal{E}'_\beta), \end{array} \quad \begin{array}{ccc} p_3^{\alpha\beta\gamma!}(\mathcal{E}_\gamma) & \xrightarrow{p_3^{\alpha\beta\gamma!}(f_\gamma)} & p_3^{\alpha\beta\gamma!}(\mathcal{E}'_\gamma) \\ \sim\downarrow\theta_{13}^{\alpha\beta\gamma} & & \sim\downarrow\theta_{13}'^{\alpha\beta\gamma} \\ p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha) & \xrightarrow{p_1^{\alpha\beta\gamma!}(f_\alpha)} & p_1^{\alpha\beta\gamma!}(\mathcal{E}'_\alpha). \end{array}$$

Definition 9.3.7.4. We denote by $\mathrm{Coh}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$ the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_\mathbb{Q}$-modules with support in X. When T is the empty divisor, we simply write $\mathrm{Coh}(X, \mathfrak{X}^\sharp/\mathfrak{S}^\sharp)$.

Lemma 9.3.7.5 (Construction of $u_0^!$)**.** *There exists a canonical functor*

$$u_0^!\colon \mathrm{Coh}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp) \to \mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$$

extending the usual functor $u_0^!$ when $X^\sharp$ has a log smooth formal $\mathfrak{S}^\sharp$-scheme lifting.

Proof. Let $\mathcal{E} \in \mathrm{Coh}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$. Thanks to Berthelot-Kashiwara's theorem (see 9.3.5.9), we get a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_\alpha/\mathfrak{S}^\sharp}({}^\dagger Z_\alpha)_\mathbb{Q}$-module by setting $\mathcal{E}_\alpha := H^0 u_\alpha^!(\mathcal{E}|\mathfrak{X}^\sharp_\alpha) \xrightarrow{\sim} u_\alpha^!(\mathcal{E}|\mathfrak{X}^\sharp_\alpha)$. Via the isomorphisms of the form τ (9.2.2.3.1), we obtain the glueing $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_{\alpha\beta}/\mathfrak{S}^\sharp}({}^\dagger Z_{\alpha\beta})_\mathbb{Q}$-linear isomorphism $\theta_{\alpha\beta}\colon p_2^{\alpha\beta!}(\mathcal{E}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta!}(\mathcal{E}_\alpha)$, as the single arrow making the following diagram commutative

$$\begin{array}{ccc} p_2^{\alpha\beta!}u_\beta^!(\mathcal{E}|_{\mathfrak{P}^\sharp_\beta}) & \xrightarrow[\sim]{\tau} & u_{\alpha\beta}^!((\mathcal{E}|_{\mathfrak{P}^\sharp_\beta})|_{\mathfrak{P}^\sharp_{\alpha\beta}}) \\ \downarrow\theta_{\alpha\beta} & & \| \\ p_1^{\alpha\beta!}u_\alpha^!(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}) & \xrightarrow[\sim]{\tau} & u_{\alpha\beta}^!((\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha})|_{\mathfrak{P}^\sharp_{\alpha\beta}}). \end{array} \tag{9.3.7.5.1}$$

Via the isomorphism τ: $p_{12}^{\alpha\beta\gamma!}u_{\alpha\beta}^!((\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha})|_{\mathfrak{P}^\sharp_{\alpha\beta}}) \xrightarrow{\sim} u_{\alpha\beta\gamma}^!((\mathcal{E}|_{\mathfrak{P}^\sharp_\beta})|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})$, by applying the functor $p_{12}^{\alpha\beta\gamma!}$ squared 9.3.7.5.1 and with 9.3.7.1.1, we get the commutative diagram:

$$\begin{array}{ccc} p_2^{\alpha\beta\gamma!}(u_\beta^!(\mathcal{E}|_{\mathfrak{P}^\sharp_\beta})) & \xrightarrow[\sim]{\tau} & u_{\alpha\beta\gamma}^!((\mathcal{E}|_{\mathfrak{P}^\sharp_\beta})|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}) \\ \sim\downarrow\theta_{12}^{\alpha\beta\gamma} & & \| \\ p_1^{\alpha\beta\gamma!}(u_\alpha^!(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha})) & \xrightarrow[\sim]{\tau} & u_{\alpha\beta\gamma}^!((\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha})|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}), \end{array}$$

where the horizontal isomorphisms are of the form τ, thanks to the transitivity formula and its commutation with extraordinary inverse images of isomorphisms of the form τ (9.2.2.3.1). Similarly, we construct the two other analogous diagrams. With these three diagrams, we verify that $u_0^!(\mathcal{E}) := ((u_\alpha^!(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ satisfies the cocycle condition $\theta_{13}^{\alpha\beta\gamma} = \theta_{12}^{\alpha\beta\gamma} \circ \theta_{23}^{\alpha\beta\gamma}$ and is thus an object of $\mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$.

Also, if $f\colon \mathcal{E} \to \mathcal{E}'$ is a morphism of $\mathrm{Coh}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$, then, by functoriality in $\mathcal{E}$ of 9.3.7.5.1 (we transform the square 9.3.7.5.1 into a cube), the family $(u_\alpha^!(f|_{\mathfrak{P}^\sharp_\alpha}))_{\alpha\in\Lambda}$ commutes with the glueing data. □

Lemma 9.3.7.6. *There exists a canonical functor*

$$u_{0+}\colon \mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp) \to \mathrm{Coh}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$$

extending the usual functor u_{0+} when $X^\sharp$ has a log smooth formal $\mathfrak{S}^\sharp$-scheme lifting.

Proof. Let $(\mathcal{E}_\alpha)_{\alpha\in\Lambda}$ be a family of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp_\alpha/\mathfrak{S}^\sharp}({}^\dagger Z_\alpha)_{\mathbb{Q}}$-modules together with a gluing data $(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}$. Let's prove that $(u_{\alpha+}(\mathcal{E}_\alpha))_{\alpha\in\Lambda}$ glues via $(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ to a coherent $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module with support in X. Let $\phi_1^{\alpha\beta}$ (resp. $\phi_2^{\alpha\beta}$) be the adjunction morphism (see the definition 9.3.6.1) of the left (resp. the right) square of

$$\begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta} & \longrightarrow & \mathfrak{P}^\sharp_\alpha \\ {\scriptstyle u_{\alpha\beta}}\uparrow & & \uparrow{\scriptstyle u_\alpha} \\ \mathfrak{X}^\sharp_{\alpha\beta} & \xrightarrow{p_1^{\alpha\beta}} & \mathfrak{X}^\sharp_\alpha, \end{array} \qquad \begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta} & \longrightarrow & \mathfrak{P}^\sharp_\beta \\ {\scriptstyle u_{\alpha\beta}}\uparrow & & \uparrow{\scriptstyle u_\beta} \\ \mathfrak{X}^\sharp_{\alpha\beta} & \xrightarrow{p_2^{\alpha\beta}} & \mathfrak{X}^\sharp_\beta. \end{array} \tag{9.3.7.6.1}$$

For every $\alpha,\beta\in\Lambda$ we define the isomorphism $\tau_{\alpha\beta}\colon (u_{\beta+}(\mathcal{E}_\beta))|_{\mathfrak{X}^\sharp_{\alpha\beta}} \xrightarrow{\sim} (u_{\alpha+}(\mathcal{E}_\alpha))|_{\mathfrak{X}^\sharp_{\alpha\beta}}$ to be the one making commutative the following diagram:

$$\begin{array}{ccc} u_{\alpha\beta+}\circ p_1^{\alpha\beta!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\phi_1^{\alpha\beta}(\mathcal{E}_\alpha)} & (u_{\alpha+}(\mathcal{E}_\alpha))|_{\mathfrak{X}^\sharp_{\alpha\beta}} \\ {\scriptstyle u_{\alpha\beta+}(\theta_{\alpha\beta})}\uparrow\wr & & \uparrow{\scriptstyle \tau_{\alpha\beta}} \\ u_{\alpha\beta+}\circ p_2^{\alpha\beta!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\phi_2^{\alpha\beta}(\mathcal{E}_\beta)} & (u_{\beta+}(\mathcal{E}_\beta))|_{\mathfrak{X}^\sharp_{\alpha\beta}}. \end{array} \tag{9.3.7.6.2}$$

It now remains to establish that the isomorphisms $\tau_{\alpha\beta}$ satisfy the condition of reattachment. To this end, note $\phi_{12}^{\alpha\beta\gamma}$ (resp. $\phi_{23}^{\alpha\beta\gamma}$ and $\phi_{13}^{\alpha\beta\gamma}$) the adjunction morphism (always 9.3.6.1) of the next left (resp. center and right) square

$$\begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta\gamma} & \longrightarrow & \mathfrak{P}^\sharp_{\alpha\beta} \\ {\scriptstyle u_{\alpha\beta\gamma}}\uparrow & & \uparrow{\scriptstyle u_{\alpha\beta}} \\ \mathfrak{X}^\sharp_{\alpha\beta\gamma} & \xrightarrow{p_{12}^{\alpha\beta\gamma}} & \mathfrak{X}^\sharp_{\alpha\beta}, \end{array} \quad \begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta\gamma} & \longrightarrow & \mathfrak{P}^\sharp_{\beta\gamma} \\ {\scriptstyle u_{\alpha\beta\gamma}}\uparrow & & \uparrow{\scriptstyle u_{\beta\gamma}} \\ \mathfrak{X}^\sharp_{\alpha\beta\gamma} & \xrightarrow{p_{23}^{\alpha\beta\gamma}} & \mathfrak{X}^\sharp_{\beta\gamma}, \end{array} \quad \begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta\gamma} & \longrightarrow & \mathfrak{P}^\sharp_{\alpha\gamma} \\ {\scriptstyle u_{\alpha\beta\gamma}}\uparrow & & \uparrow{\scriptstyle u_{\alpha\gamma}} \\ \mathfrak{X}^\sharp_{\alpha\beta\gamma} & \xrightarrow{p_{13}^{\alpha\beta\gamma}} & \mathfrak{X}^\sharp_{\alpha\gamma}, \end{array} \tag{9.3.7.6.3}$$

and $\phi_1^{\alpha\beta\gamma}$ (resp. $\phi_2^{\alpha\beta\gamma}$ and $\phi_3^{\alpha\beta\gamma}$) that of the diagram from the left (resp. from the center and from the right):

$$\begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta\gamma} & \longrightarrow & \mathfrak{P}^\sharp_{\alpha} \\ {\scriptstyle u_{\alpha\beta\gamma}}\uparrow & & \uparrow{\scriptstyle u_{\alpha}} \\ \mathfrak{X}^\sharp_{\alpha\beta\gamma} & \xrightarrow{p_{1}^{\alpha\beta\gamma}} & \mathfrak{X}^\sharp_{\alpha}, \end{array} \quad \begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta\gamma} & \longrightarrow & \mathfrak{P}^\sharp_{\beta} \\ {\scriptstyle u_{\alpha\beta\gamma}}\uparrow & & \uparrow{\scriptstyle u_{\beta}} \\ \mathfrak{X}^\sharp_{\alpha\beta\gamma} & \xrightarrow{p_{2}^{\alpha\beta\gamma}} & \mathfrak{X}^\sharp_{\beta}, \end{array} \quad \begin{array}{ccc} \mathfrak{P}^\sharp_{\alpha\beta\gamma} & \longrightarrow & \mathfrak{P}^\sharp_{\gamma} \\ {\scriptstyle u_{\alpha\beta\gamma}}\uparrow & & \uparrow{\scriptstyle u_{\gamma}} \\ \mathfrak{X}^\sharp_{\alpha\beta\gamma} & \xrightarrow{p_{3}^{\alpha\beta\gamma}} & \mathfrak{X}^\sharp_{\gamma}. \end{array} \tag{9.3.7.6.4}$$

Consider the following commutative diagram

$$\begin{array}{ccccc} u_{\alpha\beta\gamma+}p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{u_{\alpha\beta\gamma+}(\tau)} & u_{\alpha\beta\gamma+}\circ p_{12}^{\alpha\beta\gamma!}(p_1^{\alpha\beta!}(\mathcal{E}_\alpha)) & \xrightarrow[\sim]{\phi_{12}^{\alpha\beta\gamma}(p_1^{\alpha\beta!}(\mathcal{E}_\alpha))} & u_{\alpha\beta+}(p_1^{\alpha\beta!}(\mathcal{E}_\alpha))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}} \\ {\scriptstyle u_{\alpha\beta\gamma+}(\theta_{12}^{\alpha\beta\gamma})}\uparrow\wr & & {\scriptstyle u_{\alpha\beta\gamma+}\circ p_{12}^{\alpha\beta\gamma!}(\theta_{\alpha\beta})}\uparrow\wr & & \wr\uparrow{\scriptstyle u_{\alpha\beta+}(\theta_{\alpha\beta})|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}} \\ u_{\alpha\beta\gamma+}p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{u_{\alpha\beta\gamma+}(\tau)} & u_{\alpha\beta\gamma+}\circ p_{12}^{\alpha\beta\gamma!}(p_2^{\alpha\beta!}(\mathcal{E}_\beta)) & \xrightarrow[\sim]{\phi_{12}^{\alpha\beta\gamma}(p_2^{\alpha\beta!}(\mathcal{E}_\beta))} & u_{\alpha\beta+}(p_2^{\alpha\beta!}(\mathcal{E}_\beta))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}. \end{array} \tag{9.3.7.6.5}$$

Thanks to 9.3.6.1.a) and 9.3.6.1.b), we then obtain

$$(\phi_1^{\alpha\beta}(\mathcal{E}_\alpha)|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})\circ\phi_{12}^{\alpha\beta\gamma}(p_1^{\alpha\beta!}(\mathcal{E}_\alpha))\circ u_{\alpha\beta\gamma+}(\tau) = \phi_1^{\alpha\beta\gamma}(\mathcal{E}_\alpha), \tag{9.3.7.6.6}$$

$$(\phi_2^{\alpha\beta}(\mathcal{E}_\beta)|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})\circ\phi_{12}^{\alpha\beta\gamma}(p_2^{\alpha\beta!}(\mathcal{E}_\beta))\circ u_{\alpha\beta\gamma+}(\tau) = \phi_2^{\alpha\beta\gamma}(\mathcal{E}_\beta), \tag{9.3.7.6.7}$$

$$(\phi_1^{\beta\gamma}(\mathcal{E}_\beta)|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})\circ\phi_{23}^{\alpha\beta\gamma}(p_1^{\beta\gamma!}(\mathcal{E}_\beta))\circ u_{\alpha\beta\gamma+}(\tau) = \phi_2^{\alpha\beta\gamma}(\mathcal{E}_\beta), \tag{9.3.7.6.8}$$

$$(\phi_2^{\beta\gamma}(\mathcal{E}_\gamma)|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})\circ\phi_{23}^{\alpha\beta\gamma}(p_2^{\beta\gamma!}(\mathcal{E}_\gamma))\circ u_{\alpha\beta\gamma+}(\tau) = \phi_3^{\alpha\beta\gamma}(\mathcal{E}_\gamma), \tag{9.3.7.6.9}$$

$$(\phi_1^{\alpha\gamma}(\mathcal{E}_\alpha)|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})\circ\phi_{13}^{\alpha\beta\gamma}(p_1^{\alpha\gamma!}(\mathcal{E}_\alpha))\circ u_{\alpha\beta\gamma+}(\tau) = \phi_1^{\alpha\beta\gamma}(\mathcal{E}_\alpha), \tag{9.3.7.6.10}$$

$$(\phi_2^{\alpha\gamma}(\mathcal{E}_\gamma)|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}})\circ\phi_{13}^{\alpha\beta\gamma}(p_2^{\alpha\gamma!}(\mathcal{E}_\gamma))\circ u_{\alpha\beta\gamma+}(\tau) = \phi_3^{\alpha\beta\gamma}(\mathcal{E}_\gamma). \tag{9.3.7.6.11}$$

By composing 9.3.7.6.2 restricted to $\mathfrak{P}^\sharp_{\alpha\beta\gamma}$ and 9.3.7.6.5, via equalities 9.3.7.6.6 and 9.3.7.6.7, we get the commutative square:

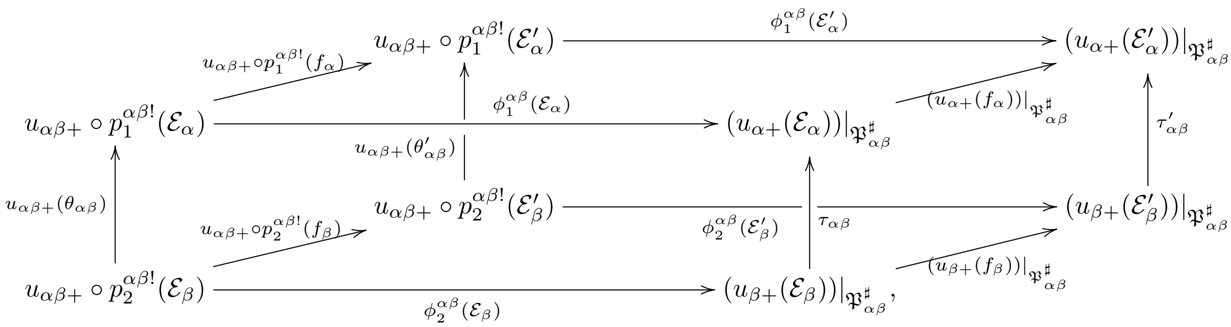
$$\begin{array}{ccc} u_{\alpha\beta\gamma+} p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\phi_1^{\alpha\beta\gamma}(\mathcal{E}_\alpha)} & (u_{\alpha+}(\mathcal{E}_\alpha))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}} \\ {\scriptstyle u_{\alpha\beta\gamma+}(\theta_{12}^{\alpha\beta\gamma})}\uparrow\wr & & \wr\uparrow{\scriptstyle \tau_{\alpha\beta}|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}} \\ u_{\alpha\beta\gamma+} p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\phi_2^{\alpha\beta\gamma}(\mathcal{E}_\beta)} & (u_{\beta+}(\mathcal{E}_\beta))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}. \end{array} \tag{9.3.7.6.12}$$

Analogously, using 9.3.7.6.8 and 9.3.7.6.9 (resp. 9.3.7.6.10 and 9.3.7.6.11) we obtain the following commutative diagrams:

$$\begin{array}{ccc} u_{\alpha\beta\gamma+}\circ p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\phi_2^{\alpha\beta\gamma}(\mathcal{E}_\beta)} & (u_{\beta+}(\mathcal{E}_\beta))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}} \\ {\scriptstyle u_{\alpha\beta\gamma+}(\theta_{23}^{\alpha\beta\gamma})}\uparrow\wr & & \wr\uparrow{\scriptstyle \tau_{\beta\gamma}|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}} \\ u_{\alpha\beta\gamma+}\circ p_3^{\alpha\beta\gamma!}(\mathcal{E}_\gamma) & \xrightarrow[\sim]{\phi_3^{\alpha\beta\gamma}(\mathcal{E}_\gamma)} & (u_{\gamma+}(\mathcal{E}_\gamma))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}, \end{array} \qquad \begin{array}{ccc} u_{\alpha\beta\gamma+}\circ p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\phi_1^{\alpha\beta\gamma}(\mathcal{E}_\alpha)} & (u_{\alpha+}(\mathcal{E}_\alpha))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}} \\ {\scriptstyle u_{\alpha\beta\gamma+}(\theta_{13}^{\alpha\beta\gamma})}\uparrow\wr & & \wr\uparrow{\scriptstyle \tau_{\alpha\gamma}|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}} \\ u_{\alpha\beta\gamma+}\circ p_3^{\alpha\beta\gamma!}(\mathcal{E}_\gamma) & \xrightarrow[\sim]{\phi_3^{-}\alpha\beta\gamma(\mathcal{E}_\gamma)} & (u_{\gamma+}(\mathcal{E}_\gamma))|_{\mathfrak{P}^\sharp_{\alpha\beta\gamma}}. \end{array} \tag{9.3.7.6.13}$$

Of these last three diagrams, since the functor $u_{\alpha\beta\gamma+}$ is (fully) faithful (for coherent modules), it follows that the isomorphisms $\theta_{\alpha\beta}$ verify the condition of cocycle if and only if the isomorphisms $\tau_{\alpha\beta}$ stick together.

Let $f=(f_\alpha)_{\alpha\in\Lambda}\colon ((\mathcal{E}_\alpha)_{\alpha\in\Lambda},(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})\to((\mathcal{E}'_\alpha)_{\alpha\in\Lambda},(\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda})$, be a morphism of $\mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda},Z/\mathfrak{S}^\sharp)$. We associate the family $u_{0+}(f)\colon (u_{\alpha+}(f_\alpha))_{\alpha\in\Lambda}$. By noting $\tau_{\alpha\beta}$ (resp. $\tau'_{\alpha\beta}$) the isomorphism making 9.3.7.6.2 commutative for $\theta_{\alpha\beta}$ (resp. $\theta'_{\alpha\beta}$), we get the cube



(9.3.7.6.14)

whose front, back, bottom and top squares are commutative by functoriality or thanks to 9.3.7.6.2. As the one on the left is (via 9.3.7.2.1), it follows that it is the same for the square on the right. The morphisms $u_{\alpha+}(f_\alpha)$ therefore stick together.

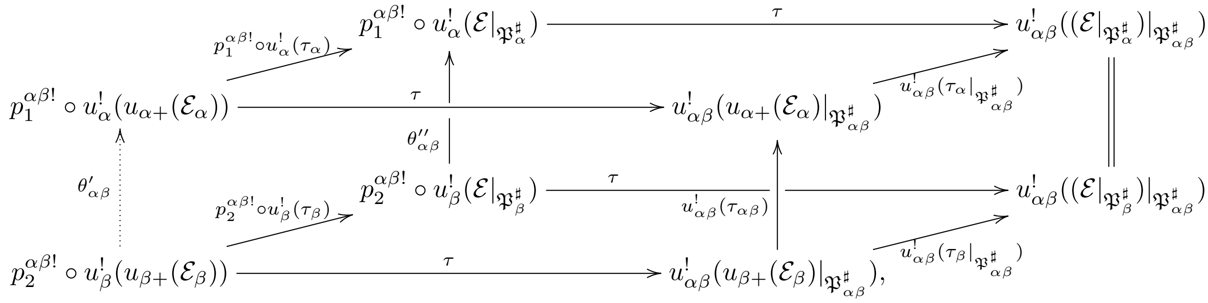
□

Theorem 9.3.7.7. *The functors $u_0^!$ and u_{0+} constructed in respectively 9.3.7.5 and 9.3.7.6 are quasi-inverse equivalences of categories between* $\mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda},Z/\mathfrak{S}^\sharp)$ *and* $\mathrm{Coh}(X^\sharp,\mathfrak{P}^\sharp,T/\mathfrak{S}^\sharp)$.

Proof. Let $((\mathcal{E}_\alpha)_{\alpha\in\Lambda},(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ an object of $\mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda},Z/\mathfrak{S}^\sharp)$. At first, it is a question of establishing a functorial isomorphism $u_0^!\circ u_{0+}((\mathcal{E}_\alpha)_{\alpha\in\Lambda},(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})\xrightarrow{\sim}((\mathcal{E}_\alpha)_{\alpha\in\Lambda},(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$.

Note $\mathcal{E}:=u_{0+}((\mathcal{E}_\alpha)_{\alpha\in\Lambda},(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$, $u_0^!(\mathcal{E})=((u_\alpha^!(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))_{\alpha\in\Lambda},(\theta''_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ and $\tau_\alpha\colon u_{\alpha+}(\mathcal{E}_\alpha)\xrightarrow{\sim}\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}$ the canonical isomorphisms satisfying $\tau_{\alpha\beta}=\tau_\alpha^{-1}|_{\mathfrak{P}^\sharp_{\alpha\beta}}\circ\tau_\beta|_{\mathfrak{P}^\sharp_{\alpha\beta}}$, where $\tau_{\alpha\beta}$ was defined via 9.3.7.6.2.

Consider the following diagram



(9.3.7.7.1)

where the arrow $\theta'_{\alpha\beta}$ is by definition the one making the front square commutative. The bottom and right squares are commutative by definition and the top and bottom are commutative by functoriality. Thanks to the remark 9.3.7.3, we deduce that the isomorphisms $\theta'_{\alpha\beta}$ satisfy the cocycle condition and we are reduced to proving that the adjunction isomorphism $\mathcal{E}_\alpha \xrightarrow{\sim} u^!_\alpha \circ u_{\alpha+}(\mathcal{E}_\alpha)$ is compatible with the respective glueing data, i.e., that the following left square

$$\begin{array}{ccccc}
p_1^{\alpha\beta!}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\mathrm{adj}_{u_\alpha}} & p_1^{\alpha\beta!}\circ u^!_\alpha \circ u_{\alpha+}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\tau} & u^!_{\alpha\beta}(u_{\alpha+}(\mathcal{E}_\alpha)|_{\mathfrak{P}^\sharp_{\alpha\beta}}) \\
{\scriptstyle \theta_{\alpha\beta}}\uparrow{\scriptstyle \sim} & & {\scriptstyle \theta'_{\alpha\beta}}\uparrow{\scriptstyle \sim} & & {\scriptstyle u^!_{\alpha\beta}(\tau_{\alpha\beta})}\uparrow{\scriptstyle \sim} \\
p_2^{\alpha\beta!}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\mathrm{adj}_{u_\beta}} & p_2^{\alpha\beta!}\circ u^!_\beta \circ u_{\beta+}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\tau} & u^!_{\alpha\beta}(u_{\beta+}(\mathcal{E}_\beta)|_{\mathfrak{P}^\sharp_{\alpha\beta}})
\end{array} \qquad (9.3.7.7.2)$$

is commutative. However, by applying the functor $u_{\alpha\beta+}$ to the diagram 9.3.7.7.2 and composing with the commutative diagram

$$\begin{array}{ccc}
u_{\alpha\beta+}u^!_{\alpha\beta}(u_{\alpha+}(\mathcal{E}_\alpha)|_{\mathfrak{P}^\sharp_{\alpha\beta}}) & \xrightarrow[\sim]{\mathrm{adj}_{u_{\alpha\beta}}} & u_{\alpha+}(\mathcal{E}_\alpha)|_{\mathfrak{P}^\sharp_{\alpha\beta}} \\
{\scriptstyle u_{\alpha\beta+}u^!_{\alpha\beta}(\tau_{\alpha\beta})}\uparrow{\scriptstyle \sim} & & {\scriptstyle \tau_{\alpha\beta}}\uparrow{\scriptstyle \sim} \\
u_{\alpha\beta+}u^!_{\alpha\beta}(u_{\beta+}(\mathcal{E}_\beta)|_{\mathfrak{P}^\sharp_{\alpha\beta}}) & \xrightarrow[\sim]{\mathrm{adj}_{u_{\alpha\beta}}} & u_{\beta+}(\mathcal{E}_\beta)|_{\mathfrak{P}^\sharp_{\alpha\beta}},
\end{array}$$

we obtain 9.3.7.6.2 (by construction of the addition arrow of the proposition 9.3.6.1), which is commutative. The functor $u_{\alpha\beta+}$ being faithful, we thus demonstrate the commutativity of the contour of 9.3.7.7.2. Since the right square of 9.3.7.7.2 is commutative (it corresponds to the front square from 9.3.7.7.1), it follows that of the left square.

Conversely, let $\mathcal{E}$ be a $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp}({}^\dagger T)_\mathbb{Q}$-coherent module with support in X. Let us check that we have the isomorphism $u_{0+}\circ u^!_0(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ functorial in $\mathcal{E}$. Note $(\theta_{\alpha\beta})_{\alpha\beta\in\Lambda}$, the gluing data of $(u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))_{\alpha\in\Lambda}$ defined in 9.3.7.5.1 and $(\tau_{\alpha\beta})_{\alpha\beta\in\Lambda}$, the glueing data of $(u_{\alpha+}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))_{\alpha\in\Lambda}$ deduced from that of $(u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))_{\alpha\in\Lambda}$ via 9.3.7.6.2. Let us now prove that the adjunction isomorphisms $u_{\alpha+}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}) \to \mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}$ are compatible with the respective glueing data, i.e., than the following square on the right (the commutativity of the other two is tautological)

$$\begin{array}{ccccccc}
u_{\alpha\beta+}\circ u^!_{\alpha\beta}(\mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}) & \xrightarrow[\sim]{u_{\alpha\beta+}(\tau)} & u_{\alpha\beta+}\circ p_1^{\alpha\beta!}(u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha})) & \xrightarrow[\sim]{\phi_1^{\alpha\beta}(u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))} & (u_{\alpha+}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))|_{\mathfrak{P}^\sharp_{\alpha\beta}} & \xrightarrow[\sim]{\mathrm{adj}_{u_\alpha}|_{\mathfrak{P}^\sharp_{\alpha\beta}}} & \mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}} \\
\| & & {\scriptstyle u_{\alpha\beta+}\theta_{\alpha\beta}}\uparrow{\scriptstyle \sim} & & {\scriptstyle \tau_{\alpha\beta}}\uparrow{\scriptstyle \sim} & & \| \\
u_{\alpha\beta+}\circ u^!_{\alpha\beta}(\mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}) & \xrightarrow[\sim]{u_{\alpha\beta+}(\tau)} & u_{\alpha\beta+}\circ p_2^{\alpha\beta!}(u^!_\beta(\mathcal{E}|_{\mathfrak{P}^\sharp_\beta})) & \xrightarrow[\sim]{\phi_2^{\alpha\beta}(u^!_\beta(\mathcal{E}|_{\mathfrak{P}^\sharp_\beta}))} & (u_{\beta+}u^!_\beta(\mathcal{E}|_{\mathfrak{P}^\sharp_\beta}))|_{\mathfrak{P}^\sharp_{\alpha\beta}} & \xrightarrow[\sim]{\mathrm{adj}_{u_\beta}|_{\mathfrak{P}^\sharp_{\alpha\beta}}} & \mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}
\end{array}$$

(9.3.7.7.3)

is commutative. Moreover, we have the following commutative diagram:

$$\begin{array}{ccccccc}
u_{\alpha\beta+}p_1^{\alpha\beta!}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}) & \xrightarrow[\sim]{\mathrm{adj}_{u_\alpha}} & u_{\alpha\beta+}p_1^{\alpha\beta!}u^!_\alpha u_{\alpha+}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}) & \xrightarrow[\sim]{u_{\alpha\beta+}\tau} & u_{\alpha\beta+}u^!_{\alpha\beta}(u_{\alpha+}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))|_{\mathfrak{P}^\sharp_{\alpha\beta}} & \xrightarrow[\sim]{\mathrm{adj}_{u_{\alpha\beta}}} & u_{\alpha+}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha})|_{\mathfrak{P}^\sharp_{\alpha\beta}} \\
\| & & {\scriptstyle \sim}\downarrow{\scriptstyle \mathrm{adj}_{u_\alpha}} & & {\scriptstyle \sim}\downarrow{\scriptstyle \mathrm{adj}_{u_\alpha}} & & {\scriptstyle \sim}\downarrow{\scriptstyle \mathrm{adj}_{u_\alpha}} \\
u_{\alpha\beta+}p_1^{\alpha\beta!}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}) & = & u_{\alpha\beta+}p_1^{\alpha\beta!}u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}) & \xrightarrow[\sim]{u_{\alpha\beta+}\tau} & u_{\alpha\beta+}u^!_{\alpha\beta}(\mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}) & \xrightarrow[\sim]{\mathrm{adj}_{u_{\alpha\beta}}} & \mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}.
\end{array}$$

(9.3.7.7.4)

Indeed, the two squares on the right of 9.3.7.7.4 are commutative by functoriality, while the one on the left is so for the same reasons as that of the second square from the left of the third row of 9.3.6.1.3. However, the morphism $\phi_1^{\alpha\beta}(u^!_\alpha(\mathcal{E}|_{\mathfrak{P}^\sharp_\alpha}))$ is, by construction (see the proof of 9.3.6.1), equal to the horizontal composite morphism from the top of 9.3.7.7.4. Via the commutativity of the diagram 9.3.7.7.4, it follows that the horizontal composite morphism from the top of 9.3.7.7.3 is the adjunct morphism $u_{\alpha\beta+}u^!_{\alpha\beta}(\mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}) \to \mathcal{E}|_{\mathfrak{P}^\sharp_{\alpha\beta}}$. Similarly, we verify that the bottom horizontal composite morphism of 9.3.7.7.3 is equal to the adjunction morphism by $u_{\alpha\beta}$. □

Definition 9.3.7.8. For any $\alpha \in \Lambda$, let $\mathcal{E}^{(\bullet)}_\alpha \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^1\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_\alpha}(Z_\alpha))$ A *glueing data* on $(\mathcal{E}^{(\bullet)}_\alpha)_{\alpha\in\Lambda}$ is the data for any α, $\beta \in \Lambda$ of an isomorphism in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^1\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_\alpha}(Z_\alpha))$ of the form

$$\theta_{\alpha\beta} \colon p_2^{\alpha\beta(\bullet)!}(\mathcal{E}^{(\bullet)}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta(\bullet)!}(\mathcal{E}^{(\bullet)}_\alpha),$$

satisfying the cocycle condition: $\theta_{13}^{\alpha\beta\gamma} = \theta_{12}^{\alpha\beta\gamma} \circ \theta_{23}^{\alpha\beta\gamma}$, where $\theta_{12}^{\alpha\beta\gamma}$, $\theta_{23}^{\alpha\beta\gamma}$ and $\theta_{13}^{\alpha\beta\gamma}$ are the isomorphisms making commutative the similar to 9.3.7.1.1 square, e.g.

$$\begin{array}{ccc} p_{12}^{\alpha\beta\gamma(\bullet)!}p_2^{\alpha\beta(\bullet)!}(\mathcal{E}^{(\bullet)}_\beta) & \xrightarrow[\sim]{\tau^{(\bullet)}} & p_2^{\alpha\beta\gamma(\bullet)!}(\mathcal{E}^{(\bullet)}_\beta) \\ \sim\downarrow p_{12}^{\alpha\beta\gamma(\bullet)!}(\theta_{\alpha\beta}) & & \downarrow \theta_{ij}^{\alpha\beta\gamma} \\ p_{12}^{\alpha\beta\gamma(\bullet)!}p_1^{\alpha\beta(\bullet)!}(\mathcal{E}^{(\bullet)}_\alpha) & \xrightarrow[\sim]{\tau^{(\bullet)}} & p_1^{\alpha\beta\gamma(\bullet)!}(\mathcal{E}^{(\bullet)}_\alpha), \end{array} \tag{9.3.7.8.1}$$

where $\tau^{(\bullet)}$ are the glueing isomorphisms defined in 9.2.2.1.

Definition 9.3.7.9. We define the category $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$ as follows:

(a) an object is a family $(\mathcal{E}^{(\bullet)}_\alpha)_{\alpha\in\Lambda}$ of objects of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^1\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_\alpha}(Z_\alpha))$ together with a glueing data $(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}$,

(b) a morphism $((\mathcal{E}^{(\bullet)}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \to ((\mathcal{E}'^{(\bullet)}_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ is a family of morphisms $f^{(\bullet)}_\alpha \colon \mathcal{E}^{(\bullet)}_\alpha \to \mathcal{E}'^{(\bullet)}_\alpha$ of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^1\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp_\alpha}(Z_\alpha))$ commuting with glueing data, i.e., such that the following diagrams are commutative:

$$\begin{array}{ccc} p_2^{\alpha\beta(\bullet)!}(\mathcal{E}^{(\bullet)}_\beta) & \xrightarrow[\sim]{\theta_{\alpha\beta}} & p_1^{\alpha\beta(\bullet)!}(\mathcal{E}^{(\bullet)}_\alpha) \\ p_2^{\alpha\beta(\bullet)!}(f^{(\bullet)}_\beta)\downarrow & & \downarrow p_1^{\alpha\beta(\bullet)!}(f^{(\bullet)}_\alpha) \\ p_2^{\alpha\beta(\bullet)!}(\mathcal{E}'^{(\bullet)}_\beta) & \xrightarrow[\sim]{\theta'_{\alpha\beta}} & p_1^{\alpha\beta(\bullet)!}(\mathcal{E}'^{(\bullet)}_\alpha). \end{array} \tag{9.3.7.9.1}$$

9.3.7.10. With notation 9.3.7.9, via the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp) \cong \mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$ (see 8.7.5.4.1), we get the (still denoted by $\underrightarrow{l}^*_{\mathbb{Q}}$) equivalence of categories

$$\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp) \cong \mathrm{Coh}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp). \tag{9.3.7.10.1}$$

Definition 9.3.7.11. We denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^1\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}(T))$ consisting of objects with support in X. When T is the empty divisor, we simply write $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{X}^\sharp/\mathfrak{S}^\sharp)$. We have the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}}$ (see 8.7.5.4.1) induces $\underrightarrow{l}^*_{\mathbb{Q}} \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp) \cong \mathrm{Coh}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$.

Theorem 9.3.7.12. *We have the following properties.*

(a) There exists a canonical functor

$$u_0^{(\bullet)!} \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}^\sharp_\alpha)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$$

extending the usual functor $u^!_0$ when $X^\sharp$ has a log smooth formal $\mathfrak{S}^\sharp$-scheme lifting. Via notation 9.3.7.5 and 9.3.7.10.1, we have the canonical isomorphism $\underrightarrow{l}^*_{\mathbb{Q}} \circ u_0^{(\bullet)!} \xrightarrow{\sim} u^!_0 \circ \underrightarrow{l}^*_{\mathbb{Q}}$.

(b) *There exists a canonical functor*

$$u_{0+}^{(\bullet)}\colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}_\alpha^\sharp)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$$

extending the usual functor u_{0+} when $X^\sharp$ has a log smooth formal $\mathfrak{S}^\sharp$-scheme lifting. Via notation 9.3.7.6 and 9.3.7.10.1, we have the canonical isomorphism $\underrightarrow{l}^*_{\mathbb{Q}} \circ u_{0+}^{(\bullet)} \xrightarrow{\sim} u_{0+} \circ \underrightarrow{l}^*_{\mathbb{Q}}$.

(c) *The functors $u_0^{(\bullet)!}$ and $u_{0+}^{(\bullet)}$ are quasi-inverse equivalences of categories between $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}_\alpha^\sharp)_{\alpha\in\Lambda}, Z/\mathfrak{S}^\sharp)$ and $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X^\sharp, \mathfrak{P}^\sharp, T/\mathfrak{S}^\sharp)$.*

Proof. Since extraordinary pullback by open immersions preserves the categories of the form $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}(T))$, using the LD-version of Berthelot-Kashiwara Theorem (see 9.3.5.13), we can copy word by word the proof of 9.3.7.5, 9.3.7.6 and 9.3.7.7. □

9.4 Stability of the coherence, base change, relative duality, Fourier transform

9.4.1 Log smooth morphisms: Spencer resolutions, stability of the coherence by pullbacks, pushforwards as relative de Rham complexes complexes

We keep notation 9.2.1 and we suppose f is a log-smooth and $\phi = \mathrm{id}$. We suppose T is a noetherian scheme of finite Krull dimension.

Notation 9.4.1.1. For all integers $m \le m'$, we set $\mathcal{D}^{(m,m')}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}(Z') := \mathcal{B}^{(m')}_{\mathfrak{X}'^\sharp}(Z') \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{D}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}$, $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'^\sharp}(Z') := \mathcal{B}^{(n_m)}_{\mathfrak{X}'^\sharp}(Z')$, $\widetilde{\omega}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}(Z') := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'^\sharp}(Z') \otimes_{\mathcal{O}_{\mathfrak{X}'}} \omega_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}(Z') := \widehat{\mathcal{D}}^{(m,n_m)}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}(Z')$.

9.4.1.2. It follows from 8.4.2.8 and from 7.5.10.1.3 that the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z')} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z') \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z', Z)$$

is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$. Hence $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z', Z) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$. Since the (induced by extension from 7.5.10.2.1) morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$

$$\widetilde{\mathrm{Sp}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}(Z', Z) \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z', Z) \tag{9.4.1.2.1}$$

is an isomorphism after applying the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z')) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z')_{\mathbb{Q}})$ (see 8.7.5.4.1), then 9.4.1.2.1 is an isomorphism (of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$).

Notation 9.4.1.3. Let $\mathcal{M}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$. The canonical map

$$\mathcal{M}'^{(\bullet)} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z')} \widetilde{\mathrm{Sp}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{X}}(Z', Z) \to \mathcal{M}'^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z')} \widetilde{\mathrm{Sp}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{X}}(Z', Z) \tag{9.4.1.3.1}$$

is an isomorphism and will be denoted by $\mathcal{M}'^{(\bullet)} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{T}^{\bullet}_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp}$.

9.4.1.4. Let us denote by $\widetilde{\mathrm{DR}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{X}}(Z, Z')$ the complex

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp} \underset{d}{\longrightarrow} \Omega^1_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp} \underset{d}{\longrightarrow} \cdots \underset{d}{\longrightarrow} \omega_{\mathfrak{X}'^\sharp/\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp},$$

where $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}$ is the 0th term. By using the similar to 9.4.1.2.1 argument, we get by extension from 7.5.10.3.1 the canonical isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$ of the form

$$\widetilde{\mathrm{DR}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{X}}(Z, Z')[d_f] \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z, Z'). \tag{9.4.1.4.1}$$

Notation 9.4.1.5. Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$. The canonical map

$$\widetilde{\mathrm{DR}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}}(Z,Z') \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')} \mathcal{E}'^{(\bullet)} \to \widetilde{\mathrm{DR}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}}(Z,Z') \widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(Z')} \mathcal{E}'^{(\bullet)} \tag{9.4.1.5.1}$$

is an isomorphism and will be denoted by $\Omega^{\bullet}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{E}'^{(\bullet)}$.

Proposition 9.4.1.6. *Assume f is a quasi-compact and quasi-separated morphism.*

(a) For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$, we have the isomorphism:

$$f^{(\bullet)}_{Z,Z',+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}f_* \left(\Omega^{\bullet}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{E}'^{(\bullet)}\right)[d_f]; \tag{9.4.1.6.1}$$

For any $\mathcal{E}' \in D({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$, we have the isomorphism:

$$f_{Z,Z',+}(\mathcal{E}') \xrightarrow{\sim} \mathbb{R}f_* \left(\Omega^{\bullet}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}^{\sharp}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{E}'\right)[d_f]; \tag{9.4.1.6.2}$$

(b) For any $\mathcal{M}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$, we have the isomorphism:

$$f^{(\bullet)}_{Z,Z',+}(\mathcal{M}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}f_* \left(\mathcal{M}'^{(\bullet)} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{T}^{\bullet}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}^{\sharp}}\right). \tag{9.4.1.6.3}$$

For any $\mathcal{M} \in D({}^{\mathrm{r}}\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$, we have the isomorphism:

$$f_{Z,Z',+}(\mathcal{M}') \xrightarrow{\sim} \mathbb{R}f_* \left(\mathcal{M}' \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{T}^{\bullet}_{\mathfrak{X}'^{\sharp}/\mathfrak{X}^{\sharp}}\right). \tag{9.4.1.6.4}$$

Proof. By using 9.2.4.2.1, the isomorphism 9.4.1.6.1 (resp. 9.4.1.6.3) is a consequence of 9.4.1.2.1 (resp. 9.4.1.4.1). Recalling the definition 9.2.4.13, we get the other isomorphisms by using the isomorphism equal to the image via $\underrightarrow{l}^*_{\mathbb{Q}}$ of 9.4.1.2.1 (resp. 9.4.1.4.1) □

Proposition 9.4.1.7. *Suppose moreover f is flat (e.g. when log structures are trivial).*

(a) The functor $f^{(\bullet)!}_{Z',Z}$ sends $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$ to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z'))$.

(b) For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}})$, we have $f^{!}_{Z',Z}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$.

(c) For any $\mathcal{E}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z))$ we have in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}})$ the canonical isomorphism:

$$\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}}({}^{\dagger}Z')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z')} f^{(m)!}_{Z',Z}(\mathcal{E}^{(m)}) \xrightarrow{\sim} f^{!}_{Z',Z}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}Z)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z)} \mathcal{E}^{(m)}). \tag{9.4.1.7.1}$$

Proof. Since f is flat, then the functor $f^{(\bullet)!}$ preserves the boundedness. It follows from 9.2.1.18 and 9.2.1.28 that we have the isomorphism $f^{(\bullet)!}_{Z',Z} \xrightarrow{\sim} ({}^{\dagger}Z') \circ f^{(\bullet)!} \circ \mathrm{forg}_Z$. Hence, the functor $f^{(\bullet)!}_{Z',Z}$ preserves boundedness. The first part is therefore a consequence of 7.5.10.8. The second and third parts are a consequence of the previous one, of 7.5.10.8 and of the isomorphism of 9.2.1.24. □

9.4.2 Pushforwards: way-out properties, stability of the coherence, tor dimension finiteness, perfectness

We keep notation 9.2.1.

Lemma 9.4.2.1. *We suppose f is a (non necessary exact) closed immersion and $\phi = \mathrm{id}$.*

(a) The left (resp. right) $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z')$-module $\widetilde{\mathcal{D}}^{(\bullet)}_{X'^{\sharp}_{\bullet}\to X^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z',Z)$ (resp. $\widetilde{\mathcal{D}}^{(\bullet)}_{X^{\sharp}_{\bullet}\leftarrow X'^{\sharp}_{\bullet}/S^{\sharp}_{\bullet}}(Z,Z')$) is locally free.

(b) The left (resp. right) $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z')$-module $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}\to\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(Z',Z)$ (resp. $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{X}'^{\sharp}/\mathfrak{S}^{\sharp}}(Z,Z')$) is flat.

Proof. This is checked similarly to 7.5.11.1. □

Proposition 9.4.2.2. We have $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z',Z) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'))$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp\leftarrow\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z,Z') \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{tdf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$.

Proof. By using 9.4.2.1, we can copy the proof of 7.5.11.2. □

Corollary 9.4.2.3. *Assume that $\mathfrak{S}$ and $\mathfrak{T}$ are noetherian of finite Krull dimension, f is quasi-compact and quasi-separated.*

$$f^{(\bullet)}_{Z,Z',\bullet+}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X'^\sharp_\bullet/S'^\sharp_\bullet}(Z')) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{X^\sharp_\bullet/S^\sharp_\bullet}(Z)), \tag{9.4.2.3.1}$$

$$f^{(\bullet)}_{Z,Z'+}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)). \tag{9.4.2.3.2}$$

Proof. This follows from 9.4.2.2, 8.5.4.14.2 and 9.2.4.2.1. □

Proposition 9.4.2.4. *Suppose f is proper and $Z' = f^{-1}(Z)$. Let $\star \in \{-, \mathrm{b}\}$ and $* \in \{\mathrm{r}, \mathrm{l}\}$.*

(a) The functor $f^{(\bullet)}_{Z,+}$ sends $\underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$ to $\underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$.

(b) For any $\mathcal{E}'^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')_{\mathbb{Q}})$, we have in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$ the canonical isomorphism:

$$\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)_{\mathbb{Q}}} f^{(m)}_{Z,+}(\mathcal{E}'^{(m)}) \xrightarrow{\sim} f_{Z,+}(\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')_{\mathbb{Q}}} \mathcal{E}'^{(m)}) \tag{9.4.2.4.1}$$

(c) For any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}'^\sharp}({}^\dagger Z')_{\mathbb{Q}})$, we have $f_{Z,+}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$.

Proof. The first part is a consequence of 7.5.11.4 and 7.5.8.14.1 (which can be applied in the case of overconvergent coefficients thanks to 9.2.1.26) and of 9.4.2.3 when $\star = \mathrm{b}$. For any $\mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}'^\sharp}({}^\dagger Z'))$. We get the isomorphism:

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z)_{\mathbb{Q}}} f^{(m)}_{Z,+}(\mathcal{E}'^{(m)}) \xrightarrow{\sim} f^{(\bullet)}_{+}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z')_{\mathbb{Q}}} \mathcal{E}'^{(m)}) \tag{9.4.2.4.2}$$

Using the equivalence of categories of 7.4.6.6.1 and 8.4.1.15, using the comparison isomorphism between both pushforwards 9.2.4.17, by applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ to 9.4.2.4.2, we get 9.4.2.4.1. The third part is a consequence of the second one. □

Proposition 9.4.2.5. *Suppose $Z' = f^{-1}(Z)$ and $\phi = \mathrm{id}$. Suppose S is a noetherian scheme of finite Krull dimension, f is quasi-compact and quasi-separated. Let $* \in \{\mathrm{r}, \mathrm{l}\}$. The functor $f^{(\bullet)}_{Z,+}$ sends $\underrightarrow{LD}_{\mathbb{Q},\mathrm{qc},\mathrm{tdf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$ to $\underrightarrow{LD}_{\mathbb{Q},\mathrm{qc},\mathrm{tdf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$ (see notation 8.5.4.11).*

Proof. This is a consequence of 7.3.2.15 (and 7.1.3.6) and 5.3.2.12 (and the fact that the tor amplitude does not depend on i). □

Proposition 9.4.2.6. *Suppose $Z' = f^{-1}(Z)$ and $\phi = \mathrm{id}$. Suppose S and X are noetherian scheme of finite Krull dimension, f is proper. Let $* \in \{\mathrm{r}, \mathrm{l}\}$. With notation 8.6.1.4, the functor $f^{(\bullet)}_{Z,+}$ sends $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$ to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$.*

Proof. Following [Sta22, 08G8], a complex is perfect if and only if it is pseudo-coherent and locally has finite tor dimension. Hence, this is a consequence of (the proofs of) 9.4.2.4 and 9.4.2.5. □

9.4.3 Projection formula: commutation of pushforwards with localization functors outside of a divisor

We keep notation 9.2.1.

Proposition 9.4.3.1. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$, and $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(Z'))$. We have the following isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$*

$$f^{(\bullet)}_{Z,Z',+}(\mathcal{E}'^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}[d_f] \xrightarrow{\sim} f^{(\bullet)}_{Z,Z',+}\left(\mathcal{E}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')} f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)})\right). \tag{9.4.3.1.1}$$

Proof. This is a consequence of 5.3.4.1. □

Corollary 9.4.3.2. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$. We have the isomorphism*

$$f^{(\bullet)}_{Z,Z',+}\left(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')\right)\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}\mathcal{E}^{(\bullet)}[d_f] \xrightarrow{\sim} f^{(\bullet)}_{Z,Z',+}\circ f^{(\bullet)!}_{Z',Z}(\mathcal{E}^{(\bullet)}). \tag{9.4.3.2.1}$$

Proof. We apply 9.4.3.1 to the case where $\mathcal{E}'^{(\bullet)} = \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')$. □

Corollary 9.4.3.3. *Let $* \in \{\mathrm{l}, \mathrm{r}\}$, $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp})$. We suppose $Z' = f^{-1}(Z)$. We have the isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z))$:*

$$f^{(\bullet)}_{Z+}\circ({}^\dagger Z')(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z)\circ f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}).$$

Proof. It follows from 9.1.1.7.4 and 9.2.4.8.2 that we reduce to the case $* = \mathrm{l}$. Using 9.4.3.1 (in the case where Z and Z' are empty and $\mathcal{E}^{(\bullet)} = \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$) and 9.2.1.26, we get the isomorphism

$$f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \xrightarrow{\sim} f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}'}}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')). \tag{9.4.3.3.1}$$

We conclude using 9.2.4.19. □

Remark 9.4.3.4. Using 9.2.4.19, the isomorphism of 9.4.3.3 could be written $f^{(\bullet)}_{+}\circ({}^\dagger Z')(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z)\circ f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)})$. Moreover, this isomorphism is also a consequence of 5.1.3.4 and 9.2.1.26.

Corollary 9.4.3.5. *With the notations of 9.4.3.1, We suppose that f is an exact closed immersion and $\phi = \mathrm{id}$. Then we have, for any $\mathcal{F}^{(\bullet)}$, $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}(Z'))$, of the canonical isomorphism*

$$f^{(\bullet)}_{Z,Z',+}(\mathcal{F}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} f^{(\bullet)}_{Z,Z',+}(\mathcal{G}^{(\bullet)})[d_f] \xrightarrow{\sim} f^{(\bullet)}_{Z,Z',+}(\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')}\mathcal{G}^{(\bullet)}). \tag{9.4.3.5.1}$$

Proof. We have the canonical isomorphism:

$$f^{(\bullet)}_{Z,Z',+}(\mathcal{F}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)} f^{(\bullet)}_{Z,Z',+}(\mathcal{G}^{(\bullet)})[d_f] \underset{9.4.3.1.1}{\xrightarrow{\sim}} f^{(\bullet)}_{Z,Z',+}(\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(Z')} f^{(\bullet)!}_{Z',Z}\circ f^{(\bullet)}_{Z,Z',+}(\mathcal{G}^{(\bullet)})).$$

Moreover, as f is a closed immersion, we deduce from 9.3.5.13.1 the canonical isomorphism $\mathcal{G}^{(\bullet)} \xrightarrow{\sim} f^{(\bullet)!}_{Z',Z}\circ f^{(\bullet)}_{Z,Z',+}(\mathcal{G}^{(\bullet)})$. Hence we are done. □

9.4.4 Base change isomorphism in the projection case and relative duality isomorphism in the projective case

Let $\mathfrak{S}^\sharp$ be a nice (see definition 3.3.1.10) fine $\mathcal{V}$-log formal scheme. Assume that $\mathfrak{S}$ is noetherian of finite Krull dimension. Let $\mathfrak{X}^\sharp$ and $\mathfrak{Q}^\sharp$ be two log smooth, quasi-compact and quasi-separated formal log schemes over $\mathfrak{S}^\sharp$, $p\colon \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{X}^\sharp$, $q\colon \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{Q}^\sharp$ be the structural maps. Let $* \in \{\mathrm{r}, \mathrm{l}\}$.

We suppose S, $S^\sharp$, X and Q are regular. Let Z_1 be a divisor of X, Z_2 be a divisor of Q and $Z := p^{-1}(Z_1)\cup q^{-1}(Z_2)$. Set $\mathfrak{R} := \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp$ and $r\colon \mathfrak{X}^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{S}^\sharp$ be the structural map.

Proposition 9.4.4.1. Let $u\colon \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ and $v\colon \mathfrak{Q}'^\sharp \to \mathfrak{Q}^\sharp$ be two morphisms of log smooth formal $\mathfrak{S}^\sharp$-schemes. Let $\mathfrak{R}'^\sharp := \mathfrak{X}'^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}'^\sharp$, and $w := (u, v)\colon \mathfrak{R}'^\sharp \to \mathfrak{R}^\sharp$ be the induced morphism. Suppose $Z_1' := u^{-1}(Z_1)$ is a divisor of X' and $Z_2' := v^{-1}(Z_2)$ is a divisor of Q'. Set $Z' := w^{-1}(Z)$.

(a) For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(Z_1))$ and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}(Z_2))$, with notation 9.2.1.15.3, we have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}'^\sharp/\mathfrak{S}^\sharp}(Z'))$ the isomorphism:

$$\mathbb{L}w_Z^{*(\bullet)}(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}u_{Z_1}^{*(\bullet)}(\mathcal{E}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z'_2}\mathbb{L}v_{Z_2}^{*(\bullet)}(\mathcal{F}^{(\bullet)}). \tag{9.4.4.1.1}$$

(b) For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'_1))$ and $\mathcal{F}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}'^\sharp/\mathfrak{S}^\sharp}(Z'_2))$, we have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}(Z))$ the isomorphism:

$$w_{Z+}^{(\bullet)}(\mathcal{E}'^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z'_2}\mathcal{F}'^{(\bullet)}) \xrightarrow{\sim} u_{Z_1+}^{(\bullet)}(\mathcal{E}'^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}v_{Z_2+}^{(\bullet)}(\mathcal{F}'^{(\bullet)}). \tag{9.4.4.1.2}$$

Proof. The first statement is a consequence of 5.3.5.2 and 9.2.5.1.5 (use also 9.2.5.5.2). The second one is a consequence of 5.3.5.13 and 9.2.5.1.5. □

Corollary 9.4.4.2. *We keep notation 9.4.4.1.*

(a) We suppose u and v smooth. For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_1)_{\mathbb{Q}})$, $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Q}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z_2)_{\mathbb{Q}})$, then we have the isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{R}'^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z')_{\mathbb{Q}})$:

$$w^*_{Z,+}(\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^\dagger_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\ \mathcal{F}) \xrightarrow{\sim} u^*_{Z_1,+}(\mathcal{E})\overset{\mathbb{L}}{\boxtimes}{}^\dagger_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z'_2}\ v^*_{Z_2,+}(\mathcal{F}). \tag{9.4.4.2.1}$$

(b) We suppose u and v proper. Then, for any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z'_1)_{\mathbb{Q}})$, $\mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Q}'^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z'_2)_{\mathbb{Q}})$, then we have the isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}({}^\dagger Z)_{\mathbb{Q}})$:

$$w_{Z,+}(\mathcal{E}'\overset{\mathbb{L}}{\boxtimes}{}^\dagger_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z'_2}\ \mathcal{F}') \xrightarrow{\sim} u_{Z_1,+}(\mathcal{E}')\overset{\mathbb{L}}{\boxtimes}{}^\dagger_{\mathcal{O}_{\mathfrak{S}},Z_1,Z_2}\ v_{Z_2,+}(\mathcal{F}'). \tag{9.4.4.2.2}$$

Proof. Via 9.2.5.13, 9.2.1.24 and 9.2.4.17, this is a consequence of Proposition 9.4.4.1. □

Corollary 9.4.4.3. *We keep notation 9.4.4.1 and we suppose v is the identity. Let $p'\colon \mathfrak{X}'^\sharp \times_{\mathfrak{S}^\sharp} \mathfrak{Q}^\sharp \to \mathfrak{X}'^\sharp$ be the natural projection. Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}^\sharp}(Z'_1))$. There exists a canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}^\sharp/\mathfrak{S}^\sharp}(Z))$ of the form:*

$$p^{(\bullet)!}_{Z,Z_1} \circ u^{(\bullet)}_{Z_1+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} w^{(\bullet)}_{Z,+} \circ p'^{(\bullet)!}_{Z',Z'_1}(\mathcal{E}'^{(\bullet)}). \tag{9.4.4.3.1}$$

Proof. We construct the isomorphism by compositing the following isomorphisms:

$$\begin{aligned} p^{(\bullet)!}_{Z,Z_1} \circ u^{(\bullet)}_{Z_1+}(\mathcal{E}'^{(\bullet)}) &\xrightarrow{\sim} u^{(\bullet)}_{Z_1+}(\mathcal{E}'^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z_2}\mathcal{B}^{(\bullet)}_{\mathfrak{Q}}(Z_2) \\ \underset{\sim}{\overset{9.4.4.1.2}{\longrightarrow}}\ & w^{(\bullet)}_{Z+}(\mathcal{E}'^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}},Z'_1,Z_2}\mathcal{B}^{(\bullet)}_{\mathfrak{Q}}(Z_2)) \xrightarrow{\sim} w^{(\bullet)}_{Z,+} \circ p'^{(\bullet)!}_{Z',Z'_1}(\mathcal{E}'^{(\bullet)}). \end{aligned}$$

□

Remark 9.4.4.4. We will prove later (see 13.2.3.7) a coherent version of Corollary 9.4.4.3. In this version, we can use for instance Berthelot-Kashiwara theorem which allow us to extend geometrically the context.

The following corollary is weaker than 9.4.5.2 but the reader can check that is the proof is much easier.

Corollary 9.4.4.5. *We assume the log structure of $\mathfrak{S}^\sharp$ is trivial (and we simply write $\mathfrak{S}$). Let $\mathfrak{X} \to \mathfrak{Y}$ be a morphism of smooth formal $\mathfrak{S}$-schemes which is the composition of a closed immersion of the form $\mathfrak{X} \hookrightarrow \widehat{\mathbb{P}}^d_{\mathfrak{Y}}$ and of the projection $\widehat{\mathbb{P}}^d_{\mathfrak{Y}} \to \mathfrak{Y}$.*

(a) For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$, we have a canonical isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}})$ of the form:

$$\mathbb{D}^{(\bullet)} \circ f^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} f^{(\bullet)}_{+} \circ \mathbb{D}^{(\bullet)}(\mathcal{E}^{(\bullet)}). \tag{9.4.4.5.1}$$

(b) Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$, and $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}})$. We have the isomorphisms

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}}(f_{+}(\mathcal{E}),\mathcal{F}) \xrightarrow{\sim} \mathbb{R}f_{*}\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E},f^{!}(\mathcal{F})), \tag{9.4.4.5.2}$$

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}}(f_{+}(\mathcal{E}),\mathcal{F}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E},f^{!}(\mathcal{F})). \tag{9.4.4.5.3}$$

Proof. The first statement is a consequence of 5.3.8.4. Similarly to 9.4.5.4, we check that 9.4.4.5.1 implies the second statement. □

Remark 9.4.4.6. Following Virrion (see [Vir04]), we have the relative duality isomorphisms and adjoint pairs $(f_{+}, f^{!})$ for proper morphisms f. In the projective case (see 9.4.4.5), we have retrieve theses theorems. They are at least four reasons for this specific study: because this case is much easier than Virrion's one, because in the case of a closed immersion we can remove the coherence hypotheses to build the adjunction morphisms and these adjunction morphisms have a clear description and will be used in chapter 9.3.5 (or also Proposition 9.3.6.1), and because the case of projective morphisms is enough for instance to check the coherence of the constant coefficient in 12.2.7.1).

9.4.5 Relative duality isomorphism in the case of a proper morphism with overconvergent singularities

We complete below Virrion's relative duality isomorphisms by adding some overconvergent singularities.

Let $\mathfrak{S}$ be a noetherian of finite Krull dimension $\mathcal{V}$-formal scheme. Let $f\colon \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of smooth formal schemes over $\mathfrak{S}$, Z and Z' be some divisors of respectively X and X' such that $Z' := f^{-1}(Z)$. We will extend later for realizable morphisms the following theorem and its first corollary (see 13.2.4.1, 13.2.4.2). Recall that we only get a priori a fully faithful functor $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}({}^{*}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z)) \to D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$ following (see 8.7.7.10).

Proposition 9.4.5.1. With notations 9.1.1.6, there exists in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{r}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z))$ a natural trace morphism:

$$f^{(\bullet)}_{Z,+}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'}(Z'))[d_{X'}] = \mathbb{R}f_{*}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'}(Z')\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}(Z')}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}(Z',Z))[d_{X'}] \to \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}}(Z)[d_{X}].$$

Proof. Following [Vir04, III.7.1], we have the trace morphism

$$f^{(\bullet)}_{+}(\omega^{(\bullet)}_{\mathfrak{X}'})[d_{X'}] = \mathbb{R}f_{*}(\omega^{(\bullet)}_{\mathfrak{X}'}\widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}})[d_{X'}] \to \omega^{(\bullet)}_{\mathfrak{X}}[d_{X}]. \tag{9.4.5.1.1}$$

Since the functor $({}^{\dagger}Z)$ commutes with pushforwards (see 9.4.3.3), since $({}^{\dagger}Z)(\omega^{(\bullet)}_{\mathfrak{X}}) \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}}(Z)$ and $({}^{\dagger}Z')(\omega^{(\bullet)}_{\mathfrak{X}'}) \xrightarrow{\sim} \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'}(Z')$, then by applying $({}^{\dagger}Z)$ to 9.4.5.1.1, this results in a morphism

$$f^{(\bullet)}_{Z+}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'}(Z'))[d_{X'}] = \mathbb{R}f_{*}(\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'}(Z')\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(\bullet)}_{\mathfrak{X}'}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}})[d_{X'}] \to \widetilde{\omega}^{(\bullet)}_{\mathfrak{X}}(Z)[d_{X}].$$

□

Theorem 9.4.5.2. *Let $* \in \{\mathrm{l},\mathrm{r}\}$. Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}})$. We have in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$ the isomorphism*

$$f_{Z+} \circ \mathbb{D}_{Z'}(\mathcal{E}') \xrightarrow{\sim} \mathbb{D}_{Z} \circ f_{Z+}(\mathcal{E}'). \tag{9.4.5.2.1}$$

Proof. We construct the morphism 9.4.5.2.1 similarly to that of 5.3.7.4: We can suppose $* = \mathrm{r}$. To simplify notations, let us drope (Z), (Z') and (Z',Z), e.g. let us denote by $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}} := \widehat{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$, $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'} := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}(Z')$, $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}} := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}(Z',Z)$. Moreover, set $\widetilde{\mathcal{B}}_{\mathfrak{X}} := \mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$, $\widetilde{\omega}_{\mathfrak{X}} := \omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$, $\widetilde{\mathcal{D}}_{\mathfrak{X}} := \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$, $\widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}} := \mathcal{D}^{\dagger}_{\mathfrak{X}'\to\mathfrak{X}/\mathfrak{S}}(Z',Z)_{\mathbb{Q}}$ (and similarly with some primes). We construct the composite morphism:

$$\begin{aligned}
f_{Z,+} \circ \mathbb{D}_{\mathfrak{X}',Z'}(\mathcal{E}') &\xrightarrow{\sim} \mathbb{R}f_{*}\left(\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}}(\mathcal{E}',\widetilde{\omega}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}\right)[d_{X'}] \\
&\underset{4.6.3.6.1}{\xrightarrow{\sim}} \mathbb{R}f_{*}\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}}(\mathcal{E}',\widetilde{\omega}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}})[d_{X'}] \\
&\to \mathbb{R}f_{*}\mathbb{R}\mathcal{H}om_{f^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}}}(\mathcal{E}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}},(\widetilde{\omega}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}})[d_{X'}] \\
&\underset{4.6.5.7.1}{\longrightarrow} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}}}\left(\mathbb{R}f_{*}(\mathcal{E}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}),\mathbb{R}f_{*}\left((\widetilde{\omega}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}\right)\right)[d_{X'}].
\end{aligned} \tag{9.4.5.2.2}$$

According to notation 8.4.1.8, we have the functors $\underset{\rightarrow X',\mathbb{Q}}{l^*}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \to D^{\mathrm{b}}({}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}_{\mathfrak{X}})$ and $\underset{\rightarrow X,\mathbb{Q}}{l^*}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \to D^{\mathrm{b}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}_{\mathfrak{X}})$. We get the morphisms

$$\mathbb{R}f_*\left((\widetilde{\omega}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}\right) \xrightarrow{\sim} \mathbb{R}f_*\circ \underset{\rightarrow X',\mathbb{Q}}{l^*}\left((\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}\right)$$
$$\xrightarrow{\sim} \underset{\rightarrow X,\mathbb{Q}}{l^*}\circ \mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}\right). \quad (9.4.5.2.3)$$

We have the ringed topoi morphisms $\underset{\leftarrow X'(\bullet)}{l}\colon (X'^{(\bullet)}_{\bullet}, f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}) \to (\mathfrak{X}'^{(\bullet)}, f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ and $\underset{\leftarrow X(\bullet)}{l}\colon (X_\bullet, \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}) \to (\mathfrak{X}, \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ such that $\mathbb{R}\underset{\leftarrow X(\bullet)*}{l}\circ \mathbb{R}f_* \xrightarrow{\sim} \mathbb{R}f_*\circ \mathbb{R}\underset{\leftarrow X'(\bullet)*}{l}$. This yields by adjunction the first morphism:

$$\mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}\right) \to \mathbb{R}f_*\circ\mathbb{R}\underset{\leftarrow X'(\bullet)*}{l}\circ\mathbb{L}\underset{\leftarrow X'(\bullet)}{l^*}\left((\widetilde{\omega}^{(\bullet)}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'\to\mathfrak{X}}\right)$$
$$\xrightarrow{\sim} \mathbb{R}\underset{\leftarrow X(\bullet)*}{l}\circ\mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}\right). \quad (9.4.5.2.4)$$

By using flat resolutions, it follows by functoriality from 4.2.6.1.1 that we have the isomorphism of complexes of right $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}$-bimodules:

$$\mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}\right) \xrightarrow{\sim} \mathbb{R}f_*\left(\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} (\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet})\right). \quad (9.4.5.2.5)$$

We have the isomorphism of left $\widetilde{\mathcal{D}}_{X'_\bullet} \otimes_{\mathcal{O}_{S_\bullet}} f^{-1}(\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet})^{\mathrm{op}} \otimes_{\mathcal{O}_{S_\bullet}} f^{-1}(\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet})^{\mathrm{op}}$-modules

$$\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet} \otimes_{\mathcal{B}_{X'_\bullet}} \mathcal{D}^{(\bullet)}_{X'_\bullet\to X_\bullet} \xrightarrow{\sim} \widetilde{f}^*_\bullet(\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet} \otimes^{\mathrm{l,l}}_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}) \underset{\gamma}{\xleftarrow{\sim}} \widetilde{f}^*_\bullet(\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet} \otimes^{\mathrm{r,l}}_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet})$$
$$\xrightarrow{\sim} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet} \otimes_{f^{-1}_\bullet\mathcal{B}_{X_\bullet}} f^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}, \quad (9.4.5.2.6)$$

where "l" and "r" mean that we choose respectively the left and the right structure to define the tensor product (we get by functoriality two others structures of $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}$-module and more precisely left, right, right $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}$-trimodules), $\widetilde{f}^*_\bullet$ of a left, right, right $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}$-trimodule gives a $({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}, {}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}, {}^{\mathrm{r}}f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet})$-trimodule and where γ is the transposition isomorphism of $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}$ (see 4.2.5.1.1). This yields

$$\mathbb{R}f_*\left(\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} (\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet})\right) \xrightarrow{\sim} \mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}) \otimes_{f^{-1}_\bullet\mathcal{B}_{X_\bullet}} f^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}\right). \quad (9.4.5.2.7)$$

By applying the projection isomorphism 7.5.7.3, we get

$$\mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}) \otimes_{f^{-1}_\bullet\mathcal{B}_{X_\bullet}} f^{-1}_\bullet\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet})\right) \xrightarrow{\sim} \mathbb{R}f_*\left(\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}\right) \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}. \quad (9.4.5.2.8)$$

By applying the functor $(-\otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet})\circ \mathbb{L}\underset{\leftarrow X(\bullet)}{l^*}$ to the trace map 9.4.5.1, we get

$$\mathbb{R}f_*\left(\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}\right) \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet} \to \widetilde{\omega}^{(\bullet)}_{X_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}[-d_{X'/X}] \quad (9.4.5.2.9)$$

By composing 9.4.5.2.5, 9.4.5.2.7, 9.4.5.2.8, 9.4.5.2.9, we get

$$\mathbb{R}f_*\left((\widetilde{\omega}^{(\bullet)}_{X'_\bullet} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet\to X_\bullet}\right) \to \widetilde{\omega}^{(\bullet)}_{X_\bullet} \otimes_{\mathcal{B}_{X_\bullet}} \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet}[-d_{X'/X}]. \quad (9.4.5.2.10)$$

By applying the functor $\underset{\rightarrow X,\mathbb{Q}}{l^*}\circ\mathbb{R}\underset{\leftarrow X(\bullet)*}{l}$ to 9.4.5.2.10 and composing it with 9.4.5.2.3, we get

$$\mathbb{R}f_*\left((\widetilde{\omega}_{\mathfrak{X}'} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}\right)[d_{X'}] \to \widetilde{\omega}_{\mathfrak{X}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}}[d_X]. \quad (9.4.5.2.11)$$

By applying the functor $\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}}}(\mathbb{R}f_*(\mathcal{E}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}), -)$ to 9.4.5.2.11 and composing it to 9.4.5.2.2, we get the first morphism:

$$f_{Z,+} \circ \mathbb{D}_{\mathfrak{X}',Z'}(\mathcal{E}') \to \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_{\mathfrak{X}}}(\mathbb{R}f_*(\mathcal{E}' \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}'}} \widetilde{\mathcal{D}}_{\mathfrak{X}'\to\mathfrak{X}}), \widetilde{\omega}_{\mathfrak{X}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}} \widetilde{\mathcal{D}}_{\mathfrak{X}})[d_X] \xrightarrow{\sim} \mathbb{D}_{\mathfrak{X},Z} \circ f_{Z,+}(\mathcal{E}'). \quad (9.4.5.2.12)$$

To verify that 9.4.5.2.12 is an isomorphism, according to 8.7.6.11, we reduce to check it above $\mathfrak{X} \setminus Z$, i.e. to the case where the divisor is empty. This then results by p-adic completion and inductive limit on the level from 5.3.7.4 (see [Vir04, IV.3.1]).

□

Remark 9.4.5.3. When the divisor Z is not empty, it is not clear that the categories used in [Vir04] correspond to complexes with bounded and coherent cohomology. This is precisely what we had led to check 9.4.5.2 for $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$.

Moreover, when f is a closed immersion, we have checked in [Car09d, 2.4.3] that 9.4.5.2 commutes with Frobenius (more precisely, with the inverse images by Frobenius F^*).

Corollary 9.4.5.4. *Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$, and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$. We have the isomorphisms*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}(f_{Z+}(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(\mathcal{E}', f^!_Z(\mathcal{E})). \quad (9.4.5.4.1)$$

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}(f_{Z+}(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(\mathcal{E}', f^!_Z(\mathcal{E})). \quad (9.4.5.4.2)$$

Proof. Similarly to 5.3.7.5, this is a formal consequence of the relative duality isomorphism 9.4.5.2: following 8.7.7.9, we have $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}) = D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}})$. Hence, using 4.6.3.6.1, we construct the canonical isomorphism:

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}(f_{Z+}(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} (\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}\left(f_{Z+}(\mathcal{E}'), \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}}\right).$$

It follows from 9.4.5.2 that we get by composition the isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}}(f_{Z+}(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} (\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} f_{Z+}\left(\mathbb{D}_Z(\mathcal{E}')\right)[-d_X]. \quad (9.4.5.4.3)$$

Using the projection formula of 9.2.3.4, the right term of 9.4.5.4.3 is isomorphic to

$$\mathbb{R}f_*\left(f^{-1}\left(\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}\right) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}\leftarrow\mathfrak{X}'}({}^\dagger Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}} \mathbb{D}_Z(\mathcal{E}')\right)[-d_X]. \quad (9.4.5.4.4)$$

Using the isomorphisms $\left(f^{-1}\left(\omega_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}\right) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}\leftarrow\mathfrak{X}'}({}^\dagger Z)_{\mathbb{Q}}\right) \otimes_{\mathcal{O}_{\mathfrak{X}'}} \omega^{-1}_{\mathfrak{X}'}[d_f] \underset{9.2.1.23.1}{\xrightarrow{\sim}} f^!_Z(\mathcal{E})$ and $\omega_{\mathfrak{X}'} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathbb{D}_Z(\mathcal{E}')[-d_{X'}] \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(\mathcal{E}', \mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$, the term of 9.4.5.4.4 is isomorphic to

$$\mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(\mathcal{E}', \mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}} f^!_Z(\mathcal{E})\right) \underset{4.6.3.6.1}{\xrightarrow{\sim}} \mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(\mathcal{E}', f^!_Z(\mathcal{E})\right).$$

□

Corollary 9.4.5.5. *We have the following properties.*

(a) Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$. We have the adjunction morphism $\mathcal{E}' \to f^!_Z f_{Z+}(\mathcal{E}')$.

(b) Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$ such that $f^!_Z(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$. We have the adjunction morphism $f_{Z+}f^!_Z(\mathcal{E}) \to \mathcal{E}$.

(c) Suppose f is proper and smooth. Then $f_{Z+}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$ is a right adjoint functor of $f^!_Z\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$.

9.4.6 Fourier transform

Let N now be an integer, $\mathfrak{X}$ be the formal projective space of dimension N over $\mathcal{V}$, Z be the hyperplane at infinity of X, and $\mathfrak{Y} = \mathfrak{X} \setminus Z$ the formal affine space of dimension N on $\mathcal{V}$. Let $t_1, \ldots, t_N$ be the canonical coordinates on the affine space, $\partial_1, \ldots, \partial_N$ the corresponding derivations. The K-algebra of global sections of $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ is then identified with the "p-adic weak completion" of the Weyl algebra:

Proposition 9.4.6.1. *There exists a canonical isomorphism of K-algebras*

$$\Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}) \xrightarrow{\sim} A_N(K)^\dagger,$$

where $A_N(K)^\dagger$ is the K-algebra of differential operators defined by

$$A_N(K)^\dagger := \left\{ \sum_{\underline{i},\underline{k}} a_{\underline{i},\underline{k}} \in K \parallel \exists c, \lambda \text{ such that } \eta < 1 \text{ and } \forall \underline{i}, \underline{k}, |a_{\underline{i},\underline{k}}| \leq c\eta^{|\underline{i}|+|\underline{k}|} \right\}.$$

9.4.6.2. Now suppose that K contains an element ϖ such that $\varpi^{p-1} = -p$. and fix such an element. We easily verify that there exists a unique automorphism ϕ of the K-algebra $A_N(K)^\dagger$ such that

$$\phi(\partial) = \varpi t_i, \phi(t_i) = -\partial_i/\varpi.$$

For all $A_N(K)^\dagger$-module E, we therefore obtain by making $A_N(K)^\dagger$ act on E via ϕ a new $A_N(K)^\dagger$-module, which we denote by $\phi_* E$. The functor which associates $\phi_* E$ with E is the naive Fourier transformation (defined by ϖ). Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$-module, and $E = \Gamma(\mathfrak{X}, \mathcal{E})$. According to 9.4.6.1, E has a canonical $A_N(K)^\dagger$-module structure, for which it is coherent according to 8.7.3.31. As the $A_N(K)^\dagger$-module $\phi_* E$ is still coherent, it exists, thanks to the theorem 8.7.3.31, a unique $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$-coherent module $\mathcal{E}^\phi$ such that $\Gamma(\mathfrak{X}, \mathcal{E}^\phi) = \phi_* E$.

The problem of the geometric Fourier transformation is to give an interpretation of the functor $\mathcal{E} \mapsto \mathcal{E}^\phi$ by means of the cohomological operations of the theory of $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-modules, in a manner analogous to the geometric Fourier transformation of Malgrange in characteristic 0 [Mal88], or to the Fourier l-adique transformation of Deligne-Katz-Laumon [KL85].

9.4.6.3. Let us first specify how the data of ϖ determines the kernel $\mathcal{N}_\varpi$ of the geometric Fourier transform. Let $\mathfrak{P}^1$ (resp. $\mathfrak{A}^1$) be the formal projective (resp. affine) space of dimension 1 on $\mathcal{V}$. We denote by $\mathcal{L}_\varpi = \mathrm{sp}_* L_\varpi$ the $\mathcal{D}^\dagger_{\mathfrak{P}^1}({}^\dagger\infty)_{\mathbb{Q}}$-coherent module defined by the Dwork F-isocrystal L_ϖ associated with ϖ: the $\mathcal{O}_{\mathfrak{P}^1}({}^\dagger\infty)_{\mathbb{Q}}$-module underlying $\mathcal{L}_\varpi$ is equal to $\mathcal{O}_{\mathfrak{P}^1}({}^\dagger\infty)_{\mathbb{Q}}$, and its structure of $\mathcal{D}^\dagger_{\mathfrak{P}^1}({}^\dagger\infty)_{\mathbb{Q}}$-module compatible with that of $\mathcal{O}_{\mathfrak{P}^1}({}^\dagger\infty)_{\mathbb{Q}}$-module is given by the formula $\partial \cdot 1 = -\varpi$ (see [Ber90]).

Let $\mathfrak{X}'$ (resp. $\mathfrak{Y}'$) be the dual projective (resp. affine) space, $Z' := X' \setminus Y'$ and let $\mu \colon \mathfrak{Y} \times \mathfrak{Y}' \to \mathfrak{A}^1$ be the duality pairing. There exists a smooth formal scheme $\mathfrak{U}$, and a projective morphism $f \colon \mathfrak{U} \to \mathfrak{X} \times \mathfrak{X}'$ such that, if we denote by $\mathfrak{W} = f^{-1}(\mathfrak{Y} \times \mathfrak{Y}')$, the morphism $\mathfrak{W} \to \mathfrak{Y} \times \mathfrak{Y}'$ induced by f is an isomorphism, and the composite morphism $\mu \circ f$ is extended to a morphism $\lambda \colon \mathfrak{U} \to \mathfrak{P}^1$. Let Z'' be the reduced divisor of $X \times X'$ which is complementary to $Y \times Y'$ and $T = f^{-1}(Z'') \subset U$. Following Katz-Laumon [KL85], we agree to consider $\mathcal{L}_\varpi$ as a complex reduced to a single module, placed in degree 1. If we apply the functor

$$\lambda^!_{T,\infty} \colon D^-_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^1}({}^\dagger\infty)_{\mathbb{Q}}) \to D^-_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{U}}({}^\dagger T)_{\mathbb{Q}}),$$

to $\mathcal{L}_\varpi[-1]$, we verify, thanks to the fact that $\mathcal{L}_\varpi$ is defined by an overconvergent isocrystal, that $\lambda^!_{T,\infty}(\mathcal{L}_\varpi[-1])$ reduces to a single coherent $\mathcal{D}^\dagger_{\mathfrak{U}}({}^\dagger T)_{\mathbb{Q}}$-module, placed in degree $2-2N$. Using the equivalence of Theorem 8.7.3.32, we identify it via $f_{Z'',T+}$ to a coherent $\mathcal{D}^\dagger_{\mathfrak{X}\times\mathfrak{X}'}({}^\dagger Z'')_{\mathbb{Q}}$-module, placed in degree $2-2N$, we will set

$$\mathcal{N}_\varpi := f_{Z'',T+}\lambda^!_{T,\infty}(\mathcal{L}_\varpi[-1]).$$

Let p_1, p_2 be the projections of $\mathfrak{X} \times \mathfrak{X}'$ onto $\mathfrak{X}$ and $\mathfrak{X}'$. The geometric Fourier transform then associates with a complex $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$ the complex

$$\phi_+\mathcal{E} := p_{2,Z',Z'',+}(\mathcal{N}_\varpi \otimes_{\mathcal{O}_{\mathfrak{X}\times\mathfrak{X}'}(Z'')_{\mathbb{Q}}} p^!_{1,Z'',Z}(\mathcal{E}[-2N])).$$

By describing the calculation of these functors, Huyghe proves that we thus obtain a transformation associating to any coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$-module a coherent $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}$-module (up to shift), and establishes the relation with the naive Fourier transformation on global sections:

Theorem 9.4.6.4. *With the previous notations, let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$-module. So:*

(a) The complex $\phi_*\mathcal{E}$ is reduced to a single $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}$-module, coherent and placed in degree $N-2$.

(b) There exists a canonical isomorphism of $\Gamma(\mathfrak{X}', \mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$-modules

$$\Gamma(\mathfrak{X}', \phi_*\mathcal{E}[2-N]) \xrightarrow{\sim} \phi_*\Gamma(\mathfrak{X}, \mathcal{E})$$

The transform $\mathcal{E}^\phi$ constructed in 9.4.6.2 is therefore identified with the module $\phi_*\mathcal{E}[2-N]$.

9.4.6.5. There is also a Fourier transform with compact supports. To define it, we provide $\mathfrak{X} \times \mathfrak{X}'$ with the divisor $Z' := p_2^{-1}(Z')$ (by abuse of notation). Huyghe then shows that, for all coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$-module the complex $\mathbb{D}_{Z''}(\mathcal{N}_\varpi \otimes_{\mathcal{O}_{\mathfrak{X}\times\mathfrak{X}'}(Z'')_{\mathbb{Q}}} p^!_{1,Z'',Z}(\mathcal{E})$ has coherent cohomology on $\mathcal{D}^\dagger_{\mathfrak{X}\times\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}$. We can then take its dual $\mathbb{D}_{Z'}$. The functor $\mathbb{D}_{Z'} \circ \mathbb{D}_{Z''}$ here plays the role of direct image with compact support for the open immersion $\mathfrak{Y} \times \mathfrak{Y}' \hookrightarrow \mathfrak{X} \times \mathfrak{Y}'$. We therefore define the Fourier transform with compact supports by setting, for $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$

$$\phi_!\mathcal{E} := p_{2,Z',+}(\mathbb{D}_{Z'} \circ \mathbb{D}_{Z''}(\mathcal{N}_\varpi \otimes_{\mathcal{O}_{\mathfrak{X}\times\mathfrak{X}'}(Z'')_{\mathbb{Q}}} p^!_{1,Z'',Z}(\mathcal{E}[-2N]))).$$

Using the relative duality theorem, we can then show, as in characteristic 0 or in l-adic cohomology, the equality of Fourier transforms with or without support conditions:

Theorem 9.4.6.6. *There exists on $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$ a functorial isomorphism*

$$\phi_!\mathcal{E} \xrightarrow{\sim} \phi_*\mathcal{E}.$$

9.4.6.7. In 1996, Rothstein and Laumon simultaneously constructed a Fourier-Mukai transform for $\mathcal{D}$-modules over a locally noetherian base of characteristic 0. This functor induces an equivalence of categories between quasi-coherent sheaves of $\mathcal{D}$-modules over an abelian variety A and quasi-coherent sheaves of $\mathcal{O}$-modules over its universal vectorial extension $A^\natural$. In the article [Vig22], F. Viguier defines a Fourier-Mukai transform for arithmetic $\mathcal{D}$-modules on an abelian formal scheme $\mathcal{A}$ over $\mathrm{Spf}(\mathcal{V})$ and get some extensions of some classical results of Fourier-Mukai transform to this arithmetic case.

9.5 Frobenius structure

9.5.1 F-complexes

Let $\mathfrak{S}$ be a flat $\mathcal{V}$-formal scheme. Let $\mathfrak{a} \subset \mathfrak{m}$ be an ideal containing p. Fix a PD-ideal $\mathfrak{b} \subset \mathfrak{a}$ (e.g. $\mathfrak{b} = (p)$). Let $m_0 \in \mathbb{N}$ be large enough such that $\mathfrak{b}$ endows $\mathfrak{a}$ with an m_0-PD-structure (see 1.2.4.2.a). Replacing m_0 by m_0+1 if necessary, we can suppose that this structure is topologically m_0-PD-nilpotent. By flatness of the structural morphism $\mathfrak{S} \to \mathrm{Spf}\,\mathcal{V}$ we get that $\mathfrak{b}\mathcal{O}_{\mathfrak{S}}$ endows $\mathfrak{a}\mathcal{O}_{\mathfrak{S}}$ with an m_0-PD-structure. For any integer $i \in \mathbb{N}$, for any flat formal $\mathfrak{S}$-scheme $\mathfrak{X}$, we denote in this subsection by X_i the reduction of $\mathfrak{S}$ modulo $\mathfrak{a}^{i+1}$ (and not $\mathfrak{m}^{i+1}$).

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. Suppose there exists an automorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining. We denote by $\mathfrak{X}' := \mathfrak{X}^\sigma$ the $\mathcal{V}$-formal scheme deduced from $\mathfrak{X}$ by the base change defined by σ. We denote by $Z' := Z^\sigma$.

Let I be the set the integers $\geq m_0$. We fix $\lambda_0 \in L(I)$. We get the inductive systems indexed by I of the form $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) := \lambda_0^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}$. Finally, we set $\mathcal{D}^{(m)}_{X_i/S_i}(Z) := \mathcal{V}/\pi^{i+1} \otimes_{\mathcal{V}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}(Z) = \mathcal{B}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i/S_i}$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i/S_i}(Z) := \widetilde{\mathcal{B}}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i/S_i}$ for any $m \in \mathbb{N}$. We use similar notation by adding some primes, e.g. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(Z') := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}'}(Z')$.

9.5.1.1. Following 6.1.5.3, for any $i \in \mathbb{N}$, we have the functor $(F^s_{X_0/S_0})^*$ (resp. $(F^s_{X_0/S_0})^\flat$) from the category of left (resp. right) $\mathcal{D}^{(m)}_{X'_i/S_i}(Z')$-modules to that of left (resp. right) $\mathcal{D}^{(m+s)}_{X_i/S_i}(Z)$-modules. To avoid confusion (since m and i vary), we denote this functor by $\widetilde{F}^{*(m)}_{X_i}$ (resp. $\widetilde{F}^{\flat(m)}_{X_i}$).

(a) We set $\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet \to X'_\bullet/S_\bullet}(Z, Z') := \widetilde{F}^{*(\bullet)}_{X_\bullet} \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z)$. Hence, $\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet \to X_\bullet/S_\bullet}(Z', Z)$ is endowed with a structure of $(\widetilde{\mathcal{D}}^{(\bullet+s)}_{X_\bullet/S_\bullet}(Z), \widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z'))$-bimodule. This yields the functor: $\widetilde{F}^{*(\bullet)}_{X_\bullet} \colon D({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z)) \to D({}^{1}\widetilde{\mathcal{D}}^{(\bullet+s)}_{X'_\bullet/S'_\bullet}(Z'))$ which is defined by setting

$$\widetilde{F}^{*(\bullet)}_{X_\bullet}(\mathcal{E}'^{(\bullet)}_\bullet) := \widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet \to X'_\bullet/S_\bullet}(Z, Z') \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z)} \mathcal{E}'^{(\bullet)}_\bullet, \tag{9.5.1.1.1}$$

where $\mathcal{E}'^{(\bullet)}_\bullet \in D(\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z))$.

(b) For any $* \in \{r, l\}$, the inverse image by Frobenius functor $\widetilde{F}^{(\bullet)}_{\mathfrak{X}} \colon D({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \to D({}^{1}\widetilde{\mathcal{D}}^{(\bullet+s)}_{\mathfrak{X}/\mathfrak{S}}(Z))$ is defined by setting

$$\widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\mathcal{E}'^{(\bullet)}) := \mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)},*} \circ \widetilde{F}^{*(\bullet)}_{X_\bullet} \circ \mathbb{L}\underleftarrow{l}^{*}_{\mathfrak{X}'^{(I)}}(\mathcal{E}'^{(\bullet)}) \tag{9.5.1.1.2}$$

for any $\mathcal{E}'^{(\bullet)} \in D({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$.

(c) We have a non-completed version of the pullback under Frobenius. With notations 8.8.1.4, we get the $(\widetilde{\mathcal{D}}^{(\bullet+s)}_{\mathfrak{X}/\mathfrak{S}}(Z), \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$-bimodule $(F^{s}_{X_0/S_0})^{*}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$, where $(F^{s}_{X_0/S_0})^{*}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$ is the inductive system with transitive maps $(F^{s}_{X_0/S_0})^{*}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \to (F^{s}_{X_0/S_0})^{*}(\widetilde{\mathcal{D}}^{(m+1)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$ given by functoriality from 8.8.1.4.1. We set

$$(F^{s}_{X_0/S_0})^{*}\mathcal{E}'^{(\bullet)} := (F^{s}_{X_0/S_0})^{*}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \otimes_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')} \mathcal{E}'^{(\bullet)}, \tag{9.5.1.1.3}$$

for any $\mathcal{E}'^{(\bullet)} \in D({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$.

(d) For any $\mathcal{E}'^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$, the canonical morphism

$$(F^{s}_{X_0/S_0})^{*}\mathcal{E}'^{(\bullet)} \to \widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\mathcal{E}'^{(\bullet)}) \tag{9.5.1.1.4}$$

is an isomorphism, i.e. both functors 9.5.1.1.2 and 9.5.1.1.3 are canonically isomorphic with coherent complexes.

Notation 9.5.1.2. We keep notation 9.5.1.1. The functors 9.5.1.1.1, 9.5.1.1.2, 9.5.1.1.3 induce

$$\widetilde{F}^{*(\bullet)}_{X_\bullet} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z')) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z)). \tag{9.5.1.2.1}$$

$$\widetilde{F}^{*(\bullet)}_{\mathfrak{X}} \colon \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \to \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)), \tag{9.5.1.2.2}$$

$$(F^{s}_{X_0/S_0})^{*} \colon \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \to \underrightarrow{LD}^{-}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)). \tag{9.5.1.2.3}$$

The functors 9.5.1.2.1 and 9.5.1.2.2 are some particular cases of the functors 9.2.2.4. This is not straightforward (but this is likely true) that the functor 9.5.1.2.3 preserves the quasi-coherence. By construction of both functors $(F^{s}_{X_0/S_0})^{*}$ of 9.5.1.2.3 and 8.8.2.4.1, we get for any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$ the canonical isomorphism

$$\underrightarrow{l}^{*}_{\mathbb{Q}} \circ (F^{s}_{X_0/S_0})^{*}\mathcal{E}'^{(\bullet)} \xrightarrow{\sim} (F^{s}_{X_0/S_0})^{*} \circ \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}'^{(\bullet)}). \tag{9.5.1.2.4}$$

Moreover, it follows from 9.5.1.1.4 that, for any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$, the canonical morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$:

$$(F^{s}_{X_0/S_0})^{*}\mathcal{E}'^{(\bullet)} \to \widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\mathcal{E}'^{(\bullet)}) \tag{9.5.1.2.5}$$

is an isomorphism.

Notation 9.5.1.3. With notation 9.5.1.2, suppose $\mathfrak{S} = \operatorname{Spf} \mathcal{V}$ and $\mathcal{V}$ satisfies properties of 8.8.3.1. Let $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$. We denote by $\mathcal{E}^{(\bullet)\sigma}_\bullet$ the object of $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z'))$ induced from $\mathcal{E}^{(\bullet)}_\bullet$ by base change via σ. We set $F^{*}\mathcal{E}^{(\bullet)}_\bullet := \widetilde{F}^{*(\bullet)}_{X_\bullet}(\mathcal{E}^{(\bullet)\sigma}_\bullet)$.

Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$. We denote by $\mathcal{E}^{(\bullet)\sigma}$ the object of $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$ induced from $\mathcal{E}^{(\bullet)}$ by base change via σ. We set $F^{*}\mathcal{E}^{(\bullet)} := \widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\mathcal{E}^{(\bullet)\sigma})$.

Definition 9.5.1.4. With notation and hypotheses of 9.5.1.3, a "F-complex (or F^s-complex if there is a risk of confusion) of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$" is the data of an object $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$ together with an isomorphism $\Phi\colon \mathcal{E}^{(\bullet)}_\bullet \xrightarrow{\sim} F^*\mathcal{E}^{(\bullet)}_\bullet$ of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$. A morphism $u\colon (\mathcal{E}^{(\bullet)}_\bullet, \Phi) \to (\mathcal{F}^{(\bullet)}_\bullet, \Psi)$ of F-complexes of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$ is a morphism of complexes of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$ of the form $u\colon \mathcal{E}^{(\bullet)}_\bullet \to \mathcal{F}^{(\bullet)}_\bullet$ such that $\Psi \circ u = F^*(u) \circ \Phi$.

Similarly a "F-complex (or F^s-complex if there is a risk of confusion) of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$" is the data of an object $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$ together with an isomorphism $\Phi\colon \mathcal{E}^{(\bullet)} \xrightarrow{\sim} F^*\mathcal{E}^{(\bullet)}$ of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$. A morphism $u\colon (\mathcal{E}^{(\bullet)}, \Phi) \to (\mathcal{F}^{(\bullet)}, \Psi)$ of F-complexes of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$ is a morphism of complexes of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$ of the form $u\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ such that $\Psi \circ u = F^*(u) \circ \Phi$.

We denote by $F\text{-}\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$ (resp. $F\text{-}\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$) the corresponding category of F-complexes. The functors $\mathbb{R}\underleftarrow{l}_{\mathfrak{X}^{(I)},*}$ and $\mathbb{L}\underleftarrow{l}^*_{\mathfrak{X}^{(I)}}$ induce equivalences of categories between $F\text{-}\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$ and $F\text{-}\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$.

With notation 8.8.3.2, by using 9.5.1.2.4 and 9.5.1.2.5, the equivalence of categories 8.4.1.15 commutes with Frobenius and we get the equivalence of categories:

$$\underrightarrow{l}^*_{\mathbb{Q}}\colon F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)) \cong F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}({}^l\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}). \tag{9.5.1.4.1}$$

Definition 9.5.1.5 (Tate twist). Let $(\mathcal{E}^{(\bullet)}, \Phi)$ be a F-complex of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$. For any integer d, we define an F-complex of $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$ denoted by $(\mathcal{E}^{(\bullet)}, \Phi)(d) = (\mathcal{E}^{(\bullet)}, \Phi(d))$ called the dth Tate twist of $(\mathcal{E}^{(\bullet)}, \Phi)$ as follows: the underlying complex is $\mathcal{E}^{(\bullet)}$ and $\Phi(d) := q^{-d}\Phi$. The F-complex $(\mathcal{E}^{(\bullet)}, \Phi)(d)$ is called the dth Tate twist of $(\mathcal{E}^{(\bullet)}, \Phi)$. The equivalence of categories 9.5.1.4.1 commutes with the Tate twist (see definition 8.8.3.3), i.e., for any $(\mathcal{E}^{(\bullet)}, \Phi) \in F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^l\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$, we have $\underrightarrow{l}^*_{\mathbb{Q}}((\mathcal{E}^{(\bullet)}, \Phi)(d)) = (\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}, \Phi))(d)$.

9.5.2 Commutations to Frobenius of pullbacks, push forwards, duality, internal or external tensor products for (quasi-)coherent complexes

Let

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{f} & \mathfrak{X} \\ \downarrow{\scriptstyle p_{\mathfrak{Y}}} & & \downarrow{\scriptstyle p_{\mathfrak{X}}} \\ \mathfrak{T} & \xrightarrow{\phi} & \mathfrak{S}, \end{array} \tag{9.5.2.0.1}$$

be a commutative diagram of $\mathcal{V}$-formal schemes, where $p_{\mathfrak{X}}$ and $p_{\mathfrak{Y}}$ are smooth morphisms. We suppose S, T are regular and of finite Krull dimension. Let Z and D be some divisors of respectively X and Y such that $f(Y \setminus D) \subset X \setminus Z$.

We suppose $\mathfrak{S}$, $\mathfrak{T}$ are p-torsion free. Let $\mathfrak{a} \subset \mathfrak{m}$ be an ideal containing p. Fix a PD-ideal $\mathfrak{b} \subset \mathfrak{a}$ (e.g. $\mathfrak{b} = (p)$). Let $m_0 \in \mathbb{N}$ be large enough such that $\mathfrak{b}$ endows $\mathfrak{a}$ with an m_0-PD-structure (see 1.2.4.2.a). Replacing m_0 by m_0+1 if necessary, we can suppose that this structure is topologically m_0-PD-nilpotent. The bottom arrow ϕ can be viewed as an m_0-PD-morphism.

For any integer $i \in \mathbb{N}$, for any flat formal $\mathfrak{V}$-scheme $\mathfrak{E}$, we denote here E_i the reduction of $\mathfrak{S}$ modulo $\mathfrak{a}^{i+1}$ (and not $\mathfrak{m}^{i+1}$).

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. Suppose there exists an automorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining. We denote by $\mathfrak{X}' := \mathfrak{X}^\sigma$ (resp. $\mathfrak{Y}' := \mathfrak{Y}^\sigma$) the $\mathcal{V}$-formal scheme deduced from $\mathfrak{X}$ (resp. $\mathfrak{Y}$) by the base change defined by σ. We denote by $Z' := (F^s_{X_0/S_0})^{-1}(Z)$, $D' := (F^s_{Y_0/T_0})(D)$, $f'\colon \mathfrak{Y}' \to \mathfrak{X}'$ the morphism induced from f by base change via σ.

Let I be the set the integers $\geq m_0$. We fix $\lambda_0 \in L(I)$. We get the inductive systems indexed by I of the form $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) := \lambda_0^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(Z)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}$. Finally, we set $\mathcal{D}^{(m)}_{X_i/S_i}(Z) := \mathcal{V}/\pi^{i+1} \otimes_{\mathcal{V}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}(Z) = \mathcal{B}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i/S_i}$ and $\widetilde{\mathcal{D}}^{(m)}_{X_i/S_i}(Z) := \widetilde{\mathcal{B}}^{(m)}_{X_i}(Z) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i/S_i}$ for any $m \in \mathbb{N}$.

We use similar notation by adding some primes, e.g. $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(Z') := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}'}(Z')$, or replacing $\mathfrak{X}$ by $\mathfrak{Y}$ and Z by D etc. We keep notation 9.5.1.2 with respect to $\mathfrak{X}/\mathfrak{S}$ and the analogous one for $\mathfrak{Y}/\mathfrak{T}$.

9.5.2.1 (Extraordinary inverse images)**.** With notation 9.5.1.2.1, it follows from 6.2.4.2.1 that for any $\mathcal{E}'^{(\bullet)}_{\bullet} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z'))$ we have the functorial commutation isomorphism in $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{Y_\bullet/T_\bullet}(D))$:

$$(f,\phi)^{(\bullet)!}_{\bullet D,Z} \circ \widetilde{F}^{*}_{X_\bullet}(\mathcal{E}'^{(\bullet)}_{\bullet}) \xrightarrow{\sim} \widetilde{F}^{*}_{Y_\bullet} \circ (f',\phi)^{(\bullet)!}_{\bullet D',Z'}(\mathcal{E}'^{(\bullet)}_{\bullet}). \tag{9.5.2.1.1}$$

Hence, when $\phi = \mathrm{id}$, $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$ and $\mathcal{V}$ satisfies properties of 8.8.3.1, the extraordinary inverse image functor 9.2.1.15.1 induces

$$f^{(\bullet)!}_{\bullet D,Z} \colon F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z)) \to F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{Y_\bullet/T_\bullet}(D)). \tag{9.5.2.1.2}$$

With notation 9.5.1.2.2, it follows from 9.5.2.1.1 that we get for any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$ the isomorphism of $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{T}}(D))$:

$$(f,\phi)^{(\bullet)!}_{D,Z} \circ \widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \widetilde{F}^{*(\bullet)}_{\mathfrak{Y}} \circ (f',\phi)^{(\bullet)!}_{D',Z'}(\mathcal{E}'^{(\bullet)}). \tag{9.5.2.1.3}$$

Suppose $\phi = \mathrm{id}$, f is smooth, $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$ and $\mathcal{V}$ satisfies properties of 8.8.3.1. We get the functor

$$f^{(\bullet)!}_{D,Z} \colon F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)) \to F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{S}}(D)). \tag{9.5.2.1.4}$$

Moreover, with 9.5.1.4.1, it follows from 9.4.1.7 that we get the functors:

$$f^{(\bullet)!}_{D,Z} \colon F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)) \to F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{T}}(D)), \tag{9.5.2.1.5}$$

$$f^{!}_{D,Z} \colon F\text{-}D_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}(Z)_{\mathbb{Q}}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{Y}/\mathfrak{T}}(D)_{\mathbb{Q}}). \tag{9.5.2.1.6}$$

9.5.2.2 (Pushforwards)**.** It follows from 6.2.6.2 that for any $\mathcal{F}'^{(\bullet)}_{\bullet} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{Y'_\bullet/T_\bullet}(D'))$ (resp. $\mathcal{F}'^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{T}}(D'))$) we have the first (resp. second) functorial commutation isomorphism in $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z'))$ (resp. $\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$):

$$(f,\phi)^{(\bullet)}_{\bullet+Z,D} \circ F^{*}_{Y_\bullet}(\mathcal{F}'^{(\bullet)}_{\bullet}) \xrightarrow{\sim} F^{*}_{X_\bullet} \circ (f',\phi)^{(\bullet)}_{\bullet+Z',D'}(\mathcal{F}'^{(\bullet)}_{\bullet}), \tag{9.5.2.2.1}$$

$$(f,\phi)^{(\bullet)}_{+Z,D} \circ \widetilde{F}^{*(\bullet)}_{\mathfrak{Y}}(\mathcal{F}'^{(\bullet)}) \xrightarrow{\sim} \widetilde{F}^{*(\bullet)}_{\mathfrak{X}} \circ (f',\phi)^{(\bullet)}_{+Z',D'}(\mathcal{F}'^{(\bullet)}). \tag{9.5.2.2.2}$$

Suppose $\phi = \mathrm{id}$, $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$ and $\mathcal{V}$ satisfies properties of 8.8.3.1. This yields the functors of 9.2.4.10 commutes with Frobenius and induce

$$f^{(\bullet)}_{\bullet+Z,D} \colon F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{Y_\bullet/T_\bullet}(D)) \to F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z));$$

$$f^{(\bullet)}_{+Z,D} \colon F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{T}}(D)) \to F\text{-}\underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)).$$

9.5.2.3 (Pushforwards: proper case)**.** Suppose $\phi = \mathrm{id}$, f is proper and $D = f^{-1}(Z)$. Since the coherence is stable under f_{+Z} (see 9.4.2.4), we get from 9.5.2.2.2 and 9.5.1.2.5 the isomorphisms:

$$f^{(\bullet)}_{+Z} \circ (F^{s}_{Y_0/S_0})^{*}(\mathcal{F}'^{(\bullet)}) \xrightarrow{\sim} (F^{s}_{X_0/S_0})^{*} \circ f'^{(\bullet)}_{+Z'}(\mathcal{F}'^{(\bullet)}), \tag{9.5.2.3.1}$$

for any $\mathcal{F}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{S}}(D'))$. By using 9.2.4.17 and the equivalence of categories of 8.4.1.15, this yields the isomorphism

$$f_{+Z} \circ (F^{s}_{Y_0/S_0})^{*}(\mathcal{F}') \xrightarrow{\sim} (F^{s}_{X_0/S_0})^{*} \circ f'_{+Z'}(\mathcal{F}'), \tag{9.5.2.3.2}$$

for any $\mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{S}}(D')_{\mathbb{Q}})$. With 9.5.1.4.1, when $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$ we get the functors:

$$f^{(\bullet)}_{+Z,D} \colon F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{S}}(D)) \to F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)) \tag{9.5.2.3.3}$$

$$f_{+Z,D} \colon F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{Y}/\mathfrak{S}}(D)_{\mathbb{Q}}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}(Z)_{\mathbb{Q}}). \tag{9.5.2.3.4}$$

9.5.2.4. The dual functor $\mathbb{D}_{\mathfrak{X},Z}$ (see notation 9.2.4.22.2 commutes canonically with Frobenius. More precisely, denoting by $\widetilde{\mathbb{D}}_{\mathfrak{X}',Z'} = \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}(Z')_\mathbb{Q}}(-,\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}(Z')_\mathbb{Q})$, which is canonically isomorphic to the composition of the functors $\mathbb{D}_{\mathfrak{X}',Z'}$ (see notation 9.2.4.22.2), and $-\otimes_{\mathcal{O}_X}\omega_\mathfrak{X}$, we have for any $\mathcal{E}'\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}(Z')_\mathbb{Q})$ the isomorphisms

$$F^\flat\mathbb{D}_{\mathfrak{X}',Z'}(\mathcal{E}') \underset{4.6.3.6.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}(Z')_\mathbb{Q}}(\mathcal{E}',F^\flat\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}(Z')_\mathbb{Q}) \underset{F^*}{\xrightarrow{\sim}}$$
$$\underset{F^*}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}(Z)_\mathbb{Q}}(F^*\mathcal{E}',F^*F^\flat\mathcal{D}^\dagger_{\mathfrak{X}'/\mathfrak{S}}(Z')_\mathbb{Q}) \xrightarrow{\sim} \widetilde{\mathbb{D}}_{\mathfrak{X},Z}(F^*\mathcal{E}'). \tag{9.5.2.4.1}$$

This yields:

$$F^*\mathbb{D}_{\mathfrak{X}',Z'}(\mathcal{E}') \overset{8.8.2.3.1}{\xrightarrow{\sim}} F^\flat\mathbb{D}_{\mathfrak{X}',Z'}(\mathcal{E}')\otimes_{\mathcal{O}_\mathfrak{X}}\omega^{-1}_{\mathfrak{X}/\mathfrak{S}} \overset{9.5.2.4.1}{\xrightarrow{\sim}} \mathbb{D}_{\mathfrak{X},Z}(F^*\mathcal{E}')\otimes_{\mathcal{O}_\mathfrak{X}}\omega^{-1}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} \mathbb{D}_{\mathfrak{X},Z}(F^*\mathcal{E}'). \tag{9.5.2.4.2}$$

9.5.2.5 (Internal tensor products)**.** It follows from 6.2.2.1.1 that for any $\mathcal{E}'^{(\bullet)}_\bullet,\mathcal{F}'^{(\bullet)}_\bullet\in \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{X'_\bullet/S_\bullet}(Z'))$, we have the functorial commutation isomorphism in $\underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))$:

$$F^*_{X_\bullet}(\mathcal{E}'^{(\bullet)}_\bullet)\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_\bullet}(Z)}F^*_{X_\bullet}(\mathcal{F}'^{(\bullet)}_\bullet)\xrightarrow{\sim}F^*_{X_\bullet}(\mathcal{E}'^{(\bullet)}_\bullet\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X'_\bullet}(Z)}\mathcal{F}'^{(\bullet)}_\bullet).$$

When $\mathfrak{S}=\operatorname{Spf}\mathcal{V}$, this yields that the functor of 9.1.1.3.1 induces the following ones:

$$-\otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{X_\bullet}(Z)}-\colon F\text{-}\underrightarrow{LD}^-_\mathbb{Q}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))\times F\text{-}\underrightarrow{LD}^-_\mathbb{Q}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z))\to F\text{-}\underrightarrow{LD}^-_\mathbb{Q}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z)), \tag{9.5.2.5.1}$$
$$-\widehat{\otimes}^{\mathbb{L}}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)}-\colon F\text{-}\underrightarrow{LD}^-_\mathbb{Q}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))\times F\text{-}\underrightarrow{LD}^-_\mathbb{Q}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))\to F\text{-}\underrightarrow{LD}^-_\mathbb{Q}({}^\mathrm{l}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)). \tag{9.5.2.5.2}$$

9.5.2.6 (External tensor products)**.** Let $\mathfrak{X}$ and $\mathfrak{Q}$ be two smooth, quasi-compact and quasi-separated formal schemes over $\mathfrak{S}=\operatorname{Spf}\mathcal{V}$, $p\colon\mathfrak{X}\times_\mathfrak{S}\mathfrak{Q}\to\mathfrak{X}$, $q\colon\mathfrak{X}\times_\mathfrak{S}\mathfrak{Q}\to\mathfrak{Q}$ be the structural maps. Let Z_1 be a divisor of X, Z_2 be a divisor of Q and $Z:=p^{-1}(Z_1)\cup q^{-1}(Z_2)$. Set $\mathfrak{R}:=\mathfrak{X}\times_\mathfrak{S}\mathfrak{Q}$ and $r\colon\mathfrak{X}\times_\mathfrak{S}\mathfrak{Q}\to\mathfrak{S}$ be the structural map.

It follows from 9.5.2.1.2 and 9.5.2.5.1 (resp. 9.5.2.1.4 and 9.5.2.5.2) that for any $\star\in\{\mathrm{qc},\mathrm{coh}\}$, the bifunctor 9.2.5.1.1 (resp. 9.2.5.1.2) induces the bifunctor:

$$-\boxtimes^{\mathbb{L}}_{\mathcal{O}_{S_\bullet},Z_1,Z_2}-\colon F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{X_\bullet/S_\bullet}(Z_1))\times F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{Q_\bullet/S_\bullet}(Z_2))\to F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{R_\bullet}(Z)); \tag{9.5.2.6.1}$$
$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_\mathfrak{S},Z_1,Z_2}-\colon F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z_1))\times F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}/\mathfrak{S}}(Z_2))\to F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}}(Z)). \tag{9.5.2.6.2}$$

Via 9.5.1.4.1, this yields the functor:

$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_\mathfrak{S},Z_1,Z_2}-\colon F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z_1)_\mathbb{Q})\times F\text{-}D^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}/\mathfrak{S}}(Z_2)_\mathbb{Q})\to F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}({}^\mathrm{l}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{R}}(Z)_\mathbb{Q}). \tag{9.5.2.6.3}$$

9.5.3 Commutation with Frobenius of the pullbacks for complexes of $\mathcal{D}^\dagger$-modules

We keep notation 9.5.2.

9.5.3.1. Let $\mathcal{E}'$ be a p-adically complete (and separated) left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')$-module.

(a) Suppose $F^s_{X_0/S_0}$ has a lifting $F\colon\mathfrak{X}\to\mathfrak{X}'$. The canonical map

$$\mathcal{O}_\mathfrak{X}\otimes_{\mathcal{O}_{\mathfrak{X}'}}\mathcal{E}'\to F^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')\otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')}\mathcal{E}'$$

is an isomorphism (where the extension $\mathcal{O}_{\mathfrak{X}'}\to\mathcal{O}_\mathfrak{X}$ is given by F). Since F is finite and $\mathcal{E}'$ is p-adically complete (and separated), then the canonical morphism

$$F^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')\otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')}\mathcal{E}'\to F^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')\widehat{\otimes}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')}\mathcal{E}' \tag{9.5.3.1.1}$$

is an isomorphism of left $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}/\mathfrak{S}}(Z')$-modules.

(b) Since this is local, we get from (a) that the canonical morphism

$$(F^s_{X_0/S_0})^* \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z') \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')} \mathcal{E}' \to (F^s_{X_0/S_0})^* \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z') \widehat{\otimes}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'/\mathfrak{S}}(Z')} \mathcal{E}' \tag{9.5.3.1.2}$$

is an isomorphism.

9.5.3.2. Since $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')$ is coherent, then following 9.5.1.2.5 we have the isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$:

$$(F^s_{X_0/S_0})^* (\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \to \widetilde{F}^{*(\bullet)}_{\mathfrak{X}} (\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$$

Since $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}(D',Z')$ is (in particular) a p-torsion free, separated complete left $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{T}}(D')$-module, then it follows from 9.5.3.1.2 that the canonical morphism

$$(F^s_{Y_0/T_0})^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{T}}(D') \otimes_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{T}}(D')} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}(D',Z') \to \widetilde{F}^{*(\bullet)}_{\mathfrak{Y}} (\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}(D',Z')), \tag{9.5.3.2.1}$$

where $\widetilde{F}^{*(\bullet)}_{\mathfrak{Y}}$ is the functor defined at 9.5.1.1.2 and $(F^s_{Y_0/T_0})^*$ is that defined at 9.5.1.2.3, is an isomorphism of $D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet+s)}_{\mathfrak{Y}/\mathfrak{T}}(D), f'^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$. We have the following isomorphisms of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{T}}(D), f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))$

$$\begin{aligned}
\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}\to\mathfrak{X}/\mathfrak{T}\to\mathfrak{S}}(D,Z) \otimes^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z)} f^{-1}(F^s_{X_0/S_0})^*(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z'))[d_f] &\xrightarrow{\sim} (f,\phi)^{(\bullet)!}_{\mathrm{alg}\,D,Z} \circ \widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \\
\underset{9.2.1.17}{\xrightarrow{\sim}} (f,\phi)^{(\bullet)!}_{D,Z} \circ \widetilde{F}^{*(\bullet)}_{\mathfrak{X}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) &\underset{9.5.2.1.3}{\xrightarrow{\sim}} \widetilde{F}^{*(\bullet)}_{\mathfrak{Y}} \circ (f',\phi)^{(\bullet)!}_{D',Z'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}}(Z')) \\
\underset{9.2.1.17}{\xleftarrow{\sim}} \widetilde{F}^{*(\bullet)}_{\mathfrak{Y}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}(D',Z'))[d_f] &\underset{9.5.3.2.1}{\xleftarrow{\sim}} (F^s_{Y_0/T_0})^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{T}}(D') \otimes_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'/\mathfrak{T}}(D')} \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}(D',Z')[d_f].
\end{aligned}$$

By applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$, with notation 8.8.3.1 we get the isomorphism:

$$\begin{aligned}
&\mathcal{D}^{\dagger}_{\mathfrak{Y}\to\mathfrak{X}/\mathfrak{T}\to\mathfrak{S}}({}^{\dagger}D,Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}/\mathfrak{T}}({}^{\dagger}D)_{\mathbb{Q}}} f^{-1}(F^s_{X_0/S_0})^* \mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}} \\
&\xrightarrow{\sim} (F^s_{Y_0/T_0})^* \mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{S}}({}^{\dagger}D')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{S}}({}^{\dagger}D')_{\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}({}^{\dagger}D',Z')_{\mathbb{Q}}.
\end{aligned} \tag{9.5.3.2.2}$$

Lemma 9.5.3.3. *The functor $(f,\phi)^!_{D,Z}$ defined at 9.2.1.21.1 commutes with Frobenius in the sense that we have (with notation 8.8.3.1) the canonical isomorphism*

$$(F^s_{Y_0/T_0})^*(f',\phi)^!_{D',Z'}(\mathcal{E}') \xrightarrow{\sim} (f,\phi)^!_{D,Z}((F^s_{X_0/S_0})^*(\mathcal{E}')), \tag{9.5.3.3.1}$$

for any $\mathcal{E}' \in D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}})$.

Proof. We build 9.5.3.3.1 by composition as follows:

$$\begin{aligned}
&(F^s_{Y_0/T_0})^*(f',\phi)^!_{D',Z'}(\mathcal{E}')[-d_f] \\
&\xrightarrow{\sim} (F^s_{Y_0/T_0})^* \mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{T}}({}^{\dagger}D')_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{T}}({}^{\dagger}D')_{\mathbb{Q}}} \left(\mathcal{D}^{\dagger}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}({}^{\dagger}D',Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f'^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}} f'^{-1}\mathcal{E}'\right) \\
&\xrightarrow{\sim} \left((F^s_{Y_0/T_0})^* \mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{T}}({}^{\dagger}D')_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{Y}'/\mathfrak{T}}({}^{\dagger}D')_{\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{Y}'\to\mathfrak{X}'/\mathfrak{T}\to\mathfrak{S}}({}^{\dagger}D',Z')_{\mathbb{Q}}\right) \otimes^{\mathbb{L}}_{f'^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}} f'^{-1}\mathcal{E}' \\
&\underset{9.5.3.2.2}{\xleftarrow{\sim}} \left(\mathcal{D}^{\dagger}_{\mathfrak{Y}\to\mathfrak{X}/\mathfrak{T}\to\mathfrak{S}}({}^{\dagger}D,Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}/\mathfrak{T}}({}^{\dagger}D)_{\mathbb{Q}}} f^{-1}(F^s_{X_0/S_0})^* \mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}\right) \otimes^{\mathbb{L}}_{f'^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}} f'^{-1}\mathcal{E}' \\
&\xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{Y}\to\mathfrak{X}/\mathfrak{T}\to\mathfrak{S}}({}^{\dagger}D,Z)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}/\mathfrak{T}}({}^{\dagger}D)_{\mathbb{Q}}} f^{-1}\left((F^s_{X_0/S_0})^* \mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S}}({}^{\dagger}Z')_{\mathbb{Q}}} \mathcal{E}'\right) \\
&\xrightarrow{\sim} (f,\phi)^!_{D,Z}((F^s_{X_0/S_0})^*(\mathcal{E}'))[-d_f].
\end{aligned}$$

□

9.5.3.4. Suppose $\phi = \mathrm{id}$, f is proper, $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$ and $\mathcal{V}$ satisfies properties of 8.8.3.1. Let $\mathcal{E} \in D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$. With notation 8.8.3.1, it follows from 9.5.3.3.1 that we have the isomorphism of $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}})$:

$$\theta_{f,D,Z} \colon F^* f^!_{D,Z}(\mathcal{E}) \xrightarrow{\sim} f^!_{D,Z}(F^*(\mathcal{E})). \tag{9.5.3.4.1}$$

When $D=f^{-1}(Z)$, we simply write $\theta_{f,Z}$.

Let $g\colon \mathfrak{P}\to\mathfrak{Y}$ be a morphism of smooth $\mathcal{V}$-formal schemes. Let T be a divisor of P such that $g(P\setminus T)\subset Y\setminus D$. Then the canonical isomorphism $g^{!}_{T,D}f^{!}_{D,Z}(\mathcal{E})\xrightarrow{\sim}(f\circ g)^{!}_{T,Z}(\mathcal{E})$ is compatible with Frobenius, i.e., we have the following commutative diagram:

$$\begin{array}{ccccc} F^{*}g^{!}_{T,D}f^{!}_{D,Z}(\mathcal{E}) & \xrightarrow{\theta_{g,T,D}\circ f^{!}_{D,Z}} & g^{!}_{T,D}F^{*}f^{!}_{D,Z}(\mathcal{E}) & \xrightarrow{g^{!}_{T,D}\circ\theta_{f,D,Z}} & g^{!}_{T,D}f^{!}_{D,Z}(F^{*}(\mathcal{E})) \\ \downarrow\wr & & & & \downarrow\wr \\ F^{*}(f\circ g)^{!}_{T,Z}(\mathcal{E}) & & \xrightarrow{\theta_{g\circ f,T,Z}} & & (f\circ g)^{!}_{T,Z}F^{*}(\mathcal{E}). \end{array} \tag{9.5.3.4.2}$$

9.5.4 Compatibility with Frobenius: relative duality isomorphism, adjunction $(f_{+},f^{!})$ for proper morphisms and coherent complexes

With notation 9.5.2, suppose $\phi=\mathrm{id}$, f is proper, $D=f^{-1}(Z)$, $\mathfrak{S}=\operatorname{Spf}\mathcal{V}$ and $\mathcal{V}$ satisfies properties of 8.8.3.1. We have already constructed the isomorphism of commutation to Frobenius of the direct image by f for coherent complexes of $\mathcal{D}^{\dagger}$-modules (see 9.5.2.3.2). We will redefine this isomorphism in paragraph 9.5.2.3.2. The aim of this change is to obtain automatically the Frobenius compatibility of adjunction morphisms between the direct image and the extraordinary inverse image (see 9.5.4.4 and 9.5.4.5) We further obtain the compatibility with the composition of the commutation to Frobenius isomorphism of the direct image (9.5.4.8).

9.5.4.1. Let $\mathcal{F}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}})$, $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$. Following 9.4.5.4.2, we have the functorial in $\mathcal{E}$ and $\mathcal{F}$ canonical adjonction isomorphism of abelian groups:

$$\mathrm{adj}_{f,Z}:\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}}(f_{Z,+}(\mathcal{F}),\mathcal{E})\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}}}(\mathcal{F},f^{!}_{Z}(\mathcal{E})), \tag{9.5.4.1.1}$$

where we set $\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}}}(-,-):=H^{0}\circ\mathbb{R}\mathrm{Hom}_{D(\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}}}(-,-)=\mathrm{Hom}_{D(\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}})}(-,-)$.

Lemma 9.5.4.2. *For any $\mathcal{F}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}})$, we uniquely define a functorial isomorphism in $\mathcal{F}$:*

$$\sigma_{f,Z}\colon F^{*}\circ f_{Z,+}(\mathcal{F})\xrightarrow{\sim}f_{Z,+}\circ F^{*}(\mathcal{F}) \tag{9.5.4.2.1}$$

as being the only one making commutative, for all $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$, the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}}}(\mathcal{F},f^{!}_{Z}\mathcal{E}) & \underset{\sim}{\xrightarrow{F^{*}}} & \mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}}}(F^{*}\mathcal{F},F^{*}\circ f^{!}_{Z}\ E) & \underset{\mathrm{Hom}(\mathrm{id},\theta_{f,Z})}{\xrightarrow{\sim}} & \mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}D)_{\mathbb{Q}}}(F^{*}\mathcal{F},f^{!}_{Z}\circ F^{*}\ E) \\ \mathrm{adj}\uparrow\wr & & & & \mathrm{adj}\uparrow\wr \\ \mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}}(f_{Z,+}\mathcal{F},\mathcal{E}) & \underset{\sim}{\xrightarrow{F^{*}}} & \mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}}(F^{*}\circ f_{Z,+}\mathcal{F},F^{*}\mathcal{E}) & \underset{\mathrm{Hom}(\sigma_{f,Z},\mathrm{id})}{\longleftarrow} & \mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}}(f_{Z,+}\circ F^{*}\mathcal{F},F^{*}\mathcal{E}) \end{array} \tag{9.5.4.2.2}$$

where all arrows are isomorphisms of abelian groups.

Proof. With $\mathcal{F}$ fixed, denote $\phi_{\mathcal{E}}$: $\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)}(F^{*}\circ f_{Z,+}\mathcal{F},F^{*}\mathcal{E})\to\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)}(f_{Z,+}\circ F^{*}\mathcal{F},F^{*}\mathcal{E})$, the unique bijection making the diagram 9.5.4.2.2 commutative. We set $\rho_{f,\,Z}:=\phi_{f_{Z,+}\mathcal{F}}\circ F^{*}(\mathrm{id}_{f_{Z,+}\mathcal{F}})$: $f_{Z,+}F^{*}\mathcal{F}\to F^{*}f_{Z,+}\mathcal{F}$. It remains to verify that $\rho_{f,\,Z}$ is an isomorphism (and then we set $\sigma_{f,\,Z}:=\rho^{-1}_{f,\,Z}$). Now, by functoriality in $\mathcal{E}$, for any morphism u: $f_{Z,+}\mathcal{F}\to\mathcal{E}$, $\phi_{\mathcal{E}}\circ F^{*}(u)$ is the composite morphism: $f_{Z,+}F^{*}\mathcal{F}\xrightarrow{\rho_{f,\,Z}}F^{*}\circ f_{Z,+}\mathcal{F}\xrightarrow{F^{*}(\phi)}F^{*}\mathcal{E}$. Furthermore, as the functor F^{*} induces an equivalence of categories, there exists an object $\mathcal{E}_{0}$ of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z))$ and an isomorphism ϵ: $f_{Z,+}F^{*}\mathcal{F}\xrightarrow{\sim}F^{*}\mathcal{E}_{0}$. There is thus a unique morphism u_{0}: $f_{Z,+}\mathcal{F}\to\mathcal{E}_{0}$ such that $F^{*}(u_{0})\circ\rho_{f,\,Z}=\epsilon$. However, noting $\tau_{f,\,Z}:=\epsilon^{-1}\circ F^{*}(u_{0})$: $F^{*}f_{Z,+}\mathcal{F}\to f_{Z,+}F^{*}\mathcal{F}$, it follows: $\tau_{f,\,Z}\circ\rho_{f,\,Z}=\mathrm{id}_{f_{Z,+}F^{*}\mathcal{F}}$. We deduce: $\rho_{f,\,Z}\circ\tau_{f,\,Z}\circ\rho_{f,\,Z}=\rho_{f,\,Z}$ and therefore $\rho_{f,\,Z}\circ\tau_{f,\,Z}=\mathrm{id}_{F^{*}f_{Z,+}\mathcal{F}}$ (by injectivity of $\phi_{f_{Z,+}\mathcal{F}}\circ F^{*}$). □

Remark 9.5.4.3. The lemma 9.5.4.2 corresponds to second way to build the commutation to Frobenius isomorphism $f_{Z+}F^*(\mathcal{E}) \xrightarrow{\sim} F^* f_{Z+}(\mathcal{E})$. This is not clear that both constructions 9.5.2.3.2 and 9.5.4.2.1 coincide. However, when f is a closed immersion, this is already known (see [Car09d, 2.5.4]). Moreover, still when f is a closed immersion, we have established (see [Car09d, 2.4.3]) that, for any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$, the relative duality isomorphism

$$f_{Z,+} \circ \mathbb{D}_D(\mathcal{E}') \xrightarrow{\sim} \mathbb{D}_Z \circ f_{Z,+}(\mathcal{E}')$$

of 9.4.5.2 is compatible with Frobenius.

Since we mainly work with proper morphisms of $\mathcal{V}$-formal scheme, we prefer in this book to use the isomorphism $f_{Z+}F^*(\mathcal{E}) \xrightarrow{\sim} F^* f_{Z+}(\mathcal{E})$ constructed in 9.5.4.2.1

Lemma 9.5.4.4. *Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Y}}({}^\dagger D)_{\mathbb{Q}})$. Let us denote by $\mathrm{adj}_{\mathcal{F}} \colon \mathcal{F} \to f^!_Z f_{Z,+}(\mathcal{F})$ the adjunction isomorphism given by 9.5.4.1.1. The morphism $\mathrm{adj}_{\mathcal{F}}$ is compatible with Frobenius, i.e. the following diagram:*

$$\begin{array}{ccc} F^*\mathcal{F} & \xrightarrow{F^*\mathrm{adj}_{\mathcal{F}}} & F^* f^!_Z f_{Z+}\mathcal{F} \\ \| & & \sim\downarrow (f^!_Z\sigma_{f,Z})\circ(\theta_{f,\,Z}f_{Z,+}) \\ F^*\mathcal{F} & \xrightarrow{\mathrm{adj}_{F^*\mathcal{F}}} & f^!_Z f_{Z+}F^*\mathcal{F} \end{array} \tag{9.5.4.4.1}$$

is commutative.

Proof. We have the commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Hom}(\mathcal{F}, f^!_Z, f_{Z+}\mathcal{F}) & \xrightarrow[\sim]{F^*} & \mathrm{Hom}(F^*\mathcal{F}, F^* f^!_Z f_{Z+}\mathcal{F}) & \xrightarrow[\sim]{} & \mathrm{Hom}(F^*\mathcal{F}, f^!_Z F^* f_{Z+}\mathcal{F}) & \xrightarrow[\sim]{} & \mathrm{Hom}(F^*\mathcal{F}, f^!_Z f_{Z+}F^*\mathcal{F}) \\ \mathrm{adj}\uparrow\sim & & & & \mathrm{adj}\uparrow\sim & & \mathrm{adj}\uparrow\sim \\ \mathrm{Hom}(f_{Z+}\mathcal{F}, f_{Z+}\mathcal{F}) & \xrightarrow[\sim]{F^*} & \mathrm{Hom}(F^* f_{Z+}\mathcal{F}, F^* f_{Z+}\mathcal{F}) & \xrightarrow[\sim]{} & \mathrm{Hom}(f_{Z+}F^*\mathcal{F}, F^* f_{Z+}\mathcal{F}) & \xrightarrow[\sim]{} & \mathrm{Hom}(f_{Z+}F^*\mathcal{F}, f_{Z+}F^*\mathcal{F}), \end{array}$$

where the top right arrow is $\mathrm{Hom}(id, f^!_Z\sigma_{f,Z})$ and the bottom right arrow is $\mathrm{Hom}(id, \sigma_{f,Z})$. Indeed, the rectangle on the left is commutative by construction of $\rho_{f,Z}$ (see the diagram 9.5.4.2.2), while the one on the right is verified by functoriality in $\mathcal{E}$ of the adjunction isomorphism $\mathrm{adj}_{f,\,Z}$ (9.5.4.1.1). The application $\mathrm{Hom}(f_{Z+}\mathcal{F}, f_{Z+}\mathcal{F}) \to \mathrm{Hom}(F^*\mathcal{F}, f^!_Z f_{Z+}F^*\mathcal{F})$ induced by the bottom and right path, sends the identity to $\mathrm{adj}_{F^*\mathcal{F}}$. The arrow $\mathrm{Hom}(f_{Z+}\mathcal{F}, f_{Z+}\mathcal{F}) \to \mathrm{Hom}(F^*\mathcal{F}, F^* f^!_Z f_{Z+}\mathcal{F})$ induced by the left vertical and the left top maps sends the identity to $F^*\mathrm{adj}_{\mathcal{F}}$. This results in the commutativity of 9.5.4.4.1. □

Lemma 9.5.4.5. *If $f^!_Z(\mathcal{E})$ has $\mathcal{D}^\dagger_{\mathfrak{Y}}({}^\dagger D)_{\mathbb{Q}}$-coherent cohomology (e.g. if f is smooth) we get from 9.5.4.1.1 the adjunction morphism $\mathrm{adj}_{\mathcal{E}} \colon f_{Z,+}f^!_Z(\mathcal{E}) \to \mathcal{E}$. The morphism $\mathrm{adj}_{\mathcal{E}}$ is compatible with Frobenius, i.e., that the following canonical diagram is commutative:*

$$\begin{array}{ccc} F^* f_{Z+} f^!_Z\mathcal{E} & \xrightarrow{F^*\mathrm{adj}_{\mathcal{E}}} & F^*\mathcal{E} \\ (f_{Z,+}\theta_{f,\,Z})\circ(\sigma_{f,Z})f^!_Z\downarrow\sim & & \| \\ f_{Z+}f^!_Z F^*\mathcal{E} & \xrightarrow{\mathrm{adj}_{F^*\mathcal{E}}} & F^*\mathcal{E}. \end{array}$$

Proof. This is checked similarly to 9.5.4.4. □

We know that the direct image of the composition of two morphisms is isomorphic to the composition of the direct image of these two morphisms. However, the compatibility with Frobenius of this isomorphism does not seem easy. In order to obtain it tautologically (see 9.5.4.8), we will first rebuild this in the context of the proposition which follows.

Proposition 9.5.4.6. Let g: $\mathfrak{P} \to \mathfrak{Y}$ and h: $\mathfrak{Q} \to \mathfrak{P}$ be two proper morphisms of $\mathcal{V}$-smooth formal schemes such that $g^{-1}(D)$ (resp. $h^{-1}g^{-1}(D)$) is a divisor of P (resp. Q). By abuse of notation, we set $D := g^{-1}(D)$ (resp. $D := h^{-1}(D)$). We further assume f smooth.

For all $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}(^\dagger D))$, there exists a unique isomorphism $(f \circ g)_{Z,+}(\mathcal{G}) \xrightarrow{\sim} f_{Z,+}g_{D,+}(\mathcal{G})$ functorial in $\mathcal{G}$ and inducing, for all $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(^\dagger Z))$, the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(^\dagger Z)}((f\circ g)_{D+}\mathcal{G},\mathcal{E}) & & \xrightarrow[\sim]{\mathrm{adj}_{f\circ g}} & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}(^\dagger D)}(\mathcal{G},(f\circ g)^!_Z\mathcal{E}) \\
\uparrow & & & & \uparrow\wr \\
\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(^\dagger Z)}(f_{Z+}g_{D+}\mathcal{G},\mathcal{E}) & \xrightarrow[\sim]{\mathrm{adj}_f} & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(^\dagger Z)}(g_{Z+}\mathcal{G},f^!_Z\mathcal{E}) & \xrightarrow[\sim]{\mathrm{adj}_g} & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}(^\dagger D)}(\mathcal{G},g^!_Df^!_Z\mathcal{E})
\end{array}$$

Furthermore, these are "transitive", i.e., if f and g are smooth then, for all $\mathcal{H} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Q}}(^\dagger D')_{\mathbb{Q}})$, the two composite morphisms $(f\circ g\circ h)_{Z,+}(\mathcal{H}) \xrightarrow{\sim} (f\circ g)_{Z,+}\circ h_{D,+}(\mathcal{H}) \xrightarrow{\sim} f_{Z,+}g_{D,+}h_{D,+}(\mathcal{H})$ and $(f\circ g\circ h)_{Z,+}(\mathcal{H}) \xrightarrow{\sim} f_{Z,+}(g\circ h)_{D,+}(\mathcal{H}) \xrightarrow{\sim} f_{Z,+}g_{D,+}h_{D,+}(\mathcal{H})$ are equal.

Proof. First of all, we notice that the diagram of 9.5.4.6 makes sense because, as f is smooth, $f^!_Z(\mathcal{E})$ is $\mathcal{D}^\dagger_{\mathfrak{Y}}(^\dagger D)_{\mathbb{Q}}$-coherent. The first assertion of the proposition is then immediate. Next, consider the following diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{Q}}(^\dagger D')_{\mathbb{Q}}}(\mathcal{H},h^!_D(f\circ g)^!_Z\mathcal{E}) & \longrightarrow & & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{Q}}(^\dagger D')_{\mathbb{Q}}}(\mathcal{H},(f\circ g\circ h)^!_Z\mathcal{E}) \\
 & \nearrow^{\mathrm{adj}_h\circ\mathrm{adj}_{f\circ g}} & \uparrow & & & \nearrow^{\mathrm{adj}_{f\circ g\circ h}} & \uparrow \\
\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}}((f\circ g)_{Z+}h_{D+}\mathcal{H},\mathcal{E}) & & \longrightarrow & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}}((f\circ g\circ h)_{Z+}\mathcal{H},\mathcal{E}) & & \\
\uparrow & & & & \uparrow & & \\
 & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{Q}}(^\dagger D')_{\mathbb{Q}}}(\mathcal{H},h^!_Dg^!_Df^!_Z\mathcal{E}) & \longrightarrow & & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{Q}}(^\dagger D')_{\mathbb{Q}}}(\mathcal{H},(g\circ h)^!_Df^!_Z\mathcal{E}) \\
 & \nearrow^{\mathrm{adj}_h\circ\mathrm{adj}_g\circ\mathrm{adj}_f} & & & & \nearrow^{\mathrm{adj}_{g\circ h}\circ\mathrm{adj}_f} & \\
\mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}}(f_{Z+}g_{D+}h_{D+}\mathcal{H},\mathcal{E}) & & \longrightarrow & & \mathrm{Hom}_{\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}}(f_{Z+}(g\circ h)_{D+}\mathcal{H},\mathcal{E}). & &
\end{array}$$

The commutativity of the left square follows from that of the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}((f\circ g)_{D+}h_{D+}\mathcal{H},\mathcal{E}) & & \xrightarrow[\sim]{\mathrm{adj}_{f\circ g}} & & \mathrm{Hom}(h_{D+}\mathcal{H},(f\circ g)^!_Z\mathcal{E}) & \xrightarrow[\sim]{\mathrm{adj}_h} & \mathrm{Hom}(\mathcal{H},h^!_D(f\circ g)^!_Z\mathcal{E}) \\
\uparrow\wr & & & & \uparrow\wr & & \uparrow\wr \\
\mathrm{Hom}(f_{Z+}g_{D+}h_{D+}\mathcal{H},\mathcal{E}) & \xrightarrow[\sim]{\mathrm{adj}_f} & \mathrm{Hom}(g_{D+}h_{D+}\mathcal{H},f^!_Z\mathcal{E}) & \xrightarrow[\sim]{\mathrm{adj}_g} & \mathrm{Hom}(h_{D+}\mathcal{H},g^!_Df^!_Z\mathcal{E}) & \xrightarrow[\sim]{\mathrm{adj}_h} & \mathrm{Hom}(\mathcal{H},h^!_Dg^!_Df^!_Z\mathcal{E}),
\end{array}$$

where, substituting $h_{D+}\mathcal{H}$ for $\mathcal{G}$, the rectangle on the left is, by construction, that of the proposition 9.5.4.6, while the commutativity of the right square is verified by functoriality of the isomorphism $g^!_Df^!_Z\mathcal{E} \xrightarrow{\sim} (f\circ g)^!_Z\mathcal{E}$.

In the same way, we see that the right, top and bottom squares are commutative. As the one at the bottom is also and the arrow at the top right is an isomorphism, this results in the commutativity of the square in front. $\square$

Remark 9.5.4.7. We keep the notations of 9.5.4.6, but we replace the hypothesis that "f smooth" by the fact property "$f^!_Zf_{Z,+}g_{D,+}(\mathcal{G})$ is $\mathcal{D}^\dagger_{\mathfrak{Y}}(^\dagger D)_{\mathbb{Q}}$-coherent". (For example, if $\mathcal{G}$ is a complex $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger D)_{\mathbb{Q}}$-overcoherent (see 15.3.6) then the condition "$f^!_Zf_{Z,+}g_{D,+}(\mathcal{G})$ is $\mathcal{D}^\dagger_{\mathfrak{Y}}(^\dagger D)_{\mathbb{Q}}$-coherent" is checked.) With this hypothesis, the diagram of 9.5.4.6 is still valid if we choose $\mathcal{E} = f_{Z,+}g_{D,+}(\mathcal{G})$. Hence, the image of the identity of $f_{Z,+}g_{D,+}(\mathcal{G})$ by the left bijection of diagram 9.5.4.6 provides a morphism $(f\circ g)_{Z,+}(\mathcal{G}) \to f_{Z,+}g_{D,+}(\mathcal{G})$. Furthermore, since $f^!_Z(f\circ g)_{Z,+}(\mathcal{G})$ is also $\mathcal{D}^\dagger_{\mathfrak{Y}}(^\dagger D)_{\mathbb{Q}}$-coherent, by taking $\mathcal{E} = (f\circ g)_{Z,+}(\mathcal{G})$ in the diagram 9.5.4.6, we construct an inverse. Modulo this condition, the proposition 9.5.4.6 (and also 9.5.4.8) then still remains valid when f and g are any two proper morphisms.

Proposition 9.5.4.8. Let g: $\mathfrak{P} \to \mathfrak{Y}$ be a proper morphism of $\mathcal{V}$-smooth formal schemes such that $g^{-1}(D)$ is a divisor of P. We assume f smooth. The isomorphism $(f\circ g)_{Z,+} \xrightarrow{\sim} f_{Z,+}\circ g_{D,+}$ constructed in 9.5.4.6 is then compatible with Frobenius, i.e. ,

$$\begin{array}{ccccc} F^* f_{Z+} g_{D,+}(\mathcal{E}) & \xrightarrow{\sigma_{f,Z}\circ g_{D,+}} & f_{Z,+}F^* g_{D,+}(\mathcal{E}) & \xrightarrow{f_{Z,+}\circ\sigma_{g,D}} & f_{Z,+}g_{D,+}F^*(\mathcal{E}) \\ \downarrow\sim & & & & \downarrow\sim \\ F^*(f\circ g)_{Z,+}(\mathcal{E}) & & \xrightarrow{\sigma_{f\circ g,Z}} & & (f\circ g)_{Z,+}F^*(\mathcal{E}) \end{array} \tag{9.5.4.8.1}$$

which can be translated (by abuse of notation) by the following equality: $(f_{Z,+}\circ\sigma_{g,D})\circ(\sigma_{f,Z}\circ g_{D,+}) = \sigma_{f\circ g,Z}$.

Proof. For all $\mathcal{G}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}D)_{\mathbb{Q}})$ and $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$, we verify that the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(F^*(f\circ g)_{Z+}\mathcal{G},F^*\mathcal{E}) & & \xrightarrow{F^*\mathrm{adj}_{f\circ g}} & & \mathrm{Hom}(F^*\mathcal{G},F^*(f\circ g)^!_Z\mathcal{E}) \\
\downarrow & & & & \downarrow \\
\mathrm{Hom}(F^*f_{Z+}g_{D+}(\mathcal{G}),F^*\mathcal{E}) & \xrightarrow{F^*\mathrm{adj}_f} & \mathrm{Hom}(F^*g_+\mathcal{G},F^*f^!_Z\mathcal{E}) & \xrightarrow{F^*\mathrm{adj}_g} & \mathrm{Hom}(F^*\mathcal{G},F^*g^!_Df^!_Z\mathcal{E}) \\
 & & \searrow & & \searrow \\
 & & & \mathrm{Hom}(g_{D+}F^*\mathcal{G},F^*f^!_Z\mathcal{E}) \xrightarrow{\mathrm{adj}_g} & \mathrm{Hom}(F^*\mathcal{G},g^!_DF^*f^!_Z\mathcal{E}) \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
\mathrm{Hom}(f_{Z+}F^*g_{D+}(\mathcal{G}),F^*\mathcal{E}) & \xrightarrow{\mathrm{adj}_f} & \mathrm{Hom}(F^*g_{D+}\mathcal{G},f^!_ZF^*\mathcal{E}) & & \\
\searrow & & \searrow & & \\
\mathrm{Hom}(f_{Z+}g_{D+}F^*\mathcal{G},F^*\mathcal{E}) & \xrightarrow{\mathrm{adj}_f} & \mathrm{Hom}(g_{D+}F^*\mathcal{G},f^!_ZF^*\mathcal{E}) & \xrightarrow{\mathrm{adj}_g} & \mathrm{Hom}(F^*\mathcal{G},g^!_Df^!_ZF^*\mathcal{E}) \\
\downarrow & & & & \downarrow \\
\mathrm{Hom}((f\circ g)_{Z+}F^*\mathcal{G},F^*\mathcal{E}) & & \xrightarrow{\mathrm{adj}_{f\circ g}} & & \mathrm{Hom}(F^*\mathcal{G},(f\circ g)^!_ZF^*\mathcal{E}),
\end{array} \tag{9.5.4.8.2}$$

is commutative. Indeed, the top and bottom rectangles are commutative by construction (9.5.4.6), the middle and left diamond as well as the right square are by functoriality while that of the left square and the right diamond results from 9.5.4.2.2. Now, the isomorphism $(f\circ g)_{Z+}F^*(\mathcal{G})\xrightarrow{\sim}F^*(f\circ g)_{Z+}(\mathcal{G})$ is the unique morphism making commutative the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}(F^*(f\circ g)_{Z+}\mathcal{G},F^*\mathcal{E}) & \underset{\sim}{\xrightarrow{F^*\mathrm{adj}_{f\circ g}}} & \mathrm{Hom}(F^*\mathcal{G},F^*(f\circ g)^!_Z\mathcal{E}) \\ \vdots & & \downarrow\sim \\ \mathrm{Hom}((f\circ g)_{Z+}F^*\mathcal{G},F^*\mathcal{E}) & \underset{\sim}{\xrightarrow{\mathrm{adj}_{f\circ g}}} & \mathrm{Hom}(F^*\mathcal{G},(f\circ g)^!_ZF^*\mathcal{E}). \end{array} \tag{9.5.4.8.3}$$

It results from the Frobenius compatibility of the canonical isomorphism $g^!_D\circ f^!_Z\xrightarrow{\sim}(f\circ g)^!_Z$ (see 9.5.3.4.2) that the right arrow of 9.5.4.8.3 is the right composite morphism of 9.5.4.8.2. Hence we are done. □

Chapter 10

Overconvergent isocrystals

Let $\mathfrak{S}$ be a $\mathcal{V}$-formal scheme locally of finite type and S be its special fiber. For rigid geometry we refer to [BGR84], [Ber96b, §0] , [LS07] or [dJ95, §7] .

10.1 Overconvergent sections

In this section, we summarize the construction and consequences of the functor $j^{\dagger}$. These results are due to Berthelot [Ber90], [Ber96b] which contain all the proofs; but see also Le Stum [LS07].

10.1.1 Specialisation morphism, tubes, strict neighborhood

10.1.1.1. For a $\mathcal{V}$-formal scheme $\mathcal{X}$ locally of finite type, let $\mathcal{X}_K$ be the set of closed subschemes Z of $\mathfrak{X}$ which are integral, finite and flat over $\mathcal{V}$. The support of such a subscheme Z is closed point of $\mathfrak{X}$, which we call the specialization of the point $x \in \mathfrak{X}$ corresponding to Z. This yields the map

$$\mathrm{sp}\colon \mathfrak{X}_K \to \mathfrak{X}. \tag{10.1.1.1.1}$$

We define on $\mathfrak{X}_K$ a structure of rigid analytic space as follows:

Proposition 10.1.1.2. *Let $\mathfrak{X}$ be a $\mathcal{V}$-formal scheme locally of finite type.*

(i) There exists on $\mathfrak{X}_K$ a unique structure of quasi-separated rigid analytic variety over K and a map $sp : \mathfrak{X}_K \to \mathfrak{X}$ satisfying the following:

- *(a) the inverse image by the map $\mathrm{sp} : \mathfrak{X}_K \to \mathfrak{X}$ of any open (resp. open covering) of $\mathfrak{X}$ is an open (resp. admissible covering) of $\mathfrak{X}_K$;*
- *(b) for any affine open $\mathfrak{U} = \mathrm{Spf}\, A \subset \mathfrak{X}$, the structure induced by $\mathfrak{X}_K$ on $\mathfrak{U}_K = \mathrm{sp}^{-1}U$ coincide with that defined by $Spm(\, A \otimes K)$.*

(ii) The application sp of 10.1.1.1.1 defines in a natural way a morphism of ringed sites ([SGA4.2, IV 4.9]) .

(iii) Let $\mathfrak{X}, \mathfrak{X}'$ be two $\mathcal{V}$-formal schemes locally of finite presentation. For any formal scheme morphism $u : \mathfrak{X} \to \mathfrak{X}'$ the diagram of ringed topoi commutes

$$\begin{array}{ccc} \mathfrak{X} & \xleftarrow{\mathrm{sp}} & \mathfrak{X}_K \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u_K} \\ \mathfrak{X}' & \xleftarrow{\mathrm{sp}} & \mathfrak{X}'_K \end{array} .$$

10.1.1.3 (Tubes)**.** Let $\mathfrak{P}$ be a $\mathcal{V}$-formal scheme locally of finite type. Let X (resp. U) be a closed (resp. open) subscheme of P. Set $Y = X \cap U$. We write $]Y[_{\mathfrak{P}}$ (or sometimes simply $]Y[$) for the inverse image $\mathrm{sp}^{-1}(Y)$ which we shall call the tube of Y where $\mathrm{sp}\colon \mathfrak{P}_K \to \mathfrak{P}$ is the specialization morphism (see 10.1.1.1.1). We say that $]Y[_{\mathfrak{P}}$ is the tube of Y in $\mathfrak{P}$. This tube is a rigid analytic subvariety of $\mathfrak{P}_K$ which has the following local description: suppose $\mathfrak{P} = \operatorname{Spf} A$ is a formal affine $\mathfrak{S}$-scheme and there exist sections $f_1, \ldots, f_r,\ g_1, \ldots, g_s \in \Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}})$ with reductions $\bar{f}_1, \ldots, \bar{f}_r,\ \bar{g}_1, \ldots, \bar{g}_s \in \Gamma(P, \mathcal{O}_P)$ such that $Y = X \cap U$ with $X = V(\bar{f}_1, \ldots, \bar{f}_r)$ and $U = D(\bar{g}_1) \cup \ldots \cup D(\bar{g}_s)$, then

$$]Y[_{\mathfrak{P}} := \{x \in \mathfrak{P}_K, |f_1(x)|, \ldots, |f_r(x)| < 1 \text{ and } \exists\, j \in \{1, \ldots, s\}, |g_j(x)| = 1\}.$$

For any real $|\pi| \leq \eta < 1$ (resp. $|\pi| \leq \eta < 1$), we can define by gluing the closed (resp. open) tube of radius η of X in P, which is a rigid analytic subvariety of $\mathfrak{P}_K$. More precisely, in the above local description, we define

$$[X]_{\mathfrak{P}\eta} := \{x \in \mathfrak{P}_K, |f_1(x)|, \ldots, |f_r(x)| \leq \eta\}$$

and

$$]X[_{\mathfrak{P}\eta} := \{x \in \mathfrak{P}_K, |f_1(x)|, \ldots, |f_r(x)| < \eta\}.$$

Since $|\pi| \leq \eta < 1$ (resp. $|\pi| \leq \eta < 1$), we check that the closed (resp. open) tube of radius η do not depend on the choice of such $f_1, \ldots, f_r$ and therefore we can glue.

Definition 10.1.1.4. We define the category of $\mathfrak{S}$-frames as follows:

(a) By an $\mathfrak{S}$-frame we shall mean a commutative diagram

$$\begin{array}{ccccc} Y & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookrightarrow} & \mathfrak{P} \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ S & = & S & \longrightarrow & \mathfrak{S} \end{array}$$

where u is an adic morphism of formal schemes of finite type, i is a closed immersion of formal schemes, j is an open immersion of S-schemes, v and w are of finite type. When u is affine we say that this an affine $\mathfrak{S}$-frame. We write this data as $(Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ or simply $(Y, X, \mathfrak{P})/\mathfrak{S}$ or even $(Y, X, \mathfrak{P})$.

(b) A morphism $f = (w, v, u)\colon (Y' \xrightarrow{j'} X' \xrightarrow{i'} \mathfrak{P}')/\mathfrak{S} \to (Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ of $\mathfrak{S}$-frames is a commutative diagram (over $\mathfrak{S}$)

$$\begin{array}{ccccc} Y' & \overset{j'}{\hookrightarrow} & X' & \overset{i'}{\hookrightarrow} & \mathfrak{P}' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ Y & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookrightarrow} & \mathfrak{P}. \end{array} \tag{10.1.1.4.1}$$

When u is smooth in a neighbourhood of Y' (resp. u is quasi-compact, resp. u is affine) we say that the morphism (w, v, u) is weakly smooth (resp. quasi-compact, resp. affine). Beware in [LS07, 3.3.5] this is called "smooth" and not weakly smooth (in order to get a coherent notion with smooth d-frames of 12.2.1.1 or of c-frames of 16.2.1.8). When v is proper (resp. finite, resp. projective), we say that f is proper (resp. finite, resp. projective).

The category of $\mathfrak{S}$-frames as the final object $(S, S, \mathfrak{S})$. By definition, an $\mathfrak{S}$-frame is weakly smooth (resp. quasi-compact, resp. affine, resp. proper) if so is its structural morphism toward the final object.

10.1.1.5. Let $(Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ be an $\mathfrak{S}$-frame. We say an admissible open subset V of $]X[$ is a "strict neighborhood of $]Y[_{\mathfrak{P}}$ in $]X[_{\mathfrak{P}}$" if $]X[= V \cup]X \setminus Y[$ is an admissible covering in the rigid topology.

Example 10.1.1.6. Let $(Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ be a quasi-compact $\mathfrak{S}$-frame and Z be the complement for X in Y Then, for $\lambda < 1$, $]X[_{\mathfrak{P}} \setminus]Z[_{\mathfrak{P}\lambda}$ is a strict neighborhood of $]Y[_{\mathfrak{P}}$ in $]X[_{\mathfrak{P}}$ (see [LS07, 3.3.1]).

The proposition 10.1.1.7 below describes strict neighborhoods.

Proposition 10.1.1.7. *Let* $(Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ *be a quasi-compact* $\mathfrak{S}$*-frame and* Z *be the complement for* X *in* Y*. For any* $\lambda, \eta < 1$*, we write*

$$V^\lambda :=]X[_{\mathfrak{P}} \backslash]Z[_{\mathfrak{P}\lambda}, \qquad V^\lambda_\eta := [X]_{\mathfrak{P}\eta} \cap V^\lambda.$$

If V *is an admissible open subset of* $]X[_{\mathfrak{P}}$*, the following are equivalent:*

(a) V *is a strict neighborhood of* $]Y[_{\mathfrak{P}}$ *in* $]X[_{\mathfrak{P}}$*.*

(b) For any affinoid open subset $W \subset]X[_{\mathfrak{P}}$*, there exists* $\lambda < 1$ *such that*

$$V^\lambda \cap W \subset V.$$

(c) For any quasi-compact admissible open subset $W \subset]X[_{\mathfrak{P}}$*, there exists* $\lambda < 1$ *such that*

$$V^\lambda \cap W \subset V.$$

(d) For any $\eta < 1$*, there exists* $\lambda < 1$ *such that*

$$V^\lambda_\eta \subset V.$$

Proof. A proof is given at [LS07, 3.3.2]. □

10.1.2 Adding overconvergent singularities: the functor $j^\dagger$

10.1.2.1. Suppose we have a frame $Y \xrightarrow{j} X \to \mathfrak{P}$. Let V be an admissible open subset of $]X[$, A be a sheaf of (not necessarily commutative) ring on V and $\mathcal{E}$ an A-module. Define

$$j^\dagger_V \mathcal{E} := \varinjlim j_{VV'*} j^{-1}_{VV'} \mathcal{E}$$

where V' runs through all the strict neighborhoods of $]Y[$ in $]X[$ and

$$j_{VV'} : V \cap V' \hookrightarrow V$$

denotes the inclusion map. This sheaf $j^\dagger_V \mathcal{E}$ on V is called the "sheaf of germs of sections of the sheaf $\mathcal{E}$ on V overconvergent along $Z := X \setminus Y$". If W is a quasi-compact open subset of V then

$$\Gamma(W, j^\dagger_V \mathcal{E}) = \varinjlim_{V' \subset V} \Gamma(W \cap V', \mathcal{E}), \tag{10.1.2.1.1}$$

where V' runs through all the strict neighborhoods of $]Y[$ in $]X[$.

Let $j_V : V \to]X[$, define $j^\dagger \mathcal{E} := j_{V*} j^\dagger_V \mathcal{E}$. Suppose we have strict neighborhoods $W \subset V$ of $]Y[$ in $]X[$. Then the functor $j^\dagger$ does not change if we replace V by W.

Example 10.1.2.2. Let $\mathfrak{X}$ be a $\mathcal{V}$-formal scheme. Let Z be a divisor of X, $\mathfrak{Y}$ be the open of $\mathfrak{X}$ complementary to the support of Z and $j \colon \mathfrak{Y} \to \mathfrak{X}$ be the canonical morphism. We have $\mathfrak{Y}_{\mathfrak{X}} =]Y[_{\mathfrak{X}}$ and $\mathfrak{X}_K =]X[_{\mathfrak{X}}$. For any $m \in \mathbb{N}$, we set $\lambda_m := p^{-1/p^{m+1}}$, $V_m := \mathfrak{X}_K \backslash]Z[_{\mathfrak{X}\lambda_m}$ and $j_m \colon V_m \to \mathfrak{X}_K$ be the canonical immersion. Then it follows from 10.1.1.7 that the family V_m forms a basis of strict neighborhoods of $\mathfrak{Y}_K$ in $\mathfrak{X}_K$.

Let V be a strict neighborhood of $]Y[_{\mathfrak{P}}$ in $]X[_{\mathfrak{P}}$ and E be an $\mathcal{O}_V$-module. Hence, by definition of the functor $j^\dagger$ (see 10.1.2.1), we get

$$j^\dagger E \xrightarrow{\sim} \varinjlim_m j_{m*}(E|V_m \cap V). \tag{10.1.2.2.1}$$

Proposition 10.1.2.3. We keep notation 10.1.2.2.

(a) There exist canonical isomorphisms of $\mathcal{O}_{\mathfrak{X}}$-algebras

$$\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{sp}_* j_{m*} j^*_m \mathcal{O}_{\mathfrak{X}_K}, \qquad \mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{sp}_* j^\dagger \mathcal{O}_{\mathfrak{X}_K}. \tag{10.1.2.3.1}$$

(b) The functor $(\mathrm{sp} \circ j_m)_*$ induces a equivalence between coherent (resp. locally free of finite type) $\mathcal{O}_{V_m}$-modules and coherent (resp. locally projective of finite type) $\mathcal{B}^{(m)}_{\mathfrak{X}}(Z)_{\mathbb{Q}}$-modules.

(c) The functors sp_* and sp^* induce quasi-inverse equivalences between coherent (resp. locally free of finite type) $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-modules and coherent (resp. locally projective of finite type) $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$-modules.

(d) Let E be a coherent $j^\dagger \mathcal{O}_{]X[_{\mathfrak{X}}}$-module, let E_m be a coherent $\mathcal{O}_{V_m}$-module. For any integer $n \neq 0$,

$$H^n \mathbb{R}(\mathrm{sp} \circ j_m)_*(E_m) = 0, \qquad H^n \mathbb{R}\mathrm{sp}_*(E) = 0. \tag{10.1.2.3.2}$$

Proof. By copying the proof of [Ber96c, 4.3.2 and 4.4.2], we get the first statements. The check of the last one is similar: when $\mathfrak{X}$ is affine and there exists $\overline{f} \in \mathcal{O}_X$ giving an equation of Z in X, then $\mathfrak{X}_K$ and V_m are affinoid. Hence, using Kiehl's theorem A and B (see [Kie67]), we get 10.1.2.3.2. □

Notation 10.1.2.4. Let $f = (w, v, u)\colon (Y' \xrightarrow{j'} X' \xrightarrow{i'} \mathfrak{P}')/\mathfrak{S} \to (Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ be a morphism of $\mathfrak{S}$-frames. We denote by $(v,u)_K \colon]X'[_{\mathfrak{P}'} \to]X[_{\mathfrak{P}}$. By abuse of notation (when u is understood), we can write v_K instead of $(v,u)_K$. This yields the inverse image by setting for any $\mathcal{O}_{]X[_{\mathfrak{P}}}$-module E

$$v_K^* E = \mathcal{O}_{]X'[_{\mathfrak{P}'}} \otimes_{v_K^{-1}\mathcal{O}_{]X[_{\mathfrak{P}}}} v_K^{-1} E. \tag{10.1.2.4.1}$$

When i and i' are the canonical inclusions of the special fibers $P \hookrightarrow \mathfrak{P}$ and $P' \hookrightarrow \mathfrak{P}'$, then $u_K^* = v_K^*$.

Proposition 10.1.2.5. *Given a frame $Y \subset X \subset \mathfrak{P}$. Let V be a strict neighbourhood of $]Y[$ in $]X[$ and E be an $\mathcal{O}_V$-module.*

(a) (i) The canonical homomorphism $E \to j_V^\dagger E$ is an epimorphism. If E is a $j_V^\dagger \mathcal{O}_V$-module then this is an isomorphism.

(ii) The canonical homomorphism $E \otimes_{\mathcal{O}_V} j_V^\dagger \mathcal{O}_V \to j_V^\dagger E$ is an isomorphism.

(iii) The functors $j_V^\dagger$ and $j^\dagger$ are exact on the category of $\mathcal{O}_V$-modules.

(b) Let $f = (w, v, u)\colon (Y' \xrightarrow{j'} X' \xrightarrow{i'} \mathfrak{P}')/\mathfrak{S} \to (Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ be a morphism of $\mathfrak{S}$-frames, V' a strict neighborhood of $]Y'[_{\mathfrak{P}'}$ in $]X'[_{\mathfrak{P}'}$ such that $u_K(V') \subset V$. Let $f_{V',V} \colon V' \to V$ be the morphism induced by u_K. Then there exist canonical $\mathcal{O}_V$-linear homomorphisms

$$j_V^\dagger E \to f_{V'V*} j'^\dagger_{V'} f_{V'V}^{-1} E, \qquad j^\dagger E \to v_{K*} j'^\dagger f_{V'V}^{-1} E. \tag{10.1.2.5.1}$$

If moreover we suppose that $Y' = Y \times_X X'$, then the induced homomorphisms

$$f_{V'V*} j_V^\dagger E \to j'^\dagger_{V'} f_{V'V}^* E, \qquad v_K^* j^\dagger E \to j'^\dagger f_{V'V}^* E \tag{10.1.2.5.2}$$

are isomorphisms.

(c) Suppose we have a morphism of frames

$$\begin{array}{ccccc} Y' & \xhookrightarrow{j'} & X' & \xhookrightarrow{i'} & \mathfrak{X}' \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ Y & \xhookrightarrow{j} & X & \xhookrightarrow{i} & \mathfrak{X} \end{array},$$

where i, i' are the canonical inclusions of the special fibers and $u : \mathfrak{X}' \to \mathfrak{X}$ is an open immersion. Let F be an $\mathcal{O}_{\mathfrak{X}'_K}$-module. Then the canonical homomorphism

$$u_{K*} j'^\dagger F \to j^\dagger u_{K*} j'^\dagger F \tag{10.1.2.5.3}$$

is an isomorphism. Moreover if $Y' = Y \cap \mathfrak{X}'$ then we have canonical isomorphism

$$j^\dagger u_{K*} F \to u_{K*} j'^\dagger F. \tag{10.1.2.5.4}$$

Proof. See [Ber96b, 2.1.3-4] and [Ber90, 4.1.2]. □

Definition 10.1.2.6. Let $f = (w, v, u)\colon (Y' \xrightarrow{j'} X' \xrightarrow{i'} \mathfrak{P}')/\mathfrak{S} \to (Y \xrightarrow{j} X \xrightarrow{i} \mathfrak{P})/\mathfrak{S}$ be a morphism of $\mathfrak{S}$-frames. With notation 10.1.2.4.1, we get a functor f_K^*, from the category of $j^\dagger \mathcal{O}_{]X[_{\mathfrak{P}}}$-modules to that of $j'^\dagger \mathcal{O}_{]X'[_{\mathfrak{P}'}}$-modules which is defined for any $j^\dagger \mathcal{O}_{]X[_{\mathfrak{P}}}$-module E by setting

$$f_K^* E := j'^\dagger v_K^* E. \qquad (10.1.2.6.1)$$

The functor f_K^* is the *(overconvergent) pullback by the morphism of frames f*. By abuse of notation (when v and w are understood), we can write $u^\dagger E$ instead of $f_K^* E$.

10.1.3 Sheaves of section supported on a tube

Proposition 10.1.3.1. *Given a frame $Y \subset X \subset \mathfrak{P}$. Let V be a strict neighbourhood of $]Y[$ in $]X[$ and E a coherent $\mathcal{O}_V$-module. Write $u :]Y[\to V$ for the immersion. Then the canonical homomorphism $j_V^\dagger E \to u_* u^* E$ is injective.*

Proof. Exercice. □

Definition 10.1.3.2. Given a frame $Y \subset X \subset \mathfrak{P}$, put $Z = X \setminus Y$. Let V be a strict neighbourhood of $]Y[$ in $]X[$, A a sheaf of rings on V and E an A-module. Define a subsheaf $\underline{\Gamma}^\dagger_{]Z[_\mathcal{P}} E$ of E by the following exact sequence

$$0 \longrightarrow \underline{\Gamma}^\dagger_{]Z[_\mathcal{P}}(E) \longrightarrow E \longrightarrow j_V^\dagger E \longrightarrow 0. \qquad (10.1.3.2.1)$$

That is $\underline{\Gamma}^\dagger_{]Z[_\mathcal{P}} E = \mathrm{Ker}(E \to j_V^\dagger E)$ and we say $\underline{\Gamma}^\dagger_{]Z[_\mathcal{P}} E$ is the sheaf of sections of E with support in $]Z[$. We can simply write $\underline{\Gamma}^\dagger_{]Z[}$ instead of $\underline{\Gamma}^\dagger_{]Z[_\mathcal{P}}$.

It follows from Proposition 10.1.2.5.a.iii) that the functor $\underline{\Gamma}^\dagger_{]Z[}$ is exact.

Lemma 10.1.3.3. *Let Y_1, Y_2 be two open subschemes of X with complements Z_1, Z_2 and let $Y = Y_1 \cup Y_2$, $Z = Z_1 \cap Z_2$, $Y' = Y_1 \cap Y_2$, $Z' = Z_1 \cup Z_2$, j_1, j_2, j' the immersions of Y_1, Y_2, Y' in X. Let V be a strict neighbourhood of $]Y[$ (so also of $]Y_1[$, $]Y_2[$, $]Y'[$) in $]X[$, A a sheaf of rings on V. Then there exists isomorphisms of functors:*
(i) $j_1^\dagger \circ j_2^\dagger = j_2^\dagger \circ j_1^\dagger = j'^\dagger$
(ii) $\underline{\Gamma}^\dagger_{]Z_1[} \circ \underline{\Gamma}^\dagger_{]Z_2[} = \underline{\Gamma}^\dagger_{]Z_2[} \circ \underline{\Gamma}^\dagger_{]Z_1[} = \underline{\Gamma}^\dagger_{]Z[}$.

Proof. Exercice. □

Proposition 10.1.3.4. *Given a frame $Y \subset X \subset \mathfrak{P}$. Let $\mathscr{Y} := \{Y_i\}_{i=1,\dots,n}$ be a finite open covering of Y. Let $j_{i_1 \cdots i_k} : Y_{i_1} \cap \dots \cap Y_{i_k} \to Y$ be the inclusions. Let V be a strict neighbourhood of $]Y[$, A a sheaf of rings on V and E an A-module. Then we have the exact sequence*

$$0 \to j^\dagger E \to \prod_i j_i^\dagger E \to \prod_{i_1 < i_j} j_{i_i i_j}^\dagger E \to \dots \to j_{1\cdots n}^\dagger E \to 0 \qquad (10.1.3.4.1)$$

Proof. Put $Z = X \setminus Y$. As the sequence is null on $]Z[$, it suffices to prove the exactness for the restriction to V. We shall keep the same notation $j_{i_i \cdots i_k}^\dagger E$ for its restriction to V. The proof will be by induction on n.

Put $Y' = Y_2 \cup \dots \cup Y_n$, $Z' = X - Y'$, $j' : Y' \to X$. Let $K^\bullet$ be the complex

$$0 \to E \to \prod_{1<i} j_i^\dagger E \to \dots \to j_{2\cdots n}^\dagger E \to 0.$$

By induction hypothesis and proposition 10.1.2.5.(a), $K^\bullet$ is a resolution of $\underline{\Gamma}^\dagger_{]Z'[} E$. According to Definition 10.1.3.2 we have an exact sequence of complexes

$$0 \to \underline{\Gamma}^\dagger_{]Z_1[} K^\bullet \to K^\bullet \to j_1^\dagger K^\bullet \to 0.$$

By 10.1.3.3 (i) we identify the complex $j_1^\dagger K^\bullet$ with the complex

$$0 \to j_1^\dagger E \to \prod_{1<i} j_{1i}^\dagger E \to \cdots \to j_{1\cdots n}^\dagger E \to 0$$

and so we identify the complex associated to the double complex $K^\bullet \to j_1^\dagger K^\bullet$ with the complex

$$0 \to E \to \prod_i j_i^\dagger E \to \cdots \to j_{1\cdots n}^\dagger E \to 0.$$

By 10.1.3.2 $\underline{\Gamma}^\dagger_{]Z_1[} K^\bullet$ is a resolution of $\underline{\Gamma}^\dagger_{]Z_1[}(\underline{\Gamma}^\dagger_{]Z'[} E)$ which is $\underline{\Gamma}^\dagger_{]Z[} E$ (10.1.3.3 (ii)). Thus the sequence

$$0 \to \underline{\Gamma}^\dagger_{]Z[} E \to E \to \prod_i j_i^\dagger E \to \cdots \to j_{1\cdots n}^\dagger E \to 0$$

is exact and the proof is completed. □

Notation 10.1.3.5. With notation 10.1.3.4, we denote by $\check{C}^{\dagger\bullet}(\mathfrak{X}, \mathscr{Y}, E)$ the unordered (by default) version of the Cech complex

$$\cdots \to 0 \to \prod_{i=1}^n j_i^\dagger E \to \prod_{1\le i_0, i_1 \le n} j_{i_0 i_1}^\dagger E \to \cdots \to \prod_{1 \le i_0, \ldots, i_h \le n} j_{i_1 \ldots i_n}^\dagger E \to \ldots, \qquad (10.1.3.5.1)$$

whose $0th$ term is $\prod_{i=1}^n j_i^\dagger E$. The exactness of 10.1.3.4.1 (i.e. of the ordered version) means that $\check{C}^{\dagger\bullet}(\mathfrak{X}, \mathscr{Y}, E)$ is a resolution of $j_Y^\dagger E$.

Proposition 10.1.3.6. *Given a frame $Y \subset X \subset \mathfrak{P}$. Let E, F be coherent $j^\dagger \mathcal{O}_{]X[}$-modules.*
(i) There exists a strict neighbourhood V of $]Y[$ and coherent $\mathcal{O}_V$-modules $\mathcal{E}, \mathscr{F}$ such that $E = j^\dagger \mathcal{E}$, $F = j^\dagger \mathscr{F}$.
(ii) We have a natural isomorphism

$$\varinjlim_{V'} \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{E}|_{V'}, \mathscr{F}|_{V'}) \to \mathrm{Hom}_{j^\dagger \mathcal{O}_{]Y[}}(E, F)$$

where V' runs through the strict neighbourhoods of $]Y[$ contained in V.

Proof. Exercice. □

10.1.4 Resolution of $j^\dagger$ by Čech complexes

Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme, E be an abelian sheaf on $\mathfrak{X}_K$, $\mathfrak{X} = \{\mathfrak{X}_i\}_{i \in I}$ be a finite open covering of $\mathfrak{X}$. Let Y be an open subscheme of X and $j \colon Y \hookrightarrow \mathfrak{X}$ be the open immersion. Let $\mathscr{Y}_i = \{Y_{ij} : j \in J_i\}$ be a finite open covering of $Y_i = Y \cap \mathfrak{X}_i$.

Fix $h, k \in \mathbb{N}$. For $\underline{i} = (i_0, \ldots, i_h) \in I^{h+1}$, put $\mathfrak{X}_{\underline{i}} = \mathfrak{X}_{i_0} \cap \cdots \cap \mathfrak{X}_{i_h}$ and $u_{\underline{i}} : \mathfrak{X}_{\underline{i}} \hookrightarrow \mathfrak{X}$. For $\underline{j} = (j_0, \ldots, j_h) \in J_{i_0} \times \cdots \times J_{i_h}$, put $Y_{\underline{i}\underline{j}} = Y_{i_0 j_0} \cap \cdots \cap Y_{i_h j_h}$. Then $\mathscr{Y}_{\underline{i}} = \{Y_{\underline{i}\underline{j}}\}_{\underline{j} \in J_{i_0} \times \cdots \times J_{i_h}}$ is a finite open covering of $Y_{\underline{i}} = Y \cap \mathfrak{X}_{\underline{i}}$. The intersection of $k+1$-opens of the covering $\mathscr{Y}_{\underline{i}}$ will be denoted by

$$Y_{\underline{i}\underline{\underline{j}}} = \cap_{\alpha=0}^k \cap_{\beta=0}^h Y_{i_\beta j_{\alpha\beta}},$$

where $\underline{\underline{j}} = (\underline{j}_0, \cdots, \underline{j}_k)$ and $\underline{j}_\alpha = (j_{\alpha 0}, \ldots, j_{\alpha h}) \in J_{i_0} \times \cdots \times J_{i_h}$ for $\alpha = 0, \ldots, h$. We denote the corresponding open immersions by

$$j_{\underline{i}} \colon Y_{\underline{i}} \hookrightarrow \mathfrak{X}_{\underline{i}}, \qquad j_{\underline{i}\underline{\underline{j}}} \colon Y_{\underline{i}\underline{\underline{j}}} \hookrightarrow \mathfrak{X}_{\underline{i}}.$$

Set $E_{\underline{i}} := u_{\underline{i}}^*(E)$. According to notation 10.1.3.5, we get a resolution for $j_{\underline{i}}^\dagger E_{\underline{i}}$ by the Čech complex on $\mathfrak{X}_{\underline{i}K}$

$$\cdots \to 0 \to \prod_{\underline{i}} j_{\underline{i}\underline{j}}^\dagger E_{\underline{i}} \to \prod_{\underline{j}_0 \underline{j}_1} j_{\underline{i}(\underline{j}_0 \underline{j}_1)}^\dagger E_{\underline{i}} \to \cdots \to \prod_{\underline{\underline{j}}} j_{\underline{i}\underline{\underline{j}}}^\dagger E_{\underline{i}} \to \cdots$$

which is denoted by $\check{C}^{\dagger\bullet}(\mathfrak{X}_{\underline{i}}, \mathscr{Y}_{\underline{i}}, E_{\underline{i}})$ or simply $\check{C}^{\dagger\bullet}(\mathscr{Y}_{\underline{i}}, E_{\underline{i}})$.

If $\underline{i} = (i_0, \ldots, i_h)$ and $\underline{i}_{\check{\alpha}} = (i_0, \ldots, \check{i}_\alpha, \ldots, i_h)$, then we can define a morphism of complexes

$$\rho^{\bullet}_{\underline{i}\,\underline{i}_{\check{\alpha}}} : u_{\underline{i}_{\check{\alpha}}K*}\check{C}^{\dagger\bullet}(\mathfrak{X}_{\underline{i}_{\check{\alpha}}}, \mathscr{Y}_{\underline{i}_{\check{\alpha}}}, E_{\underline{i}_{\check{\alpha}}}) \to u_{\underline{i}K*}\check{C}^{\dagger\bullet}(\mathfrak{X}_{\underline{i}}, \mathscr{Y}_{\underline{i}}, E_{\underline{i}})$$

extending the morphism $u_{\underline{i}_{\check{\alpha}}K*}E_{\underline{i}_{\check{\alpha}}} \to u_{\underline{i}K*}E_{\underline{i}}$. For any $\underline{\underline{j}} = (\underline{j}_0, \cdots, \underline{j}_h)$, we put $\underline{\underline{j}}_{\check{\alpha}} = (\underline{j}_{0\check{\alpha}}, \cdots, \underline{j}_{h\check{\alpha}})$. For any section s of $u_{\underline{i}_{\check{\alpha}}K*}\check{C}^{\dagger\bullet}(\mathfrak{X}_{\underline{i}_{\check{\alpha}}}, \mathscr{Y}_{\underline{i}_{\check{\alpha}}}, E_{\underline{i}_{\check{\alpha}}})$, we put

$$\rho^{\bullet}_{\underline{i}\,\underline{i}_{\check{\alpha}}}(s)_{\underline{i}\underline{\underline{j}}} := \rho_{\underline{\underline{j}}\,\underline{\underline{j}}_{\check{\alpha}}}(s_{\underline{i}_{\check{\alpha}}\underline{\underline{j}}_{\check{\alpha}}})$$

where $\rho_{\underline{\underline{j}}\,\underline{\underline{j}}_{\check{\alpha}}}$ comes from the immersion $\mathfrak{X}_{\underline{i}} \hookrightarrow \mathfrak{X}_{\underline{i}_{\check{\alpha}}}$ which carries $Y_{\underline{i}\underline{\underline{j}}}$ into $Y_{\underline{i}_{\check{\alpha}}\underline{\underline{j}}_{\check{\alpha}}}$. As $\underline{i}$ varies, these morphisms give rise to a bicomplex $\check{C}^{\dagger\bullet\bullet}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$, which has the following term in degree (h,k) -

$$\check{C}^{\dagger hk}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) = \prod_{\underline{i}=(i_0,\ldots,i_h)} u_{\underline{i}K*}(\prod_{\underline{\underline{j}}=(\underline{j}_0,\cdots,\underline{j}_h)} j^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}}).$$

Examples 10.1.4.1. Let us give the two extreme examples. On one hand, when J_i has only one element for any $i \in I$, then $\check{C}^{\dagger\bullet\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ is equal to the usual topological Check complex $\check{C}^{\dagger\bullet}(\mathscr{X}, E)$ given by $\check{C}^{\dagger h}(\mathscr{X}, E) := \prod_{\underline{i}\in I^{1+h}} u_{\underline{i}K*}u^*_{\underline{i}K}(E)$. On the other hand, when $I = \{i\}$ has only one element, the complex $\check{C}^{\dagger\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ is equal to the complex $\check{C}^{\dagger\bullet}(\mathfrak{X}, \mathscr{Y}_i, E)$.

Proposition 10.1.4.2. *Let E be an abelian sheaf on $\mathfrak{X}_K$. The simple complex associated to the Čech bicomplex $\check{C}^{\dagger\bullet\bullet}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$ is a resolution of $j^\dagger E$.*

Proof. Consider the complex $j^\dagger E \to j^\dagger E \to j^\dagger E \to \cdots$ in which the differential in even degree is 0 and in odd degree is $Id_{j^\dagger E}$. A morphism from this complex to the complex $\check{C}^{\dagger 0*}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$ is given by the canonical morphisms $j^\dagger E \to \check{C}^{\dagger 0k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$ which are defined by the family of morphisms $j^\dagger E \to u_{iK*}j^\dagger_{i\underline{j}}E_i$. Thus we only need to show that

$$\check{C}^{\dagger 0k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \to \check{C}^{\dagger 1k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \to \cdots \tag{10.1.4.2.1}$$

is a resolution of $j^\dagger E$.

As the $\mathfrak{X}_{iK} \overset{u_{iK}}{\hookrightarrow} \mathfrak{X}_K$ form an admissible covering of $\mathfrak{X}_K$, it suffices to prove the assertion on $\mathfrak{X}_{iK}$. We have $u^*_{iK}(j^\dagger E) = j^\dagger_i E_i$ and the functor $\check{C}^{\dagger\bullet}(\mathscr{Y}_i, -)$ applied to the complex $u^*_{iK}\check{C}^{\dagger\bullet k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$ gives a bicomplex $\check{C}^{\dagger\bullet}(\mathscr{Y}_i, u^*_{iK}(\check{C}^{\dagger\bullet k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)))$. By Proposition 10.1.3.4, for any sheaf F on $\mathfrak{X}_{iK}$, $\check{C}^{\dagger\bullet}(\mathscr{Y}_i, F)$ is a resolution of $j^\dagger_i F$. The simple complex associated to the bicomplex $\check{C}^{\dagger\bullet}(\mathscr{Y}_i, u^*_{iK}(\check{C}^{\dagger\bullet k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)))$ is quasi-isomorphic to $j^\dagger_i u^*_{iK}\check{C}^{\dagger\bullet k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$. Similarly, the complex $\check{C}^{\dagger\bullet}(\mathscr{Y}_i, j^\dagger_i E_i)$ is a resolution of $j^\dagger_i j^\dagger_i E_i$. By Proposition 10.1.2.5.(a), for any $j^\dagger_i\mathcal{O}_{\mathfrak{X}_{iK}}$- module F, we have $F \xrightarrow{\sim} j^\dagger_i F$ and so it remains to prove the following assertions:
(a) for any i, h, k, we have

$$u^*_{iK}\check{C}^{\dagger hk}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \xrightarrow{\sim} j^\dagger_i u^*_{iK}\check{C}^{\dagger hk}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$$

(b) for all $\underline{j} = (j_0, \ldots, j_k)$ the complex

$$j^\dagger_{i\underline{j}} u^*_{iK}\check{C}^{\dagger 0k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \to j^\dagger_{i\underline{j}} u^*_{iK}\check{C}^{\dagger 1k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \to \cdots$$

is a resolution of $j^\dagger_{i\underline{j}} j^\dagger_i E_i$.

Let $\underline{i} = (i_0, \ldots, i_h)$, $\underline{i'} = (i, i_0, \ldots, i_h)$, $u'_{\underline{i}K} : \mathfrak{X}_{\underline{i'}K} = \mathfrak{X}_{iK} \cap \mathfrak{X}_{\underline{i}K} \to \mathfrak{X}_{iK}$ the inclusion, $u'_{iK} : \mathfrak{X}_{\underline{i'}K} \hookrightarrow \mathfrak{X}_{\underline{i}K}$, $Y'_{\underline{i}\underline{\underline{j}}} = \mathfrak{X}_{iK} \cap Y_{\underline{i}\underline{\underline{j}}}$, and $j'_{\underline{i}\underline{\underline{j}}} : Y'_{\underline{i}\underline{\underline{j}}} \to \mathfrak{X}_{\underline{i'}K}$. Then, it follows from 10.1.2.5.3 that we get the isomorphisms

$$j^\dagger_i u^*_{iK}\check{C}^{\dagger hk}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \simeq \prod_{\underline{i}=(i_0,\ldots,i_h)} \prod_{\underline{\underline{j}}=(\underline{j}_0,\cdots,\underline{j}_h)} j^\dagger_i u'_{\underline{i}K*} u'^*_{iK} j^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}})$$

$$11 \simeq \prod_{\underline{i}=(i_0,\ldots,i_h)} \prod_{\underline{\underline{j}}=(\underline{j}_0,\cdots,\underline{j}_h)} j^\dagger_i u'_{\underline{i}K*} j'^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}} \simeq \prod_{\underline{i}=(i_0,\ldots,i_h)} \prod_{\underline{\underline{j}}=(\underline{j}_0,\cdots,\underline{j}_h)} u'_{\underline{i}K*} j'^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}} \simeq u^*_{iK}\check{C}^{\dagger hk}(\mathfrak{X}, (\mathscr{Y}_i)_i, E).$$

This implies the assertion (a) holds.

To prove (b) we construct a homotopy on the complex

$$0 \to j^\dagger_{\underline{i}\underline{\underline{j}}} j^\dagger_i E_i \to j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK} \check{C}^{\dagger 0k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \to j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK} \check{C}^{\dagger 1k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E) \to \cdots$$

which is going to be a endomorphism κ on $j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK}(\prod_{\underline{i}=(i_0,\ldots,i_h)} u_{\underline{i}K*}(\prod_{\underline{\underline{j}}=(\underline{j}_0,\cdots,\underline{j}_h)} j^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}}))$.

Claim: For any $\underline{\underline{j}} \in (J_{i_0} \times \cdots \times J_{i_h})^{k+1}$, the next morphism is an isomorphism

$$j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK} u_{\underline{i}K*} j^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}} \longrightarrow j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK} u_{\underline{i'}K*} j^\dagger_{\underline{i'}\underline{\underline{j'}}} E_{\underline{i'}}$$

First we have

$$j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK} u_{\underline{i}K*} j^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}} \xrightarrow{\sim} j^\dagger_{\underline{i}\underline{\underline{j}}} u'_{\underline{i}K*} u'^*_{iK} j^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i}} \xrightarrow{\sim} j^\dagger_{\underline{i}\underline{\underline{j}}} u'_{\underline{i}K*} j'^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i'}},$$

and also

$$j^\dagger_{\underline{i}\underline{\underline{j}}} u^*_{iK} u_{\underline{i'}K*} j^\dagger_{\underline{i'}\underline{\underline{j'}}} E_{\underline{i'}} \xrightarrow{\sim} j^\dagger_{\underline{i}\underline{\underline{j}}} u'_{\underline{i}K*} j^\dagger_{\underline{i'}\underline{\underline{j'}}} E_{\underline{i'}}.$$

As $Y_{\underline{i'}\underline{\underline{j'}}} = Y'_{\underline{i}\underline{j}} \cap Y'_{\underline{i}\underline{\underline{j}}}$, we get $j^\dagger_{\underline{i'}\underline{\underline{j'}}} \simeq j'^\dagger_{\underline{i}\underline{j}} j'^\dagger_{\underline{i}\underline{\underline{j}}}$. Using (10.1.2.5.4) we get the isomorphisms

$$j^\dagger_{\underline{i}\underline{\underline{j}}} u'_{\underline{i}K*} j^\dagger_{\underline{i'}\underline{\underline{j'}}} E_{\underline{i'}} \xrightarrow{\sim} j^\dagger_{\underline{i}\underline{\underline{j}}} u'_{\underline{i}K*} j'^\dagger_{\underline{i}\underline{j}} j'^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i'}} \xrightarrow{\sim} j^\dagger_{\underline{i}\underline{\underline{j}}} u'_{\underline{i}K*} j'^\dagger_{\underline{i}\underline{\underline{j}}} E_{\underline{i'}}$$

This proves the claim. Hence we can define for $s \in j^\dagger_{\underline{i}\underline{\underline{j}}} \check{C}^{\dagger h+1\, k}(\mathfrak{X}, (\mathscr{Y}_i)_i, E)$, $\kappa(s)_{\underline{i}\underline{\underline{j}}} := s_{\underline{i'}\underline{\underline{j'}}}$. It remains to check that κ is a well-defined homotopy, which is easy. □

10.2 Isocrystals

10.2.1 Stratification, integrable connection and left $\mathcal{D}$-module structure

Notation 10.2.1.1 (nth infinitesimal neighborhood)**.** Let $p\colon V \to \mathfrak{S}_K$ be a morphism of rigid varieties. Let $\delta : V \hookrightarrow V \times_{\mathfrak{S}_K} V$ be the diagonal embedding. For any $n \in \mathbb{N}$, write $V^{(n)}$ for the n-th infinitesimal neighborhood of V in $V \times_{\mathfrak{S}_K} V$ so that we have the factorisation $V \to V^{(n)} \to V \times_{\mathfrak{S}_K} V$ and projections $p^{(n)}_1, p^{(n)}_2 : V^{(n)} \to V$. Remark that $V^{(n)}$ depends a priori on the base $\mathfrak{S}_K$ but since the base is fixed we do not care in the notation. For any $n \in \mathbb{N}$, write $V^{(n)}_2$ for the n-th infinitesimal neighborhood of V in $V \times_{\mathfrak{S}_K} V \times_{\mathfrak{S}_K} V$, and $p_{ij} : V^{(n)}_2 \to V^{(n)}$ is the homomorphism corresponding to the projection of $V \times_{\mathfrak{S}_K} V \times_{\mathfrak{S}_K} V$ to $V \times_{\mathfrak{S}_K} V$ along the i and j factors.

10.2.1.2 (Stratification, integrable connection and $\mathcal{D}$-module)**.** Let $p\colon V \to \mathfrak{S}_K$ be a *smooth* morphism of rigid varieties . We denote by $\mathcal{D}_{V/\mathfrak{S}_K}$ (or simply $\mathcal{D}_V$ is there is no doubt on the base), the sheaf of differential operators on $V/\mathfrak{S}_K$. Let $\mathcal{E}$ be an $\mathcal{O}_V$-module.

(a) A *relative to* $V/\mathfrak{S}_K$ *stratification* on $\mathcal{E}$ is a compatible sequence of linear isomorphisms called the Taylor isomorphisms

$$\{\varepsilon^{(n)} : p^{(n)*}_2 \mathcal{E} \cong p^{(n)*}_1 \mathcal{E}\}_{n \in \mathbb{N}}$$

on $V^{(n)}$ with $\varepsilon^{(0)} = \mathrm{id}_{\mathcal{E}}$ that satisfy the following cocycle condition: for all $n \in \mathbb{N}$, we have

$$p^*_{12}(\varepsilon^{(n)}) \circ p^*_{23}(\varepsilon^{(n)}) = p^*_{13}(\varepsilon^{(n)}).$$

(b) A *relative to* $V/\mathfrak{S}_K$ *connection* on $\mathcal{E}$ is an $\mathcal{O}_{\mathfrak{S}_K}$-linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega^1_{V/\mathfrak{S}_K}$$

satisfying the Leibniz rule. Say it is an integrable connection if $\nabla^2 = 0$.

Similarly to 2.3.2.6 and 2.1.1.5, the following data are equivalent:

(i) a relative to $V/\mathfrak{S}_K$ stratification on $\mathcal{E}$

(ii) a relative to $V/\mathfrak{S}_K$ integrable connection on $\mathcal{E}$;

(iii) a left $\mathcal{D}_{V/\mathfrak{S}_K}$-module structure on $\mathcal{E}$ which extends its structure of $\mathcal{O}_V$-module.

Notation 10.2.1.3 (MIC($V/\mathfrak{S}_K$))**.** A ∇-module on $V/\mathfrak{S}_K$ is defined to be a coherent $\mathcal{O}_V$-module $\mathcal{E}$ on V, equipped with a relative to $\mathfrak{S}_K$ integrable connection. Denote by MIC($V/\mathfrak{S}_K$) the category of ∇-module on $V/\mathfrak{S}_K$.

10.2.1.4 (MIC($Y, X, \mathfrak{P}/\mathfrak{S}$))**.** Let $(Y \overset{j}{\hookrightarrow} X \hookrightarrow \mathfrak{P})$ be a weakly smooth $\mathfrak{S}$-frame. Let E be coherent $j^\dagger\mathcal{O}_{]X[_P}$-module. A "relative to $(Y, X, \mathfrak{P}/\mathfrak{S})$ integrable connection" on E is the data of

(i) a strict neighbourhood V of $]Y[_P$ in $]X[_P$, a coherent $\mathcal{O}_V$-module $\mathcal{E}$ such that $V/\mathfrak{S}_K$ is smooth, $E = j_Y^\dagger \mathcal{E}$,

(ii) an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega^1_{V/\mathfrak{S}_K}$.

This is equivalent to saying that E has a structure of left $j^\dagger\mathcal{D}_{V/\mathfrak{S}_K}$-module which extends its structure of $j^\dagger\mathcal{O}_{]X[_P}$-module. The category of such modules E will be denoted by MIC($Y, X, \mathfrak{P}/\mathfrak{S}$).

10.2.2 Overconvergent stratification, overconvergent connections

Definition 10.2.2.1. Let $(Y \hookrightarrow X \hookrightarrow \mathfrak{P})$ be a weakly smooth $\mathfrak{S}$-frame and Z the complement of Y in X. Let V be a strict neighborhood of $]Y[_\mathfrak{P}$ in $]X[_\mathfrak{P}$ which is smooth over $\mathfrak{S}_K$. With notation 10.2.1.1, for any $n \in \mathbb{N}$, $V'' := (V \times_{\mathfrak{S}_K} V) \cap]X[_{\mathfrak{P}\times_\mathfrak{S}\mathfrak{P}}$ is a strict neighborhood of $]Y[_{\mathfrak{P}\times_\mathfrak{S}\mathfrak{P}}$ in $]X[_{\mathfrak{P}\times_\mathfrak{S}\mathfrak{P}}$, and $V'' \cap]X[^{(n)}_\mathfrak{P} = V^{(n)}$ (see [LS07, 4.3.2]). Let $\mathcal{E}$ be an $\mathcal{O}_V$-module.

(a) An *overconvergent (along Z) stratification* on $\mathcal{E}$ is a relative to $V/\mathfrak{S}_K$ stratification $\{\varepsilon^{(n)}\}_{n\in\mathbb{N}}$ on $\mathcal{E}$ for which there exists a strict neighbourhood

$$V' \subset V'' = (V \times_{\mathfrak{S}_K} V) \cap]X[_{\mathfrak{P}\times_\mathfrak{S}\mathfrak{P}}$$

of $]Y[_{\mathfrak{P}\times_\mathfrak{S}\mathfrak{P}}$ in $]X[_{\mathfrak{P}\times_\mathfrak{S}\mathfrak{P}}$ and an isomorphism

$$\epsilon : (p_2^*\mathcal{E})|_{V'} \cong (p_1^*\mathcal{E})|_{V'}$$

such that the Taylor isomorphisms of $\mathcal{E}$ is induced on $U_n := V' \cap]X[^{(n)}_\mathfrak{P}$ by ϵ for each n, i.e., $\varepsilon^{(n)}|_{U_n} = \epsilon|_{U_n}$. When $X = Y$ we say the stratification is "convergent".

(b) An *overconvergent (along Z) connection* on $\mathcal{E}$ is a relative to $V/\mathfrak{S}_K$ integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega^1_{V/\mathfrak{S}_K}$ whose associated stratification (see 10.2.1.2) is overconvergent along Z. When $X = Y$ we say the connection is "convergent".

Definition 10.2.2.2. A "local frame" on $\mathfrak{S}$ is a (weakly smooth) affine $\mathfrak{S}$-frame $(Y \subset X \subset \mathfrak{P})$ endowed with some elements $t_1, \ldots, t_m \in \Gamma(\mathfrak{P}, \mathcal{O}_\mathfrak{P})$ which induce étale coordinates on $\mathfrak{U}/\mathfrak{S}$, where $\mathfrak{U}$ is an open of $\mathfrak{P}$ containing Y. We will simply say that $t_1, \ldots, t_m$ is a set of coordinates on the local frame. The local frame is said to be "strictly local" if the closed complement Z of Y in X is a hypersurface.

10.2.2.3. Any weakly smooth $\mathfrak{S}$-frame $(Y \subset X \subset \mathfrak{P})$ has an open covering

$$\begin{array}{ccccc} Y_i & \hookrightarrow & X_i & \hookrightarrow & \mathfrak{P}_i \\ \downarrow & & \downarrow & & \downarrow \\ Y & \hookrightarrow & X & \hookrightarrow & \mathfrak{P} \end{array}$$

by strictly local frame (see [LS07, 4.3.8]).

Theorem 10.2.2.4. *Let $(Y \subset X \subset \mathfrak{P})$ be a strictly local $\mathfrak{S}$-frame with coordinates $t_1, \ldots, t_m$ (see definition 10.2.2.2) and corresponding derivations $\partial_1, \ldots, \partial_m$. Let Z be the closed complement of Y in X and $\lambda, \eta < 1$. We write*

$$V^\lambda :=]X[_\mathfrak{P} \setminus]Z[_{\mathfrak{P}\lambda}, \qquad V^\lambda_\eta := [X]_{\mathfrak{P}\eta} \cap V^\lambda.$$

Let V be a smooth strict neighbourhood of $]Y[_{\mathfrak{P}}$ in $]X[_{\mathfrak{P}}$ and $\mathcal{E}$ a coherent $\mathcal{O}_V$-module with an integrable connection ∇.

Then ∇ is overconvergent if and only if for each $\eta < 1$, there exists $\eta \leq \delta < \lambda_0 < 1$ such that for all $\lambda_0 \leq \lambda < 1$, we have

$$\forall\, s \in \Gamma(V_\delta^\lambda, \mathcal{E}), \qquad \|\underline{\partial}^{[\underline{k}]}(s)\|\eta^{|k|} \to 0$$

with $\|\cdot\|$ a Banach norm on $\Gamma(V_\delta^\lambda, \mathcal{E})$.

If $t_1, \ldots, t_m$ are actually étale in a neighborhood of X (i.e. are étale coordinates on $\mathfrak{U}/\mathfrak{S}$, where $\mathfrak{U}$ is an open of $\mathfrak{P}$ containing X), we may choose $\delta = \eta$.

Proof. See a proof at [LS07, Theorem 4.3.9]. □

Corollary 10.2.2.5. *Let $\mathfrak{P}$ be smooth affine formal $\mathfrak{S}$-scheme with étale coordinates in the neighbourhood of a closed subvariety X of P. An integrable connection ∇ on a coherent $\mathcal{O}_{]X[_{\mathfrak{P}}}$-module $\mathcal{E}$ is convergent if and only if for each $\eta < 1$, we have*

$$\forall\, s \in \Gamma([X]_{\mathfrak{P}\eta}, \mathcal{E}), \qquad \|\underline{\partial}^{[\underline{k}]}(s)\|\eta^{|\underline{k}|} \to 0, \text{ as } |\underline{k}| \to \infty$$

Proof. This is simply the particular case $Y = X$. □

10.2.2.6 ($\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K)$)**.** Let $(Y \overset{j}{\hookrightarrow} X \hookrightarrow \mathfrak{P})$ be a weakly smooth $\mathfrak{S}$-frame. Let E be a coherent $j^\dagger\mathcal{O}_{]X[_P}$-module.

(a) A "(relative to $(Y, X, \mathfrak{P}/\mathfrak{S})$) overconvergent stratification" on E is an isomorphism of $j^\dagger\mathcal{O}_{]X[_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}}}$-modules $\epsilon\colon p_2^*E \xrightarrow{\sim} p_1^*E$ such that the cocycle condition $p_{13}^*(\epsilon) = p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon)$ holds, where $p_i\colon]X[_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}} \to]X[_{\mathfrak{P}}$ is the ith projection and $p_{ij}\colon]X[_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}} \to]X[_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}}$ is the projection along the i and j factors.

(b) A "(relative to $(Y, X, \mathfrak{P}/\mathfrak{S})$) overconvergent connection" on E is the data of

  (i) a strict neighborhood V of $]Y[_P$ in $]X[_P$, a coherent $\mathcal{O}_V$-module $\mathcal{E}$ such that $V/\mathfrak{S}_K$ is smooth, $E = j_Y^\dagger\mathcal{E}$,

  (ii) an overconvergent connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega^1_{V/\mathfrak{S}_K}$.

We have a canonical bijection between the data of an overconvergent stratification on E and of an overconvergent connexion on E (this is almost by definition 10.2.2.1). We denoted by $\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K)$ the category of coherent $j^\dagger\mathcal{O}_{]X[_P}$-module endowed with a (relative to $(Y, X, \mathfrak{P}/\mathfrak{S})$) overconvergent connection.

Notation 10.2.2.7. Let $(Y \overset{j}{\hookrightarrow} X \hookrightarrow \mathfrak{P})$ be a weakly smooth $\operatorname{Spf}\mathcal{V}$-frame. We write $\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ instead of $\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/(\operatorname{Spf}\mathcal{V})_K)$. Then (similarly to 1.1.4.5) any object of $\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ is a locally free $j^\dagger\mathcal{O}_{]X[_P}$-module (finite type). Beware this is not true for more general $\mathfrak{S}$. Since the category $\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ is independent (up to canonical isomorphism) on the weakly smooth $\operatorname{Spf}\mathcal{V}$-frame enclosing $Y \overset{j}{\hookrightarrow} X$, then we can simply $\mathrm{MIC}^\dagger(Y, X/K)$.

10.2.2.8. Let $f = (a, b, u)\colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$ be a morphism of weakly smooth $\mathfrak{S}$-frames. The functor 10.1.2.6.1 induces the functor $f_K^*\colon \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K) \to \mathrm{MIC}^\dagger(Y', X', \mathfrak{P}'/\mathfrak{S}_K)$, *the pullback by* f, which is defined for an object $E_{\mathfrak{P}} \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K)$ by setting

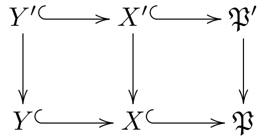
$$f_K^*(E_{\mathfrak{P}}) := j'^\dagger\mathcal{O}_{]X'[_{\mathfrak{P}'}} \otimes_{v_K^{-1}j^\dagger\mathcal{O}_{]X[_{\mathfrak{P}}}} v_K^{-1}E_{\mathfrak{P}}, \tag{10.2.2.8.1}$$

where $v_K\colon]X'[_{\mathfrak{P}'} \to]X[_{\mathfrak{P}}$ is the morphism of ringed spaces induced by f. When $X' = u^{-1}(X)$ and $Y' = u^{-1}(Y)$, the functor f_K^* can simply be denoted by u_K^*.

10.2.3 Isocrystals

Definition 10.2.3.1. According to [LS07, 7.1.1], a finitely presented overconvergent isocrystal E on an $\mathfrak{S}$-frame $(Y \hookrightarrow X \hookrightarrow \mathfrak{P})$ consists of the following data:

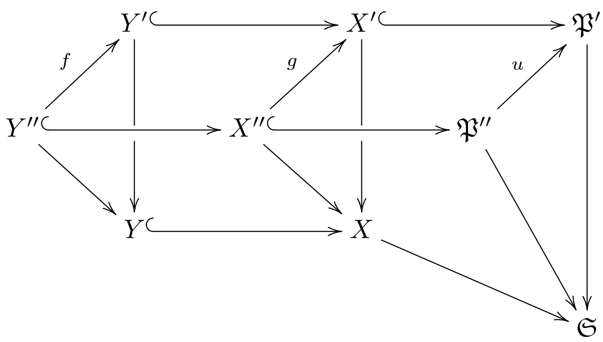
(i) a family of coherent $j^\dagger\mathcal{O}_{]X'[_{\mathfrak{P}'}}$-modules $E_{\mathfrak{P}'}$ for each morphism of $\mathfrak{S}$-frames



(ii) a family of isomorphisms

$$\varphi_u \colon u^\dagger E_{\mathfrak{P}'} \cong E_{\mathfrak{P}''} \tag{10.2.3.1.1}$$

for each commutative diagram



subject to the cocyle condition

$$\varphi_{v\circ u} = \varphi_v \circ v^*\varphi_u. \tag{10.2.3.1.2}$$

We call $E_{\mathfrak{P}'}$ the realization of E on $(Y' \hookrightarrow X' \hookrightarrow \mathfrak{P}'/\mathfrak{S})$.

A morphism of overconvergent isocrystals is a family of compatible morphisms of $j^\dagger\mathcal{O}_{]X'[_{\mathfrak{P}}}$-modules. When $X = Y$ we say E is a convergent isocrystal.

The category of finitely presented overconvergent isocrystal on $(Y \hookrightarrow X \hookrightarrow \mathfrak{P})$ is denoted by $\mathrm{Isoc}^\dagger(Y, X, \mathfrak{P}/K)$.

Proposition 10.2.3.2. *If $(Y \subset X \subset \mathfrak{P})$ is a weakly smooth $\mathfrak{S}$-frame, then the realisation functor on the frame $(Y \subset X \subset \mathfrak{P})/\mathfrak{S}$ induces an equivalence of categories*

$$\mathrm{real}_{(Y\subset X\subset\mathfrak{P})/\mathfrak{S}} \colon \mathrm{Isoc}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K) \cong \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K). \tag{10.2.3.2.1}$$

We can simply write this functor $\mathrm{real}_{\mathfrak{P}}$.

Proof. This realization functor is constructed as follows. Let $E \in \mathrm{Isoc}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K)$ and $E_{\mathfrak{P}} := \mathrm{real}_{(Y\subset X\subset\mathfrak{P})\mathfrak{S}}(E)$. Let $q_0, q_1 \colon \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{P} \to \mathfrak{P}$ be the left and the right projections. This yields the morphisms of frames $p_0 := (id, id, q_0), p_1 := (id, id, q_1) \colon (Y, X, \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{P}) \to (Y, X, \mathfrak{P})$. We denote by $E_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}}$ the realization of E on the frame $(Y, X, \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{P})$. We get the isomorphisms $\phi_{p_0} \colon p_0^* E_{\mathfrak{P}} \xrightarrow{\sim} E_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}}$ and $\phi_{p_1} \colon p_1^* E_{\mathfrak{P}} \xrightarrow{\sim} E_{\mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}}$ (see notation 10.2.3.1.1). This yields the Taylor isomorphism of $E_{\mathfrak{P}}$

$$\epsilon = \phi_{p_0}^{-1} \circ \phi_{p_1} \colon p_1^* E_{\mathfrak{P}} \xrightarrow{\sim} p_0^* E_{\mathfrak{P}}, \tag{10.2.3.2.2}$$

which corresponds to the overconvergent stratification on $E_{\mathfrak{P}}$. The check that this is an equivalence is left to the reader. ☐

10.2.3.3. Let $f = (a, b, u) \colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$ be a morphism of weakly smooth $\mathfrak{S}$-frames. This yields a functor

$$f^* \colon \mathrm{Isoc}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K) \to \mathrm{Isoc}^\dagger(Y', X', \mathfrak{P}'/\mathfrak{S}_K). \tag{10.2.3.3.1}$$

The functor $\mathrm{real}_{\mathfrak{P}}$ of 10.2.3.2.1 commutes with both pullbacks by f (see 10.2.2.8.1): for any $E \in \mathrm{Isoc}^\dagger(Y, X, \mathfrak{P}/\mathfrak{S}_K)$ we have the natural isomorphism $f_K^* \circ \mathrm{real}_{\mathfrak{P}}(E) \xrightarrow{\sim} \mathrm{real}_{\mathfrak{P}'} \circ f^*(E)$ which can also be written $f_K^*(E_{\mathfrak{P}}) \xrightarrow{\sim} f^*(E)_{\mathfrak{P}'}$.

10.2.4 Glueing data

10.2.4.1. Let $f = (a, b, u)$ and $g = (a, b, v)$ be two morphisms of weakly smooth $\mathfrak{S}$-frames of the form $(Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$.

(a) The morphism $(u, v)\colon \mathfrak{P}' \to \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{P}$ induce the morphism of frames $\delta_{u,v} = (b, a, (u, v))\colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{P})$. We get the morphisms of frames $f = p_0 \circ \delta_{u,v}$ and $g = p_1 \circ \delta_{u,v}$. Let $E_{\mathfrak{P}} \in \mathrm{MIC}^{\dagger}(Y, X, \mathfrak{P}/K)$. From the isomorphism $\epsilon\colon p_1^* E_{\mathfrak{P}} \xrightarrow{\sim} p_0^* E_{\mathfrak{P}}$ (see the construction at 10.2.3.2.2), we get the glueing isomorphism

$$\epsilon_{u,v} := \delta_{u,v}^*(\epsilon)\colon g^* E_{\mathfrak{P}} \xrightarrow{\sim} f^* E_{\mathfrak{P}}. \qquad (10.2.4.1.1)$$

Using the transitivity of the isomorphism ϕ (see 10.2.3.1.2), we have the equality $\epsilon_{u,v} := \delta_{u,v}^*(\epsilon) = \phi_f^{-1} \circ \phi_g$. In particular, $\epsilon_{u,u} = \mathrm{id}$.

(b) Let $w\colon \mathfrak{P}' \to \mathfrak{P}$ be a third morphism of formal schemes over $\mathfrak{S}$ such that we get the morphism of smooth $\mathfrak{S}$-frames $h := (b,\, a,\, w)\colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$. We have the transitive formula $\epsilon_{u,\,w} = \epsilon_{u,\,v} \circ \epsilon_{v,\,w}$.

(c) Let $f' = (a', b', u')$ and $g' = (a', b', v')$ be two morphisms of weakly smooth $\mathfrak{S}$-frames of the form $(Y'', X'', \mathfrak{P}'') \to (Y', X', \mathfrak{P}')$. We check the formulas $\epsilon_{u \circ u', v \circ u'} = f_K'^* \circ \epsilon_{u,v}$ and $\epsilon_{u',v'} \circ f_K^* = \epsilon_{u \circ u', u \circ v'}$.

10.2.4.2. We suppose $\mathfrak{S} = \mathrm{Spf}\, \mathcal{V}$. Let $(Y \hookrightarrow X \hookrightarrow \mathfrak{P})$ be an $\mathfrak{S}$-frame and Z the complement of Y in X. We suppose $\mathfrak{P}/\mathfrak{S}$ and X/S are smooth and $\mathfrak{P}$ is separated. Let $(\mathfrak{P}_\alpha)_{\alpha \in \Lambda}$ be an open covering of $\mathfrak{P}$. We set $\mathfrak{P}_{\alpha\beta} := \mathfrak{P}_\alpha \cap \mathfrak{P}_\beta$, $\mathfrak{P}_{\alpha\beta\gamma} := \mathfrak{P}_\alpha \cap \mathfrak{P}_\beta \cap \mathfrak{P}_\gamma$, $X_\alpha := X \cap P_\alpha$, $X_{\alpha\beta} := X_\alpha \cap X_\beta$ and $X_{\alpha\beta\gamma} := X_\alpha \cap X_\beta \cap X_\gamma$. We denote by $Y_\alpha := X_\alpha \cap Y$, $Y_{\alpha\beta} := Y_\alpha \cap Y_\beta$, $Y_{\alpha\beta\gamma} := Y_\alpha \cap Y_\beta \cap Y_\gamma$. $j_\alpha\colon Y_\alpha \hookrightarrow X_\alpha$, $j_{\alpha\beta}\colon Y_{\alpha\beta} \hookrightarrow X_{\alpha\beta}$ and $j_{\alpha\beta\gamma}\colon Y_{\alpha\beta\gamma} \hookrightarrow X_{\alpha\beta\gamma}$ the canonical open immersions. We suppose that for every $\alpha \in \Lambda$, X_α is affine, (for instance when the covering $(\mathfrak{P}_\alpha)_{\alpha \in \Lambda}$ is affine).

For any triple $(\alpha,\, \beta,\, \gamma) \in \Lambda^3$, fix $\mathfrak{X}_\alpha$ (resp. $\mathfrak{X}_{\alpha\beta}$, $\mathfrak{X}_{\alpha\beta\gamma}$) some smooth formal $\mathfrak{S}$-schemes lifting X_α (resp. $X_{\alpha\beta}$, $X_{\alpha\beta\gamma}$), $p_1^{\alpha\beta}\colon \mathfrak{X}_{\alpha\beta} \to \mathfrak{X}_\alpha$ (resp. $p_2^{\alpha\beta}\colon \mathfrak{X}_{\alpha\beta} \to \mathfrak{X}_\beta$) some flat lifting of $X_{\alpha\beta} \to X_\alpha$ (resp. $X_{\alpha\beta} \to X_\beta$).

Similarly, for any $(\alpha,\, \beta,\, \gamma) \in \Lambda^3$, fix some lifting $p_{12}^{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \to \mathfrak{X}_{\alpha\beta}$, $p_{23}^{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \to \mathfrak{X}_{\beta\gamma}$, $p_{13}^{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \to \mathfrak{X}_{\alpha\gamma}$, $p_1^{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \to \mathfrak{X}_\alpha$, $p_2^{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \to \mathfrak{X}_\beta$, $p_3^{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \to \mathfrak{X}_\gamma$, $u_\alpha\colon \mathfrak{X}_\alpha \hookrightarrow \mathfrak{P}_\alpha$, $u_{\alpha\beta}\colon \mathfrak{X}_{\alpha\beta} \hookrightarrow \mathfrak{P}_{\alpha\beta}$ and $u_{\alpha\beta\gamma}\colon \mathfrak{X}_{\alpha\beta\gamma} \hookrightarrow \mathfrak{P}_{\alpha\beta\gamma}$.

Definition 10.2.4.3. With notation 10.2.4.2, we define the category $\mathrm{MIC}^{\dagger}(Y, (\mathfrak{X}_\alpha)_{\alpha \in \Lambda}/K)$ as follows.

- An object of $\mathrm{MIC}^{\dagger}(Y, (\mathfrak{X}_\alpha)_{\alpha \in \Lambda}/K)$ is a family $(E_\alpha)_{\alpha \in \Lambda}$ of objects E_α of $\mathrm{MIC}^{\dagger}((Y_\alpha, X_\alpha, \mathfrak{X}_\alpha)/K)$ together with a *glueing data*, i.e., a collection of isomorphisms in $\mathrm{MIC}^{\dagger}((Y_{\alpha\beta}, X_{\alpha\beta}, \mathfrak{X}_{\alpha\beta})/K)$ of the form $\eta_{\alpha\beta}\ \ p_{2K}^{\alpha\beta *}(E_\beta) \xrightarrow{\sim} p_{1K}^{\alpha\beta *}(E_\alpha)$ satisfying the cocycle condition: $\eta_{13}^{\alpha\beta\gamma} = \eta_{12}^{\alpha\beta\gamma} \circ \eta_{23}^{\alpha\beta\gamma}$, where $\eta_{12}^{\alpha\beta\gamma}$, $\eta_{23}^{\alpha\beta\gamma}$ and $\eta_{13}^{\alpha\beta\gamma}$ are defined so that the following diagrams

$$\begin{array}{ccc} p_{12K}^{\alpha\beta\gamma *} p_{2K}^{\alpha\beta *}(E_\beta) & \underset{\sim}{\xrightarrow{\epsilon}} & p_{2K}^{\alpha\beta\gamma *}(E_\beta) \\ \sim\downarrow p_{12K}^{\alpha\beta\gamma *}(\eta_{\alpha\beta}) & & \downarrow \eta_{12}^{\alpha\beta\gamma} \\ p_{12K}^{\alpha\beta\gamma *} p_{1K}^{\alpha\beta *}(E_\alpha) & \underset{\sim}{\xrightarrow{\epsilon}} & p_{1K}^{\alpha\beta\gamma *}(E_\alpha), \end{array} \quad \begin{array}{ccc} p_{23K}^{\alpha\beta\gamma *} p_{2K}^{\beta\gamma *}(E_\gamma) & \underset{\sim}{\xrightarrow{\epsilon}} & p_{3K}^{\alpha\beta\gamma *}(E_\gamma) \\ \sim\downarrow p_{23K}^{\alpha\beta\gamma *}(\eta_{\beta\gamma}) & & \downarrow \eta_{23}^{\alpha\beta\gamma} \\ p_{23K}^{\alpha\beta\gamma *} p_{1K}^{\beta\gamma *}(E_\beta) & \underset{\sim}{\xrightarrow{\epsilon}} & p_{2K}^{\alpha\beta\gamma *}(E_\beta), \end{array} \quad \begin{array}{ccc} p_{13K}^{\alpha\beta\gamma *} p_{2K}^{\alpha\gamma *}(E_\gamma) & \underset{\sim}{\xrightarrow{\epsilon}} & p_{3K}^{\alpha\beta\gamma *}(E_\gamma) \\ \sim\downarrow p_{13K}^{\alpha\beta\gamma *}(\eta_{\alpha\gamma}) & & \downarrow \eta_{13}^{\alpha\beta\gamma} \\ p_{13K}^{\alpha\beta\gamma *} p_{1K}^{\alpha\gamma *}(E_\alpha) & \underset{\sim}{\xrightarrow{\epsilon}} & p_{1K}^{\alpha\beta\gamma *}(E_\alpha), \end{array} \qquad (10.2.4.3.1)$$

  where the isomorphisms ϵ are those of the form 10.2.4.1.1, are commutative.

- A morphism $f = (f_\alpha)_{\alpha \in \Lambda}\colon ((E_\alpha)_{\alpha \in \Lambda},\, (\eta_{\alpha\beta})_{\alpha,\beta \in \Lambda}) \to ((E'_\alpha)_{\alpha \in \Lambda},\, (\eta'_{\alpha\beta})_{\alpha,\beta \in \Lambda})$ of $\mathrm{MIC}^{\dagger}(Y, (\mathfrak{X}_\alpha)_{\alpha \in \Lambda}/K)$ is by definition a family of morphisms $f_\alpha\colon E_\alpha \to E'_\alpha$ commuting with glueing data.

Proposition 10.2.4.4. *With notation 10.2.4.3, there exists a canonical equivalence of categories*

$$u_{0K}^*\colon \mathrm{MIC}^{\dagger}(Y, X, \mathfrak{P}/K) \cong \mathrm{MIC}^{\dagger}(Y, (\mathfrak{X}_\alpha)_{\alpha \in \Lambda}/K). \qquad (10.2.4.4.1)$$

Proof. 1) Let $\phi_\alpha := (id, id, u_\alpha)\colon (Y_\alpha, X_\alpha, \mathfrak{X}_\alpha) \to (Y_\alpha, X_\alpha, \mathfrak{P}_\alpha)$ be the proper morphism of frames. We remark that ϕ_α is the composition of the morphism of frames $\gamma_{u_\alpha}\colon (Y_\alpha, X_\alpha, \mathfrak{X}_\alpha) \to (Y_\alpha, X_\alpha, \mathfrak{X}_\alpha \times_{\mathfrak{S}} \mathfrak{P}_\alpha)$ induced by the graph of u_α and of $p_2\colon (Y_\alpha, X_\alpha, \mathfrak{X}_\alpha \times_{\mathfrak{S}} \mathfrak{P}_\alpha) \to (Y_\alpha, X_\alpha, \mathfrak{P}_\alpha)$ induced by the second projection. Since p_2 is proper and weakly smooth (see Definitions 10.1.1.4), then following Theorem

[LS07, 7.1.8] we get the equivalence of categories $p_{2K}^*\colon \mathrm{MIC}^\dagger((Y_\alpha, X_\alpha, \mathfrak{P}_\alpha)/K) \cong \mathrm{MIC}^\dagger((Y_\alpha, X_\alpha, \mathfrak{X}_\alpha \times_{\mathfrak{S}} \mathfrak{P}_\alpha)/K)$. Since γ_{u_α} has a retraction (the first projection), then using Corollary [LS07, 7.1.7] we get that $\gamma_{u_\alpha K}^*$ is also an equivalence of categories. Hence, by composition, so is $\phi_{\alpha K}^*\colon \mathrm{MIC}^\dagger((Y_\alpha, X_\alpha, \mathfrak{P}_\alpha)/K) \cong \mathrm{MIC}^\dagger((Y_\alpha, X_\alpha, \mathfrak{X}_\alpha)/K)$.

1') Let $\phi_{\alpha\beta} := (id, id, u_{\alpha\beta})\colon (Y_{\alpha\beta}, X_{\alpha\beta}, \mathfrak{X}_{\alpha\beta}) \to (Y_{\alpha\beta}, X_{\alpha\beta}, \mathfrak{P}_{\alpha\beta})$ be the morphism of frames. Using the same arguments than in 1), we get the equivalence of categories $\phi_{\alpha\beta K}^*\colon \mathrm{MIC}^\dagger((Y_{\alpha\beta}, X_{\alpha\beta}, \mathfrak{P}_{\alpha\beta})/K) \cong \mathrm{MIC}^\dagger((Y_{\alpha\beta}, X_{\alpha\beta}, \mathfrak{X}_{\alpha\beta})/K)$.

2) Let $E_{\mathfrak{P}} \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$. Using the properties of the glueing isomorphisms 10.2.4.1.1, we get canonically on the object $(\phi_{\alpha K}^*(E_{\mathfrak{P}}|_{]X_\alpha[_{\mathfrak{P}_\alpha}}))_{\alpha\in\Lambda}$ a glueing data making it an object of $\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$. The functoriality is obvious and this yields the canonical functor $u_{0K}^*\colon \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) \cong \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$. Since $\phi_{\alpha K}^*$ is fully faithful and $\phi_{\alpha\beta K}^*$ is faithful, we easily check that the functor u_{0K}^* is fully faithful. The proof of the essential surjectivity is left as an exercise for the reader. □

10.3 Log-isocrystals

10.3.1 Shiho's convergent log-isocrystals

10.3.1.1. Let (X, M) be a log scheme of finite type over k. This yields $(X, M) \to \operatorname{Spec} k \hookrightarrow \operatorname{Spf} \mathcal{V}$.[1] Let τ be the Zariski or the etale topology. We denote by $I_{\mathrm{conv},\tau}((X, M)/\operatorname{Spf}\mathcal{V})$ the category of isocrystals on the log convergent site $((X, M)/\operatorname{Spf}\mathcal{V})_{\mathrm{conv},\tau}$ (see [Shi02, Definition 2.1.5]. We denote by $I_{\mathrm{conv}}((X, M)/\operatorname{Spf}\mathcal{V})$ the category of isocrystals on the log convergent site $((X, M)/\operatorname{Spf}\mathcal{V})_{\mathrm{conv}}$ defined at [Shi00]. Following [Shi02, 2.1.7], we have the equivalences of categories

$$I_{\mathrm{conv}}((X, M)/\operatorname{Spf}\mathcal{V}) \cong I_{\mathrm{conv,et}}((X, M)/\operatorname{Spf}\mathcal{V}).$$

10.3.1.2 (Logarithmic tube)**.** Let (X, M) be a fine log scheme over k, and let $i : (X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L})$ be a closed immersion into a noetherian fine log formal scheme $(\mathfrak{P}, \mathfrak{L})$ over $\operatorname{Spf}\mathcal{V}$ and P_k is of finite type over k. Shiho defines the logarithmic tube $](X, M)[^{\log}_{(\mathfrak{P},\mathfrak{L})}$ as follows.

(a) Assume that there exists a factorization of i of the form

$$(X, M) \xrightarrow{i'} (\mathfrak{P}', \mathfrak{L}') \xrightarrow{f'} (\mathfrak{P}, \mathfrak{L}) \qquad (10.3.1.2.1)$$

in which i' is an exact closed immersion and f is a formally log etale morphism. Let $\mathfrak{Q}$ be the completion of $\mathfrak{P}'$ along X. Then following [Shi02, Lemma 2.2.2], the rigid analytic space $\mathfrak{Q}_K$ is independent of the choice of the factorization, up to canonical isomorphism. Denote $\mathfrak{Q}_K$ by $](X, M)[^{\log}_{(\mathfrak{P},\mathfrak{L})}$.

(b) Following [Shi02, Remark 2.2.6], since our log structures are of Zariski type, then Zariski locally in $\mathfrak{P}$, there exists a factorization of i of the form of 10.3.1.2.1. A. Shiho checked at [Shi02, Proposition 2.2.4] that the tube $](X, M)[^{\log}_{(\mathfrak{P},\mathfrak{L})}$ admits a natural sheafification for the Zariski topology, i.e. we get by gluing the construction of $](X, M)[^{\log}_{(\mathfrak{P},\mathfrak{L})}$.

(c) By abuse of notation, we might simply write $]X[^{\log}_{\mathfrak{P}}$ instead of $](X, M)[_{(\mathfrak{P},\mathfrak{L})}$.

10.3.1.3. Suppose given the following commutative diagram

$$\begin{array}{ccc} (X, M) & \xrightarrow{\ i\ } & (\mathfrak{P}, \mathfrak{L}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{Spec}\ k & \xrightarrow{\ \iota\ } & \operatorname{Spf}\ \mathcal{V} \end{array} \qquad (10.3.1.3.1)$$

where the bottom row has trivial log structure, $(\mathfrak{P}, \mathfrak{L})$ is a fine log formal scheme over $\operatorname{Spf}\mathcal{V}$, its structural morphism g is formally log smooth, i is a (non necessarily exact) closed immersion. For an integer j, let

[1] We might add logarithmic structures on $\mathcal{V}$ but we will not need it here.

$(\mathfrak{P}(j), \mathfrak{L}(j))$ denote the $(j+1)$-th fibre product of $(\mathfrak{P}, \mathfrak{L})$ over Spf $\mathcal{V}$, and let $i(j) : (X, M) \to (\mathfrak{P}(j), \mathfrak{L}(j))$ be the locally closed immersion induced by i (and the diagonal $\mathfrak{P} \to \mathfrak{P}(j)$). Moreover, the projections

$$p'_i \colon (\mathfrak{P}(1), \mathfrak{L}(1)) \to (\mathfrak{P}, \mathfrak{L}) \qquad (i = 1, 2),$$
$$p'_{ij} \colon (\mathfrak{P}(2), \mathfrak{L}(2)) \to (\mathfrak{P}(1), \mathfrak{L}(1)) \qquad (1 \leq i < j \leq 3),$$

and the diagonal morphism

$$\Delta' \colon (\mathfrak{P}, \mathfrak{L}) \to (\mathfrak{P}(1), \mathfrak{L}(1)),$$

induce the morphisms of rigid analytic spaces

$$p_i \colon]X[^{\log}_{\mathfrak{P}(1)} \to]X[^{\log}_{\mathfrak{P}} \qquad (i = 1, 2),$$
$$p_{ij} \colon]X[^{\log}_{\mathfrak{P}(2)} \to]X[^{\log}_{\mathfrak{P}(1)} \qquad (1 \leq i < j \leq 3),$$
$$\Delta \colon]X[^{\log}_{\mathfrak{P}} \to]X[^{\log}_{\mathfrak{P}(1)}.$$

Following [Shi02, 2.2.6], we denote by $\mathrm{Str}''((X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V})$ the category of pairs (E, ϵ), where E is a coherent $\mathcal{O}_{]X[^{\log}_{\mathfrak{P}}}$-module and $\epsilon \colon p_2^*(\mathcal{E}) \xrightarrow{\sim} p_1^*(\mathcal{E})$ is an isomorphism of $\mathcal{O}_{]X[^{\log}_{\mathfrak{P}(1)}}$-modules such that $\Delta^*(\epsilon) = id_{\mathcal{E}}$, and the cocycle condition $p_{12}^*(E) \circ p_{23}^*(E) = p_{13}^*(E)$ holds on $]X[^{\log}_{\mathfrak{P}(2)}$. Following [Shi02, 2.2.7], we have the canonical and functorial equivalence of categories

$$I_{\mathrm{conv}}((X, M) / \operatorname{Spf} \mathcal{V}) \cong \mathrm{Str}''((X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V}).$$

In particular, the category $\mathrm{Str}''((X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V})$ does not depend, up to canonical equivalence of categories, on the choice of the diagram 10.3.1.3.1 factorising $(X, M) \to \operatorname{Spec} k \hookrightarrow \operatorname{Spf} \mathcal{V}$. The objects of $\mathrm{Str}''((X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V})$ are called "convergent log-isocrystals on $(X, M) / \operatorname{Spf} \mathcal{V}$ with respect to i" or "convergent log-isocrystals on $(X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V}$".

If (E, ϵ) is a pair of $\mathrm{Str}''((X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V})$ such that E is a locally free $\mathcal{O}_{]X[^{\log}_{\mathfrak{P}}}$-module, then (E, ϵ) is called a "locally free convergent log-isocrystal on $(X, M) / \operatorname{Spf} \mathcal{V}$ with respect to i" or "locally free convergent log-isocrystal on $(X, M) \hookrightarrow (\mathfrak{P}, \mathfrak{L}) / \operatorname{Spf} \mathcal{V}$". The corresponding isocrystal on the log convergent site $((X, M) / \operatorname{Spf} \mathcal{V})_{\mathrm{conv}, \tau}$ will be said to be "locally free". Beware that following Kedlaya's terminology a convergent log-isocrystal is by definition locally free (see [Ked07, 6.1.7]), but we prefer to add "locally free" to remember this property.

10.3.2 Log-isocrystal with overconvergent singularities: the lifted case of relative strict normal crossing divisors

We will mostly need later, specially when working with the arithmetic $\mathcal{D}$-module analogue, to deal with convergent log-isocrystals in the specific context where the log structure is given by relative SNCD. We will also need to add overconvergent singularities.

We will fix some notation: let $g : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of smooth $\mathcal{V}$-formal schemes, of pure relative dimension d, let $\mathfrak{Z}$ be a relative $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor, let $\mathfrak{X}^*$ be the complement of $\mathfrak{Z}$ in $\mathfrak{X}$, let T be a divisor of X and $\mathfrak{Y}$ the complement of T in $\mathfrak{X}$. Let $\mathfrak{X}^\sharp = (\mathfrak{X}, M(\mathfrak{Z}))$ be the logarithmic formal $\mathcal{V}$-scheme with the logarithmic structure associated to $\mathfrak{Z}$ (see 4.5.2.14), $j \colon Y \hookrightarrow X$ the open immersion and $\mathfrak{Y}^\sharp$ the restriction of $\mathfrak{X}^\sharp$ on $\mathfrak{Y}$.

10.3.2.1 (Specialization and log connections)**.** We have the specialization morphism of locally ringed spaces: $\mathrm{sp} \colon \mathfrak{X}_K \to \mathfrak{X}$. For any coherent $\mathcal{O}_{\mathfrak{X}_K}$-module E, the $\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}$-module $\mathrm{sp}_* E$ is coherent and the adjunction morphism $\mathrm{sp}^* \mathrm{sp}_* E \to E$ is an isomorphism. Indeed, a coherent sheaf on an affinoid rigid space (resp. an affine formal scheme) is determined by its global sections, and for any affine open $\mathfrak{U} = \operatorname{Spf} A \subset \mathfrak{X}$, $\mathrm{sp}^{-1}(\mathfrak{U}) = \operatorname{Spm} A \otimes K$ is affinoid. For any coherent $\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}$-module $\mathcal{E}$, the $\mathcal{O}_{\mathfrak{X}_K}$-module $\mathrm{sp}^* E$ is coherent and the adjunction morphism $\mathcal{E} \to \mathrm{sp}_* \mathrm{sp}^* \mathcal{E}$ is an isomorphism. Moreover, the functors sp_* and sp^* induce quasi-inverse equivalences between locally free of finite type $\mathcal{O}_{\mathfrak{X}_K}$-modules and locally projective of finite type $\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}$-modules.

We get the locally free of finite type $\mathcal{O}_{\mathfrak{X}_K}$-module $\Omega^\bullet_{\mathfrak{X}^\sharp_K / \mathfrak{S}_K} := \mathrm{sp}^*(\mathcal{O}_{\mathfrak{X}, \mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^\bullet_{\mathfrak{X}^\sharp / \mathfrak{S}})$. From the morphism $d_{\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}} \colon \mathcal{O}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}^\sharp / \mathfrak{S}}$ (see notation 4.7.1.5.(c)), we get the map $d \colon \mathcal{O}_{\mathfrak{X}_K} \to \Omega^1_{\mathfrak{X}^\sharp_K / \mathfrak{S}_K}$.

Let V be an open of $\mathfrak{X}_K$. We set $\Omega^1_{V^\sharp/\mathfrak{S}_K} := \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|V$. Let E be an $\mathcal{O}_V$-module. A "logarithmic connection relative to $V^\sharp/\mathfrak{S}_K$" on E is an additive map $\nabla : E \to E \otimes_{\mathcal{O}_V} \Omega^1_{V^\sharp/\mathfrak{S}_K}$ such that for any open $W \subseteq V$, the map $\nabla_W : E(W) \to (E \otimes_{\mathcal{O}_V} \Omega^1_{V^\sharp/\mathfrak{S}_K})(W)$ satisfies the condition : for $x \in E(W)$, $a \in \mathcal{O}_V(W)$ we have

$$\nabla(ax) = a\nabla(x) + x \otimes d(a). \tag{10.3.2.1.1}$$

When $V = \mathfrak{X}_K$, the functors sp_* and sp^* induce quasi-inverse equivalences between coherent (resp. locally free of finite type) $\mathcal{O}_{\mathfrak{X}_K}$-modules together with a logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ and coherent (resp. locally projective of finite type) $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-modules together with a logarithmic connection relative to $\mathfrak{X}^\sharp/\mathfrak{S}$ with coefficients in $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ (compare 4.7.1.8 and 10.3.2.1.1). Via this correspondance, we say that a logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ on E is integrable if the connexion on $\mathrm{sp}_*(E, \nabla)$ is integrable.

10.3.2.2 (Integrable log-connections with overconvergent singularities)**.** Let E be a coherent $j^\dagger\mathcal{O}_{\mathfrak{X}_K}$-module. A "logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ with overconvergent singularities along T" on E is an additive map $\nabla : E \to E \otimes_{j^\dagger\mathcal{O}_{\mathfrak{X}_K}} j^\dagger\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}$ such that for any open $V \subseteq \mathfrak{X}_K$, the map $\nabla_V : E|V \to E|V \otimes_{j^\dagger_V\mathcal{O}_V} j^\dagger\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|V$ satisfies the condition : for $x \in E(V)$, $b \in j^\dagger\mathcal{O}_{\mathfrak{X}_K}(V)$ we have

$$\nabla(bx) = b\nabla(x) + x \otimes d(b), \tag{10.3.2.2.1}$$

where $d : j^\dagger\mathcal{O}_{\mathfrak{X}_K} \to j^\dagger\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}$ is the map induced by applying $j^\dagger$ to $d : \mathcal{O}_{\mathfrak{X}_K} \to \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}$

The functors sp_* and sp^* induce exact quasi-inverse equivalences between coherent (resp. locally free of finite type) $j^\dagger\mathcal{O}_{\mathfrak{X}_K}$-modules together with a logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ with overconvergent singularities along T and coherent (resp. locally projective of finite type) $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_\mathbb{Q}$-modules together with a logarithmic connection relative to $\mathfrak{X}^\sharp/\mathfrak{S}$ with coefficients in $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_\mathbb{Q}$ (use 10.1.2.3 and compare 4.7.1.8 and 10.3.2.1.1). Via this correspondance, we say that a logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ with overconvergent singularities along T on E is integrable if so is the connexion on $\mathrm{sp}_*(E, \nabla)$.

Definition 10.3.2.3 (Convergent connections: local context)**.** Suppose T is empty. Suppose $\mathfrak{S}$ is affine, $\mathfrak{X}$ is affine and endowed with nice coordinates $t_1, \dots, t_n$ (see definition 4.5.2.15) such that $\mathfrak{Z}$ is the zero locus of $t_1 \dots t_m$ on $\mathfrak{X}$ with $m \leq n$ and let us keep the notations of 4.5.2.18. The $\mathcal{O}_{\mathfrak{X}_K}$-module $\Omega^1_{\mathfrak{X}^\sharp_K/K}$ is free with the basis $d\log t_1, \dots, d\log t_m, dt_{m+1}, \dots, dt_n$.

Let (E, ∇) be a coherent $\mathcal{O}_{\mathfrak{X}_K}$-module E endowed with an integrable logarithmic connection ∇ relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$. The logarithmic connection $\nabla : E \to \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} E$ is "convergent" if it satisfies the following convergence condition: for any $\eta \in |K^\times|_\mathbb{Q} \cap]0, 1[$, for any $e \in \Gamma(\mathfrak{X}_K, E)$, we have

$$\| \underline{\partial}^{[\underline{k}]}_\sharp e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty, \tag{10.3.2.3.1}$$

with $\| \cdot \|$ a Banach norm on $\Gamma(\mathfrak{X}_K, \mathcal{E})$.

Definition 10.3.2.4 (Overconvergent connections: local context)**.** Suppose $\mathfrak{S}$ is affine, $\mathfrak{X}$ is affine and endowed with nice coordinates $t_1, \dots, t_n$ (see definition 4.5.2.15) such that $\mathfrak{Z}$ is the zero locus of $t_1 \dots t_m$ on $\mathfrak{X}$ with $m \leq n$ and let us keep the notations of 4.5.2.18. The $\mathcal{O}_{\mathfrak{X}_K}$-module $\Omega^1_{\mathfrak{X}^\sharp_K/K}$ is free with the basis $d\log t_1, \dots, d\log t_m, dt_{m+1}, \dots, dt_n$. We suppose T is a divisor which is defined by $f = 0$ in X for $f \in \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$. For any $0 \leq \lambda < 1$, set $Y_\lambda := \{x \in \mathfrak{X}_K \mid |f(x)| \geq \lambda\}$.

Let (E, ∇) be a coherent $j^\dagger\mathcal{O}_{\mathfrak{X}_K}$-module E endowed with an integrable logarithmic connection ∇ relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ with overconvergent singularities along T. The logarithmic connection $\nabla : E \to j^\dagger\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} E$ is "overconvergent" if there exist a strict neighborhood V of $]Y[_\mathfrak{X}$ in $]X[_\mathfrak{X}$ and a coherent $\mathcal{O}_V$-module $\mathcal{E}_V$ of finite type furnished with an integrable logarithmic connection $\nabla_V : \mathcal{E}_V \to (\Omega^1_{V^\sharp/\mathfrak{S}_K}) \otimes_{\mathcal{O}_V} \mathcal{E}_V$ such that $j^\dagger(\mathcal{E}_V, \nabla_V) = (E, \nabla)$, which satisfies the following overconvergence condition: for any $\eta \in |K^\times|_\mathbb{Q} \cap]0, 1[$, there exist $0 \leq \lambda_\eta < 1$ such that such that $Y_{\lambda_\eta} \subset V$ and such that for any $\lambda_\eta \leq \lambda < 1$ and each section $e \in \Gamma(Y_\lambda, E)$, we have

$$\| \underline{\partial}^{[\underline{k}]}_\sharp e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty, \tag{10.3.2.4.1}$$

with $\| \cdot \|$ a Banach norm on $\Gamma(Y_\lambda, \mathcal{E})$.

Remark that the convergent condition of 10.3.2.3 is simply the overconvergent condition in the case where the divisor T is empty.

Definition 10.3.2.5. Let (E, ∇) be a coherent $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-module endowed with an integrable logarithmic connection relative to $\mathfrak{X}^\sharp_K/K$ with overconvergent singularities along T. We say the logarithmic connection $\nabla : E \to j^\dagger \Omega^1_{\mathfrak{X}^\sharp_K/K} \otimes_{j^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E$ is "overconvergent over T" if for any affine open $\mathfrak{S}'$ of $\mathfrak{S}$ and $\mathfrak{X}'$ of $\mathfrak{X}$ such that $\mathfrak{X}' \subset f^{-1}(\mathfrak{S}')$ and $T \cap X'$ is defined by the vanishing of a section of $\mathcal{O}_X(X')$, the connection $\nabla | \mathfrak{X}'_K$ is overconvergent in the sense of 10.3.2.4.

A "locally free log-isocrystal on $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ overconvergent along T" is by definition a coherent $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-module endowed with an overconvergent along T logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ with overconvergent singularities along T.

We denote by $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}_K)$ the category of log-isocrystals on $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ overconvergent along T. When T is empty, we simply write $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K/\mathfrak{S}_K)$ and its objects are called "convergent log-isocrystal on $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$". An object (E, ∇) of $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}_K)$ is said to be locally free (of finite type) if E is locally free.

When T is empty, we say the connection or the log-isocrystal is "convergent" to say it is overconvergent along the empty set.

Example 10.3.2.6. We keep notation 10.3.2.5.

(a) It follows from [Ked07, 6.3.4 and 6.4.1] (see definitions [Ked07, 2.3.7 and 6.3.1]), when T is empty and when $\mathfrak{S} = \operatorname{Spf} \mathcal{V}$, the full subcategory of $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K/\mathfrak{S}_K)$ consisting in locally free objects is equivalent to full subcategory of $I_{\mathrm{conv}}(X^\sharp/\operatorname{Spf} \mathcal{V})$ consisting in locally free isocrystals on the log convergent site $((X, M)/\operatorname{Spf} \mathcal{V})_{\mathrm{conv}}$ (see 10.3.1.1).

(b) When $\mathfrak{Z}$ is empty, the category $\mathrm{MIC}^\dagger(\mathfrak{X}_K, T/\mathfrak{S}_K)$ is equivalent to the category $\mathrm{MIC}^\dagger(Y, X, \mathfrak{X}/\mathfrak{S}_K)$ of coherent $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-module endowed with a (relative to $(Y, X, \mathfrak{X}/\mathfrak{S})$) overconvergent connection (indeed, the overconvergent conditions of 10.2.2.4 and 10.3.2.4 are the same).

Proposition 10.3.2.7. *Suppose T is empty. Let (E, ∇) be a locally free of finite type $\mathcal{O}_{\mathfrak{X}_K}$-module endowed with an integrable logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$. The logarithmic connection $\nabla : E \to \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{\mathcal{O}_{\mathfrak{X}_K}} E$ is convergent if and only if $\nabla | \mathfrak{X}^*_K : E | \mathfrak{X}^*_K \to \Omega^1_{\mathfrak{X}^*_K/\mathfrak{S}_K} \otimes_{\mathcal{O}_{\mathfrak{X}^*_K}} E$ is convergent in the sense of 10.2.2.5.*

Proof. If ∇ is convergent, then this is obvious that so is $\nabla | \mathfrak{X}^*_K$. Let us check the converse. Since this is local, we can suppose $\mathfrak{S}$ is affine, $\mathfrak{X}$ is affine and endowed with nice coordinates $t_1, \dots, t_n$ (see definition 4.5.2.15) such that $\mathfrak{Z}$ is the zero locus of $t_1 \dots t_m$ on $\mathfrak{X}$ with $m \leq n$ and let us keep the notations of 4.5.2.18. The $\mathcal{O}_{\mathfrak{X}_K}$-module $\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}$ is free with the basis $d \log t_1, \dots, d \log t_m, dt_{m+1}, \dots, dt_n$. Let $\eta \in |K^\times|_\mathbb{Q} \cap]0, 1[$. By hypotheses, the connection $\nabla | \mathfrak{X}^*_K : \mathcal{E} | \mathfrak{X}^*_K \to \Omega^1_{\mathfrak{X}^*_K/\mathfrak{S}_K} \otimes_{\mathcal{O}_{\mathfrak{X}^*_K}} \mathcal{E} | \mathfrak{X}^*_K$ satisfies the condition 10.3.2.3.1, i.e., for any $e \in \Gamma(\mathfrak{X}^*_K, \mathcal{E})$, we have

$$\| \underline{\partial}^{[\underline{k}]} e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty, \tag{10.3.2.7.1}$$

with $\| \cdot \|$ a Banach norm on $\Gamma(\mathfrak{X}^*_K, \mathcal{E})$. Since the logarithmic coordinates $t_1, \dots, t_n$ are invertible on $\mathfrak{X}^*$, then it follows from 3.2.3.9.4 that we can replace $\| \underline{\partial}^{[\underline{k}]} e \|$ by $\| \underline{\partial}^{[\underline{k}]}_\sharp e \|$ in the convergent condition 10.3.2.7.1. Since X^* is dense in X, this yields that for any $e \in \Gamma(\mathfrak{X}_K, \mathcal{E})$, we have

$$\| \underline{\partial}^{[\underline{k}]}_\sharp e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty, \tag{10.3.2.7.2}$$

with $\| \cdot \|$ a Banach norm on $\Gamma(\mathfrak{X}_K, \mathcal{E})$. □

Remark 10.3.2.8. Beware that the case where T is not empty, the proposition 10.3.2.7 seems false (e.g. take $T = Z$).

10.3.2.9 (Local computations, residues, exponents)**.** Suppose we are in the local situation 10.3.2.4. Let E be a locally free log-isocrystal on $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ overconvergent along T. Let $\mathfrak{Z}_i = V(t_i)$ be an irreducible component of $\mathfrak{Z}$ which is not included in T. For any $i = 1, \ldots, m$, by composition ∇ with the projection $E \otimes_{j^\dagger \mathcal{O}_{\mathfrak{X}_K}} j^\dagger \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \to E \otimes_{j^\dagger \mathcal{O}_{\mathfrak{X}_K}} d \log t_i j^\dagger \mathcal{O}_{\mathfrak{X}_K} \xrightarrow{\sim} E$ we get the map $\partial_{\sharp i} \colon E \to E$ given by $x \mapsto \partial_{\sharp i}(x)$. Since $\partial_{\sharp i} t_i = t_i \partial_{\sharp i} + t_i$, then the map $\partial_{\sharp i} \colon E \to E$ preserves $t_i E$.

(a) The "residue of ∇ along $V(t_i)$" is the map $\mathrm{res}_i \colon E/t_iE \to E/t_iE$, given by the reduction modulo t_i of the map $\partial_{\sharp i} \colon E \to E$ corresponding to the action of $\partial_{\sharp i}$.

(b) The eigenvalues of the residue of ∇ along $\mathfrak{Z}_{iK}$, i.e., the eigenvalues of the matrix of res_i contained in an algebraic closure of the field of fractions of $\Gamma(\mathfrak{Z}_{iK}, j^\dagger_{Z_i \cap Y} \mathcal{O}_{\mathfrak{Z}_{iK}})$, are called "exponent of E along Z_i".

(c) We say "E has nilpotent residues" if the residue of ∇ along each irreducible component of $\mathfrak{Z}$ which is not contained in T is nilpotent. An "exponent of E" is an exponent of E along some irreducible component of $\mathfrak{Z}$ which is not contained in T. Any exponent is contained in $\mathbb{Z}_p$ by 10.3.2.4.1.

Definition 10.3.2.10. Let X be a smooth scheme over $\operatorname{Spec} k$, let Z be a strict normal crossing divisor of X, $X^\sharp = (X, M_Z)$ be the logarithmic $\operatorname{Spec} k$-scheme with the logarithmic structure associated to Z. Let E be a locally free convergent log-isocrystal of $I_{\mathrm{conv}}(X^\sharp/\operatorname{Spf} \mathcal{V})$.

(a) Suppose there exists a lifting of X (resp. Z) in a smooth over $\operatorname{Spf} \mathcal{V}$ formal scheme $\mathfrak{X}$ (resp. $\mathfrak{Z}$ in a relative strict normal crossing divisor of $\mathfrak{X}$ over $\operatorname{Spf} \mathcal{V}$. Let $\mathfrak{X}^\sharp = (\mathfrak{X}, M(\mathfrak{Z}))$ be the logarithmic formal $\mathcal{V}$-scheme (see 4.5.2.14). We say that E has nilpotent residues if so is the corresponding object (via the equivalence 10.3.2.6.(a)) of $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K/K)$ in the sense of 10.3.2.9. Moreover, an "exponent of E" is an exponent of the corresponding object of $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K/K)$ in the sense of 10.3.2.9.

(b) Locally, such liftings of X (resp. Z) exist. We say that E has nilpotent residues, if E has locally nilpotent residues. An exponent of E is an exponent of the restriction of E on some open subset (where it has a meaning).

10.3.3 Kedlaya's semistable reduction theorem

We recall the following Kedlaya's definitions (see [Ked08, 3.2.1, 3.2.4]):

Definition 10.3.3.1. Let X be a smooth irreducible variety over $\operatorname{Spec} k$, Z be a strict normal crossing divisor of X, M_Z the induced log-structure and let E be a convergent isocrystal on $X \setminus Z$. We say that E is *log-extendable* on X if there exists a locally free convergent isocrystal of $I_{\mathrm{conv}}((X, M_Z)/\operatorname{Spf} \mathcal{V})$ having nilpotent residues (in the sense of 10.3.2.10) whose induced convergent isocrystal on $X \setminus Z$ is E.

Definition 10.3.3.2. Let Y be a smooth irreducible variety over $\operatorname{Spec} k$, let X be a partial compactification of Y, and let E be an F-isocrystal on Y overconvergent along $X \setminus Y$. We say that E *admits semistable reduction* if there exists

(a) a proper, surjective, generically étale morphism $f \colon X_1 \to X$,

(b) an open immersion $X_1 \hookrightarrow \overline{X}_1$ into a smooth projective variety over k such that $T_1 := f^{-1}(X \setminus Y) \cup (\overline{X}_1 \setminus X_1)$ is a strict normal crossing divisor of $\overline{X}_1$

such that the isocrystal $f^*(E)$ on $Y_1 := f^{-1}(Y)$ overconvergent along $T_1 \cap X_1$ is log-extendable on X_1 (see 10.3.3.1).

With the previous definitions, Kedlaya has proved in [Ked11, 2.4.4] (see also [Ked07], [Ked08], [Ked09]) the following theorem which answers positively to Shiho's conjecture in [Shi02, 3.1.8]:

Theorem 10.3.3.3 (Kedlaya)**.** *Let Y be a smooth irreducible k-variety, X be a partial compactification of Y, $Z := X \setminus Y$, E be an F-isocrystal on Y overconvergent along Z. Then E admits semistable reduction.*

Remark 10.3.3.4. This conjecture was previously checked by Tsuzuki when E is unit-root in [Tsu02].

10.3.4 A comparison theorem between relative log-rigid cohomology and relative rigid cohomology

Let $g : \mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of smooth $\mathcal{V}$-formal schemes, of pure relative dimension d, let $\mathfrak{Z}$ be a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor, let $\mathfrak{X}^*$ be the complement of $\mathfrak{Z}$ in $\mathfrak{X}$, let D be a closed subscheme of X and $\mathfrak{Y}$ the complement of D in $\mathfrak{X}$. Let $\mathfrak{X}^\sharp = (\mathfrak{X}, M(\mathfrak{Z}))$ be the logarithmic formal $\mathcal{V}$-scheme with the logarithmic structure associated to $\mathfrak{Z}$ (see 4.5.2.14), and $\mathfrak{Y}^\sharp$ the restriction of $\mathfrak{X}^\sharp$ on $\mathfrak{Y}$. If $\mathfrak{U}$ is an open of $\mathfrak{X}$, we denote by $j_U : U \hookrightarrow X$ the induced open immersion and $j_U^\dagger E$ the sheaf of germs of sections of a sheaf E on $\mathfrak{X}_K$ overconvergent along $X \setminus U$ (see 10.1.2.1).

Let $\mathcal{I}_{\mathfrak{Z}}$ be the sheaf of ideals of $\mathfrak{Z}$ in $\mathfrak{X}$. Since $\mathcal{I}_{\mathfrak{Z}}$ is invertible, $\mathcal{I}_{\mathfrak{Z},\mathbb{Q}}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp,\mathbb{Q}}$-module which is an invertible $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module. Hence, $I_{\mathfrak{Z},\mathbb{Q}} = \mathrm{sp}^*\mathcal{I}_{\mathfrak{Z},\mathbb{Q}}$ is a convergent isocrystal on X/K with logarithmic pole along Z. Let E be a locally free log-isocrystal on $Y^\sharp/\mathfrak{S}_K$ overconvergent along D. For an integer m, we put

$$E(m\mathfrak{Z}) = E \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} j_Y^\dagger I_{\mathfrak{Z},\mathbb{Q}}^{\otimes -m}.$$

$E(m\mathfrak{Z})$ is a locally free overconvergent log-isocrystal and the exponents of $E(m\mathfrak{Z})$ are the exponents of E minus m. Then there is a natural commutative diagram

$$\begin{array}{ccc} E & \overset{\subset}{\longrightarrow} & E(m\mathfrak{Z}) \\ =\downarrow & & \downarrow \\ E & \longrightarrow & j^\dagger_{X^*\cap Y} E \end{array} \tag{10.3.4.0.1}$$

for any nonnegative integer m.

Definition 10.3.4.1. A p-adic integer α is a "p-adic Liouville number" if the radius of convergence of formal power series, either $\sum_{n\in\mathbb{Z}_{\geq 0}, n\neq\alpha} x^n/(n-\alpha)$ or $\sum_{n\in\mathbb{Z}_{\geq 0}, n\neq -\alpha} x^n/(n+\alpha)$, is less than 1. Note that (1) a p-adic integer which is an algebraic number is not a p-adic Liouville number and (2) a p-adic integer α is a p-adic Liouville number if and only if so is $-\alpha$ (resp. $\alpha + m$ for any integer m). For p-adic Liouville numbers, we refer to [DGS94, VI, 1] and [BC92, 1.2].

Theorem 10.3.4.2. *With the above notation, let E be a locally free log-isocrystal on $\mathfrak{X}_K^\sharp/\mathfrak{S}_K$ overconvergent along D. Suppose that*

(a) *none of differences of exponents of E is a p-adic Liouville number, and*

(b) *none of exponents of E is a p-adic Liouville number*

along each irreducible component Z_i of Z such that $Z_i \not\subset D$. Let c be the nonnegative integer defined by

$c = \max\{e \,|\, e$ *is a positive integral exponent of E along some irreducible component Z_i of Z such that $Z_i \not\subset D$*$\} \cup \{0\}$

Then the diagram 10.3.4.0.1 induces an isomorphism

$$\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}_K^\sharp/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E) \cong \mathbb{R}g_{K*}\mathrm{Cone}\left(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}_K^\sharp/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E \to j_Y^\dagger \Omega^\bullet_{\mathfrak{X}_K^\sharp/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathcal{Z})\right)[-1] \tag{10.3.4.2.1}$$

for any $m \geq c$. In particular, if none of exponents along each irreducible component Z_i of Z such that $Z_i \not\subset D$ is a positive integer, then the restriction induces an isomorphism

$$\mathbb{R}g_{K*}(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}_K^\sharp/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E) \xrightarrow{\sim} \mathbb{R}g_{K*}(j^\dagger_{X^*\cap Y} \Omega^\bullet_{\mathfrak{X}_K/\mathfrak{S}_K} \otimes_{j^\dagger_{X^*\cap Y} \mathcal{O}_{]X[_\mathfrak{X}}} j^\dagger_{X^*\cap Y} E). \tag{10.3.4.2.2}$$

Remark 10.3.4.3. (a) In fact, we will see in 18.3.1.13 that the comparison homomorphism corresponding to 10.3.4.2.2 is an isomorphism on the formal scheme side without the functor $g_+^\sharp$. But the first step towards this result is to establish 10.3.4.2.

(b) Note that $j^\dagger_{X^*\cap Y} E$ is a locally free isocrystal on $X^* \cap Y/\mathfrak{S}_K$ overconvergent along $Z \cup D$ and the right handside of the isomorphism in the theorem above is a relative rigid cohomology with respect to the closed immersion $S \to \mathfrak{S}$. It is independent of the choice of $\mathfrak{X}$ which is smooth over $\mathfrak{S}$ around Y [CT03, sect. 10]. The left handside of 10.3.4.2.2 in the theorem above is regarded as a relative logarithmic rigid cohomology.

(c) This type of comparison theorem between p-adic cohomology with logarithmic pole and rigid cohomology was studied in [BC94, 3.1], [Tsu99, 3.5.1], [Shi02, 2.2.4 and 2.2.13] (see also the definition [Shi02, 2.1.5]) and [BB04, A.1]BC94. They suppose that an overconvergent isocrystal is locally free on the formal side or for [Shi02, 2.2.4 and 2.2.13] it concerns the absolute case. In the theorem above we relax this assumption and suppose that an overconvergent isocrystal is locally free only on the analytic side.

(d) One can also prove the comparison theorem in the case g is smooth around Y replacing 10.3.4.9 and 10.3.4.19 (the weak fibration theorem) by the strong forms (the strong fibration theorem) with modifications.

Remark 10.3.4.4. For a locally free log-isocrystal E on $\mathfrak{X}^{\sharp}_K/\mathfrak{S}_K$ overconvergent along D, we denote by $\mathrm{Exp}(E) \subset \mathbb{Z}_p$ (resp. $\mathrm{Exp}(E)^{\mathrm{gr}} \subset \mathbb{Z}_p$) the monoid (resp. abelian group) generated by all exponents along irreducible components Z_i of Z such that $Z_i \not\subset D$. $\mathrm{Exp}(E)$ and $\mathrm{Exp}(E)^{\mathrm{gr}}$ do not depend on the choice of local coordinates.

(a) Let $\mathfrak{X}^{\sharp} = (\mathfrak{X}, \mathfrak{Z})$ and $\mathfrak{X}'^{\sharp} = (\mathfrak{X}', \mathfrak{Z}')$ be smooth $\mathcal{V}$-formal schemes with relative strict normal crossing divisors over $\mathfrak{S}$, let $\mathfrak{Y}, D, \mathfrak{Y}^{\sharp}, \mathfrak{Y}', D', \mathfrak{Y}'^{\sharp}$ as above, and let $h : \mathfrak{X}' \to \mathfrak{X}$ be a morphism over $\mathfrak{S}$ such that $h^{-1}(D \cup Z) \subset D' \cup Z'$. Suppose that h induces a log-morphism $(h|_{\mathfrak{Y}'})^{\sharp} : \mathfrak{Y}'^{\sharp} \to \mathfrak{Y}^{\sharp}$. Then the inverse image $h^{\sharp *}_K E$ is a locally free log-isocrystal on $\mathfrak{X}'^{\sharp}_K/\mathfrak{S}_K$ overconvergent along D' because h_K induces a log-morphism of rigid analytic spaces between suitable strict neighborhoods by our assumption. Suppose furthermore that none of elements in $\mathrm{Exp}(E)$ (resp. $\mathrm{Exp}(E)^{\mathrm{gr}}$) is a p-adic Liouville number. Then the same holds for the inverse image $h^{\sharp *}_K E$. Indeed, for a suitable choice of local coordinates z_i $(1 \leq i \leq s)$ and z'_j $(1 \leq j \leq s')$ along the normal crossing divisors $\mathfrak{Z}$ and $\mathfrak{Z}'$ of $\mathfrak{X}$ and $\mathfrak{X}'$ respectively, we have $z_i = u_i {z'_1}^{m_{i1}} \cdots {z'_{s'}}^{m_{is'}}$ locally at a generic point of $\mathfrak{Z}'$. Here u_i is a unit of $\mathcal{O}_{\mathfrak{Y}'}$ and m_{ij} is a nonnegative integer. Since the residues of E with respect to Z_{i_1} and Z_{i_2} commute with each other by the integrability of the log-connection and $dz_i/z_i \equiv \sum_j m_{ij} dz'_j/z'_j \pmod{\Omega^1_{\mathfrak{Y}'/\mathfrak{S}}}$, $\mathrm{Exp}(h^{\sharp *}_K E)$ is a submonoid of $\mathrm{Exp}(E)$. (See [AB01, 6.2.5].)

Even if any exponent of E is not a positive integer, it might happen that some exponent of the inverse image $h^{\sharp *}_K E$ is a positive integer. Since $\mathrm{Exp}(E) \cap \mathbb{Q}_{\geq 0}$ is finitely generated as a monoid where $\mathbb{Q}_{\geq 0}$ is the monoid consisting of nonnegative rational numbers, $\mathrm{Exp}(E(m\mathfrak{Z}))$ does not contain any positive rational numbers for a sufficiently large integer m. Therefore, none of exponents of an arbitrary inverse image $h^{\sharp *}_K E(m\mathfrak{Z})$ is a positive integer.

(b) Let $h^{\sharp} : \mathfrak{X}'^{\sharp} \to \mathfrak{X}^{\sharp}$ be a log-morphism such that $h^{-1}(D) = D'$ and $h^{-1}(\mathfrak{Z}) = \mathfrak{Z}'$. Suppose that the underlying morphism h is finite étale. Note that local parameters of $\mathfrak{X}^{\sharp}$ becomes local parameters of $\mathfrak{X}'^{\sharp}$. Then, for a locally free log-isocrystal E' on $\mathfrak{X}'^{\sharp}_K/\mathfrak{S}_K$ overconvergent along D', $h^{\sharp}_{K*}E'$ is a locally free log-isocrystal on $\mathfrak{X}^{\sharp}_K/\mathfrak{S}_K$ overconvergent along D. Moreover, for an irreducible component Z_i of Z such that $Z_i \not\subset D$, the exponents of $h^{\sharp}_{K*}E'$ along Z_i coincide with the exponents of E' along $h^{-1}(Z)$ (including multiplicities). In particular, $\mathrm{Exp}(h^{\sharp}_{K*}E') = \mathrm{Exp}(E')$. (See [AB01, 6.5.4].) The first part easily follows from our geometric situation and we have $\mathrm{rank}_{j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}}} h^{\sharp}_{K*}E' = \deg(h)\mathrm{rank}_{j^{\dagger}\mathcal{O}_{]\overline{X}'[_{\mathfrak{X}'}}} E'$, where $\deg(h)$ is the degree of the underlying morphism of h. The second part is a problem only along the generic point of $\mathfrak{Z}_i$. We may assume that $\mathfrak{X}$ and $\mathfrak{X}'$ are affine, Z is irreducible and is not included in D. Let $\widehat{(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}})}_{\mathfrak{Z}}$ be the completion along $\mathfrak{Z}_K$. Then there is a natural K-algebra homomorphism from the ring of global sections of $\widehat{(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}})}_{\mathfrak{Z}}$ into $K(\mathfrak{Z})[[z]]$, where $K(\mathfrak{Z})$ is the field of fractions of $\Gamma(\mathfrak{Z}, j^{\dagger}\mathcal{O}_{\mathfrak{Z}})$ and z is a local coordinate of $\mathfrak{Z}$. This K-algebra homomorphism naturally extends to a K-algebra homomorphism from the ring of global sections of $\widehat{(j^{\dagger}\mathcal{O}_{]\overline{X}'[_{\mathfrak{X}'}})}_{\mathfrak{Z}'}$ into a direct sum of finite unramified extensions of $K(\mathfrak{Z})[[z]]$. We may replace the residue field $K(\mathfrak{Z})$ of $K(\mathfrak{Z})[[z]]$ by its algebraic closure $\overline{K(\mathfrak{Z})}$ since all exponents are contained in $\mathbb{Z}_p$ and invariant under any automorphism of $\overline{K(\mathfrak{Z})}$. Hence, $\widehat{(j^{\dagger}\mathcal{O}_{]\overline{X}'[_{\mathfrak{X}'}})}_{\mathfrak{Z}'}$ goes to a direct sum of $\deg(h)$ copies of $\overline{K(\mathfrak{Z})}[[z]]$. Now our second assertion is clear.

First we prove a special case.

Proposition 10.3.4.5. *Under the hypothesis in 10.3.4.2, suppose that $\mathfrak{Z}$ is irreducible such that $Z \not\subset D$, and that the composition $g \circ i : \mathfrak{Z} \to \mathfrak{S}$ of the closed immersion $i : \mathfrak{Z} \to \mathfrak{X}$ and $g : \mathfrak{X} \to \mathfrak{S}$ is an isomorphism. If we define $S \cap Y = Z \cap Y$ through the isomorphism $g \circ i : Z \to S$, then $g_{K*}\nabla : g_{K*}(E(m\mathfrak{Z})/E) \to g_{K*}(j_Y^\dagger \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathfrak{Z})/E)$ is a $j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}}$-homomorphism of locally free $j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}}$-modules of finite type and the natural morphism 10.3.4.2.1 induces an isomorphism*

$$\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E) \cong \left[g_{K*}(E(m\mathfrak{Z})/E) \overset{g_{K*}\nabla}{\longrightarrow} g_{K*}(j_Y^\dagger \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathfrak{Z})/E)\right][-1] \tag{10.3.4.5.1}$$

for any $m \geq c$ in the derived category of complexes of $j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}}$-modules. Here $[A \to B]$ means a complex consisting of the terms of degree 0 and degree 1.

We will see that, in 10.3.4.23, the overconvergence of the induced Gauss-Manin connection on $g_{K*}(E(m\mathfrak{Z})/E)$ in the relative case. An example such that the cokernel of $g_{K*}\nabla : g_{K*}(E(m\mathfrak{Z})/E) \to g_{K*}(j_Y^\dagger \Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathfrak{Z})/E)$ is not locally free is also given in 10.3.4.24.

Proof. At first we shall define $j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}}$-module structures we both sides of 10.3.4.5.1.

We shall prove that $\mathbb{R}^q g_{K*}(E(\mathfrak{Z})/E) = 0$ for $q \neq 0$ and the locally freeness of $g_{K*}(E(\mathfrak{Z})/E)$. Since $i^{-1}(X \setminus Y) = Z \setminus Y$ as underlying topological spaces, $i_K^* E(\mathfrak{Z}) = j^\dagger_{Z\cap Y}\mathcal{O}_{]Z[_\mathfrak{Z}} \otimes_{i_K^{-1} j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} i_K^{-1}E(\mathfrak{Z})$ is a locally free $j^\dagger_{Z\cap Y}\mathcal{O}_{]Z[_\mathfrak{Z}}$-module of finite type and the adjoint gives an isomorphism $i_{K*}i_K^*E(\mathfrak{Z}) \cong E(\mathfrak{Z})/E$. Because i is a closed immersion, $i_K :]Z[_\mathfrak{Z} \to]X[_\mathfrak{X}$ is an affinoid morphism. Hence $\mathbb{R}i_{K*}\mathcal{M} = i_{K*}\mathcal{M}$ for any coherent $j^\dagger_{Z\cap Y}\mathcal{O}_{]Z[_\mathfrak{Z}}$-module $\mathcal{M}$ by $i^{-1}(X \setminus Y) = Z \setminus Y$ [CT03, 5.2.2]. Since $g \circ i$ is an isomorphism, we have

$$\mathbb{R}g_{K*}(E(\mathfrak{Z})/E) = \mathbb{R}g_{K*}(i_{K*}i_K^*E(\mathfrak{Z})) = \mathbb{R}g_{K*}\mathbb{R}i_{K*}i_K^*E(\mathfrak{Z}) = \mathbb{R}(g \circ i)_{K*}i_K^*E(\mathfrak{Z}) = (g \circ i)_{K*}i_K^*E(\mathfrak{Z})$$

and the two assertions above. Therefore, we show, for $m \geq 0$, $\mathbb{R}^q g_{K*}(E(m\mathfrak{Z})/E) = 0$ for $q \neq 0$ and $g_{K*}(E(m\mathfrak{Z})/E)$ is a locally free $j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}}$-module of finite type by induction on m.

For a $j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}$-module $\mathcal{H}$, the $j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}$-module $\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \mathcal{H})$ is not a priori a $g_K^{-1}(j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}})$-module because $Y \subset g^{-1}(S \cap Y)$ might not hold. The following lemma says that $\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \mathcal{H})$ has a $g_K^{-1}(j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}})$-module structure. Hence the left hand side of 10.3.4.5.1 belongs to the derived category of complexes of $j^\dagger_{S\cap Y}\mathcal{O}_{]S[_\mathfrak{S}}$-modules.

Lemma 10.3.4.6. *Ynder the hypothesis in 10.3.4.5, let us put $Y' = g^{-1}(S \cap Y) \cap Y$. If $\mathcal{A}$ is a sheaf of rings on $]X[_\mathfrak{X}$, then the restriction morphism*

$$\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \mathcal{H}) \to \underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_{Y'}^\dagger \mathcal{H})$$

is an isomorphism for any $\mathcal{A}$-module $\mathcal{H}$.

Proof. Since $S \cap Y = Z \cap Y$ via $g \circ i$ and $X^* \cap Y = Y \setminus Z$, we have $(Z \cap Y) \subset Y'$ and $Y = Y' \cup (X^* \cap Y)$. Hence, the natural morphism $[j_Y^\dagger \mathcal{H} \to j^\dagger_{X^*\cap Y}\mathcal{H}] \to [j^\dagger_{Y'}\mathcal{H} \to j^\dagger_{X^*\cap Y'}\mathcal{H}]$ of complexes is an isomorphism by [Ber96b, 2.1.8]. □

We divide the proof of 10.3.4.5 into 7 parts.

0° *Reduce to the case where none of the exponents of E along $\mathfrak{Z}$ is a positive integer, that is, $c = 0$.*

Since the natural morphism $j^\dagger_{X^*\cap Y}E \to j^\dagger_{X^*\cap Y}E(m\mathfrak{Z})$ is an isomorphism, the natural morphism of complexes induces a triangle

$$\begin{array}{c} \mathbb{R}g_{K*}\mathrm{Cone}\left(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E \to j_Y^\dagger \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathfrak{Z})\right)[-1] \\ \swarrow \qquad\qquad \nwarrow +1 \\ \mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E) \to \mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathfrak{Z})) \end{array}$$

for any $m \geq 0$. If we prove the vanishing $\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_{Y}\Omega^{\bullet}_{\mathfrak{X}^{\sharp}_{K}/\mathfrak{S}_{K}} \otimes_{j^{\dagger}_{Y}\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$ for $c = 0$, then, for any c, the triangle above induces the desired isomorphism when $m \geq c$. Hence, we may assume $m = c = 0$ and we shall prove the vanishing.

1° *Local problem on X and Y.*

By the Čech spectral sequences associated to a finite open covering $\{\mathfrak{X}_i\}$ of $\mathfrak{X}$ (resp. a finite open covering $\{\mathfrak{Y}_{ij}\}$ of each $\mathfrak{X}_i \cap \mathfrak{Y}$) 10.1.4.2 [CT03, 8.3.3], the vanishing is local on X and Y. Since the vanishing of $\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_{Y}\Omega^{\bullet}_{\mathfrak{X}^{\sharp}_{K}/\mathfrak{S}_{K}} \otimes_{j^{\dagger}_{Y}\mathcal{O}_{]X[_{\mathfrak{X}}}} E)$ is trivial in the case where $Z = \emptyset$, we may assume that $\mathfrak{X}$ is affine, D is defined by a single equation $f = 0$ in X for some $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and there is a coordinate z of $\mathfrak{X}$ over $\mathfrak{S}$ such that $\mathfrak{Z}$ is defined by $z = 0$ in $\mathfrak{X}$. Indeed, it is enough to take a certain covering consisting of $\mathfrak{X} \setminus \mathfrak{Z}$ and a covering of $\mathfrak{Z}$.

2° *Reduction to the local case by rigid analytic geometry.*

Let us add some notation. Let us put $]Y[_{\mathfrak{X},\lambda}= \{x \in]X[_{\mathfrak{X}} \,|\, |f(x)| \geq \lambda\}$ (resp. $]X^*[_{\mathfrak{X},\lambda}= \{x \in]X[_{\mathfrak{X}} \,|\, |z(x)| \geq \lambda\}$, resp. $]Z \cap Y[_{\mathfrak{Z},\lambda}= \{x \in]Z[_{\mathfrak{Z}} \,|\, |\overline{f}(x)| \geq \lambda\}$, resp. $[Z]_{\mathfrak{X},\lambda} = \{x \in]Z[_{\mathfrak{X}} \,|\, |z(x)| \leq \lambda\}$) for $\lambda \in |K^{\times}|_{\mathbb{Q}} \cap]0, 1[$, where $\overline{f}$ is the reduction of f in $\Gamma(\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}})$. We define $]S \cap Y[_{\mathfrak{S},\lambda}=]Z \cap Y[_{\mathfrak{Z},\lambda}$ by the identification through $g \circ i$. Note that the set $\{]Y[_{\mathfrak{X},\lambda}\}_{\lambda \in |K^{\times}|_{\mathbb{Q}} \cap]0,1[}$ forms a fundamental system of strict neighborhoods of $]Y[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$. Let $\alpha_V : V \to]X[_{\mathfrak{X}}$ denote the canonical morphism for admissible open subsets V in $]X[_{\mathfrak{X}}$.

Take $\nu \in |K^{\times}|_{\mathbb{Q}} \cap]0, 1[$ such that there is a locally free $\mathcal{O}_{]Y[_{\mathfrak{X},\nu}}$-module $\mathcal{E}$ of finite type endowed with a logarithmic connection $\nabla : \mathcal{E} \to (\Omega^1_{\mathfrak{X}^{\sharp}_{K}/\mathfrak{S}_{K}}|_{]Y[_{\mathfrak{X},\nu}}) \otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\nu}}} \mathcal{E}$ which satisfies the overconvergence condition 10.3.2.4.1. Hence, there exist a strictly increasing sequence $\underline{\xi} = (\xi_l)$ in $|K^{\times}|_{\mathbb{Q}} \cap]0, 1[$ with $\xi_l \to 1^-$ as $l \to \infty$ and an increasing sequence $\underline{\lambda} = (\lambda_l)$ in $|K^{\times}|_{\mathbb{Q}} \cap [\nu, 1[$ such that, for any l,

$$||\partial^{[n]}_{\sharp}(e)||\xi^n_l \to 0 \ \text{(as } n \to \infty) \tag{10.3.4.6.1}$$

for any section $e \in \Gamma(]Y[_{\mathfrak{X},\lambda_l}, \mathcal{E})$. Here $\partial_{\sharp} = \nabla(z\frac{d}{dz})$ and $\partial^{[l]}_{\sharp} = \frac{1}{l!}\prod_{j=0}^{l-1}(\partial_{\sharp} - j)$.

Let $\mathcal{A}$ be a sheaf of rings on $]X[_{\mathfrak{X}}$. Let $\eta \in |K^{\times}|_{\mathbb{Q}} \cap]0, 1[$. We define a functor $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}$ from the category of $\mathcal{A}$-modules to itself by the exact sequence

$$0 \longrightarrow \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow \lim_{\mu \to \eta^-} \alpha_{]X^*[_{\mathfrak{X},\mu}*}(\mathcal{H}|_{]X^*[_{\mathfrak{X},\mu}}) \longrightarrow 0 \tag{10.3.4.6.2}$$

for any $\mathcal{A}$-module $\mathcal{H}$. Here the morphism $\mathcal{H} \to \lim_{\mu \to \eta^-} \alpha_{]X^*[_{\mathfrak{X},\mu}*}(\mathcal{H}|_{]X^*[_{\mathfrak{X},\mu}})$ is an epimorphism for the same reason than for the epimorphism $\mathcal{H} \to j^{\dagger}_{X^*}\mathcal{H}$. One can easily see that $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\mathcal{H})|_{]X^*[_{\mathfrak{X},\eta}} = 0$ and $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}$ is an exact functor by the snake lemma. For $\xi \in |K^{\times}|_{\mathbb{Q}} \cap [\eta, 1[$, the restriction induces a morphism

$$\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\mathcal{H}) \to \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\xi}(\mathcal{H})$$

of $\mathcal{A}$-modules. By definition we have

Proposition 10.3.4.7. *With the same notation as above, the inductive system induces an isomorphism*

$$\lim_{\eta \to 1^-} \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\mathcal{H}) \cong \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(\mathcal{H}).$$

Proposition 10.3.4.8. *Let $\lambda \in |K^{\times}|_{\mathbb{Q}} \cap]0, 1[$.*

(a) The functor $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}$ commutes with direct limits. Also, for any $\mathcal{A}$-module $\mathcal{H}$, the natural morphism

$$\alpha_{]Y[_{\mathfrak{X},\lambda}*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\mathcal{H})|_{]Y[_{\mathfrak{X},\lambda}}) \to \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}(\mathcal{H}|_{]Y[_{\mathfrak{X},\lambda}}))$$

is an isomorphism. Moreover, $j^{\dagger}_{Y}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta} = \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}j^{\dagger}_{Y}$.

(b) For any coherent $\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}$-module $\mathcal{H}_{\lambda}$ and any $q \geq 1$ we have $\mathbb{R}^q\alpha_{]Y[_{\mathfrak{X},\lambda}*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}},\eta}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}\mathcal{H}_{\lambda})|_{]Y[_{\mathfrak{X},\lambda}}) = 0$.

Proof. (1) Since the morphism $\alpha_{]X^*[_{\mathfrak{X},\mu}}$ is quasi-compact and quasi-separated, we obtain from 10.3.4.6.2 the first assertion. By applying the functor $\alpha_{]Y[_{\mathfrak{X},\lambda}*}\alpha^{-1}_{]Y[_{\mathfrak{X},\lambda}}$ to the exact sequence 10.3.4.6.2, we get the sequence

$$0 \longrightarrow \alpha_{]Y[_{\mathfrak{X},\lambda}*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H})|_{]Y[_{\mathfrak{X},\lambda}}) \longrightarrow \alpha_{]Y[_{\mathfrak{X},\lambda}*}(\mathcal{H}|_{]Y[_{\mathfrak{X},\lambda}}) \longrightarrow \alpha_{]Y[_{\mathfrak{X},\lambda}*}\left(\left(\varinjlim_{\mu\to\eta^-}\alpha_{]X^*[_{\mathfrak{X},\mu}*}(\mathcal{H}|_{]X^*[_{\mathfrak{X},\mu}})\right)|_{]Y[_{\mathfrak{X},\lambda}}\right) \longrightarrow 0,$$

which is exact by a similar proof to that of [Ber96b, 2.1.3.(i)]. The quasi-compactness and quasi-separateness of $\alpha_{]Y[_{\mathfrak{X},\lambda}}$ implies the assertions.

(2) Because $\mathcal{H}_\lambda$ is a coherent $\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}$-module and both $]Y[_{\mathfrak{X},\lambda}$ and $]X^*[_{\mathfrak{X},\mu}$ are affinoid subdomains of the affinoid $]X[_{\mathfrak{X}}$, $\mathbb{R}^q\alpha_{]Y[_{\mathfrak{X},\lambda}*}(\mathcal{H}_\lambda) = 0$ and $\mathbb{R}^q\alpha_{]Y[_{\mathfrak{X},\lambda}*}\left(\left(\varinjlim_{\mu\to\eta^-}\alpha_{]X^*[_{\mathfrak{X},\mu}*}(\mathcal{H}_\lambda|_{]X^*[_{\mathfrak{X},\mu}})\right)|_{]Y[_{\mathfrak{X},\lambda}}\right) = 0$ for $q \geq 1$ by Kieh the Theorem B [Kie67, 2.4]. These facts and the exactness of the sequence in the proof of (1) imply the vanishing of higher direct images. □

Since g_K is an affinoid morphism, it is quasi-compact and $\mathbb{R}g_{K*}$ commutes with direct limits [Ber96b, 0.1.8]. Hence we have

$$\begin{aligned}
&\mathbb{R}^q g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_Y\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}\otimes_{j^{\dagger}_Y\mathcal{O}_{]X[_{\mathfrak{X}}}} E)\\
&\qquad\cong \mathbb{R}^q g_{K*}\left(\varinjlim_{\eta\to 1^-}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(j^{\dagger}_Y(\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}\otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}}\alpha_{]Y[_{\mathfrak{X},\nu}*}\mathcal{E}))\right)\\
&\qquad\cong \varinjlim_{\eta\to 1^-}\mathbb{R}^q g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}\left(\varinjlim_{\lambda\to 1^-}\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}})\right)\\
&\qquad\cong \varinjlim_{\eta\to 1^-}\varinjlim_{\lambda\to 1^-}\mathbb{R}^q g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))\\
&\qquad\cong \varinjlim_{\eta,\lambda\to 1^-}\mathbb{R}^q g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))
\end{aligned}$$

for any q. Indeed, the first isomorphism follows from 10.3.4.7 and the other ones from the commutation of the functors $\mathbb{R}g_{K*}$ and $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}$ (by 10.3.4.8) with direct limits. We will consider the family of open subsets indexed by the directed set

$$\Lambda_{\underline{\xi},\underline{\lambda}} = \left\{(\lambda,\eta)\in\left(|K^\times|_{\mathbb{Q}}\cap]0,1[\right)^2 \;\middle|\; \begin{array}{l}\lambda>\eta, \lambda\geq\max\{\lambda_l,\nu\},\\ \eta<\xi_l \text{ for some } l\end{array}\right\}. \qquad (10.3.4.8.1)$$

Here the condition $\lambda\rangle\eta$ comes from 10.3.4.9 (2). This family is cofinal for $\eta,\lambda\to 1^-$, so that the limit with respect to $\Lambda_{\underline{\xi},\underline{\lambda}}$ is the same as the original one.

Let $g_\lambda:]Y[_{\mathfrak{X},\lambda}\to]S[_{\mathfrak{S}}$ and $g_{\lambda,\eta}:]Y[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}\to]S[_{\mathfrak{S}}$ denote the restrictions of g for $(\lambda,\eta)\in\Lambda_{\underline{\xi},\underline{\lambda}}$. Then

$$\begin{aligned}
&\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))\\
&\qquad\cong \mathbb{R}g_{\lambda*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))|_{]Y[_{\mathfrak{X},\lambda}})
\end{aligned}$$

by 10.3.4.8. Since $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E})|_{]X^*[_{\mathfrak{X},\eta}} = 0$ and $\{]Y[_{\mathfrak{X},\lambda}\cap]X^*[_{\mathfrak{X},\eta},]Y[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}\}$ is an admissible covering of $]Y[_{\mathfrak{X},\lambda}$, we have

$$\begin{aligned}
&\mathbb{R}g_{\lambda*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))|_{]Y[_{\mathfrak{X},\lambda}})\\
&\qquad\cong \mathbb{R}g_{\lambda,\eta*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))|_{]Y[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}}).
\end{aligned}$$

Hence, in order to prove the vanishing $\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_Y\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}\otimes_{j^{\dagger}_Y\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$, we have only to prove the vanishing

$$\mathbb{R}g_{\lambda,\eta*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]Y[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]Y[_{\mathfrak{X},\lambda}}))|_{]Y[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}}) = 0 \qquad (10.3.4.8.2)$$

for any $(\lambda,\eta)\in\Lambda_{\underline{\xi},\underline{\lambda}}$.

3° *Reduce to the local computations.*

Let us denote the 1-dimensional open (resp. closed) unit disk over $\operatorname{Spm} K$ of radius $\eta\in|K^\times|_{\mathbb{Q}}$ by $D(0,\eta^-)$ (resp. $D(0,\eta^+)$). Since $Z\not\subset D$, we have the lemma below by the weak fibration theorem [Ber96b, 1.3.1, 1.3.2] (see also [BC94, 4.3].).

Lemma 10.3.4.9. *With the notation as above, we have*

(a) There is an admissible covering $\{V_\beta\}_\beta$ *of* $]S[_{\mathfrak{S}}$ *such that there exists an isomorphism*

$$g_K^{-1}(V_\beta)\cap]Z[_{\mathfrak{X}} \cong V_\beta \times_{\mathrm{Spm}\,K} D(0,1^-)$$

of rigid analytic K-vector spaces, under which the coordinate of $D(0,1^-)$ *is* z *as above.*

(b) Under the isomorphism in (1),

$$g_{\lambda,\eta}^{-1}(V_\beta) \;\cong\; (V_\beta\cap]S\cap Y[_{\mathfrak{S},\lambda}) \times_{\mathrm{Spm}\,K} D(0,\eta^+)$$

for any $\lambda,\eta \in |K^\times|_{\mathbb{Q}}\cap]0,1[$ *with* $\lambda\rangle\eta$.

In order to prove 10.3.4.9 (2), the condition $\lambda\rangle\eta$ is needed because of using $\overline{f}$ for the definition of $]S\cap Y[_{\mathfrak{S},\lambda}$.

Let $S = \mathrm{Spm}\,R$ be an integral smooth K-affinoid subdomain of $V_\beta\cap]S\cap Y[_{\mathfrak{S},\lambda}$ with a complete K-algebra norm $|\text{-}|_R$ on R. Since R is an integral K-Banach algebra, all complete K-algebra norms are equivalent [BGR84, 3.8.2, Cor. 4]. In order to prove the vanishing 10.3.4.8.2, it is sufficient to prove the vanishing

$$\mathbb{R}\Gamma\left(g_{\lambda,\eta}^{-1}(S),\, \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}},\eta}\left(\left[\mathcal{E} \xrightarrow{\nabla} (\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\nu}}) \otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\nu}}} \mathcal{E}\right]\right)\right) = \mathbb{R}\Gamma\left(g_{\lambda,\eta}^{-1}(S),\, \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}},\eta}\left(\left[\mathcal{E} \xrightarrow{\partial_\sharp} \mathcal{E}\right]\right)\right) = 0$$

of hypercohomology for any such S by 10.3.4.9 (2) since $]S[_{\mathfrak{S}}=]Z[_{\mathfrak{Z}}$ is integral and smooth and $\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}$ is a free $\mathcal{O}_{]X[_{\mathfrak{X}}}$-module of rank 1 generated by $\frac{dz}{z}$. The hypercohomology above can be computation ated by

$$\mathbb{R}^q\Gamma\left(g_{\lambda,\eta}^{-1}(S), \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}},\eta}\left(\left[\mathcal{E} \xrightarrow{\partial_\sharp} \mathcal{E}\right]\right)\right) \cong H^q\left(\mathrm{Tot}\begin{bmatrix} \Gamma(g_{\lambda,\eta}^{-1}(S),\mathcal{E}) & \to & \lim\limits_{\mu\to\eta^-} \Gamma(g_{\lambda,\eta}^{-1}(S)\cap]X^*[_{\mathfrak{X},\mu},\mathcal{E}) \\ \partial_\sharp\downarrow & & \downarrow\partial_\sharp \\ \Gamma(g_{\lambda,\eta}^{-1}(S),\mathcal{E}) & \to & \lim\limits_{\mu\to\eta^-} \Gamma(g_{\lambda,\eta}^{-1}(S)\cap]X^*[_{\mathfrak{X},\mu},*\mathcal{E}) \end{bmatrix}\right).$$

Here Tot means the total complex induced by the commutative bicomplex, the left top item in the bicomplex is located at degree $(0,0)$ and the horizontal arrows in the bicomplex are the natural injections. Indeed, the cohomological functor commutes with filtered direct limits since $g_{\lambda,\eta}$ is an affinoid morphism, and the vanishings $H^q(g_{\lambda,\eta}^{-1}(S),\mathcal{E}) = 0$ and $H^q(g_{\lambda,\eta}^{-1}(S)\cap]X^*[_{\mathfrak{X},\mu},\mathcal{E}) = 0$ for $q \geq 1$ hold by Kieh the Theorem B [Kie67, 2.4] since $g_{\lambda,\eta}^{-1}(S)$ and $g_{\lambda,\eta}^{-1}(S)\cap]X^*[_{\mathfrak{X},\mu}$ are affinoid.

More explicitly, the following formula 10.3.4.9.1 holds when $\mathcal{E}|_{g_{\lambda,\eta}^{-1}(S)}$ is a free $\mathcal{O}_{g_{\lambda,\eta}^{-1}(S)}$-module of rank r. We will prove the freeness in the next step 4°. Put R-algebras

$$\begin{aligned}
\mathcal{A}_R(\eta) &= \Gamma(g_{\lambda,\eta}^{-1}(S),\mathcal{O}_{]X[_{\mathfrak{X}}}) = \left\{\sum_{n=0}^\infty a_n z^n \,\middle|\, a_n\in R,\ |a_n|_R\eta^n \to 0 \text{ as } n\to\infty\right\} \\
\mathcal{A}_R(\eta^-) &= \Gamma\left(\bigcup_{\mu<\eta} g_{\lambda,\mu}^{-1}(S),\mathcal{O}_{]X[_{\mathfrak{X}}}\right) = \left\{\sum_{n=0}^\infty a_n z^n \,\middle|\, a_n\in R,\ |a_n|_R\mu^n \to 0 \text{ as } n\to\infty \text{ for any } \mu<\eta\right\} \\
\mathcal{R}_R(\eta) &= \lim_{\mu\to\eta^-} \Gamma(g_{\lambda,\eta}^{-1}(S), \alpha_{]X^*[_{\mathfrak{X},\mu}*}\mathcal{O}_{]X^*[_{\mathfrak{X}.\mu}}) \\
&= \left\{\sum_{n=-\infty}^\infty a_n z^n \,\middle|\, a_n\in R,\ \begin{array}{l}|a_n|_R\eta^n \to 0 \text{ as } n\to\infty \\ |a_n|_R\mu^n \to 0 \text{ as } n\to-\infty \text{ for some } \mu<\eta\end{array}\right\},
\end{aligned}$$

and define a norm on $\mathcal{A}_R(\eta)$ by $|\sum_n a_n z^n|_{\mathcal{A}_R(\eta)} = \sup_n |a_n|\eta^n$. $\mathcal{A}_R(\eta), \mathcal{A}_R(\eta^-)$ and $\mathcal{R}_R(\eta)$ are independent of the choice of complete K-algebra norms on R since there exist positive real numbers ρ_1 and ρ_2 such that $\rho_1|\text{-}| \leq |\text{-}|' \leq \rho_2|\text{-}|$ for equivalent norms $|\text{-}|$ and $|\text{-}|'$ by [BGR84, 2.1.8, Cor. 4]. Let $\underline{v}$ be a basis of vectors of $\Gamma(g_{\lambda,\eta}^{-1}(S),\mathcal{E})$ over $\mathcal{A}_R(\eta)$ such that the derivation along z is given by $\partial_\sharp(\underline{v}) = \underline{v}G$ for a matrix G with entries in $\mathcal{A}_R(\eta)$. Then we have

$$\begin{aligned}
\mathbb{R}^q\Gamma\left(g_{\lambda,\eta}^{-1}(S), \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}},\eta}\left(\left[\mathcal{E} \xrightarrow{\partial_\sharp} \mathcal{E}\right]\right)\right) &\cong H^q\left(\mathrm{Tot}\begin{bmatrix} \mathcal{A}_R(\eta)^r & \to & \mathcal{R}_R(\eta)^r \\ \partial_\sharp + G\downarrow & & \downarrow \partial_\sharp + G \\ \mathcal{A}_R(\eta)^r & \to & \mathcal{R}_R(\eta)^r \end{bmatrix}\right) \\
&\cong H^q\left(\left[(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r \xrightarrow{\partial_\sharp+G} (\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r\right][-1]\right).
\end{aligned} \tag{10.3.4.9.1}$$

4° *Local classification of logarithmic connections along a smooth divisor.*

Proposition 10.3.4.10. *Let $S = \operatorname{Spm} R$ be a smooth integral K-affinoid variety, and let $W = S \times_{\operatorname{Spm} K} D(0, \xi^-)$ be a quasi-Stein space over S for some $\xi \in |K^\times|_{\mathbb{Q}} \cap]0, 1]$. Let $\mathcal{M}$ be a locally free $\mathcal{O}_W$-module of finite type furnished with an R-derivation $\partial_\sharp = z\frac{d}{dz} : M \to M$, where $M = \Gamma(W, \mathcal{M})$, such that*

(i) *for any $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$, if $W_\eta = S \times_{\operatorname{Spm} K} D(0, \eta^+)$ is an affinoid subdomain of W and if $||\text{-}||$ is a Banach $\mathcal{A}_R(\eta)$-norm on $M_\eta = \Gamma(W_\eta, \mathcal{M})$, then $||\frac{1}{n!}\prod_{j=0}^{n-1}(\partial_\sharp - j)(e)||\mu^n \to 0\,(n \to \infty)$ for any $e \in M_\eta$ and $0 < \mu < 1$, and*

(ii) *any difference of exponents of $(\mathcal{M}, \partial_\sharp)$ along $z = 0$ is neither a p-adic Liouville number nor a non-zero integer.*

Then there are a projective R-module L of finite type furnished with a linear R-operator $N : L \to L$ such that $||\frac{1}{n!}\prod_{j=0}^{n-1}(N - j)(e)||\mu^n \to 0\,(n \to \infty)$ for any $e \in L$ and $0 < \mu < 1$, where $||\text{-}||$ is a Banach R-norm on L, and an isomorphism $(\mathcal{M}, \partial_\sharp) \cong (\mathcal{O}_W \otimes_R L, \partial_{\sharp N})$ in which the R-derivation $\partial_{\sharp N}$ on $\mathcal{O}_W \otimes_R L$ is defined by $\partial_{\sharp N}(a \otimes e) = \partial_\sharp(a) \otimes e + a \otimes N(e)$.

If $\mathcal{M}$ is a free $\mathcal{O}_W$-module in the proposition above, then the assertion is a part of Christo the transfer theorem [Chr84, Thm. 2] and its generalization in [BC92]. Christol's transfer theorem is in the case where R is a field K. By the argument in [BC92, 4.1], the transfer theorem also works on an integral K-affinoid algebra R. A part means that we consider solutions not in meromorphic functions but only in holomorphic functions. When M is free, one has a formal matrix solution by the hypothesis that any difference of exponents is not an integer except 0, and then all entries are contained in $\mathcal{A}_R(\xi^-)$ because of the conditions (i) and (ii).

Lemma 10.3.4.11. *Let R be an integral K-affinoid algebra.*

(a) There exists a finite injective morphism $T_l \to R$ of K-affinoid algebras from a free Tate K-algebra T_l of some dimension l.

(b) Suppose furthermore that R is Cohen-Macaulay. Then, for any finite injective morphism $T_l \to R$ of K-affinoid algebras, R is projective of finite type over T_l. Moreover, if M is a projective R-module of finite type, then M is free over T_l.

Proof. (1) The assertion is the Noether normalization theorem [BGR84, 6.1.2 Cor. 2].

(2) Since T_l is regular and R is Cohen-Macaulay,[Ked04a] R is projective over T_l by [Nag, 25.16]. If M is a projective R-module of finite type, then M is also projective of finite type over T_l, hence M is free over T_l by [Ked04a, 6.5]. □

With the notation as in 10.3.4.10, let us fix a finite injective morphism $T_l \to R$ of K-affinoid algebras 10.3.4.11 (1). Considering the norm on R which is defined by the maximum of norms of tuple under an identification $R \cong T_l^m$ by 10.3.4.11 (2), we regard M_η as an $\mathcal{A}_{T_l}(\eta)[\partial_\sharp]$-module by the natural finite injective morphism $\mathcal{A}_{T_l}(\eta) \to \mathcal{A}_R(\eta)$ of K-affinoid algebras for $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$. Moreover, $\mathcal{A}_{T_l}(\eta)[\partial_\sharp]$-module M_η satisfies the hypothesis in 10.3.4.10 (see b) and M_η is a free $\mathcal{A}_{T_l}(\eta)$-module 10.3.4.11 (2). Fix a basis $\underline{v}$ of M_η over $\mathcal{A}_{T_l}(\eta)$ and let G_η be a matrix with entries in $\mathcal{A}_{T_l}(\eta)$ such that $\partial_\sharp(\underline{v}) = \underline{v}G_\eta$. By applying a generalization of Christol's transfer theorem (as we explain after 10.3.4.10), there is an invertible matrix Y with entries in $\mathcal{A}_{T_l}(\eta^-)$ such that

$$\partial_\sharp Y + G_\eta Y = Y G_\eta(0), \tag{10.3.4.11.1}$$

where $G_\eta(0) = G_\eta \,(\operatorname{mod} z\mathcal{A}_{T_l}(\eta))$ is a matrix with entries in T_l. Then there is a free T_l-module L_η with a T_l-linear homomorphism N_η defined by the matrix $G_\eta(0)$ such that $(\mathcal{A}_{T_l}(\eta^-) \otimes_{\mathcal{A}_{T_l}(\eta)} M_\eta, \partial_\sharp) \cong (\mathcal{A}_{T_l}(\eta^-) \otimes_{T_l} L_\eta, \partial_{\sharp N_\eta})$. If we put $H^0(M_\eta) = \operatorname{Ker}(\partial_\sharp : M_\eta \to M_\eta)$, then $H^0(M_\eta) \cong \operatorname{Ker}(N_\eta : L_\eta \to L_\eta)$.

Lemma 10.3.4.12. *With the notation as above, the followings hold.*

(a) The pair (L_η, N_η) is independent of the choices of $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$ up to canonical isomorphisms. Moreover, $(M, \partial_\sharp) \cong (\mathcal{A}_{T_l}(\xi^-) \otimes_{T_l} L_\eta, \partial_{\sharp N_\eta})$ for any η.

(b) If we put $H^0(M) = \operatorname{Ker}(\partial_\sharp : M \to M)$, then the natural R-homomorphism $H^0(M) \to H^0(M_\eta)$ (not only the T_l structure) induced by the restriction is an isomorphism.

Proof. (1) For $\eta' \leq \eta$, there is an invertible matrix Q with entries in $\mathcal{A}_{T_l}(\eta')$ such that $\partial_\sharp Q + G_{\eta'}(0)Q = QG_\eta(0)$ by the restriction. Since none of the differences of exponents is an integer except 0, Q is an invertible matrix with entries in T_l. Hence the pair is independent of the choices of η. Note that $\{W_\eta\}_{\eta \in |K^\times|_{\mathbb{Q}} \cap]0,\xi[}$ is an affinoid covering of the quasi-Stein space W and M is the projective limit of M_η ($\eta \in |K^\times|_{\mathbb{Q}} \cap]0,\xi[$). Therefore, the assertion holds.

(2) follows from (1). □

Lemma 10.3.4.13. *Let R be an integral domain over $\mathbb{Q}_p$ with field of fractions F, and let (L, N) be a pair such that L is a free R-module of finite rank and $N : L \to L$ is an R-linear endomorphism. Suppose that $e_1, \cdots, e_s$ are distinct eigenvalues of $N \otimes F$ with multiplicities $m_1, \cdots, m_s$, respectively, such that $e_1, \cdots, e_s$ are contained in $\mathbb{Z}_p$ and let $\varphi_N(x) = (x - e_1)^{m_1} \cdots (x - e_s)^{m_s} \in \mathbb{Z}_p[x]$ the characteristic polynomial of N. If we put $L(e_i) = \varphi_i(N)L$ where $\varphi_i(x) = \varphi_N(x)/(x - e_i)^{m_i}$, then L is a direct sum of R-sub module $L(e_1), \cdots, L(e_s)$ of L such that all eigenvalues of $N|_{L(e_i)} \otimes F$ are e_i for any i. Such a decomposition is unique.*

Lemma 10.3.4.14. *With the notation in 10.3.4.10, let $e_1, \cdots, e_s$ be distinct exponents of $(M, \partial_\sharp)$ along $z = 0$. Then M is a direct sum of $\mathcal{A}_R(\xi^-)[\partial_\sharp]$-submodules $M(e_1), \cdots, M(e_s)$ of M such that all exponents of $(M(e_i), \partial_\sharp)$ are e_i for any i.*

Proof. With the notation in 10.3.4.12 and 10.3.4.13, take a free T_l-module L of finite type furnished with a T_l-linear homomorphism N such that $(M, \partial_\sharp) \cong (\mathcal{A}_{T_l}(\xi^-) \otimes_{T_l} L, \partial_{\sharp N})$. Since $L(e_i)$ is a direct summand of the free T_l-module L, $L(e_i)$ is free. Put $M(e_i) = (\mathcal{A}_{T_l}(\xi^-) \otimes_{T_l} L(e_i), \partial_{\sharp N|_{L(e_i)}})$. Then M is a direct sum of $M(e_1), \cdots, M(e_s)$ as $\mathcal{A}_{T_l}(\xi^-)[\partial_\sharp]$-modules. Since any $\mathcal{A}_{T_l}(\xi^-)[\partial_\sharp]$-homomorphism between $M(e_i)$ and $M(e_j)$ for $i \neq j$ is a zero map, $M(e_i)$ is an $\mathcal{A}_R(\xi^-)[\partial_\sharp]$-module for all i. Hence, the decomposition is the desired one. □

Lemma 10.3.4.15. *Let $S = \operatorname{Spm} R$ be a K-affinoid variety, $W = S \times_{\operatorname{Spm} K} D(0, \xi^+)$ for some $\xi \in |K^\times|_{\mathbb{Q}}$, and let $\mathcal{M}$ be a locally free $\mathcal{O}_W$-module of finite type. Then there exist a finite affinoid covering $\{S_i\}$ of S and a real number $\xi' \in |K^\times|_{\mathbb{Q}} \cap]0, \xi]$ such that, if $W_{S_i,\xi'}$ denotes the affinoid subdomain $S_i \times D(0, \xi'^+)$ of W, then $\mathcal{M}|_{W_{S_i,\xi'}}$ is a free $\mathcal{O}_{W_{S_i,\xi'}}$-module for all i.*

Proof. Since $\mathcal{M}/z\mathcal{M}$ is regarded as a locally free $\mathcal{O}_S$-module, there is a finite affinoid covering $\{S_i\}$ of S such that $(\mathcal{M}/z\mathcal{M})|_{S_i}$ is a free $\mathcal{O}_{S_i}$-module for all i. Since $W_i = S_i \times_{\operatorname{Spm} K} D(0, \xi^+)$ is an affinoid, $\mathcal{M}/z\mathcal{M}$ is generated by $\Gamma(W_i, \mathcal{M})$ by Kieh the Theorems A and B [Kie67, 2.4]. Let $v_1, \cdots, v_r \in \Gamma(W_i, \mathcal{M})$ be elements whose reductions form a basis of $(\mathcal{M}/z\mathcal{M})|_{S_i}$ over $\mathcal{O}_{S_i}$. The support of $\mathcal{M}|_{W_i}/(v_1, \cdots, v_r)$ is an analytic closed subset of W_i which does not intersect with the closed subspace defined by $z = 0$. Since $\mathcal{M}$ is locally free, there is a real number $\xi'_i \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$ such that $\mathcal{M}|_{S_i \times_{\operatorname{Spm} K} D(0, \xi'^+_i)}$ is free and is generated by $v_1, \cdots, v_r$ because of the maximum modulus principle [BGR84, 6.2.1, Prop.4]. Then it is enough to take $\xi' = \min_i \xi'_i$. □

Proof of 10.3.4.10. We may assume that any exponent of $\mathcal{M}$ along $z = 0$ is 0 by 10.3.4.14 and by twisting by an object of rank 1 with a suitable exponent. We may also assume that $\mathcal{M}|_{W_{\xi'}}$ is a free $\mathcal{O}_{W_{\xi'}}$-module for some $\xi' \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$ by 10.3.4.15. By applying the transfer theorem 10.3.4.10 for the free cases with the conditions (i) and (ii), if one takes an $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi'[$, then there is a free R-module L furnished with an R-linear operator $N : L \to L$ such that $\beta_\eta : (\mathcal{M}, \partial_\sharp)|_{W_\eta} \xrightarrow{\sim} (\mathcal{O}_{W_\eta} \otimes_R L, \partial_{\sharp N})$. Denote the dual of $\mathcal{M}$ by $(\mathcal{M}^\vee, -\partial_\sharp)$. Then we have a natural commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{O}_W[\partial_\sharp]}(\mathcal{M}, \mathcal{O}_W \otimes_R L) & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_{W_\eta}[\partial_\sharp]}(\mathcal{M}|_{W_\eta}, \mathcal{O}_{W_\eta} \otimes_R L) \\ \cong\downarrow & & \downarrow\cong \\ H^0(M^\vee \otimes_R L) & \xrightarrow{\sim} & H^0(M^\vee_\eta \otimes_R L), \end{array}$$

where the vertical arrows are isomorphisms since $\mathcal{M}$ is locally free and the bottom horizontal arrow is an isomorphism by 10.3.4.12 (2) since all differences of exponents of $(\mathcal{M}^\vee \otimes_R L, -\partial_\sharp \otimes 1 + 1 \otimes \partial_N)$ along $z = 0$ are 0.

Let $\beta : (\mathcal{M}, \partial_\sharp) \to (\mathcal{O}_W \otimes_R L, \partial_{\sharp N})$ be the $\mathcal{O}_W[\partial_\sharp]$-homomorphism corresponding to β_η via the isomorphisms above. We will prove that β is an isomorphism. In the case where R is a field, β is an isomorphism since the support of an $\mathcal{A}_R(\xi^-)[\partial_\sharp]$-module, which is finitely generated over $\mathcal{A}_R(\xi^-)$, is either W or one point $z = 0$ by Bézout property of $\mathcal{A}_R(\xi^-)$ [Cre98, 4.6]. Let us return to the case of general R. For a maximal ideal x of R, the induced homomorphism $\beta \pmod{x}$ is an isomorphism by the case where R is a field. Hence, β is an isomorphism around $x \times_{\mathrm{Spm}\, K} D(0, \xi^-)$ by Nakayama's lemma. Since both sides of β are coherent, β is an isomorphism [BGR84, 9.4.2, Corollary 7]. □

5° *The vanishing 10.3.4.8.2 in special cases: any difference of exponents is neither a p-adic Liouville number nor an integer except* 0.

Let us first suppose that (ii) in 10.3.4.10 and $c = 0$ for the exponents along $z = 0$ by 0°.

Lemma 10.3.4.16. *With the notation in 10.3.4.13, the followings hold.*

(a) Let j be an integer. Then there is a monic polynomial $g_j(x) \in \mathbb{Z}_p[x]$ of degree $r-1$ such that $(N-j)g_j(N) + \varphi_N(j)I_L = 0$. Here I_L is the identity of L.

(b) If all of $e_1, \cdots, e_s$ are neither p-adic Liouville numbers nor positive integers, then $(N-j)$ is invertible and, for any $0 < \eta < 1$, $|\varphi_N(j)^{-1}|\eta^j \to 0$ as $j \to \infty$

Take $(\lambda, \eta) \in \Lambda_{\underline{\xi},\underline{\lambda}}$ such that $\lambda \geq \lambda_m$ and $\eta < \xi_m$ for some m. Then the restriction $(\mathcal{E}, \partial_\sharp)$ on $S \times_{\mathrm{Spm} K} D(0, \xi_m^-)$ for an integral smooth K-affinoid $S = \mathrm{Spm}\, R$ in $V_\beta \cap]Z \cap Y[_{3,\lambda_m}$ satisfies the assumption of 10.3.4.10 by the overconvergence condition in 2°. Considering an admissible affinoid covering of S, we may assume that there is a basis of $\Gamma(g_{\lambda,\eta}^{-1}(S), \mathcal{E})$ over $\mathcal{A}_R(\eta)$ such that G is a matrix with entries in R.

Since any eigenvalue of G is not a positive integer, $\partial_\sharp + G$ is injective on $(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r$. Since any eigenvalue of G is neither a p-adic Liouville nor a positive integer, $\partial_\sharp + G$ is surjective on $(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r$. Indeed, with the notation in 10.3.4.16 (1), $\partial_\sharp + G$ maps $-\sum_{j=1}^\infty \varphi_G(j)^{-1} g_j(G)\underline{a}_j z^{-j}$ to $\sum_{j=1}^\infty \underline{a}_j z^{-j}$ and $\sum_{j=1}^\infty \varphi_G(j)^{-1} g_j(G)\underline{a}_j z^{-j}$ is contained in $(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r$ by 10.3.4.16 (2). Hence, the cohomology groups in 10.3.4.9.1 vanish for any q and it implies the vanishing 10.3.4.8.2.

6° *The vanishing 10.3.4.8.2 in general cases: any difference of exponents is not a p-adic Liouville number.*

Let us suppose the conditions (a) in 10.3.4.2 and $c = 0$ for the exponents along $z = 0$ by 0°.

Proposition 10.3.4.17. *With the notation as in 10.3.4.10, we assume the conditions (i) in 10.3.4.10, (a) in 10.3.4.2 and $c = 0$ for exponents of $(\mathcal{M}, \partial_\sharp)$ along $z = 0$. Then there is a locally free $\mathcal{O}_W$-submodule $\mathcal{M}'$ of $\mathcal{M}$ which is stable under $\partial_\sharp$ such that (1) $(\mathcal{M}', \partial_\sharp)$ satisfies the conditions (i) and (ii) in 10.3.4.10, (2) none of exponents of $(\mathcal{M}', \partial_\sharp)$ along $z = 0$ is a positive integer, (3) the support of $\mathcal{M}/\mathcal{M}'$ is included in the closed subset defined by $z = 0$ and it is a locally free O_S-module of finite type, and (4) the induced homomorphism $\overline{\partial}_\sharp : \mathcal{M}/\mathcal{M}' \to \mathcal{M}/\mathcal{M}'$ is an isomorphism.*

Lemma 10.3.4.18. *Let R be an integral K-affinoid algebra and let $\eta \in |K^\times|_{\mathbb{Q}}$. Suppose that M is a free $\mathcal{A}_R(\eta)$-module of finite rank furnished with an R-derivation $\partial_\sharp = z\frac{d}{dz} : M \to M$ such that $e_1, \cdots, e_s$ are distinct exponents of $(M, \partial_\sharp)$ along $z = 0$ with multiplicities $m_1, \cdots, m_s$, respectively.*

(a) There exists a basis $\underline{v}$ of M such that, if G is the matrix with entries in $\mathcal{A}_R(\xi)$ defined by $\partial_\sharp(\underline{v}) = \underline{v}G$, then $G(0) = \begin{pmatrix} G_1(0) & & 0 \\ & \ddots & \\ 0 & & G_s(0) \end{pmatrix}$ and all eigenvalues of the R-matrix $G_i(0)$ of degree m_i are e_i for any i.

(b) Let $\underline{v}_i$ be the part bottom is as in (1) corresponding to the i-th direct summand modulo z, that is, $\partial_\sharp(\underline{v}_i) \equiv \underline{v}_i G_i(0) \pmod{z\mathcal{A}_R(\eta)}$. Let M' be the $\mathcal{A}_R(\eta)$-submodule of M generated by $z\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_s$. Then M' is stable under $\partial_\sharp$ with exponents $e_1 + 1, e_2, \cdots, e_s$ and multiplicities $m_1, m_2, \cdots, m_s$, respectively. Moreover, M/M' is a free R-module of rank m_1, and, if $e_1 \neq 0$, then the induced R-homomorphism $\overline{\partial}_\sharp : M/M' \to M/M'$ is an isomorphism.

Proof. (1) follows from 10.3.4.13.

(2) The stability follows from (1). If we denote the matrix which represents the derivation of M' by G', then

$$G' = P^{-1} z \frac{d}{dz} P + P^{-1} G P \equiv \begin{pmatrix} G_1(0) + I_{m_1} & & & * \\ & G_2(0) & & \\ & & \ddots & \\ 0 & & & G_s(0) \end{pmatrix} \pmod{z\mathcal{A}_R(\eta)}$$

for $P = \begin{pmatrix} zI_{m_1} & 0 \\ 0 & I_{r-m_1} \end{pmatrix}$. Here $r = m_1 + \cdots + m_s$ and I_t is the identity matrix of degree t. The induced R-homomorphism $\overline{\partial}_\sharp : M/M' \to M/M'$ is given by the matrix $G_1(0)$. □

Proof of 10.3.4.17. We use the induction on the largest integral difference of exponents and its multiplicity. By 10.3.4.15 we may assume that $\mathcal{M}|_{W_\eta}$ is free for some $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$. We have an $\mathcal{O}_{W_\eta}$-submodule $\mathcal{M}'_\eta$ of $\mathcal{M}|_{W_\eta}$ such that exponents are improved by 10.3.4.18. Indeed, we apply 10.3.4.18 to an exponent which is neither a positive integer nor 0 because of the condition $c = 0$. Since the support of $\mathcal{M}|_{W_\eta}/\mathcal{M}'_\eta$ is included in $z = 0$, one can glue $\mathcal{M}'_\eta$ and $\mathcal{M}|_{W \setminus \{z=0\}}$. Hence, the induction works. □

We use the same notation in 5°. Considering an admissible affinoid covering of S, we may assume that $\mathcal{E}|_{g^{-1}_{\lambda,\mu}(S)}$ is free for some $\mu \in |K^\times|_{\mathbb{Q}} \cap]0, \xi_m]$ by 10.3.4.15 and, then, we can apply 10.3.4.17. Let $\mathcal{E}'$ be a locally free $\mathcal{O}_{g^{-1}_{\lambda,\xi_m}(S)}$-submodule of $\mathcal{E}|_{g^{-1}_{\lambda,\xi_m}(S)}$ which is stable under $\partial_\sharp$ such that it satisfies the condition (1), (2) and (3) in 10.3.4.17. Now we computation ate the difference of the local computation of cohomology between $\mathcal{E}$ and $\mathcal{E}'$ by the module version of the second form of 10.3.4.8.2. If $E_\eta = \Gamma(g^{-1}_{\lambda,\eta}(S), \mathcal{E})$ and $E'_\eta = \Gamma(g^{-1}_{\lambda,\eta}(S), \mathcal{E})$, then $E' \otimes \mathcal{R}_R(\eta) = E \otimes \mathcal{R}_R(\eta)$ by the condition (2) on the support of $\mathcal{E}/\mathcal{E}'$. The difference is computation ated by the complex

$$\mathrm{Tot} \begin{bmatrix} E'_\eta & \to & E_\eta \\ \partial_\sharp \downarrow & & \downarrow \partial_\sharp \\ E'_\eta & \to & E_\eta \end{bmatrix} \cong \left[E_\eta/E'_\eta \xrightarrow{\partial_\sharp} E_\eta/E'_\eta \right],$$

and it is 0 by (3). Hence, the vanishing 10.3.4.8.2 for $\mathcal{E}$ follows from the vanishing for $\mathcal{E}'$ by 5°.

This completes the proof of Proposition 10.3.4.5. □

Proof of Theorem 10.3.4.2. By the same reason of 0° in the proof of 10.3.4.5, we may assume $c = 0$ and have only to prove the vanishing $\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_Y \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j^\dagger_Y \mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$.

By the Čech spectral sequence the problem of the vanishing is local on X and Y as in 1° in the proof of 10.3.4.5. We may assume that $\mathfrak{X}$ is affine, D is defined by a single equation $f = 0$ in X for some $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and there is a system of relative local coordinates $z_1, z_2, \cdots, z_d \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of $\mathfrak{X}$ over $\mathfrak{S}$ such that each irreducible component $\mathfrak{Z}_i$ of the relative strict normal crossing divisor $\mathfrak{Z} = \cup_{i=1}^s \mathfrak{Z}_i$ is defined by $z_i = 0$. Let us denote by Z_i (resp. X_i^*) the closed subscheme of X defined by $z_i = 0$ (resp. the complement of Z_i in X).

Let us define $]Y[_{\mathfrak{X},\lambda}$ (resp. $]X_i^*[_{\mathfrak{X},\lambda}$, resp. $[Z_i]_{\mathfrak{X},\lambda}$) as in 2° of the proof of 10.3.4.5 (resp. replacing $\mathfrak{Z}$, Z by $\mathfrak{Z}_i$, Z_i).

By the hypothesis on (E, ∇) there exist a strict neighborhood $]Y[_{\mathfrak{X},\nu}$ of $]Y[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$ for some $\nu \in |K^\times|_{\mathbb{Q}} \cap]0, 1[$ and a locally free $\mathcal{O}_{]Y[_{\mathfrak{X},\nu}}$-module $\mathcal{E}$ of finite type furnished with a logarithmic connection $\nabla : \mathcal{E} \to (\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K}|_{]Y[_{\mathfrak{X},\nu}}) \otimes_{\mathcal{O}_{]Y[_{\mathfrak{X},\nu}}} \mathcal{E}$ such that $j^\dagger_Y(\mathcal{E}, \nabla) = (E, \nabla)$, which satisfies the overconvergence condition 10.3.2.4.1.

7° *Induction on the number s of irreducible components of the strict normal crossing divisor Z.*

If $s = 0$, then the assertion is trivial. Put $Z' = \cup_{i=2}^s Z_i$. Applying the natural exact sequence

$$0 \longrightarrow \underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(\mathcal{H}) \longrightarrow \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(\mathcal{H}) \longrightarrow \underline{\Gamma}^\dagger_{]Z'[_{\mathfrak{X}}}(j^\dagger_{X_1^*}\mathcal{H}) \longrightarrow 0$$

for a sheaf $\mathcal{H}$ of abelian groups on $]X[_{\mathfrak{X}}$ (see the proof of [Ber96b, 2.1.7]), we have a triangle

$$\begin{array}{ccc} & \mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z'[_{\mathfrak{X}}}(j^\dagger_{X_1^*\cap Y}\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j^\dagger_{X_1^*\cap Y}\mathcal{O}_{]X[_{\mathfrak{X}}}} j^\dagger_{X_1^*\cap Y}E) & \\ & {}^{+1}\swarrow \qquad \nwarrow & \\ \mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j^\dagger_Y\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j^\dagger_Y\mathcal{O}_{]X[_{\mathfrak{X}}}} E) & \longrightarrow & \mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_Y\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j^\dagger_Y\mathcal{O}_{]X[_{\mathfrak{X}}}} E). \end{array}$$

Hence we have only to prove the vanishing

$$\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j^\dagger_Y\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j^\dagger_Y\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$$

by the induction on s. If $Z_1 \subset D$, the vanishing is trivial. Hence, we may assume that Z_1 is not included in D.

8° *Reduction to the case of sections.*

Let us denote the formal affine space of relative dimension r over $\mathfrak{S}$ by $\widehat{\mathbb{A}}^r_{\mathfrak{S}}$. By our hypothesis there is a commutative diagram

$$\begin{array}{ccccc} \mathfrak{Z}_1 & \longrightarrow & \mathfrak{X} & & \\ \downarrow & & \downarrow & & \\ \widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}} & \longrightarrow & \widehat{\mathbb{A}}^{d}_{\mathfrak{S}} & \longrightarrow & \widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}} \end{array} \tag{10.3.4.18.1}$$

of $\mathcal{V}$-formal schemes such that the vertical arrow $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathfrak{S}}$, which is étale, (resp. $\mathfrak{Z}_1 \to \widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}}$) is induced by $z_1, \cdots, z_d$ (resp. $z_2, \cdots, z_d$) and the composition of bottom arrows is the identity. Since the diagonal morphism $\Delta : \mathfrak{Z}_1 \to \mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}}} \mathfrak{Z}_1$ is étale and a closed immersion, $\widetilde{\mathfrak{X}} = \mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}}} \mathfrak{X} \setminus (\mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}}} \mathfrak{Z}_1 \setminus \Delta(\mathfrak{Z}_1))$ is an open formal subscheme of $\mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}}} \mathfrak{X}$. Let now us consider the commutative diagram

$$\begin{array}{ccccc} & & \widetilde{\mathfrak{X}} & & \\ & \overset{\Delta}{\nearrow} & \downarrow & \overset{h}{\searrow} & \\ \mathfrak{Z}_1 & \overset{\Delta}{\longrightarrow} & \mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \mathfrak{X} & \underset{\mathrm{pr}_2}{\longrightarrow} & \mathfrak{X} \\ =\downarrow & & \downarrow & & \\ \mathfrak{Z}_1 & \underset{\mathrm{pr}_1}{\longleftarrow} & \mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \widehat{\mathbb{A}}^d_{\mathcal{V}} & & \end{array} \tag{10.3.4.18.2}$$

of formal $\mathfrak{S}$-schemes, and define $h : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ (resp. $\widetilde{g}_1 : \widetilde{\mathfrak{X}} \to \mathfrak{Z}_1$, $\widetilde{g}' : \mathfrak{Z}_1 \to \mathfrak{S}$, resp. $\widetilde{g} = \widetilde{g}' \circ \widetilde{g}_1$) as in the diagram (resp. by the composition $\widetilde{\mathfrak{X}} \to \mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \mathfrak{X} \to \mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \widehat{\mathbb{A}}^d_{\mathcal{V}} \to \mathfrak{Z}_1$, resp. by the canonical morphism, resp. by the composition). We identify $\Delta(\mathfrak{Z}_1)$ (resp. $\Delta(Z_1)$) with $\mathfrak{Z}_1$ (resp. Z_1), and denote the special fiber of $\widetilde{\mathfrak{X}}$ (resp. the complement of Z_1, resp. the inverse image of Y by h) by $\widetilde{X}$ (resp. $\widetilde{X}_1^*$, resp. $\widetilde{Y}$). $\mathfrak{Z}_1$ is a smooth divisor over $\mathfrak{S}$ and note that, étale locally, $h^{-1}(\mathfrak{Z})$ is a relative normal crossing divisor. $\widetilde{\mathfrak{X}}^\sharp_K$ denotes the $\mathcal{V}$-formal scheme with the logarithmic structure over $\mathfrak{S}_K$ which is induced by the logarithmic structure of $\mathfrak{X}^\sharp_K$, and $\Omega^1_{\widetilde{\mathfrak{X}}^\sharp_K/\mathfrak{S}_K}$ denotes the sheaf of logarithmic Kähler differentials on $\widetilde{\mathfrak{X}}^\sharp_K$ over $\mathfrak{S}_K$. Then $h_K^* \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \cong \Omega^\bullet_{\widetilde{\mathfrak{X}}^\sharp_K/\mathfrak{S}_K}$.

Let us define $]\widetilde{Y}[_{\widetilde{\mathfrak{X}},\lambda}$ (resp. $]\widetilde{X}_1^*[_{\widetilde{\mathfrak{X}},\lambda}$, resp. $[Z_1]_{\widetilde{\mathfrak{X}},\lambda}$) as in 2° of the proof of 10.3.4.5.

Lemma 10.3.4.19. *With the notation as above, we have*

(a) $h_K^{-1}(]Z_1[_{\mathfrak{X}}) =]Z_1[_{\widetilde{\mathfrak{X}}}$.

(b) The restriction of h_K *gives an isomorphism* $]Z_1[_{\widetilde{\mathfrak{X}}} \xrightarrow{\sim}]Z_1[_{\mathfrak{X}}$.

(c) Under the isomorphism in (2),

$$]\widetilde{Y}[_{\widetilde{\mathfrak{X}},\lambda} \cap [Z_1]_{\widetilde{\mathfrak{X}},\eta} \xrightarrow{\sim}]Y[_{\mathfrak{X},\lambda} \cap [Z_1]_{\mathfrak{X},\eta}$$

for any $\lambda, \eta \in |K^\times|_{\mathbb{Q}} \cap]0,1[$.

Proof. Since $(\mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathfrak{S}}} \mathfrak{Z}_1 \setminus \Delta(\mathfrak{Z}_1))$ is removed, we get (1). The other assertion (2) (resp. (3)) follow from [Ber96b, 1.3.1] and the fact that h is étale (resp. and $Z_1 \not\subset D$). □

Proposition 10.3.4.20. *With the notation as above, we have the followings.*

(a) If $\mathcal{H}$ *is a sheaf of Abelian groups on* $]\widetilde{X}[_{\widetilde{\mathfrak{X}}}$*, then*

$$\mathbb{R} h_{K*} \underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}) \cong h_{K*} \underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}).$$

(b) *Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaf of rings on $]X[_{\mathfrak{X}}$ and $]\widetilde{X}[_{\widetilde{\mathfrak{X}}}$, respectively, with a morphism $h_K^{-1}\mathcal{A} \to \mathcal{B}$ of rings such that $\mathcal{A}|_{]Z_1[_{\mathfrak{X}}} \xrightarrow{\sim} \mathcal{B}|_{]Z_1[_{\widetilde{\mathfrak{X}}}}$ under the isomorphism in 10.3.4.19 (2). If $\mathcal{H}$ is an $\mathcal{A}$-module, then the adjoint map*

$$\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}) \to h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{B} \otimes_{h_K^{-1}\mathcal{A}} h_K^{-1}\mathcal{H}).$$

is an isomorphism of $\mathcal{A}$-modules.

Proof. Let us define a functor

$$\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{X}},\eta}(\mathcal{H}) = \mathrm{Ker}\left(\mathcal{H} \to \lim_{\mu\to\eta^-} \alpha_{]\widetilde{X}_1^*[_{\widetilde{\mathfrak{X}},\mu}}{}_*(\mathcal{H}|_{]\widetilde{X}_1^*[_{\widetilde{\mathfrak{X}},\mu}})\right)$$

as in 2° of the proof of 10.3.4.5, where $\alpha_{]\widetilde{X}_1^*[_{\widetilde{\mathfrak{X}},\mu}} :]\widetilde{X}_1^*[_{\widetilde{\mathfrak{X}},\mu} \to]\widetilde{X}[_{\widetilde{\mathfrak{X}}}$ is the canonical open immersion. Then the analog of 10.3.4.7 and 10.3.4.8 hold.

(1) Since $\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{X}},\eta}(\mathcal{H})|_{]X_1^*[_{\widetilde{X},\eta}} = 0$, we have $\mathbb{R}^q h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{X}},\eta}(\mathcal{H}) = 0$ for any $q \geq 1$ by 10.3.4.19 (2). Because the cohomological functor $\mathbb{R}^q h_{K*}$ commutes with filtered inductive limits by the quasi-compactness and quasi-separateness of h_K, we have

$$\mathbb{R}^q h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}) \cong \mathbb{R}^q h_{K*}\left(\lim_{\eta\to 1^-}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}},\eta}(\mathcal{H})\right) \cong \lim_{\eta\to 1^-}\mathbb{R}^q h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}},\eta}(\mathcal{H}) = 0$$

for any $q \geq 1$ by 10.3.4.7.

(2) Since $\mathcal{H}|_{[Z_1]_{\mathfrak{X},\eta}} \xrightarrow{\sim} (\mathcal{B} \otimes_{h_K^{-1}\mathcal{A}} h_K^{-1}\mathcal{H})|_{[Z_1]_{\widetilde{\mathfrak{X}},\eta}}$, the assertion follows from 10.3.4.7 and 10.3.4.19. □

Let $(\widetilde{E}, \widetilde{\nabla})$ be the inverse image of (E, ∇) by h_k, i.e.,

$$\widetilde{E} = h_K^* E = j_{\widetilde{Y}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}} \otimes_{h_K^{-1}(j_Y^\dagger \mathcal{O}_{]X[_{\mathfrak{X}}})} h_K^{-1}E$$
$$\widetilde{\nabla} : \widetilde{E} \to j_{\widetilde{Y}}^\dagger \Omega^1_{\widetilde{\mathfrak{X}}_K^\sharp/\mathfrak{S}_K} \otimes_{j_{\widetilde{Y}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E},$$

where $\widetilde{\nabla}$ is the induced $\mathcal{O}_{]S[_{\mathfrak{S}}}$-linear connection by ∇ because of the étaleness of h. We also denote the induced basis of $\Omega^1_{\widetilde{\mathfrak{X}}_K^\sharp/\mathfrak{S}_K}$ by $\frac{dz_1}{z_1}, \cdots, \frac{dz_s}{z_s}, dz_{s+1}, \cdots, dz_d$ and the dual basis of derivations by $z_1\frac{\partial}{\partial z_1}, \cdots, z_s\frac{\partial}{\partial z_s}, \frac{\partial}{\partial z_{s+1}}, \cdots, \frac{\partial}{\partial z_d}$.

Proposition 10.3.4.21. *(a) If we put $(\widetilde{\mathcal{E}}, \widetilde{\nabla}) = h_K^*(\mathcal{E}, \nabla)$, then the natural morphism $j_{\widetilde{Y}}^\dagger(\widetilde{\mathcal{E}}, \widetilde{\nabla}) \to (\widetilde{E}, \widetilde{\nabla})$ is an isomorphism.*

(b) The derivation $\widetilde{\partial}_{\sharp 1} = \nabla(z_1\frac{\partial}{\partial z_1})$ on $\widetilde{\mathcal{E}}$ satisfies the overconvergence condition 10.3.4.6.1.

Proof. (1) easily follows from the fact $\mathcal{E}$ is locally free.

(2) It is enough to check the overconvergence condition for $\mathrm{pr}_{2K}^*(\mathcal{E}, \nabla)$ along $z_1 = 0$. Fix a complete K-algebra norm on the affinoid algebra associated to $]X[_{\mathfrak{X}}$. Then one can take a contractive complete K-algebra norm on the affinoid algebra associated to $]Z_1 \times_{\mathbb{A}_k^{d-1}} X[_{\mathfrak{Z}_1 \times_{\widehat{\mathbb{A}}_{\mathfrak{S}}^{d-1}} \mathfrak{X}}$ [BGR84, 6.1.3, Prop. 3]. The induced norms $||\text{-}||_{\mathfrak{X}}$ on $\Gamma(]Y[_{\mathfrak{X},\lambda}, \mathcal{E})$ and $||\text{-}||_{\mathfrak{Z}_1\times\mathfrak{X}}$ on $\Gamma\left(\mathrm{pr}_{2K}^{-1}(]Y[_{\mathfrak{X},\lambda}), \mathrm{pr}_{2K}^*\mathcal{E}\right)$ satisfy the inequality $||e||_{\mathfrak{Z}_1\times\mathfrak{X}} \leq ||e||_{\mathfrak{X}}$ for any $e \in \Gamma(]Y[_{\mathfrak{X},\lambda}, \mathcal{E})$. The overconvergence condition for $\mathrm{pr}_{2K}^*(\mathcal{E}, \nabla)$ along $z_1 = 0$ follows from the inequality. □

Remark 10.3.4.22. The connection $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ satisfies the overconvergence condition 10.3.2.4.1. It should be called a locally free log-isocrystal on $\mathfrak{X}_K^\sharp/\mathfrak{S}_K$ overconvergent along $\widetilde{D}$.

Since $(j_Y^\dagger \mathcal{O}_{]X[_{\mathfrak{X}}})|_{]Z_1[_{\mathfrak{X}}} \xrightarrow{\sim} (j_{\widetilde{Y}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}})|_{]Z_1[_{\widetilde{\mathfrak{X}}}}$, we have

$$\begin{aligned}
\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_Y^\dagger \Omega^\bullet_{\mathfrak{X}_K^\sharp/\mathfrak{S}_K} \otimes_{j_Y^\dagger \mathcal{O}_{]X[_{\mathfrak{X}}}} E) &\cong \mathbb{R}g_{K*}(h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{Y}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\sharp/\mathfrak{S}_K} \otimes_{j_{\widetilde{Y}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E})) \\
&\cong \mathbb{R}g_{K*}\mathbb{R}h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{Y}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\sharp/\mathfrak{S}_K} \otimes_{j_{\widetilde{Y}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}) \\
&\cong \mathbb{R}\widetilde{g}_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{Y}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\sharp/\mathfrak{S}_K} \otimes_{j_{\widetilde{Y}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E})
\end{aligned}$$

by 10.3.4.20. Hence we have only to prove the vanishing

$$\mathbb{R}\widetilde{g}_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{S}_K}\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}) = 0.$$

9° *An argument of Gauss-Manin type.*

Let Ω^q_0 (resp. Ω^q_1) be the free $\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}$-submodule of $\Omega^q_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{S}_K}$ generated by wedge products of $\frac{dz_2}{z_2},\cdots,\frac{dz_s}{z_s},dz_{s+1},\cdots,dz_d$ (resp. $\frac{dz_1}{z_1}\wedge\omega$ for $\omega\in\Omega^{q-1}_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{S}_K}$). Then $\Omega^q_0 \xrightarrow{\sim} \Omega^{q+1}_1$ by $\omega\mapsto\frac{dz_1}{z_1}\wedge\omega$. Define

$$\begin{aligned}\widetilde{\nabla}_0 &= \textstyle\sum_{i=2}^{s}\frac{dz_i}{z_i}\otimes\partial_{\sharp i}+\sum_{i=s+1}^{d}dz_i\otimes\partial_i : \widetilde{E}\to j^{\dagger}_{\widetilde{Y}}\Omega^1_0\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}\\ \widetilde{\nabla}_1 &= \mathrm{id}\otimes\partial_{\sharp 1} : j^{\dagger}_{\widetilde{Y}}\Omega^q_0\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}\to j^{\dagger}_{\widetilde{Y}}\Omega^q_1\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E},\end{aligned} \qquad (10.3.4.22.1)$$

where id is the identity of $j^{\dagger}_{\widetilde{Y}}\Omega^q_0$. The definition of $\widetilde{\nabla}_0$ and $\widetilde{\nabla}_1$ is independent of the choices of local parameters $z_1,z_2,\cdots,z_d$ of $\mathfrak{X}$ over $\mathfrak{S}$ as above. Then the exterior power of $j^{\dagger}_{\widetilde{Y}}\Omega^1_0$ induces a complex $(j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_0\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E},\widetilde{\nabla}_0)$ and there is an isomorphism

$$j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{S}_K}\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}\xrightarrow{\sim}\left[(j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_0\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E},\widetilde{\nabla}_0)\xrightarrow{\widetilde{\nabla}_1}(j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_1\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E},\frac{dz_1}{z_1}\wedge\widetilde{\nabla}_0)\right] \quad (10.3.4.22.2)$$

of complexes of $\mathcal{O}_{]S[_{\mathfrak{S}}}$-modules. Note that $\widetilde{\nabla}_1$ is the relative connection $\widetilde{E}\to j^{\dagger}_{\widetilde{Y}}\Omega^1_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{Z}^{\sharp}_{1K}}\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}$ induced by $\widetilde{\nabla}$.

One can easily see that $(\widetilde{E},\widetilde{\nabla}_1)$ satisfies the hypothesis (a) and (b) along $z_1=0$ in 10.3.4.2 by 10.3.4.19 and the overconvergence condition in 10.3.4.5, so that

$$\mathbb{R}\widetilde{g}_{1K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}\left(\left[j^{\dagger}_{\widetilde{Y}}\Omega^q_0\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}\xrightarrow{\widetilde{\nabla}_1}j^{\dagger}_{\widetilde{Y}}\Omega^{q+1}_1\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E}\right]\right)=0$$

for any q by 10.3.4.5. Hence,

$$\mathbb{R}\widetilde{g}_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{S}_K}\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E})=\mathbb{R}\widetilde{g}'_{K*}\mathbb{R}\widetilde{g}_{1K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j^{\dagger}_{\widetilde{Y}}\Omega^{\bullet}_{\widetilde{\mathfrak{X}}^{\sharp}_K/\mathfrak{S}_K}\otimes_{j^{\dagger}_{\widetilde{Y}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}}\widetilde{E})=0.$$

This completes the proof of 10.3.4.2. □

Proposition 10.3.4.23. *With the notation in 10.3.4.2, we assume furthermore that $g:\mathfrak{X}\to\mathfrak{S}$ factors through an irreducible component $\mathfrak{Z}_1$ of $\mathfrak{Z}$ by a smooth morphism $g_1:\mathfrak{X}\to\mathfrak{Z}_1$ over $\mathfrak{S}$ such that the composition $g_1\circ i_1:\mathfrak{Z}_1\to\mathfrak{Z}_1$ of the closed immersion $i_1:\mathfrak{Z}_1\to\mathfrak{X}$ and g_1 is the identity of $\mathfrak{Z}_1$ and that the inverse image of the relative strict normal crossing divisor $\mathfrak{Z}'_1=\cup_{i=2}^{s}\mathfrak{Z}_1\cap\mathfrak{Z}_i$ of $\mathfrak{Z}_1$ by g_1 is $\cup_{i=2}^{s}\mathfrak{Z}_i$. Let E be a locally free log-isocrystal on $\mathfrak{X}^{\sharp}/\mathfrak{S}_K$ overconvergent along D. Then, for any nonnegative integer m, $g_{1K*}\widetilde{\nabla}_0$ (resp. $g_{1K*}(\frac{dz_1}{z_1}\wedge\widetilde{\nabla}_0)$) in 10.3.4.22.1 induces an integrable logarithmic $\mathcal{O}_{]S[_{\mathfrak{S}}}$-connection of the locally free $j^{\dagger}_{Z_1\cap Y}\mathcal{O}_{]Z_1[_{\mathfrak{Z}_1}}$-module $g_{1K*}(E(m\mathfrak{Z}_1)/E)$ (resp. $g_{1K*}(j^{\dagger}_Y\Omega^1_{\mathfrak{X}^{\sharp}_K/\mathfrak{Z}^{\sharp}_{1K}}\otimes_{j^{\dagger}_Y\mathcal{O}_{]X[_{\mathfrak{X}}}}E(m\mathfrak{Z}_1)/E)$) of finite type on $\mathfrak{Z}^{\sharp}_{1K}:=(\mathfrak{Z}_{1K},\mathfrak{Z}'_{1K})/\mathfrak{S}_K$ which satisfies the overconvergence condition as a log-isocrystal on $(Z_1\cap Y)^{\sharp}/\mathfrak{S}_K$ overconvergent along $Z_1\cap D$.*

Suppose furthermore that $Z_1\not\subset D$ and that E satisfies the conditions (a) and (b) in 10.3.4.2. Then

$$\mathbb{R}g_{1K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}(j^{\dagger}_Y\Omega^{\bullet}_{\mathfrak{X}^{\sharp}_K/\mathfrak{Z}^{\sharp}_{1K}}\otimes_{j^{\dagger}_Y\mathcal{O}_{]X[_{\mathfrak{X}}}}E)\cong\left[g_{1K*}(E(m\mathfrak{Z}_1)/E)\xrightarrow{g_{1K*}\nabla}g_{1K*}(j^{\dagger}_Y\Omega^1_{\mathfrak{X}^{\sharp}_K/\mathfrak{Z}^{\sharp}_{1K}}\otimes_{j^{\dagger}_Y\mathcal{O}_{]X[_{\mathfrak{X}}}}E(m\mathfrak{Z}_1)/E)\right][-1] \quad (10.3.4.23.1)$$

and $g_{1K*}(E(m\mathfrak{Z}_1)/E)$ (resp. $g_{1K*}(j^{\dagger}_Y\Omega^1_{\mathfrak{X}^{\sharp}_K/\mathfrak{Z}^{\sharp}_{1K}}\otimes_{j^{\dagger}_Y\mathcal{O}_{]X[_{\mathfrak{X}}}}E(m\mathfrak{Z}_1)/E)$) also satisfies the same conditions (a) and (b) for any $m\geq\max\{e\,|\,e$ is a positive integral exponent of ∇ along $Z_1\}\cup\{0\}$.

Proof. The locally freeness has been already proved at the beginning of the proof of 10.3.4.5. From the definition of $\widetilde{\nabla}_0$ in 10.3.4.22.1, it induces an integrable connection. Since $\mathfrak{Z}_1$ is a section of $\mathfrak{X}$ over $\mathfrak{S}$, one can use on an affinoid open subset of $]Z_1[_{\mathfrak{Z}_1}$ a Banach norm induced by a Banach norm on some affinoid open subset of $]X[_{\mathfrak{X}}$. Hence the logarithmic connections on $g_{1K*}(E(m\mathfrak{Z}_1)/E)$ and $g_{1K*}(j_Y^\dagger\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\sharp_{1K}} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathfrak{Z}_1)/E)$ satisfy the overconvergence condition. Their exponents along Z_i are m copies of those of E by the definition of $\widetilde{\nabla}_0$ for $i \neq 1$. Therefore, the conditions (a) and (b) also hold. □

Example 10.3.4.24. Let $\mathfrak{X}$ be the formal projective scheme $\widehat{\mathbb{P}}^1_{\mathcal{V}} \times_{\operatorname{Spf}\mathcal{V}} \widehat{\mathbb{P}}^1_{\mathcal{V}}$ over $\mathfrak{S} = \operatorname{Spf}\mathcal{V}$ with homogeneous coordinates $(x_0, x_1), (y_0, y_1)$, let $\mathfrak{Z}_1$ (resp. $\mathfrak{Z}_2$) be the divisor defined by $x_1 = 0$ (resp. $y_1 = 0$) in $\mathfrak{X}$, and put $\mathfrak{Z} = \mathfrak{Z}_1 \cup \mathfrak{Z}_2$ and $\mathfrak{X}^\sharp = (\mathfrak{X}, \mathfrak{Z})$. Let X (resp. Z, resp. Z_1, resp. Z_2) be the special fiber of $\mathfrak{X}$ (resp. $\mathfrak{Z}$, resp. $\mathfrak{Z}_1$, resp. $\mathfrak{Z}_2$), let D be a closed subscheme of X defined by $x_0 = 0$ or $y_0 = 0$, put $Y = X \setminus D$, and let $z_1 = x_1/x_0, z_2 = y_1/y_0$ be the affine coordinates. For integers $e > 0$ and $h \geq 0$, we define a locally free log-isocrystal E on $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ of rank 2 overconvergent along D ($E = j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}v_1 \oplus j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}v_2$) by

$$\nabla(v_1, v_2) = (v_1, v_2)\begin{pmatrix} e & z_2^h \\ 0 & e \end{pmatrix}\frac{dz_1}{z_1} + (v_1, v_2)\begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}\frac{dz_2}{z_2}$$

for some strict neighborhood of $]Y[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$. Indeed, since the exponents along Z_1 (resp. Z_2) are e and e (resp. 0 and h), the logarithmic connection satisfies the overconvergence condition and is overconvergent along D. Moreover, it satisfies the conditions (a) and (b) in 10.3.4.2. If $g_1 : \mathfrak{X} \to \mathfrak{Z}_1$ is the second projection (note that the coordinate of $\mathfrak{Z}_1 \cap \mathfrak{Y}$ is z_2), then

$$\begin{aligned}
&\mathbb{R}g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_Y^\dagger\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\sharp_{1K}} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \\
&\qquad \cong \left[g_{1K*}\left(E(m\mathfrak{Z}_1)/E\right) \overset{g_{K*}(\frac{dz_1}{z_1}\otimes\partial_{\sharp 1})}{\longrightarrow} g_{1K*}(j_Y^\dagger\Omega^1_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\sharp_{1K}} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathfrak{Z}_1)/E)\right][-1]
\end{aligned}$$

for $m \geq e$ by 10.3.4.5. Hence $\mathbb{R}^q g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_Y^\dagger\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\sharp_{1K}} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$ for $q \neq 1, 2$ and

$$\mathbb{R}^q g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_Y^\dagger\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\sharp_{1K}} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \cong \begin{cases} j^\dagger_{Z_1\cap Y}\mathcal{O}_{]Z_1[_{\mathfrak{Z}_1}} z_1^{-e}v_1 & \text{if } q = 1, \\ (j^\dagger_{Z_1\cap Y}\mathcal{O}_{]Z_1[_{\mathfrak{Z}_1}}/z_2^h j^\dagger_{Z_1\cap Y}\mathcal{O}_{]Z_1[_{\mathfrak{Z}_1}})z_1^{-e}v_1 \oplus j^\dagger_{Z_1\cap Y}\mathcal{O}_{]Z_1[_{\mathfrak{Z}_1}} z_1^{-e}v_2 & \text{if } q = 2. \end{cases}$$

Therefore, $\mathbb{R}^2 g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_Y^\dagger\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\sharp_{1K}} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E)$ is not always locally free. By 10.3.4.22.2 and using a spectral sequence, the dimensions of total cohomology groups are as follow:

$$\dim_K \mathbb{H}^q(]X[_{\mathfrak{X}}, \underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_Y^\dagger\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{S}_K} \otimes_{j_Y^\dagger\mathcal{O}_{]X[_{\mathfrak{X}}}} E)) = \begin{cases} 1 & \text{if } q = 1, \\ 2 & \text{if } q = 2, \\ 1 & \text{if } q = 3, \\ 0 & \text{if } q \neq 1, 2, 3. \end{cases}$$

10.4 Theorems of full faithfulness for overconvergent isocrystals

10.4.1 Pullback -restriction functor

The proper descent theorem of Shiho (see [Shi07, 7.3]) implies that the canonical functor of the theorem 10.4.1.1 below is fully faithful. When we have a Frobenius structure, we will establish that this functor induced an equivalence of categories (see 10.4.1.3).

Theorem 10.4.1.1. *Let $a\colon X^{(0)} \to X$ be a proper surjective morphism of k-varieties, Y a dense open subset of X, $j\colon Y \hookrightarrow X$ the open immersion, $Y^{(0)} := a^{-1}(Y)$. The canonical functor*

$$(a^*, j^*)\colon \mathrm{MIC}^\dagger(Y, X/K) \to \mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K) \times_{\mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K)} \mathrm{MIC}^\dagger(Y, Y/K) \qquad (10.4.1.1.1)$$

is fully faithful.

Proof. Since the canonical functor $\mathrm{MIC}^\dagger(Y, X/K) \to \mathrm{MIC}^\dagger(Y, Y/K)$ is faithful, this yields that so is the canonical functor of 10.4.1.1.1.

Let E_1, E_2 two objects of $\mathrm{MIC}^\dagger(Y, X/K)$. Denote by $E_1^{(0)}, E_2^{(0)}$ (resp. $\widehat{E}_1$, $\widehat{E}_2$) the associated objects of $\mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K)$ (resp. $\mathrm{MIC}^\dagger(Y, Y/K)$). Let $\alpha^{(0)} : E_1^{(0)} \to E_2^{(0)}$ be a morphism of $\mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K)$, $\beta : \widehat{E}_1 \to \widehat{E}_2$ be a morphism of $\mathrm{MIC}^\dagger(Y, Y/K)$ such that $\alpha^{(0)}$ and β induce canonically the same morphism of $\mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K)$ of the form $b^*(\widehat{E}_1) \to b^*(\widehat{E}_2)$. We have to build a morphism $\alpha : E_1 \to E_2$ of $\mathrm{MIC}^\dagger(Y, X/K)$ inducing $\alpha^{(0)}$ and β.

Denote by $b : Y^{(0)} \to Y$ the morphism induced by a and, for $i = 1, 2$, $a_i : X^{(0)} \times_X X^{(0)} \to X^{(0)}$ and $b_i : Y^{(0)} \times_Y Y^{(0)} \to Y^{(0)}$ the respectively left and right canonical projections. Consider the diagram of $\mathrm{MIC}^\dagger(Y^{(0)} \times_Y Y^{(0)}, X^{(0)} \times_X X^{(0)}/K)$ of left below

$$\begin{array}{ccccccc} a_1^*a^*(E_1) & = & a_1^*(E_1^{(0)}) & \xrightarrow{a_1^*(\alpha^{(0)})} & a_1^*(E_2^{(0)}) & = & a_1^*a^*(E_2) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ a_2^*a^*(E_1) & = & a_2^*(E_1^{(0)}) & \xrightarrow{a_2^*(\alpha^{(0)})} & a_2^*(E_2^{(0)}) & = & a_2^*a^*(E_2), \end{array} \quad \begin{array}{ccc} b_1^*b^*(\widehat{E}_1) & \xrightarrow{b_1^*b^*(\beta)} & b_1^*b^*(\widehat{E}_2) \\ \downarrow\wr & & \downarrow\wr \\ b_2^*b^*(\widehat{E}_1) & \xrightarrow{b_2^*b^*(\beta)} & b_2^*b^*(\widehat{E}_2) \end{array} \tag{10.4.1.1.2}$$

whose vertical isomorphisms follow from the equality $a \circ a_1 = a \circ a_2$. By applying the canonical functor of restriction $\mathrm{MIC}^\dagger(Y^{(0)} \times_Y Y^{(0)}, X^{(0)} \times_X X^{(0)}/K) \to \mathrm{MIC}^\dagger(Y^{(0)} \times_Y Y^{(0)}, Y^{(0)} \times_Y Y^{(0)}/K)$ to the left diagram of 10.4.1.1.2, we get the diagram to its right which is commutative by functoriality. Since this functor is faithful, this yields then the commutativity of the left diagram of 10.4.1.1.2. By using proper descent theorem of Shiho (see [Shi07, 7.3]), this yields the existence of one of the morphism $\alpha : E_1 \to E_2$ of $\mathrm{MIC}^\dagger(Y, X/K)$ inducing $\alpha^{(0)}$ and β. □

We will need later de Jong 's desingularisation theorem in the following form:

10.4.1.2 (Desingularization of de Jong)**.** Let X be a k-variety (always reduced by convention). Denote by $X_1, \ldots, X_r$ the irreducible components of X and $i \in \{1, \ldots, r\}$. Let Z be a subvariety of X that do not contain any X_i. Since k is perfect, since X_i is integral, following the de Jong 's desingularisation theorem (see [dJ96] or [Ber97a]), then there exists a projective, surjective, generically finite and etale morphism $f_i : X_i' \to X_i$ such that X_i' is integral and smooth and $f^{-1}(Z \cap X_i)$ is a normal crossing divisor of X_i'. Denoting by X' the disjoint union of X_i', this yields a projective, surjective morphism of the form $f : X' \to X$ which is moreover generically finite and etale (i.e, there exists a dense open set Y of X such that $f^{-1}(Y) \to Y$ is finite and etale) and such that $f^{-1}(Z)$ is the support a normal crossing divisor strict of X'.

With some Frobenius structures, we get thanks to theorem of full faithfulness of Kedlaya the theorem below which extends [Éte02, Theorem 2]:

Theorem 10.4.1.3. *Let $a : X^{(0)} \to X$ be a proper surjective morphism of k-varieties, Y an open set of X, $Y^{(0)} := a^{-1}(Y)$. The canonical functor*

$$(a^*, j^*) : F\text{-}\mathrm{MIC}^\dagger(Y, X/K) \to F\text{-}\mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K) \times_{F\text{-}\mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K)} F\text{-}\mathrm{MIC}^\dagger(Y, Y/K) \tag{10.4.1.3.1}$$

is an equivalence of categories.

Proof. (I) following 10.4.1.1, the functor is fully faithful. Denote by $X^{(1)} := X^{(0)} \times_X X^{(0)}$, $Y^{(1)} := Y^{(0)} \times_Y Y^{(0)}$, $X^{(2)} := X^{(0)} \times_X X^{(0)} \times_X X^{(0)}$, $Y^{(2)} := Y^{(0)} \times_Y Y^{(0)} \times_Y Y^{(0)}$ (following our conventions, these fibered products are computed In the category of schemes reduced). For $i = 1, 2$, let us denote by $a_i : X^{(1)} \to X^{(0)}$ the respectively left and right canonical projections.

(II) Let us establish now that this functor is essentiellement surjective.

Let $(E^{(0)}, \widehat{E}, \rho)$ an object of $F\text{-}\mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K) \times_{F\text{-}\mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K)} F\text{-}\mathrm{MIC}^\dagger(Y, Y/K)$. Following the de Jong 's desingularisation theorem (of the form 10.4.1.2 and by recalling that $X^{(1)}$ is reduced by convention), there exists $a' : X'' \to X^{(1)}$ be a morphism projective, surjective such that X'' is smooth. Denote by $Y'' := X'' \setminus a'^{-1}(Y^{(1)})$, $b' : Y'' \to Y^{(1)}$ the morphism induced by a', $j'' : Y'' \subset X''$ and, for $i = 0, 1, 2$, $j^{(i)} : Y^{(i)} \subset X^{(i)}$ the canonical open immersions. We denote by $j''^* : F\text{-}\mathrm{MIC}^\dagger(Y'', X''/K) \to F\text{-}\mathrm{MIC}^\dagger(Y'', Y''/K)$ and $j^{(i)*} : F\text{-}\mathrm{MIC}^\dagger(Y^{(i)}, X^{(i)}/K) \to F\text{-}\mathrm{MIC}^\dagger(Y^{(i)}, Y^{(i)}/K)$ the functors restrictions.

i) Construction of the glueing isomorphism $\theta\colon a_2^*(E^{(0)}) \xrightarrow{\sim} a_1^*(E^{(0)})$. To do so, we use the full faithfulness (checked in the part (I) of the denoting by) of the canonical functor below denoted by

$$\phi := (a'^*, j^{(1)*})\colon F\text{-MIC}^\dagger(Y^{(1)}, X^{(1)}/K) \to F\text{-MIC}^\dagger(Y'', X''/K) \times_{F\text{-MIC}^\dagger(Y'',Y''/K)} F\text{-MIC}^\dagger(Y^{(1)}, Y^{(1)}/K). \tag{10.4.1.3.2}$$

For $i = 1, 2$, denote by ρ_i the canonical isomorphism $j''^* \circ a'^*[a_i^*(E^{(0)})] \xrightarrow{\sim} b'^* \circ j^{(1)*}[a_i^*(E^{(0)})]$. Hence, the image of $a_i^*(E^{(0)})$ by ϕ is $\phi(a_i^*(E^{(0)})) = (a'^*[a_i^*(E^{(0)})], j^{(1)*}[a_i^*(E^{(0)})], \rho_i)$.

We define canonically the isomorphisms $\theta_{Y^{(1)}}\colon j^{(1)*}[a_2^*(E^{(0)})] \xrightarrow{\sim} j^{(1)*}[a_1^*(E^{(0)})]$ and $\widehat{\theta}\colon b_2^*(j^{(0)*}(E^{(0)})) \xrightarrow{\sim} b_1^*(j^{(0)*}(E^{(0)}))$ from ρ as that making commutative the canonical diagram below:

$$\begin{array}{ccccc} j^{(1)*}[a_2^*(E^{(0)})] & \xrightarrow{\sim} & b_2^*(j^{(0)*}(E^{(0)})) & \xrightarrow[b_2^*(\rho)]{\sim} & b_2^*(b^*(\widehat{E})) \\ \downarrow{\scriptstyle \theta_{Y^{(1)}}} & & \downarrow{\scriptstyle \widehat{\theta}} & & \downarrow{\scriptstyle \sim} \\ j^{(1)*}[a_1^*(E^{(0)})] & \xrightarrow{\sim} & b_1^*(j^{(0)*}(E^{(0)})) & \xrightarrow[b_1^*(\rho)]{\sim} & b_1^*(b^*(\widehat{E})). \end{array} \tag{10.4.1.3.3}$$

We define canonically the isomorphism $\theta_{Y''}\colon j''^* a'^*[a_2^*(E^{(0)})] \xrightarrow{\sim} j''^* a'^*[a_1^*(E^{(0)})]$ from ρ via the commutative diagram :

$$\begin{array}{ccc} j''^* a'^*[a_2^*(E^{(0)})] & \xrightarrow[\rho_2]{\sim} & b'^* j^{(1)*}[a_2^*(E^{(0)})] \\ \downarrow{\scriptstyle \theta_{Y''}} & & {\scriptstyle b'^*(\theta_{Y^{(1)}})}\downarrow{\scriptstyle \sim} \\ j''^* a'^*[a_1^*(E^{(0)})] & \xrightarrow[\rho_1]{\sim} & b'^* j^{(1)*}[a_1^*(E^{(0)})]. \end{array} \tag{10.4.1.3.4}$$

Since X'' is smooth, following the theorem [Ked08, 4.2.1], the functor j''^* is fully faithful. Then there exists one and only one isomorphism $\theta''\colon a'^* a_2^*(E^{(0)}) \xrightarrow{\sim} a'^* a_1^*(E^{(0)})$ such that $j''^*(\theta'') = \theta_{Y''}$. It follows from the commutativity of 10.4.1.3.4, that $(\theta'', \theta_{Y^{(1)}})$ induced an isomorphism of the form $\phi(a_2^*(E^{(0)})) \xrightarrow{\sim} \phi(a_1^*(E^{(0)}))$. By full faithfulness of ϕ, we get an isomorphism $\theta\colon a_2^*(E^{(0)}) \xrightarrow{\sim} a_1^*(E^{(0)})$ such that $a'^*(\theta) = \theta''$ and $j^{(1)*}(\theta) = \theta_{Y^{(1)}}$.

ii) Let us check now that θ satisfies the cocycle condition. Moreover, it is sufficient to check after applying the functor $j^{(2)*}$ since this is faithful. Since $j^{(1)*}(\theta) = \theta_{Y^{(1)}}$, this is a consequence of the construction of $\theta_{Y^{(1)}}$ (see 10.4.1.3.3).

iii) By using proper descent theorem of Shiho (see [Shi07, 7.3]), this yields the existence of an object $E \in F\text{-MIC}^\dagger(Y, X/K)$ (unique up to canonical isomorphism) and of an isomorphism $\rho_{(Y^{(0)},X^{(0)})}\colon a^*(E) \xrightarrow{\sim} E^{(0)}$ which can be included in the commutative diagram

$$\begin{array}{ccc} a_2^* a^*(E) & \xrightarrow[a_2^*\rho_{(Y^{(0)},X^{(0)})}]{\sim} & a_2^*(E^{(0)}) \\ \downarrow{\scriptstyle \sim} & & {\scriptstyle \theta}\downarrow{\scriptstyle \sim} \\ a_1^* a^*(E) & \xrightarrow[a_1^*\rho_{(Y^{(0)},X^{(0)})}]{\sim} & a_1^*(E^{(0)}). \end{array} \tag{10.4.1.3.5}$$

We define the isomorphism $\rho_{(Y^{(0)},Y^{(0)})}\colon b^* j^*(E) \xrightarrow{\sim} b^*\widehat{E}$ via the commutative diagram :

$$\begin{array}{ccc} j^{(0)*} a^*(E) & \xrightarrow[j^{(0)*}\rho_{(Y^{(0)},X^{(0)})}]{\sim} & j^{(0)*}(E^{(0)}) \\ {\scriptstyle \rho_{\mathrm{can}}}\downarrow{\scriptstyle \sim} & & {\scriptstyle \rho}\downarrow{\scriptstyle \sim} \\ b^* j^*(E) & \xrightarrow[\rho_{(Y^{(0)},Y^{(0)})}]{\sim} & b^*\widehat{E}, \end{array} \tag{10.4.1.3.6}$$

where ρ_{can} is the canonical isomorphism. Consider now the "cube A" whose face of the front side is the image by the functor b_2^* of 10.4.1.3.6, the face of the back is the image by the functor b_1^* of 10.4.1.3.6, the right face corresponds to the right square of 10.4.1.3.3 and whose isomorphisms of the left face are canonical. The left face of the cube A is canonically commutative. The right side (resp. of the front side, resp. of the back side) the is by definition. To ensure the commutativity of the cube A, then it is sufficient to validate that of the top face. For this Consider the "cube B" whose face of the bottom is the face of the top of the cube A, whose face of the top is the image by $j^{(1)*}$ of the commutative square 10.4.1.3.5, whose face of the front side (resp. of the back side) is the commutative square by functoriality in the isomorphism $j^{(1)*} \circ a_2^* \xrightarrow{\sim} b_2^* \circ j^{(0)*}$ (resp. $j^{(1)*} \circ a_1^* \xrightarrow{\sim} b_1^* \circ j^{(0)*}$). We remark that the right face of the cube B is the left square of 10.4.1.3.3 (this has a meaning since $j^{(1)*}(\theta) = \theta_{Y^{(1)}}$). Hence it is commutative. That of the left face of the cube B can be checked canonically. The cube B is then commutative. Hence so is the cube A. Hence, the face of the bottom of the cube A is commutative, i.e., the isomorphism $\rho_{(Y^{(0)},Y^{(0)})} \colon b^* j^*(E) \xrightarrow{\sim} b^* \widehat{E}$ commutes with glueing data. This yields there exists an isomorphism $\rho_{(Y,Y)} \colon j^*(E) \xrightarrow{\sim} \widehat{E}$ such that $\rho_{(Y^{(0)},Y^{(0)})} = b^*(\rho_{(Y,Y)})$. The commutativity of the diagram 10.4.1.3.6 means then that we have the isomorphism $(\rho_{(Y^{(0)},X^{(0)})}, \rho_{(Y,Y)}) \colon (a^*(E), j^*(E), \rho_{\mathrm{can}}) \xrightarrow{\sim} (E^{(0)}, \widehat{E}, \rho)$. Hence we are done.

□

10.4.2 Restriction functor

Theorem 10.4.2.1. *Let X be a k-variety, Y an open set of X and $j \colon Y \hookrightarrow X$ the corresponding open immersion. The functor $j^* \colon F\text{-MIC}^\dagger(Y, X/K) \to F\text{-MIC}^\dagger(Y, Y/K)$ is fully faithful.*

Proof. Since the faithfulness is well known, then it remains to prove that this one is full. Following the de Jong 's desingularisation theorem (of the form 10.4.1.2), there exists a morphism projective, surjective, generically finite and etale of k-varieties of the form $a \colon X^{(0)} \to X$ such that $X^{(0)}$ is smooth. Let $Y^{(0)} := a^{-1}(Y)$. Let $E_1, E_2 \in F\text{-MIC}^\dagger(Y, X/K)$ and $\widehat{\phi} \colon j^* E_1 \to j^* E_2$. Moreover, since $X^{(0)}$ is smooth, the canonical functor $F\text{-MIC}^\dagger(Y^{(0)}, X^{(0)}/K) \to F\text{-MIC}^\dagger(Y^{(0)}, Y^{(0)}/K)$ is fully faithful (see [Ked08, 4.2.1]). The morphism $\widehat{\phi}$ induced then canonically a morphism $\phi^{(0)} \colon a^*(E_1) \to a^*(E_2)$. Following theorem 10.4.1.3 there exists a morphism $\phi \colon E_1 \to E_2$ inducing $\phi^{(0)}$ and $\widehat{\phi}$. □

Remark 10.4.2.2. N. Tsuzuki conjecture (see the conjecture [Tsu02, 1.2.1]) that the functor j^* of 10.4.2.1 remains fully faithful without Frobenius structure.

10.4.3 Localisation and inverse image-localisation functors

10.4.3.1 (Reminders over the functor extension)**.** Let X be a variety over k, Y an open set of X with Y normal, $\widetilde{Y}$ a dense open subset of Y, $\widetilde{j} \colon \widetilde{Y} \hookrightarrow X$ the corresponding open immersion. Tsuzuki has proved in [Tsu09] t (see also [Tsu02, 4.1.1], [Ked07, 5.2.1] for previous versions) hat the canonical functor $\widetilde{j}^\dagger \colon \mathrm{MIC}^\dagger(Y, X/K) \to \mathrm{MIC}^\dagger(\widetilde{Y}, X/K)$ is fully faithful. This theorem was previously established by Tsuzuki when X is smooth (see [Tsu02, 4.1.1]), next extended by Kedlaya when Y is smooth in [Ked07, 5.2.1] (we will not need that in the case where the variety is smooth). We denote by $\mathrm{MIC}^\dagger(\widetilde{Y} \subset Y, X/K)$ the essential image of the functor $\widetilde{j}^\dagger \colon \mathrm{MIC}^\dagger(Y, X/K) \to \mathrm{MIC}^\dagger(\widetilde{Y}, X/K)$.

The lemma below is obvious considering the paragraph 10.4.3.1 and of its notations.

Lemma 10.4.3.2. *Let $a \colon X^{(0)} \to X$ be a proper surjective generically finite and etale morphism of integral k-varieties, Y of X, $\widetilde{Y}$ a dense open subset of Y, $\widetilde{j} \colon \widetilde{Y} \hookrightarrow X$ the corresponding open immersion, $Y^{(0)} := a^{-1}(Y)$, $\widetilde{Y}^{(0)} := a^{-1}(\widetilde{Y})$. We suppose moreover Y and $Y^{(0)}$ smooth. The canonical functor*

$$(a^*, \widetilde{j}^\dagger) \colon \mathrm{MIC}^\dagger(Y, X/K) \to \mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K) \times_{\mathrm{MIC}^\dagger(\widetilde{Y}^{(0)}, X^{(0)}/K)} \mathrm{MIC}^\dagger(\widetilde{Y} \subset Y, X/K) \quad (10.4.3.2.1)$$

is an equivalence of categories.

Chapter 11

Arithmetic $\mathcal{D}$-modules associated with overconvergent isocrystals: the lifted case with divisorial singularities

11.1 Convergent log-isocrystals as arithmetic $\mathcal{D}$-modules

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{Y}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is p-torsion free, noetherian of finite Krull dimension. For any integer $i \geq 0$, set $Y_i^\sharp := \mathfrak{Y}^\sharp \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$.

11.1.1 The category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S})$ of convergent log-isocrystals

Proposition 11.1.1.1. *Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-module. The following conditions are equivalent.*

(a) The structure of left $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module extends to a structure of left $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules.

(b) For any open $\mathfrak{U}^\sharp$ endowed with logarithmic coordinates, for each $\eta < 1$, for all $e \in \Gamma(\mathfrak{U}^\sharp, \mathcal{E})$, the following convergent condition holds

$$\|\underline{\partial}_\sharp^{[\underline{k}]}(e)\|\eta^{|\underline{k}|} \to 0, \text{ as } |\underline{k}| \to \infty, \qquad (11.1.1.1.1)$$

where $\Gamma(\mathfrak{U}^\sharp, \mathcal{E})$ is endowed with the topology given by its structure of $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$-module of finite type.

Proof. Suppose (b) holds. Let $\mathfrak{U}^\sharp \subset \mathfrak{Y}^\sharp$ be an affine open endowed with logarithmic coordinates. Let $P \in \Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$. Then P can be written of the form $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}_\sharp^{[\underline{k}]}$ with $a_{\underline{k}} \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$ satisfying the growth condition of Proposition 8.7.1.8.(c). Hence, for any $e \in \Gamma(\mathfrak{U}, \mathcal{E})$, the convergence condition 11.1.1.1.1 implies that $a_{\underline{k}} \underline{\partial}_\sharp^{[\underline{k}]}(e) \to 0$ as $|\underline{k}| \to \infty$. This yields we can define $P(e)$ as the sum of the serie $P(e) = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}_\sharp^{[\underline{k}]}(e)$. This gives a pairing

$$\Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \times \Gamma(\mathfrak{U}, \mathcal{E}) \to \Gamma(\mathfrak{U}, \mathcal{E})$$

and a canonical structure of $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module on the given $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-module $\mathcal{E}$ which extends the structure of $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module.

Suppose (a) holds. Take $\eta < 1$ and let $\mathfrak{U}^\sharp$ be an affine open endowed with logarithmic coordinates. By 8.7.1.7 and with notation 1.2.1.3.1, there exists $m \in \mathbb{N}$ such that $\eta^{|\underline{k}|} < c_{\underline{k}} |q_{\underline{k}}^{(m)}!|$, with $c_{\underline{k}} \to 0$. As $q_{\underline{k}}^{(m)}! \underline{\partial}_\sharp^{[\underline{k}]} = \underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}$, it suffices to show that, for any m and any $e \in \Gamma(\mathfrak{U}, \mathcal{E}) = E$, the elements $\underline{\partial}_\sharp^{\langle \underline{k} \rangle_{(m)}}(e)$, $\underline{k} \in \mathbb{N}^d$, form a bounded family in E for its natural topology of an $A_{\mathbb{Q}}$-module of finite type.

Write $\widehat{D}^{(m)}_{\mathfrak{U}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} = \Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})$. Take $P = \sum a_{\underline{k}} \underline{\partial}_\sharp^{\langle \underline{k}\rangle}$ with $a_{\underline{k}} \to 0$. We extend the Banach norm $\|\cdot\|$ on $A_\mathbb{Q} = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$ to $\widehat{D}^{(m)}_{\mathfrak{U}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ by setting $\|P\| = \sup \|a_{\underline{k}}\|$. As the topology on E as $A_\mathbb{Q}$-module of finite type coincides with its topology as $\widehat{D}^{(m)}_{\mathfrak{U}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module of finite type (see 7.5.2.6), the elements $\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$ form a bounded family in $\widehat{D}^{(m)}_{\mathfrak{U}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$, the same holds for their images under the map $\widehat{D}^{(m)}_{\mathfrak{U}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \to E$ given by $P \mapsto P(e)$. □

Corollary 11.1.1.2. *Suppose that $\mathfrak{S} \to \operatorname{Spf} \mathcal{V}$ is smooth, and suppose there exist a smooth $\mathfrak{S}$-formal scheme $\mathfrak{Y}$, a relative strict normal crossing divisor $\mathfrak{Z}$ of $\mathfrak{Y}$ over $\mathfrak{S}$, such that $\mathfrak{Y}^\sharp = (\mathfrak{Y}, M_\mathfrak{Z})$ is the logarithmic $\mathcal{V}$-formal scheme whose underlying logarithmic structure $M_\mathfrak{Z}$ is the one associated with $\mathfrak{Z}$. We have the morphism of ringed spaces $\mathrm{sp}\colon (\mathfrak{Y}_K, \mathcal{O}_{\mathfrak{Y}_K}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$ induced by the specialization morphism. We get the inverse image functor sp^* by setting $\mathrm{sp}^*(\mathcal{E}) := \mathcal{O}_{\mathfrak{Y}_K} \otimes_{\mathrm{sp}^{-1}\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} \mathrm{sp}^{-1}(\mathcal{E})$, for any $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-modules $\mathcal{E}$.*

The functors sp_* and sp^* induce quasi-inverse equivalences between the category of (locally free) coherent $\mathcal{O}_{\mathfrak{Y}_K}$-modules together with a convergent logarithmic connection relative to $\mathfrak{Y}^\sharp_K/\mathfrak{S}_K$ and that of left $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S},\mathbb{Q}}$-modules which are (locally projective) coherent as $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-module.

Proof. This is a consequence of 11.1.1.1 and of the fact that the convergent condition 10.3.2.5 corresponds to that of 11.1.1.1.1.. □

Notation 11.1.1.3. We denote by $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$ the full subcategory of the category of $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules consisting of $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp,\mathbb{Q}}$-modules which are $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Its objects are called "convergent log-isocrystals on $\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp$". When log structures are trivial, we drop *log*, i.e. we say "convergent isocrystals on $\mathfrak{Y}/\mathfrak{S}$". Remark that any morphism $\mathcal{E} \to \mathcal{E}'$ of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$ is $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-linear. Indeed, for any affine open formal subscheme $\mathfrak{U} \subset \mathfrak{Y}$, the morphism $\Gamma(\mathfrak{U}, \mathcal{E}) \to \Gamma(\mathfrak{U}, \mathcal{E}')$ is continuous for the Banach space topology.

Proposition 11.1.1.4. *Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is locally projective of finite type over $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$. The object $\mathcal{E}$ is a convergent log-isocrystal on $\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp$ if and only if the structure of $\mathcal{D}_{\mathfrak{Y}^*,\mathbb{Q}}$-module of $\mathcal{E}|_{\mathfrak{Y}^*}$ extends to a structure of coherent $\mathcal{D}^\dagger_{\mathfrak{Y}^*,\mathbb{Q}}$-module.*

Proof. By using the translation 11.1.1.1.(b) of the objects of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$, we reduce to copy 10.3.2.7. □

Proposition 11.1.1.5. *Let $\mathcal{E}$ be a $\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module, coherent as $\mathcal{O}_\mathfrak{Y}$-module.*

(a) If $\mathfrak{Y}$ is affine then $\mathcal{E}$ is globally of finite presentation on $\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$.

(b) The sheaf $\mathcal{E}$ is coherent on $\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$.

(c) The canonical homomorphism $\mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{E}$ is an isomorphism.

Proof. This is a particular case of 7.5.2.1. □

Proposition 11.1.1.6. *Let $\mathcal{E}$ be an object of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. Let m be an integer.*

(a) There exists a p-torsion free $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module $\overset{\circ}{\mathcal{E}}$, coherent over $\mathcal{O}_\mathfrak{Y}$ together with a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-linear isomorphism $\overset{\circ}{\mathcal{E}}_\mathbb{Q} \xrightarrow{\sim} \mathcal{E}$.

(b) The sheaf $\mathcal{E}$ is a coherent $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module and a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module. Moreover, the canonical homomorphism $\mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}$ is an isomorphism.

(c) The sheaf $\mathcal{E}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module and the canonical homomorphisms $\mathcal{E} \to \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}$ is an isomorphism.

Proof. The assertion (a) (resp. (b)) is a particular case of 7.5.2.8.(a) (resp. 7.5.2.8.(b)). Taking the inductive limit on the level, we get the last statement. □

Proposition 11.1.1.7 (Theorem A)**.** *We assume that $\mathfrak{Y}^\sharp$ is affine and $\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp$ has coordinates. Set $A := \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$, $D_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} := \Gamma(\mathfrak{Y}, \mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ and $D^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} := \Gamma(\mathfrak{Y}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. We denote by* $\mathrm{MIC}^\dagger(A_K/K)$ *the category of coherent $D^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,K}$-module which are also coherent as A_K-module.*

(a) The functors $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{D^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,K}} -$ and $\Gamma(\mathfrak{Y}, -)$ are quasi-inverse equivalences between $\mathrm{MIC}^\dagger(A_K/K)$ *and* $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$.

(b) For any $E \in \mathrm{MIC}^\dagger(A_K/K)$, the canonical morphisms

$$E \to D^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{D_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} E, \qquad \mathcal{O}_{\mathfrak{Y},\mathbb{Q}} \otimes_{A_K} E \to \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{D^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} E \tag{11.1.1.7.1}$$

are isomorphisms.

Proof. 1) Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{U}/\mathcal{V})$. It follows from Theorem of type A for coherent $\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}}$-modules and coherent $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-modules that $\Gamma(\mathfrak{U}, \mathcal{E}) \in \mathrm{MIC}^\dagger(A_K/K)$ (recall also definition 11.1.1.3).

2) Let $E \in \mathrm{MIC}^\dagger(A_K/K)$. Let us check that both morphisms of 11.1.1.7.1 are isomorphisms. By copying the demonstration of 11.1.1.1, we prove that for all $\eta < 1$, for all $e \in E$, we have

$$\| \underline{\partial}^{[\underline{k}]} \cdot e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty, \tag{11.1.1.7.2}$$

where $\| - \|$ denotes a Banach norm on E defined by its structure of $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$-module of finite type. This condition 11.1.1.7.2 is the convergence condition 10.3.2.4.1 (still valid without log structures). By copying the proof of 11.1.1.6, we then deduce that, for all $m \in \mathbb{N}$, there exists a $\Gamma(\mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$-module $\mathring{E}^{(m)}$, which is furthermore a $\Gamma(Y, \mathcal{O}_{\mathfrak{Y}})$-module of finite type together with an isomorphism $\mathring{E}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} E$. Now, it follows by p-adic completion from 4.3.4.6.1 that the canonical morphism

$$\mathcal{O}_{\mathfrak{Y}} \widehat{\otimes}_{\Gamma(Y,\mathcal{O}_{\mathfrak{Y}})} \mathring{E}^{(m)} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}_{\Gamma(\mathfrak{Y},\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})} \mathring{E}^{(m)}$$

is an isomorphism. Since $\mathring{E}^{(m)}$ is of finite type as $\Gamma(Y, \mathcal{O}_{\mathfrak{Y}})$-module and therefore as $\Gamma(\mathfrak{Y}, \mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$-module, the morphisms

$$\mathcal{O}_{\mathfrak{Y}} \otimes_{\Gamma(Y,\mathcal{O}_{\mathfrak{Y}})} \mathring{E}^{(m)} \to \mathcal{O}_{\mathfrak{Y}} \widehat{\otimes}_{\Gamma(Y,\mathcal{O}_{\mathfrak{Y}})} \mathring{E}^{(m)}, \ \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\Gamma(\mathfrak{Y},\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})} \mathring{E}^{(m)} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}_{\Gamma(\mathfrak{Y},\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})} \mathring{E}^{(m)}$$

are also isomorphisms. The morphism $\mathcal{O}_{\mathfrak{Y}} \otimes_{\Gamma(Y,\mathcal{O}_{\mathfrak{Y}})} \mathring{E}^{(m)} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\Gamma(\mathfrak{Y},\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})} \mathring{E}^{(m)}$ is therefore an isomorphism. Tensoring this isomorphism with $\mathbb{Q}$ and taking the inductive limit on m, this implies the canonical morphism:

$$\mathcal{O}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\Gamma(Y,\mathcal{O}_{\mathfrak{Y},\mathbb{Q}})} E \to \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}} \otimes_{\Gamma(\mathfrak{Y},\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})} E \tag{11.1.1.7.3}$$

is an isomorphism. Now, as $\mathring{E}^{(m)}$ is separated complete and of finite type as $\Gamma(\mathfrak{Y}, \mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$-module, the morphism $\mathring{E}^{(m)} \to \Gamma(\mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \otimes_{\Gamma(\mathfrak{Y},\mathcal{D}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})} \mathring{E}^{(m)}$ is an isomorphism. Tensoring with $\mathbb{Q}$ and taking the inductive limit on m, it follows that the canonical morphism

$$E \to \Gamma(\mathfrak{Y}, \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}) \otimes_{\Gamma(\mathfrak{Y},\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})} E \tag{11.1.1.7.4}$$

is an isomorphism and we have checked the first isomorphism of 11.1.1.7.1. From 11.1.1.7.3 and 11.1.1.7.4, we get the canonical morphism

$$\mathcal{O}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\Gamma(Y,\mathcal{O}_{\mathfrak{Y},\mathbb{Q}})} E \to \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\Gamma(\mathfrak{Y},\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}})} E \tag{11.1.1.7.5}$$

is an isomorphism, and we are done.

3) Let $E \in \mathrm{MIC}^\dagger(A_K/K)$. Thanks to the isomorphism 11.1.1.7.1, it follows from Theorem of type A for coherent $\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}}$-modules and coherent $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-modules that $\mathcal{E} := \mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}} \otimes_{D^\dagger_{\mathfrak{U},K}} E \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. □

Corollary 11.1.1.8. *We assume that $\mathfrak{Y}^\sharp$ is affine and $\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp$ has coordinates. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module. Then $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent if and only if $\Gamma(\mathfrak{Y},\mathcal{E})$ is a $\Gamma(\mathfrak{Y},\mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$-module of finite type.*

Proposition 11.1.1.9. *Suppose that $\mathfrak{S}\to \operatorname{Spf}\mathcal{V}$ is smooth, and suppose there exist a smooth $\mathfrak{S}$-formal scheme $\mathfrak{Y}$, a relative strict normal crossing divisor $\mathfrak{Z}$ of $\mathfrak{Y}$ over $\mathfrak{S}$, such that $\mathfrak{Y}^\sharp=(\mathfrak{Y},M_{\mathfrak{Z}})$ is the logarithmic $\mathcal{V}$-formal scheme whose underlying logarithmic structure $M_{\mathfrak{Z}}$ is the one associated with $\mathfrak{Z}$.*

Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S},\mathbb{Q}}$-module which is locally projective of finite type on $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$. The following conditions are equivalent:

(a) $\mathcal{E}$ is a convergent log-isocrystal on $\mathfrak{Y}^\sharp/\mathfrak{S}$ (in the sense of 11.1.1.3) ;

(b) the structure of $\mathcal{O}_{\mathfrak{Y}}({}^\dagger Z)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}}\mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S},\mathbb{Q}}$-module of $\mathcal{O}_{\mathfrak{Y}}({}^\dagger Z)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}}\mathcal{E}$ extends to a structure of coherent $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}$-module.

Proof. Since $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}|\mathfrak{Y}^*=\mathcal{D}^\dagger_{\mathfrak{Y}^*/\mathfrak{S},\mathbb{Q}}$, then by using 11.1.1.4 we get the implication $(b)\Rightarrow(a)$. Conversely, suppose $\mathcal{E}$ be an object of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S})$. Since $\mathcal{E}$ is a topologically nilpotent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S},\mathbb{Q}}$-module which is coherent over $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$ (use 11.1.1.6 and 7.5.2.7), then it follows from 7.5.2.8.2 that the canonical homomorphism

$$\mathcal{B}^{(m)}_{\mathfrak{Y}}(Z)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}}\mathcal{E}\to(\mathcal{B}^{(m)}_{\mathfrak{Y}}(Z)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}})_{\mathbb{Q}}\otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S},\mathbb{Q}}}\mathcal{E}$$

is an isomorphism. Taking the limit, this yields that the canonical homomorphism

$$\mathcal{O}_{\mathfrak{Y}}({}^\dagger Z)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}}\mathcal{E}\to\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}\otimes_{\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S},\mathbb{Q}}}\mathcal{E}$$

is an isomorphism. Hence, we are done. □

We finish the section with the following Lemmas which completes Theorems 11.1.1.2 and 11.1.1.6. They will also be useful in order to prove Berthelot's Theorem 11.2.1.12.

Lemma 11.1.1.10. *Let $m_0\in\mathbb{N}$, $\mathcal{E}^{(m_0)}$ be a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module. We put for any $m\geq m_0$, $\mathcal{E}^{(m)}:=\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}\otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}}\mathcal{E}^{(m_0)}$, and $\mathcal{E}:=\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}\otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}}\mathcal{E}^{(m_0)}$.*

If $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent, then there exists $m_1\geq m_0$ such that for any $m\geq m_1$ the canonical homomorphism $\mathcal{E}^{(m)}\to\mathcal{E}$ is an isomorphism.

Proof. This is a consequence of Proposition 8.4.1.11 and of 11.1.1.6. □

Lemma 11.1.1.11. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent, and $\mathring{\mathcal{E}}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module together with a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-linear isomorphism of the form $\mathcal{E}\xrightarrow{\sim}\mathring{\mathcal{E}}_{\mathbb{Q}}$. Then $\mathring{\mathcal{E}}$ is $\mathcal{O}_{\mathfrak{Y}}$-coherent and $\mathring{\mathcal{E}}/\pi^{i+1}\mathring{\mathcal{E}}$ is a nilpotent $\widehat{\mathcal{D}}^{(m)}_{Y_i^\sharp/S_i^\sharp}$-module (see definition 4.2.1.11).*

Proof. Since $\mathcal{E}$ is a topologically nilpotent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module and is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent (use 11.1.1.6), this is a particular case of 7.5.2.9. □

11.1.2 The category $\mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$

Notation 11.1.2.1. Below are some categories:

(a) According to 8.1.5.1, we denote by $M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$ the category of $\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}$-modules. We get a canonical functor $\mathrm{cst}\colon M(\mathcal{O}_{\mathfrak{Y}})\to M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$ defined by $\mathcal{F}\mapsto\mathcal{F}^{(\bullet)}$ so that $\mathcal{F}^{(m)}\to\mathcal{F}^{(m+1)}$ is the identity of $\mathcal{F}$. Since this functor is exact, this yields the t-exact functor $\mathrm{cst}\colon D(\mathcal{O}_{\mathfrak{Y}})\to D(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$. According to 8.1.5.1, we have the notion of ind-isogenies (resp. of lim-ind-isogenies) of $M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$. Following 8.4.3.4, we denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$ the category localized by lim-ind-isogenies. We remark that $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$ is the subcategory of $\underrightarrow{LM}_{\mathbb{Q}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$ consisting of objects which are locally isomorphic to an object of the form $\mathrm{cst}(\mathcal{G})$ where $\mathcal{G}$ is a coherent $\mathcal{O}_{\mathfrak{Y}}$-module (use 8.4.2.1 and 8.4.3.2).

(b) Following notation 11.1.1.3, we denote by $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$ the category of $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules which are also $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Recall these objects are necessarily $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-coherent (see 11.1.1.6), and when $\mathfrak{S}^\sharp = \operatorname{Spf}\mathcal{V}$ and log structures are trivial, they are $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-locally projective of finite type. We denote by $\mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ consisting of objects $\mathcal{E}^{(\bullet)}$ such that $\underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)}$ are $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent.

Remark 11.1.2.2. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. Let $\widetilde{\mathcal{D}} := \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ or $\widetilde{\mathcal{D}} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$. Let $\mathcal{D} := \mathcal{D}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$ or $\mathcal{D} := \widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$. By using the isomorphisms of 11.1.1.6.(b-c), we check that both morphisms $\mathcal{E} \to \widetilde{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{E} \to \mathcal{E}$ are isomorphisms. This yields that the first morphism is in fact $\widetilde{\mathcal{D}}$-linear. Hence, if $\mathcal{F}$ is a $\widetilde{\mathcal{D}}$-module, then any $\mathcal{D}$-linear morphism $\mathcal{E} \to \mathcal{F}$ is necessarily $\widetilde{\mathcal{D}}$-linear.

Lemma 11.1.2.3. *Let $\mathcal{F}^{(m)}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module and $f\colon \mathcal{F}^{(m)} \to \mathcal{F}^{(m)}$ be a $\mathcal{V}$-linear morphism such that $f_{\mathbb{Q}}\colon \mathcal{F}^{(m)}_{\mathbb{Q}} \to \mathcal{F}^{(m)}_{\mathbb{Q}}$ is equal to $p^N \mathrm{id}$ for some $N \in \mathbb{N}$. Then, for $N' \in \mathbb{N}$ large enough, we have $p^{N'} f = p^{N'+N}\mathrm{id}$.*

Proof. Since $\mathfrak{X}$ is quasi-compact and $\mathcal{F}^{(m)}$ is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module, then the p-torsion part of $\mathcal{F}^{(m)}$ is killed by some power of p. Hence, we are done. □

Proposition 11.1.2.4. *Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. Let $\mathcal{F}^{(0)}$ be a $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module, coherent over $\mathcal{O}_{\mathfrak{Y}}$ together with an isomorphism of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules of the form $\mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. For any $m \in \mathbb{N}$, let $\mathcal{G}^{(m)}$ be the quotient of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ by its p-torsion part. The following conditions hold.*

(a) The module $\mathcal{G}^{(m)}$ is $\mathcal{O}_{\mathfrak{Y}}$-coherent.

(b) The first (resp. second) canonical morphism

$$\mathcal{F}^{(0)} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)} \to \mathcal{G}^{(m)}$$

is an isogeny in the category of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules (resp. of coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules).

(c) $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$ and $\underrightarrow{l}^*_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}) \xrightarrow{\sim} \mathcal{E}$.

Proof. a) Following 11.1.2.2, the canonical morphism $\mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}$ is an isomorphism of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules. The isomorphism $\mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules induces the last isomorphism of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules $\mathcal{G}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}$. Using 11.1.1.11, this yields $\mathcal{G}^{(m)}$ is $\mathcal{O}_{\mathfrak{Y}}$-coherent.

b) i) Let us denote by $\alpha^{(m)}\colon \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)} \to \mathcal{G}^{(m)}$ the canonical epimorphism of coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules. Since $\alpha^{(m)}$ is a morphism of coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules which is an isomorphism after tensoring by $\mathbb{Q}$, then this is an isogeny in the category of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules (use 7.4.5.2).

ii) Let $\iota^{(m)}\colon \mathcal{F}^{(0)} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ be the canonical morphism. It remains to check that $\iota^{(m)}$ is an isogeny. Using b.i) in the case $m = 0$, we get a morphism $\beta^{(0)}\colon \mathcal{G}^{(0)} \to \mathcal{F}^{(0)}$ of coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules such that $\alpha^{(0)} \circ \beta^{(0)} = p^N \mathrm{id}$ and $\beta^{(0)} \circ \alpha^{(0)} = p^N \mathrm{id}$ for some integer N. Since the canonical morphism $\mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}$ is an isomorphism, then the canonical morphism $\mathcal{G}^{(0)}_{\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{G}^{(0)}_{\mathbb{Q}}$ is an isomorphism. Since the canonical morphism $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{G}^{(0)}_{\mathbb{Q}} \to \mathcal{G}^{(m)}_{\mathbb{Q}}$ is an isomorphism, this yields by composition that the canonical $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-linear morphism $\mathcal{G}^{(0)} \to \mathcal{G}^{(m)}$ is an isomorphism after tensoring by $\mathbb{Q}$. Let us denote by $\gamma^{(m)}\colon \mathcal{G}^{(0)} \to \mathcal{G}^{(m)}$

this morphism and by $\gamma^{(m)}_{\mathbb{Q}}\colon \mathcal{G}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}^{(m)}_{\mathbb{Q}}$ the induced isomorphism. Since $\gamma^{(m)}$ is $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-linear (and then $\mathcal{O}_{\mathfrak{Y}}$-linear), since $\mathcal{G}^{(m)}$ is $\mathcal{O}_{\mathfrak{Y}}$-coherent and $\mathcal{G}^{(0)}$ is p-torsion free, then, for N' large enough, $p^{N'}(\gamma^{(m)}_{\mathbb{Q}})^{-1}$ induces the morphism $\delta^{(m)}\colon \mathcal{G}^{(m)} \to \mathcal{G}^{(0)}$ of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules. We get $\kappa^{(m)} := \beta^{(0)} \circ \delta^{(m)} \circ \alpha^{(m)}\colon \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)} \to \mathcal{F}^{(0)}$. Using the Lemma 11.1.2.3, increasing N or N' if necessary in the construction of $\kappa^{(m)}$, we get $\iota^{(m)} \circ \kappa^{(m)} = p^{N+N'}\,\mathrm{id}$ and $\kappa^{(m)} \circ \iota^{(m)} = p^{N+N'}\,\mathrm{id}$. Hence, this morphism $\iota^{(m)}\colon \mathcal{F}^{(0)} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ is an isogeny in the category of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-modules.

c) Finally $\underrightarrow{l}^*_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp},\mathbb{Q}} \mathcal{F}^{(0)}_{\mathbb{Q}}$. Following the first part, the canonical morphism $\mathcal{F}^{(0)}_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp},\mathbb{Q}} \mathcal{F}^{(0)}_{\mathbb{Q}}$ is an isomorphism. We endow $\mathcal{F}^{(0)}_{\mathbb{Q}}$ with the structure of $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module making $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-linear the isomorphism $\mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. Following the remark 11.1.2.2, this yields that the canonical isomorphism $\mathcal{F}^{(0)}_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp},\mathbb{Q}} \mathcal{F}^{(0)}_{\mathbb{Q}}$ is in fact $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-linear. Hence, we get the isomorphism $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp},\mathbb{Q}} \mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. □

Corollary 11.1.2.5. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. The following conditions are equivalent:*

(a) The object $\mathcal{E}^{(\bullet)}$ belongs to $\mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$.

(b) There exists a $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{F}^{(0)}$, coherent over $\mathcal{O}_{\mathfrak{Y}}$ such that

(i) $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ to $\mathcal{E}^{(\bullet)}$

(ii) and the canonical morphism $\mathrm{cst}(\mathcal{F}^{(0)}) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ is an ind-isogeny in $M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$.

Moreover, when $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$, we can choose such $\mathcal{F}^{(0)}$ without p-torsion.

Proof. Let $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)}$ be the corresponding $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module. If such $\mathcal{F}^{(0)}$ exists, then $\mathcal{E}$ is in particular $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent and then by definition $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. Conversely, suppose $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. Then $\mathcal{E}$ is a $\mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is also $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Using 11.1.1.6, there exists a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{F}^{(0)}$ without p-torsion, coherent over $\mathcal{O}_{\mathfrak{Y}}$ together with an isomorphism of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules $\mathcal{E} \xrightarrow{\sim} \mathcal{F}^{(0)}_{\mathbb{Q}}$. Hence, we conclude by using 11.1.2.4. □

11.1.3 Stability under inverse images of convergent log-isocrystals

Let $f\colon \mathfrak{Y}'^\sharp \to \mathfrak{Y}^\sharp$ be a morphism of log smooth over $\mathfrak{S}^\sharp$ log formal schemes.

Remark 11.1.3.1. This subsection is a particular case of 11.2.3 (when the divisor is empty). We have also stability under tensor products and duality of convergent isocrystals (see 11.2.4, 11.2.5 in the particular case where the divisor is empty). The convergent statements are left to the reader.

11.1.3.2. Let $\mathcal{E}^{(\bullet)} \in M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. With similar to 9.2.1.8.1 notation (with locally projective isocrystals, we prefer to work with f^* instead of $f^!$), we get the functor $f^{*(\bullet)}_{\mathrm{alg}}\colon M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \to M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$ by setting $f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{E}^{(\bullet)}) := \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{S}}} f^{-1}\mathcal{E}^{(\bullet)}$. By left deriving the functor $f^{*(\bullet)}_{\mathrm{alg}}$, this yields the functor $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}\colon D^-(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \to D^-(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$, defined by setting $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) := \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}^{(\bullet)}$ for any $\mathcal{F}^{(\bullet)} \in D^-(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. Since it preserves lim-ind-isogenies, this induces the functor $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}\colon \underrightarrow{LD}^-_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \to \underrightarrow{LD}^-_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$. Remark $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}} = f^{(\bullet)!}_{\mathrm{alg}}[-d_f]$ where $f^{(\bullet)!}_{\mathrm{alg}}$ is the functor defined at 9.2.4.11.1 in the case where the divisors are empty and $\phi = \mathrm{id}$.

Following notation 9.2.1.15.3, we set $\mathbb{L}f^{*(\bullet)}(\mathcal{F}^{(\bullet)}) := \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}^{(\bullet)}$, for any $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$. We get the morphism $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to \mathbb{L}f^{*(\bullet)}(\mathcal{F}^{(\bullet)})$ (beware the notation is slightly misleading since $\mathbb{L}f^{*(\bullet)}$ is not necessarily the left derived functor of a functor).

Lemma 11.1.3.3. *We have the following properties.*

(a) *Let* $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$*. With notation 9.2.1.15.3, the canonical morphism*

$$\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{F}^{(\bullet)} \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}^{(\bullet)} = \mathbb{L}f^{*(\bullet)}(\mathcal{F}^{(\bullet)})$$

is an isomorphism.

(b) *Let* $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$*. Then, the canonical morphism*

$$\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{G}^{(\bullet)} \to \mathcal{O}^{(\bullet)}_{\mathfrak{Y}'}\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{G}^{(\bullet)}$$

is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'})$*.*

Proof. By using 4.3.4.6.1, we get (a). We check (b) similarly to Lemma 9.2.1.17. □

Proposition 11.1.3.4. *We have the following properties.*

(a) *Let* $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$*. Then the canonical morphism*

$$\mathcal{O}_{\mathfrak{Y}',\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}} f^{-1}\mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} f^{-1}\mathcal{E}$$

is an isomorphism. Hence, we can set $f^*(\mathcal{E}) := \mathcal{D}^{\dagger}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} f^{-1}\mathcal{E}$ *without ambiguity. We have also* $f^*(\mathcal{E}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp)$*.*

(b) *Let* $\mathcal{F}$ *be a* $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$*-module, coherent over* $\mathcal{O}_{\mathfrak{Y}}$*. Then the morphisms*

$$\mathcal{O}_{\mathfrak{Y}'} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{Y}}} f^{-1}\mathcal{F} \to \mathcal{O}_{\mathfrak{Y}'}\widehat{\otimes}_{f^{-1}\mathcal{O}_{\mathfrak{Y}}} f^{-1}\mathcal{F} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{f^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}$$
$$\leftarrow \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{f^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}$$

are isomorphisms. Hence, we can set $f^*(\mathcal{F}) := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{f^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}$ *without ambiguity. Moreover,* $f^*(\mathcal{F})$ *is a* $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}$*-module, coherent over* $\mathcal{O}_{\mathfrak{Y}'}$*.*

Proof. 1) Following 11.1.2.5, there exists $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}) \cap \underrightarrow{LD}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ such that $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}$. By applying Lemmas 9.2.1.17 and 11.1.3.3, this yields that the canonical morphism

$$\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{E}^{(\bullet)} \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp\to\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{E}^{(\bullet)}$$

is an isomorphism. By applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$, we get the desired isomorphism. Since $f^*(\mathcal{E})$ is $\mathcal{O}_{\mathfrak{Y}',\mathbb{Q}}$-coherent, this yields $f^*(\mathcal{E}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp)$.

2) Since $\mathcal{F}$ is both $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-coherent and $\mathcal{O}_{\mathfrak{Y}}$-coherent, the first and the last morphisms are isomorphisms. Since the modulo π^{n+1} reduction of the middle morphism is an isomorphism for any $n \in \mathbb{N}$ (see 4.3.4.6.1), since this is a morphism of separated complete modules for the p-adic topology, this implies that the middle morphism is an isomorphism. □

Proposition 11.1.3.5. *Let* $\mathcal{F}^{(0)}$ *be a* $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$*-module, coherent over* $\mathcal{O}_{\mathfrak{Y}}$ *and such that the canonical morphism* $\mathrm{cst}(\mathcal{F}^{(0)}) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)} =: \mathcal{F}^{(\bullet)}$ *is an ind-isogeny in* $M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$*. For any* $m \in \mathbb{N}$*, let* $\mathcal{G}^{(m)}$ *be the quotient of* $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ *by its p-torsion part.*

(a) *The canonical morphism* $\mathrm{cst}(f^*(\mathcal{F}^{(0)})) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} f^*(\mathcal{F}^{(0)})$ *is an ind-isogeny of* $M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'})$*. In particular,* $\mathrm{cst}(f^*(\mathcal{F}^{(0)}))$ *belongs to* $\mathrm{MIC}^{(\bullet)}(\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp)$*.*

(b) *The canonical morphisms* $f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)})$*, and* $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} f^*(\mathcal{F}^{(0)}) \to f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)})$ *are ind-isogenies of* $M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$.

(c) *The canonical morphism* $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to \mathbb{L}f^{*(\bullet)}(\mathcal{F}^{(\bullet)})$ *is an isomorphism of* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$.

(d) *If* $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})$ *is flat as* $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$*-module, then the canonical morphism* $\mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)})$ *is an isomorphism of* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$.

Proof. a) Following 11.1.2.5, $\mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. Set $\mathcal{F} := \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. By applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ to the canonical morphism $\mathrm{cst}(\mathcal{F}^{(0)}) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)} =: \mathcal{F}^{(\bullet)}$, we get the canonical morphism $\mathcal{F}^{(0)}_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}}$ is an isomorphism. Via this isomorphism, we can view $\mathcal{F}^{(0)}_{\mathbb{Q}}$ as an object of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$. This yields that $f^*(\mathcal{F}^{(0)})$ is a $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}$-module, coherent over $\mathcal{O}_{\mathfrak{Y}'}$ and such that $\left(f^*(\mathcal{F}^{(0)})\right)_{\mathbb{Q}} \xrightarrow{\sim} f^*(\mathcal{F}^{(0)}_{\mathbb{Q}})$ (see both notation 11.1.3.4.a and 11.1.3.4.b) is an object of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp)$. Hence, via 11.1.2.4, we get the first statement.

b) Since $\mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ is an ind-isogeny of $M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$, then $f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)})$ is an isogeny of $M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp})$. We have the commutative diagram with canonical morphisms

$$\begin{array}{ccccc} f^*(\mathcal{F}^{(0)}) & \longrightarrow & \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} f^*(\mathcal{F}^{(0)}) & \longrightarrow & f^*(\mathcal{G}^{(m)}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ f^{*(0)}_{\mathrm{alg}}(\mathcal{F}^{(0)}) & \longrightarrow & \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} f^{*(0)}_{\mathrm{alg}}(\mathcal{F}^{(0)}) & \longrightarrow & f^{*(m)}_{\mathrm{alg}}(\mathcal{G}^{(m)}), \end{array} \tag{11.1.3.5.1}$$

where f^* is the functor defined at 11.1.3.4.(b). Since $\mathcal{F}^{(0)}$ and $\mathcal{G}^{(m)}$ are $\mathcal{O}_{\mathfrak{Y}}$-coherent (see 11.1.2.4, then following 11.1.3.4.(b), the vertical arrows are isomorphisms. From the first statement, the left horizontal arrows are isogenies of $\mathcal{O}_{\mathfrak{Y}'}$-modules. Since $\mathcal{F}^{(0)} \to \mathcal{G}^{(m)}$ are also isogenies, then the morphisms of the diagram 11.1.3.5.1 become isomorphisms after tensoring by $\mathbb{Q}$. Since $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} f^*(\mathcal{F}^{(0)}) \to f^{*(m)}_{\mathrm{alg}}(\mathcal{G}^{(m)})$ is a morphism of coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}$-modules, this yields the second statement.

c) By using 11.1.3.3 and 9.2.1.17, since $\mathcal{F}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$ and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$, three arrows of the diagram

$$\begin{array}{ccc} \mathbb{L}f^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathbb{L}f^{*(\bullet)}(\mathcal{F}^{(\bullet)}) \\ \uparrow & & \uparrow \wr \\ \mathcal{O}^{(\bullet)}_{\mathfrak{Y}'} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & \mathcal{O}^{(\bullet)}_{\mathfrak{Y}'} \widehat{\otimes}^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{F}^{(\bullet)} \end{array}$$

are isomorphisms. Hence so is the forth.

d) It remains to check that the canonical morphism $\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{F}^{(\bullet)} \to \mathcal{O}^{(\bullet)}_{\mathfrak{Y}'} \otimes_{f^{-1}\mathcal{O}^{(\bullet)}_{\mathfrak{Y}}} f^{-1}\mathcal{F}^{(\bullet)}$ is an isomorphism under the flatness assumption. Since this is a morphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}'})$, we reduce to check it after applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$, which is a consequence of the flatness as $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-module. □

Corollary 11.1.3.6. *Let* $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp)$ *such that* $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)}$ *is flat as* $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$*-module.*

(a) $\mathbb{L}f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp)$ *(i.e. is isomorphic to such an object) and* $\underrightarrow{l}^*_{\mathbb{Q}}\mathbb{L}f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} f^*(\mathcal{E})$.

(b) Choose a $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{F}^{(0)}$, coherent over $\mathcal{O}_{\mathfrak{Y}}$ such that $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp})$ to $\mathcal{E}^{(\bullet)}$ and such that the canonical morphism $\mathrm{cst}(\mathcal{F}^{(0)}) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp}} \mathcal{F}^{(0)}$ is an ind-isogeny in $M(\mathcal{O}^{(\bullet)}_{\mathfrak{Y}})$. Then $\mathbb{L}f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Y}'^\sharp/\mathfrak{S}^\sharp}} f^*(\mathcal{F}^{(0)})$.

11.2 Overconvergent log-isocrystals as arithmetic $\mathcal{D}$-modules

11.2.1 $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$: Overconvergent connections and arithmetic $\mathcal{D}$-modules

Let $\mathfrak{S}^\sharp$ be a nice fine $\mathcal{V}$-log formal scheme (see definition 3.3.1.10). Moreover, let $\mathfrak{X}^\sharp \to \mathfrak{S}^\sharp$ be a log smooth morphism of log formal schemes. We suppose the underlying formal scheme $\mathfrak{X}$ is p-torsion free, noetherian of finite Krull dimension. For any integer $i \geq 0$, set $X_i^\sharp := \mathfrak{X}^\sharp \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}(\mathcal{V}/\pi^{i+1}\mathcal{V})$. We suppose X_0 is regular. Let T be a divisor of X_0 and $\mathfrak{Y}^\sharp$ the open subset of $\mathfrak{X}^\sharp$ complementary to the support of T.

Definition 11.2.1.1 (Overconvergent connections)**.** Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module.

(a) Suppose $\mathfrak{X}$ is affine and endowed with logarithmic coordinates. We say that the induced connection of $\mathcal{E}$ is "overconvergent along T" if the following conditions holds: there exists $n_0 \in \mathbb{N}$, and a $\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{E}_0$ which is coherent as $\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module together with a $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism

$$\varinjlim_{n \geq n_0} \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}, \tag{11.2.1.1.1}$$

satisfying the following condition: for any $\eta \in |K^\times|_{\mathbb{Q}} \cap]0,1[$, there exist $n_0 \leq n_\eta$ such that for any $n_\lambda \leq n$ and each section $e \in \Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_0)$, we have

$$\| \underline{\partial}_\sharp^{[\underline{k}]} e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty, \tag{11.2.1.1.2}$$

with $\| \cdot \|$ a Banach norm on $\Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_0)$. Moreover, the isomorphism 11.2.1.1.1 is $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear.

(b) In general, the connection of $\mathcal{E}$ is overconvergent along T if and only if it is overconvergent on any affine open of $\mathfrak{X}^\sharp$ having logarithmic coordinates.

Definition 11.2.1.2. Suppose T is empty. Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module. We say that induced connection of $\mathcal{E}$ is "convergent" if it is overconvergent along the empty set. We retrieve the convergent condition 11.1.1.1.1. In other word, the connection of $\mathcal{E}$ is convergent if and only if the structure of left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module extends to a structure of left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-modules.

11.2.1.3. Suppose that $\mathfrak{S} \to \mathrm{Spf}\,\mathcal{V}$ is smooth, and suppose there exist a smooth $\mathfrak{S}$-formal scheme $\mathfrak{X}$, a relative strict normal crossing divisor $\mathfrak{Z}$ of $\mathfrak{X}$ over $\mathfrak{S}$, such that $\mathfrak{X}^\sharp = (\mathfrak{X}, M_{\mathfrak{Z}})$ is the logarithmic $\mathcal{V}$-formal scheme whose underlying logarithmic structure $M_{\mathfrak{Z}}$ is the one associated with $\mathfrak{Z}$.

We have the morphism of ringed spaces $\mathrm{sp}\colon (\mathfrak{X}_K, j^\dagger \mathcal{O}_{\mathfrak{X}_K}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$ induced by the specialization morphism. We get the inverse image functor sp^* by setting $\mathrm{sp}^*(\mathcal{E}) := j^\dagger \mathcal{O}_{\mathfrak{X}_K} \otimes_{\mathrm{sp}^{-1}\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathrm{sp}^{-1}(\mathcal{E})$, for any $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules $\mathcal{E}$. The overconvergent condition 11.2.1.1 corresponds to that of 10.3.2.5. More precisely, the functors sp_* and sp^* induce quasi-inverse equivalences between (locally free) coherent $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-modules together with an overconvergent logarithmic connection relative to $\mathfrak{X}^\sharp_K/\mathfrak{S}_K$ with overconvergent singularities along T and (locally projective) coherent $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules together with an overconvergent logarithmic connection relative to $\mathfrak{X}^\sharp/\mathfrak{S}$ with coefficients in $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$

Notation 11.2.1.4. Let $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ be the category of $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules which are coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module and such that the underlying connection is overconvergent along T. Its objects are called " overconvergent along T log-isocrystal on $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$". When T is empty we remove it in the notation. When $\mathfrak{S}^\sharp = \mathrm{Spf}\,\mathcal{V}$, we also write $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/K)$.

Proposition 11.2.1.5. *Suppose that $\mathfrak{S} \to \operatorname{Spf} \mathcal{V}$ is smooth, and suppose there exist a smooth $\mathfrak{S}$-formal scheme $\mathfrak{X}$, a relative strict normal crossing divisor $\mathfrak{Z}$ of $\mathfrak{X}$ over $\mathfrak{S}$, such that $\mathfrak{X}^\sharp = (\mathfrak{X}, M_{\mathfrak{Z}})$ is the logarithmic $\mathcal{V}$-formal scheme whose underlying logarithmic structure $M_{\mathfrak{Z}}$ is the one associated with $\mathfrak{Z}$.*

(a) The functors sp_* and sp^* (see 11.2.1.3) induce quasi-inverse equivalences of categories between $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}^\sharp_K)$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. Moreover, an object $\mathcal{E}$ of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module if and only if $\mathrm{sp}^*(\mathcal{E})$ is a locally free $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-module.

(b) Suppose $\mathfrak{S} = \operatorname{Spf} \mathcal{V}$ (and the divisor T is empty). The full subcategory of $I_{\mathrm{conv}}(X^\sharp/\operatorname{Spf}\mathcal{V})$ consisting in locally free isocrystals on the log convergent site $((X, M)/\operatorname{Spf}\mathcal{V})_{\mathrm{conv}}$ (see 10.3.1.1) is equivalent to the full subcategory of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp/\mathfrak{S}^\sharp)$ consisting of locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-modules.

Proof. With notations 10.3.2.5 and 11.2.1.4, the first statement is a rewriting of 11.2.1.3. We get the last part with 10.3.2.6.(a). ☐

Remark 11.2.1.6. Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is locally projective of finite type as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module. It $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ then $\mathcal{E}|\mathfrak{X}^* \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^*, T \cap X^*/\mathfrak{S}^*)$. Beware that contrary to 11.1.1.4 the converse seems false in general.

Proposition 11.2.1.7. *Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module. The following are equivalent:*

(a) $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$;

(b) for $\lambda \in L(\mathbb{N})$ large enough, denoting by $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^* \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, there exist a $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module $\mathcal{E}^{(0)}$ which is coherent as $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module satisfying the following both conditions:

(i) $\mathcal{E}^{(0)}$ is endowed with a $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism of the form

$$\varinjlim_m \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)} \xrightarrow{\sim} \mathcal{E}, \tag{11.2.1.7.1}$$

(ii) the structure of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module on $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)}$ (uniquely) extends to a structure of topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module so that the homomorphisms

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)} \to \widetilde{\mathcal{B}}^{(m+1)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)} \tag{11.2.1.7.2}$$

are $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-linear.

Proof. 1) Let us check $(a) \to (b)$. Suppose the connection is overconvergent along T. By using 8.4.1.11, there exists $n_{-1} \in \mathbb{N}$, and a coherent $\mathcal{B}^{(n_{-1})}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module $\mathcal{E}_{-1}$ together with an $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism

$$\varinjlim_{n \geq n_{-1}} \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_{-1})}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{-1} \xrightarrow{\sim} \mathcal{E}, \tag{11.2.1.7.3}$$

By using 11.2.1.1.1 (and 8.4.1.11), increasing n_{-1} if necessary, we can suppose that $\mathcal{E}_{-1}$ is endowed with a structure of left $\mathcal{B}^{(n_{-1})}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module inducing its structure of $\mathcal{B}^{(n_{-1})}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module and the isomorphism 11.2.1.7.3 is $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear.

Let $m \in \mathbb{N}$. Following 8.7.1.7, there exists $\eta < 1$, $c \in \mathbb{R}$ such that $|q^{(m)}_k!| \leq c\eta^k$ for all $k \in \mathbb{N}$. By using the condition 11.2.1.1.2, there exist $n_{-1} \leq n_m$ such that for any $n_m \leq n$, for any affine open $\mathfrak{U}^\sharp$ of $\mathfrak{X}^\sharp$ and endowed with logarithmic coordinates, and each section $e \in \Gamma(\mathfrak{U}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_0)$, we have

$$\| \underline{\partial}^{[\underline{k}]}_\sharp e \| \, \eta^{|\underline{k}|} \to 0 \text{ for } |\underline{k}| \to \infty. \tag{11.2.1.7.4}$$

Let $n_m \leq n$. Let $\mathfrak{U}^\sharp$ be an affine open of $\mathfrak{X}^\sharp$ and endowed with logarithmic coordinates. Let $P \in \Gamma(\mathfrak{U}^\sharp, (\mathcal{B}^{(n)}_{\mathfrak{X}}(T)\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathbb{Q}})$. We can write $P = \sum_{\underline{k}} b_{\underline{k}}\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$, with $b_{\underline{k}} \in \Gamma(\mathfrak{U}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}})$ converging to 0 when $|\underline{k}|$ goes to infinity. Let $e \in \Gamma(\mathfrak{U}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n-1)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{-1})$. We get

$$\| b_{\underline{k}}\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}(e) \| = \| b_{\underline{k}} q^{(m)}_{\underline{k}}!\underline{\partial}_\sharp^{[\underline{k}]}(e) \| \leq c \| b_{\underline{k}} \| \| \underline{\partial}_\sharp^{[\underline{k}]}(e) \| \eta^{|\underline{k}|} \to 0$$

and we can define $P(e)$ as the image in $\Gamma(\mathfrak{U}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n-1)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{-1})$ of the converging sum of the series of general terms $b_{\underline{k}}\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}(e)$. This yields a structure of $\Gamma(\mathfrak{U}^\sharp, (\mathcal{B}^{(n)}_{\mathfrak{X}}(T)\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathbb{Q}})$-module on $\Gamma(\mathfrak{U}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n-1)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{-1})$. It is clear that this does not depend on the choice of the logarithmic coordinates. Hence, this yields a structure of $(\mathcal{B}^{(n)}_{\mathfrak{X}}(T)\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathbb{Q}}$-module on $\mathcal{B}^{(n)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n-1)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{-1}$.

We can suppose the sequence (n_m) is increasing. We set $\mathcal{E}^{(m)} := \mathcal{B}^{(n_{m+1})}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n-1)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{-1}$. We get a element $\lambda \in L(\mathbb{N})$ by setting $\lambda(m) = n_{m+1}$. Following 7.5.2.7, $\mathcal{E}^{(m)}$ is endowed with a structure of topologically nilpotent $(\mathcal{B}^{(n_{m+1})}_{\mathfrak{X}}(T)\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_{\mathbb{Q}}$-module and is a coherent $\mathcal{B}^{(n_{m+1})}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module. Hence, we are done.

2) Conversely, suppose conditions 11.2.1.7.1 and 11.2.1.7.2 hold. Let $\mathfrak{U}^\sharp$ be an affine open of $\mathfrak{X}^\sharp$ and endowed with logarithmic coordinates. Take $\eta < 1$. By 8.7.1.7 and with notation 1.2.1.3.1, there exists $m \in \mathbb{N}$ such that $\eta^{|\underline{k}|} < c_{\underline{k}}|q^{(m)}_{\underline{k}}!|$, with $c_{\underline{k}} \to 0$. As $q^{(m)}_{\underline{k}}!\partial^{[\underline{k}]} = \partial^{\langle \underline{k}\rangle_{(m)}}$, it suffices to show that, for any m and any $e \in \Gamma(\mathfrak{U}, \mathcal{E}^{(m)})$, the elements $\partial^{\langle \underline{k}\rangle_{(m)}}(e)$ with $\underline{k} \in \mathbb{N}^d$, form a bounded family in $\Gamma(\mathfrak{U}, \mathcal{E}^{(m)})$ for its natural topology of $\Gamma(\mathfrak{U}, \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}})$-module of finite type. Following 7.5.2.6, the action of $\Gamma(\mathfrak{U}^\sharp, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}})$ on $\Gamma(\mathfrak{U}, \mathcal{E}^{(m)})$ is continuous. Since the elements $\underline{\partial}_\sharp^{\langle \underline{k}\rangle_{(m)}}$ form a bounded family of $\Gamma(\mathfrak{U}^\sharp, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}})$, then we are done. □

Remark 11.2.1.8. Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module. When the connection of $\mathcal{E}$ has an overconvergent along T connection, then the structure of left $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$ of $\mathcal{E}$ extends (uniquely) to a structure of left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module so that 11.2.1.7.1 is $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear. But, contrary to Proposition 11.1.1.1, this is not clear that if $\mathcal{E}$ is a left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module then the connection of $\mathcal{E}$ is overconvergent.

Theorem 11.2.1.9. *Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. For $\lambda \in L(\mathbb{N})$ large enough, denoting by $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, let $\mathcal{E}^{(0)}$ be a topologically nilpotent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module, coherent as $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module satisfying both conditions of 11.2.1.7.(b). We get the $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module by setting $\mathcal{E}^{(\bullet)} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)}$.*

(a) The canonical homomorphism

$$\mathcal{E}^{(m)} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)} \tag{11.2.1.9.1}$$

is an isomorphism.

(b) The sheaf $\mathcal{E}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module.

(c) The sheaf $\mathcal{E}$ is $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-coherent and the canonical morphism

$$\mathcal{E} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \tag{11.2.1.9.2}$$

is an isomorphism.

Proof. a) Since $\mathcal{E}^{(m)}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q}$-module, then following 7.5.2.8.2 (resp. 7.5.2.8.1) the above (resp. below) canonical homomorphism

$$\mathcal{E}^{(m)} = \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_\mathbb{Q} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_\mathbb{Q}} \mathcal{E}^{(0)} \to (\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_\mathbb{Q} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q}} \mathcal{E}^{(0)}$$

$$\mathcal{E}^{(m)} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q} \otimes_{(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})_\mathbb{Q}} \mathcal{E}^{(m)}$$

is an isomorphism. This yields 11.2.1.9.1 is an isomorphism.

b) As $\mathcal{E}^{(0)}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q}$-module, coherent on $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_\mathbb{Q}$, then following 7.5.2.8 it is coherent on $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q}$. Let

$$(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q})^s \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q} \to \mathcal{E}^{(0)}$$

be a local presentation of $\mathcal{E}^{(0)}$. Thanks to 11.2.1.9.1, this yields by extension the presentations

$$(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q})^s \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q} \to \mathcal{E}^{(m)}$$

which provide an analogous presentation of $\mathcal{E}$ on $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}$ by passing to the inductive limit.

c) As $\mathcal{E}^{(0)}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_\mathbb{Q}$-module, coherent on $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_\mathbb{Q}$, then following 7.5.2.8 $\mathcal{E}^{(0)}$ is $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T) \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-coherent. Moreover, via 4.3.4.6.1 and 11.2.1.7.1, we obtain

$$\mathcal{D}_\mathfrak{X}({}^\dagger T)_\mathbb{Q} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}} \mathcal{E}^{(0)} \xrightarrow{\sim} \mathcal{E}.$$

The sheaf $\mathcal{E}$ is then $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_\mathbb{Q}$-coherent.

d) Using b) and c), we get that the map 11.2.1.9.2 is a morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}$-modules. Following 11.1.1.6.(c), the morphism 11.2.1.9.2 is an isomorphism apart from T. We conclude via 8.7.6.11. □

Remark 11.2.1.10. Following the theorem 11.2.1.9, a log-isocrystal on $\mathfrak{X}^\sharp/\mathfrak{S}^\sharp$ overconvergent along T is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}$-module, coherent as $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module.

11.2.1.11 (Flatness, locally freeness, projectivity for coherent modules)**.** It follows from 1.4.3.21.(1) that a flat coherent $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module is therefore a locally projective of finite type $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module.

For any $x \in \mathfrak{X}$, beware that the sheaf $(\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q})_x$ is not a local ring (even when T is empy). Hence, we cannot apply 1.4.3.21.(2) and this is not clear that a flat coherent $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module is therefore a locally free of finite type $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module.

Theorem 11.2.1.12. *Suppose the log structures are trivial. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_\mathbb{Q}$-module. The following conditions are equivalent.*

(a) $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathfrak{S})$ and is locally projective of finite type over $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module

(b) $\mathcal{E}|\mathfrak{Y}$ is locally projective of finite type over $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$.

Proof. 0) The implication $(a) \Rightarrow (b)$ is obvious.

I) Conversely, suppose $\mathcal{E}|\mathfrak{Y}$ is locally projective of finite type over $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$. For $n \geq m$, we set $\widehat{\mathcal{D}}^{(m,n)}_{\mathfrak{X}/\mathfrak{S}}(T) := \mathcal{B}^{(n)}_{\mathfrak{X}}(T)\widehat{\otimes}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}$, and $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}(T) := \widehat{\mathcal{D}}^{(m,m)}_{\mathfrak{X}/\mathfrak{S}}(T)$. For m_0 large enough, there exists a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{X}/\mathfrak{S}}(T)_\mathbb{Q}$-module $\mathcal{E}^{(m_0)}$ together with a $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_\mathbb{Q}$-linear isomorphism $\mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_\mathbb{Q} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{X}/\mathfrak{S}}(T)} \mathcal{E}^{(m_0)} \xrightarrow{\sim} \mathcal{E}$ (use 8.4.1.11 and the isomorphism $\varinjlim_m \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}/\mathfrak{S}}(T) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_\mathbb{Q}$). Following 7.4.5.2, there exists a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{X}/\mathfrak{S}}(T)$-module $\mathcal{G}^{(m_0)}$ without p-torsion together with a $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{X}/\mathfrak{S}}(T)_\mathbb{Q}$-linear isomorphism $\mathcal{G}^{(m_0)}_\mathbb{Q} \xrightarrow{\sim} \mathcal{E}^{(m_0)}$. For any $n \geq m \geq m_0$, we put $\mathcal{G}^{(m,n)} := \widehat{\mathcal{D}}^{(m,n)}_{\mathfrak{X}/\mathfrak{S}}(T) \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{X}/\mathfrak{S}}(T)} \mathcal{G}^{(m_0)}/p\text{-torsion}$, $\mathcal{G}^{(m)} := \mathcal{G}^{(m,m)}$. Following 7.4.5.1, $\mathcal{G}^{(m,n)}$ is $\widehat{\mathcal{D}}^{(m,n)}_{\mathfrak{X}/\mathfrak{S}}(T)$-coherent. From 11.1.1.6, we get $\mathcal{G}^{(m,n)}_\mathbb{Q}|\mathfrak{Y} \xrightarrow{\sim} \mathcal{E}|\mathfrak{Y}$. From 11.1.1.11, this yields that $\mathcal{G}^{(m,n)}|\mathfrak{Y}$ is $\mathcal{O}_\mathfrak{Y}$-coherent.

II) Fix $m \geq m_0$. We will now prove that for n large enough $\mathcal{G}^{(m,n)}$ is $\mathcal{B}^{(n)}_{\mathfrak{X}}(T)$-coherent.

1) Since this is local, we can suppose $\mathfrak{X} = \mathrm{Spf}\, A$ is affine, there exist $f \in A$ such that $T = \mathrm{Spec}\, \overline{A}/(\overline{f})$ and local coordinates $t_1, \dots, t_d \in A$ of $\mathfrak{X}/\mathfrak{S}$. Let $\partial_1, \dots, \partial_d$ be the induced derivations. Following 7.5.2.3, we reduce to check that for n large enough, $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)})$ is a $\Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T))$-module of finite type.

Put $\mathcal{D}^{(m)}_{X/S}(T) := \mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}(T)/\pi\mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}}(T)$, $\overline{\mathcal{G}}^{(m)} := \mathcal{G}^{(m)}/\pi\mathcal{G}^{(m)}$, and $\overline{\mathcal{G}}^{(m,n)} := \mathcal{G}^{(m,n)}/\pi\mathcal{G}^{(m,n)}$. Let $\overline{x}_1, \dots, \overline{x}_r \in \Gamma(X, \overline{\mathcal{G}}^{(m_0)})$ which generate $\overline{\mathcal{G}}^{(m_0)}$ as $\mathcal{D}^{(m_0)}_{X/S}(T)$-module.

From Lemma 11.1.1.11, $\overline{\mathcal{G}}^{(m)}|Y$ is a nilpotent $\mathcal{D}^{(m)}_{Y/S}$-module. Hence, there exists $h \in \mathbb{N}$ large enough so that we get in $\Gamma(Y, \overline{\mathcal{G}}^{(m)})$ the relation

$$\forall i = 1, \dots, r, \forall j = 1, \dots, d, \forall l = 1, \dots, m,\ (\partial_j^{[p^l]})^h \cdot \overline{x}_i = 0,$$

where by abuse of notation we still denote by $\overline{x}_i$ (resp. $(\partial_j^{[p^l]})^h$) the image of $\overline{x}_i$ (resp. $(\partial_j^{[p^l]})^h$) via the canonical map $\Gamma(X, \overline{\mathcal{G}}^{(m_0)}) \to \Gamma(Y, \overline{\mathcal{G}}^{(m)})$ (resp. $\Gamma(X, \mathcal{D}^{(m)}_{X/S}) \to \Gamma(Y, \mathcal{D}^{(m)}_{Y/S})$). Hence, for $n_m > m$ large enough, we get in $\Gamma(X, \overline{\mathcal{G}}^{(m)})$ the relation

$$\forall i = 1, \dots, r, \forall j = 1, \dots, d, \forall l = 1, \dots, m,\ \overline{f}^{p^{n_m}} (\partial_j^{[p^l]})^h \cdot \overline{x}_i = 0.$$

Fix such n_m. Since $n_m > m$, then following 1.4.4.1 the section $\overline{f}^{p^{n_m}}$ is in the center of $\Gamma(X, \mathcal{D}^{(m)}_{X/S})$. Let $P = \prod_{j=1}^d \prod_{l=1}^m (\partial_j^{[p^l]})^{h_{jl}} \in \Gamma(X, \mathcal{D}^{(m)}_{X/S})$ where $h_{jl} \in \mathbb{N}$. Since $\overline{f}^{p^{n_m}}$ is in the center of $\Gamma(X, \mathcal{D}^{(m)}_{X/S})$, if there exist j_0 and l_0 such that $h_{j_0 l_0} \geq h$, then we have in $\Gamma(X, \overline{\mathcal{G}}^{(m)})$ the relation $\overline{f}^{p^{n_m}} P \cdot \overline{x}_i = 0$, for any i.

2) Let $x_1, \dots, x_r \in \Gamma(\mathfrak{X}, \mathcal{G}^{(m_0)})$ be some sections lifting respectively $\overline{x}_1, \dots, \overline{x}_r$.

i) Let $P = \prod_{j=1}^d \prod_{l=1}^m (\partial_j^{[p^l]})^{h_{jl}} \in \Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$ where $h_{jl} \in \mathbb{N}$ are such that there exist j_0 and l_0 satisfying $h_{j_0 l_0} \geq h$. Then, we get in $\Gamma(\mathfrak{X}, \mathcal{G}^{(m)})$ the relation

$$\forall i = 1, \dots, r,\ f^{p^{n_m}} P \cdot x_i \in p\Gamma(\mathfrak{X}, \mathcal{G}^{(m)}),$$

where by abuse of notation we still denote by x_i the image of x_i via the canonical map $\Gamma(\mathfrak{X}, \mathcal{G}^{(m_0)}) \to \Gamma(\mathfrak{X}, \mathcal{G}^{(m)})$. Let $T_{n_m-1} \in \mathcal{B}^{(n_m-1)}_{\mathfrak{X}}(T)$ be the element such that $f^{p^{n_m}} T_{n_m-1} = p$. Since the $\mathcal{B}^{(n_m-1)}_{\mathfrak{X}}(T)$-module $\mathcal{G}^{(m,n_m-1)}$ is π-torsion free then it is f-torsion free. Hence, for such P, we get in $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n_m-1)})$:

$$\forall i = 1, \dots, r,\ P \cdot x_i \in T_{n_m-1}\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n_m-1)}). \tag{11.2.1.12.1}$$

ii) Let $y_1, \dots, y_s$ be the elements of the form $\left(\prod_{j=1}^d \prod_{l=1}^m (\partial_j^{[p^l]})^{h_{jl}}\right) \cdot x_i$ where $h_{jl} \in \{0, \dots, h-1\}$ for any j and l (beware that these elements and their number depend on m but remark that $y_1, \dots, y_s \in \Gamma(\mathfrak{X}, \mathcal{E}^{(m_0)})$). Following 1.4.2.10, $\Gamma(\mathfrak{X}, \mathcal{D}^{(m)}_{\mathfrak{X}/\mathfrak{S}})$ is generated as $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$-module (for its left or right structure) by the elements of the form $\prod_{j=1}^d \prod_{l=1}^m (\partial_j^{[p^l]})^{h_{jl}}$, where $h_{jl} \in \mathbb{N}$. Since $\overline{x}_1, \dots, \overline{x}_r$ generate $\overline{\mathcal{G}}^{(m_0)}$ as $\mathcal{D}^{(m_0)}_{X/S}(T)$-module, then for any $n \geq n_m - 1$, $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)})$ is generated as $\Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T))$-module by $p\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)})$ and by the elements of the form $\left(\prod_{j=1}^d \prod_{l=1}^m (\partial_j^{[p^l]})^{h_{jl}}\right) \cdot x_i$ where $h_{jl} \in \mathbb{N}$. Since T_{n_m-1} divides p, since $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)})$ has no p-torsion, then by using 11.2.1.12.1 we get

$$\forall n \geq n_m - 1,\ \Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)}) = \sum_{i=1}^s \Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)) \cdot y_i + T_{n_m-1}\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)}).$$

By iteration, this yields

$$\forall n \geq n_m - 1,\ \Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)}) = \sum_{i=1}^s \Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)) \cdot y_i + T^p_{n_m-1}\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)}).$$

For any $n \geq n_m$, we have $T^p_{n_m-1} = p^{p-1} T_{n_m}$. We get,

$$\forall n \geq n_m,\ \Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)}) = \sum_{i=1}^s \Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T)) \cdot y_i + p\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)}).$$

Since $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)})$ is p-adically separated and complete, this yields that $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n)})$ is generated as $\Gamma(\mathfrak{X}, \mathcal{B}^{(n)}_{\mathfrak{X}}(T))$-module by $y_1, \ldots, y_s$.

III) We can suppose that the sequence $(n_m)_m$ is increasing. Set $\mathcal{E}_m := \mathcal{G}^{(m,n_m)}_{\mathbb{Q}}$. Following II.2), the $\widehat{\mathcal{D}}^{(m,n_m)}_{\mathfrak{X}/\mathfrak{S}}(T)_{\mathbb{Q}}$-module $\mathcal{E}_m$ is $\mathcal{B}^{(n_m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-coherent. Since $\mathcal{E}_{m_0}|\mathfrak{Y} \xrightarrow{\sim} \mathcal{E}_m|\mathfrak{Y}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{Y}^\sharp/\mathfrak{S}^\sharp,\mathbb{Q}}$-module which is locally projective of finite type over $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$, then, with the remark 8.7.6.9, increasing m_0 is necessary, $\mathcal{E}_{m_0}$ is a projective $\mathcal{B}^{(n_{m_0})}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module of finite type.

Following 11.2.1.7, it is sufficient to check that the canonical homomorphism

$$\mathcal{B}^{(n_m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{m_0} \to \mathcal{E}_m$$

is an isomorphism for any $m \geq m_0$. Fix $m \geq m_0$ and set $B_m := \Gamma(\mathfrak{X}, \mathcal{B}^{(n_m)}_{\mathfrak{X}}(T)_{\mathbb{Q}})$, $E_m := \Gamma(\mathfrak{X}, \mathcal{E}_m)$, $E_{m_0,m} := \Gamma(\mathfrak{X}, \mathcal{B}^{(n_m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}_{m_0})$. We get the morphism $E_{m_0,m} \to E_m$ of B_m-modules of finite type. We end the proof by check this morphism $E_{m_0,m} \to E_m$ is an isomorphism:

Following the part II.2) and its notations, $\Gamma(\mathfrak{X}, \mathcal{G}^{(m,n_m)})$ is generated as $\Gamma(\mathfrak{X}, \mathcal{B}^{(n_m)}_{\mathfrak{X}}(T))$-module by $y_1, \ldots, y_s$. We remark that $y_1, \ldots, y_s \in E_{m_0}$. Hence, the morphism $E_{m_0,m} \to E_m$ is surjective. After applying $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}}) \otimes_{B_m} -$ to the morphism $E_{m_0,m} \to E_m$, we get an isomorphism. Since $E_{m_0,m}$ is a projective B_m-module of finite type and since $B_m \to \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$ is injective, we get the injectivity of $E_{m_0,m} \to E_m$. We are done. □

Corollary 11.2.1.13. *Suppose the log structures are trivial. Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module which is locally projective of finite type as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module. The following conditions are equivalent:*

(a) The structure of left $\mathcal{D}_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module of $\mathcal{E}$ extends to a structure of coherent left $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-modules ;

(b) $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathfrak{S})$.

Proof. The implication $(b) \Rightarrow (a)$ is 11.2.1.9. Conversely, suppose (a) holds. By using 11.2.1.12, since $\mathcal{E}$ is locally projective of finite type as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module, then we reduce to check $\mathcal{E}|\mathfrak{Y} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathfrak{S})$, i.e. we can suppose first to the case where the divisor is empty. In this case, this is 11.1.1.1. □

Proposition 11.2.1.14. *Suppose the log structures are trivial and $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$. Let $\mathcal{E}$ be a left $\mathcal{D}_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module.*

(a) The object $\mathcal{E}$ is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module of finite type.

(b) If $\mathfrak{X}$ is affine, then $\Gamma(\mathfrak{X}, \mathcal{E})$ is a projective $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$-module of finite type.

(c) We have $\mathcal{E} = 0$ if and only if there exists a dense open subset $\mathfrak{U}$ of $\mathfrak{X}$ such that $\mathcal{E}|\mathfrak{U} = 0$.

(d) The category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathcal{V})$ is equal to the full category of that of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-modules consisting of objects which are coherent as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module.

(e) Set $\mathfrak{Y} := \mathfrak{X} \setminus T$. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module. We have $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathcal{V})$ if and only if $\mathcal{E}|\mathfrak{Y} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathcal{V})$.

Proof. a) and b) By using 10.2.2.7 and 11.2.1.3, we get (a). More precisely, this is a consequence of (b) which is checked as follows. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ affine. Since $\mathfrak{Y}_K$ is affinoid, since $\mathrm{sp}^*(\mathcal{E})$ is a locally free $\mathcal{O}_{\mathfrak{Y}_K}$-module of finite type (see 10.2.2.7), then $\Gamma(\mathfrak{Y}, \mathcal{E}) = \Gamma(\mathfrak{Y}_K, \mathrm{sp}^*(\mathcal{E}))$ is a projective $\Gamma(\mathfrak{Y}_K, \mathcal{O}_{\mathfrak{Y}_K}) = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$-module of finite type. Using theorem of type A concerning coherent $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules, $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})} \Gamma(\mathfrak{X}, \mathcal{E}) \to \Gamma(\mathfrak{Y}, \mathcal{E})$ is an isomorphism. Since $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$, since $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \to \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$ is faithfully flat (this is checked in the proof of 8.7.6.8), then this implies that $\Gamma(\mathfrak{X}, \mathcal{E})$ is a projective $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$-module of finite type.

c) Since the third part is local on $\mathfrak{X}$, we can suppose $\mathfrak{X}$ is affine and that $\mathcal{E}$ is a direct summand (in the category of coherent $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules) of a free $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module $\mathcal{L}$ of finite type. Suppose $\mathcal{E}|\mathfrak{U} = 0$. Replacing $\mathfrak{U}$ by a smaller open subset, we reduce to the case where $\mathfrak{U}$ is a principal open subset (i.e. given by a global section of $\mathfrak{X}$). Since $\Gamma(\mathfrak{X}, \mathcal{L}) \to \Gamma(\mathfrak{U}, \mathcal{L})$ is injective, we get $\Gamma(\mathfrak{X}, \mathcal{E}) = 0$. Using theorem of type A, this yields that $\mathcal{E} = 0$. The converse is obvious.

d) This is a consequence of 11.2.1.13 and of (a).

e) The last statement follows from a) and d) and from 11.2.1.12. □

11.2.2 $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$

We keep notation 11.2.1.

Notation 11.2.2.1. We denote by $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$ consisting of objects $\mathcal{E}^{(\bullet)}$ such that $\underrightarrow{l}^*_{\mathbb{Q}}\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ (see notation 11.2.1.4). Since the functor $\underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^{(\bullet)}) \to \mathrm{Coh}(\mathcal{D}^\dagger_{\mathbb{Q}})$ is an equivalence of category (e.g. see 8.4.5.9), then we get the equivalence of categories

$$\underrightarrow{l}^*_{\mathbb{Q}}\colon \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp) \cong \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp). \qquad (11.2.2.1.1)$$

Proposition 11.2.2.2. *Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. For $\lambda \in L(\mathbb{N})$ large enough, denoting by $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, let $\mathcal{E}^{(0)}$ be a topologically nilpotent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module, coherent as $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module satisfying both conditions of 11.2.1.7.(b). We get a $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module by setting $\mathcal{E}^{(\bullet)} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)}$.*

1. *There exists $\mathcal{F}^{(0)}$ a $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-module, coherent over $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$ together with an isomorphism of $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-modules of the form $\mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(0)}$.*
2. *Let $\mathcal{F}^{(m)} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}^{(0)}$, let $\mathcal{G}^{(m)} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}^{(0)}$ and let $\mathcal{H}^{(m)}$ be the quotient of $\mathcal{F}^{(m)}$ by its p-torsion part for any $m \in \mathbb{N}$. The following conditions hold.*
   - (a) *We have the isomorphisms $\mathcal{G}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{H}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(m)}$ of topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-modules.*
   - (b) *The module $\mathcal{H}^{(m)}$ is $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$-coherent for any integer m.*
   - (c) *The canonical morphism $\mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$. The canonical morphism $\mathcal{G}^{(\bullet)} \to \mathcal{H}^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$.*
   - (d) *We have $\mathcal{G}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ and the $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism $\underrightarrow{l}^*_{\mathbb{Q}}\,\mathcal{G}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$.*
   - (e) *The canonical morphism*
     $$\mathcal{G}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} \mathcal{G}^{(\bullet)} \qquad (11.2.2.2.1)$$
     is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$.
   - (f) *We have $\mathcal{F}^{(0)} \in D^{\mathrm{b}}_{\mathrm{perf}}((\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)))$ and the canonical morphism*
     $$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}^{(0)} \to \mathcal{G}^{(\bullet)} \qquad (11.2.2.2.2)$$
     is an isomorphism of $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$. In particular, $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$ (see notation 8.6.1.4)

Proof. 1) The existence of $\mathcal{F}^{(0)}$ follows from 7.5.2.8.(a).

2)a) Since the canonical morphisms $\mathcal{G}^{(m)}_{\mathbb{Q}} \to \mathcal{H}^{(m)}_{\mathbb{Q}} \to \mathcal{E}^{(m)}$ are isomorphisms (use 11.2.1.9.1 for the second isomorphism), since $\mathcal{E}^{(m)}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module then we get the assertion.

b) Using 7.5.2.9, the part 2.a) of the proof implies that $\mathcal{H}^{(m)}$ is $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$-coherent for any integer m.

c) Since the canonical epimorphism $\mathcal{G}^{(m)} \to \mathcal{H}^{(m)}$ of coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-modules is an isomorphism after tensoring by $\mathbb{Q}$, then $\mathcal{G}^{(\bullet)} \to \mathcal{H}^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$ (use 8.4.2.8).

Since the canonical morphism $\mathcal{F}^{(m)} \to \mathcal{H}^{(m)}$ of coherent $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$-modules is an isomorphism after tensoring by $\mathbb{Q}$ (this follows from 11.2.1.9.1), then the canonical morphism $\mathcal{F}^{(\bullet)} \to \mathcal{H}^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$ (use 8.4.2.8). Hence so is $\mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$.

d) It follows from c) that the canonical morphisms $\mathcal{F}^{(\bullet)}_{\mathbb{Q}} \to \mathcal{G}^{(\bullet)}_{\mathbb{Q}} \to \mathcal{H}^{(\bullet)}_{\mathbb{Q}} \to \mathcal{E}^{(\bullet)}$ are isomorphisms. Hence, $\mathcal{G}^{(\bullet)}_{\mathbb{Q}}$ satisfies both conditions of 11.2.1.7.(b) and is endowed with a $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-linear isomorphism $\underrightarrow{l}^{*}_{\mathbb{Q}}\, \mathcal{G}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$.

e) By using the part b) and 7.5.2.8.(b), we get that the canonical morphism

$$\mathcal{H}^{(\bullet)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T) \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}} \mathcal{H}^{(\bullet)} \tag{11.2.2.2.3}$$

is an isomorphism of $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)$-modules. We conclude with c).

f) It follows from 4.7.3.7 that $\mathcal{F}^{(0)} \in D^{\mathrm{b}}_{\mathrm{perf}}((\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)))$. By flatness of $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}}$, the morphism 11.2.2.2.2 of $D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$ becomes an isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}})$. Hence, by using 8.4.2.9 we get that the morphism 11.2.2.2.2 becomes an isomorphism in $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$. Since the left term of 11.2.2.2.2 is an object of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$, then so is $\mathcal{G}^{(\bullet)}$.

□

Corollary 11.2.2.3. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$. The following conditions are equivalent:*

(a) The object $\mathcal{E}^{(\bullet)}$ belongs to $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^{\sharp}, T/\mathfrak{S}^{\sharp})$.

(b) For $\lambda \in L(\mathbb{N})$ large enough, denoting by $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^{*}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$, there exist $\mathcal{F}^{(0)}$ a p-torsion free $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)$-module, coherent over $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$ such that

(i) $\mathcal{F}^{(\bullet)} := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)} \mathcal{F}^{(0)}$ is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$ to $\mathcal{E}^{(\bullet)}$

(ii) the canonical morphism

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)} \mathcal{F}^{(0)}$$

is an ind-isogeny in $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$,

(iii) $\mathcal{F}^{(m)}_{\mathbb{Q}}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}}$-module for any m.

Proof. The implication $(a) \Rightarrow (b)$ follows from 11.2.2.2. Conversely, if such $\lambda \in L(\mathbb{N})$ and $\mathcal{F}^{(0)}$ exist, then using properties (ii) and (iii) we can check that $\mathcal{F}^{(\bullet)}_{\mathbb{Q}}$ satisfies both conditions of 11.2.1.7.(b). Hence, it follows from 11.2.1.7 that $\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^{\sharp}, T/\mathfrak{S}^{\sharp})$. The property (i) implies that $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$ and $\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^{\sharp}, T/\mathfrak{S}^{\sharp})$. Hence $\mathcal{E}^{(\bullet)}$ belongs to $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^{\sharp}, T/\mathfrak{S}^{\sharp})$.

□

11.2.2.4. Let $\lambda \in L(\mathbb{N})$, $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^{*}\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}$, $\mathcal{F}^{(0)}$ be a p-torsion free $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)$-module, coherent over $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$ such that the canonical morphism

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)} \mathcal{F}^{(0)} = \mathcal{F}^{(\bullet)}$$

is an ind-isogeny in $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$.

It follows from 8.4.2.1 that $\mathcal{F}^{(\bullet)}$ is a $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)$-module having global finite presentation. It follows from 8.4.2.5 that $\mathcal{F}^{(\bullet)}$ is a $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)$-module having global finite presentation *up to an ind-isogeny*. In particular $\mathcal{F}^{(\bullet)}$ is an object of $\underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)) \cap \underrightarrow{M}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$

11.2.2.5. Let us denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$, the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$ of complexes whose image by the forgetful functor $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$ is in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$. By replacing "$\underrightarrow{LD}^{\mathrm{b}}$" by "$\underrightarrow{LM}$", we define similarly $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)) \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$.

Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. With 8.4.3.2, 8.4.3.3 and 11.2.2.2, we get from 11.2.2.4 that $\mathcal{E}^{(\bullet)}$ is coherent up to a lim-ind-isogeny both as $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-module and as $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$-module, i.e. we have the fully faithful functor (the identity on the objects):

$$\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp) \subset \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)) \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}^\sharp}(T)). \tag{11.2.2.5.1}$$

When log structures are trivial and $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$, it follows from 11.2.1.14.(d) that the functor 11.2.2.5.1 is essentially surjective, i.e. that we have the equality:

$$\mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T/K) = \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathcal{V}}(T)) \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)). \tag{11.2.2.5.2}$$

11.2.3 Stability by inverse images of isocrystals

We keep notation 9.2.1 except concerning the divisors which are denoted here respectively by T and T' (instead of Z and Z'). In particular, we fix $\lambda_0 \in L(\mathbb{N})$ and we set $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda_0^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ and $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$ etc.

11.2.3.1. Let $\mathcal{E}^{(\bullet)} \in M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$. With similar to 9.2.1.8.1 notation (with locally projective isocrystals, we prefer to work with $\widetilde{f}^*$ instead of $\widetilde{f}^!$), we get the functor $\widetilde{f}^{*(\bullet)}_{\mathrm{alg}} \colon M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)) \to M(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp})$ by setting $\widetilde{f}^{(\bullet)!}_{\mathrm{alg}}(\mathcal{E}^{(\bullet)}) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}} f^{-1}\mathcal{F}^{(\bullet)}$. By left deriving the functor $\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}$, this yields the functor $\mathbb{L}\widetilde{f}^{*(\bullet)}_{\mathrm{alg}} \colon D^-(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)) \to D^-(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T'))$, defined by setting $\mathbb{L}\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T) \otimes^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathcal{F}^{(\bullet)}$ for any $\mathcal{F}^{(\bullet)} \in D^-(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$. Since it preserves lim-ind-isogenies, this induces the functor $\mathbb{L}\widetilde{f}^{*(\bullet)}_{\mathrm{alg}} \colon \underrightarrow{LD}^-_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)) \to \underrightarrow{LD}^-_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T'))$.

Following notation 9.2.1.15.3, we set $\mathbb{L}\widetilde{f}^{*(\bullet)}(\mathcal{F}^{(\bullet)}) := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T)\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathcal{F}^{(\bullet)}$, for any $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$. We get the morphism $\mathbb{L}\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to \mathbb{L}\widetilde{f}^{*(\bullet)}(\mathcal{F}^{(\bullet)})$ (beware the notation is slightly misleading since $\mathbb{L}\widetilde{f}^{*(\bullet)}$ is not necessarily the left derived functor of a functor).

Lemma 11.2.3.2. *Let $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$. Then, the canonical morphism*

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T') \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} f^{-1}\mathcal{G}^{(\bullet)} \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T')\widehat{\otimes}^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} f^{-1}\mathcal{G}^{(\bullet)}$$

is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T'))$.

Proof. This is checked similarly to Lemma 9.2.1.17. □

Proposition 11.2.3.3. *We have the following assertions.*

(a) Let $\mathcal{F}$ be a $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-module, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$. Then the morphism

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)} f^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathcal{F}$$

is an isomorphism. In particular, the inverse image $\widetilde{f}^{(m)*}_{\mathrm{alg}}(\mathcal{F}) := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathcal{F}$ is a (coherent) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')$-module, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')$.

(b) Let $\mathcal{F}$ be a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$. Then the morphism

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} f^{-1}\mathcal{F} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T)_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} f^{-1}\mathcal{F}$$

is an isomorphism. The inverse image $\widetilde{f}^{(m)*}_{\mathrm{alg}}(\mathcal{F}) := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T', T)_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} f^{-1}\mathcal{F}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')_{\mathbb{Q}}$-module, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}$.

Proof. Let us check the second statement. It follows from 7.5.2.8.(a) that there exists a p-torsion free $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-module $\mathring{\mathcal{F}}$, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$ together with an $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{F}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{F}$. Then the canonical morphisms

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T') \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)} f^{-1}\mathring{\mathcal{F}} \to \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')\widehat{\otimes}_{f^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)} f^{-1}\mathring{\mathcal{F}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T',T)\widehat{\otimes}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathring{\mathcal{F}}$$
$$\leftarrow \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T',T) \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathring{\mathcal{F}}$$

are isomorphisms. Indeed, since $\mathring{\mathcal{F}}$ is both $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-coherent and $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$ -coherent, the first and the last morphisms are isomorphisms. Since the modulo π^{n+1} reduction of the middle morphism is an isomorphism for any $n \in \mathbb{N}$ (see 5.1.1.6), since this is a morphism of separated complete modules for the p-adic topology, this implies that the middle morphism is an isomorphism. It follows from 4.4.2.10 that the modulo π^{n+1} reduction of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T',T)\widehat{\otimes}_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} f^{-1}\mathring{\mathcal{F}}$ is nilpotent module for any n. Hence, tensoring with $\mathbb{Q}$, we get that $\widetilde{f}^{(m)*}_{\mathrm{alg}}(\mathcal{F})$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')_{\mathbb{Q}}$-module.

We proceed similarly (without taking care of topological nilpotence) for the first statement. □

Proposition 11.2.3.4. *Let $\mathcal{F}^{(0)}$ be a $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-module, coherent over $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$ such that the canonical morphism*

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}^{(0)} =: \mathcal{F}^{(\bullet)} \tag{11.2.3.4.1}$$

is an ind-isogeny in $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$ and such that $\mathcal{F}^{(m)}_{\mathbb{Q}}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module for any m.

(a) The canonical morphism $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}}\mathcal{F}^{(0)} \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)})$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T'))$.

(b) The canonical morphism

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T') \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}}\mathcal{F}^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}}\mathcal{F}^{(0)} \tag{11.2.3.4.2}$$

is an ind-isogeny of $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T'))$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}}\mathcal{F}^{(0)}$ is a topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')_{\mathbb{Q}}$-module for any m. In particular, $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}}\mathcal{F}^{(0)}$ belongs to $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}'^\sharp, T'/\mathfrak{S}'^\sharp)$.

(c) The canonical morphism $\mathbb{L}\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to \mathbb{L}\widetilde{f}^{*(\bullet)}(\mathcal{F}^{(\bullet)})$ is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T'))$.

(d) If $\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})$ is flat as $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module, then the canonical morphism $\mathbb{L}\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)})$ is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp}(T'))$.

Proof. 1) Since the canonical morphism $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}} = \mathcal{F}^{(m)}_{\mathbb{Q}}$ is an isomorphism, then the coherent $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}}$ is canonically endowed with a structure of topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module (extending its structure of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module). This yields the isomorphisms

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}}\mathcal{F}^{(0)}_{\mathbb{Q}} \underset{11.2.3.3}{\xleftarrow{\sim}} \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \left(\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} f^{-1}\mathcal{F}^{(0)}_{\mathbb{Q}}\right)$$
$$\xrightarrow{\sim} \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} f^{-1}\left(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}}\right)$$
$$\underset{11.2.3.3}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp\to\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T',T)_{\mathbb{Q}} \otimes_{f^{-1}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} f^{-1}\left(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}}\right)$$
$$= f^{(m)*}_{\mathrm{alg}}\left(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}}\right) \overset{11.2.3.4.1}{\xrightarrow{\sim}} f^{(m)*}_{\mathrm{alg}}(\mathcal{F}^{(m)}_{\mathbb{Q}}). \tag{11.2.3.4.3}$$

Hence, with 11.2.3.3, $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}}$ is therefore canonically endowed with a structure of topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')_{\mathbb{Q}}$-module and is also $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}$-coherent (for the induced structure of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module). This yields the isomorphisms:

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow[7.5.2.8.2]{\sim} (\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T') \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}'}} \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T))_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}}$$

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \xrightarrow[7.5.2.8.1]{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T') \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}'}} \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T))_{\mathbb{Q}}} \left(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \right).$$

Hence, the canonical map

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \tag{11.2.3.4.4}$$

is an isomorphism. With 11.2.3.4.3 and 11.2.3.4.5 we get that the canonical map

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \to f^{(m)*}_{\mathrm{alg}}(\mathcal{F}^{(m)}_{\mathbb{Q}}) \tag{11.2.3.4.5}$$

is an isomorphism.

2) For any $m \in \mathbb{N}$, let $\mathcal{G}^{(m)}$ be the quotient of $\mathcal{F}^{(m)}$ by its p-torsion part. i) Let us prove the part (a). Since $\mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T))$, then $\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{F}^{(\bullet)}) \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)})$ is an isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T'))$. Hence, to prove the part (a) we reduce to check that the canonical morphism

$$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)} \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)}) \tag{11.2.3.4.6}$$

is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T'))$. It follows from 11.2.3.3 that $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}$ is therefore a coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')$-module. Using 7.5.2.8, $\mathcal{G}^{(m)}$ is a (coherent) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}(T)$-module, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$. Hence, with 11.2.3.3 $\widetilde{f}^{(m)*}_{\mathrm{alg}} \mathcal{G}^{(m)}$ is a (coherent) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')$-module, coherent over $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')$. We get the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)} \to \widetilde{f}^{(m)*}_{\mathrm{alg}} \mathcal{G}^{(m)}$ of coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')$-modules. Hence, by using 8.4.2.8, to check that $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T') \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)} \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)})$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T'))$, we reduce to check that $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp}}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)}_{\mathbb{Q}})$ is an isomorphism, which is a consequence of the isomorphism 11.2.3.4.5.

ii) To check that 11.2.3.4.2 is an ind-isogeny, we reduce to check that its composition with the ind-isogeny 11.2.3.4.6

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T') \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)} \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)}) \tag{11.2.3.4.7}$$

is an ind-isogeny. Since $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T') \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}$ and $\widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)})$ are both $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}'}(T')$-coherent, by using 8.4.2.8 we reduce to prove that $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T')_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}'}(T')_{\mathbb{Q}}} \widetilde{f}^{(0)*}_{\mathrm{alg}} \mathcal{F}^{(0)}_{\mathbb{Q}} \to \widetilde{f}^{*(\bullet)}_{\mathrm{alg}}(\mathcal{G}^{(\bullet)}_{\mathbb{Q}})$ is an isomorphism, which follows from 11.2.3.4.3.

3) The part c) follows from 9.2.1.17 and the coherence of $\mathcal{F}^{(\bullet)}$.

4) It remains to check that the canonical morphism $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T') \otimes^{\mathbb{L}}_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} f^{-1}\mathcal{F}^{(\bullet)} \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}'}(T') \otimes_{f^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} f^{-1}\mathcal{F}^{(\bullet)}$ is an isomorphism under the flatness assumption. Since this is a morphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}'}(T'))$, we reduce to check it after applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$, which is a consequence of the flatness as $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$-module. □

Corollary 11.2.3.5. *We have the following properties.*

(a) Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^{\sharp}, T/\mathfrak{S}^{\sharp})$. *The canonical morphism*

$$\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}T')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{O}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}} f^{-1}\mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\sharp}/\mathfrak{S}'^{\sharp} \to \mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}T', T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}} f^{-1}\mathcal{E}$$

is an isomorphism. We have $\widetilde{f}^*(\mathcal{E}) := \mathcal{D}^\dagger_{\mathfrak{X}'^\sharp/\mathfrak{S}'^\sharp \to \mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T', T)_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}} f^{-1}\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}'^\sharp, T'/\mathfrak{S}'^\sharp)$, *which induces the functor* $\widetilde{f}^* \colon \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}'^\sharp, T'/\mathfrak{S}'^\sharp)$.

(b) Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. *We have the isomorphism*

$$\underrightarrow{l}^*_{\mathbb{Q}} \circ \mathbb{L}\widetilde{f}^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{L}\widetilde{f}^* \circ \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}).$$

(c) Suppose log structures are trivial and $\mathfrak{S} = \mathrm{Spf}\,\mathcal{V}$. *The functor* $\mathbb{L}\widetilde{f}^{*(\bullet)}$ *factors through*

$$\mathbb{L}\widetilde{f}^{*(\bullet)} \colon \mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T/\mathfrak{S}) \to \mathrm{MIC}^{(\bullet)}(\mathfrak{X}', T'/\mathfrak{S}). \tag{11.2.3.5.1}$$

Proof. We can suppose that $\mathcal{E}^{(\bullet)}$ satisfies the hypotheses of 11.2.2.3.(b). Hence, the part (a) and (b) are a corollary of 11.2.3.4. The part (c) is a consequence of 11.2.1.14 and 11.2.3.4. □

11.2.4 Stability by tensor product of isocrystals

We keep notation 11.2.1.

Proposition 11.2.4.1. *Let* $\mathcal{E}, \mathcal{E}' \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. *For* $\lambda \in L(\mathbb{N})$ *large enough, denoting by* $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, *let* $\mathcal{E}^{(0)}$, $\mathcal{E}'^{(0)}$ *be tow topologically nilpotent* $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-*module, coherent as* $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-*module satisfying both conditions of 11.2.1.7.(b) for respectively* $\mathcal{E}$ *and* $\mathcal{E}'$. *We get the* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-*module by setting* $\mathcal{E}^{(\bullet)} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)}$, $\mathcal{E}'^{(\bullet)} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}'^{(0)}$.

(a) $\mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}' \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$

(b) $\mathcal{E}^{(\bullet)} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}'^{(\bullet)}$ *both conditions of 11.2.1.7.(b) for* $\mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}'$.

Proof. This follows from 7.5.2.10. □

Proposition 11.2.4.2. *Let* $\lambda \in L(\mathbb{N})$. *Set* $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda_0^*\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$. *Let* $\mathcal{F}^{(0)}$ *and* $\mathcal{F}'^{(0)}$ *be two p-torsion free* $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$-*modules, coherent over* $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$ *such that the canonical morphism*

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}^{(0)} =: \mathcal{F}^{(\bullet)},$$

and similarly for $\mathcal{F}'^{(0)}$, *is an ind-isogeny in* $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$ *and such that* $\mathcal{F}^{(m)}_{\mathbb{Q}}$ *and* $\mathcal{F}'^{(m)}_{\mathbb{Q}}$ *are topologically nilpotent* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-*modules for any* m. *Set* $\mathcal{F}''^{(0)} := \mathcal{F}^{(0)} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(0)}$, $\mathcal{F}''^{(\bullet)} := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}''^{(0)}$.

(a) $\mathcal{F}''^{(m)}_{\mathbb{Q}}$ *are topologically nilpotent* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-*modules for any* m.

(b) The canonical morphism

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}''^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}''^{(0)} = \mathcal{F}''^{(\bullet)} \tag{11.2.4.2.1}$$

is an ind-isogeny of $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$.

(c) $\mathcal{F}''^{(\bullet)}$ *belongs to* $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$.

(d) The canonical morphism

$$\mathcal{F}^{(\bullet)} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(\bullet)} \to \mathcal{F}^{(\bullet)}\widehat{\otimes}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)}\mathcal{F}'^{(\bullet)}. \tag{11.2.4.2.2}$$

is an ind-isogeny of $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp})$.

(e) *The canonical morphism*

$$\mathcal{F}''^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}''^{(0)} \to \mathcal{F}^{(\bullet)} \widehat{\otimes}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(\bullet)} \tag{11.2.4.2.3}$$

is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$*. In particular,* $\mathcal{F}^{(\bullet)} \widehat{\otimes}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(\bullet)}$ *belongs to* $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$.

Proof. a) We have the isomorphism

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} (\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}^{(0)}_{\mathbb{Q}}) \otimes_{\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} (\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}'^{(0)}_{\mathbb{Q}})$$

By using 7.5.2.10, this yields that $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}}$ is therefore canonically endowed with a structure of topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module and is also $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-coherent (for the induced structure of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module). Hence, we get the isomorphisms:

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}} \underset{7.5.2.8.2}{\xrightarrow{\sim}} (\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}}$$

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}} \underset{7.5.2.8.1}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))_{\mathbb{Q}}} \left(\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}}\right).$$

This implies that the canonical map

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}} \mathcal{F}''^{(0)}_{\mathbb{Q}} \tag{11.2.4.2.4}$$

is an isomorphism.

b) Set $\mathcal{E}''^{(\bullet)} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}''^{(0)}$. For any $m \in \mathbb{N}$, let $\mathcal{G}''^{(m)}$ be the quotient of $\mathcal{F}''^{(m)}$ by its p-torsion part. Since $\mathcal{F}''^{(\bullet)} \to \mathcal{G}''^{(\bullet)}$ is an ind-isogeny, to check that $\mathcal{E}''^{(\bullet)} \to \mathcal{F}''^{(\bullet)}$ is an ind-isogeny we reduce to prove that so is the composition $\mathcal{E}''^{(\bullet)} \to \mathcal{G}''^{(\bullet)}$.

Using 7.5.2.8, the part 1) of the proof implies that $\mathcal{G}''^{(m)}$ is $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$-coherent for any integer m. Similarly, $\mathcal{F}^{(0)}$ and $\mathcal{F}'^{(0)}$ are $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$-coherent. This yields that $\mathcal{E}''^{(m)} = \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}''^{(0)}$ is $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$-coherent. Hence, by using 8.4.2.8, to check that $\mathcal{E}''^{(\bullet)} \to \mathcal{G}''^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$, we reduce to check that $\mathcal{E}''^{(\bullet)}_{\mathbb{Q}} \to \mathcal{G}''^{(\bullet)}_{\mathbb{Q}}$ is an isomorphism. Since $\mathcal{E}''^{(\bullet)}_{\mathbb{Q}} \to \mathcal{F}''^{(\bullet)}_{\mathbb{Q}}$ is an isomorphism (see 11.2.4.2.4) and $\mathcal{F}''^{(\bullet)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}''^{(\bullet)}_{\mathbb{Q}}$, then we are done.

c) By using 11.2.2.3, this is a consequence of (a) and (b).

d) Let $\mathcal{G}^{(m)}$ (resp. $\mathcal{G}'^{(m)}$) be the quotient of $\mathcal{F}^{(m)}$ (resp. $\mathcal{F}'^{(m)}$) by its p-torsion part for any $m \in \mathbb{N}$. Following 7.5.2.9, $\mathcal{G}^{(m)}$ (resp. $\mathcal{G}'^{(m)}$) are $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)$-coherent for any integer m. Hence, so is $\mathcal{G}^{(m)} \otimes_{\mathcal{B}^{(m)}_{\mathfrak{X}}(T)} \mathcal{G}'^{(m)}$. By applying 7.5.2.1, this yields that the canonical morphism

$$\mathcal{G}^{(\bullet)} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{G}'^{(\bullet)} \to \mathcal{G}^{(\bullet)} \widehat{\otimes}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)} \mathcal{G}'^{(\bullet)}. \tag{11.2.4.2.5}$$

is an isomorphism. Hence, 11.2.4.2.2 is an ind-isogeny.

e) It follows from 8.4.2.8 that to check that the map

$$\mathcal{F}''^{(\bullet)} = \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}''^{(0)} \to \mathcal{G}^{(\bullet)} \widehat{\otimes}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{G}'^{(\bullet)} \tag{11.2.4.2.6}$$

is an ind-isogeny, we reduce to prove $\mathcal{F}''^{(\bullet)}_{\mathbb{Q}} \to \mathcal{G}^{(\bullet)}_{\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{G}'^{(\bullet)}_{\mathbb{Q}}$ is an isomorphism. Since $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}''^{(0)} \xrightarrow{\sim} \mathcal{G}^{(\bullet)}_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{G}'^{(\bullet)}_{\mathbb{Q}}$, this follows from 11.2.4.2.1 and 11.2.4.2.5. □

Proposition 11.2.4.3. *Let* $\mathcal{F}^{(\bullet)}$ *and* $\mathcal{F}'^{(\bullet)}$ *two objects of* $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$.

(a) *The canonical morphism of* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T))$ *:*

$$\mathcal{F}^{(\bullet)} \otimes^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(\bullet)} \to \mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{P}}(T)} \mathcal{F}'^{(\bullet)}. \tag{11.2.4.3.1}$$

is an isomorphism. This yields $\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{P}}(T)} \mathcal{F}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathfrak{X}^\sharp}(T))$.

(b) *Set* $\mathcal{F} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}^{(\bullet)})$, $\mathcal{F}' := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{F}'^{(\bullet)})$. *The canonical morphism*

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{F}' \to \mathcal{F} \otimes^{\mathbb{L}\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{F}'. \tag{11.2.4.3.2}$$

is an isomorphism.

(c) *We have* $\mathcal{F}^{(\bullet)} \widehat{\otimes}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. *By abuse of notation, we can simply write this isocrystal* $\mathcal{F}^{(\bullet)} \otimes_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)} \mathcal{F}'^{(\bullet)}$.

Proof. By using 11.2.2.5.1, we get that 11.2.4.3.1 is an isomorphism. By applying $\underrightarrow{l}^*_{\mathbb{Q}}$ to 11.2.4.3.1, we get 11.2.4.3.2. The last assertion follows from 9.1.1.16, 11.2.2.3 and 11.2.4.2. □

Corollary 11.2.4.4. *Suppose* $\mathfrak{S}^\sharp = \mathrm{Spf}\,\mathcal{V}$ *and log structures are trivial. For* $i = 1, 2$, *let* $\mathfrak{X}_i$ *be a noetherian of finite Krull dimension* $\mathcal{V}$*-smooth formal scheme, For* $i = 1, 2$, *let* X_i *be a* k*-smooth closed subscheme of* P, T_i *be a divisor of* P_i *such that* $Z_i := T_i \cap X_i$ *is a divisor of* X_i. *Let* $\mathfrak{X} := \mathfrak{X}_1 \times_{\mathfrak{S}} \mathfrak{X}_2$. *For* $i = 1, 2$, *let* $p_i \colon \mathfrak{X} \to \mathfrak{X}_i$ *be the natural projection,* $T := p_1^{-1}(T_1) \cup p_2^{-1}(T_2)$, $\mathcal{E}_i^{(\bullet)}$ *be an object of* $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}_i, T_i/K)$. *Then* $\mathcal{E}_1^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{E}_2^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T/K)$.

Proof. Following Lemma 9.2.5.9 (see also 9.2.5.3), we already know $\mathcal{E}_1^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{E}_2^{(\bullet)} \xrightarrow{\sim} \mathcal{E}_1^{(\bullet)} \widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{E}_2^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(T))$. This follows from 11.2.4.3 and 11.2.3.5.1. □

11.2.5 Stability by $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)$-linear duality of isocrystals

Proposition 11.2.5.1. *Let* $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. *For* $\lambda \in L(\mathbb{N})$ *large enough, denoting by* $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) := \lambda^* \mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$ *and* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$, *let* $\mathcal{E}^{(0)}$ *be a topologically nilpotent* $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$*-module, coherent as* $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$*-module satisfying both conditions of 11.2.1.7.(b). We get the* $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$*-module by setting* $\mathcal{E}^{(\bullet)} := \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{(0)}$.

(a) $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$;

(b) $\mathcal{E}^{\vee(\bullet)} := \mathcal{H}om_{\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}}}(\mathcal{E}^{(\bullet)}, \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T)_{\mathbb{Q}})$ *satisfies both conditions of 11.2.1.7.(b) for* $\mathcal{E}^\vee$.

Proof. Since $\mathcal{E}^{\vee(0)}$ is a left $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-module, then so is $\mathcal{E}^{\vee(0)}$. Via the canonical isomorphism of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{\vee(0)} \xrightarrow{\sim} \mathcal{E}^{\vee(m)}$, we get the morphism of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}$-modules $\mathcal{E}^{\vee(m)} \to \mathcal{E}^{\vee(m+1)}$. We get the $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism

$$\varinjlim_m \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} \mathcal{E}^{\vee(0)} \xrightarrow{\sim} \mathcal{E}^\vee. \tag{11.2.5.1.1}$$

Since $\mathcal{E}^{(m)}$ (uniquely) extends to a structure of topologically nilpotent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-module then so is $\mathcal{E}^{\vee(m)}$ (see 7.5.2.10). By continuity, we get that the homomorphisms

$$\mathcal{E}^{\vee(m)} \to \mathcal{E}^{\vee(m+1)} \tag{11.2.5.1.2}$$

are $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$-linear. □

Proposition 11.2.5.2. *Let* $\mathcal{F}^{(0)}$ *be a* $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)$*-module, coherent over* $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)$ *such that the canonical morphism*

$$\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T) \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)} \mathcal{F}^{(0)} \to \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}^{(0)} =: \mathcal{F}^{(\bullet)} \tag{11.2.5.2.1}$$

is an ind-isogeny in $M(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(T))$ *and such that* $\mathcal{F}^{(m)}_{\mathbb{Q}}$ *is a topologically nilpotent* $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}}$*-module for any* m. *Set* $\mathcal{F}^{\vee(0)} := \mathcal{H}om_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)}(\mathcal{F}^{(0)}, \widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T))$, $\mathcal{F}^{\vee(\bullet)} := \widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T) \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}(T)} \mathcal{F}^{\vee(0)}$.

(a) $\mathcal{F}_{\mathbb{Q}}^{\vee(m)} \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}}}(\mathcal{F}_{\mathbb{Q}}^{(m)}, \widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}})$ *is a topologically nilpotent* $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(m)}(T)_{\mathbb{Q}}$*-modules for any* m*.*

(b) The canonical morphism

$$\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(\bullet)}(T) \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)} \mathcal{F}^{\vee(0)} \to \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(\bullet)}(T) \otimes_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(0)}(T)} \mathcal{F}^{\vee(0)} = \mathcal{F}^{\vee(\bullet)} \tag{11.2.5.2.2}$$

is an ind-isogeny of $M(\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(\bullet)}(T))$*.*

(c) $\mathcal{F}^{\vee(\bullet)}$ *belongs to* $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$*.*

Proof. a) We have the isomorphism

$$\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)} \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}}}(\mathcal{F}_{\mathbb{Q}}^{(m)}, \widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}})$$

By using 7.5.2.10, this yields that $\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)}$ is therefore canonically endowed with a structure of topologically nilpotent $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(m)}(T)_{\mathbb{Q}}$-module and is also $\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}}$-coherent. Hence, we get the isomorphisms:

$$\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)} \underset{7.5.2.8.2}{\xrightarrow{\sim}} (\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(0)}(T))_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)}$$

$$\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)} \underset{7.5.2.8.1}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(m)}(T) \otimes_{(\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(0)}(T))_{\mathbb{Q}}} \left(\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)}\right).$$

This implies that the canonical map

$$\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)} \to \widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(m)}(T)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(0)}(T)_{\mathbb{Q}}} \mathcal{F}_{\mathbb{Q}}^{\vee(0)} \tag{11.2.5.2.3}$$

is an isomorphism.

By tensoring with $\mathbb{Q}$ the ind-isogeny 11.2.5.2.1, we get an isomorphism which yields the isomorphism $\mathcal{F}_{\mathbb{Q}}^{\vee(m)} \xrightarrow{\sim} \mathcal{H}om_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}}}(\mathcal{F}_{\mathbb{Q}}^{(m)}, \widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)_{\mathbb{Q}})$ of $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(m)}(T)_{\mathbb{Q}}$-modules for any m. Since $\mathcal{F}_{\mathbb{Q}}^{(m)}$ is a topologically nilpotent $\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(m)}(T)_{\mathbb{Q}}$-module then so is $\mathcal{F}^{\vee(m)}$ (see 7.5.2.10).

b) Set $\mathcal{E}^{(\bullet)} := \widetilde{\mathcal{B}}_{\mathfrak{X}}^{(\bullet)}(T) \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)} \mathcal{F}^{\vee(0)}$. For any $m \in \mathbb{N}$, let $\mathcal{G}^{(m)}$ be the quotient of $\mathcal{F}^{\vee(m)}$ by its p-torsion part. Since $\mathcal{F}^{\vee(\bullet)} \to \mathcal{G}^{(\bullet)}$ is an ind-isogeny, to check that $\mathcal{E}^{(\bullet)} \to \mathcal{F}^{\vee(\bullet)}$ is an ind-isogeny we reduce to prove that so is the composition $\mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$.

Using 7.5.2.8, the part a) of the proof implies that $\mathcal{G}^{(m)}$ is $\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)$-coherent for any integer m. Similarly, $\mathcal{F}^{\vee(0)}$ is $\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)$-coherent. This yields that $\mathcal{E}''^{(m)} = \widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T) \otimes_{\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(0)}(T)} \mathcal{F}^{\vee(0)}$ is $\widetilde{\mathcal{B}}_{\mathfrak{X}}^{(m)}(T)$-coherent. Hence, by using 8.4.2.8, to check that $\mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)}$ is an ind-isogeny of $M(\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}^{(\bullet)}(T))$, we reduce to check that $\mathcal{E}_{\mathbb{Q}}^{(\bullet)} \to \mathcal{G}_{\mathbb{Q}}^{(\bullet)}$ is an isomorphism. Since $\mathcal{E}_{\mathbb{Q}}^{(\bullet)} \to \mathcal{F}_{\mathbb{Q}}^{(\bullet)}$ is an isomorphism (see 11.2.5.2.3) and $\mathcal{F}_{\mathbb{Q}}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}_{\mathbb{Q}}^{(\bullet)}$, then we are done.

□

11.2.6 Comparison with the $\mathcal{D}$-linear duality

We keep notation 11.2.1. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ which is locally projective of finite type on $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$.

Remark 11.2.6.1. Specially when T is not empty (see 8.7.7.7 to get example when T is empty), we do not know a priori if $\mathcal{D}_{\mathfrak{X}^\sharp}^\dagger({}^\dagger T)_{\mathbb{Q}}$ has finite tor dimension. Be careful because standard isomorphisms concerning the sheaves of homomorphisms require to work with perfect complexes instead of complexes with bounded and coherent cohomology.

Proposition 11.2.6.2. *We have* $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$*,* $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}_{\mathfrak{X}^\sharp}^\dagger({}^\dagger T)_{\mathbb{Q}})$*.*

Proof. Following 4.7.3.7.1 (see also the remark 4.7.3.15), the first Spencer sequence $\mathrm{Sp}^{\bullet}_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E})$ associated to the trivial filtration is exact. As $\mathcal{E}$ is locally projective of finite type on $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$, then this sequence gives a finite resolution of $\mathcal{E}$ by of locally projective of finite type left $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules. Hence, $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$. Since the extension $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$ is flat, with 11.2.1.9.2, this yields that $\mathcal{E} \in D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$. □

11.2.6.3. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ (see notation 11.2.1.4) which is locally projective of finite type on $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$. Following 4.6.7.1, with notation 9.2.4.22.1 we have the canonical isomorphism of the form

$$\theta^{\mathrm{alg}} \colon \mathbb{D}^{\mathrm{alg}}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)}(\mathcal{E},\, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)) \to \mathbb{D}^{\mathrm{alg}}_T(\mathcal{E}). \tag{11.2.6.3.1}$$

Following 4.6.7.1, since $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}} \in D_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}})$ and $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$ (because it is locally projective of finite type on $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)$), then the map 11.2.6.3.1 is an isomorphism.

Since $\mathcal{E}$ is locally projective of finite type on $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)$, the morphism $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)}(\mathcal{E},\, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)) \to \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)}(\mathcal{E},\, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T))$ is an isomorphism. Following 8.7.7.5.c), we have a canonical isomorphism $\mathbb{D}^{\mathrm{alg}}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$. Hence, we get the isomorphism of $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}})$ (see 11.2.6.2):

$$\mathcal{E}^\vee \xrightarrow{\sim} \mathbb{D}^{\mathrm{alg}}_T(\mathcal{E}). \tag{11.2.6.3.2}$$

In particular, for any $i \neq 0$, $H^i(\mathbb{D}^{\mathrm{alg}}_T(\mathcal{E})) = 0$ and we can see $\mathbb{D}^{\mathrm{alg}}_T(\mathcal{E})$ as a $\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is locally projective of finite type on $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)$.

Following 11.2.5.1, $\mathcal{E}^\vee \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. Hence, so is $\mathbb{D}^{\mathrm{alg}}_T(\mathcal{E})$. This yields from of 11.2.1.9.2 that the canonical morphisms $\mathcal{E} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}$ and $\mathbb{D}^{\mathrm{alg}}_T(\mathcal{E}) \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathbb{D}^{\mathrm{alg}}_T(\mathcal{E})$ is an isomorphism. With notation 9.2.4.22.2, this yields the isomorphism:

$$\rho_T \colon \mathbb{D}^{\mathrm{alg}}_T(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathbb{D}^{\mathrm{alg}}_T(\mathcal{E}) \underset{4.6.4.7.1}{\xrightarrow{\sim}} \mathbb{D}_T(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}) \xrightarrow{\sim} \mathbb{D}_T(\mathcal{E}), \tag{11.2.6.3.3}$$

Hence, from 11.2.6.3.2 we get the $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp/\mathfrak{S}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism:

$$\mathcal{E}^\vee \xrightarrow{\sim} \mathbb{D}_T(\mathcal{E}). \tag{11.2.6.3.4}$$

In particular, for any $r \in \mathbb{Z} \setminus \{0\}$ we have $H^r(\mathbb{D}_T(\mathcal{E})) = 0$.

11.2.6.4. Beware the isomorphism 11.2.6.3.4 is not compatible with Frobenius. The key point is that the isomorphism $\mathbb{D}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ [1] is not compatible. Hence, the "twist" $\mathbb{D}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$ is essential to obtain this one. To get a compatible with Frobenius morphism, let us consider the isomorphism $\theta \colon \mathbb{D}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}^\vee \xrightarrow{\sim} \mathbb{D}_T(\mathcal{E})$ making commutative the following diagram:

$$\begin{array}{ccc} \mathbb{D}^{\mathrm{alg}}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}^\vee & \xrightarrow[\sim]{\theta^{\mathrm{alg}}} & \mathbb{D}^{\mathrm{alg}}_T(\mathcal{E}) \\ \sim \downarrow \rho\otimes\mathrm{id} & & \sim \downarrow \rho \\ \mathbb{D}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}^\vee & \overset{\theta}{\underset{\sim}{\dashrightarrow}} & \mathbb{D}_T(\mathcal{E}). \end{array} \tag{11.2.6.4.1}$$

We will check later its compatibility with Frobenius (see **??**).

Corollary 11.2.6.5. *Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$. The following assertions are equivalent :*

(a) For any $r \in \mathbb{Z}$, $H^r(\mathcal{F}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ and is locally projective of finite type on $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$.

(b) For any $r \in \mathbb{Z}$, $H^r(\mathbb{D}_T(\mathcal{F})) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ and is locally projective of finite type on $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ for any $r \in \mathbb{Z}$.

If $\mathcal{F}$ satisfies one of these equivalent conditions, then, for any r, $s \in \mathbb{Z}$ such that $r \neq 0$, $H^r(\mathbb{D}_T(H^s(\mathcal{F})) = 0$ and $\mathbb{D}_T(H^s(\mathcal{F})) \xrightarrow{\sim} H^s(\mathbb{D}_T(\mathcal{F}))$.

Proof. This is a consequence of 11.2.6.3.4. □

[1] See Abe's paper

11.2.7 Commutation of sp_* with inverse images, glueing isomorphisms and duality

11.2.7.1. Let $\mathfrak{S} \to \operatorname{Spf} \mathcal{V}$ be a smooth morphism. Let $\mathfrak{X}$ (resp. $\mathfrak{X}'$) be a smooth $\mathfrak{S}$-scheme, $\mathfrak{Z}$ (resp. $\mathfrak{Z}'$) be a relative strict normal crossing divisor of $\mathfrak{X}$ (resp. $\mathfrak{X}'$) over $\mathfrak{S}$, $\mathfrak{X}^\sharp := (\mathfrak{X}, M_{\mathfrak{Z}})$ (resp. $\mathfrak{X}'^\sharp := (\mathfrak{X}', M_{\mathfrak{Z}'})$) be the logarithmic $\mathcal{V}$-formal scheme induced by the logarithmic structure $M_{\mathfrak{Z}}$ (resp. $M_{\mathfrak{Z}'}$) associated with $\mathfrak{Z}$ (resp. $\mathfrak{Z}'$). Let T (resp. T') be a divisor of X_0 (resp. X_0') and $\mathfrak{Y}^\sharp$ (resp. $\mathfrak{Y}'^\sharp$) the open subset of $\mathfrak{X}^\sharp$ (resp. $\mathfrak{X}'^\sharp$) complementary to the support of T (resp. T'), let $j\colon \mathfrak{Y}^\sharp \setminus \mathfrak{X}^\sharp$ (resp. $j\colon \mathfrak{Y}'^\sharp \setminus \mathfrak{X}'^\sharp$) be the induced open immersion.

Let $u\colon \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ be a morphism of formal schemes such that $T' \supset u_0^{-1}(T)$. This yields the morphism of smooth frames $f := (b, a, u)\colon (Y', X', \mathfrak{X}') \to (Y, X, \mathfrak{X})$. According to notation 10.1.2.6.1, we have the functor

$$f_K^* := j'^\dagger u_K^*, \tag{11.2.7.1.1}$$

which induces the functor $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}_K) \to \mathrm{MIC}^\dagger(\mathfrak{X}'^\sharp_K, T'/\mathfrak{S}_K)$. If this do not cause too much confusion (the tilde symbol indicates the divisors T and T'), we will write $\widetilde{u}_K^*$ instead of f_K^*.

Following 11.2.1.5.(a), the functors sp_* and sp^* (see 11.2.1.3) induce quasi-inverse equivalences of categories between $\mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}^\sharp_K)$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$. With notation 11.2.3.5, we have the functor $\widetilde{u}^*\colon \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/K) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}', T'/K)$ which is compatible with $\widetilde{u}_K^*$, i.e. for any $E \in \mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}^\sharp_K)$ and $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$, there exist canonical isomorphisms of respectively $\mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{X}', T'/K)$ and $\mathrm{MIC}^\dagger(Y', X', \mathfrak{X}'/K)$ of the form

$$\mathrm{sp}_* \widetilde{u}_K^*(E) \xrightarrow{\sim} \widetilde{u}^* \mathrm{sp}_*(E), \qquad \widetilde{u}_K^* \mathrm{sp}^*(\mathcal{E}) \xrightarrow{\sim} \mathrm{sp}^* \widetilde{u}^*(\mathcal{E}). \tag{11.2.7.1.2}$$

We leave as an exercice the check that these isomorphisms are transitive with respect to the composition of morphisms.

Example 11.2.7.2. With notation 11.2.7.1, suppose $u = \mathrm{id}$. For any $E \in \mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}^\sharp_K)$, following 11.2.3.5 and 11.2.7.1.2, we have the isomorphism:

$$\mathrm{sp}_*(j'^\dagger E) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T')_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathrm{sp}_*(E) = ({}^\dagger T')(\mathrm{sp}_*(E)). \tag{11.2.7.2.1}$$

Proposition 11.2.7.3. *We keep notation 11.2.7.1 and suppose log structures are trivial. Let $u'\colon \mathfrak{X}'^\sharp \to \mathfrak{X}^\sharp$ be a morphism of formal log smooth $\mathfrak{S}$-schemes such that $u_0' = u_0$. Then, the following diagrams*

$$\begin{array}{ccc} \mathrm{sp}_* \widetilde{u}'^*_K(E) & \xrightarrow[\sim]{\mathrm{sp}_*(\epsilon_{u,\,u'})} & \mathrm{sp}_* \widetilde{u}^*_K(E) \\ \sim\downarrow 11.2.7.1.2 & & \sim\downarrow 11.2.7.1.2 \\ \widetilde{u}'^* \mathrm{sp}_*(E) & \xrightarrow[\sim]{\tau_{u,u'}} & \widetilde{u}^* \mathrm{sp}_*(E), \end{array} \qquad \begin{array}{ccc} \widetilde{u}'^*_K \mathrm{sp}^*(\mathcal{E}) & \xrightarrow[\sim]{\epsilon_{u,\,u'}} & \widetilde{u}^*_K \mathrm{sp}^*(\mathcal{E}) \\ \sim\downarrow 11.2.7.1.2 & & \sim\downarrow 11.2.7.1.2 \\ \mathrm{sp}^* \widetilde{u}'^*(\mathcal{E}) & \xrightarrow[\sim]{\mathrm{sp}^*(\tau_{u,u'})} & \widetilde{u}^* \mathrm{sp}_*(\mathcal{E}), \end{array}$$

where the glueing isomorphisms $\epsilon_{u,u'}$ and $\tau_{u,u'}$ are that of 10.2.4.1.1 and 9.2.2.3.1, are commutative.

Proof. This follows from the fact that both glueing isomorphisms $\epsilon_{u,u'}$ and $\tau_{u,u'}$ are built similarly using some factorization via the closed imbedding $(u, u')\colon \mathfrak{X}' \hookrightarrow (\mathfrak{X})^{(n)}$ where $(\mathfrak{X})^{(n)}$ is the nth infinitesimal neighborhood of the diagonal immersion $\mathfrak{X} \hookrightarrow \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$. □

Proposition 11.2.7.4. *Let $\mathfrak{X} \to \mathfrak{S} \to \operatorname{Spf} \mathcal{V}$ be two smooth morphisms, $\mathfrak{Z}$ be a relative strict normal crossing divisor of $\mathfrak{X}$ over $\mathfrak{S}$, $\mathfrak{X}^\sharp := (\mathfrak{X}, M_{\mathfrak{Z}})$ be the logarithmic $\mathcal{V}$-formal scheme induced by the logarithmic structure $M_{\mathfrak{Z}}$ associated with $\mathfrak{Z}$. Let T be a divisor of X_0. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ which is locally projective of finite type as $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)$-module. Let $E \in \mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}_K)$ which is locally free of finite type $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-module. We have the canonical isomorphisms (compatibility with Frobenius when log structures are trivial):*

$$\mathrm{sp}^* \mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)}(\mathcal{E}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)) \xrightarrow{\sim} \mathcal{H}om_{j^\dagger \mathcal{O}_{\mathfrak{X}_K}}(\mathrm{sp}^* \mathcal{E}, j^\dagger \mathcal{O}_{\mathfrak{X}_K}) \tag{11.2.7.4.1}$$

$$\mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)}(\mathrm{sp}_* E, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)) \xrightarrow{\sim} \mathrm{sp}_*(\mathcal{H}om_{j^\dagger \mathcal{O}_{\mathfrak{X}_K}}(E, j^\dagger \mathcal{O}_{\mathfrak{X}_K})). \tag{11.2.7.4.2}$$

Proof. Since $\mathcal{E}$ is an $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)$-module locally projective of finite type, then the canonical homomorphism

$$\mathrm{sp}^* \mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)}(\mathcal{E}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger T)) \to \mathcal{H}om_{j^\dagger \mathcal{O}_{\mathfrak{X}_K}}(\mathrm{sp}^* \mathcal{E}, j^\dagger \mathcal{O}_{\mathfrak{X}_K})$$

is an isomorphism. By using the equivalence of categories 11.2.1.5, since $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(^\dagger T)}(\mathcal{E}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(^\dagger T))$ is an isocrystal which is locally projective of finite type as $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(^\dagger T)$-module, by applying the functor sp_* to 11.2.7.4.1, we get the isomorphism 11.2.7.4.2. □

Corollary 11.2.7.5. *With notation and hypotheses of 11.2.7.4, we have the canonical isomorphism*

$$\mathrm{sp}_*(E^\vee) \xrightarrow{\sim} \mathbb{D}_T(\mathrm{sp}_* E), \tag{11.2.7.5.1}$$

where $E^\vee = \mathcal{H}om_{j^\dagger \mathcal{O}_{\mathfrak{X}_K}}(E, j^\dagger \mathcal{O}_{\mathfrak{X}_K})$.

Proof. This is a consequence of 11.2.6.3.4 and 11.2.7.4. □

Remark 11.2.7.6. The isomorphism 11.2.7.5.1 is not compatible with Frobenius. To get a compatible with Frobenius version we need to add a twist (see 11.3.5.3).

11.3 Compatibility with Frobenius of the comparison isomorphism between both dual functors of isocrystals

We suppose there exists a lifting $\sigma\colon \mathcal{V} \to \mathcal{V}$ of the power s-th of Frobenius of k. Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme, $\mathfrak{X}'$ be the $\mathcal{V}$-formal scheme induced by $\mathfrak{X}$ by the base change relative to σ, T be a divisor X and T' the divisor of X' induced by T by the base change relative to σ. We prove in this section that the isomorphism θ constructed at 11.2.6.4.1 is compatible with Frobenius (see 11.3.4.6.1).

11.3.1 Construction of $\widehat{\theta}^{(m)}$

Let m and i_0 be two nonnegative integers, X will be either $\mathfrak{X}$ or X_{i_0} (the case $X = \mathfrak{X}$ can be seen as the case where $i_0 = \infty$) and S will be either $\mathrm{Spf}\,\mathcal{V}$ or $\mathrm{Spec}\,\mathcal{V}/\mathfrak{m}\mathcal{V}^{i+1}$, with $\mathfrak{m}$ the ideal maximal of $\mathcal{V}$.

The $\mathcal{D}_X^{(m)}$-linear functor, denoted by $\mathbb{D}^{(m)}\colon D_{\mathrm{perf}}(\mathcal{D}_X^{(m)}) \to D_{\mathrm{perf}}(\mathcal{D}_X^{(m)})$, is defined by setting for any $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{D}_X^{(m)})$

$$\mathbb{D}_X^{(m)}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X^{(m)}}(\mathcal{E}, \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \omega_{X/S}^{-1}))[d_X]. \tag{11.3.1.0.1}$$

If there is no risk of confusion, we simply write $\mathbb{D}^{(m)}$ instead of $\mathbb{D}_X^{(m)}$.

The $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}$-linear functor, denoted by $\widehat{\mathbb{D}}^{(m)}\colon D_{\mathrm{perf}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}) \to D_{\mathrm{perf}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)})$, is defined by setting for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)})$

$$\widehat{\mathbb{D}}_{\mathfrak{X}}^{(m)}(\mathcal{E}) := \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}}(\mathcal{E}, \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}/\mathfrak{S}}^{-1}))[d_X]. \tag{11.3.1.0.2}$$

If there is no risk of confusion, we simply write $\widehat{\mathbb{D}}^{(m)}$ instead of $\widehat{\mathbb{D}}_{\mathfrak{X}}^{(m)}$. Similarly to 8.7.7.3, we construct for any $\mathcal{E} \in D(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)})$ the canonical morphism

$$\mathcal{E} \to \widehat{\mathbb{D}}_{\mathfrak{X}}^{(m)} \circ \widehat{\mathbb{D}}_{\mathfrak{X}}^{(m)}(\mathcal{E}) \tag{11.3.1.0.3}$$

which is an isomorphism when $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)})$.

In this subsection, we construct (see 11.3.1.16.2) , for any $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}}) \cap D({}^{1}\mathcal{D}_{\mathfrak{X}}^{(m)})$, the isomorphism

$$\widehat{\theta}^{(m)}\colon \widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\vee \to \widehat{\mathbb{D}}^{(m)}(\mathcal{E}),$$

where $\mathcal{E}^\vee := \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{E}, \mathcal{O}_{\mathfrak{X}})$. In order to get its compatibility with Frobenius (see 11.3.2.9.1), the idea is to take the p-adic completion of the isomorphism $\theta^{(m)}\colon \mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\vee \to \mathbb{D}^{(m)}(\mathcal{E})$ constructed at 6.3.4.15. To make later the link with the construction of θ (by extension from $\theta^{(m)}$: see 11.2.6.4.1), we also check that this morphism $\widehat{\theta}^{(m)}$ is the one induced by extension from $\theta^{(m)}$ (see below 11.3.1.19).

Remark 11.3.1.1. Let us recall that $\dim \mathrm{coh}\, \mathcal{D}_{\mathfrak{X}}^{(0)} = \dim \mathrm{coh}\, \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)} = 2d_{\mathfrak{X}} + 1$ (8.7.7.6). As the sheaves $\mathcal{D}_{\mathfrak{X}}^{(m)}$ and $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}$ are coherent, we obtain $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}}^{(0)}) = D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathfrak{X}}^{(0)})$, $D^{\mathrm{b}}_{\mathrm{perf}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}) = D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)})$ and $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_{\mathfrak{X}}^{(m)}) = D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\mathfrak{X}}^{(m)}) \cap D_{\mathrm{tdf}}(\mathcal{D}_{\mathfrak{X}}^{(m)})$.

Via 11.1.1.5, this yields that every object $\mathcal{E}$ of $D(\mathcal{D}^{(m)}_{\mathfrak{X}}) \cap D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}})$ is also an object of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{\mathfrak{X}})$ and of $D^{\mathrm{b}}_{\mathrm{perf}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}})$. Moreover, since the extension $\mathcal{D}^{(m)}_{\mathfrak{X}} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}$ is flat, since $\mathcal{E} \in D_{\mathrm{tdf}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}})$ then $\mathcal{E} \in D_{\mathrm{tdf}}(\mathcal{D}^{(m)}_{\mathfrak{X}})$. Hence, $\mathcal{E}$ belongs to $D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}^{(m)}_{\mathfrak{X}})$.

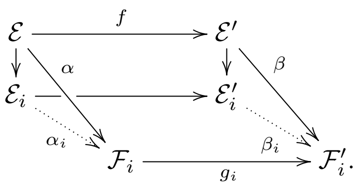
Remark 11.3.1.2. Following 4.7.3.7 and 4.7.3.14, we have the canonical isomorphisms $\mathbb{D}^{(0)}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}$, $\widehat{\mathbb{D}}^{(0)}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}$ and $\mathbb{D}^{(0)}(\mathcal{O}_{X_i}) \xrightarrow{\sim} \mathcal{O}_{X_i}$ for any nonnegative integer i. By Frobenius increasing of the level (use 6.2.7.2), since the pullback under Frobenius of the constant coefficient is the constant coefficient, then we get $\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}$ and for any positive integer i, $\mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \xrightarrow{\sim} \mathcal{O}_{X_i}$. However, we do not know if we have an isomorphism $\mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}$ nor if $\mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}})$ or $\mathcal{O}_{\mathfrak{X}} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}^{(m)}_{\mathfrak{X}})$.

Notation 11.3.1.3. Let $\mathcal{D} = \mathcal{D}^{(m)}_{X}$ or $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}$. For any integer $i \leq i_0$ and every $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{D})$, the canonical morphism

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E} \tag{11.3.1.3.1}$$

is an isomorphism and we set $\mathcal{E}_i := \mathcal{D}^{(m)}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{E}$.

11.3.1.4. Let $f\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of $D(\mathcal{D}^{(m)}_{X})$, $g_i\colon \mathcal{F}_i \to \mathcal{F}'_i$ be a morphism of $D(\mathcal{D}^{(m)}_{X_i})$, $\alpha\colon \mathcal{E} \to \mathcal{F}_i$ and $\beta\colon \mathcal{E}' \to \mathcal{F}'_i$ be two morphisms of $D(\mathcal{D}^{(m)}_{X})$ such that $\beta \circ f = g_i \circ \alpha$. The morphisms α and β factors through the morphisms $\alpha_i\colon \mathcal{E}_i \to \mathcal{F}_i$ and $\beta_i\colon \mathcal{E}'_i \to \mathcal{F}_i$. We get also the following commutative diagram :



11.3.1.5. The goal of this paragraph is to construct the functor 11.3.1.5.1. Let $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X})$, $\mathcal{G} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$. Let $\mathcal{I} \to \mathcal{J}$ be a quasi-isomorphism of K-injective complexes of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$. Then, the morphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ \mathcal{J})$ is a quasi-isomorphism of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$ because it is canonically isomorphic to $\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{I}) \to \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{J})$ and because $\mathcal{I} \to \mathcal{J}$ is also (by flatness of $\mathcal{O}_{X_i} \to \mathcal{D}^{(m)}_{X_i}$) a quasi-isomorphism of K-injective complexes of $K(\mathcal{O}_{X_i})$. Hence, following [Sta22, 13.14.15], we get the functor $\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ -)\colon D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i}) \to D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$, i.e. the right derived functor (with respect to quasi-isomophisms of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$) of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ -)\colon K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i}) \to D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$ exists and is computed by K-injective complexes. Moreover, the functor $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X}) \to D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$, given by $\mathcal{E} \mapsto \mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ \mathcal{G})$ has a derived functor (with respect to quasi-isomophisms of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X})$). Indeed, following [Sta22, 13.14.15], it is sufficient to check that if $\mathcal{P} \to \mathcal{P}'$ is a quasi-isomorphism of K-flat complexes of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X})$ (and therefore a K-flat complexes of $K(\mathcal{O}_X)$), then $\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}',\ \mathcal{G}) \to \mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\ \mathcal{G})$ is an isomorphism. Let $\mathcal{I}$ be a K-injective complex of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$ representing $\mathcal{G}$. Then, the map $\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}',\ \mathcal{G}) \to \mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\ \mathcal{G})$ corresponds to the map $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}',\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\ \mathcal{I})$ induced by the quasi-isomorphism $\mathcal{P} \to \mathcal{P}'$. Moreover, this latter is canonically isomorphic to $\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{P}',\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{P},\ \mathcal{I})$. Since $\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{P} \to \mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{P}'$ is a quasi-isomorphism of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$, since $\mathcal{I}$ is a K-injective complex of $K({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$ then $\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{P}',\ \mathcal{I}) \to \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes_{\mathcal{O}_X} \mathcal{P},\ \mathcal{I})$ is an isomorphism. We denote by $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})\colon D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X})^{\mathrm{op}} \to D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i})$ the derived functor. Since this is functorial in $\mathcal{G}$, this yields the bifunctor

$${}^{2}\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(-,-)\colon D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X})^{\mathrm{op}} \times D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i}) \to D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{X_i}). \tag{11.3.1.5.1}$$

We remark that $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})$ could have been directly defined by setting $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{I})$ where $\mathcal{P}$ is a K-flat complex representing $\mathcal{E}$ and $\mathcal{I}$ is a K-injective complex representing $\mathcal{G}$

[2] This can still be defined by adding log structure and coefficients

and by checking its independence relatively to the choices (it is a way of defining analogous to the one of [Sta22, 20.38.7]). By using [Sta22, 13.14.16], we check that we have the 2-commutative diagram

$$\begin{array}{ccc} D({}^{1}\mathcal{D}_{X}^{(m)})^{\mathrm{op}} \times D({}^{1}\mathcal{D}_{X_i}^{(m)}) & \xrightarrow{\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(-,-)} & D({}^{1}\mathcal{D}_{X_i}^{(m)}) \\ \uparrow & & \| \\ D({}^{1}\mathcal{D}_{X}^{(m)})^{\mathrm{op}} \times D({}^{1}\mathcal{D}_{X}^{(m)}) & \xrightarrow{\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(-,-)} & D({}^{1}\mathcal{D}_{X}^{(m)}). \end{array} \quad (11.3.1.5.2)$$

Hence, this is not confusing to simply denote by $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(-,-)\colon D({}^{1}\mathcal{D}_{X}^{(m)})^{\mathrm{op}} \times D({}^{1}\mathcal{D}_{X_i}^{(m)}) \to D({}^{1}\mathcal{D}_{X_i}^{(m)})$, instead of $\mathbb{R}_I\mathbb{R}_{II}\mathcal{H}om_{\mathcal{O}_X}(-,-)$.

Lemma 11.3.1.6. *Let $\mathcal{E} \in D({}^{1}\mathcal{D}_{X}^{(m)})$, $0 \le i \le i_0$ be an integer, $\mathcal{G} \in D({}^{1}\mathcal{D}_{X_i}^{(m)})$. We have the canonical isomorphism of $D({}^{1}\mathcal{D}_{X_i}^{(m)})$:*

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ \mathcal{G}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}, \mathcal{G}) \quad (11.3.1.6.1)$$

This isomorphism is transitive, i.e. for any integer $0 \le j \le i$ and any $\mathcal{H} \in D({}^{1}\mathcal{D}_{X_j}^{(m)})$ we have the commutative diagram

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\ \mathcal{H}) & \xrightarrow{11.3.1.6.1} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}, \mathcal{H}) \\ \downarrow{\scriptstyle 11.3.1.6.1} & & \downarrow{\scriptstyle 11.3.1.6.1} \\ \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_j}}(\mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}, \mathcal{H}) & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_j}}(\mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}), \mathcal{H}). \end{array} \quad (11.3.1.6.2)$$

Proof. We construct the morphism 11.3.1.6.1 by choosing a K-flat complex of $K({}^{1}\mathcal{D}_{X}^{(m)})$ representing $\mathcal{E}$ and a K-injective complex of $K({}^{1}\mathcal{D}_{X_i}^{(m)})$ representing $\mathcal{G}$. The transitivity is left to the reader as an easy exercice. □

Lemma 11.3.1.7. *For any $\mathcal{E} \in D({}^{1}\mathcal{D}_{X}^{(m)})$, $\mathcal{F} \in D({}^{1}\mathcal{D}_{X}^{(m)})$, for any integer $0 \le i \le i_0$, we have the canonical morphism of $D({}^{1}\mathcal{D}_{X_i}^{(m)})$:*

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i} \to \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i}). \quad (11.3.1.7.1)$$

This one is an isomorphism if $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_X)$. Moreover, this is transitive, i.e., if $j \le i \le i_0$ is a nonnegative integer, we have the following commutative diagram :

$$\begin{array}{ccc} (\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i}) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_j} & \underset{\sim}{\xrightarrow{11.3.1.7.1}} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i}) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_j} \\ \sim\downarrow & & \sim\downarrow{\scriptstyle 11.3.1.7.1} \\ \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_j} & \underset{\sim}{\xrightarrow{11.3.1.7.1}} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_j}). \end{array}$$

Proof. It follows from (the right version of) 4.6.4.1.1 (and 11.3.1.3.1) that the morphism $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i} \to \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i})$ induces 11.3.1.7.1. The transitivity is an easy exercice. □

Proposition 11.3.1.8. *Let $\mathcal{E} \in D({}^{1}\mathcal{D}_{X}^{(m)})$ and $\mathcal{F} \in D({}^{1}\mathcal{D}_{X}^{(m)})$.*

(i) For any $i \le i_0$, with notation 11.3.1.3 we have the canonical morphisms of $D({}^{1}\mathcal{D}_{X_i}^{(m)})$:

$$\alpha\colon \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i} \to \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i, \mathcal{F}_i), \quad (11.3.1.8.1)$$

which is an isomorphism when $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_X)$.

(ii) They are transitive, i.e., if $j \leq i \leq i_0$ is a nonnegative integer, the following diagram

$$\begin{array}{ccc} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_j} & \xrightarrow{\alpha \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_j}} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\,\mathcal{F}_i) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_j} \\ \downarrow & & \downarrow \alpha \\ \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_j} & \xrightarrow{\alpha} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_j}}(\mathcal{E}_j,\,\mathcal{F}_j) \end{array} \tag{11.3.1.8.2}$$

is commutative.

Proof. We construct the morphism 11.3.1.8.1 by composition as follows

$$\alpha\colon \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\,\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{X_i} \overset{11.3.1.7.1}{\longrightarrow} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\,\mathcal{F}_i) \overset{11.3.1.6.1}{\underset{\sim}{\longrightarrow}} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\,\mathcal{F}_i).$$

□

Proposition 11.3.1.9. *Let $i \leq i_0$ be a nonnegative integer and $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_X) \cap D(\mathcal{D}^{(m)}_X)$. There exists the canonical isomorphisms*

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{D}^{(m)}(\mathcal{O}_X) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}^\vee] \xrightarrow{\sim} \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^\vee, \tag{11.3.1.9.1}$$

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\vee] \xrightarrow{\sim} \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^\vee. \tag{11.3.1.9.2}$$

Moreover, they are transitive.

Proof. First recall that thanks to 11.3.1.2 some $\mathbb{L}$ can be removed in both isomorphisms of the proposition. Let us treat the isomorphism 11.3.1.9.1. Following 4.6.4.7, we have the last isomorphism $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathbb{D}^{(m)}(\mathcal{O}_X) \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} \mathbb{D}^{(m)}(\mathcal{O}_X) \xrightarrow{\sim} \mathbb{D}^{(m)}(\mathcal{O}_{X_i})$. Following 11.3.1.8, we have the isomorphisms $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}^\vee \xrightarrow{\sim} \mathcal{E}_i^\vee$. We get then

$$\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{D}^{(m)}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee] \xrightarrow{\sim} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathbb{D}^{(m)}(\mathcal{O}_X)) \otimes_{\mathcal{O}_{X_i}} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}^\vee) \xrightarrow{\sim} \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^\vee.$$

Since the isomorphism $\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [- \otimes^{\mathbb{L}}_{\mathcal{O}_X} -] \xrightarrow{\sim} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} -) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} (\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} -)$ is also transitive, we obtain the transitivity by functoriality.

The second case is checked in a similar way □

Lemma 11.3.1.10. *Let $i \leq i_0$ be a nonnegative integer, $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_X) \cap D(\mathcal{D}^{(m)}_X)$, $\mathcal{F} \in D(\mathcal{D}^{(m)}_X)$ and $\mathcal{G} \in D(\mathcal{D}^{(m)}_X)$. The following canonical diagram of $D(\mathcal{D}^{(m)}_{X_i})$*

$$\begin{array}{ccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G}] & \underset{\sim}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{G}_i \\ \sim \downarrow 6.3.4.12 & & \sim \downarrow 6.3.4.12 \\ \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G})] & \underset{\sim}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{G}_i) \end{array} \tag{11.3.1.10.1}$$

is commutative.

Proof. Let us consider the diagram

$$\begin{array}{ccccc} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G} & \longrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}_i) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G} & \underset{\sim}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{G}_i \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G}) & \longrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}_i \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{G}) & \underset{\sim}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{G}_i). \end{array} \tag{11.3.1.10.2}$$

By functoriality, the left one is commutative. To check the right one, let $\mathcal{I}_i$ be a K-injective complex of $\mathcal{D}^{(m)}_{X_i}$-modules representing $\mathcal{F}_i$, $\mathcal{P}$ (resp. $\mathcal{P}'$) be a K-flat complex of $\mathcal{D}^{(m)}_{X^\sharp}$-modules representing $\mathcal{E}$ (resp.

$\mathcal{G}$) and $\mathcal{I}'_i$ be K-injective complex of $\mathcal{D}^{(m)}_{X_i}$-modules representing $\mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{P}'$ ($\xrightarrow{\sim} \mathcal{I}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{P}'_i$). Let us recall that theses allow to compute the derived functors of the right square of the diagram of 11.3.1.10.2 (see 4.6.4.2). We compute that the canonical diagram

$$\begin{array}{ccccc} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\mathcal{I}_i) \otimes_{\mathcal{O}_X} \mathcal{P}' & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{P}') & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P},\mathcal{I}'_i) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}_i,\mathcal{I}_i) \otimes_{\mathcal{O}_{X_i}} \mathcal{P}'_i & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}_i,\mathcal{I}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{P}'_i) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}_i,\mathcal{I}'_i) \end{array}$$

is commutative. Hence so is the right diagram of 11.3.1.10.2. We conclude thanks to 11.3.1.4 (and 11.3.1.3.1). □

Lemma 11.3.1.11. *Let $i \leq i_0$ be a nonnegative integer, $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_X)$ and $\mathcal{F} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_X)$. The canonical diagram*

$$\begin{array}{ccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{E},\mathcal{F})] & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i) \\ \sim\downarrow 6.3.4.14 & & \sim\downarrow 6.3.4.14 \\ \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{O}_X,\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}))] & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X_i}}(\mathcal{O}_{X_i},\mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i)) \end{array} \tag{11.3.1.11.1}$$

is commutative.

Proof. By functoriality, the left square of the following diagram

$$\begin{array}{ccccc} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{E},\mathcal{F}) & \longrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{E},\mathcal{F}_i) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i) \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{O}_X,\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})) & \to & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{O}_X,\mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}_i)) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X_i}}(\mathcal{O}_{X_i},\mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}_i,\mathcal{F}_i)) \end{array} \tag{11.3.1.11.2}$$

is commutative. Let us now check the right one. Let $\mathcal{P}$ (resp. $\mathcal{P}'$) be a K-flat complex of $\mathcal{D}^{(m)}_{X^\sharp}$-modules representing $\mathcal{O}_X$ (resp. $\mathcal{E}$), $\mathcal{I}_i$ be a K-injective complex of $\mathcal{D}^{(m)}_{X_i}$-modules representing $\mathcal{F}_i$. We remark that $\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{P}'$ is then a K-flat complex of $\mathcal{D}^{(m)}_{X^\sharp}$-modules representing $\mathcal{E}$. Moreover, as $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}',\mathcal{I}_i) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}'_i,\mathcal{I}_i)$, following 4.2.4.6, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}',\mathcal{I}_i)$ is a K-injective complex of $\mathcal{D}^{(m)}_{X_i}$-modules. Hence, they allow us to compute the terms of the right square of the diagram of 11.3.1.11.2 (for its top arrow, recall the construction of 4.6.4.2.1). Its commutativity is then a consequence of that of the square :

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{P}',\mathcal{I}_i) & \longrightarrow & \mathcal{H}om_{\mathcal{D}^{(m)}_{X_i}}(\mathcal{P}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{P}'_i,\mathcal{I}_i) \\ \downarrow & & \downarrow \\ \mathcal{H}om_{\mathcal{D}^{(m)}_{X^\sharp}}(\mathcal{P},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}',\mathcal{I}_i)) & \longrightarrow & \mathcal{H}om_{\mathcal{D}^{(m)}_{X_i}}(\mathcal{P}_i,\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}'_i,\mathcal{I}_i)). \end{array}$$

Finally, via 11.3.1.4 (and 11.3.1.3.1), the commutativity of 11.3.1.11.1 follows from that of 11.3.1.11.2. □

Lemma 11.3.1.12. *For any nonnegative integer $i \leq i_0$, for any $\mathcal{E} \in D({}^{\mathrm{l}}\mathcal{D}^{(m)}_X)$, we have the commutative diagram :*

$$\begin{array}{ccc} \mathcal{D}^{(m)}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} [\mathcal{D}^{(m)}_X \otimes_{\mathcal{O}_X} \mathcal{E}] & \xrightarrow[\sim]{\mathcal{D}^{(m)}_{X_i} \otimes \gamma_{\mathcal{E}}} & \mathcal{D}^{(m)}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_X} [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^{(m)}_X] \\ \sim\downarrow & & \sim\downarrow \\ \mathcal{D}^{(m)}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i & \xrightarrow[\sim]{\gamma_{\mathcal{E}_i}} & \mathcal{E}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i}, \end{array}$$

where γ is the logarithmic transposition isomorphism (see 4.2.5.1).

Proof. Let $\mathcal{P}$ be a K-flat complex of left $\mathcal{D}_X^{(m)}$-modules representing $\mathcal{E}$. Then $\mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \mathcal{P}$ is a K-flat complex of left $\mathcal{D}_X^{(m)}$-modules (for the underlying structure) representing $\mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E}$. It follows from the logarithmic transposition isomorphism that $\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)}$ is a K-flat complex of left $\mathcal{D}_X^{(m)}$-modules (for the underlying structure) representing $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{(m)}$. Hence, we reduce to check the commutativity of the diagram where $\mathcal{E}$ is replaced by $\mathcal{P}$ and without $\mathbb{L}$. By $\mathcal{D}_{X_i}^{(m)}$-linearity, it is sufficient to check that for any both path, for any $x \in \mathcal{P}$, $1 \otimes [1 \otimes x]$ is sent on $(1 \otimes x) \otimes 1$, which is straightforward. □

Lemma 11.3.1.13. *Let $i \leq i_0$ be a nonnegative integer and $\mathcal{E} \in D({}^1\mathcal{D}_X^{(m)})$. The following diagram*

$$\begin{array}{ccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{D}_X^{(m)}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}] & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{D}_{X_i}^{(m)}) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{E}_i \\ \downarrow \sim & & \downarrow \sim \\ \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_X} [\mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E})] & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{D}_{X_i}^{(m)} \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i), \end{array} \quad (11.3.1.13.1)$$

where the horizontal isomorphisms follow from 4.6.4.7 and the vertical from 6.3.4.8, is commutative.

Proof. Since $\mathcal{O}_X \in D_{\mathrm{perf}}({}^1\mathcal{D}_X^{(m)})$ and $\mathcal{O}_{X_i} \in D_{\mathrm{perf}}({}^1\mathcal{D}_{X_i}^{(m)})$ (see 11.3.1.2), the vertical arrows are therefore isomorphisms. Moreover, we check by functoriality the commutativity of the left diagram below.

$$\begin{array}{ccccc} \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{D}_X^{(m)}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E} & \longrightarrow & \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{D}_{X_i}^{(m)}) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{D}_{X_i}^{(m)}) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{E}_i \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{D}_X^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) & \longrightarrow & \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{D}_{X_i}^{(m)} \otimes_{\mathcal{O}_X} \mathcal{E}) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{D}_{X_i}^{(m)} \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i). \end{array}$$

(11.3.1.13.2)

Let $\mathcal{I}_i$ be a K-injective complex of left $\mathcal{D}_{X_i}^{(m)} \otimes_{\mathcal{O}_{S_i}} \mathcal{D}_{X_i}^{(m)}$-modules representing $\mathcal{D}_{X_i}^{(m)}$, $\mathcal{P}$ be a K-flat complex of $\mathcal{D}_{X^\sharp}^{(m)}$-modules representing $\mathcal{E}$ and $\mathcal{I}'_i$ be a K-injective complex of $\mathcal{D}_{X_i}^{(m)} \otimes_{\mathcal{O}_{S_i}} \mathcal{D}_{X_i}^{(m)}$-modules representing $\mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{P}$ ($\xrightarrow{\sim} \mathcal{I}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{P}_i$). The commutativity of the right diagram of 11.3.1.13.2 comes from that of

$$\begin{array}{ccccc} \mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{I}_i) \otimes_{\mathcal{O}_X} \mathcal{P} & \longrightarrow & \mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{P}) & \longrightarrow & \mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{I}'_i) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{I}_i) \otimes_{\mathcal{O}_{X_i}} \mathcal{P}_i & \longrightarrow & \mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{I}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{P}_i) & \longrightarrow & \mathcal{H}om_{{}^1\mathcal{D}_{X_i}^{(m)}}(\mathcal{O}_{X_i}, \mathcal{I}'_i). \end{array}$$

(11.3.1.13.3)

Moreover, if $\phi \in \mathcal{H}om_{{}^1\mathcal{D}_X^{(m)}}(\mathcal{O}_X, \mathcal{I}_i)$ and $x \in \mathcal{P}$, we compute that $\phi \otimes x$ is sent via both possible ways of the left diagram of 11.3.1.13.3, on $P_i \otimes a \mapsto P_i[\phi(a) \otimes (1 \otimes x)] = (P_i\phi(a)) \otimes (1 \otimes x)$, with $P_i \in \mathcal{D}_{X_i}^{(m)}$ and $a \in \mathcal{O}_X$. As the right diagram of 11.3.1.13.3 is commutative by functoriality, this yields that of 11.3.1.13.3.

Finally, with the remark 11.3.1.4, we deduce from the commutativity of 11.3.1.13.2 that of 11.3.1.13.1. □

Notation 11.3.1.14. For any $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_X) \cap D({}^1\mathcal{D}_X^{(m)})$, we denote by $\mathcal{E}^\vee := \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ and by $\theta^{(m)} \colon \mathbb{D}^{(m)}(\mathcal{O}_X) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathbb{D}^{(m)}(\mathcal{E})$ and $\theta_i^{(m)} \colon \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^\vee \xrightarrow{\sim} \mathbb{D}^{(m)}(\mathcal{E}_i)$ the homomorphism constructed via 6.3.4.15. Following 11.3.1.2, $\theta_i^{(m)}$ do is an isomorphism.

Proposition 11.3.1.15. *Let $i \leq i_0$ be a nonnegative integer and $\mathcal{E} \in D_{\mathrm{perf}}(\mathcal{O}_X) \cap D({}^1\mathcal{D}_X^{(m)})$. The following diagram*

$$\begin{array}{ccc} \mathcal{D}_{X_i}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{D}_X^{(m)}} [\mathbb{D}^{(m)}(\mathcal{O}_X) \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{E}^\vee] & \xrightarrow[\sim]{} & \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^\vee \\ \downarrow 1\otimes\theta^{(m)} & & \sim \downarrow \theta_i^{(m)} \\ \mathcal{D}_{X_i}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{D}_X^{(m)}} [\mathbb{D}^{(m)}(\mathcal{E})] & \xrightarrow[\sim]{} & \mathbb{D}^{(m)}(\mathcal{E}_i), \end{array}$$

where the horizontal isomorphisms come from 4.6.4.7 (see also the example 4.6.4.5) and of 11.3.1.9, is commutative.

Proof. By construction of $\theta^{(m)}$ and $\theta_i^{(m)}$ (6.3.4.15), this is a consequence of lemmas 11.3.1.10, 11.3.1.11.1, 11.3.1.12 and 11.3.1.13.1. $\square$

11.3.1.16 (Construction of $\widehat{\theta}^{(m)}$)**.** For any nonnegative integer $i \leq i_0$, for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}}) \cap D({}^{\mathrm{l}}\mathcal{D}^{(m)}_{\mathfrak{X}})$, let $\theta_i'^{(m)}$ be the morphism making commutative by definition the diagram

$$\begin{array}{ccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow[\sim]{\theta_i'^{(m)}} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{E})] \\ \sim\downarrow & & \sim\downarrow \\ \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee} & \xrightarrow[\sim]{\theta_i^{(m)}} & \mathbb{D}^{(m)}(\mathcal{E}_i), \end{array} \tag{11.3.1.16.1}$$

where the vertical isomorphisms are constructed thanks to 4.6.4.7 (see also the remark 4.6.4.5.(i)) and to 11.3.1.9. For any $j \leq i \leq i_0$, let us consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\theta_j'^{(m)}} & \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}\mathcal{E}] \\ \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \theta_i'^{(m)}} & \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}\mathcal{E}] \\ \mathbb{D}^{(m)}(\mathcal{O}_{X_j}) \otimes_{\mathcal{O}_{X_j}} \mathcal{E}_j^{\vee} & \xrightarrow{\theta_j^{(m)}} & \mathbb{D}^{(m)}(\mathcal{E}_j) \\ \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} [\mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee}] & \xrightarrow{\mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} \theta_i^{(m)}} & \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{X_i}} [\mathbb{D}^{(m)}\mathcal{E}_i]. \end{array}$$

The squares of the front side and of the back side are commutative by construction of $\theta_i'^{(m)}$ (11.3.1.16.1). Moreover, thanks to 11.3.1.15 (resp. 4.6.4.7 and 4.6.4.5.(i), resp. 11.3.1.9.2), so is the bottom (resp. right, resp. left) one. Since the arrows are isomorphisms, then it follows that the top square is commutative. In other words, the family of isomorphisms $(\theta_i'^{(m)})_{i\in\mathbb{N}}$ induces the isomorphism of $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_{X_\bullet})$:

$$\mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] \xrightarrow{\sim} \mathcal{O}_{X_\bullet} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{E})].$$

By using 7.3.2.15 and 7.3.3.4, this yields the construction of an isomorphism

$$\widehat{\theta}^{(m)} \colon \widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee} \to \widehat{\mathbb{D}}^{(m)}(\mathcal{E})$$

inducing the following diagram

$$\begin{array}{ccccc} \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & = & \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow[\sim]{} & \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee} \\ \sim\downarrow \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\theta}^{(m)} & & \downarrow \theta_j'^{(m)} & & \sim\downarrow \theta_i^{(m)} \\ \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}\mathcal{E}] & = & \mathcal{O}_{X_j} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}\mathcal{E}] & \xrightarrow[\sim]{} & \mathbb{D}^{(m)}(\mathcal{E}_i). \end{array} \tag{11.3.1.16.2}$$

Notation 11.3.1.17. For any $\mathcal{E} \in D(\mathcal{D}^{(m)}_{\mathfrak{X}}) \cap D_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}})$, we denote by $\rho^{(m)}_{\mathcal{E}}$ or $\rho^{(m)}$ the morphism

$$\rho^{(m)}_{\mathcal{E}} \colon \mathbb{D}^{(m)}(\mathcal{E}) \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}}} \mathbb{D}^{(m)}(\mathcal{E}) \underset{4.6.4.7.1}{\xrightarrow{\sim}} \widehat{\mathbb{D}}^{(m)}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}}} \mathcal{E}) \xrightarrow{\sim} \widehat{\mathbb{D}}^{(m)}(\mathcal{E}), \tag{11.3.1.17.1}$$

the latter isomorphism coming from 11.1.1.5.

Lemma 11.3.1.18. *Let $f \colon \mathcal{E} \to \mathcal{E}'$ be a morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{\mathfrak{X}})$, $g \colon \mathcal{F} \to \mathcal{F}'$ be a morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}})$, $\alpha \colon \mathcal{E} \to \mathcal{F}$ and $\beta \colon \mathcal{E}' \to \mathcal{F}$ be two morphisms of $D(\mathcal{D}^{(m)}_{\mathfrak{X}})$.*

The left following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{E}' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathcal{F} & \xrightarrow{g} & \mathcal{F}', \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} & \xrightarrow{\mathrm{id} \otimes f} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}' \\ \downarrow{\scriptstyle id \otimes \alpha} & & \downarrow{\scriptstyle \mathrm{id} \otimes \beta} \\ \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F} & \xrightarrow{\mathrm{id} \otimes g} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}' \end{array} \tag{11.3.1.18.1}$$

is commutative if and only if, for any nonnegative integer i, so is the right one.

Proof. The morphisms α and β factors through $\alpha' \colon \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}}} \mathcal{E} \to \mathcal{F}$ and $\beta' \colon \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}}} \mathcal{E}' \to \mathcal{F}'$. By considering the diagram

$$\begin{array}{ccccc} \mathcal{E} & & \xrightarrow{f} & & \mathcal{E}' \\ \downarrow & \searrow{\scriptstyle \alpha} & & & \downarrow \quad \searrow{\scriptstyle \beta} \\ \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}}} \mathcal{E} & & \xrightarrow{1 \otimes f} & & \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}}} \mathcal{E}' \\ & \searrow{\scriptstyle \alpha'} & & & \searrow{\scriptstyle \beta'} \\ & \mathcal{F} & \xrightarrow{g} & & \mathcal{F}', \end{array} \tag{11.3.1.18.2}$$

we check that the left square of 11.3.1.18.1 is commutative if and only if so is the horizontal square of 11.3.1.18.2. We conclude via 7.3.2.15 and 7.3.3.4. □

Proposition 11.3.1.19. *For any $\mathcal{E} \in D(\mathcal{D}^{(m)}_{\mathfrak{X}}) \cap D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}})$, the following diagram*

$$\begin{array}{ccc} \mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee} & \xrightarrow{\theta^{(m)}} & \mathbb{D}^{(m)}(\mathcal{E}) \\ \downarrow{\scriptstyle \rho^{(m)}_{\mathcal{O}_{\mathfrak{X}}} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \rho^{(m)}_{\mathcal{E}}} \\ \widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee} & \underset{\sim}{\xrightarrow{\widehat{\theta}^{(m)}}} & \widehat{\mathbb{D}}^{(m)}(\mathcal{E}) \end{array}$$

is commutative.

Proof. By using 11.3.1.18, it is sufficient to prove, for any nonnegative integer i, that the central square of the diagram

$$\begin{array}{ccccccc} \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee} & \underset{\sim}{\longleftarrow} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\theta^{(m)}} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\mathbb{D}^{(m)}(\mathcal{E})] & \underset{\sim}{\longrightarrow} & \mathbb{D}^{(m)}(\mathcal{E}_i) \\ \| & & \downarrow{\scriptstyle \mathrm{id} \otimes \rho^{(m)}_{\mathcal{O}_{\mathfrak{X}}} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \rho^{(m)}_{\mathcal{E}}} & & \| \\ \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee} & \underset{\sim}{\longleftarrow} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \underset{\sim}{\xrightarrow{\mathrm{id} \otimes \widehat{\theta}^{(m)}}} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{E})] & \underset{\sim}{\longrightarrow} & \mathbb{D}^{(m)}(\mathcal{E}_i) \end{array} \tag{11.3.1.19.1}$$

is commutative. Following 11.3.1.15, the composition of the morphisms of the top of 11.3.1.19.1 is equal to $\theta_i^{(m)}$. By construction (11.3.1.16.2), so is that bottom. Since the thorizontal arrows at the end of the diagram 11.3.1.19.1 are isomorphisms, to get the commutativity of the square of the center, it is sufficient to prove that of the right and left squares. The right square of 11.3.1.19.1 corresponds to the outer of the following diagram

$$\begin{array}{ccccccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbb{D}(\mathcal{E}) & \longrightarrow & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathcal{D}}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathbb{D}(\mathcal{E})] & \longrightarrow & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathbb{D}}(\widehat{\mathcal{D}}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{E}) & \longrightarrow & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathbb{D}}(\mathcal{E}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}_{\mathfrak{X}}} \mathbb{D}(\mathcal{E}) & \longrightarrow & \mathcal{D}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{X}}} [\widehat{\mathcal{D}}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathbb{D}(\mathcal{E})] & \longrightarrow & \mathcal{D}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{X}}} \widehat{\mathbb{D}}(\widehat{\mathcal{D}}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{E}) & \longrightarrow & \mathcal{D}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{X}}} \widehat{\mathbb{D}}(\mathcal{E}) \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbb{D}(\mathcal{D}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}_{\mathfrak{X}}} \mathcal{E}) & & \longleftarrow & & \mathbb{D}(\mathcal{D}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{X}}} \widehat{\mathcal{D}}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{E}) & \longrightarrow & \mathbb{D}(\mathcal{D}_{X_i} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{X}}} \mathcal{E}) \\ \downarrow & & & & & & \downarrow \\ \mathbb{D}(\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) & & = & = & = & & \mathbb{D}(\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}), \end{array} \tag{11.3.1.19.2}$$

where we have omitted to write the indices (m). Moreover, we compute that the top right square (we choose a K-flat complex representing $\mathbb{D}^{(m)}(\mathcal{E})$) and the bottom rectangle (by forgetting $\mathbb{D}^{(m)}$, we choose a K-flat complex representing $\mathcal{E}$) are commutative. Moreover, so is the rectangle of the middle thanks to the transitivity of 4.6.4.7 (with also the remark 4.6.4.5.(i)). The commutativity of the other squares is functorial. Hence we get that of the right diagram of 11.3.1.19.1.

Finally, the commutativity of the left square of 11.3.1.19.1 is a consequence of that of the diagram below :

$$\begin{array}{ccccc}
\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\sim} & [\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}})] \otimes_{\mathcal{O}_{X_i}} [\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\sim} & \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee} \\
\downarrow{\scriptstyle \mathrm{id} \otimes \rho^{(m)} \otimes \mathrm{id}} & & \downarrow{\scriptstyle (id \otimes \rho^{(m)}) \otimes \mathrm{id}} & & \| \\
\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\sim} & [\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}})] \otimes_{\mathcal{O}_{X_i}} [\mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee}] & \xrightarrow{\sim} & \mathbb{D}^{(m)}(\mathcal{O}_{X_i}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E}_i^{\vee},
\end{array}$$

whose that of the right square can be checked by functoriality thanks to that of the right square of 11.3.1.19.1. □

11.3.2 Compatibility with Frobenius of $\widehat{\theta}^{(m)}$

For simplicity suppose [3] there exists $F\colon \mathfrak{X} \to \mathfrak{X}'$ a morphism of smooth formal $\mathfrak{S}$-schemes which is a lifting of the relative Frobenius $F^s_{X_0/S_0}\colon X_0 \to X_0^{(s)}$. We prove in this subsection that the isomorphism

$$\widehat{\theta}^{(m)}\colon \widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^{\vee} \to \widehat{\mathbb{D}}^{(m)}(\mathcal{E})$$

is compatible with Frobenius (see 11.3.2.9.1). The idea it to check it by p-adic completion.

11.3.2.1. The functor $\widehat{\mathbb{D}}^{(m)}$ commutes canonically with Frobenius. More precisely, denoting by $\widetilde{\mathbb{D}}^{(m)} = \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(-, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})[d_{X'/S}]$, which is canonically isomorphic to the composition of the functors $\widehat{\mathbb{D}}^{(m)}$ and $- \otimes_{\mathcal{O}_X} \omega_{\mathfrak{X}}$, we have for any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})$ the isomorphisms

$$F^{\flat}\widetilde{\mathbb{D}}^{(m)}(\mathcal{E}') \underset{4.6.3.6.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \underset{F^*}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \underset{8.8.1.1.1}{\xrightarrow{\sim}} \widetilde{\mathbb{D}}^{(m+s)}(F^*\mathcal{E}'). \tag{11.3.2.1.1}$$

This yields:

$$F^*\widehat{\mathbb{D}}^{(m)}(\mathcal{E}') \overset{8.8.1.3.1}{\xrightarrow{\sim}} F^{\flat}\widetilde{\mathbb{D}}^{(m)}(\mathcal{E}') \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}/\mathfrak{S}} \overset{11.3.2.1.1}{\xrightarrow{\sim}} \widetilde{\mathbb{D}}^{(m+s)}(F^*\mathcal{E}') \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} \widehat{\mathbb{D}}^{(m+s)}(F^*\mathcal{E}'). \tag{11.3.2.1.2}$$

Remark, replacing $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$ by $\mathcal{D}^{(m)}_{X'_i}$ and $\widehat{\mathcal{D}}^{(m+s)}_{X_i}$ by $\mathcal{D}^{(m+s)}_{X_i}$, we get the construction of the commutation with Frobenius of the dual functor of 6.2.7.2. The purpose of the following lemmas and facts is to check the compatibility with base change of this isomorphism 11.3.2.1.1 (i.e. the commutativity of the diagram 11.3.2.7.1).

11.3.2.2. It follows by functoriality from 8.8.1.3 that the functors F^* and $F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} -$ induce exact canonically quasi-inverse equivalences between the category of complexes of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules (i.e. $(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})$-bimodules whose both induced structures of $\mathcal{V}$-algebras coincide) and that of complexes of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}})^{\mathrm{op}}$-modules (i.e. $(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}, \widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}})$-bimodules whose both induced structures of $\mathcal{V}$-algebras coincide).

Similarly, by functoriality, we can check that the functors $F^{\flat}$ and $- \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} F^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$ induce exact canonically quasi-inverse equivalences between the category of complexes of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules and that of complexes of left $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules.

11.3.2.3. 1) Let $\mathcal{M}'$ be a K-flat complex of right $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules. The functor F^* induces an equivalence of categories between acyclic complexes of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules and acyclic complexes of left $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules. For any acyclic complex $\mathcal{E}'$ of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules, we get

$$F^{\flat}\mathcal{M}' \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} F^*(\mathcal{E}') \xrightarrow{\sim} \mathcal{M}' \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} F^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} \mathcal{E}' \xrightarrow{\sim} \mathcal{M}' \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} \mathcal{E}'.$$

[3] This hypothesis is useless since we can define by glueing F^* etc.

This yields that $F^\flat \mathcal{M}'$ is a K-flat complex of right $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules.

Similarly, if $\mathcal{E}'$ is a K-flat complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules then $F^*\mathcal{E}'$ is a K-flat complex of left $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules.

1') Similarly, let $\mathcal{M}'$ be a K-flat complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules. Then, $F^\flat \mathcal{M}'$ is a complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}})^{\mathrm{op}}$-modules which is both a K-flat complex of right $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules and a K-flat complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules. Indeed, the first property is checked as above. Let us check the second one. For any acyclic complex $\mathcal{E}'$ of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules, we get

$$\mathcal{E}' \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \mathcal{M}' \xrightarrow{\sim} \left(\mathcal{E}' \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} \mathcal{M}'\right) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}.$$

Since $F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$ is a flat right $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-module, then we can conclude.

Moreover, similarly, we can check that $F^*\mathcal{M}'$ is a complex of left $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules which is both a K-flat complex of left $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules and a K-flat complex of right $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules.

2) Let $\mathcal{I}'$ be a K-injective complex of right $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules. The functor $F^\flat$ induces an equivalence of categories between acyclic complexes of right $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules and acyclic complexes of right $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules. This yields that $F^\flat \mathcal{I}'$ is a K-injective complex of right $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules. Indeed, for any acyclic complex $\mathcal{N}'$ of right $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules, we have the isomorphism

$$\mathrm{Hom}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{N}', \mathcal{I}') \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^\flat \mathcal{N}', F^\flat \mathcal{I}').$$

Hence, we are done.

Similarly (use 11.3.2.2), if $\mathcal{I}'$ is a K-injective complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules. then $F^\flat \mathcal{I}'$ is a K-injective complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} \otimes_{\mathcal{V}} (\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}'})^{\mathrm{op}}$-modules. In particular, it is both a K-injective complex of right $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$-modules and a K-injective complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules.

To prove 11.3.2.7, that we will use in the proof of the theorem 11.3.2.9 of commutation to Frobenius of $\widehat{\theta}^{(m)}$, we will need the three lemmas below.

Lemma 11.3.2.4. *Let i be a nonnegative integer and $\mathcal{E}' \in D_{\mathrm{perf}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})$. The following diagram of $D_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}^{(m+s)}_{X_i})$*

$$\begin{array}{ccc}
\left(\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}\right) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow[\sim]{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\mathcal{D}^{(m)}_{X'_i}}(\mathcal{E}'_i, \mathcal{D}^{(m)}_{X'_i}) \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^\flat \mathcal{D}^{(m)}_{X'_i} \\
\sim\downarrow 4.6.3.6.1 & & \sim\downarrow 4.6.3.6.1 \\
\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow[\sim]{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\mathcal{D}^{(m)}_{X'_i}}(\mathcal{E}'_i, F^\flat \mathcal{D}^{(m)}_{X'_i}),
\end{array} \tag{11.3.2.4.1}$$

is commutative.

Proof. By using the remark 11.3.1.4, we reduce to check the commutativity of (the outer of) the diagram:

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} & \xrightarrow[\sim]{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{D}^{(m)}_{X'_i}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} & \xrightarrow[\sim]{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{D}^{(m)}_{X'_i} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\mathcal{D}^{(m)}_{X'_i}}(\mathcal{E}'_i, \mathcal{D}^{(m)}_{X'_i}) \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^\flat \mathcal{D}^{(m)}_{X'_i} & \xrightarrow[\sim]{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{{}^{\mathrm{l}}\mathcal{D}^{(m)}_{X'_i}}(\mathcal{E}'_i, F^\flat \mathcal{D}^{(m)}_{X'_i}).
\end{array} \tag{11.3.2.4.2}$$

By functoriality, we can check the commutativity of the top square. Let $\mathcal{P}'$ be a K-flat complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules representing $\mathcal{E}'$, let $\mathcal{I}'_i$ be a K-injective complex of left $\mathcal{D}^{(m)}_{X'_i} \otimes_{\mathcal{V}} (\mathcal{D}^{(m)}_{X'_i})^{\mathrm{op}}$-modules representing $\mathcal{D}^{(m)}_{X'_i}$. We set $\mathcal{P}'_i := \mathcal{D}^{(m)}_{X'_i} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} \mathcal{P}'$. We easily compute the commutativity of the diagram:

$$\begin{array}{ccc}
\mathcal{H}om_{{}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{P}', \mathcal{I}'_i) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'} & \longrightarrow & \mathcal{H}om_{{}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{P}', \mathcal{I}'_i \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \\
\downarrow & & \downarrow \\
\mathcal{H}om_{{}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{P}', \mathcal{I}'_i) \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^{\flat}\mathcal{D}^{(m)}_{X'_i} & \longrightarrow & \mathcal{H}om_{{}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{P}', \mathcal{I}'_i \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^{\flat}\mathcal{D}^{(m)}_{X'_i}) \\
\downarrow & & \downarrow \\
\mathcal{H}om_{{}^1\mathcal{D}^{(m)}_{X'_i}}(\mathcal{P}'_i, \mathcal{J}'_i) \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^{\flat}\mathcal{D}^{(m)}_{X'_i} & \xrightarrow{\sim} & \mathcal{H}om_{{}^1\mathcal{D}^{(m)}_{X'_i}}(\mathcal{P}'_i, \mathcal{J}'_i \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^{\flat}\mathcal{D}^{(m)}_{X'_i}),
\end{array} \quad (11.3.2.4.3)$$

Since $\mathcal{J}'_i \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^{\flat}\mathcal{D}^{(m)}_{X'_i} = F^{\flat}\mathcal{J}'_i$ is a K-injective complex of left $\mathcal{D}^{(m)}_{X'_i} \otimes_{\mathcal{V}} (\mathcal{D}^{(m+s)}_{X_i})^{\mathrm{op}}$-modules representing $F^{\flat}\mathcal{D}^{(m)}_{X'_i}$ (see 11.3.2.3), this yields the commutativity of the bottom square of 11.3.2.4.2. □

Lemma 11.3.2.5. *Let i be a nonnegative integer, $\mathcal{E}' \in D({}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})$ and $\mathcal{F}' \in D({}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}, \widehat{\mathcal{D}}^{(m+s)\mathrm{r}}_{\mathfrak{X}})$ The following diagram*

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{F}') \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X'_i}}(\mathcal{E}'_i, \mathcal{F}'_i) \\
\sim \downarrow F^*\otimes \mathrm{id} & & \sim \downarrow F^* \\
\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*\mathcal{F}') \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m+s)}_{X_i}}(F^*\mathcal{E}'_i, F^*\mathcal{F}'_i)
\end{array}$$

is commutative.

Proof. By using the remark 11.3.1.4, it is sufficient to prove the commutativity of the diagram below

$$\begin{array}{ccccc}
\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{F}') & \longrightarrow & \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{F}'_i) & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_{X'_i}}(\mathcal{E}'_i, \mathcal{F}'_i) \\
\sim\downarrow F^* & & \sim\downarrow F^* & & \sim\downarrow F^* \\
\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*\mathcal{F}') & \longrightarrow & \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*\mathcal{F}'_i) & \xrightarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m+s)}_{X_i}}(F^*\mathcal{E}'_i, F^*\mathcal{F}'_i).
\end{array}$$

That of the left square is proved by functoriality, whereas we can check the right one by choosing a K-injective complex of left $\mathcal{D}^{(m)}_{X'_i} \otimes_{\mathcal{O}_{S_i}} \mathcal{D}^{(m+s)}_{X_i}$-modules representing $\mathcal{F}'_i$ and a K-flat complex of left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-modules representing $\mathcal{E}'$ (the functor F^*, by increasing the level, preserves the K-flatness and the K-injectivity). □

Lemma 11.3.2.6. *Let i be a nonnegative integer. The following canonical diagram*

$$\begin{array}{ccc}
\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m+s)}_{X_i}}(F^*\mathcal{E}'_i, F^*F^{\flat}\mathcal{D}^{(m)}_{X'_i}) \\
\sim\downarrow 8.8.1.1.1 & & \sim\downarrow 6.1.3.2.1 \\
\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', \widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow{4.6.4.7.1} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m+s)}_{X_i}}(F^*\mathcal{E}'_i, \mathcal{D}^{(m+s)}_{X_i})
\end{array}$$

is commutative.

Proof. Exercice. □

Proposition 11.3.2.7. *Let i be a nonnegative integer and $\mathcal{E}' \in D_{\mathrm{perf}}({}^1\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'})$. The following canonical diagram*

$$\begin{array}{ccccc}
\mathcal{D}^{(m+s)}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} [F^*\widehat{\mathbb{D}}^{(m)}(\mathcal{E}')] & \xrightarrow{\sim} & F^*[\mathcal{D}^{(m)}_{X'_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} \widehat{\mathbb{D}}^{(m)}(\mathcal{E}')] & \underset{\sim}{\xrightarrow{4.6.4.7.1}} & F^*\mathbb{D}^{(m)}(\mathcal{E}'_i) \\
\sim\downarrow 11.3.2.1.2 & & & & \sim\downarrow 11.3.2.1.2 \\
\mathcal{D}^{(m+s)}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} [\widehat{\mathbb{D}}^{(m+s)}F^*(\mathcal{E}')] & \underset{\sim}{\xrightarrow{4.6.4.7.1}} & \mathbb{D}^{(m+s)}(\mathcal{D}^{(m+s)}_{X_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} F^*(\mathcal{E}')) & \xrightarrow{\sim} & \mathbb{D}^{(m+s)}F^*(\mathcal{E}'_i),
\end{array}$$

(11.3.2.7.1)

is commutative.

Proof. To get 11.3.2.7.1, we have just to By composing the commutative diagrams 11.3.2.4 with 11.3.2.5 in the case where $\mathcal{F}' = F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$ and next with 11.3.2.6, we get the commutative diagram

$$\begin{array}{ccc} \left(\widetilde{\mathbb{D}}^{(m)}(\mathcal{E}') \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}} F^\flat \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}\right) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow[\sim]{4.6.4.7.1} & \widetilde{\mathbb{D}}^{(m)}(\mathcal{E}'_i) \otimes_{\mathcal{D}^{(m)}_{X'_i}} F^\flat \mathcal{D}^{(m)}_{X'_i} \\ \sim\downarrow 11.3.2.1.1 & & \sim\downarrow 11.3.2.1.1 \\ \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}}(F^*\mathcal{E}', \widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}} \mathcal{D}^{(m+s)}_{X_i} & \xrightarrow{4.6.4.7.1} & \widetilde{\mathbb{D}}^{(m+s)}(F^*\mathcal{E}'_i) \end{array} \qquad (11.3.2.7.2)$$

By applying the switching functor $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}/\mathfrak{S}}$ to 11.3.2.7.2, we get 11.3.2.7.1. □

Proposition 11.3.2.8. *For any nonnegative integer i, for any $\mathcal{E}', \mathcal{F}' \in D(\mathcal{D}^{(m)}_{\mathfrak{X}'})$, the diagram*

$$\begin{array}{ccc} \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} F^*\mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{F}') & \xrightarrow[\sim]{} & \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*\mathcal{F}') \\ \downarrow & & \downarrow \\ F_i^*\mathbb{R}\mathcal{H}om_{\mathcal{O}_{X'_i}}(\mathcal{E}'_i, \mathcal{F}'_i) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(F_i^*\mathcal{E}'_i, \mathcal{F}'_i) \end{array}$$

is commutative.

Proof. By using the remark 11.3.1.3.1 and 11.3.1.3.1, it is sufficient to prove the commutativity of the diagram below

$$\begin{array}{ccc} F^*\mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{F}') & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*\mathcal{F}') \\ \downarrow & & \downarrow \\ F^*\mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{F}'_i) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*\mathcal{F}'_i) \\ \downarrow & & \downarrow \\ F^*\mathbb{R}\mathcal{H}om_{\mathcal{O}_{X'_i}}(\mathcal{E}'_i, \mathcal{F}'_i) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X_i}}(F_i^*\mathcal{E}'_i, F_i^*\mathcal{F}'_i). \end{array} \qquad (11.3.2.8.1)$$

The commutativity of the top square follows by functoriality. Choose a K-flat complex $\mathcal{P}'$ of left $\mathcal{D}^{(m)}_{\mathfrak{X}'}$-modules representing $\mathcal{E}'$ and a K-injective complex $\mathcal{I}'_i$ of left $\mathcal{D}^{(m)}_{X'_i}$-modules representing $\mathcal{F}'_i$. Since $F^*\mathcal{P}'$ is a K-flat complex of left $\mathcal{D}^{(m+s)}_{\mathfrak{X}}$-modules representing $f^*\mathcal{E}'$, since $F^*\mathcal{P}'_i$ is a K-flat complex of left $\mathcal{D}^{(m+s)}_{X_i}$-modules representing $F^*\mathcal{E}'_i$ since $F^*\mathcal{I}'_i$ is a K-injective complex of left $\mathcal{D}^{(m+s)}_{X_i}$-modules representing $F^*\mathcal{F}'_i$, to check the commutativity of the bottom square by replacing $\mathcal{E}'$ by $\mathcal{P}'$ and $\mathcal{F}'$ by $\mathcal{I}'$, we can remove $\mathbb{R}$. Hence, the check of the commutativity is elementary. □

Theorem 11.3.2.9. *For any $\mathcal{E}' \in D(\mathcal{D}^{(m)}_{\mathfrak{X}'}) \cap D_{\mathrm{perf}}(\mathcal{O}_{\mathfrak{X}'})$, the diagram below*

$$\begin{array}{ccc} F^*(\mathcal{E}'^\vee \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}} \widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'})) & \xrightarrow[\sim]{F^*\widehat{\theta}^{(m)}} & F^*\widehat{\mathbb{D}}^{(m)}(\mathcal{E}') \\ \sim\downarrow & & \sim\downarrow \\ (F^*\mathcal{E}')^\vee \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathbb{D}}^{(m+s)}(\mathcal{O}_{\mathfrak{X}}) & \xrightarrow[\sim]{\widehat{\theta}^{(m+s)}} & \widehat{\mathbb{D}}^{(m+s)}(F^*\mathcal{E}') \end{array} \qquad (11.3.2.9.1)$$

is commutative.

Proof. Following 7.3.2.15 and 7.3.3.4, it is sufficient to prove that for any nonnegative integer i, the square of the front side of the cube

$$\begin{array}{ccccc}
 & F^*(\mathcal{E}_i'^{\vee}\otimes_{\mathcal{O}_{X_i'}}\mathbb{D}^{(m)}\mathcal{O}_{X_i'}) & \xrightarrow{F^*\theta_i^{(m)}} & F^*\mathbb{D}^{(m)}(\mathcal{E}_i') & \\
\mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}[F^*(\mathcal{E}'^{\vee}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'}))] & \xrightarrow{\mathrm{id}\otimes F^*\widehat{\theta}^{(m)}} & \mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}[F^*\widehat{\mathbb{D}}^{(m)}(\mathcal{E}')] & & \\
 & (F^*\mathcal{E}_i')^{\vee}\otimes_{\mathcal{O}_{X_i}}\mathbb{D}^{(m+s)}\mathcal{O}_{X_i} & \xrightarrow{\theta_i^{(m+s)}} & \mathbb{D}^{(m+s)}(F^*\mathcal{E}_i') & \\
\mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}[(F^*\mathcal{E}')^{\vee}\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathbb{D}}^{(m+s)}(\mathcal{O}_{\mathfrak{X}})] & \xrightarrow{\mathrm{id}\otimes\widehat{\theta}^{(m+s)}} & \mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}\widehat{\mathbb{D}}^{(m+s)}(F^*\mathcal{E}') & &
\end{array} \tag{11.3.2.9.2}$$

is commutative. Following 11.3.2.7, this is the case for the right square. Moreover, by using 11.3.2.7 and of 11.3.2.8 so is the left one. For the bottom square, this is a consequence (modulo the isomorphism $F^*\mathcal{E}_i'\xrightarrow{\sim}\mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}F^*\mathcal{E}'$) of the definitions (11.3.1.16.2). The square of the top corresponds to the outer of

$$\begin{array}{ccccc}
\mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}[F^*(\mathcal{E}'^{\vee}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'}))] & \xrightarrow[\sim]{} & F^*[\mathcal{O}_{X_i'}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}(\mathcal{E}'^{\vee}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'}))] & \xrightarrow[\sim]{} & F^*(\mathcal{E}_i'^{\vee}\otimes_{\mathcal{O}_{X_i'}}\mathbb{D}^{(m)}\mathcal{O}_{X_i'}) \\
\sim\downarrow \mathrm{id}\otimes F^*(\widehat{\theta}^{(m)}) & & \sim\downarrow F^*(id\otimes\widehat{\theta}^{(m)}) & & \sim\downarrow F^*\theta_i^{(m)} \\
\mathcal{O}_{X_i}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}[F^*\widehat{\mathbb{D}}^{(m)}(\mathcal{E}')] & \xrightarrow[\sim]{} & F^*[\mathcal{O}_{X_i'}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}'}}\widehat{\mathbb{D}}^{(m)}(\mathcal{E}')] & \xrightarrow[\sim]{} & F^*\mathbb{D}^{(m)}(\mathcal{E}_i').
\end{array} \tag{11.3.2.9.3}$$

The commutativity of the left square of 11.3.2.9.3 is functorial and that of the right square follows from 11.3.1.16.2. Finally, that of the back square of 11.3.2.9.2 is due to the fact that $\theta_i^{(m)}$ is compatible with Frobenius (6.3.4.15.(ii)). The arrows of 11.3.2.9.2 being isomorphisms, since we check that five of its faces are commutative, then so is the front side one. □

11.3.3 Construction of $\theta_{\mathbb{Q}}^{(m)}$ and its compatibility with Frobenius

11.3.3.1. Let $\mathcal{E}'$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$-coherent (i.e., it corresponds to a convergent isocrystal on X') and, for any nonnegative integer m, $\mathring{\mathcal{E}}'^{(m)}$ be a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$-module which is $\mathcal{O}_{\mathfrak{X}'}$-coherent and such that there exists a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}'^{(m)}_{\mathbb{Q}}\xrightarrow{\sim}\mathcal{E}'$.

We denote by $\mathbb{D}^{\mathrm{alg}}_{\mathfrak{X}}$ or $\mathbb{D}^{\mathrm{alg}}$, the $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}$-linear dual functor, and by $\mathbb{D}^{(m)}_{\mathfrak{X},\mathbb{Q}}$ or $\mathbb{D}^{(m)}_{\mathbb{Q}}$, the $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}$-linear dual. We remark that it follows from 8.7.7.5 that for any nonnegative integer m we have the isomorphisms $\mathbb{D}^{\mathrm{alg}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}})\xrightarrow{\sim}\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ and $\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}})\xrightarrow{\sim}\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$, this latter is probably not compatible with Frobenius.

We construct the isomorphism $\theta^{(m)}_{\mathbb{Q}}$, via the commutative diagram :

$$\begin{array}{ccc}
\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}})\otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}}\mathcal{E}'^{\vee} & \overset{\theta^{(m)}_{\mathbb{Q}}}{\underset{\sim}{\dashrightarrow}} & \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \\
\uparrow\sim & & \uparrow\sim \\
(\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'})\otimes_{\mathcal{O}_{\mathfrak{X}'}}\mathring{\mathcal{E}}'^{(m)\vee})\otimes_{\mathbb{Z}}\mathbb{Q} & \xrightarrow[\sim]{\widehat{\theta}^{(m)}\otimes\mathbb{Q}} & (\widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)}))\otimes_{\mathbb{Z}}\mathbb{Q}.
\end{array} \tag{11.3.3.1.1}$$

We notice that thanks to 11.3.3.3, this one is independent of the choice of $\mathring{\mathcal{E}}'^{(m)}$.

11.3.3.2. For any coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent $\mathcal{E}$ (similarly by replacing $\mathfrak{X}$ by another smooth $\mathcal{V}$-formal scheme, like by example $\mathfrak{X}'$), we denote by $\rho^{(m)}_{\mathbb{Q}}$, the composite isomorphism :

$$\rho^{(m)}_{\mathbb{Q}}\colon\mathbb{D}^{\mathrm{alg}}(\mathcal{E})\xrightarrow{\sim}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}\otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}}\mathbb{D}^{\mathrm{alg}}(\mathcal{E})\underset{4.6.4.7.1}{\xrightarrow{\sim}}\mathbb{D}^{(m)}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}\otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}}\mathcal{E})\xrightarrow{\sim}\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}). \tag{11.3.3.2.1}$$

According to 11.2.6.3.1, we denote by θ^{alg}, the isomorphism $\mathbb{D}^{\mathrm{alg}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}})\otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}}\mathcal{E}^{\vee}\xrightarrow{\sim}\mathbb{D}^{\mathrm{alg}}(\mathcal{E})$ constructed via 6.3.4.15.

Theorem 11.3.3.3. *With the notations of 11.3.3.1, the following diagram*

$$\begin{array}{ccc}
\mathbb{D}^{\mathrm{alg}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}'^{\vee} & \xrightarrow[\sim]{\theta^{\mathrm{alg}}} & \mathbb{D}^{\mathrm{alg}}(\mathcal{E}') \\
\sim \downarrow {\scriptstyle \rho^{(m)}_{\mathbb{Q}} \otimes \mathrm{id}} & & \sim \downarrow {\scriptstyle \rho^{(m)}_{\mathbb{Q}}} \\
\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}'^{\vee} & \xrightarrow[\sim]{\theta^{(m)}_{\mathbb{Q}}} & \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}')
\end{array} \tag{11.3.3.3.1}$$

is commutative

Proof. Let us consider the cube

$$\begin{array}{ccccc}
 & (\mathbb{D}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathring{\mathcal{E}}'^{(m)\vee}) \otimes \mathbb{Q} & \xrightarrow{\theta^{(m)} \otimes \mathbb{Q}} & & \mathbb{D}^{(m)}(\mathring{\mathcal{E}}'^{(m)}) \otimes \mathbb{Q} \\
\swarrow {\scriptstyle 4.6.4.7} & \downarrow {\scriptstyle \rho^{(m)} \otimes id \otimes \mathbb{Q}} & & \swarrow {\scriptstyle 4.6.4.7} & \downarrow {\scriptstyle \rho^{(m)} \otimes \mathbb{Q}} \\
\mathbb{D}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}'^{\vee} & & \xrightarrow{\theta^{\mathrm{alg}}} & \mathbb{D}(\mathcal{E}') & \\
\downarrow {\scriptstyle \rho^{(m)}_{\mathbb{Q}} \otimes \mathrm{id}} & (\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathring{\mathcal{E}}'^{(m)\vee}) \otimes \mathbb{Q} & \xrightarrow{\widehat{\theta}^{(m)} \otimes \mathbb{Q}} & \downarrow {\scriptstyle \rho^{(m)}_{\mathbb{Q}}} & (\widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)})) \otimes \mathbb{Q} \\
\swarrow {\scriptstyle 4.6.4.7} & & & \swarrow {\scriptstyle 4.6.4.7} & \\
\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}'^{\vee} & & \xrightarrow{\theta^{(m)}_{\mathbb{Q}}} & \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}'), &
\end{array} \tag{11.3.3.3.2}$$

where the four horizontal isomorphisms from the back square towards the front one can be constructed via 4.6.4.7 and the remark 4.6.4.5(ii) (in the category of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}$-modules, the functor $-\otimes_{\mathbb{Z}} \mathbb{Q}$ is canonically isomorphic to $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}} -$ and similarly by replacing $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}$ by $\mathcal{D}^{(m)}_{\mathfrak{X}}$). The bottom square is commutative by definition (11.3.3.1.1). Moreover, the right one corresponds to the diagram below

$$\begin{array}{ccccccc}
\mathbb{D}^{(m)}(\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}} & \longrightarrow & \widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}} \widehat{\mathcal{D}} \otimes_{\mathcal{D}} \mathbb{D}^{(m)}(\mathring{\mathcal{E}}'^{(m)}) & \xrightarrow{4.6.4.7} & \widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}} \widehat{\mathbb{D}}^{(m)}(\widehat{\mathcal{D}} \otimes_{\mathcal{D}} \mathring{\mathcal{E}}'^{(m)}) & \longrightarrow & \widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle 4.6.4.7} \quad \searrow & & \downarrow \\
\mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{D}^{(m)}(\mathring{\mathcal{E}}'^{(m)}) & \longrightarrow & \widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathbb{Q}}} \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{D}^{(m)}(\mathring{\mathcal{E}}'^{(m)}) & & \mathbb{D}^{(m)}_{\mathbb{Q}}(\widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}} \widehat{\mathcal{D}} \otimes_{\mathcal{D}} \mathring{\mathcal{E}}'^{(m)}) & & \widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}} \widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)}) \\
\downarrow & & \downarrow {\scriptstyle 4.6.4.7} & & \downarrow \quad \searrow & & \downarrow \\
\mathbb{D}(\mathcal{E}') & \longrightarrow & \widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathbb{Q}}} \mathbb{D}(\mathcal{E}') & \xrightarrow{4.6.4.7} & \mathbb{D}^{(m)}_{\mathbb{Q}}(\widehat{\mathcal{D}}_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathbb{Q}}} \mathcal{E}') & \longrightarrow & \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}'),
\end{array} \tag{11.3.3.3.3}$$

where the ring $\mathcal{D}$ is $\mathcal{D}^{(m)}_{\mathfrak{X}}$. By transitivity of 4.6.4.7, we check that the rectangle of the middle is commutative. This yields that 11.3.3.3.3 is commutative. Then so is the right diagram of 11.3.3.3.2. Moreover, the commutativity of the right square of 11.3.3.3.2 implies that of the left square. For the back side one, this is a consequence of 11.3.1.19. Moreover, similarly to 11.3.1.15 (i.e., we check that the four morphisms that we use in the construction of $\theta^{(m)}$ and θ^{alg} commute with the functors $-\otimes_{\mathbb{Z}} \mathbb{Q}$), we prove that the top square is commutative. As the arrows of 11.3.3.3.2 are isomorphisms, we get the commutativity of the front side. □

11.3.3.4. The functor $\mathbb{D}^{(m)}_{\mathbb{Q}}$ commutes canonically with Frobenius. More precisely, denoting by $\widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}} = \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}}(-, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}})$, which is canonically isomorphic to the composition of the functors $\widehat{\mathbb{D}}^{(m)}_{\mathbb{Q}}$ and $-\otimes_{\mathcal{O}_X} \omega_{\mathfrak{X}}$, we have for any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}})$ the isomorphisms

$$F^{\flat} \widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \underset{4.6.3.6.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}}(\mathcal{E}', F^{\flat} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}) \underset{F^*}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}}(F^*\mathcal{E}', F^* F^{\flat} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}) \xrightarrow{\sim} \widetilde{\mathbb{D}}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}'). \tag{11.3.3.4.1}$$

This yields:

$$F^* \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \overset{8.8.1.3.1}{\xrightarrow{\sim}} F^{\flat} \widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}/\mathfrak{S}} \overset{11.3.3.4.1}{\xrightarrow{\sim}} \widetilde{\mathbb{D}}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\sim} \mathbb{D}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}'). \tag{11.3.3.4.2}$$

Theorem 11.3.3.5. *With the notations of 11.3.3.1, the following canonical diagram*

$$\begin{array}{ccc} \mathbb{D}^{(m+s)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} (F^*\mathcal{E}')^\vee & \xrightarrow[\sim]{\theta^{(m+s)}_{\mathbb{Q}}} & \mathbb{D}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \\ {\scriptstyle 11.3.3.4.2}\uparrow\wr & & {\scriptstyle 11.3.3.4.2}\uparrow\wr \\ F^*[\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}})\otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}}(\mathcal{E}')^\vee] & \xrightarrow[\sim]{F^*\theta^{(m)}_{\mathbb{Q}}} & F^*[\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}')] \end{array} \qquad (11.3.3.5.1)$$

is commutative.

Proof. Let us consider the following cube :

$$\begin{array}{ccccc} & & (\widehat{\mathbb{D}}^{(m+s)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} (F^*\mathring{\mathcal{E}}'^{(m)})^\vee)_{\mathbb{Q}} & \xrightarrow{\widehat{\theta}^{(m+s)}\otimes\mathbb{Q}} & \widehat{\mathbb{D}}^{(m+s)}(F^*\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}} \\ & \nearrow & \downarrow & \nearrow & \downarrow \\ F^*[(\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'}) \otimes_{\mathcal{O}_{\mathfrak{X}'}} (\mathring{\mathcal{E}}'^{(m)})^\vee)_{\mathbb{Q}}] & \xrightarrow{F^*(\widehat{\theta}^{(m)}\otimes\mathbb{Q})} & & F^*[\widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}}] & \\ \downarrow & & \mathbb{D}^{(m+s)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} (F^*\mathcal{E}')^\vee & \xrightarrow{\theta^{(m+s)}_{\mathbb{Q}}} & \mathbb{D}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \\ & \nearrow & & \downarrow\ \nearrow & \\ F^*[\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}} (\mathcal{E}')^\vee] & \xrightarrow{F^*\theta^{(m)}_{\mathbb{Q}}} & & F^*[\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}')]. & \end{array}$$

(11.3.3.5.2)

The commutativity of the squares of the front face and of the back face follows from the construction of $\theta^{(m)}_{\mathbb{Q}}$ (and $\theta^{(m+s)}_{\mathbb{Q}}$) given at 11.3.3.1.1. We denote by $\mathcal{D}' := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'}$ and $\mathcal{D} := \widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X}}$. We will put a tilde above $\mathcal{D}$ or $\mathcal{D}'$ to mean that we apply to them the functors $\otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}}$ or $\otimes_{\mathcal{O}_{\mathfrak{X}'}} \omega^{-1}_{\mathfrak{X}'}$. The commutativity of the right square of 11.3.3.5.2 can be translated by the commutativity of the diagram below

$$\begin{array}{ccc} \boxed{F^*\mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{D}'_{\mathbb{Q}}} \mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{D}'} \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}')} & \xrightarrow[\sim]{} & \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} F^*\mathcal{D}' \otimes_{\mathcal{D}'} \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}') \\ \sim\downarrow & & \sim\downarrow \\ F^*\mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{D}'_{\mathbb{Q}}} \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}'_{\mathbb{Q}}) & & \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{r}}\widetilde{\mathcal{D}}') \\ \sim\downarrow & \swarrow\sim & \sim\downarrow \\ \boxed{F^*\mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{D}'_{\mathbb{Q}}} \mathbb{R}\mathcal{H}om_{\mathcal{D}'_{\mathbb{Q}}}(\mathcal{E}', \widetilde{\mathcal{D}}'_{\mathbb{Q}})} & & \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{l}}F^*_{\mathrm{r}}\widetilde{\mathcal{D}}') \\ \sim\downarrow & & \sim\downarrow \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}'_{\mathbb{Q}}}(\mathcal{E}', F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) & & \boxed{\mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}})} \\ \sim\downarrow & \swarrow\sim & \sim\downarrow \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}(F^*\mathcal{E}', F^*_{\mathrm{l}}F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) & & \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}_{\mathbb{Q}}) \\ \sim\downarrow & & \sim\downarrow \\ \boxed{\mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}(F^*\mathcal{E}', \widetilde{\mathcal{D}}_{\mathbb{Q}})} & \xleftarrow{\sim} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}((F^*\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}}, \widetilde{\mathcal{D}}_{\mathbb{Q}}). \end{array} \qquad (11.3.3.5.3)$$

We remark that the commutativity of the trapeze (at the top) of 11.3.3.5.3 is a consequence of that of the diagram

$$\begin{array}{ccc} F^*\mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{D}'_{\mathbb{Q}}} \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) \\ \sim\downarrow & & \sim\downarrow \\ F^*\mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{D}'_{\mathbb{Q}}} \mathbb{R}\mathcal{H}om_{\mathcal{D}'_{\mathbb{Q}}}(\mathcal{E}', \widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xrightarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}'_{\mathbb{Q}}}(\mathcal{E}', F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) \end{array} \qquad (11.3.3.5.4)$$

and of the transitivity 6.3.4.1. To check that of 11.3.3.5.4, we choose a K-flat complex of left $\mathcal{D}'$-modules representing $\mathring{\mathcal{E}}'^{(m)}$, a K-injective complex of left $\mathcal{D}'_{\mathbb{Q}} \otimes_{\mathcal{V}} \mathcal{D}'_{\mathbb{Q}}$-modules representing $\widetilde{\mathcal{D}}'_{\mathbb{Q}}$.

The losange of 11.3.3.5.3 corresponds to the outer of

$$\begin{array}{ccccc} \mathbb{R}\mathcal{H}om_{\mathcal{D}'_{\mathbb{Q}}}(\mathcal{E}', F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xleftarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xleftarrow[\sim]{} & \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{R}\mathcal{H}om_{\mathcal{D}'}(\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{r}}\widetilde{\mathcal{D}}') \\ \sim\downarrow F^* & & \sim\downarrow F^* & & \sim\downarrow \mathrm{id}\otimes F^* \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}(F^*\mathcal{E}', F^*_{\mathrm{l}}F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xleftarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{l}}F^*_{\mathrm{r}}\widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xleftarrow[\sim]{} & \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, F^*_{\mathrm{l}}F^*_{\mathrm{r}}\widetilde{\mathcal{D}}') \\ \sim\downarrow & & \sim\downarrow & & \downarrow \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{Q}}}(F^*\mathcal{E}', \widetilde{\mathcal{D}}_{\mathbb{Q}}) & \xleftarrow[\sim]{} & \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}'_{\mathbb{Q}}) & \xleftarrow[\sim]{} & \mathcal{D}_{\mathbb{Q}} \otimes_{\mathcal{D}} \mathbb{R}\mathcal{H}om_{\mathcal{D}}(F^*\mathring{\mathcal{E}}'^{(m)}, \widetilde{\mathcal{D}}), \end{array}$$

(11.3.3.5.5)

whose commutativity is straightforward. Finally, as the (bottom) triangle of 11.3.3.5.3 is also commutative, we conclude that the right square of 11.3.3.5.2 is commutative.

We proceed similarly to prove that of the left square of 11.3.3.5.2.

Finally, the top square of 11.3.3.5.2 corresponds to the following outer:

$$\begin{array}{ccccc} F^*[(\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'}) \otimes_{\mathcal{O}_{\mathfrak{X}'}} (\mathring{\mathcal{E}}'^{(m)})^\vee)_{\mathbb{Q}}] & \xrightarrow{\sim} & [F^*(\widehat{\mathbb{D}}^{(m)}(\mathcal{O}_{\mathfrak{X}'}) \otimes_{\mathcal{O}_{\mathfrak{X}'}} (\mathring{\mathcal{E}}'^{(m)})^\vee)]_{\mathbb{Q}} & \xrightarrow{\sim} & (\widehat{\mathbb{D}}^{(m+s)}(\mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} (F^*\mathring{\mathcal{E}}'^{(m)})^\vee)_{\mathbb{Q}} \\ \sim\downarrow F^*(\widehat{\theta}^{(m)}\otimes\mathbb{Q}) & & \sim\downarrow (F^*\widehat{\theta}^{(m)})\otimes\mathbb{Q} & & \sim\downarrow \widehat{\theta}_{\mathbb{Q}}^{(m+s)}\otimes\mathbb{Q} \\ F^*[\widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}}] & \xrightarrow{\sim} & [F^*\widehat{\mathbb{D}}^{(m)}(\mathring{\mathcal{E}}'^{(m)})]_{\mathbb{Q}} & \xrightarrow{\sim} & \widehat{\mathbb{D}}^{(m+s)}(F^*\mathring{\mathcal{E}}'^{(m)})_{\mathbb{Q}}. \end{array} \tag{11.3.3.5.6}$$

The left square of 11.3.3.5.6 is commutative by functoriality. It follows from 11.3.2.9.1 that the right square of 11.3.3.5.6 is commutative. Hence, we have checked that five of the squares of 11.3.3.5.2 are commutative. Then so is the bottom one. □

11.3.4 Compatibility with Frobenius of θ

11.3.4.1. For any coherent $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module $\mathcal{E}$ which is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent, let us denote by $\sigma^{(m)}_{\mathbb{Q}}$ the isomorphism

$$\sigma^{(m)}_{\mathbb{Q}} \colon \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}) \underset{4.6.4.7.1}{\xrightarrow{\sim}} \mathbb{D}(\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}) \xrightarrow{\sim} \mathbb{D}(\mathcal{E}). \tag{11.3.4.1.1}$$

Proposition 11.3.4.2. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent. The square*

$$\begin{array}{ccc} \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}^\vee & \xrightarrow[\sim]{\theta^{(m)}_{\mathbb{Q}}} & \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}) \\ \sim\downarrow \sigma^{(m)}_{\mathbb{Q}}\otimes\mathrm{id} & & \sim\downarrow \sigma^{(m)}_{\mathbb{Q}} \\ \mathbb{D}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}^\vee & \xrightarrow[\sim]{\theta} & \mathbb{D}(\mathcal{E}), \end{array}$$

where θ was defined in 11.2.6.4.1, is commutative.

Proof. Thanks to the commutative diagrams 11.2.6.4.1 and 11.3.3.3.1, it is about proving the equality $\sigma^{(m)}_{\mathbb{Q}} \circ \rho^{(m)}_{\mathbb{Q}} = \rho$, where ρ was constructed at 11.2.6.3.3. It amounts to saying that the outer of the diagram

$$\begin{array}{ccccccc} \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}) & \longrightarrow & \mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}) & \longrightarrow & \mathbb{D}(\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}) & \longrightarrow & \mathbb{D}(\mathcal{E}) \\ \uparrow & & \uparrow & & \uparrow & \nearrow & \\ \mathbb{D}^{(m)}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}) & \longrightarrow & \mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}}} \mathbb{D}^{(m)}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}) & \longrightarrow & \mathbb{D}(\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}) & & \\ \uparrow & & \uparrow & \nearrow & & & \\ \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}} \mathbb{D}(\mathcal{E}) & \longrightarrow & \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}} \mathbb{D}(\mathcal{E}) & & & & \\ \uparrow & \nearrow & & & & & \\ \mathbb{D}(\mathcal{E}) & & & & & & \end{array} \tag{11.3.4.2.1}$$

is commutative. Moreover, that of squares is checked by functoriality and that of the top and bottom triangles are straightforward. Finally, that of the triangle of the middle follows from the transitivity of 4.6.4.7 (see also the remark 4.6.4.5.(ii)). □

11.3.4.3. With the notations of 11.3.3.1, we have a morphism $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*\mathcal{E}' \to F^*(\mathcal{D}^\dagger_{\mathfrak{X}'\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}')$ making commutative the following diagram

$$F^*\mathcal{E}' \longrightarrow \mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*\mathcal{E}' \dashrightarrow F^*(\mathcal{D}^\dagger_{\mathfrak{X}'\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}'). \tag{11.3.4.3.1}$$

By completion, tensorizing by $\mathbb{Q}$ and taking the limit, it follows from 6.2.3.2.1 that this morphism is an isomorphism.

We will need the following lemma to prove the proposition below.

Lemma 11.3.4.4. *For any $\mathcal{E}' \in D_{\mathrm{perf}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}})$, the following diagram*

$$\begin{array}{ccccc}
F^*\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}') & \underset{11.3.4.3.1}{\overset{\sim}{\longleftarrow}} & \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}') & \underset{\sim}{\overset{11.3.3.4.2}{\longrightarrow}} & \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} \mathbb{D}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \\
\sim\downarrow & & & & \downarrow\sim \\
F^*\mathbb{D}(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}') \xrightarrow[\sim]{9.5.2.4.2} & & \mathbb{D}F^*(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}') & \xrightarrow[\sim]{11.3.4.3} & \mathbb{D}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*\mathcal{E}'),
\end{array}$$

is commutative.

Proof. A tilde above $\mathbb{D}$ means that the functor is tensorized by $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}$ (or $- \otimes_{\mathcal{O}_{\mathfrak{X}'}} \omega_{\mathfrak{X}'}$).

We denote by τ, the composite isomorphism:

$$\tau\colon F^{\flat}\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \underset{8.8.1.1.2}{\overset{\sim}{\longleftarrow}} F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*F^{\flat}\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \overset{\mathrm{id}\otimes\alpha}{\longrightarrow} F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}},$$

and where α is the inverse of the canonical isomorphism $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \overset{\sim}{\longrightarrow} F^*F^{\flat}\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$ (see 8.8.2.2.1). The structure of $(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}, \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$-bimodule of $F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$ extends then to a structure of $(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}, \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$-bimodule. The morphism τ is the inverse of the morphism of $(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}, \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$-bimodules that we deduce by extension from $F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}} \to F^{\flat}\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$. Hence, we reduce to prove the commutativity of the diagram:

$$\begin{array}{ccccc}
\widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} F^{\flat}\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} & \underset{\sim}{\overset{\mathrm{id}\otimes\tau}{\longrightarrow}} & \widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} F^{\flat}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} & \underset{\sim}{\overset{11.3.3.4.2}{\longrightarrow}} & \widetilde{\mathbb{D}}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \\
\sim\downarrow & & & & \downarrow\sim \\
F^{\flat}\widetilde{\mathbb{D}}(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}') & \underset{\sim}{\overset{9.5.2.4.1}{\longrightarrow}} & \widetilde{\mathbb{D}}(F^*\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}') & \underset{\sim}{\overset{11.3.4.3}{\longrightarrow}} & \widetilde{\mathbb{D}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*\mathcal{E}').
\end{array} \tag{11.3.4.4.1}$$

We will write $\mathcal{D}^{\dagger}$ (resp. $\mathcal{D}_{\mathfrak{X}'}$, resp. $\mathcal{D}_{\mathfrak{X}}$) instead of $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$ or $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$ (resp. $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}$, resp. $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}$). We set $\mathcal{E}'^{\dagger} := \mathcal{D}^{\dagger} \otimes_{\mathcal{D}_{\mathfrak{X}'}} \mathcal{E}'$ and $(F^*\mathcal{E}')^{\dagger} = \mathcal{D}^{\dagger} \otimes_{\mathcal{D}_{\mathfrak{X}}} F^*\mathcal{E}'$. Let us consider the diagram

$$\begin{array}{ccccc}
\boxed{\widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \otimes_{\mathcal{D}_{\mathfrak{X}'}} F^{\flat}\mathcal{D}^{\dagger}} & \overset{\mathrm{id}\otimes\tau}{\longrightarrow} & \widetilde{\mathbb{D}}^{(m)}_{\mathbb{Q}}(\mathcal{E}') \otimes_{\mathcal{D}_{\mathfrak{X}'}} F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger} & \longrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}'}}(\mathcal{E}', F^{\flat}\mathcal{D}_{\mathfrak{X}'}) \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger} \\
\downarrow & & & & \downarrow \\
\mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}'}}(\mathcal{E}', \mathcal{D}^{\dagger}) \otimes_{\mathcal{D}^{\dagger}} F^{\flat}\mathcal{D}^{\dagger} & \longrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}'}}(\mathcal{E}', F^{\flat}\mathcal{D}^{\dagger}) & \overset{\tau}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}'}}(\mathcal{E}', F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathbb{D}}(\mathcal{E}'^{\dagger}) \otimes_{\mathcal{D}^{\dagger}} F^{\flat}\mathcal{D}^{\dagger} & \longrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(\mathcal{E}'^{\dagger}, F^{\flat}\mathcal{D}^{\dagger}) & \overset{\tau}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(\mathcal{E}'^{\dagger}, F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) \\
\Vert & & \downarrow F^* & & \downarrow F^* \\
\Vert & & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(F^*\mathcal{E}'^{\dagger}, F^*F^{\flat}\mathcal{D}^{\dagger}) & \overset{\tau}{\longrightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(F^*\mathcal{E}'^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) \\
\Vert & & \downarrow & & \downarrow \\
\boxed{\widetilde{\mathbb{D}}(\mathcal{E}'^{\dagger}) \otimes_{\mathcal{D}^{\dagger}} F^{\flat}\mathcal{D}^{\dagger}} & \dashrightarrow & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}((F^*\mathcal{E}')^{\dagger}, F^*F^{\flat}\mathcal{D}^{\dagger}) & \overset{\tau}{\rightarrow} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}((F^*\mathcal{E}')^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}),
\end{array} \tag{11.3.4.4.2}$$

where the left bottom arrow is defined in order to make commutative the left bottom rectangle. By composing 11.3.4.4.2 with the following diagram

$$\begin{array}{ccccc}
\mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}'}}(\mathcal{E}', F^{\flat}\mathcal{D}_{\mathfrak{X}'}) \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger} & \xrightarrow{F^*} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'}) \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger} & \xrightarrow{\alpha^{(m)}} & \boxed{\widetilde{\mathbb{D}}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}'}}(\mathcal{E}', F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & \xrightarrow{F^*} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}}}(F^*\mathcal{E}', F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & \xrightarrow{\alpha^{(m)}} & \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}}}(F^*\mathcal{E}', \mathcal{D}^{\dagger}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(\mathcal{E}'^{\dagger}, F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & & & & \\
\downarrow F^* & & & & \\
\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(F^*\mathcal{E}'^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & & & & \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}((F^*\mathcal{E}')^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & = & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}((F^*\mathcal{E}')^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & \xrightarrow{\alpha^{(m)}} & \boxed{\widetilde{\mathbb{D}}((F^*\mathcal{E}')^{\dagger})},
\end{array} \tag{11.3.4.4.3}$$

where $\alpha^{(m)}$ is the inverse of the canonical isomorphism $\mathcal{D}_{\mathfrak{X}} \xrightarrow{\sim} F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'}$ of 8.8.1.1.1, we get the square 11.3.4.4.1. Indeed, concerning the top, right and left arrows, this is straightforward. To establish that so is the bottom one, we have to check that the composition arrow of the top of the commutative diagram below

$$\begin{array}{ccccc}
\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(F^*\mathcal{E}'^{\dagger}, F^*F^{\flat}\mathcal{D}^{\dagger}) & \xrightarrow{\tau} & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}(F^*\mathcal{E}'^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & \xrightarrow{\alpha^{(m)}} & \widetilde{\mathbb{D}}(F^*\mathcal{E}'^{\dagger}) \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}}((F^*\mathcal{E}')^{\dagger}, F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}) & \xrightarrow{\alpha^{(m)}} & \widetilde{\mathbb{D}}((F^*\mathcal{E}')^{\dagger}).
\end{array} \tag{11.3.4.4.4}$$

is the canonical morphism induced by α. This latter fact is a consequence of the commutativity of

$$\begin{array}{ccccc}
F^*F^{\flat}\mathcal{D}^{\dagger} & \longrightarrow & F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} F^*F^{\flat}\mathcal{D}^{\dagger}) & \xrightarrow{\alpha^{(m)}\otimes \mathrm{id}} & \mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} F^*F^{\flat}\mathcal{D}^{\dagger} \\
 & \searrow^{F^*\tau} & \downarrow \mathrm{id}\otimes\alpha & & \downarrow \mathrm{id}\otimes\alpha \\
 & & F^*F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger} & \xrightarrow{\alpha^{(m)}\otimes \mathrm{id}} & \mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}.
\end{array} \tag{11.3.4.4.5}$$

By using the commutativity of the rectangle of the top of 6.2.3.3.1 (we complete, we tensorize by $\mathbb{Q}$ and we apply it with $\mathcal{E}' = F^{\flat}\mathcal{D}_{\mathfrak{X}'}$), we get that of the top diagram of 11.3.4.4.5. The one of the square of 11.3.4.4.5 is functorial and that of the triangle is tautological.

It remains now to prove that the diagrams 11.3.4.4.2 and 11.3.4.4.3 are commutative. The one of the top rectangle of 11.3.4.4.2 follows, via the remark 4.6.4.5.(ii), from the transitivity of 4.6.4.7. We can check that of the left square of the second row of 11.3.4.4.2 by proceeding similarly to 11.3.3.5.4. To check the commutativity of the rectangle at the bottom left of 11.3.4.4.3, it is sufficient to choose a K-injective complex of $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}} \otimes_{\mathcal{V}} \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-modules representing $F^{\flat}\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{D}^{\dagger}$ and a K-flat complex of left $\mathcal{D}_{\mathfrak{X}'}$-modules representing $\mathcal{E}'$. That of the top right square of 11.3.4.4.3 can be computed by choosing a K-flat complex of left $\mathcal{D}_{\mathfrak{X}'}$-modules representing $\mathcal{E}'$ and a K-injective complex of left $\mathcal{D}_{\mathfrak{X}'} \otimes_{\mathcal{V}} \mathcal{D}_{\mathfrak{X}}$-modules representing $F^{\flat}\mathcal{D}_{\mathfrak{X}'}$. The commutativity of the other squares or of the rectangle is checked by functoriality. □

Proposition 11.3.4.5. *Let $\mathcal{E}'$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$-coherent. The diagram*

$$\begin{array}{ccc}
F^*[\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}')] & \longrightarrow & \mathbb{D}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \\
\downarrow \sigma^{(m)}_{\mathbb{Q}} & & \downarrow \sigma^{(m+s)}_{\mathbb{Q}} \\
F^*[\mathbb{D}(\mathcal{E}')] & \longrightarrow & \mathbb{D}(F^*\mathcal{E}')
\end{array}$$

is commutative.

Proof. By construction of the isomorphism $\mathcal{D}^{\dagger}_{\mathfrak{X}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{X},\mathbb{Q}}} F^*\mathcal{E} \xrightarrow{\sim} F^*(\mathcal{D}^{\dagger}_{\mathfrak{X}'\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E})$ (see 11.3.4.3), the first and the latter isomorphism of 11.3.4.1.1 are compatible with Frobenius. For the middle one, this corresponds to 11.3.4.4. □

Theorem 11.3.4.6. *Let $\mathcal{E}'$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}$-module, $\mathcal{O}_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}$-coherent. The diagram*

$$\begin{array}{ccc} F^*[\mathbb{D}_{T'}(\mathcal{O}_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}} \mathcal{E}'^{\vee}] & \xrightarrow{F^*\theta} & F^*\mathbb{D}_{T'}(\mathcal{E}') \\ \downarrow & & \downarrow \\ \mathbb{D}_T(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} (F^*\mathcal{E}')^{\vee} & \xrightarrow{\theta} & \mathbb{D}_T(F^*\mathcal{E}'). \end{array} \tag{11.3.4.6.1}$$

is commutative.

Proof. As the edges of 11.3.4.6.1 are morphisms of coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules, thanks to 8.7.6.11, it is sufficient to prove the commutativity of 11.3.4.6.1 when T is empty. Let us consider the following cube

$$\begin{array}{ccccccc} & & (\mathbb{D}^{(m+s)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} (F^*\mathcal{E}')^{\vee}) & & \xrightarrow{\theta^{(m+s)}_{\mathbb{Q}}} & & \mathbb{D}^{(m+s)}_{\mathbb{Q}}(F^*\mathcal{E}') \\ & \nearrow & & & & \nearrow & \\ F^*[(\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}} (\mathcal{E}')^{\vee})] & & \downarrow & \xrightarrow{F^*(\theta^{(m)}_{\mathbb{Q}})} & F^*[\mathbb{D}^{(m)}_{\mathbb{Q}}(\mathcal{E}')] & & \downarrow \\ & & \mathbb{D}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} (F^*\mathcal{E}')^{\vee} & & \xrightarrow{\theta^{\dagger}} & & \mathbb{D}(F^*\mathcal{E}') \\ \downarrow & \nearrow & & & \downarrow & \nearrow & \\ F^*[\mathbb{D}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}} (\mathcal{E}')^{\vee}] & & & \xrightarrow{F^*\theta^{\dagger}} & F^*[\mathbb{D}(\mathcal{E}')], & & \end{array} \tag{11.3.4.6.2}$$

where the vertical arrows are induced by the morphisms of the form $\sigma^{(m)}_{\mathbb{Q}}$. Following 11.3.3.5, the top square is commutative. Moreover, thanks to 11.3.4.2, so are the front and back squares. Via 11.3.4.5, the right and left squares are commutative. As the morphisms are isomorphisms, it follows the commutativity of the bottom square. □

Theorem 11.3.4.7. *Let E' be an isocrystal on $X' \setminus T'$ overconvergent along X'. The isomorphism*

$$\mathbb{D}_{T'}(\mathcal{O}_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}} \mathrm{sp}_*(E'^{\vee}) \xrightarrow{\sim} \mathbb{D}_{T'}(\mathrm{sp}_*(E'))$$

is compatible with Frobenius.

Proof. Following 11.2.7.4, we have the isomorphism $\mathrm{sp}_*(E'^{\vee}) \xrightarrow{\sim} (\mathrm{sp}_*(E'))^{\vee}$ which is compatible with Frobenius. By using 11.3.4.6, we are done. □

11.3.5 Commutation of the duality with pullbacks via a smooth morphism, adjunction

We collect here some results concerning the compatibility with Frobenius and its corollaries. Following [Abe14a, 3.10], we have the following theorem.

Theorem 11.3.5.1 (Abe)**.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of smooth $\mathcal{V}$-formal schemes of relative dimension d_f. Let us denote by d the relative dimension of X_0 over Y_0. For any divisors T of X and W of Y such that $f(X \setminus T) \subset Y \setminus W$, for any integers $0 \leq m \leq n_m$, for any $\mathcal{E}^{(m)} \in F$-$D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{B}^{(n_m)}_{\mathfrak{Y}}(W)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}})$, denoting by $\mathcal{E} := \mathcal{D}^\dagger_{\mathfrak{Y}}({}^\dagger W)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_m)}_{\mathfrak{Y}}(W)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}}} \mathcal{E}^{(m)}$, and using notation 8.8.3.3 we have the canonical isomorphism*

$$f^!_{T,W} \circ \mathbb{D}_{\mathfrak{Y},W}(\mathcal{E}) \xrightarrow{\sim} \mathbb{D}_{\mathfrak{X},T} \circ f^!_{T,W}(\mathcal{E})(d_f)[2d_f]. \tag{11.3.5.1.1}$$

In the case where $\mathfrak{Y} = \mathrm{Spf}\,\mathcal{V}$, this yields:

Corollary 11.3.5.2. *We have the isomorphism of F-$\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules:*

$$\mathbb{D}_{\mathfrak{X},Z}(\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}(-d). \tag{11.3.5.2.1}$$

Corollary 11.3.5.3. *We have the compatible to Frobenius isomorphism*

$$\mathrm{sp}_*(E'^{\vee})(-d) \xrightarrow{\sim} \mathbb{D}_{T'}(\mathrm{sp}_*(E')).$$

Proof. This follows from 11.3.4.7 and 11.3.5.2.1. □

11.3.5.4. With notation 11.3.5.1, set $f^{+}_{T,W} := \mathbb{D}_{\mathfrak{X},T} \circ f^{!}_{T,W} \circ \mathbb{D}_{\mathfrak{Y},W}$. We get the isomorphisms:

$$f^{!}_{T,W}(\mathcal{E}) \xrightarrow{\sim} f^{!}_{T,W} \circ \mathbb{D}_{\mathfrak{X},T}(\mathbb{D}_{\mathfrak{X},T}(\mathcal{E})) \underset{11.3.5.1.1}{\xrightarrow{\sim}} \mathbb{D}_{\mathfrak{X},T} \circ f^{!}_{T,W}(\mathbb{D}_{\mathfrak{X},T}(\mathcal{E}))(d_f)[2d_f] = f^{+}_{T,W}(\mathcal{E})(d_f)[2d_f]. \tag{11.3.5.4.1}$$

The twisted version of 11.3.5.1 is:

$$f^{!}_{T,W}(\omega_{\mathfrak{Y}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathbb{D}_{\mathfrak{Y},W}(\mathcal{E})) \underset{9.2.1.23.1}{\xrightarrow{\sim}} \omega_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}}} f^{!}_{T,W} \circ \mathbb{D}_{\mathfrak{Y},W}(\mathcal{E}) \underset{11.3.5.1.1}{\xrightarrow{\sim}} \omega_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathbb{D}_{\mathfrak{X},T} \circ f^{!}_{T,W}(\mathcal{E})(d_f)[2d_f]. \tag{11.3.5.4.2}$$

Corollary 11.3.5.5. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of smooth $\mathcal{V}$-formal schemes of relative dimension d_f. Let us denote by d the relative dimension of X_0 over Y_0. For any divisors T of X and W of Y such that $f(X \setminus T) \subset Y \setminus W$, for any integers $0 \le m \le n_m$, for any $\mathcal{E}^{(m)} \in F\text{-}D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{B}^{(n_m)}_{\mathfrak{Y}}(W) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}})$, denoting by $\mathcal{E} := \mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}} \otimes_{\mathcal{B}^{(n_m)}_{\mathfrak{Y}}(W) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}}} \mathcal{E}^{(m)}$, for any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$. we have the canonical isomorphism*

$$\mathbb{R}f_* \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}}(f^{+}_{T,W}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}}(\mathcal{E}, f_{T,W,+}(\mathcal{F})), \tag{11.3.5.5.1}$$

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}}(f^{+}_{T,W}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}}(\mathcal{E}, f_{T,W,+}(\mathcal{F})). \tag{11.3.5.5.2}$$

Proof. From the twisted version 11.3.5.4.2, we get:

$$\begin{aligned} f^{-1}(\omega_{\mathfrak{Y}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathbb{D}_{\mathfrak{Y},W}(\mathcal{E})) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{Y}\leftarrow\mathfrak{X}}({}^{\dagger}W)_{\mathbb{Q}}[d_f] &\xrightarrow{\sim} f^{!}_{T,W}(\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}))[d_Y] \\ &\underset{11.3.5.4.2}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}}(f^{!}_{T,W}(\mathcal{E}), \mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}))[d_X](d_f)[2d_f] \\ &\underset{11.3.5.4.1}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}}(f^{+}_{T,W}(\mathcal{E}), \mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}))[d_X]. \end{aligned} \tag{11.3.5.5.3}$$

Since $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}})$ (resp. $f^{+}_{T,W}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$), then we get the last (resp. first) isomorphism:

$$\begin{aligned} \mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}}(f^{+}_{T,W}(\mathcal{E}), \mathcal{F})\right) &\xrightarrow{\sim} \mathbb{R}f_*\left(\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}}(f^{+}_{T,W}(\mathcal{E}), \mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} \mathcal{F}\right) \\ &\underset{11.3.5.5.3}{\xrightarrow{\sim}} \mathbb{R}f_*\left(f^{-1}(\omega_{\mathfrak{Y}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathbb{D}_{\mathfrak{Y},W}(\mathcal{E})[-d_Y]) \otimes^{\mathbb{L}}_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{Y}\leftarrow\mathfrak{X}}({}^{\dagger}W)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} \mathcal{F}\right) \\ &\xrightarrow{\sim} \omega_{\mathfrak{Y}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathbb{D}_{\mathfrak{Y},W}(\mathcal{E})[-d_Y] \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} \mathbb{R}f_*\left(\mathcal{D}^{\dagger}_{\mathfrak{Y}\leftarrow\mathfrak{X}}({}^{\dagger}W)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} \mathcal{F}\right) \\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}} f_{T,W,+}(\mathcal{F}) \\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{Y}}({}^{\dagger}W)_{\mathbb{Q}}}(\mathcal{E}, f_{T,W,+}(\mathcal{F})). \end{aligned}$$

Hence, we get 11.3.5.5.1. Applying $\mathbb{R}\Gamma(\mathfrak{Y}, -)$, this yields 11.3.5.5.2.

□

Chapter 12

Differential coherence of the constant coefficient

Put $\mathfrak{S} = \operatorname{Spf} \mathcal{V}$.

12.1 Local cohomology with support in a smooth closed subscheme

12.1.1 Arithmetic $\mathcal{D}$-modules associated with overconvergent isocrystals in the lifted case via the functor $\mathbb{R}\mathrm{sp}_*$

Contrary to the situation of chapter 11, the locus of overconvergent singularities is not the support of a divisor. In that case, the functor sp_* is not exact. The purpose of this section is to compute $\mathbb{R}\mathrm{sp}_*$ in the category of $\mathcal{D}^\dagger$-modules.

12.1.1.1. Let $\mathfrak{X}$ be an affine smooth $\mathcal{V}$-formal scheme. Let Z be a divisor of X, $\mathfrak{Y}$ be the open of $\mathfrak{X}$ complementary to the support of Z and $j\colon \mathfrak{Y} \to \mathfrak{X}$ be the canonical morphism. For any $m \in \mathbb{N}$, we set $\lambda_m := p^{-1/p^{m+1}}$, $V_m := \mathfrak{X}_K \backslash]Z[_{\mathfrak{X}\lambda_m}$ and $j_m\colon V_m \to \mathfrak{X}_K$ be the canonical immersion. Following 10.1.1.7, the family V_m forms a basis of strict neighborhoods of $\mathfrak{Y}_K$ in $\mathfrak{X}_K$.

Let E be a $j^\dagger\mathcal{O}_{\mathfrak{X}_K}$-module of finite presentation. Then, for m_0 large enough, there exists $\mathcal{E}_0$ an $\mathcal{O}_{V_{m_0}}$-module of finite presentation together with an isomorphism $E \xrightarrow{\sim} j^\dagger\mathcal{E}_0 = j^\dagger\mathcal{O}_{V_{m_0}} \otimes_{\mathcal{O}_{V_{m_0}}} \mathcal{E}_0$. With 10.1.2.2.1, this yields the isomorphism

$$E \xrightarrow{\sim} \varinjlim_{m \geq m_0} j_{m*}(\mathcal{E}_0|V_m).$$

Let $r \geq 1$, $m \geq m_0$ be two integers, U be an affinoid subset of $\mathfrak{X}_K$. Set $\mathcal{E}_m := \mathcal{E}_0|V_m$. Since V_m is an affinoid subset of $\mathfrak{X}_K$, then $j_m^{-1}(U) = U \cap Y_m$ is also an affinoid subset of $\mathfrak{X}_K$ and then is an affinoid subset of V_m. Hence, by using theorem B of Kiehl's (see [Kie67]), we get $H^r(j_m^{-1}(U), \mathcal{E}_m) = 0$. This yields that $R^r j_{m*}(\mathcal{E}_m) = 0$ (see [Gro61, 0.12.2.1]). This means that the canonical morphism

$$j_{m*}(\mathcal{E}_m) \to \mathbb{R}j_{m*}(\mathcal{E}_m) \qquad (12.1.1.1.1)$$

is an isomorphism.

The following Lemma will be useful to check the resolution 12.1.1.4.1.

Lemma 12.1.1.2. *Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine smooth formal $\mathfrak{S}$-scheme. Let $g \in A$, $\mathfrak{Y} := D(g) \overset{j}{\hookrightarrow} \mathfrak{X}$ be the corresponding open immersion. Let $u\colon \mathfrak{X} \hookrightarrow \mathfrak{X}'$ be an open immersion of separated smooth formal $\mathfrak{S}$-schemes. Let $u_K\colon \mathfrak{X}_K \hookrightarrow \mathfrak{X}'_K$ be the induced morphism of rigid spaces.*

(a) Let E be a $j^\dagger\mathcal{O}_{\mathfrak{X}_K}$-module of finite presentation.

(i) The module E is acyclic for the functor u_{K*}, i.e. the canonical morphism

$$u_{K*}(E) \to \mathbb{R}u_{K*}(E) \qquad (12.1.1.2.1)$$

is an isomorphism.

(ii) The module $u_{K*}(E)$ is acyclic for sp_*, i.e., the canonical morphism

$$\mathrm{sp}_* u_{K*}(E) \to \mathbb{R}\mathrm{sp}_* u_{K*}(E) \qquad (12.1.1.2.2)$$

is an isomorphism.

(b) Let E be a coherent $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$-module which is equipped with an integrable connection overconvergent along $X \setminus Y$.

(i) The sheaf $\mathrm{sp}_* u_{K*} E$ is equipped with a canonical structure of $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}$-module functorial in E.

(ii) If $\mathfrak{X}_1$ is an affine open subscheme of $\mathfrak{X}$ and if $f_1 \in \Gamma(\mathfrak{X}_1, \mathcal{O}_{\mathfrak{X}_1})$ is so that the induced open immersion $\mathfrak{Y}_1 = D(f_1) \stackrel{j_1}{\hookrightarrow} \mathfrak{X}_1$ factors through $\mathfrak{Y}_1 \subset \mathfrak{Y}$, then the canonical homomorphism

$$\mathrm{sp}_*(u_{K*}E) \to \mathrm{sp}_*(u_{1K*} j_1^\dagger (E|\mathfrak{X}_{1K})),$$

where $u_1 \colon \mathfrak{X}_1 \hookrightarrow \mathfrak{X}'$ is the open immersion, is $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}$-linear.

Proof. 0) This yields, since $\mathfrak{X}'$ is separated, that we can suppose both $\mathfrak{X}$ and $\mathfrak{X}'$ are affine (and $\mathfrak{Y}$ is still a standard open formal subscheme of $\mathfrak{X}$) and we use notations of 12.1.1.1.

a)i) Let us check 12.1.1.2.1. This is local on $\mathfrak{X}'_K$. Let $r \geq 1$ be an integer, U' be an affinoid subset of $\mathfrak{X}'_K$. Since $\mathfrak{X}_K$ and Y_m are affinoid spaces, then $U := (u_K)^{-1}(U')$ is an affinoid subset of $\mathfrak{X}_K$, and $j_m^{-1}(U)$ is an affinoid subset of Y_m. By using theorem B of Kiehl's (see [Kie67]), this implies $H^r((u_K \circ j_m)^{-1}(U'), \mathcal{E}_m) = 0$. This yields that the canonical morphism $u_{K*} \circ j_{m*}(\mathcal{E}_m) \to \mathbb{R}(u_{K*} \circ j_{m*})(\mathcal{E}_m)$ is an isomorphism. From the isomorphism 12.1.1.1.1, we get $\mathbb{R}(u_{K*} \circ j_{m*})(\mathcal{E}_m) \xrightarrow{\sim} \mathbb{R}u_{K*}(j_{m*}(\mathcal{E}_m))$. Hence the canonical morphism $u_{K*}(j_{m*}\mathcal{E}_m) \to \mathbb{R}u_{K*}(j_{m*}\mathcal{E}_m)$ is an isomorphism, i.e. $R^r u_{K*}(j_{m*}\mathcal{E}_m) = 0$ for any $r \geq 1$. Since u_K is a coherent morphism of coherent topological spaces, then inductive limits commutes with $R^r u_{K*}$ (see [SGA4.2, VI.5.1], or also [FK18, 0.3.1.9]). Hence, taking the inductive limit, this yields $R^r u_{K*}(j_Y^\dagger \mathcal{O}_{\mathfrak{X}_K}) = 0$ for any $r \geq 1$, i.e. that the canonical morphism 12.1.1.2.2 is an isomorphism.

ii) Using the same arguments than in the first part (i.e. theorem B of Kiehl's and next taking the inductive limits), we check the canonical morphism

$$(\mathrm{sp}_* \circ u_{K*})(j_Y^\dagger \mathcal{O}_{\mathfrak{X}_K}) \to \mathbb{R}(\mathrm{sp}_* \circ u_{K*})(j_Y^\dagger \mathcal{O}_{\mathfrak{X}_K}) \qquad (12.1.1.2.3)$$

is an isomorphism. Using 12.1.1.2.1 and 12.1.1.2.3, we get 12.1.1.2.2.

b) This follows from 11.2.1.5. □

Notation 12.1.1.3. Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme. Let $\mathrm{sp} \colon \mathfrak{X}_K \to \mathfrak{X}$ be the specialization morphism. Let Z be a closed subscheme of X and let $Y := X \setminus Z$ and $j \colon Y \subset X$ be the open immersion. Let $\mathscr{X} := (\mathfrak{X}_i)_{i \in I}$ be a finite affine open covering of $\mathfrak{X}$. For any $i \in I$, let $\mathscr{Y}_i := (Y_{i\,j_i})_{j_i \in J_i}$ be a finite open covering of $Y_i := Y \cap \mathfrak{X}_i$ such that there exists $f_{i\,j_i} \in \Gamma(\mathfrak{X}_i, \mathcal{O}_{\mathfrak{X}})$ satisfying $Y_{i\,j_i} = D(f_{i\,j_i}) \cap X_i$. We get the divisor $T_{i\,j_i} := V(f_{i\,j_i})$ of X_i such that $Y_{i\,j_i} = X_i \setminus T_{i\,j_i}$. Let E be an abelian sheaf on $\mathfrak{X}_K$.

Fix $h, l \in \mathbb{N}$. Let $\underline{i} = (i_0, \ldots, i_h) \in I^{1+h}$. We set $\mathfrak{X}_{\underline{i}} := \mathfrak{X}_{i_0} \cap \cdots \cap \mathfrak{X}_{i_h}$, $Y_{\underline{i}} := Y \cap \mathfrak{X}_{\underline{i}}$, $u_{\underline{i}} \colon \mathfrak{X}_{\underline{i}} \to \mathfrak{X}$, $u_{\underline{i}K} \colon \mathfrak{X}_{\underline{i}K} \to \mathfrak{X}_K$, and $J_{\underline{i}} := J_{i_0} \times \cdots \times J_{i_h}$. For any $\underline{j} = (j_{i_0}, \ldots, j_{i_h}) \in J_{\underline{i}}$, we set $Y_{\underline{i}\,\underline{j}} := Y_{i_0\,j_{i_0}} \cap \cdots \cap Y_{i_h\,j_{i_h}}$, $f_{\underline{i}\,\underline{j}} := f_{i_0\,j_{i_0}}|_{\mathfrak{X}_{\underline{i}}} \cdots f_{i_h\,j_{i_h}}|_{\mathfrak{X}_{\underline{i}}}$. Denoting by $T_{\underline{i}\,\underline{j}} := V(f_{\underline{i}\,\underline{j}})$ the divisor of $X_{\underline{i}}$, we have $Y_{\underline{i}\,\underline{j}} = X_{\underline{i}} \setminus T_{\underline{i}\,\underline{j}}$.

We get the covering $\mathscr{Y}_{\underline{i}} := (Y_{\underline{i}\,\underline{j}})_{\underline{j} \in J_{\underline{i}}}$ of $Y_{\underline{i}}$. For any $\underline{\underline{j}} = (\underline{j}_0, \ldots, \underline{j}_l) \in (J_{\underline{i}})^{1+l}$, we set $Y_{\underline{i},\underline{\underline{j}}} := Y_{\underline{i}\,\underline{j}_0} \cap \cdots \cap Y_{\underline{i}\,\underline{j}_l}$, $f_{\underline{i},\underline{\underline{j}}} := f_{\underline{i}\,\underline{j}_0} \cdots f_{\underline{i}\,\underline{j}_l}$. We denote the corresponding open immersions by

$$j \colon Y \hookrightarrow \mathfrak{X}, \qquad j_{\underline{i}} \colon Y_{\underline{i}} \hookrightarrow \mathfrak{X}_{\underline{i}}, \qquad j_{\underline{i}\underline{\underline{j}}} \colon Y_{\underline{i}\underline{\underline{j}}} \hookrightarrow \mathfrak{X}_{\underline{i}}.$$

Denoting by $T_{\underline{i}\underline{\underline{j}}} := V(f_{\underline{i}\underline{\underline{j}}})$ the divisor of $X_{\underline{i}}$, we have $Y_{\underline{i}\underline{\underline{j}}} = X_{\underline{i}} \setminus T_{\underline{i}\underline{\underline{j}}}$. We set

$$\check{C}^{\dagger hl}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E) := \prod_{\underline{i}\in I^{1+h}} u_{\underline{i}K*} \check{C}^{\dagger l}(\mathfrak{X}_{\underline{i}}, \mathscr{Y}_{\underline{i}}, u^*_{\underline{i}K}(E)) = \prod_{\underline{i}\in I^{1+h}} u_{\underline{i}K*}\left(\prod_{\underline{\underline{j}}\in J_{\underline{i}}^{1+l}} j^\dagger_{\underline{i},\underline{\underline{j}}} u^*_{\underline{i}K}(E)\right),$$

where $\check{C}^{\dagger l}(\mathfrak{X}_{\underline{i}}, \mathscr{Y}_{\underline{i}}, u^*_{\underline{i}K}(E))$ is defined in 10.1.3.5.1. We get the Cech bicomplexes $\check{C}^{\dagger\bullet\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ associated with the coverings $\mathscr{X}, \mathscr{Y}_{\underline{i}}$ of E (see 10.1.4). We denote by $\check{C}^{\dagger\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ the total complex of $\check{C}^{\dagger\bullet\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$.

Proposition 12.1.1.4. *Let E be a coherent $j^\dagger\mathcal{O}_{\mathfrak{X}_K}$-module which is equipped with an integrable connection overconvergent along $X \setminus Y$. We keep notation 12.1.1.3*

(a) The complex $\check{C}^{\dagger\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ gives a resolution of E.

(b) For any h, l, the module $\check{C}^{\dagger hl}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ is acyclic for the functor sp_*.

(c) $\mathrm{sp}_*\check{C}^{\dagger\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E)$ *is a complex of left $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-modules.*

(d) We have in $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$ the canonical isomorphism

$$\mathbb{R}\mathrm{sp}_*(E) \xrightarrow{\sim} \mathrm{sp}_*\check{C}^{\dagger\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E). \qquad (12.1.1.4.1)$$

(e) Let T be a closed subscheme of X and $j'\colon Y \setminus T \hookrightarrow X$ be the open immersion. We have the exact triangle in $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$:

$$\mathbb{R}\mathrm{sp}_*(\underline{\Gamma}^\dagger_{]T[}(E)) \to \mathbb{R}\mathrm{sp}_*(E) \to \mathbb{R}\mathrm{sp}_*(j'^\dagger E) \to +1. \qquad (12.1.1.4.2)$$

Proof. Since, $j^\dagger E = E$, the first statement is a consequence of 10.1.4.2. Since $Y_{\underline{i},\underline{\underline{j}}} = D(f_{\underline{i},\underline{\underline{j}}})$, following 12.1.1.2.2, $u_{\underline{i}K*}\left(j^\dagger_{\underline{i},\underline{\underline{j}}} u^*_{\underline{i}K}(E)\right)$ is acyclic for the functor sp_*. Hence,

$$\check{C}^{\dagger hl}(\mathscr{X}, (\mathscr{Y}_i)_{i\in I}, E) = \prod_{\underline{i}\in I^{1+h}} u_{\underline{i}K*}\left(\prod_{\underline{\underline{j}}\in J_{\underline{i}}^{1+l}} \left(j^\dagger_{\underline{i},\underline{\underline{j}}} u^*_{\underline{i}K}(E)\right)\right)$$

is acyclic for the functor sp_*. This yields the isomorphism 12.1.1.4.1. It follows from 12.1.1.2.2 that by applying the functor $\mathbb{R}\mathrm{sp}_*$ to the map $E \to j'^\dagger E$ we get a morphism in $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$. Hence, by applying the functor $\mathbb{R}\mathrm{sp}_*$ to the exact sequence

$$0 \longrightarrow \underline{\Gamma}^\dagger_{]T[_{\mathcal{P}}}(E) \longrightarrow E \to j'^\dagger E \longrightarrow 0. \qquad (12.1.1.4.3)$$

we get the exact triangle 12.1.1.4.2 of $D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$. □

12.1.2 Differential coherence of $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ when Z is a strict normal crossing divisor

Lemma 12.1.2.1. *Let $\mathfrak{X} = \mathrm{Spf}\, A$ be a reduced affine $\mathcal{V}$-formal scheme of finite type. Suppose $\|\cdot\|$ is a Banach norm on $A \otimes K$, $f \in A$ and $\{a_n\}_{n\geq 0}$ is a sequence of elements of $A \otimes K$ satisfying the following conditions:*

(i) There exist real constants c, η with $\eta < 1$ such that $\|a_n\| \leq c\eta^n$ for all n.

(ii) $\sum_{n\geq 0} a_n f^{-n} = 0$ in $B := A_{\{f\}} \otimes K$ where $A_{\{f\}}$ denotes the p-adic separated completion of A_f.

Then for all $j \geq 0$ there exist c' and $\eta' < 1$ such that

$$\| \sum_{n=0}^{j} a_n f^{j-n} \| \leq c' {\eta'}^j.$$

Proof. If the lemma holds for one norm on $A \otimes K$ then it holds also for any equivalent norm. As $A \otimes K$ is reduced, the spectral semi-norm on $\mathrm{Spm}(A \otimes K)$ is a Banach norm and is equivalent to any other Banach norm ([BGR84] §6.2.4, th. 1). Thus we shall take $\| \cdot \|$ to be the spectral norm. Let η' be a real number a power of which is in the value group of K such that $\eta < \eta' < 1$. Consider now a covering of $\mathfrak{X}_K$ by open affinoids

$$V_1 := \{x \in \mathfrak{X}_K : |f(x)| \leq \eta'\}, \qquad V_2 := \{x \in \mathfrak{X}_K : |f(x)| \geq \eta'\}.$$

It suffices to bound $b_j := \sum_{n=0}^{j} a_n f^{j-n}$ on V_1 and on V_2. For $x \in V_1$ we have

$$|b_j(x)| \leq \sup_{n \leq j} \|a_n\| {\eta'}^{j-n} \leq c{\eta'}^j.$$

Next we consider V_2. $A \otimes K$ is reduced implies $C := \Gamma(V_2, \mathcal{O}_{\mathfrak{X}_K})$ is reduced ([BGR84] §7.3.2, cor. 10). Thus the spectral norm is a Banach norm on C. In C we have $|a_n(x) f(x)^{-n}| \leq c(\eta/\eta')^n$. The series $\sum_n a_n f^{-n}$ converges to say b in C. As b is 0 in B by assumption, we get $|f(x)| < 1$ at all points in the support of b, and then the maximal modulus principle ([BGR84] §3.8.1 p. 171, §6.2.1. prop 4) implies that, increasing η' if necessary, we can suppose that $\sum_n a_n f^{-n} = 0$ in C. Then on V_2 we have

$$|b_j(X)| = | - \sum_{n \rangle j} a_n(x) f(x)^{j-n}| \leq \sup_{n \rangle j} \|a_n\| {\eta'}^{j-n} \leq c{\eta'}^j.$$

□

Proposition 12.1.2.2. *Let $\mathfrak{X}$ be a smooth separated $\mathcal{V}$-formal scheme with reduction X on k, Z a smooth divisor in X and $j : Y = X \setminus Z \to X$ the open immersion. Then:*

(a) The $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ is coherent.

(b) If $\mathfrak{U}$ is an open formal subscheme of $\mathfrak{X}$ such that there exist local coordinates $t_1, \ldots, t_d$ satisfying $Z = V(t_1)$ modulo $\mathfrak{m}$, and $\partial_1, \ldots, \partial_d$ are the corresponding derivations. Then we have the exact sequence

$$(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})^d \xrightarrow{\psi} \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}} \xrightarrow{\phi} \mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \to 0, \qquad (12.1.2.2.1)$$

where $\phi(P) = P \cdot (1/t_1)$, and ψ is defined by

$$\psi(P_1, \ldots, P_d) = P_1 \partial_1 t_1 + \sum_{i=2}^{d} P_i \partial_i. \qquad (12.1.2.2.2)$$

Proof. Part (a) follows from (b) as $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}$ is coherent. It suffices to check (b) for an affine open subscheme $\mathfrak{U}$. Put $A = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$, $D^\dagger = \Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}})$. We claim that the map $\phi : D^\dagger \to \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}) = A[1/t_1]^\dagger \otimes_{\mathcal{V}} K$ is surjective. Indeed, an element f of $A[1/t_1]^\dagger$ can be written as $f = \sum_{n \geq 0} a_n t_1^{-n-1}$ with $a_n \in A$ and $\|a_n\| < c\eta^n$ for some $\eta < 1$. Take $P = \sum_{n \geq 0} (-1)^n a_n \partial^{[n]} \in D^\dagger$, then $P \cdot (1/t_1) = f$.

Next we consider $\mathrm{Ker}\, \phi$. Take $P = \sum_{\underline{n}} a_{\underline{n}} \partial^{[\underline{n}]} \in D^\dagger$ such that $P \cdot (1/t_1) = 0$. Break P into two sums $\sum'$ and $\sum''$ - in $\sum'$ we include those terms with $\underline{n}$ such that $n_i = 0$ for $i > 1$; and in $\sum''$ we include those terms with $\underline{n}$ such that there is an $i > 1$ for which $n_i \neq 0$. According to 8.7.1.10, the operator $\sum''_{\underline{n}} a_{\underline{n}} \partial^{[\underline{n}]}$ lies in the ideal generated by $\partial_2, \ldots, \partial_d$ and thus it is in the image of ψ. We can thus suppose that P is of the form $P = \sum_{n \geq 0} a_n \partial_1^{[n]}$. Since $P \in \mathrm{Ker}\, \phi$, we get in $A[1/t_1]^\dagger$ the relation

$$\sum_{n \geq 0} (-1)^n a_n t_1^{-n-1} = 0. \qquad (12.1.2.2.3)$$

For $j \geq 0$ put

$$b_j := (-1)^{j+1} \sum_{n=0}^{j-1} (-1)^n a_n t_1^{j-n-1}.$$

As $P \in D^\dagger$, there exist c and $\eta < 1$ such that $\|a_n\| < c\eta^n$ for some $\eta < 1$. By lemma 12.1.2.1, there exist c' and $\eta' < 1$ such that $\|b_j\| < c'\eta'^j$. Now put $Q := \sum_{j\geq 0} b_j \partial_1^{[j]}$. We claim: $Qt_1 = P$. It suffices to check in $\Gamma(\mathfrak{U} \cap Y, \mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}) \supset D^\dagger$. Using (12.1.2.2.3) we have $b_j = (-1)^j \sum_{n \geq j} (-1)^n a_n t_1^{j-n-1}$. From this we get

$$\begin{aligned} Qt_1 &= \sum_{j\geq 0} (\sum_{n\geq j} (-1)^{n-j} a_n t_1^{j-n-1}) \partial_1^{[j]} t_1 = \sum_{n \geq 0} a_n (\sum_{j=0}^{n} (-1)^{n-j} t_1^{j-n-1} \partial_1^{[j]}) t_1 \\ &= \sum_{n\geq 0} a_n (\partial_1^{[n]} t_1^{-1}) t_1 = P. \end{aligned}$$

Moreover as $b_0 = 0$, by 8.7.1.10, we can write $Q = Q_1 \partial_1$ for some $Q_1 \in D^\dagger$. □

Proposition 12.1.2.3. *Let $\mathfrak{X}$ be a smooth separated $\mathcal{V}$-formal scheme with reduction X on k, Z be a strict normal crossing divisor in X. Then:*

(a) The $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ is coherent.

(b) If $\mathfrak{U}$ is an affine open formal subscheme of $\mathfrak{X}$ such that there exist local coordinates $t_1, \ldots, t_d$ satisfying $t_1, \ldots, t_r \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})$ with $r \leq d$, and $Z \cap U = V(\bar{t}_1 \cdots \bar{t}_r)$ where $\bar{t}_1, \ldots, \bar{t}_r$ is image of $t_1, \ldots, t_r$ in $\Gamma(U, \mathcal{O}_U)$. We have the exact sequence

$$(\mathcal{D}^\dagger_{\mathfrak{U}/\mathfrak{S},\mathbb{Q}})^d \xrightarrow{\psi} \mathcal{D}^\dagger_{\mathfrak{U}/\mathfrak{S},\mathbb{Q}} \xrightarrow{\phi} \mathcal{O}_{\mathfrak{U}}({}^\dagger Z \cap U)_{\mathbb{Q}} \to 0, \tag{12.1.2.3.1}$$

where $\phi(P) = P \cdot (1/t_1 \cdots t_r)$, and ψ is defined by

$$\psi(P_1, \ldots, P_d) = \sum_{i=1}^{r} P_i \partial_i t_i + \sum_{i=r+1}^{d} P_i \partial_i. \tag{12.1.2.3.2}$$

Proof. See [Ber90, 4.3.2]. □

12.1.3 Differential coherence of the local cohomology with support in a smooth closed subscheme of the constant coefficient

Notation 12.1.3.1. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathfrak{S}$. Let Z be a closed subscheme of X and $j_Z \colon X \setminus Z \to X$ be the open immersion. We set

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} := \mathbb{R}\mathrm{sp}_* j_Z^\dagger(\mathcal{O}_{\mathfrak{X}_K}), \qquad \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} := \mathbb{R}\mathrm{sp}_* \underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathfrak{X}_K}).$$

By definition, $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ is the local cohomology with support in Z of the constant coefficient $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. Remark that when Z is moreover a divisor of X, then we retrieve the definition 8.7.3.5.2. The exact sequence $0 \to \underline{\Gamma}^\dagger_Z(\mathcal{O}_{\mathfrak{X}_K}) \to \mathcal{O}_{\mathfrak{X}_K} \to j_Z^\dagger(\mathcal{O}_{\mathfrak{X}_K}) \to 0$ induces the exact triangle

$$\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \to \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}[1]. \tag{12.1.3.1.1}$$

For any integer $i \in \mathbb{Z}$, we set $\mathcal{H}^{\dagger i}_Z(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) := H^i \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$.

Notation 12.1.3.2 (Cech complexes with divisors for the constant coefficient)**.** Let $\mathfrak{X}$ be a smooth formal scheme. Let $\mathscr{T} := (T_i)_{i\in I}$ be a finite set of divisors of X and $Z = \cap_{i\in I} T_i$.

For each $h \in I$, for any $(i_0, \ldots, i_h) \in I$, put $T_{i_0,\ldots,i_h} := T_{i_0} \cup \cdots \cup T_{i_h}$. For each $h \in I$, set

$$\check{C}^{\dagger h}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}) := \prod_{(i_0,i_1,\ldots,i_h)\in I^{1+h}} \mathcal{O}_{\mathfrak{X}}({}^\dagger T_{i_0\ldots i_h})_{\mathbb{Q}}. \tag{12.1.3.2.1}$$

Let $\alpha \in \check{C}^{\dagger h}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}})$. For any $h \in I$, for any $(i_0, \dots, i_h) \in I$, we denote by $\alpha_{i_0,\dots,i_h}$ the coefficient of α in $\mathcal{O}_{\mathfrak{X}}({}^\dagger T_{i_0 \dots i_h})_{\mathbb{Q}}$.

We define the coboundary map $d\colon \check{C}^{\dagger h}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \to \check{C}^{\dagger h+1}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}})$ by setting

$$(d\alpha)_{i_0,\dots,i_{h+1}} := \sum_{j=0}^{h+1} (-1)^j \alpha_{i_0,\dots,\widehat{i_j},\dots,i_{h+1}}.$$

This yields the complex

$$\cdots 0 \to \check{C}^{\dagger 0}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \to \check{C}^{\dagger 1}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \to \cdots \to \check{C}^{\dagger h}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \to 0 \cdots$$

that we will denote by $\check{C}^{\dagger \bullet}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}})$.

Let $Y_i := X \setminus T_i$ the open subscheme of X and $Y := \cup_{i \in I} Y_i$. We get the finite open covering $\mathscr{Y} := (Y_i)_{i=1,\dots,r}$ of Y (i.e. $Y = X \setminus Z$). Since $\mathrm{sp}_*(j^\dagger_{Y_i} \mathcal{O}_{\mathfrak{X}_K}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^\dagger T_i)_{\mathbb{Q}}$, then

$$\mathrm{sp}_* \check{C}^{\dagger \bullet}(\mathfrak{X}, \mathscr{Y}, \mathcal{O}_{\mathfrak{X}_K}) \xrightarrow{\sim} \check{C}^{\dagger \bullet}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}), \tag{12.1.3.2.2}$$

where $\check{C}^{\dagger \bullet}(\mathfrak{X}, \mathscr{Y}, \mathcal{O}_{\mathfrak{X}_K})$ is defined in 10.1.3.5.

Notation 12.1.3.3. We keep notation 12.1.1.3 and for any $i \in I$ we denote by $\mathscr{T}_i := (T_{i\, j_i})_{j_i \in J_i}$ the finite set of divisors of X_i. For any $h, l \in \mathbb{N}$, for any $\underline{i} = (i_0, \dots, i_h) \in I^{1+h}$, we denote by $\mathscr{T}_{\underline{i}} := (T_{\underline{i}\, \underline{\underline{j}}})_{\underline{\underline{j}} \in (J_{\underline{i}})^{1+l}}$ the finite set of divisors of $X_{\underline{i}}$. For any $h, l \in \mathbb{N}$, we get the left $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module

$$\check{C}^{\dagger hl}(\mathscr{X}, (\mathscr{T}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}}) := \prod_{\underline{i} \in I^{1+h}} u_{\underline{i}*} \check{C}^{\dagger l}(\mathfrak{X}_{\underline{i}}, \mathscr{T}_{\underline{i}}, \mathcal{O}_{\mathfrak{X}_{\underline{i}},\mathbb{Q}}) \overset{12.1.3.2.1}{=} \prod_{\underline{i} \in I^{1+h}} u_{\underline{i}K*} \left(\prod_{\underline{\underline{j}} \in J^{1+l}_{\underline{i}}} \mathcal{O}_{\mathfrak{X}_{\underline{i}}}({}^\dagger T_{\underline{i}\,\underline{\underline{j}}})_{\mathbb{Q}} \right).$$

Similarly to 10.1.4, we get the bicomplex $\check{C}^{\dagger \bullet\bullet}(\mathscr{X}, (\mathscr{T}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}})$. We denote by $\check{C}^{\dagger \bullet}(\mathscr{X}, (\mathscr{T}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}})$ its total complex.

Proposition 12.1.3.4. *With notation 12.1.3.3, we have the isomorphism:*

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} = \mathbb{R}\mathrm{sp}_*(j^\dagger \mathcal{O}_{\mathfrak{X}_K}) \xrightarrow{\sim} \check{C}^{\dagger \bullet}(\mathscr{X}, (\mathscr{T}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}}).$$

Proof. Since $\mathrm{sp}_* u_{\underline{i}K*} \xrightarrow{\sim} u_{\underline{i}*} \mathrm{sp}_*$, then we get

$$\begin{aligned} \mathrm{sp}_* \check{C}^{\dagger hl}(\mathscr{X}, (\mathscr{Y}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}_K}) &\xrightarrow{\sim} \prod_{\underline{i} \in I^{1+h}} \mathrm{sp}_* u_{\underline{i}K*} \check{C}^{\dagger l}(\mathfrak{X}_{\underline{i}}, \mathscr{Y}_{\underline{i}}, \mathcal{O}_{\mathfrak{X}_{\underline{i}K}}) \xrightarrow{\sim} \prod_{\underline{i} \in I^{1+h}} u_{\underline{i}*} \mathrm{sp}_* \check{C}^{\dagger l}(\mathfrak{X}_{\underline{i}}, \mathscr{Y}_{\underline{i}}, \mathcal{O}_{\mathfrak{X}_{\underline{i}K}}) \\ &\underset{12.1.3.2.2}{\xrightarrow{\sim}} \prod_{\underline{i} \in I^{1+h}} u_{\underline{i}*} \check{C}^{\dagger l}(\mathfrak{X}_{\underline{i}}, \mathscr{T}_{\underline{i}}, \mathcal{O}_{\mathfrak{X}_{\underline{i}},\mathbb{Q}}) = \check{C}^{\dagger hl}(\mathscr{X}, (\mathscr{T}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}}). \end{aligned} \tag{12.1.3.4.1}$$

Since these isomorphisms commute with the transition maps, this yields the last isomorphism

$$\mathbb{R}\mathrm{sp}_*(j^\dagger \mathcal{O}_{\mathfrak{X}_K}) \overset{10.1.4.2}{\xrightarrow{\sim}} \mathrm{sp}_* \check{C}^{\dagger \bullet\bullet}(\mathscr{X}, (\mathscr{Y}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}_K}) \xrightarrow{\sim} \check{C}^{\dagger \bullet\bullet}(\mathscr{X}, (\mathscr{T}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}}).$$

□

Example 12.1.3.5. Within notation 12.1.1.3 suppose I has only one element and remove i in the notation, i.e. suppose $\mathfrak{X}$ is an affine smooth $\mathcal{V}$-formal scheme, $\mathscr{T} := (T_j)_{j \in J}$ is a finite set of divisors of X such that there exists $f_j \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ satisfying $T_j = V(f_j)$. By setting $Z = \cap_{j \in J} T_j$, we get

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \xrightarrow{\sim} \check{C}^{\dagger \bullet}(\mathfrak{X}, \mathscr{T}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}}).$$

Proposition 12.1.3.6. *Let $\mathfrak{X}$ be a smooth formal scheme over $\mathfrak{S}$. Let $u\colon Z \to X$ be a closed immersion of k-smooth schemes purely of codimension r.*

(a) We have $\mathcal{H}^{\dagger i}_Z(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) = 0$ for any $i \neq r$.

(b) *Let $x \in X$. Choose an affine open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}$ containing x such that there exist coordinates $t_1, \dots, t_d \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})$ such that $Z \cap U = V(\bar{t}_1, \dots, \bar{t}_r)$ where $r \leq d$ and $\bar{t}_1, \dots, \bar{t}_r$ are the image of $t_1, \dots, t_r$ in $\Gamma(U, \mathcal{O}_U)$. We have the exact sequence*

$$(\mathcal{D}^\dagger_{\mathfrak{U}/\mathfrak{S},\mathbb{Q}})^d \xrightarrow{\vartheta} \mathcal{D}^\dagger_{\mathfrak{U}/\mathfrak{S},\mathbb{Q}} \xrightarrow{\phi} \mathcal{H}^{\dagger r}_{Z\cap U}(\mathcal{O}_{\mathfrak{U},\mathbb{Q}}) \to 0, \tag{12.1.3.6.1}$$

where $\phi(P) = P \cdot (1/t_1 \cdots t_r)$, and ϑ is defined by

$$\vartheta(P_1, \dots, P_d) = \sum_{i=1}^{r} P_i t_i + \sum_{i=r+1}^{d} P_i \partial_i. \tag{12.1.3.6.2}$$

(c) *The complexes $\mathcal{O}_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}, \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ are in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$.*

Proof. See [Ber90, 4.3.4]. □

Remark 12.1.3.7. With the notation 12.1.3.6, suppose, $\mathfrak{U} = \mathfrak{X}$. For $i = 1, \dots, r$, put $Z_i := V(\bar{t}_i)$, and $Z_{i_0,\dots,i_k} := Z_{i_0} \cup \dots \cup Z_{i_k}$ (i.e. $V(\bar{t}_{i_0} \cdots \bar{t}_{i_k}) = Z_{i_0,\dots,i_k}$). Then it follows from 12.1.3.5 that $\mathcal{O}_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$ is represented by the (ordered version) Cech complex

$$\prod_{i=1}^{d} \mathcal{O}_{\mathfrak{X}}(^\dagger Z_i)_{\mathbb{Q}} \to \prod_{i_0<i_1} \mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0 i_1})_{\mathbb{Q}} \to \dots \to \mathcal{O}_{\mathfrak{X}}(^\dagger Z_{1\dots r})_{\mathbb{Q}} \to 0, \tag{12.1.3.7.1}$$

whose first term is at degree 0. This yields that $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ is represented by the complex

$$\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \prod_{i=1}^{d} \mathcal{O}_{\mathfrak{X}}(^\dagger Z_i)_{\mathbb{Q}} \to \prod_{i_0<i_1} \mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0 i_1})_{\mathbb{Q}} \to \dots \to \mathcal{O}_{\mathfrak{X}}(^\dagger Z_{1\dots r})_{\mathbb{Q}} \to 0, \tag{12.1.3.7.2}$$

whose first term is at degree 0. Using 12.1.2.3, this is how Berthelot checked in [Ber90, 4.3.4] that $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$.

Corollary 12.1.3.8. *Let $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ be a closed immersion of smooth formal schemes over $\mathfrak{S}$.*

(a) *We have $u^!(\mathcal{O}_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}) = 0$, i.e. by applying the functor $u^!$ to the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$, we get an isomorphism.*

(b) *We have the isomorphisms $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \xrightarrow{\sim} u_+ u^!(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \xrightarrow{\sim} u_+(\mathcal{O}_{\mathfrak{Z},\mathbb{Q}})[d_u]$.*

Proof. 0) Since this is local in $\mathfrak{X}$ and $\mathfrak{Z}$, we can suppose u ispurely of codimension r. Let $\mathfrak{Y}$ be the open formal subscheme of $\mathfrak{X}$ complementary to Z. First we check that $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}|\mathfrak{Y} = 0$. Since this is local, we can suppose with the notation of Proposition 12.1.3.6 that $\mathfrak{U} = \mathfrak{X}$. In that case, we compute that the restriction to $\mathfrak{Y}$ of the map ϑ of the exact sequence 12.1.3.6.1 is surjective. Hence, $\mathcal{H}^{\dagger r}_Z(\mathcal{O}_{\mathfrak{X},\mathbb{Q}})|\mathfrak{Y} = 0$ (or we can also compute the rth cohomological space of the complex 12.1.3.7.2).

a) Now, let us check that $u^!(\mathcal{O}_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}) = 0$. Since this local, we can suppose with the notation of Proposition 12.1.3.6 that $\mathfrak{U} = \mathfrak{X}$, and we use notation 12.1.3.7. Using the resolution 12.1.3.7.1 of $\mathcal{O}_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}}$, we reduce to check $\mathbb{L}u^*(\mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0\dots i_k})_{\mathbb{Q}}) = 0$. Let $u_i\colon \mathfrak{Z}_i \hookrightarrow \mathfrak{X}$ be the closed immersion of formal schemes whose corresponding ideal is generated by t_i. If we choose $i \in \{i_0, \dots, i_k\}$, then the multiplication by $t_i\colon \mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0\dots i_k})_{\mathbb{Q}} \xrightarrow{t_i} \mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0\dots i_k})_{\mathbb{Q}}$ is an isomorphism. Hence, $\mathbb{L}u_i^*(\mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0\dots i_k})_{\mathbb{Q}}) = 0$. This yields $\mathbb{L}u^*(\mathcal{O}_{\mathfrak{X}}(^\dagger Z_{i_0\dots i_k})_{\mathbb{Q}}) = 0$.

b) Hence, by applying the functor $u^!$ to the exact triangle 12.1.3.1.1, using part b) of the proof, we get the isomorphism $u^! \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \xrightarrow{\sim} u^! \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. Hence, $u_+ u^! \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \xrightarrow{\sim} u_+ u^! \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. Since $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ has his support in Z, we get from Theorem 9.3.5.9 the isomorphism $\mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \xrightarrow{\sim} u_+ u^! \mathbb{R}\underline{\Gamma}^\dagger_Z \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. Hence, we are done by composition. □

12.1.4 Local cohomology with support in a smooth closed subscheme for (quasi-)coherent complexes of $\mathcal{D}$-modules

Notation 12.1.4.1. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathfrak{S}$. Let Z be a k-smooth closed subscheme of X. Let $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{O}^{(\bullet)}_{\mathfrak{X}}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ such that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{O}^{(\bullet)}_{\mathfrak{X}}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$, where this latter complex is defined at 12.1.3.1 (and is coherent thanks to 12.1.3.6). Then we can define the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ by setting for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)}) := \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{O}^{(\bullet)}_{\mathfrak{X}})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)}. \tag{12.1.4.1.1}$$

Let T be a divisor of X. Following 8.4.1.15, we have the equivalence of categories: $\underrightarrow{l}^{*}_{X,\mathbb{Q}} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(T) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$. Then we get the functor

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}) \tag{12.1.4.1.2}$$

so that for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(T))$ we get by definition the isomorphism:

$$\underrightarrow{l}^{*}_{X,\mathbb{Q}}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\underrightarrow{l}^{*}_{X,\mathbb{Q}}(\mathcal{E}^{(\bullet)})).$$

By using the map $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$ (see 12.1.3.1.1), we get the arrow $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{O}^{(\bullet)}_{\mathfrak{X}}) \to \mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$. This yields for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$ the morphism of $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) \to \mathcal{E}. \tag{12.1.4.1.3}$$

Remark 12.1.4.2. Beware that in our work first we do need to use the functor 12.1.4.1.2 and the arrow 12.1.4.1.3 in order to prove the $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-coherence of $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ when Z is a divisor (more precisely, see the proof of 12.2.7.1). This coherence result is fundamental in order to be able to define local cohomology with support in any closed subscheme in the context of quasi-coherent complexes of the next chapter. We will see via 13.1.3.10 that both local cohomologies coincide, which justifies adopting the notation 12.1.3.1 and 12.1.4.1.

Lemma 12.1.4.3. *Let $u \colon \mathfrak{Z} \to \mathfrak{X}$ be a closed immersion of smooth formal $\mathfrak{S}$-schemes. For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$, we have the isomorphism*

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}), \tag{12.1.4.3.1}$$

where by abuse of notation we denote $u(Z)$ by Z.

Proof. Using 9.4.3.2.1 and 12.1.4.1.1, we reduce to the case where $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{X}}$. Then the Lemma follows from 12.1.3.8. □

12.2 Arithmetic $\mathcal{D}$-modules associated with overconvergent isocrystals on completely smooth d-frames

12.2.1 The categories of isocrystals $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$, $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$ on completely smooth d-frames and their stability

Definition 12.2.1.1. We define the category of d-frames as follows:

(a) A "d-frame over $\mathcal{V}$" is the data of a separated and smooth $\mathcal{V}$-formal scheme $\mathfrak{P}$, of a closed subscheme X of P, of a divisor T of P such that $Y = X \setminus T$. Such a d-frame over $\mathcal{V}$ is denoted by $(Y, X, \mathfrak{P}, T)/\mathcal{V}$. We say that $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ is d-frame over $\mathcal{V}$ enclosing (Y, X). A d-frame $(Y, X, \mathfrak{P}, T)$ is "completely smooth" (resp. "smooth") if X (resp. Y) is k-smooth and if $T \cap X$ is a divisor of X (resp. and $T \cap Y$ is a divisor of Y).

(b) A morphism $\theta\colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ of (resp. completely smooth, resp. smooth) d-frames (over $\mathcal{V}$) is the data of a morphism $f\colon \mathfrak{P}' \to \mathfrak{P}$ of $\mathfrak{S}$-formal schemes such that such that $X' \subset f^{-1}X$ and $T' \supset f^{-1}(T)$. When $T' = f^{-1}(T)$, we say that the morphism is *strict*. If $a\colon X' \to X$, $b\colon Y' \to Y$ are the morphisms induced by f, the morphism θ is also denoted by (b, a, f).

Notation 12.2.1.2. Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a d-frame over $\mathcal{V}$. Then, we can simply write it $(X, \mathfrak{P}, T)/\mathcal{V}$ and even $(X, \mathfrak{P}, T)$ if $\mathcal{V}$ is understood.

Remark 12.2.1.3. We will introduce later the notion of frames (see 16.2.1.8) which extends that of d-frames of 12.2.1.1.

Notation 12.2.1.4. Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a completely smooth d-frame. We denote by $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ the full subcategory of $\mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V})$ (see notation 9.3.7.4) whose objects $\mathcal{E}$ satisfy the following condition: for any affine open formal subscheme $\mathfrak{P}'$ of $\mathfrak{P}$, for any morphism of formal schemes $v\colon \mathfrak{X}' \hookrightarrow \mathfrak{P}'$ whose reduction modulo π is the closed imbedding $X \cap P' \hookrightarrow P'$, the sheaf $v^!(\mathcal{E}|\mathfrak{P}')$ is $\mathcal{O}_{\mathfrak{X}'}({}^\dagger T \cap X')_{\mathbb{Q}}$-coherent. When T is empty, we remove it in the notation. When T is empty, we remove it in the notation. When $X = P$, we get $\mathrm{MIC}^{\dagger\dagger}(P, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$ (see notation 11.2.1.4).

Proposition 12.2.1.5. *Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a completely smooth d-frame. Let $\mathcal{E} \in \mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V})$. The following properties are equivalent:*

(a) $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$;

(b) $\mathcal{E}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$.

Proof. This follows from 11.2.1.14.(e). □

Notation 12.2.1.6. Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a completely smooth d-frame. We denote by $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$ the full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ consisting of objects $\mathcal{E}^{(\bullet)}$ with support in X and such that $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ where $\underrightarrow{l}^*_{\mathbb{Q}}$ is the equivalence of categories of 8.4.5.6, and where $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ is defined in 12.2.1.4. By definition, we get the equivalence of categories

$$\underrightarrow{l}^*_{\mathbb{Q}}\colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \cong \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}). \tag{12.2.1.6.1}$$

When T is empty, we remove it in the notation. When $X = P$, we get $\mathrm{MIC}^{(\bullet)}(P, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{(\bullet)}(\mathfrak{P}, T/\mathcal{V})$ (see notation 11.2.2.1).

12.2.1.7 (Berthelot-Kashiwara isocrystal variation)**.** Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a completely smooth d-frame such that $X \hookrightarrow \mathfrak{P}$ lifts to a morphism $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ of smooth $\mathcal{V}$-formal schemes. It follows from 9.3.5.9 (resp. 9.3.5.13) that the functors u_+ and $u^!$ (resp. $u_+^{(\bullet)}$ and $u^{(\bullet)!}$) induce quasi-inverse equivalences of categories between $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T \cap X/\mathcal{V})$ (resp. $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$ and $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T \cap X/\mathcal{V})$).

Since $\mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T \cap X/\mathcal{V}) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(T \cap X))$ (see 11.2.2.2) then it follows from 9.4.2.6 that

$$\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T)). \tag{12.2.1.7.1}$$

Proposition 12.2.1.8. *For $i = 1, 2$, let $(Y_i, X_i, \mathfrak{P}_i, T_i)/\mathcal{V}$ be a completely smooth d-frame, let $\mathcal{E}_i^{(\bullet)}$ be an object of $\mathrm{MIC}^{(\bullet)}(X_i, \mathfrak{P}_i, T_i/\mathcal{V})$. Let $\mathfrak{P} := \mathfrak{P}_1 \times_{\mathfrak{S}} \mathfrak{P}_2$, $p_i\colon \mathfrak{P} \to \mathfrak{P}_i$ be the natural projection for $i = 1, 2$, $X := X_1 \times_k X_2$, $T := p_1^{-1}(T_1) \cup p_2^{-1}(T_2)$. Then $\mathcal{E}_1^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{E}_2^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$.*

Proof. Following Lemma 9.2.5.9 (see also 9.2.5.3), we already know $\mathcal{E}_1^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{E}_2^{(\bullet)} \xrightarrow{\sim} \mathcal{E}_1^{(\bullet)} \widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}}} \mathcal{E}_2^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. Since the proposition is local, by using 12.2.1.7 and 9.4.4.1, we reduce to the case where $X = P$ and $Y = Q$. Hence, this follows from 11.2.4.4. □

Proposition 12.2.1.9. Let $\theta := (b,\, a,\, f)\colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ be a morphism of completely smooth d-frames (see 12.2.1.1) such that f is smooth.

(i) For any $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$, we have $\theta^{(\bullet)*} := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}_{T',T} \mathcal{E}^{(\bullet)}[-d_{X'/X}] \in \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V})$. We get the functor:

$$\theta^{(\bullet)*} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}', T'/\mathcal{V}) \qquad (12.2.1.9.1)$$

(ii) If a and b are the identities, then the functor $\theta^{(\bullet)*}$ of 12.2.1.9.1 is an equivalence of categories.

Proof. (i) The fact that $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}\mathcal{E}^{(\bullet)}[-d_{X'/X}] \in \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V})$ is local on $\mathfrak{P}'$. Hence, we can suppose there exists a closed immersion of smooth formal $\mathfrak{S}$-schemes of the form $u \colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ (resp. $u' \colon \mathfrak{X}' \hookrightarrow \mathfrak{P}'$, resp. $a \colon \mathfrak{X}' \to \mathfrak{X}$) which reduces modulo π to the given morphism $X \hookrightarrow P$ (resp. $X' \hookrightarrow P'$, resp. a). Hence, it follows from 9.2.2.1 that we have the glueing isomorphism

$$\tau \colon u'^{(\bullet)!} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} a^{(\bullet)!} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}). \qquad (12.2.1.9.2)$$

This yields

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \underset{12.1.4.3.1}{\xrightarrow{\sim}} u'^{(\bullet)}_{+} \circ u'^{(\bullet)!} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \underset{12.2.1.9.2}{\xrightarrow{\sim}} u'^{(\bullet)}_{+} \circ a^{(\bullet)!} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}). \qquad (12.2.1.9.3)$$

Since $u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T \cap X/\mathcal{V})$ (see 12.2.1.7), then $\mathbb{L}a^{*(\bullet)} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}', T' \cap X'/\mathcal{V})$ (see 11.2.3.5.1)). Since $\mathbb{L}a^{*(\bullet)} = a^{(\bullet)!}[-d_{X'/X}]$, we get the first statement (again, use 12.2.1.7).

(ii) The fact that this is an equivalence of categories is local in $\mathfrak{P}$. Hence, we can suppose that $\mathfrak{P}$ is affine, in which case there exists $u \colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ (resp. $u' \colon \mathfrak{X} \hookrightarrow \mathfrak{P}'$) which reduces modulo π to the given morphism $X \hookrightarrow P$ (resp. $X \hookrightarrow P'$). Following 12.2.1.9.3, we get

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u'^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}).$$

We conclude by using Berthelot-Kashiwara theorem of the form 12.2.1.7. □

12.2.1.10 (Independance of the d-frame enclosing (Y, X))**.** Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a completely smooth d-frame over $\mathcal{V}$. Then the category $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$ do not depend on the completely smooth d-frame over $\mathcal{V}$ enclosing (Y, X). Indeed let $(Y, X, \mathfrak{P}', T')/\mathcal{V}$ be a second choice of completely smooth d-frame enclosing (Y, X). Let $\mathfrak{P}'' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{P}'$, $p_0 \colon \mathfrak{P}'' \to \mathfrak{P}$ and $p_1 \colon \mathfrak{P}'' \to \mathfrak{P}'$ be the projections and $T'' := p_0^{-1}(T) \cup p_1^{-1}(T')$. We get the morphisms of d-frames $\theta_0 := (\mathrm{id}, \mathrm{id}, p_0) \colon (Y, X, \mathfrak{P}'', T'')/\mathcal{V} \to (Y, X, \mathfrak{P}, T)/\mathcal{V}$ and $\theta_1 := (\mathrm{id}, \mathrm{id}, p_1) \colon (Y, X, \mathfrak{P}'', T'')/\mathcal{V} \to (Y, X, \mathfrak{P}', T')/\mathcal{V}$. Since $\theta_0^{(\bullet)*}$ and $\theta_1^{(\bullet)*}$ are equivalence of categories, then we are done.

Corollary 12.2.1.11. *Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ and $(Y', X', \mathfrak{P}', T')/\mathcal{V}$ be two completely smooth d-frames. Suppose we have the following commutative diagram:*

$$\begin{array}{ccccccc} Y' & \xrightarrow{j'} & X' & \xrightarrow{i'} & P' & \longrightarrow & \mathfrak{P}' \\ \downarrow h & & \downarrow g & & \downarrow f_0 & & \\ Y & \xrightarrow{j} & X & \xrightarrow{i} & P & \longrightarrow & \mathfrak{P}, \end{array} \qquad (12.2.1.11.1)$$

where j and j' are the underlying open embeddings, i and i' are the underlying closed embeddings. Set $\theta := (h,\, g,\, f_0)$.

(a) We get a natural functor

$$\theta^{(\bullet)*} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V}) \qquad (12.2.1.11.2)$$

satisfying the condition (b):

(b) Suppose f_0 has a lifting $f \colon \mathfrak{P}' \to \mathfrak{P}$. The functor 12.2.1.11.2 is equal (up to equivalence of categories) to the functor $(h,\, g,\, f)^{(\bullet)*}$ of 12.2.1.9.1. In particular, the functor $(h,\, g,\, f)^{(\bullet)*}$ does not depend (up to equivalence of categories) on the lifting f of f_0.

Proof. Let $\mathfrak{P}'' := \mathfrak{P}' \times_{\mathfrak{S}} \mathfrak{P}$, $p_0 \colon \mathfrak{P}'' \to \mathfrak{P}'$ and $p_1 \colon \mathfrak{P}'' \to \mathfrak{P}$ be the projections and $T'' := p_0^{-1}(T) \cup p_1^{-1}(T)$. Since $\mathcal{P}$ is separated then the morphism $i'' := (i', i \circ g) \colon X' \to \mathcal{P}''$ is a closed immersion. We get the morphisms of d-frames $\theta_0 := (\mathrm{id}, \mathrm{id}, p_0) \colon (Y', X', \mathfrak{P}'', T'')/\mathcal{V} \to (Y', X', \mathfrak{P}', T')/\mathcal{V}$ and $\theta_1 :=$

$(h,g,p_1)\colon (Y',X',\mathfrak{P}'',T'') \to (Y,X,\mathfrak{P},T)/\mathcal{V}$. Since $\theta_0^{(\bullet)*}$ is an equivalence of categories, then we get the unique (up to equivalence of categories) functor $\theta^{(\bullet)*}$ so that

$$\theta_0^{(\bullet)*} \circ \theta^{(\bullet)*} \cong \theta_1^{(\bullet)*}.$$

□

Remark 12.2.1.12. With notation 12.2.1.11, Berthelot constructed in the same way (see [Ber96b, 2.3.2.(iv)]) the inverse image functor:

$$(h,\, g,\, f_0)^*\colon \mathrm{MIC}^\dagger(Y,X,\mathfrak{P}/\mathcal{V}) \to \mathrm{MIC}^\dagger(Y',X',\mathfrak{P}'/\mathcal{V}). \qquad (12.2.1.12.1)$$

In [Car09a, 2.3.1], we have constructed a functor of the form 12.2.1.12.1 in another way by glueing local data (which allows us to reduce to the case where f_0 has a lifting). We checked that both functors coincide.

Proposition 12.2.1.13. Let $(Y,X,\mathfrak{P},T)/\mathcal{V}$ be a completely smooth d-frame over $\mathcal{V}$. Let $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ be two objects of $\mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V})$.

(a) We have $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}[d_{X/P}] \in \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V})$, i.e. for any $j \neq d_{X/P}$, $\mathcal{H}^j(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) = 0$ and $\mathcal{H}^{d_{X/P}}(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V})$.

(b) The functor $\mathbb{D}^{(\bullet)}_T$ of 9.2.4.20 induces the involution:

$$\mathbb{D}^{(\bullet)}_T\colon \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V}). \qquad (12.2.1.13.1)$$

Proof. a) Let $\mathfrak{R} := \mathfrak{P}\times_{\mathfrak{S}}\mathfrak{P}$, $p_0,p_1\colon \mathfrak{Q} \to \mathfrak{P}$ be respectively the natural left and right projection, $Q := X\times_k X$, $D := p_0^{-1}(T)\cup p_1^{-1}(T)$. Following 12.2.1.8, $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(Q,\mathfrak{R},D/\mathcal{V})$. By using 12.2.1.9 in the case of the diagonal morphism $(X,\mathfrak{P},T/\mathcal{V}) \to (Q,\mathfrak{R},D/\mathcal{V})$, we get (a) from 9.2.5.15.1.

b) Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V})$. Following 12.2.1.7.1, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ and then $\mathbb{D}^{(\bullet)}_T(\mathcal{E}^{(\bullet)})$ is well defined. We reduce to check that $\underrightarrow{l}^*_{\mathbb{Q}}\mathbb{D}^{(\bullet)}_T(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V})$, where $\underrightarrow{l}^*_{\mathbb{Q}}$ is the equivalence of categories of 8.4.5.6. Since this is local, we reduce to the case where there exists a morphism of smooth $\mathfrak{S}$-formal schemes $\mathfrak{X} \to \mathfrak{P}$ which is a lifting of $X \to \mathfrak{P}$. This is then the consequence of 11.2.6.3.4 and of the relative duality isomorphism 9.4.5.2.1. □

12.2.1.14. Via the equivalence of categories 12.2.1.6.1, the results of 12.2.1.8, 12.2.1.11 and 12.2.1.13 are still valid by replacing $\mathrm{MIC}^{(\bullet)}$ with $\mathrm{MIC}^{\dagger\dagger}$. More precisely, with notation 12.2.1.11, we get the functors

$$\theta^*\colon \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(X',\mathfrak{P}',T'/\mathcal{V}), \qquad (12.2.1.14.1)$$

$$-\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}} - [d_{X/P}]\colon \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}) \times \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}), \qquad (12.2.1.14.2)$$

$$\mathbb{D}_{\mathfrak{P},T}\colon \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}). \qquad (12.2.1.14.3)$$

Moreover, with notation 12.2.1.8, we get the functor

$$-\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},T_1,T_2} -\colon \mathrm{MIC}^{\dagger\dagger}(X_1,\mathfrak{P}_1,T_1/\mathcal{V}) \times \mathrm{MIC}^{\dagger\dagger}(X_2,\mathfrak{P}_2,T_2/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V}). \qquad (12.2.1.14.4)$$

12.2.1.15. With notation 12.2.1.11, we can check that the inverse image functor 12.2.1.11.2 is equal to that of 9.2.2.4.3. In particular, when f_0 is the relative Frobenius, then this is equal to the functor 9.5.1.2.2. Since our objects are coherent then 9.5.1.1.3 via the isomorphisms 9.5.1.2.5 and 9.5.1.2.4, we get the functor $(F^s_{X_0/S_0})^*$ constructed at 8.8.2.4.1 is equal to the functor $(F^s_{Y_0/S_0}, F^s_{X_0/S_0}, F^s_{P_0/S_0})^*$ of 12.2.1.14.1.

12.2.2 Equivalence of categories via the functor sp_+

Let $\mathfrak{P}$ be a separated smooth formal scheme over $\mathfrak{S}$. Let $u_0 : X \to P$ be a closed immersion of smooth k-schemes. Let T be a divisor of P such that $Z := T \cap X$ is a divisor of X. We set $Y := X \setminus Z$. Choose $(\mathfrak{P}_\alpha)_{\alpha\in\Lambda}$ an affine open covering of $\mathfrak{P}$ and let us use the corresponding notation and lifting choices of 9.3.7 (which are compatible with that of 10.2.4.2).

Notation 12.2.2.1. We denote by $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$ the full subcategory of $\mathrm{Coh}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$ (see the notation of 9.3.7.2) whose objects $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ for all $\alpha \in \Lambda$, $\mathcal{E}_\alpha$ is $\mathcal{O}_{\mathfrak{X}_\alpha}({}^\dagger Z_\alpha)_{\mathbb{Q}}$-coherent.

Lemma 12.2.2.2. *With notation 10.2.4.3 and 12.2.2.1, we have the canonical functor*

$$\mathrm{sp}_* : \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K) \to \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}).$$

Proof. Let us sketch the construction (see the proof of [Car09a, 2.5.9.i)] for the details). Let us pick the object $((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$. Let $\theta_{\alpha\beta}$ be the isomorphism making commutative the diagram

$$\begin{array}{ccc} \mathrm{sp}_* p_{1K}^{\alpha\beta !}(E_\alpha) & \xrightarrow[\sim]{11.2.7.1.2} & p_1^{\alpha\beta !}\mathrm{sp}_*(E_\alpha) \\ {\scriptstyle \mathrm{sp}_*\eta_{\alpha\beta}}\uparrow\wr & & \uparrow{\scriptstyle \theta_{\alpha\beta}} \\ \mathrm{sp}_* p_{2K}^{\alpha\beta !}(E_\beta) & \xrightarrow[\sim]{11.2.7.1.2} & p_2^{\alpha\beta !}\mathrm{sp}_*(E_\beta). \end{array} \qquad (12.2.2.2.1)$$

Using 11.2.7.3, we can check that $\mathrm{sp}_*((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) := ((\mathrm{sp}_* E_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ is functorially an object of $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$. □

Lemma 12.2.2.3. *We have the canonical functor* $\mathrm{sp}^* : \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) \to \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$.

Proof. The construction is similar to that of 12.2.2.2 (see the proof of [Car09a, 2.5.9.i)] for the details). □

Proposition 12.2.2.4. *The functors* sp_* *and* sp^* *are quasi-inverse equivalences of categories between the categories* $\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$ *and* $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$.

Proof. Let us sketch the proof (see the proof of [Car09a, 2.5.9)] for the details). For any object $((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ of $\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$, we check that the adjunction isomorphisms $\mathrm{sp}^*\mathrm{sp}_*(E_\alpha) \xrightarrow{\sim} E_\alpha$ commute with glueing data. Similarly, for any object $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ of $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$ the adjunction isomorphisms $\mathcal{E}_\alpha \xrightarrow{\sim} \mathrm{sp}_*\mathrm{sp}^*(\mathcal{E}_\alpha)$ commute with glueing data. □

12.2.2.5. The functors $u_0^!$ and u_{0+} constructed in respectively 9.3.7.5 and 9.3.7.6 induce quasi-inverse equivalences of categories between $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ and $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$, i.e., we have the commutative diagram

$$\begin{array}{ccc} \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}) & \hookrightarrow & \mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V}) \\ \cong \uparrow u_{0+} \quad u_0^! \downarrow \cong & & \cong \uparrow u_{0+} \quad u_0^! \downarrow \cong \\ \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) & \hookrightarrow & \mathrm{Coh}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}). \end{array} \qquad (12.2.2.5.1)$$

Notation 12.2.2.6. We get the canonical equivalence of categories

$$\mathrm{sp}_{X\hookrightarrow\mathfrak{P}, T+} : \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) \cong \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}) \qquad (12.2.2.6.1)$$

by composition of the equivalences

$$\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) \underset{u_{0K}^*}{\overset{10.2.4.4.1}{\xrightarrow{\sim}}} \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K) \underset{\mathrm{sp}_*}{\xrightarrow{\sim}} \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) \underset{u_{0+}}{\xrightarrow{\sim}} \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}).$$

When there is no doubt on $(X, \mathfrak{P}, T/\mathcal{V})$, we simply write sp_+.

12.2.3 sp_+ of the constant coefficient

12.2.3.1. Let $\mathfrak{P}$ be a separated smooth formal scheme over $\mathfrak{S}$. Let $u_0\colon X \hookrightarrow P$ be a closed immersion of k-smooth schemes purely of codimension r. Let T be a divisor of P such that $Z := T \cap X$ is a divisor of X. We set $\mathfrak{U} := \mathfrak{P} \setminus T$, $Y := X \setminus Z$, $v_0\colon Y \to U$ be the morphism induced by u_0. Choose $(\mathfrak{P}_\alpha)_{\alpha\in\Lambda}$ an affine open covering of $\mathfrak{P}$ and let us use the corresponding notation and lifting choices of 9.3.7 (which are compatible with that of 10.2.4.2).

(a) Denoting by $f := (v_0, u_0, id)\colon (Y, X, \mathfrak{P}) \to (U, P, \mathfrak{P})$ the morphism of frames, we get the inverse image $f_K^* = |_{]X[_\mathfrak{P}}\colon \mathrm{MIC}^\dagger(U, P, \mathfrak{P}/K) \to \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ (see notation 10.2.2.8.1). Hence, we get the functor $u_{0K}^* \circ |_{]X[_\mathfrak{P}}\colon \mathrm{MIC}^\dagger(U, P, \mathfrak{P}/K) \to \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$ (where u_{0K}^* is the functor 10.2.4.4.1).

(b) Similarly to the construction of $u_0^!\colon \mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{Coh}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$ of 9.3.7.5, we define the functor
$$u_0^*\colon \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$$
as follows. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$, i.e. a coherent $\mathcal{D}^\dagger_\mathfrak{P}({}^\dagger T)_\mathbb{Q}$ which is also $\mathcal{O}_\mathfrak{P}({}^\dagger T)_\mathbb{Q}$-coherent. We set $\mathcal{E}_\alpha := u_\alpha^*(\mathcal{E}|\mathfrak{P}_\alpha) := H^r u_\alpha^!(\mathcal{E}|\mathfrak{P}_\alpha) \xrightarrow{\sim} u_\alpha^!(\mathcal{E}|\mathfrak{P}_\alpha)[r]$. Then it follows from 11.2.1.7 (and 11.2.1.14) that $\mathcal{E}_\alpha$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}_\alpha}({}^\dagger Z_\alpha)_\mathbb{Q}$-module, which is also $\mathcal{O}_{\mathfrak{X}_\alpha}({}^\dagger Z_\alpha)_\mathbb{Q}$-coherent. Via the isomorphisms of the form τ (9.2.2.3.1), we obtain the glueing $\mathcal{D}^\dagger_{\mathfrak{X}_{\alpha\beta}}({}^\dagger Z_{\alpha\beta})_\mathbb{Q}$-linear isomorphism $\theta_{\alpha\beta}\, p_2^{\alpha\beta !}(\mathcal{E}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta !}(\mathcal{E}_\alpha)$, satisfying the cocycle condition: $\theta_{13}^{\alpha\beta\gamma} = \theta_{12}^{\alpha\beta\gamma} \circ \theta_{23}^{\alpha\beta\gamma}$.

Beware we have three different functors u_0^*, $u_0^!$ and u_{0K}^* (defined at 10.2.4.4.1) whose notation are very similar.

Proposition 12.2.3.2. *With the notation 12.2.3.1, we have the canonical isomorphism*
$$\mathrm{sp}_* \circ u_{0K}^* \circ |_{]X[_\mathfrak{P}} \xrightarrow{\sim} u_0^* \circ \mathrm{sp}_*$$
of functors $\mathrm{MIC}^\dagger(U, P, \mathfrak{P}/K) \to \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$.

Proof. Using 11.2.7.3, we check that glueing data are compatible. □

Corollary 12.2.3.3. *Let $\mathfrak{P}$ be a separated smooth formal scheme over $\mathfrak{S}$. Let X be a closed smooth k-subvariety of P purely of codimension r. We have the isomorphism of* $\mathrm{Coh}(X, \mathfrak{P}/K)$ *of the form*
$$\mathrm{sp}_+(\mathcal{O}_{]X[_\mathfrak{P}}) \xrightarrow{\sim} \mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}.$$

Proof. 0) Choose $(\mathfrak{P}_\alpha)_{\alpha\in\Lambda}$ an affine open covering of $\mathfrak{P}$ and let us use the corresponding notation and lifting choices of 9.3.7 (which are compatible with that of 10.2.4.2). Choosing a finer covering if necessarily, we can suppose furthermore that there exist local coordinates $t_{\alpha 1}, \ldots, t_{\alpha d} \in \Gamma(\mathfrak{P}_\alpha, \mathcal{O}_\mathfrak{P})$ such that $\mathfrak{X}_\alpha = V(t_{\alpha 1}, \ldots, t_{\alpha r})$.

1) Following 12.1.3.6, $\mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}} \in \mathrm{Coh}(X, \mathfrak{P}/K)$ and $\mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}[r]$. By functoriality, we have the commutative diagram
$$\begin{array}{ccc}
p_2^{\alpha\beta !} u_\beta^!(\mathbb{R}\underline{\Gamma}^\dagger_{X_\beta} \mathcal{O}_{\mathfrak{P}_\beta,\mathbb{Q}})[-r] & \longrightarrow & p_2^{\alpha\beta !} u_\beta^!(\mathcal{O}_{\mathfrak{P}_\beta,\mathbb{Q}}[-r]) \\
\tau \downarrow \wr & & \tau \downarrow \wr \\
p_1^{\alpha\beta !} u_\alpha^!(\mathbb{R}\underline{\Gamma}^\dagger_{X_\alpha} \mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}}[-r]) & \longrightarrow & p_1^{\alpha\beta !} u_\alpha^!(\mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}}[-r]),
\end{array}$$
where the horizontal morphisms are induced by $\mathbb{R}\underline{\Gamma}^\dagger_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{P},\mathbb{Q}}$ and where the vertical isomorphisms are the canonical glueing ones (i.e. of the form 9.2.2.3.1). By applying $\mathcal{H}^0$, since the glueing isomorphisms induced by $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$ are the identity, we get the commutative diagram
$$\begin{array}{ccc}
p_2^{\alpha\beta !} u_\beta^!(\mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}|\mathfrak{P}_\beta) & \longrightarrow & \mathcal{O}_{\mathfrak{X}_{\alpha\beta},\mathbb{Q}} \\
\tau \downarrow \wr & & \| \\
p_1^{\alpha\beta !} u_\alpha^!(\mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}|\mathfrak{P}_\alpha) & \longrightarrow & \mathcal{O}_{\mathfrak{X}_{\alpha\beta},\mathbb{Q}},
\end{array} \tag{12.2.3.3.1}$$

where we omit indicating $\mathcal{H}^0$ to simplify notation. Set $\mathcal{E} := \mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}} \in \mathrm{Coh}(X,\mathfrak{P}/K)$, and $\mathcal{E}_\alpha := H^0 u^!_\alpha(\mathcal{E}|\mathfrak{P}_\alpha)$. We denote by $\theta_{\alpha\beta}$ $p_2^{\alpha\beta !}(\mathcal{E}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta !}(\mathcal{E}_\alpha)$, the glueing $\mathcal{D}^\dagger_{\mathfrak{X}_{\alpha\beta}}({}^\dagger Z_{\alpha\beta})_{\mathbb{Q}}$-linear isomorphism (which is equal to the left arrow of 12.2.3.3.1).

2) For $i = 1,\dots,r$, put $X_{\alpha i} := V(\bar{t}_{\alpha i})$, and $X_{\alpha i_0,\dots,i_k} := X_{\alpha i_0} \cup \dots \cup X_{\alpha i_k}$. Consider the diagram

$$\begin{array}{ccccccccccc}
\mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}} & \longrightarrow & \prod_{i=1}^d \mathcal{O}_{\mathfrak{P}_\alpha}({}^\dagger X_i)_{\mathbb{Q}} & \longrightarrow & \prod_{i_0<i_1} \mathcal{O}_{\mathfrak{P}_\alpha}({}^\dagger X_{i_0 i_1})_{\mathbb{Q}} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_{\mathfrak{P}_\alpha}({}^\dagger X_{1\dots r})_{\mathbb{Q}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0.
\end{array} \tag{12.2.3.3.2}$$

We denote by $\mathcal{F}^\bullet_\alpha$ the complex of the top of 12.2.3.3.2 such that $\mathcal{F}^0_\alpha = \mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}}$. Then, $\mathcal{F}^\bullet_\alpha[r]$ is a left resolution of $\mathcal{E}|\mathfrak{P}_\alpha$ by coherent $\mathcal{D}^\dagger_{\mathfrak{P}_\alpha}({}^\dagger T_\alpha)_{\mathbb{Q}}$-modules which are $\mathcal{O}_{\mathfrak{P}_\alpha}({}^\dagger T_\alpha)_{\mathbb{Q}}$-flat, and the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_{X_\alpha} \mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}} \to \mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}}$ is represented by $\mathcal{E}|\mathfrak{P}_\alpha[-r] \xleftarrow{\sim} \mathcal{F}^\bullet_\alpha \to \mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}}$, where the first arrow is a quasi-isomorphism and the second one corresponds to the vertical morphism of complexes of 12.2.3.3.2. Hence, since $H^0 u^!_\alpha(\mathcal{E}|\mathfrak{P}_\alpha) = H^r \mathbb{L} u^*_\alpha(\mathcal{E}|\mathfrak{P}_\alpha)$ (see 9.2.2.5), then we get the isomorphism $\mathcal{E}'_\alpha \xrightarrow{\sim} \mathcal{E}_\alpha$, where $\mathcal{E}'_\alpha := H^0 \mathbb{L} u^*_\alpha(\mathcal{F}^\bullet_\alpha)$. Then, we denote by $\theta'_{\alpha\beta}$ $p_2^{\alpha\beta !}(\mathcal{E}'_\beta) \xrightarrow{\sim} p_1^{\alpha\beta !}(\mathcal{E}'_\alpha)$, the glueing $\mathcal{D}^\dagger_{\mathfrak{X}_{\alpha\beta}}({}^\dagger Z_{\alpha\beta})_{\mathbb{Q}}$-linear isomorphism making commutative the diagram

$$\begin{array}{ccc}
p_2^{\alpha\beta !}(\mathcal{E}'_\beta) & \xrightarrow{\sim} & p_2^{\alpha\beta !}(\mathcal{E}_\beta) \\
{\scriptstyle \theta'_{\alpha\beta}} \downarrow \wr & & {\scriptstyle \theta_{\alpha\beta}} \downarrow \wr \\
p_1^{\alpha\beta !}(\mathcal{E}'_\alpha) & \xrightarrow{\sim} & p_1^{\alpha\beta !}(\mathcal{E}_\alpha).
\end{array} \tag{12.2.3.3.3}$$

Since $\theta_{\alpha\beta}$ satisfy the cocycle condition, then so are $\theta'_{\alpha\beta}$ (this is just a matter of writing some commutative cubes). Hence, $u^!_0(\mathcal{E})$ is isomorphic to $((\mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda})$.

3) For any $s \neq 0$, we have $u^*_\alpha(\mathcal{F}^s_\alpha) = 0$. Moreover, $\mathcal{E}'_\alpha = u^*_\alpha(\mathcal{F}^0_\alpha) = \mathcal{O}_{\mathfrak{X}_{\alpha\beta},\mathbb{Q}}$. Hence, by applying the functor $\mathbb{L}u^*_\alpha$ to $\mathcal{F}^\bullet_\alpha \to \mathcal{O}_{\mathfrak{P}_\alpha,\mathbb{Q}}$ we get the identity $\mathcal{O}_{\mathfrak{X}_\alpha,\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X}_\alpha,\mathbb{Q}}$. This yields that composing 12.2.3.3.1 with 12.2.3.3.3, we get a square whose morphisms are the identity of $\mathcal{O}_{\mathfrak{X}_{\alpha\beta},\mathbb{Q}}$. In particular $\theta'_{\alpha\beta}$ is the identity of $\mathcal{O}_{\mathfrak{X}_{\alpha\beta},\mathbb{Q}}$. By construction of the functor u^*_0 (see 12.2.3.1) this means that $((\mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda}) = u^*_0(\mathcal{O}_{\mathfrak{P},\mathbb{Q}})$.

4) Using 2) and 3), we get the canonical isomorphism

$$u^*_0(\mathcal{O}_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} u^!_0(\mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$$

of $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$. Using 12.2.3.2 and its notation, since $\mathrm{sp}_*(\mathcal{O}_{\mathfrak{P}_K}) = \mathcal{O}_{\mathfrak{P},\mathbb{Q}}$, then we get $\mathrm{sp}_+(\mathcal{O}_{]X[_\mathfrak{P}}) = u_{0+}\mathrm{sp}_* u^*_{0K}(\mathcal{O}_{]X[_\mathfrak{P}}) \underset{12.2.3.2}{\xrightarrow{\sim}} u_{0+}u^*_0 \mathrm{sp}_*(\mathcal{O}_{\mathfrak{P}_K}) = u_{0+}u^*_0(\mathcal{O}_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} u_{0+}u^!_0(\mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}) \underset{9.3.7.7}{\xrightarrow{\sim}} \mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}$. $\square$

12.2.4 Commutation of sp_+ with pullbacks, compatibility with Frobenius

Proposition 12.2.4.1. *Let $(Y,X,\mathfrak{P},T)/\mathcal{V}$ and $(Y',X',\mathfrak{P}',T')/\mathcal{V}$ be two completely smooth d-frames. Suppose we have the following commutative diagram:*

$$\begin{array}{ccccccc}
Y' & \xrightarrow{j'} & X' & \xrightarrow{i'} & P' & \longrightarrow & \mathfrak{P}' \\
\downarrow h & & \downarrow g & & \downarrow f_0 & & \\
Y & \xrightarrow{j} & X & \xrightarrow{i} & P & \longrightarrow & \mathfrak{P},
\end{array} \tag{12.2.4.1.1}$$

where j and j' are the underlying open embeddings, i and i' are the underlying closed embeddings. Set $\theta := (h, g, f_0)$. Let $E \in \mathrm{MIC}^\dagger(Y,X,\mathfrak{P}/K)$ (see notation 10.2.2.7). We have the isomorphism in $\mathrm{MIC}^{\dagger\dagger}(X',\mathfrak{P}',T'/\mathcal{V})$ (see notation 12.2.1.4):

$$\mathrm{sp}_+(\theta^*(E)) \xrightarrow{\sim} \theta^* \mathrm{sp}_+(E), \tag{12.2.4.1.2}$$

where θ^* are the inverse functors constructed respectively at 12.2.1.14.1 and 12.2.1.12.1.

Proof. 0) By construction of both of the functors of the form θ^*, we reduce to the case where there exists a *smooth* morphism $f\colon \mathfrak{P}' \to \mathfrak{P}$ of smooth $\mathcal{V}$-formal schemes whose reduction modulo π is f_0.

1) First suppose the squares of the diagram 12.2.4.1.1 are cartesian. Let $(\mathfrak{P}_\alpha)_{\alpha\in\Lambda}$ be an affine open covering of $\mathfrak{P}$. We fix some liftings of $X \cap P_\alpha$ etc. and we use notation 9.3.7. Moreover, we denote by $\mathfrak{P}'_\alpha := f^{-1}(\mathfrak{P}_\alpha)$, $\mathfrak{X}'_\alpha := \mathfrak{P}'_\alpha \times_{\mathfrak{P}_\alpha} \mathfrak{X}_\alpha$, $a_\alpha\colon \mathfrak{X}'_\alpha \to \mathfrak{X}_\alpha$ the projection, and similarly for other notations. If $((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ is an object of $\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$, we get canonically an object of $\mathrm{MIC}^\dagger(Y', (\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}/K)$ of the form $((a^*_{\alpha K}E_\alpha)_{\alpha\in\Lambda}, (\eta'_{\alpha\beta})_{\alpha,\beta\in\Lambda})$. This yields the functor $\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K) \to \mathrm{MIC}^\dagger(Y', (\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}/K)$ that we will denote by a^*_K. Similarly, we construct the functor $a^*\colon \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}, Z'/\mathcal{V})$. Consider the following diagram.

$$\begin{array}{ccc}
\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) & \xrightarrow{f^*_K} & \mathrm{MIC}^\dagger(Y', X', \mathfrak{P}'/K) \\
u^*_{0K}\big\downarrow 10.2.4.4 & & u'^*_{0K}\big\downarrow 10.2.4.4 \\
\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K) & \xrightarrow{a^*_K} & \mathrm{MIC}^\dagger(Y', (\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}/K) \\
12.2.2.3\ \big\uparrow \mathrm{sp}^* \quad \mathrm{sp}_* \big\downarrow\ 12.2.2.2 & & 12.2.2.3\ \big\uparrow \mathrm{sp}^* \quad \mathrm{sp}_* \big\downarrow\ 12.2.2.2 \\
\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) & \xrightarrow{a^*} & \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}, Z'/\mathcal{V}) \\
12.2.2.5.1\ \big\uparrow u^!_0 \quad u_{0+} \big\downarrow\ 12.2.2.5.1 & & 12.2.2.5.1\ \big\uparrow u'^!_0 \quad u'_{0+} \big\downarrow\ 12.2.2.5.1 \\
\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}) & \xrightarrow{f^*} & \mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/\mathcal{V}).
\end{array} \qquad (12.2.4.1.3)$$

By transitivity of the inverse image with respect to the composition, the top square is commutative up to canonical isomorphism. For the same reason, the middle square involving sp^* is commutative up to canonical isomorphism. Since sp_* and sp^* are canonically quasi-inverse equivalences of categories, this yields the middle square involving sp_* is commutative up to canonical isomorphism. Using similar arguments, we check the commutativity up to canonical isomorphism of the bottom square.

2) Now suppose $f = \mathrm{id}$ and a is a closed immersion and the left square of the diagram 12.2.4.1.1 is cartesian. Let $(\mathfrak{P}_\alpha)_{\alpha\in\Lambda}$ be an open covering of $\mathfrak{P}$ by affine subschemes. Then, we fix some liftings (separately) for both u and u' (for the later case, add some primes in notation) and we use notation 9.3.7. Then choose some lifting morphisms $a_\alpha\colon \mathfrak{X}'_\alpha \to \mathfrak{X}_\alpha$, and similarly for other notations. Consider the following diagram.

$$\begin{array}{ccc}
\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) & \xrightarrow{|_{]X'[_{\mathfrak{P}}}} & \mathrm{MIC}^\dagger(Y', X', \mathfrak{P}/K) \\
u^*_{0K}\big\downarrow 10.2.4.4 & & u'^*_{0K}\big\downarrow 10.2.4.4 \\
\mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K) & \xrightarrow{a^*_K} & \mathrm{MIC}^\dagger(Y', (\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}/K) \\
12.2.2.2\ \big\uparrow \mathrm{sp}^* \quad \mathrm{sp}_* \big\downarrow\ 12.2.2.2 & & 12.2.2.2\ \big\uparrow \mathrm{sp}^* \quad \mathrm{sp}_* \big\downarrow\ 12.2.2.2 \\
\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) & \xrightarrow{a^*} & \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}'_\alpha)_{\alpha\in\Lambda}, Z'/\mathcal{V}) \\
12.2.2.5.1\ \big\uparrow u^!_0 \quad u_{0+} \big\downarrow\ 12.2.2.5.1 & & 12.2.2.5.1\ \big\uparrow u'^!_0 \quad u'_{0+} \big\downarrow\ 12.2.2.5.1 \\
\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}) & \xrightarrow{\mathbb{R}\underline{\Gamma}^\dagger_{X'}[-d_{X'/X}]} & \mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}, T/\mathcal{V}).
\end{array} \qquad (12.2.4.1.4)$$

The commutativity up to canonical isomorphism of the top and middle squares of 12.2.4.1.4 is checked as for 12.2.4.1.3. It remains to look at the bottom square. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$. The canonical morphism

$$u'^!_\alpha\left(\mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E})|\mathfrak{P}_\alpha\right)[-d_{X'/X}] \to u'^!_\alpha(\mathcal{E}|\mathfrak{P}_\alpha)[-d_{X'/X}]$$

is an isomorphism. Moreover, $u'^!_\alpha(\mathcal{E}|\mathfrak{P}_\alpha)[-d_{X'/X}] \xrightarrow{\sim} a^!_\alpha u^!_\alpha(\mathcal{E}|\mathfrak{P}_\alpha)[-d_{X'/X}] \xrightarrow{\sim} a^*_\alpha\left(u^!_\alpha(\mathcal{E}|\mathfrak{P}_\alpha)\right)$. These isomorphisms glue, hence we get the commutativity up to canonical isomorphism of the bottom square.

3) The case where $f = \mathrm{id}$ and $a = \mathrm{id}$ is checked similarly (in the diagram 12.2.4.1.4, we replace $|_{]X'[_{\mathfrak{P}}}$ by $j'^\dagger$ and $\mathbb{R}\underline{\Gamma}^\dagger_{X'}[-d_{X'/X}]$ by $({}^\dagger T')$). This yields the general case by splitting the diagram 12.2.4.1.1. $\square$

Remark 12.2.4.2. The isomorphism $\mathrm{sp}_+(\mathcal{O}_{]X[_{\mathfrak{P}}}) \xrightarrow{\sim} \mathcal{H}^{\dagger,r}_X \mathcal{O}_{\mathfrak{P},\mathbb{Q}}$ of 12.2.3.3 can be viewed as a particular case of 12.2.4.1 (in the case $\theta\colon (X, X, \mathfrak{P}/\mathcal{V}) \to (P, \mathfrak{P}, /\mathcal{V})$ and for the constant coefficient). Notice that in order to give a meaning of the isomorphism 12.2.4.1.2, first we had to construct the local cohomological functor with support over a k-smooth closed subscheme In order to define our local cohomological functor in its general context (see 13.1), we do need the coherent theorem 12.2.7.1. But, the check of this latter theorem only uses the case where the closed subscheme is k-smooth, which explains why we have given a preliminary different construction of the local cohomological functor in this k-smooth context.

Notation 12.2.4.3. Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. Suppose there exists an automorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining. We keep notation 8.8.3.1. Let $\mathfrak{P}$ be a separated smooth formal scheme over $\mathfrak{S}$. Let $u_0\colon X \hookrightarrow P$ be a closed immersion of k-smooth schemes. Let T be a divisor of P such that $Z := T \cap X$ is a divisor of X.

(a) We denote by $F\text{-Coh}(X, \mathfrak{P}, T/\mathcal{V})$ the category of coherent $F\text{-}\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger Z)_{\mathbb{Q}}$-modules $(\mathcal{E}, \Phi)$ such that $\mathcal{E}$ has its support in X (see notation 8.8.3.2).

(b) We denote by $F\text{-MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ the full subcategory of $F\text{-Coh}(X, \mathfrak{P}, T/\mathcal{V})$ consisting of objects $(\mathcal{E}, \Phi)$ such that $\mathcal{E}$ is an object of $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$.

(c) When T is the empty divisor (or when $X = \mathcal{P}$), e.g. we simply write $F\text{-Coh}(X, \mathfrak{P}/\mathcal{V})$ or $F\text{-MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$.

Corollary 12.2.4.4. *We keep notation 12.2.4.3 and set* $Y := X \setminus T$.

(a) For any $E \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$*, we have the functorial in* E *isomorphism:*

$$\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\,T\,+}(F^*E) \xrightarrow{\sim} F^*\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\,T\,+}(E).$$

(b) The functor $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\,T\,+}$ *induces the equivalence of categories:*

$$\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\,T\,+}\colon F\text{-MIC}^\dagger(Y,\, X/K) \cong F\text{-MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}).$$

Proof. This follows from 12.2.4.1 and 12.2.1.15. □

12.2.5 Commutation of sp_+ with duality

We keep notation 12.2.2.

12.2.5.1. With notation 10.2.4.3, let $((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$. The $j^\dagger\mathcal{O}_{\mathfrak{X}_{\alpha K}}$-linear dual of E_α is denoted by $E_\alpha^\vee := \mathcal{H}om_{j^\dagger\mathcal{O}_{\mathfrak{X}_{\alpha K}}}(E, j^\dagger\mathcal{O}_{\mathfrak{X}_{\alpha K}})$. Since the $j^\dagger\mathcal{O}$-linear dual commutes with pullbacks, the inverse of the isomorphism $(\eta_{\alpha\beta})^\vee$ is canonically isomorphic to an isomorphism of the form $p_{2K}^{\alpha\beta*}((E_\beta)^\vee) \xrightarrow{\sim} p_{1K}^{\alpha\beta*}((E_\alpha)^\vee)$ that we denote by $\eta^*_{\alpha\beta}$. These isomorphisms satisfy the cocycle condition (for more details, see [Car09a, 4.3.1]). Hence, we get the dual functor $(-)^\vee\colon \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K) \to \mathrm{MIC}^\dagger(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$ defined by setting

$$((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda})^\vee := (((E_\alpha)^\vee)_{\alpha\in\Lambda}, (\eta^*_{\alpha\beta})_{\alpha,\beta\in\Lambda}).$$

Let $E \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$, and $E^\vee$ be its dual. We check the isomorphism

$$u^*_{0K}(E^\vee) \xrightarrow{\sim} (u^*_{0K}(E))^\vee,$$

i.e. that the isomorphisms coming from the commutation of the dual with the pullbacks are compatible with glueing data, which is easy.

12.2.5.2. Let $f\colon \mathfrak{X}' \to \mathfrak{X}$ be an open immersion of smooth formal schemes over $\mathfrak{S}$. Let Z be a divisor of X and $Z' := f^{-1}(Z)$. According to notation 9.2.1.21, the functor $f^!_Z\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}})$ is equal to $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}} f^{-1}(-)$. Hence, for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}})$, we get the isomorphism:

$$\begin{aligned} \xi\ \ f^!_Z\mathbb{D}_Z(\mathcal{E}) &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(f^!_Z(\mathcal{E}), f^!_{Z\mathrm{r}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X},\mathbb{Q}}))[d_X] \\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}}(f^!_Z(\mathcal{E}), (\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \omega^{-1}_{\mathfrak{X}'/\mathfrak{S}})_{\mathrm{t}})[d_X] \underset{\beta}{\xrightarrow{\sim}} \mathbb{D}_{Z'} f^!_Z(\mathcal{E}), \end{aligned} \qquad (12.2.5.2.1)$$

where β is the transposition isomorphism exchanging both structures of left $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}$-modules of $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \omega^{-1}_{\mathfrak{X}'/\mathfrak{S}}$.

12.2.5.3. With notation 9.3.7, let $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$. Via the isomorphisms 12.2.5.2.1, the inverse of the isomorphism $\mathbb{D}_{Z_{\alpha\beta}}(\theta_{\alpha\beta})$ is canonically isomorphic to an isomorphism of the form $p_2^{\alpha\beta!}(\mathbb{D}_{Z_\beta}(\mathcal{E}_\beta)) \xrightarrow{\sim} p_1^{\alpha\beta!}(\mathbb{D}_{Z_\alpha}(\mathcal{E}_\alpha))$ and denoted by $\theta^*_{\alpha\beta}$. These isomorphisms satisfy the cocycle condition (for more details, see [Car09a, 4.3.1]). Hence, we get the dual functor

$$\mathbb{D}_Z\colon \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$$

defined by $\mathbb{D}_Z((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) := ((\mathbb{D}_{Z_\alpha}(\mathcal{E}_\alpha))_{\alpha\in\Lambda}, (\theta^*_{\alpha\beta})_{\alpha,\beta\in\Lambda})$.

12.2.5.4. With notation 9.3.7, let $((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^{\dagger}(Y, (\mathfrak{X}_\alpha)_{\alpha\in\Lambda}/K)$. Following 11.2.7.5, we have the canonical isomorphism $\mathrm{sp}_*(E_\alpha^\vee) \xrightarrow{\sim} \mathbb{D}(\mathrm{sp}_*(E_\alpha))$ of $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$. These isomorphisms satisfy the cocycle condition (for more details, see [Car09a, 4.3.1]). Hence, we get the isomorphism

$$\mathrm{sp}_*(((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda})^\vee) \xrightarrow{\sim} \mathbb{D}\circ \mathrm{sp}_*((E_\alpha)_{\alpha\in\Lambda}, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda}).$$

12.2.5.5. With notation 9.3.7, let $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, Z/\mathcal{V})$. From the relative duality isomorphism (see 9.4.5.2), we have the isomorphism $u_{\alpha+}\circ \mathbb{D}_{Z_\alpha}(\mathcal{E}_\alpha) \xrightarrow{\sim} \mathbb{D}_{T_\alpha}\circ u_{\alpha+}(\mathcal{E}_\alpha)$ These isomorphisms satisfy the cocycle condition (for more details, see [Car09a, 4.3.1]), i.e. we get the commutation isomorphism:

$$u_{0+}\circ \mathbb{D}_Z((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \xrightarrow{\sim} \mathbb{D}_T\circ u_{0+}((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})).$$

Proposition 12.2.5.6. *We keep notation 12.2.2. Let $E\in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$, and $E^\vee$ be its dual. We have the functorial canonical isomorphism in E: $\mathrm{sp}_+(E^\vee) \xrightarrow{\sim} \mathbb{D}_T\circ \mathrm{sp}_+(E)$.*

Proof. Since $\mathrm{sp}_+ = u_{0+}\circ\mathrm{sp}_*\circ u^*_{0K}$, the proposition is a consequence of 12.2.5.1, 12.2.5.4, and 12.2.5.5. □

Proposition 12.2.5.7. *Let $\theta\colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ be a morphism of completely smooth d-frames over $\mathcal{V}$. We have the isomorphism of $\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V})$ of the form*

$$\mathbb{D}^{(\bullet)}_{T'}\left(\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^{(\bullet)!}\mathcal{E}^{(\bullet)}[-d_{X'/X}]\right) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{X'} f^{(\bullet)!}(\mathbb{D}^{(\bullet)}_T\mathcal{E}^{(\bullet)})[-d_{X'/X}]. \tag{12.2.5.7.1}$$

Proof. Following 12.2.1.9 and 12.2.1.13, the objects appearing in 12.2.5.7.1 belong to $\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V})$. Hence, it is sufficient to check the isomorphism 12.2.5.7.1 in $\mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/\mathcal{V})$ (i.e.after applying the equivalence functor $\underrightarrow{l}^*_{\mathbb{Q}}$ of 12.2.1.6.1). This follows from 12.2.5.6 and 12.2.4.1.2 and from the isomorphism $\theta^*(E^\vee) \xrightarrow{\sim} \theta^*(E)^\vee$. □

12.2.6 Commutation of the exterior or internal tensor product with sp_+

We keep notation 12.2.2.

Proposition 12.2.6.1. *With the notations of 12.2.2.1, we define the bifunctor tensor product*

$$-\otimes-\quad \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/\mathcal{V}) \times \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/\mathcal{V}), \tag{12.2.6.1.1}$$

by setting, for any $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda})$, $((\mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/\mathcal{V})$,

$$((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \otimes ((\mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda}) := ((\mathcal{E}_\alpha \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}({}^\dagger Z_\alpha)_{\mathbb{Q}}} \mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta''_{\alpha\beta})_{\alpha,\beta\in\Lambda}),$$

where $\theta''_{\alpha\beta}$ is the unique morphism inducing the commutative diagram :

$$\begin{array}{ccc} p_2^{\alpha\beta!}(\mathcal{E}_\beta \otimes_{\mathcal{O}_{\mathfrak{X}_\beta}({}^\dagger Z_\beta)_{\mathbb{Q}}} \mathcal{E}'_\beta) & \xrightarrow[\sim]{} & p_2^{\alpha\beta!}(\mathcal{E}_\beta) \otimes_{\mathcal{O}_{\mathfrak{X}_{\alpha\beta}}({}^\dagger Z_{\alpha\beta})_{\mathbb{Q}}} p_2^{\alpha\beta!}(\mathcal{E}'_\beta) \\ \sim\downarrow \theta''_{\alpha\beta} & & \sim\downarrow \theta_{\alpha\beta}\otimes\theta'_{\alpha\beta} \\ p_1^{\alpha\beta!}(\mathcal{E}_\alpha \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}({}^\dagger Z_\alpha)_{\mathbb{Q}}} \mathcal{E}'_\alpha) & \xrightarrow[\sim]{} & p_1^{\alpha\beta!}(\mathcal{E}_\alpha) \otimes_{\mathcal{O}_{\mathfrak{X}_{\alpha\beta}}({}^\dagger Z_{\alpha\beta})_{\mathbb{Q}}} p_1^{\alpha\beta!}(\mathcal{E}'_\alpha), \end{array} \tag{12.2.6.1.2}$$

whose horizontal isomorphisms are constructed by using the equivalence of categories 12.2.1.6.1 from the commutation of tensor products with extraordinary inverse images (see 9.2.1.27.1) and from the isomorphisms 11.2.4.3.1.

Proof. To check that this tensor product bifunctor is well defined, we have to establish that the isomorphisms $\theta''_{\alpha\beta}$ satisfy to the cocycle condition (see 9.3.7.1). Let us consider the commutative diagram:

$$\begin{array}{ccccccc}
p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta\otimes\mathcal{E}'_\beta) & \underset{\sim}{\xrightarrow{\tau}} & p_{12}^{\alpha\beta!}p_2^{\alpha\beta!}(\mathcal{E}_\beta\otimes\mathcal{E}'_\beta) & \underset{\sim}{\longrightarrow} & p_{12}^{\alpha\beta!}p_2^{\alpha\beta!}(\mathcal{E}_\beta)\otimes p_{12}^{\alpha\beta!}p_2^{\alpha\beta!}(\mathcal{E}'_\beta) & \underset{\tau\otimes\tau}{\xleftarrow{\sim}} & p_2^{\alpha\beta\gamma!}(\mathcal{E}_\beta)\otimes p_2^{\alpha\beta\gamma!}(\mathcal{E}'_\beta) \\
\sim\downarrow\theta''^{\alpha\beta\gamma}_{12} & & \sim\downarrow p_{12}^{\alpha\beta!}(\theta''_{\alpha\beta}) & & \sim\downarrow p_{12}^{\alpha\beta!}(\theta_{\alpha\beta})\otimes p_{12}^{\alpha\beta!}(\theta'_{\alpha\beta}) & & \sim\downarrow\theta_{12}^{\alpha\beta\gamma}\otimes\theta'^{\alpha\beta\gamma}_{12} \\
p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha\otimes\mathcal{E}'_\alpha) & \underset{\sim}{\xrightarrow{\tau}} & p_{12}^{\alpha\beta!}p_1^{\alpha\beta!}(\mathcal{E}_\alpha\otimes\mathcal{E}'_\alpha) & \underset{\sim}{\longrightarrow} & p_{12}^{\alpha\beta!}p_1^{\alpha\beta!}(\mathcal{E}_\alpha)\otimes p_{12}^{\alpha\beta!}p_1^{\alpha\beta!}(\mathcal{E}'_\alpha) & \underset{\tau\otimes\tau}{\xleftarrow{\sim}} & p_1^{\alpha\beta\gamma!}(\mathcal{E}_\alpha)\otimes p_1^{\alpha\beta\gamma}(\mathcal{E}'_\alpha),
\end{array} \tag{12.2.6.1.3}$$

where, following 9.3.7.1.1 and with its notations, the right and left squares are commutative by definition. By applying the functor $p_{12}^{\alpha\beta!}$ to 12.2.6.1.2 and next by functoriality of the commutation of the tensor product with extraordinary inverse images, we get the square of the middle, which is then commutative. Moreover, It follows from 9.2.2.2.1 that the composition of the horizontal isomorphisms of 12.2.6.1.3 are the canonical isomorphisms of commutation of extraordinary inverse images with tensor products. With the two other diagrams similar to 12.2.6.1.3, as the family of isomorphisms $\theta_{\alpha\beta}$ and $\theta'_{\alpha\beta}$ satisfy to the cocycle conditions, so is $\theta''_{\alpha\beta}$. □

Lemma 12.2.6.2. *Let $\mathcal{E}$, $\mathcal{E}'\in \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V})$. We have the canonical isomorphism of $\mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},Z/\mathcal{V})$ which commutes with Frobenius:*

$$u_0^!(\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}\mathcal{E}'[d_{Y/P}]) \xrightarrow{\sim} u_0^!(\mathcal{E})\otimes u_0^!(\mathcal{E}'), \tag{12.2.6.2.1}$$

where $u_0^!\colon \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V})\cong \mathrm{MIC}^{\dagger\dagger}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},Z/\mathcal{V})$ is the functor defined at 12.2.2.5.

Proof. 0) Following 12.2.1.14.2, $\mathcal{E}'':=\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}\mathcal{E}'[d_{Y/P}]\in \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V})$.

1) For any $\alpha\in\Lambda$, let us construct the isomorphism

$$u_\alpha^!(\mathcal{E}''|_{\mathfrak{P}_\alpha}) \xrightarrow{\sim} u_\alpha^!(\mathcal{E}|_{\mathfrak{P}_\alpha})\otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}({}^\dagger Z_\alpha)_{\mathbb{Q}}} u_\alpha^!(\mathcal{E}'|_{\mathfrak{P}_\alpha}). \tag{12.2.6.2.2}$$

Choose $\mathcal{E}^{(\bullet)}$, $\mathcal{E}'^{(\bullet)}\in \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V})$ such that $\mathcal{E}\xrightarrow{\sim}\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}^{(\bullet)}$ and $\mathcal{E}'\xrightarrow{\sim}\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}'^{(\bullet)}$. Let us denote by $\mathcal{E}''^{(\bullet)}:=\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{P}}(T)}\mathcal{E}'^{(\bullet)}[d_{Y/P}]$. Hence, $\mathcal{E}''\xrightarrow{\sim}\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}''^{(\bullet)}$. We have isomorphisms:

$$\begin{aligned} u_\alpha^{(\bullet)!}(\mathcal{E}''^{(\bullet)}|_{\mathfrak{P}_\alpha}) &\underset{9.2.1.27.1}{\xrightarrow{\sim}} u_\alpha^{(\bullet)!}(\mathcal{E}^{(\bullet)}|_{\mathfrak{P}_\alpha})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}_\alpha}(Z_\alpha)}u_\alpha^{(\bullet)!}(\mathcal{E}'^{(\bullet)}|_{\mathfrak{P}_\alpha}) \\ &\underset{11.2.4.3.1}{\xrightarrow{\sim}} u_\alpha^{(\bullet)!}(\mathcal{E}^{(\bullet)}|_{\mathfrak{P}_\alpha})\otimes_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}_\alpha}(Z_\alpha)} u_\alpha^{(\bullet)!}(\mathcal{E}'^{(\bullet)}|_{\mathfrak{P}_\alpha}). \end{aligned} \tag{12.2.6.2.3}$$

By applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ to 12.2.6.2.3, we obtain the desired isomorphism 12.2.6.2.2.

2) It remains now to check that the isomorphisms 12.2.6.2.2 commute with glueing isomorphisms. Let us consider the following diagram

$$\begin{array}{ccc}
p_2^{\alpha\beta(\bullet)!}u_\beta^{(\bullet)!}(\mathcal{E}''^{(\bullet)}|_{\mathfrak{P}_\beta}) & \underset{\sim}{\xrightarrow{\tau}} & p_1^{\alpha\beta(\bullet)!}u_\alpha^{(\bullet)!}(\mathcal{E}''^{(\bullet)}|_{\mathfrak{P}_\alpha}) \\
\sim\downarrow 9.2.1.27.1 & & \sim\downarrow 9.2.1.27.1 \\
p_2^{\alpha\beta(\bullet)!}(u_\beta^{(\bullet)!}(\mathcal{E}^{(\bullet)}|_{\mathfrak{P}_\beta})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}_\beta}(Z_\beta)}(\mathcal{E}'^{(\bullet)}|_{\mathfrak{P}_\beta})) & & p_1^{\alpha\beta(\bullet)!}(u_\alpha^{(\bullet)!}(\mathcal{E}^{(\bullet)}|_{\mathfrak{P}_\alpha})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}_\alpha}(Z_\alpha)}(\mathcal{E}'^{(\bullet)}|_{\mathfrak{P}_\alpha})) \\
\sim\downarrow 9.2.1.27.1 & & \sim\downarrow 9.2.1.27.1 \\
p_2^{\alpha\beta(\bullet)!}(u_\beta^{(\bullet)!}(\mathcal{E}^{(\bullet)}|_{\mathfrak{P}_\beta}))\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}_{\alpha\beta}}(Z_{\alpha\beta})}p_2^{\alpha\beta(\bullet)!}(u_\beta^{(\bullet)!}(\mathcal{E}'^{(\bullet)}|_{\mathfrak{P}_\beta})) & \underset{\sim}{\xrightarrow{\tau\otimes\tau}} & p_1^{\alpha\beta(\bullet)!}(u_\alpha^{(\bullet)!}(\mathcal{E}^{(\bullet)}|_{\mathfrak{P}_\alpha}))\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}^{(\bullet)}_{\mathfrak{X}_{\alpha\beta}}(Z_{\alpha\beta})}p_1^{\alpha\beta(\bullet)!}(u_\alpha^{(\bullet)!}(\mathcal{E}'^{(\bullet)}|_{\mathfrak{P}_\alpha}))
\end{array} \tag{12.2.6.2.4}$$

Modulo the identifications $p_2^{\alpha\beta(\bullet)!}\circ u_\beta^{(\bullet)!}\xrightarrow{\sim}(u_\beta\circ p_2^{\alpha\beta})^{(\bullet)!}$ and $p_1^{\alpha\beta(\bullet)!}\circ u_\alpha^{(\bullet)!}\xrightarrow{\sim}(u_\alpha\circ p_1^{\alpha\beta})^{(\bullet)!}$ and by transitivity of isomorphisms 9.2.1.27.1 (see 9.2.1.27.2), the rectangle 12.2.6.2.4 is of the form 9.2.2.2.1. Hence it is commutative. By applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ to the rectangle 12.2.6.2.4, modulo the canonical isomorphisms of the form 11.2.4.3.2, we get the outer of the diagram

$$\begin{array}{ccc} p_2^{\alpha\beta !} u_\beta^!(\mathcal{E}''|_{\mathfrak{P}_\beta}) & \xrightarrow[\sim]{\tau} & p_1^{\alpha\beta !} u_\alpha^!(\mathcal{E}''|_{\mathfrak{P}_\alpha}) \\ \sim \downarrow 12.2.6.2.2 & & \sim \downarrow 12.2.6.2.2 \\ p_2^{\alpha\beta !}(u_\beta^!(\mathcal{E}|_{\mathfrak{P}_\beta}) \otimes_{\mathcal{O}_{\mathfrak{X}_\beta}(^\dagger Z_\beta)_\mathbb{Q}} u_\beta^!(\mathcal{E}'|_{\mathfrak{P}_\beta})) & \xrightarrow[\sim]{} & p_1^{\alpha\beta !}(u_\alpha^!(\mathcal{E}|_{\mathfrak{P}_\alpha}) \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}(^\dagger Z_\alpha)_\mathbb{Q}} u_\alpha^!(\mathcal{E}'|_{\mathfrak{P}_\alpha})) \\ \sim \downarrow & & \sim \downarrow \\ p_2^{\alpha\beta !} u_\beta^!(\mathcal{E}|_{\mathfrak{P}_\beta}) \otimes_{\mathcal{O}_{\mathfrak{X}_{\alpha\beta}}(^\dagger Z_{\alpha\beta})_\mathbb{Q}} p_2^{\alpha\beta !} u_\beta^!(\mathcal{E}|_{\mathfrak{P}_\beta}) & \xrightarrow[\sim]{\tau\otimes\tau} & p_1^{\alpha\beta !} u_\alpha^!(\mathcal{E}|_{\mathfrak{P}_\alpha}) \otimes_{\mathcal{O}_{\mathfrak{X}_{\alpha\beta}}(^\dagger Z_{\alpha\beta})_\mathbb{Q}} p_1^{\alpha\beta !} u_\alpha^!(\mathcal{E}|_{\mathfrak{P}_\alpha}) \end{array} \tag{12.2.6.2.5}$$

whose middle horizontal isomorphism is defined such that the bottom square is commutative. By definition, this isomorphism is the canonical glueing data structure of the family $\left(u_\alpha^!(\mathcal{E}|_{\mathfrak{P}_\alpha}) \otimes_{\mathcal{O}_{\mathfrak{X}_\alpha}(Z_\alpha)_\mathbb{Q}} u_\alpha^!(\mathcal{E}'|_{\mathfrak{P}_\alpha})\right)_\alpha$ (see 12.2.6.1.2). Hence we have to check that the top square of the diagram 12.2.6.2.5 is commutative, which follows from that of its outer and of the bottom square. □

12.2.6.3. With the notations of 10.2.4.3, in a similar way to 12.2.6.1.1, we define the bifunctor tensor product

$$-\otimes - \quad \mathrm{MIC}^\dagger((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/K) \times \mathrm{MIC}^\dagger((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/K) \to \mathrm{MIC}^\dagger((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/K).$$

Similarly to 12.2.6.2, we then construct, for any $E, E' \in \mathrm{MIC}^\dagger(X, \mathfrak{P}, T/K)$, the canonical isomorphism commuting to Frobenius:

$$u_{0K}^*(E \otimes_{j^\dagger \mathcal{O}_{]X[_\mathfrak{P}}} E') \xrightarrow{\sim} u_{0K}^*(E) \otimes u_{0K}^*(E'). \tag{12.2.6.3.1}$$

Lemma 12.2.6.4. *Let* $((E_\alpha)_{\alpha\in\Lambda},\, (\eta_{\alpha\beta})_{\alpha,\beta\in\Lambda}), ((E'_\alpha)_{\alpha\in\Lambda},\, (\eta'_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^\dagger((\mathfrak{X}_\alpha)_{\alpha\in\Lambda},\, Z/K)$. *With the notations of 10.2.4.3 and 12.2.6.3, we have the canonical isomorphism which commutes with Frobenius:*

$$\mathrm{sp}_*((E_\alpha,\, \eta_{\alpha\beta}) \otimes (E'_\alpha,\, \eta'_{\alpha\beta})) \xrightarrow{\sim} \mathrm{sp}_*(E_\alpha,\, \eta_{\alpha\beta}) \otimes \mathrm{sp}_*(E'_\alpha,\, \eta'_{\alpha\beta}).$$

Proof. Following [Car09a, 2.5.9], denoting by $(\mathcal{E}_\alpha,\, \theta_{\alpha\beta}) := \mathrm{sp}_*(E_\alpha,\, \eta_{\alpha\beta})$ and similarly with some primes, it comes back to the same to establish the isomorphism

$$\mathrm{sp}^*((\mathcal{E}_\alpha,\, \theta_{\alpha\beta}) \otimes (\mathcal{E}'_\alpha,\, \theta'_{\alpha\beta})) \xrightarrow{\sim} \mathrm{sp}^*(\mathcal{E}_\alpha,\, \theta_{\alpha\beta}) \otimes \mathrm{sp}^*(\mathcal{E}'_\alpha,\, \theta'_{\alpha\beta}).$$

For this purpose, we check that the canonical isomorphisms $\mathrm{sp}^*(\mathcal{E}_\alpha \otimes \mathcal{E}'_\alpha) \xrightarrow{\sim} \mathrm{sp}^*(\mathcal{E}_\alpha) \otimes \mathrm{sp}^*(\mathcal{E}'_\alpha)$ commute with the respective glueing isomorphisms. □

Proposition 12.2.6.5. *For any* $E, E' \in \mathrm{MIC}^\dagger(X, \mathfrak{P}, T/K)$, *we have the canonical isomorphism in* $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$*:*

$$\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}(E \otimes_{j^\dagger \mathcal{O}_{]X[_\mathfrak{P}}} E') \xrightarrow{\sim} \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}(E) \overset{\mathbb{L}}{\otimes}{}^\dagger_{\mathcal{O}_\mathfrak{P}(^\dagger T)_\mathbb{Q}} \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}(E')[d_{Y/P}] \tag{12.2.6.5.1}$$

which commutes with Frobenius.

Proof. By using 12.2.1.14.2, we get that the right term of 12.2.6.5.1 do is a element of $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$. The construction of 12.2.6.5.1 is equivalent to that of an isomorphism of the form

$$u_0^! \circ \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}(E \otimes_{j^\dagger \mathcal{O}_{]X[_\mathfrak{P}}} E') \xrightarrow{\sim} u_0^!(\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}(E) \overset{\mathbb{L}}{\otimes}{}^\dagger_{\mathcal{O}_\mathfrak{P}(^\dagger T)_\mathbb{Q}} \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}(E')[d_{Y/P}]). \tag{12.2.6.5.2}$$

Moreover, we have the canonical isomorphism $u_0^! \circ \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+} \xrightarrow{\sim} \mathrm{sp}_* \circ u_{0K}^*$ (see the construction of $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$ of 12.2.2.6 and use the equivalences of 12.2.2.5), this one commuting to actions of Frobenius. Following 12.2.6.2.2, 12.2.6.3.1 and 12.2.6.4, then we get the isomorphism 12.2.6.5.2.

□

Notation 12.2.6.6. Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ and $(Y', X', \mathfrak{P}', T')/\mathcal{V}$ be two completely smooth d-frames over $\mathcal{V}$. We set $\mathfrak{P}'' := \mathfrak{P} \times \mathfrak{P}'$, $X'' := X \times X'$, $Y'' := Y \times Y'$, $j \colon Y \subset X$, $j' \colon Y' \subset X'$ and $j'' \colon Y'' \subset X''$ the canonical inclusions. We denote by $\theta = (a, b, p) \colon (Y'', X'', \mathfrak{P}'', T'') \to (Y, X, \mathfrak{P}, T)$ and $\theta' = (a', b', p') \colon (Y'', X'', \mathfrak{P}'', T'') \to (Y', X', \mathfrak{P}', T')$ the morphisms of d-frames induced by the canonical projections, where $T'' = p^{-1}(T) \cup p'^{-1}(T')$. Let $E \in \mathrm{MIC}^{\dagger}(X, \mathfrak{P}, T/K)$ and $E' \in \mathrm{MIC}^{\dagger}(X', \mathfrak{P}', T'/K)$. With the notations of 12.2.1.12.1, we define the bifunctor $- \boxtimes - \colon \mathrm{MIC}^{\dagger}(X, \mathfrak{P}, T/K) \times \mathrm{MIC}^{\dagger}(X', \mathfrak{P}', T'/K) \to \mathrm{MIC}^{\dagger}(X'', \mathfrak{P}'', T''/K)$ by setting

$$E \boxtimes E' := \theta^*(E) \otimes_{j''^{\dagger}\mathcal{O}_{]X''[_{\mathfrak{P}''}}} \theta'^*(E').$$

Proposition 12.2.6.7. *With the notations 12.2.6.6, we have the canonical isomorphism commuting to Frobenius in* $\mathrm{MIC}^{\dagger\dagger}(X'', \mathfrak{P}'', T''/\mathcal{V})$*:*

$$\mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'', +}(E \boxtimes E') \xrightarrow{\sim} \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E) \overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}}, T, T'} \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}', T', +}(E'). \tag{12.2.6.7.1}$$

Proof. Following 12.2.1.14.4, the right term do belong to $\mathrm{MIC}^{\dagger\dagger}(X'', \mathfrak{P}'', T''/\mathcal{V})$. Let us denote by $\mathcal{E} := \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E)$ and $\mathcal{E}' := \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}', T', +}(E')$. Following 12.2.4.1.2, we have

$$\mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'', +}(\theta^*(E)) \xrightarrow{\sim} \theta^*(\mathcal{E}) := \mathbb{R}\underline{\Gamma}^{\dagger}_{X''} p^{!}_{T'', T}(\mathcal{E}[-d_{Y'}]),$$
$$\mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'', +}(\theta'^*(E')) \xrightarrow{\sim} \theta'^*(\mathcal{E}') := \mathbb{R}\underline{\Gamma}^{\dagger}_{X''} p'^{!}_{T'', T'}(\mathcal{E}'[-d_{Y}]).$$

By using 12.2.6.5, this yields the isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}''}({}^{\dagger}T'')_{\mathbb{Q}})$

$$\mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'', +}(E \boxtimes E') \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X''} p^{!}_{T'', T}(\mathcal{E}) \overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}''}({}^{\dagger}T'')_{\mathbb{Q}}} \mathbb{R}\underline{\Gamma}^{\dagger}_{X''} p'^{!}_{T'', T'}(\mathcal{E}')[-d_P - d_{P'}]. \tag{12.2.6.7.2}$$

Moreover, following 9.2.5.16.1, we have the canonical isomorphism

$$\mathcal{E} \overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}}, T, T'} \mathcal{E}' \xrightarrow{\sim} p^{!}_{T'', T}(\mathcal{E}) \overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}''}({}^{\dagger}T'')_{\mathbb{Q}}} p'^{!}_{T'', T'}(\mathcal{E})[-d_P - d_{P'}]. \tag{12.2.6.7.3}$$

As the right term of 12.2.6.7.3 has its support in X'', then it is isomorphic to the right term of 12.2.6.7.2. Hence we are done. ☐

12.2.7 Differential coherence of $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ when Z is a divisor

Theorem 12.2.7.1. *Let $\mathfrak{X}$ be a smooth formal scheme over $\mathfrak{S}$. Let Z be a divisor of X. Then $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}, \mathbb{Q}}$-module.*

Proof. We can adapt the proof of Berthelot of [Ber96a] as follows. 0) Using [Gro66, 8.8.2, 8.10.5] and [Gro67, 17.7.8], it follows from Theorem [dJ96, 4.1] (this is also explained in [dJ96, 4.5]), that there exists a finite extension l of k satisfying the following property: for any irreducible component $\widetilde{X}$ of $X \times_k l$, setting $\widetilde{Z} := \widetilde{X} \cap (Z \times_k l)$, there exist a smooth integral l-variety X', a projective morphism of l-varieties $\phi \colon X' \to \widetilde{X}$ which is generically finite and étale such that X' is quasi-projective and $Z' := \phi^{-1}(\widetilde{Z})$ is a strict normal crossing divisor of X'.

1) Using Lemma 9.2.7.4, we can suppose $k = l$ and X integral.

2) i) There exists a closed immersion of the form $u_0 \colon X' \hookrightarrow \mathbb{P}^n_X$ whose composition with the projection $\mathbb{P}^n_X \to X$ is ϕ. Let $\mathfrak{P} := \widehat{\mathbb{P}}^n_{\mathfrak{X}}$, $f \colon \mathfrak{P} \to \mathfrak{X}$ be the projection. Since f is proper and smooth, we have the adjoint morphism $f_+ \circ f^!(\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}) \to \mathcal{O}_{\mathfrak{X}, \mathbb{Q}}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}, \mathbb{Q}})$ (see 9.4.5.5). Following 12.1.3.1.1 and 12.1.3.6, we have in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}, \mathbb{Q}})$ the morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{O}_{\mathfrak{P}, \mathbb{Q}}) \to \mathcal{O}_{\mathfrak{P}, \mathbb{Q}}$. Since $f^!(\mathcal{O}_{\mathfrak{X}, \mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{P}, \mathbb{Q}}[n]$, then we get the morphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}, \mathbb{Q}})$

$$f_+(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \mathcal{O}_{\mathfrak{P}, \mathbb{Q}}[n]) \to \mathcal{O}_{\mathfrak{X}, \mathbb{Q}}. \tag{12.2.7.1.1}$$

ii) In this step, we construct the morphism $\mathcal{O}_{\mathfrak{X}, \mathbb{Q}} \to f_+(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \mathcal{O}_{\mathfrak{P}, \mathbb{Q}}[n])$ as follows: we have

$$\mathbb{D}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \mathcal{O}_{\mathfrak{P}, \mathbb{Q}}[n]) \underset{12.2.3.3}{\xrightarrow{\sim}} \mathbb{D}(\mathrm{sp}_+(\mathcal{O}_{]X'[_{\mathfrak{P}}})) \underset{12.2.5.6}{\xrightarrow{\sim}} \mathrm{sp}_+((\mathcal{O}_{]X'[_{\mathfrak{P}}})^{\vee}) \xrightarrow{\sim} \mathrm{sp}_+(\mathcal{O}_{]X'[_{\mathfrak{P}}}) \underset{12.2.3.3}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \mathcal{O}_{\mathfrak{P}, \mathbb{Q}}[n]. \tag{12.2.7.1.2}$$

This yields

$$\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \underset{11.2.6.3.4}{\xrightarrow{\sim}} \mathbb{D}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}) \underset{12.2.7.1.1}{\longrightarrow} \mathbb{D}f_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]) \underset{9.4.5.2}{\xrightarrow{\sim}} f_{+}\mathbb{D}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]) \underset{12.2.7.1.2}{\xrightarrow{\sim}} f_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]).$$

iii) The composite morphism $\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to f_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]) \to \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$ is an isomorphism. Indeed, using the third part of Proposition 11.2.1.14.c, since this composition is a morphism of the abelian category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}/\mathcal{V})$, we reduce to check that its restriction to a dense open subset is an isomorphism. Hence, we can suppose that $\phi\colon X' \to X$ is finite and étale, which is easy.

3) Following the step 2), $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ is a direct summand of $f_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n])$ in the category $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}})$. This yields that $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ is a direct summand of $({}^{\dagger}Z)f_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n])$ in the category $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}})$. Using 9.4.3.3, we get in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S}}({}^{\dagger}Z)_{\mathbb{Q}})$ morphism

$$({}^{\dagger}Z)f_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]) \xrightarrow{\sim} f_{Z,+}\circ({}^{\dagger}f^{-1}(Z))(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]).$$

Hence, it is sufficient to check that this latter object is $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-coherent. Since f is proper and since $({}^{\dagger}f^{-1}(Z))(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n])$ is already known to be $\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}}({}^{\dagger}f^{-1}(Z))_{\mathbb{Q}}$-coherent, using the remark of 9.2.4.19.(b), we reduce to check that $({}^{\dagger}f^{-1}(Z))(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n])$ is $\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}$-coherent. Since this is local on $\mathfrak{P}$, we can suppose $\mathfrak{P}$ affine. Hence, there exists a morphism $u\colon \mathfrak{X}' \to \mathfrak{P}$ of smooth formal schemes over $\mathfrak{S}$ which is $u_0\colon X' \to P$ modulo π. We get

$$({}^{\dagger}f^{-1}(Z))(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}_{\mathfrak{P},\mathbb{Q}}[n]) \underset{12.1.3.8}{\xrightarrow{\sim}} ({}^{\dagger}f^{-1}(Z))(u_{+}(\mathcal{O}_{\mathfrak{X}',\mathbb{Q}})) \underset{9.4.3.3}{\xrightarrow{\sim}} u_{f^{-1}(Z),+}(\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}\phi^{-1}(Z))_{\mathbb{Q}}).$$

Since $\phi^{-1}(Z)$ is a strict normal crossing divisor of X', then following 12.1.2.3 $\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}\phi^{-1}(Z))_{\mathbb{Q}})$ is $\mathcal{D}^{\dagger}_{\mathfrak{X}'/\mathfrak{S},\mathbb{Q}}$-coherent. Hence, using the remark 9.2.4.19, $u_{f^{-1}(Z),+}(\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}\phi^{-1}(Z))_{\mathbb{Q}}) \xrightarrow{\sim} u_{+}(\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}\phi^{-1}(Z))_{\mathbb{Q}})$ is $\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}$-coherent.

□

Corollary 12.2.7.2. *We have* $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}) \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$.

Proof. We already know that $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$. Following 12.2.7.1, $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}} = \underrightarrow{l}^{*}_{\mathbb{Q}}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}}(Z)$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}/\mathfrak{S},\mathbb{Q}}$-module. Using 9.1.6.3, we can conclude. □

We will need later the following proposition.

Proposition 12.2.7.3. *With notation 9.2.7.4, let* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$. *Let* $\mathcal{E}'^{(\bullet)} := \mathcal{V}' \otimes_{\mathcal{V}} \mathcal{E}^{(\bullet)}$. *If* $({}^{\dagger}Z')(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}'})$, *then* $({}^{\dagger}Z)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$.

Proof. Using 9.1.6.3, this is a consequence of Lemma 9.2.7.4. □

Chapter 13

Local cohomological operations and applications

13.1 Local cohomological functors

Let $\mathfrak{P}$ a separated, smooth and quasi-compact $\mathcal{V}$-formal scheme, $\mathfrak{D}$ a relative to $\mathfrak{P}/\mathfrak{S}$ strict normal crossing divisor and $\mathfrak{P}^\sharp := (\mathfrak{P}, M(\mathfrak{D}))$ the $\mathfrak{S}$-log smooth log formal scheme whose log-structure is given by $\mathfrak{D}$ (see 4.5.2.14). If $\mathfrak{U}$ is an open of $\mathfrak{P}$, we will write $\mathfrak{U}^\sharp := (\mathfrak{U}, M(\mathfrak{D} \cap \mathfrak{U}))$.

We fix $\lambda_0 \colon \mathbb{N} \to \mathbb{N}$ an increasing map such that $\lambda_0(m) \geq m$. For any divisor T of P, we then set $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{P}}(T) := \widehat{\mathcal{B}}^{(\lambda_0(m))}_{\mathfrak{P}}(T)$ and $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp}(T) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{P}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{P}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp}$. We will also see a posteriori (see 9.1.1.16.(c), that the hypothesis $\lambda_0 = \mathrm{id}$ does not harm generality. Finally, if $f \colon \mathfrak{X} \to \mathfrak{P}$ (resp. $f^\sharp \colon \mathfrak{X}^\sharp \to \mathfrak{P}^\sharp$) is a morphism of smooth formal $\mathfrak{S}$-schemes (resp. $\mathfrak{S}$-log-smooth log formal schemes), for any integer $i \in \mathbb{N}$, we denote by $f_i \colon X_i \to P_i$ (resp. $f^\sharp_i \colon X^\sharp_i \to P^\sharp_i$) the induced morphism modulo π^{i+1}. We finally set $\mathcal{D}^{(m)}_{P^\sharp_i}(T) := \mathcal{V}/\pi^{i+1} \otimes_{\mathcal{V}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}^\sharp}(T) = \mathcal{B}^{(m)}_{P_i}(T) \otimes_{\mathcal{O}_{P_i}} \mathcal{D}^{(m)}_{P^\sharp_i}$ and $\widetilde{\mathcal{D}}^{(m)}_{P^\sharp_i}(T) := \widetilde{\mathcal{B}}^{(m)}_{P_i}(T) \otimes_{\mathcal{O}_{P_i}} \mathcal{D}^{(m)}_{P^\sharp_i}$.

13.1.1 Local cohomological functor with strict support over a divisor

Let T be a divisor of P. We have already defined in 9.1.1.5.3 the localisation functor $({}^\dagger T)$ outside T. In this subsection, we define and study the local cohomological functor with support in T, which we denote by $\mathbb{R}\underline{\Gamma}^\dagger_T$.

Lemme 13.1.1.1. Consider a distinguished triangle in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ of the form

$$\mathcal{F}^{(\bullet)} \to \mathcal{E}^{(\bullet)} \to ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \to \mathcal{F}^{(\bullet)}[1], \tag{13.1.1.1.1}$$

where the second morphism is the canonical morphism. For any divisor $T \subset T'$, we then have the isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ of the form $({}^\dagger T')(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} 0$.

Proof. As according to 9.1.3.2 the canonical morphism $({}^\dagger T')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger T')(({}^\dagger T)(\mathcal{E}^{(\bullet)}))$ is an isomorphism, then by applying the functor $({}^\dagger T')$ to the distinguished triangle 13.1.1.1.1, one of the axioms on triangulated categories allows us to conclude. □

Lemme 13.1.1.2. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$. We further assume that we have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ of the isomorphism $({}^\dagger T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$. Then $\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})}(\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}) = 0$.

Proof. Let $\phi \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$. As the canonical morphism $\mathcal{F}^{(\bullet)} \to ({}^\dagger T)(\mathcal{F}^{(\bullet)})$ is an isomorphism, then the morphism ϕ is canonically factorized by $({}^\dagger T)(\phi)$. Now, as $({}^\dagger T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$, then $({}^\dagger T)(\phi) = 0$. We deduce that $\phi = 0$. Hence the result. □

13.1.1.3. Following 8.3.4.4, we have the bifunctor (which is the standard construction bifunctor of homomorphims of the abelian category $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$):

$$\mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))}(-,-)\colon K(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))^{\circ}\times K(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))\to K(\mathbb{Z}).$$

Following 8.3.4.8, the bifunctor $\mathrm{Hom}^{\bullet}_{\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))}(-,-)$ is right localizable. We get the bifunctor

$$\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))}(-,-)\colon D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))\to D(\mathbb{Z}).$$

Let $\mathfrak{Ab}$ be the category of abelian groups. Following 8.3.4.8.3, we have the isomorphism of bifunctors $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))^{\circ}\times D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))\to\mathfrak{Ab}$ of the form:

$$H^0(\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))}(-,-))\xrightarrow{\sim}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))}(-,-). \qquad (13.1.1.3.1)$$

13.1.1.4. Let $T\subset T'$ be a second divisor. Suppose we have the commutative diagram in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ of the form

$$\begin{array}{ccccccc}
\mathcal{F}^{(\bullet)} & \to & \mathcal{E}^{(\bullet)} & \to & ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) & \to & \mathcal{F}^{(\bullet)}[1] \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle({}^{\dagger}T)(\phi)} & & \\
\mathcal{F}'^{(\bullet)} & \to & \mathcal{E}'^{(\bullet)} & \to & ({}^{\dagger}T)(\mathcal{E}'^{(\bullet)}) & \to & \mathcal{F}'^{(\bullet)}[1]
\end{array} \qquad (13.1.1.4.1)$$

where middle horizontal morphisms are the canonical ones and where both horizontal triangles are distinguished. Modulo the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))\cong D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))$ (see 8.1.5.14.1) which allows us to see 13.1.1.4.1 as a diagram of $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))$, we have

$$H^{-1}(\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))}(\mathcal{F}^{(\bullet)},({}^{\dagger}T)(\mathcal{E}'^{(\bullet)})))\underset{13.1.1.3.1}{\xrightarrow{\sim}}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)))}(\mathcal{F}^{(\bullet)},({}^{\dagger}T)(\mathcal{E}'^{(\bullet)})[-1])\underset{13.1.1.2}{=}0.$$

Following [BBD82, 1.1.9], this implies there exists a unique morphism $\mathcal{F}^{(\bullet)}\to\mathcal{F}'^{(\bullet)}$ making commutative in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ the diagram:

$$\begin{array}{ccccccc}
\mathcal{F}^{(\bullet)} & \to & \mathcal{E}^{(\bullet)} & \to & ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) & \to & \mathcal{F}^{(\bullet)}[1] \\
\downarrow{\scriptstyle\exists!} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle({}^{\dagger}T)(\phi)} & & \downarrow{\scriptstyle\exists!} \\
\mathcal{F}'^{(\bullet)} & \to & \mathcal{E}'^{(\bullet)} & \to & ({}^{\dagger}T)(\mathcal{E}'^{(\bullet)}) & \to & \mathcal{F}'^{(\bullet)}[1].
\end{array} \qquad (13.1.1.4.2)$$

Similarly to [BBD82, 1.1.10], this implies that the cone of $\mathcal{E}^{(\bullet)}\to({}^{\dagger}T)(\mathcal{E}^{(\bullet)})$ is unique up to canonical isomorphism. Hence, such a complex $\mathcal{F}^{(\bullet)}$ is unique up to canonical isomorphism. We denote it by $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})$. Moreover,the complex $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})$ is functorial in $\mathcal{E}^{(\bullet)}$.

Definition 13.1.1.5. With notation 13.1.1.4, the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}\colon\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})\to\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ is the "local cohomological functor with strict support over the divisor T". For $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$, we denote by $\Delta_T(\mathcal{E}^{(\bullet)})$ the canonical exact triangle

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})\to\mathcal{E}^{(\bullet)}\to({}^{\dagger}T)(\mathcal{E}^{(\bullet)})\to\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})[1]. \qquad (13.1.1.5.1)$$

Lemma 13.1.1.6. *Let $T\subset T'$ be a second divisor, and $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. There exists a unique morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})\to\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E}^{(\bullet)})$ making commutative the following diagram*

$$\begin{array}{ccccccc}
\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)} & \to & ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})[1] \\
\downarrow{\scriptstyle\exists!} & & \| & & \downarrow & & \downarrow{\scriptstyle\exists!} \\
\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)} & \to & ({}^{\dagger}T')(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E}^{(\bullet)})[1].
\end{array} \qquad (13.1.1.6.1)$$

In other words, $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}^{(\bullet)})$ is functorial in T.

Proof. This follows from 13.1.1.4. □

13.1.1.7. Let $T \subset T'$ be a second divisor. Let's denote by $\mathrm{forg}_\sharp$ the canonical forgetful functors of the form $\mathrm{forg}_\sharp\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$ or of the form $\mathrm{forg}_\sharp\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{P_\bullet}(T)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{P^\sharp_\bullet/\mathfrak{S}}(T))$ We deduce from 9.1.1.5.2 that the location functors $({}^\dagger T', T)$ of 9.1.1.5.3 or of 9.1.1.12.1 do not depend on the log- structure, i.e., we have the canonical isomorphism

$$({}^\dagger T', T) \circ \mathrm{forg}_\sharp \xrightarrow{\sim} \mathrm{forg}_\sharp \circ ({}^\dagger T', T) \tag{13.1.1.7.1}$$

of functors of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T'))$ or of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{P_\bullet}(T)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{P^\sharp_\bullet}(T))$.

13.1.1.8. We deduce from 13.1.1.7 that the functor $\mathbb{R}\underline{\Gamma}^\dagger_T$ does not depend on the log-structure, i.e., we have the canonical isomorphism

$$\mathbb{R}\underline{\Gamma}^\dagger_T \circ \mathrm{forg}_\sharp \xrightarrow{\sim} \mathrm{forg}_\sharp \circ \mathbb{R}\underline{\Gamma}^\dagger_T$$

of functors of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

Example 13.1.1.9. Let $T \subset T'$ be a second divisor, $\mathfrak{U}'$ be the complementary to T' open of $\mathfrak{X}$ and $j'\colon \mathfrak{U}' \to \mathfrak{X}$ be the open immersion. Let $E \in \mathrm{MIC}^\dagger(\mathfrak{X}^\sharp_K, T/\mathfrak{S}^\sharp_K)$. By applying the functor $\mathbb{R}\mathrm{sp}_*$ to an exact sequence of the form 10.1.3.2.1, we get the exact triangle (and with 10.1.2.3.2):

$$\mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^\dagger_{]T'[_{\mathfrak{X}}}(E) \longrightarrow \mathrm{sp}_*(E) \longrightarrow \mathrm{sp}_*(j'^\dagger(E)) \longrightarrow \mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^\dagger_{]T'[_{\mathfrak{X}}}(E)[1].$$

Since $\mathrm{sp}_*(E) \to \mathrm{sp}_*(j'^\dagger(E))$ is canonically isomorphic to $\mathcal{E} \to \mathcal{E}({}^\dagger T')$ (see 11.2.7.2.1), it follows from the exact triangle of localization of $\mathrm{sp}_*(E)$ with respect to T' (see 13.1.5.6.3), then using [BBD82, 1.1.9–10] we get the canonical isomorphism:

$$\mathbb{R}\underline{\Gamma}^\dagger_{T'}(\mathrm{sp}_*(E)) \xrightarrow{\sim} \mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^\dagger_{]T'[_{\mathfrak{X}}}(E). \tag{13.1.1.9.1}$$

13.1.2 Commutation between localisations and local functors in the case of a divisor

13.1.2.1 (Commutation of location and local functors to the tensor product)**.** Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. By commutativity and associativity of tensor products, we have the canonical isomorphisms

$$({}^\dagger T)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} ({}^\dagger T)(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T)(\mathcal{F}^{(\bullet)}).$$

Hence, there exists a unique isomorphism of the form $\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}$ (resp. $\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{F}^{(\bullet)})$) and making commutative the following diagram

$$\begin{array}{ccccccc}
\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & \mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}[1] \\
\uparrow{\exists!} & & \| & & \uparrow{\sim} & & \uparrow{\exists!} \\
\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T)(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)})[1] \\
\downarrow{\exists!} & & \| & & \downarrow{\sim} & & \downarrow{\exists!} \\
\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{F}^{(\bullet)})[1].
\end{array} \tag{13.1.2.1.1}$$

These isomorphisms are functorial in $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$, T (for the meaning of the functoriality in T, we can look at 13.1.1.6.1). To check the functoriality in $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$, T, it is a question of writing the corresponding diagrams in three dimensions, which is left to the reader.

13.1.2.2 (Commutation between local cohomological functors and localization functors)**.** Let T_1, T_2 be two divisors of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

(a) By commutativity of the tensor product, we have the functorial in T_1, T_2 and $\mathcal{E}^{(\bullet)}$ canonical isomorphism
$$({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)}). \tag{13.1.2.2.1}$$

(b) There then exists a unique isomorphism $({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})$ inducing the canonical morphism of distinguished triangles $({}^\dagger T_2)(\Delta_{T_1}(\mathcal{E}^{(\bullet)}))\to\Delta_{T_1}(({}^\dagger T_2)(\mathcal{E}^{(\bullet)}))$, i.e. of the form:
$$\begin{array}{ccccccc}
({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})[1] \\
\downarrow\exists! & & \| & & \downarrow\sim & & \downarrow\exists! \\
\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})[1],
\end{array} \tag{13.1.2.2.2}$$
whose middle square is indeed commutative by functoriality in T_1 of the isomorphism 13.1.2.2.1. By writing the diagrams in three dimensions (i.e., we write the parallelepiped whose face in front is 13.1.2.2.2, the back face is 13.1.2.2.2 with T_1' replacing T_1, the morphisms from front to back are the functoriality morphisms induced by $T_1\subset T_1'$; ditto to validate the functoriality in T_2 or $\mathcal{E}^{(\bullet)}$), we check that the isomorphism $({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})$ is functorial in T_1, T_2, $\mathcal{E}^{(\bullet)}$.

(c) Likewise, there then exists a unique isomorphism $\mathbb{R}\underline{\Gamma}^\dagger_{T_2}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{E}^{(\bullet)})$ functorial in T_1, T_2, $\mathcal{E}^{(\bullet)}$ inducing the canonical morphism of distinguished triangles $\Delta_{T_2}(\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}))\to\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\Delta_{T_2}(\mathcal{E}^{(\bullet)}))$, i.e. of the form:
$$\begin{array}{ccccccc}
\mathbb{R}\underline{\Gamma}^\dagger_{T_2}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{T_2}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})[1] \\
\downarrow\exists! & & \| & & \downarrow\sim & & \downarrow\exists! \\
\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{E}^{(\bullet)})[1],
\end{array} \tag{13.1.2.2.3}$$
whose middle square is indeed commutative by functoriality in T_2 of the isomorphism $({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})$.

13.1.2.3 (Compatibility of commutation isomorphisms with the tensor product)**.** The three isomorphisms of 13.1.2.2 are compatible with that of 13.1.2.1. More precisely, let T, T_1, T_2 be divisors of P, $\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. By composing the vertical isomorphisms of the diagram 13.1.2.1.1, we obtain the canonical isomorphisms $({}^\dagger T)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T)(\mathcal{F}^{(\bullet)})$ and $\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{F}^{(\bullet)})$ functorials in T, $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$. We verify in this paragraph that these isomorphisms are compatible with the three isomorphisms of commutation of local functors and localization of 13.1.2.2.

(a) Compatibility with isomorphisms 13.1.2.2.1 means that the diagram
$$\begin{array}{ccccc}
({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & ({}^\dagger T_1)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) \\
{\scriptstyle 13.1.2.2.1}\downarrow\sim & & & & \sim\uparrow{\scriptstyle 13.1.2.2.1} \\
({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_1)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{F}^{(\bullet)}),
\end{array} \tag{13.1.2.3.1}$$
is commutative, which is easy.

(b) Now let's check that the diagram
$$\begin{array}{ccccc}
({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) \\
\downarrow\sim & & & & \sim\uparrow \\
\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{F}^{(\bullet)}),
\end{array} \tag{13.1.2.3.2}$$

is commutative. To do this, consider the diagram:

$$\begin{array}{ccccccc}
({}^\dagger T_2)\circ \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & +1\\
\downarrow & & \downarrow & & \downarrow\wr & & \\
\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & +1,
\end{array} \tag{13.1.2.3.3}$$

whose right (resp. left) vertical arrow is the top composite of 13.1.2.3.1 (resp. 13.1.2.3.2). As these morphisms are functorial in T_1, the left (resp right) square of the diagram 13.1.2.3.3 is commutative. This diagram 13.1.2.3.3 therefore corresponds to a morphism of distinguished triangles of the form $({}^\dagger T_2)(\Delta_{T_1}(\mathcal{E}^{(\bullet)}))\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} \to \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\Delta_{T_1}(({}^\dagger T_2)(\mathcal{E}^{(\bullet)}))$. In the same way, we check the commutativity of the middle squares of the diagram

$$\begin{array}{ccccccc}
({}^\dagger T_2)\circ \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & +1\\
\downarrow & & \| & & \downarrow\wr & & \\
\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & ({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \to & +1\\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)\circ({}^\dagger T_1)(\mathcal{F}^{(\bullet)}) & \to & +1\\
\downarrow & & \| & & \downarrow & & \\
\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}({}^\dagger T_1)\circ({}^\dagger T_2)(\mathcal{F}^{(\bullet)}) & \to & +1,
\end{array} \tag{13.1.2.3.4}$$

whose top (resp. bottom) triangle morphism is the image by the functor $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}$ (resp. $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}-$) of the morphism of distinguished triangles 13.1.2.2.2. The commutativity of the diagram 13.1.2.3.1 means that the right (resp. middle) vertical composite arrow of the 13.1.2.3.4 diagram is the right (resp. middle) vertical arrow of 13.1.2.3.3. For uniqueness (thanks to [BBD82, 1.1.9]), the same applies to the left vertical arrows. Hence the result.

(c) By tracing the proof of the commutativity of 13.1.2.3.2 from 13.1.2.3.1, we verify from the commutativity of 13.1.2.3.2 that of the diagram below:

$$\begin{array}{ccccc}
\mathbb{R}\underline{\Gamma}^\dagger_{T_2}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{F}^{(\bullet)})\\
\downarrow\wr & & & & \wr\uparrow\\
\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} & \xrightarrow{\sim} & \mathbb{R}\underline{\Gamma}^\dagger_{T_2}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_2}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{F}^{(\bullet)}).
\end{array} \tag{13.1.2.3.5}$$

13.1.2.4. Let $T_1,\ldots,T_r$ and $T'_1,\ldots,T'_s$ be divisors of P, $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. Using s-times the morphisms of the form 13.1.2.3.5, we can check that we have the canonical functorial isomorphisms in $T_1,\ldots,T_r$, $T'_1,\ldots,T'_s$, $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$ of the form

$$\begin{aligned}
\mathbb{R}\underline{\Gamma}^\dagger_{T'_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T'_s}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) &\xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{T'_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T'_s}(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{F}^{(\bullet)}))\\
&\xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{T'_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T'_s}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{F}^{(\bullet)}),
\end{aligned} \tag{13.1.2.4.1}$$

these do not depend, up to canonical isomorphism, on the order of $T_1,\ldots,T_r$ or $T'_1,\ldots,T'_s$. In particular, by taking $\mathcal{F}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{P}}$ and the empty $T'_1,\ldots,T'_s$, we obtain the canonical functorial isomorphism in $T_1,\ldots,T_r$, $\mathcal{E}^{(\bullet)}$ of the form:

$$\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \tag{13.1.2.4.2}$$

and which does not depend, up to canonical isomorphism, on the order of $T_1,\ldots,T_r$.

13.1.3 Local cohomological functor with strict support in a closed subscheme

We will need the following Lemmas (e.g. see the construction of 13.1.3.8 or Proposition 13.1.4.8) in the next section.

Lemma 13.1.3.1. *Let Z, T be two divisors of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$, $\mathfrak{U}$ the open of $\mathfrak{P}$ complementary to the support of Z. The following assertions are equivalent:*

(a) We have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}^\sharp}(T\cap U))$ the isomorphism $\mathcal{E}^{(\bullet)}|\mathfrak{U} \xrightarrow{\sim} 0$.

(b) The canonical morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$.

(c) We have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}(T))$ the isomorphism $({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$.

Proof. The equivalence between (b) and (c) is tautological (i.e., we have the distinguished triangle 13.1.1.5.1). The assertion $(c) \to (a)$ is trivial. Conversely, let's prove $(a) \to (c)$. Since $({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z)\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger T\cup Z)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(Z\cup T))$, as the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ is fully faithful on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}(Z\cup T))$, it is then equivalent to prove that $\underrightarrow{l}^*_{\mathbb{Q}}({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$. Since the cohomological spaces of $\underrightarrow{l}^*_{\mathbb{Q}}({}^\dagger Z)(\mathcal{E}^{(\bullet)})$ are coherent $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp}({}^\dagger T\cup Z)_{\mathbb{Q}}$-modules with support in Z (see the remark 9.3.5.15) and therefore in $T\cup Z$, the latter is indeed zero thanks to 8.7.6.11. □

Lemma 13.1.3.2. *Let $T_1,\ldots,T_r,T$ be divisors of P. Then $\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$.*

Proof. As according to 9.1.3.2 we have the canonical isomorphism $({}^\dagger T_1)(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T_1\cup T)$, then we obtain the distinguished triangle:

$$\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T) \to \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T_1\cup T) \to \mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T))[1]. \tag{13.1.3.2.1}$$

According to 12.2.7.2, it follows that the distinguished triangle 13.1.3.2.1 is a triangle in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. We deduce by induction on r (we apply the functor $\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_2}$ to the distinguished triangle 13.1.3.2.1) the result. □

Corollary 13.1.3.3. *For $r\in\mathbb{N}$, let $T_1,\ldots,T_r$ be some divisors of P (by convention, $r=0$ means there is no divisors). Let T be a divisor of P. Then there exists a canonical isomorphism*

$$\mathbb{R}\mathrm{sp}_*\left(\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K})\right) \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T))$$

which is functorial in T_i and T, i.e. making commutative the diagram of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S},\mathbb{Q}})$

$$\begin{array}{ccc}
\mathbb{R}\mathrm{sp}_*\left(\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(\mathcal{O}_{\mathfrak{P}_K})\right) & \xrightarrow{\sim} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{sp}_*\left(\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K})\right) & \xrightarrow{\sim} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{sp}_*\left(\underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K})\right) & \xrightarrow{\sim} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)),
\end{array}$$

where the vertical arrows are the canonical ones induced by $\mathcal{O}_{\mathfrak{P}_K}\to j^\dagger_T\mathcal{O}_{\mathfrak{P}_K}$, $\mathcal{O}^{(\bullet)}_{\mathfrak{P}}\to\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)$, $\underline{\Gamma}^\dagger_{T_r}\to\mathrm{id}$, $\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\to\mathrm{id}$, and where $\underrightarrow{l}^*_{\mathbb{Q}}$ is the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S},\mathbb{Q}})$ (see 8.4.5.6).

Proof. This is checked by induction on $r\in\mathbb{N}$. When $r=0$, this corresponds to the functorial in T isomorphism $\mathbb{R}\mathrm{sp}_*(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}} \xrightarrow{\sim} \underrightarrow{l}^*_{\mathbb{Q}}\,\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)$. Since $j^\dagger_{T_r}j^\dagger_T\mathcal{O}_{\mathfrak{P}_K} = j^\dagger_{T_r\cup T}\mathcal{O}_{\mathfrak{P}_K}$, we get the exact sequence

$$0\to\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K}) \to \underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K}) \to \underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_{T_r\cup T}\mathcal{O}_{\mathfrak{P}_K}) \to 0. \tag{13.1.3.3.1}$$

Hence, by induction hypothesis, we get a unique (using again [BBD82, 1.1.9]) isomorphism making commutative the diagram of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S},\mathbb{Q}})$:

$$\begin{array}{ccc}
\mathbb{R}\mathrm{sp}_*\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K}) & \overset{\sim}{\dashrightarrow} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{sp}_*\underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K}) & \xrightarrow{\sim} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{sp}_*\underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_{T_r\cup T}\mathcal{O}_{\mathfrak{P}_K}) & \xrightarrow{\sim} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_{r-1}}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T_r\cup T)) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{sp}_*\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\underline{\Gamma}^\dagger_{T_1}(j^\dagger_T\mathcal{O}_{\mathfrak{P}_K})[1] & \overset{\sim}{\dashrightarrow} & \underrightarrow{l}^*_{\mathbb{Q}}\,\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T))[1]
\end{array} \qquad (13.1.3.3.2)$$

where the triangle given by the right vertical arrows is distinguished since we have $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T_r\cup T)\xrightarrow{\sim}({}^\dagger T_r)(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T))$ (use 9.1.3.2).

□

Lemma 13.1.3.4. *Let X be a closed subscheme of P, $T_1,\dots,T_r$ (resp. $T'_1,\dots,T'_s$) be some divisors of P such that $X=\cap_{l=1,\dots,r}T_l$ (resp. $X=\cap_{l=1,\dots,s}T'_l$) and $\mathcal{E}^{(\bullet)}\in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp})$. There then exists an canonical isomorphism (compatible with Frobenius when $\mathfrak{D}$ is empty)*

$$\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathbb{R}\underline{\Gamma}^\dagger_{T'_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T'_s}\mathcal{E}^{(\bullet)}.$$

Proof. Since $\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ thanks to 13.1.3.1, the morphism canonical $\mathbb{R}\underline{\Gamma}^\dagger_{T'_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T'_s}\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})\to\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})$ is an isomorphism (compatible with Frobenius when $\mathfrak{D}$ is empty). By symmetry and noting that the proposition 13.1.2.1 implies that the functors $\mathbb{R}\underline{\Gamma}^\dagger_{T_i}$ and $\mathbb{R}\underline{\Gamma}^\dagger_{T'_j}$ commute canonically, we deduce an isomorphism $\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})\xrightarrow{\sim}\mathbb{R}\underline{\Gamma}^\dagger_{T'_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T'_s}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})$.

Now, it follows, by induction on r, from 13.1.2.1 that we have a canonical isomorphism $\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{E}^{(\bullet)}\xrightarrow{\sim}\mathbb{R}\underline{\Gamma}^\dagger_{T_1}\circ\cdots\circ\mathbb{R}\underline{\Gamma}^\dagger_{T_r}(\mathcal{E}^{(\bullet)})$. Likewise, replacing T with T' and r with s. Hence the result. □

13.1.3.5. Let U be an open subset of P. The maps $D\mapsto\overline{D}$ and $T\mapsto T\cap U$ are reciprocal bijections between the set of irreducible divisors of U and that of the irreducible divisors of P meeting U. The maps $D\mapsto\overline{D}$ and $T\mapsto T\cap U$ are reciprocal bijections between the set of divisors of U and that of the divisors of P whose irreducible components meet U. These bijections preserve the number of irreducible components.

Lemma 13.1.3.6. *Suppose P integral. Let X be a reduced closed subscheme of P. Then, X is a finite intersection of divisors of P.*

Proof. When $X=P$, we can see X as the intersection of 0 divisors of P. Suppose $X\neq P$. By using the formula commuting the intersection and union of the form $\cup_{i=1}^r\cap_{j=1}^s D_{i,j}=\cap_{1\le j_1,\dots,j_r\le s}\cup_{i=1}^r D_{i,j_i}$ for a family of subsets $D_{i,j}$ (divisors or P) of P, since the union of divisors is a divisor, then we reduce to the case where X is irreducible. Let $(P_i)_{i\in I}$ be a family of affine and dense open sets in P, such that the family $(X\cap P_i)_{i\in I}$ covers X and $X\cap P_i$ is non-empty for all $i\in I$. Now, there exists a finite number of divisors $(T_{i,j_i})_{j_i\in J_i}$ of P_i such that $X\cap P_i=\cap_{j_i}T_{i,j_i}$. As $X\cap P_i$ is dense in X, we then notice that $X=\cap_{i,j_i}\overline{T_{i,j_i}}$, where $\overline{T_{i,j_i}}$ is the closure of T_{i,j_i} in X. □

Remark 13.1.3.7. Beware that if P has at least two connected components, if X is one on its connected components then X is not a finite intersection of divisors of P.

Definition 13.1.3.8. Let X be a reduced closed subscheme of P. We define the local cohomological functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}$ with strict support in X as follows. Since P is the sum of its irreducible components, it suffices to define it in the case where P is integral.

1. If $X = P$, then the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}$ is by definition the identity.

2. Now suppose $X \neq P$. According to 13.1.3.6, there exists $T_1, \dots, T_r$ some divisors of P such that $X = \cap_{i=1}^{r} T_i$, then, for all $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$, the complex $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) := \mathbb{R}\underline{\Gamma}^{\dagger}_{T_r} \circ \dots \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{T_1}(\mathcal{E}^{(\bullet)})$ does not canonically depend on the choice of such divisors $T_1, \dots, T_r$ satisfying $X = \cap_{i=1}^{r} T_i$ (see 13.1.3.4).

13.1.3.9. Let X be a closed subscheme of P. We deduce from 13.1.1.8 that the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}$ does not depend on the log-structure, i.e., we have the canonical isomorphism

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X} \circ \mathrm{forg}_{\sharp} \xrightarrow{\sim} \mathrm{forg}_{\sharp} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X} \tag{13.1.3.9.1}$$

of functors of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$.

Proposition 13.1.3.10. *Let X be a smooth closed subscheme of P. The complex $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}\mathcal{O}_{\mathfrak{P},\mathbb{Q}} := \mathbb{R}\mathrm{sp}_{*}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}_{\mathfrak{P}_K})$ defined at 12.1.3.1 is canonically isomorphic to $\underrightarrow{l}^{*}_{\mathbb{Q}}\, \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$, which confirms the compatibility of our notation.*

Proof. By the construction explained in 13.1.3.8, this is a consequence of 13.1.3.3. □

Proposition 13.1.3.11. *Let X, X' be two closed subschemes of P, $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$.*

(a) We have the canonical isomorphism functorial in $\mathcal{E}^{(\bullet)}$, X, and X':

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X\cap X'}(\mathcal{E}^{(\bullet)}). \tag{13.1.3.11.1}$$

(b) We have the canonical isomorphism functorial in $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$, X, and X':

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X\cap X'}(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{F}^{(\bullet)}). \tag{13.1.3.11.2}$$

Proof. The first statement is obvious by construction of the local cohomological functor with strict support. The last one results from the canonical isomorphisms 13.1.2.4.1. □

Example 13.1.3.12. Taking $X' = P$ (and $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{P}}$) in 13.1.3.11.2, we get

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}, \qquad \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}. \tag{13.1.3.12.1}$$

13.1.4 Localisation outside a closed subscheme functor

Lemma 13.1.4.1. *Let $X \subset X'$ be two closed subschemes of P, $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$. We further assume that we have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$ the isomorphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} 0$. Then*

$$\mathrm{Hom}_{\underrightarrow{LD}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}), \mathcal{F}^{(\bullet)}) = 0.$$

Proof. Let $\phi\colon \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \to \mathcal{F}^{(\bullet)}$ be a morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$. Since the canonical morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)})) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)})$ is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$(see 13.1.3.11.1), then the morphism ϕ is canonically factorized by $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\phi)$. Now, since $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{F}^{(\bullet)}) \xleftarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} 0$, then $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\phi) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$. This implies $\phi = 0$. Hence the result. □

Lemma 13.1.4.2. *Let X be a closed subscheme of P and a distinguished triangle in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ of the form*

$$\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)} \to \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})[1], \tag{13.1.4.2.1}$$

where the first morphism is the canonical morphism. We then have the isomorphism $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

Proof. Since the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})) \to \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})$ is an isomorphism (see 13.1.3.11.1), by applying the functor $\mathbb{R}\underline{\Gamma}^\dagger_X$ to the distinguished triangle 13.1.4.2.1, one of the axioms on triangulated categories allows us to conclude. □

13.1.4.3. Let $X \subset X'$ be two closed subschemes of P. Suppose given the commutative diagram in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ of the form

$$\begin{array}{ccccccc} \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) & \longrightarrow & \mathcal{E}^{(\bullet)} & \longrightarrow & \mathcal{F}^{(\bullet)} & \longrightarrow & \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})[1] \\ \downarrow{\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_X(\phi)} & & \downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_X(\phi)} \\ \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}'^{(\bullet)}) & \longrightarrow & \mathcal{E}'^{(\bullet)} & \longrightarrow & \mathcal{F}'^{(\bullet)} & \longrightarrow & \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}'^{(\bullet)})[1] \end{array} \tag{13.1.4.3.1}$$

whose left horizontal arrows are the canonical morphisms and whose two horizontal triangles are distinguished. According to lemmas 13.1.4.1 and 13.1.4.2, we then obtain

$$H^{-1}(\mathbb{R}\mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}))}(\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}), \mathcal{F}'^{(\bullet)})) = \mathrm{Hom}_{D(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp}))}(\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}), \mathcal{F}'^{(\bullet)}[-1]) = 0.$$

We deduce, thanks to [BBD82, 1.1.9], that there therefore exists a unique morphism $\mathcal{F}^{(\bullet)} \to \mathcal{F}'^{(\bullet)}$ inducing in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ the commutative diagram:

$$\begin{array}{ccccccc} \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) & \longrightarrow & \mathcal{E}^{(\bullet)} & \longrightarrow & \mathcal{F}^{(\bullet)} & \longrightarrow & \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})[1] \\ \downarrow{\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_X(\phi)} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \exists !} & & \downarrow{\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_X(\phi)} \\ \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}'^{(\bullet)}) & \longrightarrow & \mathcal{E}'^{(\bullet)} & \longrightarrow & \mathcal{F}'^{(\bullet)} & \longrightarrow & \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}'^{(\bullet)})[1]. \end{array} \tag{13.1.4.3.2}$$

As for [BBD82, 1.1.10], this implies that the cone of $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is unique up to canonical isomorphism. We will denote it $({}^\dagger X)(\mathcal{E}^{(\bullet)})$. We check that $({}^\dagger X)(\mathcal{E}^{(\bullet)})$ is functorial in X, and $\mathcal{E}^{(\bullet)}$, e.g. we get the morphism of the form $\mathcal{E}^{(\bullet)} \to ({}^\dagger X)(\mathcal{E}^{(\bullet)})$. We have by construction the distinguished triangle

$$\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to ({}^\dagger X)(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})[1]. \tag{13.1.4.3.3}$$

We deduce from 13.1.5.2 that the functor $({}^\dagger X)$ does not depend on the log-structure, i.e., we have the canonical isomorphism $({}^\dagger X) \circ \mathrm{forg}_\sharp \xrightarrow{\sim} \mathrm{forg}_\sharp \circ ({}^\dagger X)$ of functors $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

Definition 13.1.4.4. With notation 13.1.4.3, the functor $({}^\dagger X)\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ is called the localisation outside X functor.

13.1.4.5. For any closed subscheme X of P, for all $\mathcal{E}^{(\bullet)},\ \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$, there exists a unique isomorphism of the form

$$({}^\dagger X)(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} ({}^\dagger X)(\mathcal{F}^{(\bullet)}) \tag{13.1.4.5.1}$$

fitting into the commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)} & \to & ({}^\dagger X)(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)})[1] \\ {\scriptstyle 13.1.3.12.1}\downarrow\wr & & \| & & \downarrow{\scriptstyle \exists !} & & {\scriptstyle 13.1.3.12.1}\downarrow\wr \\ \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathcal{F}^{(\bullet)} & \to & \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} ({}^\dagger X)(\mathcal{F}^{(\bullet)}) & \to & \mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{F}^{(\bullet)})[1]. \end{array}$$

As usual (by writing parallelepipeds), we check that it is functorial by X, $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$.

Theorem 13.1.4.6. *Let X, X' be two closed subschemes of P. We have $({}^\dagger X')\circ\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$.*

Proof. By devissage (use 13.1.4.3.3), this is a straightforward consequence of lemma 13.1.3.2. □

13.1.4.7. Let X, X' be to closed subschemes of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. There then exists a unique isomorphism

$$({}^\dagger X') \circ \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_X \circ ({}^\dagger X')(\mathcal{E}^{(\bullet)}) \tag{13.1.4.7.1}$$

functorial in X, X', $\mathcal{E}^{(\bullet)}$ fitting into the commutative diagram of the form

$$\begin{array}{ccccccc}
\mathbb{R}\underline{\Gamma}^\dagger_{X'} \circ \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) & \to & ({}^\dagger X') \circ \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_{X'} \circ \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)})[1] \\
\downarrow\sim & & \| & & \downarrow\exists! & & \downarrow\sim \\
\mathbb{R}\underline{\Gamma}^\dagger_X \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_X \circ ({}^\dagger X')(\mathcal{E}^{(\bullet)}) & \to & \mathbb{R}\underline{\Gamma}^\dagger_X \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)})[1].
\end{array}$$

Let us now generalize 13.1.3.1.

Proposition 13.1.4.8. *Let Z be a divisor of P and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(Z))$. Let X be a closed subscheme of P, $\mathfrak{U}$ be the open subset of $\mathfrak{P}$ complementary to X. The following assertions are equivalent:*

(a) We have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}^\sharp/\mathfrak{S}}(Z\cap U))$ the isomorphism $\mathcal{E}^{(\bullet)}|\mathfrak{U} \xrightarrow{\sim} 0$.

(b) The canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is an isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(Z))$.

(c) We have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(Z))$ the isomorphism $({}^\dagger X)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$.

Proof. We can suppose P is integral. Let's prove $(a) \to (b)$. If T is a divisor containing X, by 13.1.3.1, the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is an isomorphism. As X is a finite intersection of divisor of P containing it we conclude $(a) \to (b)$. The converse $(b) \to (a)$ is obvious while the localization triangle at X leads to the equivalence between the assertions (b) and (c). □

Corollary 13.1.4.9. *Let T be a divisor of P, X a closed subscheme of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$ Then $({}^\dagger X)(\mathcal{E}^{(\bullet)}) = 0$ if and only if for any integer r, $({}^\dagger X)(\mathcal{H}^r(\mathcal{E}^{(\bullet)})) = 0$.*

Remark 13.1.4.10. Let X be a closed subscheme of P and $\mathcal{E}^{(\bullet)}$ be a complex of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$ such that $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) = 0$. In general, it is false that its cohomology spaces satisfies $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{H}^r(\mathcal{E}^{(\bullet)})) = 0$.

For example, if X is a smooth closed subscheme of P of pure codimension 2, 13.1.4.11 gives $\mathbb{R}\underline{\Gamma}^\dagger_X(({}^\dagger X)(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})) = 0$. Now, using the location triangle at X of $\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}$, we obtain $\mathcal{H}^1(({}^\dagger X)(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})) \xrightarrow{\sim} \mathcal{H}^{\dagger,2}_X(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})$. Finally, since $\mathcal{H}^{\dagger,2}_X(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})$ is an object of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ with support in X, then $\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{H}^1(({}^\dagger X)(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}))) \xrightarrow{\sim} \mathcal{H}^{\dagger,2}_X(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \neq 0$.

Proposition 13.1.4.11. Let $X \subset X'$ be two closed subschemes of X and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. We then have $\mathbb{R}\underline{\Gamma}^\dagger_X \circ ({}^\dagger X')(\mathcal{E}^{(\bullet)})$ =0.

Proof. By applying the local cohomological functor $\mathbb{R}\underline{\Gamma}^\dagger_X$ to the distinguished triangle $\mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)}) \to \mathcal{E} \to ({}^\dagger X')(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)})[1]$, the theorem 13.1.3.11.1 allows us to conclude. □

Lemme 13.1.4.12. Let X and X' be two closed subschemes of P. We have: $({}^\dagger X)\circ({}^\dagger X')\circ\mathbb{R}\underline{\Gamma}^\dagger_{X\cup X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) = 0$.

Proof. Let us denote by $\mathcal{E}^{(\bullet)} = ({}^\dagger X) \circ ({}^\dagger X') \circ \mathbb{R}\underline{\Gamma}^\dagger_{X\cup X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})$, $\mathfrak{U} = \mathfrak{P} \setminus X$ and $|_{\mathfrak{U}}$ the restriction functor to $\mathfrak{U}$. We have $|_{\mathfrak{U}} \circ ({}^\dagger X) \xrightarrow{\sim} |_{\mathfrak{U}}$, $|_{\mathfrak{U}} \circ ({}^\dagger X') \xrightarrow{\sim} ({}^\dagger X' \cap U) \circ |_{\mathfrak{U}}$ and $|_{\mathfrak{U}} \circ \mathbb{R}\underline{\Gamma}^\dagger_{X\cup X'} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{X'\cap U} \circ |_{\mathfrak{U}}$. This implies that $\mathcal{E}^{(\bullet)}|_{\mathfrak{U}} = 0$. It follows by devissage from 13.1.4.6, 13.1.4.7.1 and 13.1.3.11.1 that $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

It then follows from 13.1.4.8 that $({}^\dagger X)(\mathcal{E}^{(\bullet)}) = 0$. However, thanks to 13.1.4.11, $({}^\dagger X) \to ({}^\dagger X) \circ ({}^\dagger X)$ is an isomorphism of functors and therefore that $\mathcal{E}^{(\bullet)} \to ({}^\dagger X)(\mathcal{E}^{(\bullet)})$ is an isomorphism. □

Theorem 13.1.4.13. *Let X, X' be two closed subschemes of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. We have the canonical isomorphism*

$$({}^\dagger X) \circ ({}^\dagger X')(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger X \cup X')(\mathcal{E}^{(\bullet)}), \tag{13.1.4.13.1}$$

functorial in X, X', $\mathcal{E}^{(\bullet)}$.

Proof. On one hand, it follows from 13.1.4.11 that the morphism $({}^\dagger X \cup X')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger X) \circ ({}^\dagger X')({}^\dagger X \cup X')(\mathcal{E}^{(\bullet)})$ is an isomorphism. On the other hand, we deduce from 13.1.4.12 that the morphism $({}^\dagger X) \circ ({}^\dagger X')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger X) \circ ({}^\dagger X') \circ ({}^\dagger X \cup X')(\mathcal{E}^{(\bullet)})$ is an isomorphism. □

In order to prove the following theorem, we will need the following lemma.

Lemma 13.1.4.14. *Let $\mathcal{E}'^{(\bullet)} \to \mathcal{E}^{(\bullet)} \to \mathcal{E}''^{(\bullet)} \to \mathcal{E}'^{(\bullet)}[1]$ be a triangle in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ and X a closed subscheme of P. This is distinguished if and only if the triangles obtained after applying the functors $\mathbb{R}\underline{\Gamma}^\dagger_X$ and $({}^\dagger X)$ are distinguished.*

Theorem 13.1.4.15. *Let X, X' be two closed subschemes of P. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$. We have the Mayer-Vietoris distinguished triangles:*

$$\mathbb{R}\underline{\Gamma}^\dagger_{X\cap X'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X}(\mathcal{E}^{(\bullet)}) \oplus \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X\cup X'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X\cap X'}(\mathcal{E}^{(\bullet)})[1], \tag{13.1.4.15.1}$$

$$({}^\dagger X \cap X')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger X)(\mathcal{E}^{(\bullet)}) \oplus ({}^\dagger X')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger X \cup X')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger X \cap X')(\mathcal{E}^{(\bullet)})[1]. \tag{13.1.4.15.2}$$

Proof. Let us just demonstrate that the triangle 13.1.4.15.1 is distinguished, the second proving itself in the same way. By using 13.1.3.12.1 and 13.1.4.5.1, it is enough to prove it for $\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}$. Now, the functors $\mathbb{R}\underline{\Gamma}^\dagger_X$ and $({}^\dagger X)$ applied to the exact triangle 13.1.4.15.1 for $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}$ give respectively

$$\mathbb{R}\underline{\Gamma}^\dagger_{X\cap X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \oplus \mathbb{R}\underline{\Gamma}^\dagger_{X\cap X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X\cap X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})[1]$$
$$0 \to 0 \oplus ({}^\dagger X)\mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \to ({}^\dagger X)\mathbb{R}\underline{\Gamma}^\dagger_{X\cup X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \to 0.$$

It is immediate that the first triangle is distinguished while the last is if and only if the morphism $({}^\dagger X)\mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}}) \to ({}^\dagger X)\mathbb{R}\underline{\Gamma}^\dagger_{X\cup X'}(\mathcal{O}^{(\bullet)}_{\mathfrak{P},\mathbb{Q}})$ is an isomorphism. Now, it follows from 13.1.4.8 that it suffices to verify it above $P \setminus X$, which is immediate. Thanks to 13.1.4.14, we conclude that 13.1.4.15.1 is distinguished. □

Example 13.1.4.16. Let $X_1, \ldots, X_r$ be r closed subschemes of P which are two by two disjoint. Let $X := X_1 \cup \cdots \cup X_r$. By induction in r, it follows from the Mayer-Vietoris exact sequence that the canonical morphism

$$\oplus_{j=1}^r \mathbb{R}\underline{\Gamma}^\dagger_{X_j}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{X}(\mathcal{E}^{(\bullet)}). \tag{13.1.4.16.1}$$

is an isomorphism.

13.1.4.17 (Support)**.** Let T be a divisor of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$. The support of $\mathcal{E}^{(\bullet)}$ is by definition the biggest closed subscheme X of P such that $({}^\dagger X)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ (one of the equivalent conditions of 13.1.4.8). We denote it by $\mathrm{Supp}\, \mathcal{E}^{(\bullet)}$.

Remark if $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$, then this is equal to the support (for the usual definition) of the coherent $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module $\underrightarrow{l}^*_{\mathbb{Q}}\, \mathcal{E}^{(\bullet)}$, which justifies the terminology. We will extend later the proposition 13.1.4.8 (see 15.3.8.2) and the notion of support (15.3.8.1) in the case where T is not a divisor but we will need the notion of partially overcoherent complexes.

13.1.4.18 (t-structure for coherent complexes)**.** For a definition of t-structure, see [HTT08, 8.1.1]. Let T be a divisor of P. Following 8.7.5.4.1, for any $* \in \{\mathrm{r}, \mathrm{l}\}$, the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ induces the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)) \cong D^{\mathrm{b}}_{\mathrm{coh}}({}^*\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}})$. Since the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ preserves distinguished triangles, then we get the canonical t-structure on $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^\sharp/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}})$ induces a t-structure on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$. For any $n \in \mathbb{N}$, the functor $H^n\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T)) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$ of 8.4.5.4 and the usual functor $H^n\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}) \to \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$ commutes with the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$. In other words, the functor H^n constructed at 8.4.5.4 is the one induced by the t-structure of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$ via the equivalence of categories $\underrightarrow{l}^{*}_{\mathbb{Q}}$.

Let $n \in \mathbb{Z}$. Let us denote by $\underrightarrow{LD}^{\leq n}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}(T))$ (resp. $\underrightarrow{LD}^{\geq n}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}(T))$) the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}(T))$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that, for any $j \geq n+1$ (resp. for any $j \leq n-1$), we have $H^j(\mathcal{E}^{(\bullet)}) = 0$. We denote by $\tau^{\star}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}(T)) \to \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}(T))$ the canonical truncation functors, where $\star$ means either $\leq n$ or $\geq n$.

Remark 13.1.4.19. Since $\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$ is an abelian category, then we have a canonical t-structure on $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}))$ given by the standard truncation functors. By using the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}) \cong D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}}))$, of 8.1.5.14.1, this yields a canonical t-structure on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$.

Since the truncation functors do not a priori preserve quasi-coherence, beware that there is no clear that $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$ can be endowed with a t-structure induced by that of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$.

13.1.4.20. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. With notation 13.1.4.18 and 13.1.4.17, we have the equality

$$\mathrm{Supp}\,\mathcal{E}^{(\bullet)} = \cup_{j\in\mathbb{Z}}\,\mathrm{Supp}\,H^j(\mathcal{E}^{(\bullet)}). \tag{13.1.4.20.1}$$

13.1.5 Local cohomological functor with strict support over a subvariety

13.1.5.1. Let X, X', T, T' be closed subschemes of P such that $X \setminus T = X' \setminus T'$. For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$, we have the canonical isomorphism:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X}({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}({}^{\dagger}T')(\mathcal{E}^{(\bullet)}). \tag{13.1.5.1.1}$$

Indeed, $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}({}^{\dagger}T)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{E}^{(\bullet)}$, and similarly with some primes. Hence, we reduce to the case $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{P}}$. Using 13.1.4.6, 13.1.3.11.1, 13.1.4.13.1, 13.1.4.8, we get the isomorphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}({}^{\dagger}T)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X\cap X'}({}^{\dagger}T\cup T')(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$. We conclude by symmetry.

Setting $Y := X \setminus T$, we denote by $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)})$ one of both complexes of 13.1.5.1.1.

13.1.5.2. Let Y be a subscheme of P. We deduce from 13.1.5.2 that the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}$ does not depend on the log-structure, i.e., we have the canonical isomorphism

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} \circ \mathrm{forg}_{\sharp} \xrightarrow{\sim} \mathrm{forg}_{\sharp} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Y} \tag{13.1.5.2.1}$$

of functors of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}/\mathfrak{S}})$.

Notation 13.1.5.3. Let Y be a subvariety of P and T be a divisor of P. Using notation 9.1.6.6, we get the functor

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} := \mathrm{Coh}_{T}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y})\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}).$$

If we want to distinguish both functors $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}$, then we will denote in this case by $\mathbb{R}\underline{\Gamma}^{(\bullet)}_{Y}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^{\sharp}})$. Finally, for any $n \in \mathbb{N}$, we denote by $\mathcal{H}^{\dagger n}_{Y} := \mathcal{H}^n \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}) \to M(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})$, where $M(\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})$ is the category of left $\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules.

Lemma 13.1.5.4. *Let X be a closed subscheme of P, T be a divisor of P, $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-module.*

(a) For any $j \leq -1$, we have $H^j(\mathcal{E}({}^{\dagger}X)) = 0$.

(b) The canonical morphism $\mathcal{H}^{\dagger 0}_{X}(\mathcal{E}) \to \mathcal{E}$ is injective.

Proof. a) We can suppose P is integral. When X is a divisor of P, this is a consequence of the exactness of the functor $({}^\dagger Z)$ on the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules. This yields the general case via the distinguished triangle of Mayer-Vietoris (see 13.1.4.15), by proceeding by induction on the minimal number of divisors whose intersection is Y (recall 13.1.3.6).

b) Considering the long exact sequence induced by the triangle of localisation of $\mathcal{E}$ with respect to X, the fact that $H^{-1}(\mathcal{E}({}^\dagger Y)) = 0$ implies the desired injectivity. □

Proposition 13.1.5.5. *We have* $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

Proof. This is a translation of 13.1.4.6. □

13.1.5.6. Let Y and Y' be two subschemes of P. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$.

(a) Using 13.1.3.11.1, 13.1.4.13.1, we get the canonical isomorphism functorial in $\mathcal{E}^{(\bullet)}$, Y, and Y':

$$\mathbb{R}\underline{\Gamma}^\dagger_Y \circ \mathbb{R}\underline{\Gamma}^\dagger_{Y'}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Y\cap Y'}(\mathcal{E}^{(\bullet)}). \tag{13.1.5.6.1}$$

(b) Using 13.1.3.11.2 and 13.1.4.5.1 we get the canonical isomorphism functorial in $\mathcal{E}^{(\bullet)}$, $\mathcal{F}^{(\bullet)}$, Y, and Y':

$$\mathbb{R}\underline{\Gamma}^\dagger_{Y\cap Y'}(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathbb{R}\underline{\Gamma}^\dagger_{Y'}(\mathcal{F}^{(\bullet)}). \tag{13.1.5.6.2}$$

(c) If Y' is an open (resp. a closed) subscheme of Y, we have the canonical homomorphism $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{Y'}(\mathcal{E}^{(\bullet)})$ (resp. $\mathbb{R}\underline{\Gamma}^\dagger_{Y'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)})$). If Y' is a closed subscheme of Y, we have the localization distinguished triangle

$$\mathbb{R}\underline{\Gamma}^\dagger_{Y'}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{Y\setminus Y'}(\mathcal{E}^{(\bullet)}) \to +1. \tag{13.1.5.6.3}$$

Proposition 13.1.5.7 (Commutation with base change)**.** *Let $\mathcal{V} \to \mathcal{V}'$ be a morphism of rings of complete discrete valuation of unequal characteristics $(0,p)$, $\mathfrak{S}' := \operatorname{Spf}\mathcal{V}'$. Let $\mathfrak{P}' := \mathfrak{P} \times_{\operatorname{Spf}(\mathcal{V})} \operatorname{Spf}(\mathcal{V}')$ be the separated, smooth and quasi-compact $\mathcal{V}$-formal scheme induced by base change, $\varpi\colon \mathfrak{X}' \to \mathfrak{X}$ the canonical projection, $\mathfrak{D}' := \varpi^{-1}(\mathfrak{D})$ the relative strict normal crossing divisor on $\mathfrak{P}'/\mathfrak{S}'$ induced by base change, $\mathfrak{P}'^\sharp := (\mathfrak{P}', \mathfrak{D}')$ the $\mathfrak{S}'$-smooth logarithmic formal scheme whose log-structure is given by $\mathfrak{D}'$ and $\varpi^\sharp\colon \mathfrak{P}'^\sharp \to \mathfrak{P}^\sharp$ the canonical projection. Let Y be a subvariety of P. Let $Y' := \varpi^{-1}(Y)$. For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{-}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}^\sharp}(Z_1))$, we have the isomorphism*

$$\mathbb{R}\underline{\Gamma}^\dagger_{Y'} \circ \varpi^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \varpi^{\sharp(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)}) \tag{13.1.5.7.1}$$

Proof. This follows by devissage and by Mayer-Vietoris exact triangles from 9.2.6.8. □

13.2 Fundamental properties

13.2.1 Commutation with local cohomological functors

Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of smooth $\mathcal{V}$-formal schemes, T and T' of the respective divisors of P and P' such that $f(P' \setminus T') \subset P \setminus T$, $\mathfrak{D}$ and $\mathfrak{D}'$ be relative to respectively $\mathfrak{P}/\mathfrak{S}$ and $\mathfrak{P}'/\mathfrak{S}$ strict normal crossing divisors such that $f^{-1}(\mathfrak{D}) \subset \mathfrak{D}'$. We set $\mathfrak{P}^\sharp := (\mathfrak{P}, M(\mathfrak{D}))$, $\mathfrak{P}'^\sharp := (\mathfrak{P}', M(\mathfrak{D}'))$ and $f^\sharp\colon \mathfrak{P}'^\sharp \to \mathfrak{P}^\sharp$ the morphism of $\mathfrak{S}$-log smooth formal logarithmic schemes induced by f. If $\mathfrak{U}$ is an open of $\mathfrak{P}$, we will write $\mathfrak{U}^\sharp := (\mathfrak{U}, M(\mathfrak{D} \cap \mathfrak{U}))$.

13.2.1.1. With the notations of 13.1.1.7, for all $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ and for all $\mathcal{E}^{(\bullet)}_\bullet \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{P_\bullet}(T))$ we verify that the canonical morphisms

$$f^{\sharp(\bullet)!}_{T',T} \circ \mathrm{forg}_\sharp(\mathcal{E}^{(\bullet)}) \to \mathrm{forg}_\sharp \circ f^{(\bullet)!}_{T',T}(\mathcal{E}^{(\bullet)}), \tag{13.2.1.1.1}$$

$$f^{\sharp(\bullet)!}_{\bullet T',T} \circ \mathrm{forg}_\sharp(\mathcal{E}^{(\bullet)}_\bullet) \to \mathrm{forg}_\sharp \circ f^{(\bullet)!}_{\bullet T',T}(\mathcal{E}^{(\bullet)}_\bullet) \tag{13.2.1.1.2}$$

are isomorphisms. Indeed, the fact that 13.2.1.1.2 is an isomorphism follows from 9.2.1.9.1. The yields so is 13.2.1.1.1.

13.2.1.2. In this paragraph, assume that $f^{-1}(\mathfrak{D}) = \mathfrak{D}'$ and $f^\sharp$ is exact. For all $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'))$, the canonical morphism

$$f^{(\bullet)}_{T,T',+} \circ \mathrm{forg}_\sharp(\mathcal{E}'^{(\bullet)}) \to \mathrm{forg}_\sharp \circ f^{\sharp(\bullet)}_{T,T',+}(\mathcal{E}'^{(\bullet)}) \tag{13.2.1.2.1}$$

is then an isomorphism. Indeed, since $f^\sharp$ is the composition of its graph with the canonical projection $\mathfrak{P}'^\sharp \times \mathfrak{P}^\sharp \to \mathfrak{P}^\sharp$, we reduce to the case where $f^\sharp$ is an exact closed immersion or a smooth (i.e. exact and log smooth) morphism. In the case where $f^\sharp$ is an exact closed immersion, this follows from an easy calculation in local coordinates (e.g. 9.3.3.6.1) and that where $f^\sharp$ is a smooth morphism, this is a consequence of 9.4.1.6 (and the fact that $\Omega^\bullet_{\mathfrak{P}'/\mathfrak{P}} = \Omega^\bullet_{\mathfrak{P}'^\sharp/\mathfrak{P}^\sharp}$).

Without the hypothesis that $f^\sharp$ is exact, we will take care of the fact that the morphism 13.2.1.2.1 is no longer an isomorphism (just consider the case $f = \mathrm{id}$).

Lemma 13.2.1.3. *Let $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ be a closed immersion of smooth formal $\mathfrak{S}$-scheme. Let T be a divisor of P such that $u(X) \subset T$. We have the canonical isomorphism: $u^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$.*

Proof. 0) Following the proposition 8.3.3.5, the Lemma is local en $\mathfrak{P}$. So, we can suppose $\mathfrak{P}$ affine and endowed with coordinates $t_1, \ldots, t_d$ such that $\mathfrak{X} = V(t_1, \ldots, t_r)$ and there exists an element f of $O_{\mathfrak{P}}$ satisfying $T = V(\overline{f})$, where $\overline{f}$ is the reduction of f modulo $\pi O_{\mathfrak{P}}$. Let us check by induction on $r \geq 1$ the lemma.

1) Let us first deal with the case $r = 1$. Thanks to 9.2.1.17, we reduce to prove the isomorphism $u^{(\bullet)!}_{\mathrm{alg}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. As $\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)$ is a sheaf of integral rings, using the exact sequence $0 \to u^{-1}\mathcal{O}_{\mathfrak{P}} \xrightarrow{t_1} u^{-1}\mathcal{O}_{\mathfrak{P}} \to \mathcal{O}_{\mathfrak{X}} \to 0$ which solves $\mathcal{O}_{\mathfrak{X}}$ by flat $u^{-1}\mathcal{O}_{\mathfrak{P}}$-modules, we verify that the canonical morphism of the form $\mathcal{O}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T) \to \mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)$ is then an isomorphism in $D(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. Thus, we have the canonical isomorphism $u^{(\bullet)!}_{\mathrm{alg}}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)[d_{X/P}]$ in $D(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ and therefore in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. Let $\chi\colon \mathbb{N} \to \mathbb{N}$ be the element of M (see the definition in paragraph 8.1.5.1) defined by $\chi(m) = 1$. It is then sufficient to verify that the canonical morphism of $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}$-modules

$$\mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T) \to \chi^*(\mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T))$$

is the null morphism (in other words, $\mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)$ is canceled by multiplication by p). As $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{P}}(T)$ does not depend, up to canonical isomorphism, from the choice of the lifting f of a local equation of T (see 8.7.3.4), we can suppose that t divides f, i.e. the image of f on $\mathcal{O}_{\mathfrak{X}}$ is zero. In this case, we calculate that $\mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{P}}(T) = \mathcal{O}_{\mathfrak{X}} \otimes_{u^{-1}\mathcal{O}_{\mathfrak{P}}} u^{-1}\mathcal{O}_{\mathfrak{P}}\{X\}/(f^{p^{m+1}}X - p) = \mathcal{O}_{\mathfrak{X}}\{X\}/(p) = \mathcal{O}_X[X]$, which is canceled by p. Hence the result.

2) Now assume the property is true for $r-1$ and prove it for r. Let us denote $\mathfrak{X}_1 := V(t_1)$, and by $u_1\colon \mathfrak{X}_1 \hookrightarrow \mathfrak{P}$ and $v_1\colon \mathfrak{X} \hookrightarrow \mathfrak{X}_1$ the closed immersions. We can suppose X_1 is integral.

i) If $T \cap X_1$ is a divisor of X, following lemma 9.2.1.26, $u_1^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} \widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}_1}(T \cap X_1))[-1]$. By using the induction hypothesis, we get the last isomorphism $u^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} v_1^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{X}_1}(T \cap X_1))[-1]) \xrightarrow{\sim} 0$, hence we are done.

ii) If $T \supset X_1$, then from the case $r = 1$, we get $u_1^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} 0$, and then $u^{(\bullet)!}(\widetilde{\mathcal{B}}^{(\bullet)}_{\mathfrak{P}}(T)) \xrightarrow{\sim} 0$. □

Theorem 13.2.1.4. *Let Y be a subscheme of P, $Y' := f^{-1}(Y)$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ and $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}})$. We have the functorial in Y isomorphisms:*

$$f^{\sharp(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Y'} \circ f^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}), \tag{13.2.1.4.1}$$

$$\mathbb{R}\underline{\Gamma}^\dagger_Y \circ f^{\sharp(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f^{\sharp(\bullet)}_+ \circ \mathbb{R}\underline{\Gamma}^\dagger_{Y'}(\mathcal{E}'^{(\bullet)}). \tag{13.2.1.4.2}$$

Proof. 1) Let us check 13.2.1.4.1. With 13.1.5.2.1 and 13.2.1.1, we can suppose logarithmic structures are trivial, i.e. $\mathfrak{D}$ and $\mathfrak{D}'$ are empty. Using 9.2.1.27.1 and 13.1.5.6.2, we reduce to the case where Y is the complement of a divisor T and $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{P}}$. The morphism f is the composition of its graph $\gamma_f\colon \mathfrak{P}' \hookrightarrow \mathfrak{P}' \times \mathfrak{P}$ with the projection $\mathfrak{P}' \times \mathfrak{P} \to \mathfrak{P}$. Since the case where f is a flat morphism follows from 9.2.1.26, we reduce to the case where f is a closed immersion. We can suppose P' is integral. Either

$T \cap P'$ is a divisor and we can conclude by using 9.2.1.26, or $T \cap P' = P'$ and then the isomorphism 13.2.1.4.1 is $0 \xrightarrow{\sim} 0$ following proposition 13.2.1.3.

2) Let us check that 13.2.1.4.2 is a consequence of 13.2.1.4.1.

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} \circ f^{\sharp(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \underset{13.1.5.6.2}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} f^{\sharp(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \underset{9.4.3.1}{\xrightarrow{\sim}} f^{\sharp(\bullet)}_{+}(f^{\sharp(\bullet)!}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}))\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}'}}\mathcal{E}'^{(\bullet)}))[-d_{P'/P}] \tag{13.2.1.4.3}$$

Using 13.2.1.4.1, we get $f^{\sharp(\bullet)!}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}))[-d_{P'/P}] \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Y'}\mathcal{O}^{(\bullet)}_{\mathfrak{P}'}$. Hence we are done. □

The following corollary means that the definitions 12.1.4.1.1 and 13.1.3.8 coincide.

Corollary 13.2.1.5. *Suppose $f^\sharp$ is an exact closed immersion. Let denote $f^\sharp$ (resp. $\mathfrak{P}'$) by $u^\sharp$ (resp. by $\mathfrak{X}$). For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$, with notation 13.1.3.8, we have the isomorphism*

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{\sharp(\bullet)}_{+} \circ u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}). \tag{13.2.1.5.1}$$

Proof. As the immersion $u^\sharp$ is exact, by 13.1.5.2.1, 13.2.1.1 and 13.2.1.2.1, we can suppose the logarithmic structures are trivial. Using 9.4.3.2.1 and 13.1.3.11.2, we reduce to the case where $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{P}}$. From Berthelot-Kashiwara's theorem 9.3.5.13, since $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$ is coherent with support in X (see 13.1.4.6), we get

$$u^{(\bullet)}_{+}u^{(\bullet)!}\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}).$$

On the other hand,

$$u^{(\bullet)!}\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \underset{13.2.1.4.1}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}u^{(\bullet)!}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \xrightarrow{\sim} u^{(\bullet)!}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}).$$

Hence $u^{(\bullet)}_{+}u^{(\bullet)!}\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \xrightarrow{\sim} u^{(\bullet)}_{+}u^{(\bullet)!}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$, and we are done. □

Corollary 13.2.1.6. *Suppose $f^\sharp$ is an exact closed immersion. Let denote $f^\sharp$ (resp. $\mathfrak{P}'$) by $u^\sharp$ (resp. by $\mathfrak{X}$). Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}^\sharp/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$ with support in X. With notation 13.1.5.3, the natural morphism*

$$\mathcal{H}^{\dagger 0}_{Z}(\mathcal{E}) \to \mathcal{E} \tag{13.2.1.6.1}$$

is an isomorphism.

Proof. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}})$ such that $\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}$. Then this follows from Berthelot-Kashiwara theorem 9.3.5.13 and from 13.2.1.5. □

Proposition 13.2.1.7 (Compatibility with Frobenius)**.** *Suppose k is perfect. Let $u\colon \mathfrak{X} \to \mathfrak{P}$ be a closed immersion of smooth $\mathcal{V}$-formal schemes. Let T be a divisor of X such that $Z := T \cap X$ is a divisor of X. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ such $u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(Z))$ (e.g. if $\mathcal{E}^{(\bullet)}$ has its support in X). Then we have a compatible with Frobenius isomorphism of the form*

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}). \tag{13.2.1.7.1}$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} u^{(\bullet)}_{+} \circ u^{(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) & \xrightarrow{\mathrm{adj}_{\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)})}} & \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \\ \downarrow & & \downarrow \\ u^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) & \xrightarrow{\mathrm{adj}_{\mathcal{E}^{(\bullet)}}} & \mathcal{E}^{(\bullet)}, \end{array} \tag{13.2.1.7.2}$$

where the horizontal arrows are the adjunction morphism of 9.5.4.5 and vertical arrows follows by functoriality in X of the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}$.

1) Following 9.5.4.5, the adjunction morphism $\mathrm{adj}_{\mathcal{E}^{(\bullet)}}$ and $\mathrm{adj}_{\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)})}$ are compatible with Frobenius.

2) It follows from 13.2.1.5.1 that $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ and has its support in X. Hence, by using Berthelot-Kashiwara theorem of the form 9.3.5.13, we can check that the adjunction morphism $\mathrm{adj}_{\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)})}$ is an isomorphism.

3) Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{X} = \mathrm{id}$ on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$, it follows from 13.2.1.4.1 that the canonical morphism $u^{(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \to u^{(\bullet)!}(\mathcal{E}^{(\bullet)})$ is an isomorphism. Hence, the left morphism of the square 13.2.1.7.2 is a isomorphism. We get the compatible with Frobenius isomorphism 13.2.1.7.1 fitting into the commutative diagram

$$\begin{array}{ccc} u^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) & \xrightarrow{\mathrm{adj}_{\mathcal{E}^{(\bullet)}}} & \mathcal{E}^{(\bullet)} \\ \sim\downarrow & & \| \\ \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) & \longrightarrow & \mathcal{E}^{(\bullet)}. \end{array} \tag{13.2.1.7.3}$$

□

13.2.2 Coherence of the localisation of convergent isocrystals

Theorem 13.2.2.1. *Let $\mathfrak{P}$ be a smooth formal scheme over $\mathfrak{S}$. Let X be a smooth closed subscheme of P, and T be a divisor of X. Let $\mathcal{E}^{(\bullet)}$ be an object of $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}/\mathcal{V})$ (as defined in 12.2.1.6). Then $({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$.*

Proof. 1) Using the inductive system version of Berthelot-Kashiwara's theorem (see 9.3.5.13), we reduce to the case where $X = P$. In this case, we write $\mathfrak{X}$ (resp. Z) instead of $\mathfrak{P}$ (resp. T) and we will use the notation of the proof of 12.2.7.1. Now, following the part 0),1) and 2) of the proof of 12.2.7.1, modulo the equivalence of categories 8.4.5.6 and the compatibility of 13.1.3.10, the object $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ is a direct summand of $f^{(\bullet)}_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}^{(\bullet)}_{\mathfrak{P}}[n])$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$. This yields that $({}^{\dagger}Z)(\mathcal{E}^{(\bullet)})$ is a direct summand of

$$\begin{aligned} &({}^{\dagger}Z)\left(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} f^{(\bullet)}_{+}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}^{(\bullet)}_{\mathfrak{P}}[n])\right) \underset{9.4.3.1.1}{\xrightarrow{\sim}} ({}^{\dagger}Z) f^{(\bullet)}_{+}\left(f^{(\bullet)!}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\mathcal{O}^{(\bullet)}_{\mathfrak{P}}\right) \\ &\underset{13.1.5.6.2}{\xrightarrow{\sim}} ({}^{\dagger}Z) f^{(\bullet)}_{+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)})\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \underset{13.2.1.4.2}{\xrightarrow{\sim}} f^{(\bullet)}_{+}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'\setminus Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}). \end{aligned} \tag{13.2.2.1.1}$$

Hence, we reduce to check that $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'\setminus Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$ is an object of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$. Since this is local on $\mathfrak{P}$, we can suppose there exists a closed immersion of smooth $\mathcal{V}$-formal schemes $u\colon \mathfrak{X}' \hookrightarrow \mathfrak{P}$ which reduces modulo π to $X' \hookrightarrow P$. Following 13.2.1.5.1, $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+} \circ u^{(\bullet)!} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$. Hence, $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'\setminus Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+} \circ ({}^{\dagger}Z') \circ u^{(\bullet)!} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$. Following 11.1.3.6, we get $\mathcal{E}'^{(\bullet)} := u^{(\bullet)!} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}'/\mathcal{V})$. Since u is proper, then $u^{(\bullet)}_{+}$ preserves the coherence. Hence, we reduce to check that $({}^{\dagger}Z')(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}})$.

2) Since this is local, we can suppose $\mathfrak{X}'$ is integral and affine. We proceed by induction on the number of irreducible component of Z'. Let Z'_1 be one irreducible component of Z' and Z'' be the union of the other irreducible components. Let $u_1\colon \mathfrak{Z}'_1 \hookrightarrow \mathfrak{X}'$ be a lifting of $Z'_1 \hookrightarrow X'$, and $Z'_2 := Z'_1 \cap Z''$.

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'_1}(({}^{\dagger}Z'')\mathcal{E}'^{(\bullet)}) \to ({}^{\dagger}Z'')(\mathcal{E}'^{(\bullet)}) \to ({}^{\dagger}Z')(\mathcal{E}'^{(\bullet)}) \to +1 \tag{13.2.2.1.2}$$

Following 11.1.3.6, we get $\mathbb{L}u^{*(\bullet)}_1(\mathcal{E}'^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{Z}'_1/\mathcal{V})$. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'_1}(({}^{\dagger}Z'')\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{1+} \circ u^{(\bullet)!}_1(({}^{\dagger}Z'')\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{1+} \circ ({}^{\dagger}Z'_2)(u^{(\bullet)!}_1\mathcal{E}'^{(\bullet)})$, by induction hypothesis and preservation of the coherence under $u^{(\bullet)}_{1+}$, this yields $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'_1}(({}^{\dagger}Z'')\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}})$. By induction hypothesis $({}^{\dagger}Z'')(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'/\mathfrak{S}})$. We conclude using the exact triangle 13.2.2.1.2.

□

13.2.3 Base change isomorphism for coherent complexes and realizable morphisms

Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of separated $\mathcal{V}$-smooth formal schemes. Let $\mathfrak{D}$ (resp. $\mathfrak{D}'$) be a relative strict normal crossing divisors of $\mathfrak{P}/\mathfrak{S}$ (resp. $\mathfrak{P}'/\mathfrak{S}$) such that $f^{-1}(\mathfrak{D}) \subset \mathfrak{D}'$. We set $\mathfrak{P}^{\sharp} = (\mathfrak{P}, \mathfrak{D})$, $\mathfrak{P}'^{\sharp} := (\mathfrak{P}', \mathfrak{D}')$ the induced $\mathfrak{S}$-log smooth log formal schemes and $f^{\sharp}\colon \mathfrak{P}'^{\sharp} \to \mathfrak{P}^{\sharp}$ the induced morphisms of $\mathfrak{S}$-log smooth log formal schemes. Let T be a divisor of P such that $T' := f^{-1}(T)$ is a divisor of P'.

Definition 13.2.3.1. We say that $f^\sharp$ is *realizable with respect to* T if there exist a proper morphism $\pi\colon \mathfrak{P}'' \to \mathfrak{P}$ of smooth $\mathcal{V}$-formal schemes $\pi^{-1}(T)$ is a divisor of P'', a relative strict normal crossing divisors $\mathfrak{D}''$ of $\mathfrak{P}''/\mathfrak{S}$ such that $\pi^{-1}(\mathfrak{D}) \subset \mathfrak{D}''$, an exact immersion $u^\sharp\colon \mathfrak{P}'^\sharp \hookrightarrow \mathfrak{P}''^\sharp$ of log formal schemes such that $f^\sharp = \pi^\sharp \circ u^\sharp$, where $\mathfrak{P}''^\sharp := (\mathfrak{P}'', \mathfrak{D}'')$ and $\pi^\sharp\colon \mathfrak{P}''^\sharp \to \mathfrak{P}^\sharp$ is the morphism induced by π.

When T is empty, we simply say that $f^\sharp$ is a realizable morphism. When $\mathfrak{P}^\sharp = \operatorname{Spf} \mathcal{V}$ and T is empty, we say that $\mathfrak{P}'^\sharp$ is a realizable $\mathfrak{S}$-log smooth log formal scheme.

Example 13.2.3.2. If f is proper, then $f^\sharp$ is a realizable morphism with respect to T. If $f^\sharp$ is an exact immersion, then $f^\sharp$ is a realizable morphism with respect to T.

When log structures are trivial, if $\mathfrak{P}'$ is realizable, then any f is a realizable morphism with respect to T.

Remark 13.2.3.3. We have the following properties.

(a) The morphism f is realizable with respect to T (in the sense of 13.2.3.1 when $\mathfrak{D}$ and $\mathfrak{D}'$ are empty) if and only if there exist a proper morphism $\pi\colon \mathfrak{P}'' \to \mathfrak{P}$ of smooth $\mathcal{V}$-formal schemes, $\pi^{-1}(T)$ is a divisor of P'', an immersion $u\colon \mathfrak{P}' \hookrightarrow \mathfrak{P}''$ of formal schemes such that $f = \pi \circ u$.

(b) If $f^\sharp$ is realizable with respect to T then so is f. The converse is not clear.

Proposition 13.2.3.4. *Suppose $f^\sharp$ is realizable with respect to T in the sense of 13.2.3.1. For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}}(T'))$ with proper support over P (i.e., if X' is the support of $\mathcal{E}'^{(\bullet)}$ in the sense 13.1.4.17 then the composite $X' \hookrightarrow P' \xrightarrow{f} P$ is proper), the object $f^{\sharp(\bullet)}_{+}(\mathcal{E}'^{(\bullet)})$ belongs to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$.*

Proof. Let X' be the support of $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}}(T'))$. By assumption, X' is proper over P via f. Let $\pi\colon \mathfrak{P}'' \to \mathfrak{P}$ be a proper morphism of smooth $\mathcal{V}$-formal schemes, $\mathfrak{D}''$ be a relative strict normal crossing divisors of $\mathfrak{P}''/\mathfrak{S}$ such that $\pi^{-1}(\mathfrak{D}) \subset \mathfrak{D}''$ and $T'' := \pi^{-1}(T)$ is a divisor of P'', let $u^\sharp\colon \mathfrak{P}'^\sharp \hookrightarrow \mathfrak{P}''^\sharp$ be an exact immersion of log formal schemes such that $f^\sharp = \pi^\sharp \circ u^\sharp$, where $\mathfrak{P}''^\sharp := (\mathfrak{P}'', M_{\mathfrak{D}''})$ and $\pi^\sharp\colon \mathfrak{P}''^\sharp \to \mathfrak{P}^\sharp$ is the morphism induced by π.

Let $v^\sharp\colon \mathfrak{P}'^\sharp \hookrightarrow \mathfrak{U}''^\sharp$ be an exact closed immersion, and $j^\sharp\colon \mathfrak{U}''^\sharp \hookrightarrow \mathfrak{P}''^\sharp$ be an open immersion such that $u^\sharp = j^\sharp \circ v^\sharp$. Since X' is proper over P via f and since π is proper, then X' is proper over P'' via u. Since v is proper, then it follows from 9.4.2.6 that we get $\mathcal{E}''^{(\bullet)} := v^{\sharp(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U''))$ with proper support over P'' via j.

a) We check in this step that $\mathbb{R}j_*\mathcal{E}''^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''^\sharp/\mathfrak{S}}(T''))$ (with support in X'). Following 8.4.5.6.2, we have the equivalence of triangulated categories

$$\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U'')) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U''))).$$

Hence, we reduce by devissage to the case where $\mathcal{E}''^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U''))$. Following 8.4.5.10, there exists $m_0 \in \mathbb{N}$ large enough such that $\mathcal{E}''^{(\bullet)}$ is isomorphic in $\underrightarrow{LM}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U''))$ to a locally finitely presented $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U'')$-module $\mathcal{G}^{(\bullet)}$. Since $\mathcal{E}''^{(\bullet)}$ is null on the open complementary to X', then, taking larger m_0 is necessary, we can suppose $\mathcal{G}^{(0)}$ is a coherent $\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U'')$-module with support in X' (i.e. $\mathcal{G}^{(0)}$ is zero on the open complementary to X'). This yields that $\mathcal{G}^{(m)}$ is a coherent $\widetilde{\mathcal{D}}^{(m+m_0)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U'')$-module with support in X'. Since X' is closed in P'', a coherent $\widetilde{\mathcal{D}}^{(m+m_0)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U'')$-module with support in X' is acyclic for the functor j_*. Moreover, the functors j_* and j^* induce quasi-inverse equivalences of categories between the category of coherent $\widetilde{\mathcal{D}}^{(m+m_0)}_{\mathfrak{U}''^\sharp/\mathfrak{S}}(T'' \cap U'')$-modules with support in X' and that of coherent $\widetilde{\mathcal{D}}^{(m+m_0)}_{\mathfrak{P}''^\sharp/\mathfrak{S}}(T'')$-modules with support in X'. This yields that the canonical morphism $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{P}''^\sharp/\mathfrak{S}}(T'') \otimes_{\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{P}''^\sharp/\mathfrak{S}}(T'')} j_*\mathcal{G}^{(0)} \to j_*\mathcal{G}^{(\bullet)}$ is an isomorphim and that $j_*\mathcal{G}^{(\bullet)}$ is a locally finitely presented $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{P}''^\sharp/\mathfrak{S}}(T'')$-module with support in X'. Moreover, since the functor j_* is exact over the category of sheaves with support in X' (because X' is closed in P'' and in U''), then the canonical morphism $j_*\mathcal{G}^{(\bullet)} \to \mathbb{R}j_*\mathcal{G}^{(\bullet)}$ is an isomorphism. Hence, $j^{\sharp(\bullet)}_{+}\mathcal{E}''^{(\bullet)} = \mathbb{R}j_*\mathcal{E}''^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''^\sharp/\mathfrak{S}}(T''))$.

b) Since π is proper, by using 9.4.2.4 and the part a) we get $\pi^{\sharp(\bullet)}_{+} j^{\sharp(\bullet)}_{+}(\mathcal{E}''^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$. By transitivity of the push forwards, we get the isomorphism $f^{\sharp(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \pi^{\sharp(\bullet)}_{+} j^{\sharp(\bullet)}_{+}(\mathcal{E}''^{(\bullet)})$. Hence, we are done. □

Remark 13.2.3.5. In the overcoherent case, we can remove the hypothesis that f realizable (see 13.2.3.4).

Corollary 13.2.3.6. *Suppose $f\colon \mathfrak{P}' \to \mathfrak{P}$ is an open immersion of smooth $\mathcal{V}$-formal schemes and $f^\sharp$ is exact. Let X' be a closed subscheme of P' such that the composite $X' \hookrightarrow P' \xrightarrow{f} P$ is proper. Then the functors $f_+^{\sharp(\bullet)}$ and $f^{\sharp(\bullet)!}$ induce quasi-inverse equivalences of categories between the subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}}(T'))$ consisting of objects with support in X' and the subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$ consisting of objects with support in X'.*

Proof. Denote by $\mathfrak{C}$ (resp. $\mathfrak{C}'$) the subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}^\sharp/\mathfrak{S}}(T))$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}}(T'))$) consisting of objects with support in X'. From 13.2.3.4, the functor $f_+^{\sharp(\bullet)} = \mathbb{R}f_*$ is well defined. This is also obvious for $f^{\sharp(\bullet)!} = f^*$. We easily check that the canonical morphism $f^*\mathbb{R}f_* \to \mathrm{id}$ of functors $\mathfrak{C}' \to \mathfrak{C}'$ is an isomorphism and that the functor $f^*\colon \mathfrak{C} \to \mathfrak{C}'$ is faithful. Hence, the morphism $\mathrm{id} \to \mathbb{R}f_* f^*$ of functors $\mathfrak{C} \to \mathfrak{C}$ is an isomorphism. □

Theorem 13.2.3.7. *Suppose $f^\sharp$ is realizable with respect to T in the sense of 13.2.3.1. Let $g\colon \mathfrak{Q} \to \mathfrak{P}$ be a smooth morphism of $\mathfrak{S}$-smooth formal schemes. We denote by $\mathfrak{Q}' := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{Q}$, by $f'\colon \mathfrak{Q}' \to \mathfrak{Q}$ and $g'\colon \mathfrak{Q}' \to \mathfrak{P}'$ the two canonical projections. We set $\mathfrak{Z} := g^{-1}(\mathfrak{D})$, $\mathfrak{Z}' := g'^{-1}(\mathfrak{D}')$, $U := g^{-1}(T)$, $U' := g'^{-1}(T')$, $\mathfrak{Q}^\sharp := (\mathfrak{Q}, M(\mathfrak{Z}))$, $\mathfrak{Q}'^\sharp := (\mathfrak{Q}', M(\mathfrak{Z}'))$) the induced $\mathfrak{S}$-log smooth log formal schemes and $g^\sharp\colon \mathfrak{Q}^\sharp \to \mathfrak{P}^\sharp$, $f'^\sharp\colon \mathfrak{Q}'^\sharp \to \mathfrak{Q}^\sharp$ and $g'^\sharp\colon \mathfrak{Q}'^\sharp \to \mathfrak{P}'^\sharp$ the induced morphisms of $\mathfrak{S}$-log smooth log formal schemes. Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'^\sharp/\mathfrak{S}}(T'))$ with proper support over P. There then exists a canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}^\sharp/\mathfrak{S}}(U))$:*

$$g^{\sharp(\bullet)!} \circ f_+^{\sharp(\bullet)}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f'^{\sharp(\bullet)}_+ \circ g'^{\sharp(\bullet)!}(\mathcal{E}'^{(\bullet)}). \tag{13.2.3.7.1}$$

Proof. With the hypotheses on f and g, it follows from 13.2.3.4 and 9.4.1.7 that the complexes of 13.2.3.7.1 are indeed coherent. The morphism g decomposes into its graph γ followed by the canonical projection $\pi\colon \mathfrak{P} \times \mathfrak{Q} \to \mathfrak{P}$. By base change via f, this yields that the morphism g' decomposes into a closed immersion $\gamma'\colon \mathfrak{Q}' \hookrightarrow \mathfrak{P}' \times \mathfrak{Q}$ followed by the canonical projection $\pi'\colon \mathfrak{P}' \times \mathfrak{Q} \to \mathfrak{P}'$. Set $\mathfrak{U} := \mathfrak{P} \times \mathfrak{Q}$, $\mathfrak{U}' := \mathfrak{P}' \times \mathfrak{Q}$. We denote by $\pi^\sharp\colon \mathfrak{U}^\sharp \to \mathfrak{P}^\sharp$ and $\gamma^\sharp\colon \mathfrak{Q}^\sharp \to \mathfrak{U}^\sharp$ the strict morphisms of formal log-schemes whose underlying formal scheme morphisms are π and γ; the same with primes. We denote by $f''^\sharp = f^\sharp \times \mathrm{id}_{\mathfrak{Q}}\colon \mathfrak{U}'^\sharp \to \mathfrak{U}^\sharp$ the morphism canonically induced. Thanks to the theorem 9.3.5.13, we come down to checking that we have of a canonical isomorphism of the form

$$\gamma_+^{\sharp(\bullet)} \circ g^{\sharp(\bullet)!} \circ f_+^{\sharp(\bullet)}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \gamma_+^{\sharp(\bullet)} \circ f'^{\sharp(\bullet)}_+ \circ g'^{\sharp(\bullet)!}(\mathcal{E}'^{(\bullet)}).$$

Concerning the left term, by transitivity of extraordinary inverse images, we obtain the canonical isomorphism $\gamma_+^{\sharp(\bullet)} \circ g^{\sharp(\bullet)!} \circ f_+^{\sharp(\bullet)}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \gamma_+^{\sharp(\bullet)} \circ \gamma^{\sharp(\bullet)!} \circ \pi^{\sharp(\bullet)!} \circ f_+^{\sharp(\bullet)}(\mathcal{E}'^{(\bullet)})$. For the one on the right, by transitivity of direct images or extraordinary inverse images and using 13.2.1.4.2 and 13.2.1.5.1 we obtain the canonical isomorphisms $\gamma_+^{\sharp(\bullet)} \circ f'^{\sharp(\bullet)}_+ \circ g'^{\sharp(\bullet)!}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f''^{\sharp(\bullet)}_+ \circ \gamma'^{\sharp(\bullet)}_+ \circ \gamma'^{\sharp(\bullet)!} \circ \pi'^{\sharp(\bullet)!}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \gamma_+^{\sharp(\bullet)} \circ \gamma^{\sharp(\bullet)!} \circ f''^{\sharp(\bullet)}_+ \circ \pi'^{\sharp(\bullet)!}(\mathcal{E}'^{(\bullet)})$. It is thus sufficient to check the isomorphism $\pi^{\sharp(\bullet)!} \circ f_+^{\sharp(\bullet)}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f''^{\sharp(\bullet)}_+ \circ \pi'^{\sharp(\bullet)!}(\mathcal{E}'^{(\bullet)})$, which is already known following Theorem 9.4.4.3. □

Remark 13.2.3.8. In the theorem 13.2.3.7, we can remove the hypothesis of realizability of the morphism g when we work with overcoherent complexes and when the residue field k is perfect (see later 16.3.3.2).

Remark 13.2.3.9. When logarithmic structures are trivial, f is proper and the residue field k is perfect, T. Abe has checked that the isomorphism of base change 13.2.3.7.1 is compatible with Frobenius (see [Abe14a, 5.7]).

13.2.4 Relative duality isomorphism and adjunction for realisable morphisms

Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a realizable with respect to T (see 13.2.3.3) morphism of separated $\mathcal{V}$-smooth formal schemes. Let T be a divisor of P such that $T' := f^{-1}(T)$ is a divisor of P'.

Theorem 13.2.4.1 (Relative duality isomorphism)**.** *For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$ with proper support over P, we have the isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ of the form*

$$f_+^{(\bullet)} \circ \mathbb{D}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{D} \circ f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)}).$$

Proof. Let X' be the support of $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$. By assumption, X' is proper over P via f. Let $\pi\colon \mathfrak{P}'' \to \mathfrak{P}$ be a proper morphism of smooth $\mathcal{V}$-formal schemes, let $u\colon \mathfrak{P}' \hookrightarrow \mathfrak{P}''$ be an immersion of formal schemes such that $f = \pi \circ u$. Let $v\colon \mathfrak{P}' \hookrightarrow \mathfrak{U}''$ be a closed immersion, and $j\colon \mathfrak{U}'' \hookrightarrow \mathfrak{P}''$ be an open immersion such that $u = j \circ v$. Since X' is proper over P via f and since π is proper, then X' is proper over P'' via u. Since v is proper, then it follows from 9.4.2.6 that we get $\mathcal{E}''^{(\bullet)} := v^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}''/\mathfrak{S}}(T'' \cap U''))$ with proper support over P'' via j.

From the relative duality isomorphism in the case of a closed immersion (see 9.3.2.8), we have $\mathbb{D}v^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} v^{(\bullet)}_{+}\mathbb{D}(\mathcal{E}'^{(\bullet)})$. Set $\mathcal{E}''^{(\bullet)} := v^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)})$. Using 13.2.3.4, we get $j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''/\mathfrak{S}}(T''))$ and it has his support in X'. Hence, $\mathbb{D}j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''/\mathfrak{S}}(T''))$ and has its support in X'. Hence, $\mathbb{D}j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)} \xrightarrow{\sim} j^{(\bullet)}_{+}j^{(\bullet)!}\mathbb{D}j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)}$ (use 13.2.3.6). Moreover, this is obvious that $j^{(\bullet)!}\mathbb{D}j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)} \xrightarrow{\sim} \mathbb{D}j^{(\bullet)!}j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)} \xrightarrow{\sim} \mathbb{D}\mathcal{E}''^{(\bullet)}$. Hence, $\mathbb{D}j^{(\bullet)}_{+}\mathcal{E}''^{(\bullet)} \xrightarrow{\sim} j^{(\bullet)}_{+}\mathbb{D}\mathcal{E}''^{(\bullet)}$. By composition we get $\mathbb{D}u^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+}\mathbb{D}(\mathcal{E}'^{(\bullet)})$. Since π is proper, from the relative duality isomorphism in the proper case (see 9.4.5.2), we obtain the first isomorphism $\mathbb{D}\pi^{(\bullet)}_{+}u^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \pi^{(\bullet)}_{+}\mathbb{D}u^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \pi^{(\bullet)}_{+}u^{(\bullet)}_{+}\mathbb{D}(\mathcal{E}'^{(\bullet)})$. Hence, we are done.

□

Corollary 13.2.4.2. *Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}}({}^{\dagger}T')_{\mathbb{Q}})$ with proper support over P, and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$. We have the isomorphisms*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}}(f_{T,+}(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} \mathbb{R}f_{*}\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}}({}^{\dagger}T')_{\mathbb{Q}}}(\mathcal{E}', f^{!}_{T}(\mathcal{E})). \tag{13.2.4.2.1}$$

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}}(f_{T,+}(\mathcal{E}'), \mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}}({}^{\dagger}T')_{\mathbb{Q}}}(\mathcal{E}', f^{!}_{T}(\mathcal{E})). \tag{13.2.4.2.2}$$

Proof. Using 13.2.4.1, we can copy word by word the proof of 9.4.5.4. □

Corollary 13.2.4.3. *We have the following properties.*

(a) Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}}({}^{\dagger}T')_{\mathbb{Q}})$ with proper support over P. We have the adjunction morphism $\mathcal{E}' \to f^{!}_{T}f_{T+}(\mathcal{E}')$.

(b) Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ such that $f^{!}_{T}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$. We have the adjunction morphism $f_{T+}f^{!}_{T}(\mathcal{E}) \to \mathcal{E}$.

(c) Suppose f is proper and smooth. Then $f_{T+}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ is a right adjoint functor of $f^{!}_{T}\colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$.

Chapter 14

On the preservation of the coherence by extraordinary inverse image of a closed immersion

In the first section of this chapter, we give some preliminaries topological concerning locally convex K-vector spaces. We recall notably the definition of spaces of type LB and we consider some points on the completed tensor products of locally convex module in the context which we will be useful in the sequence of this work. In the second section, we naturally endow the overconvergent isocrystals and the sheaf of differential operators of finite level with overconvergent singularities with a canonical structure of space of type LB. After some topological properties on pushforwards and extraordinary pullbacks by a closed immersion, we establish in the next section the main result described in the beginning of introduction of this paper. We end with some applications of the main theorem with overconvergent log-isocrystals satisfying some non-Liouville type properties.

14.1 Topological preliminaries

Let us denote by $\mathfrak{C}$ the category of locally convex topological K-vector spaces. Let us denote by $\mathfrak{D}$ the full subcategory of $\mathfrak{C}$ consisting of separated and complete K-vector spaces.

14.1.1 LB-spaces

We aggregate everything we will need on K-vector spaces of LB-type, above all of the lemma 14.1.1.7 but also of its proof (see the step 2 of the proof of 14.3.3.8). Recall that since $\mathfrak{C}$ has filtrant inductive limits and projective limits, then a strict morphism of $\mathfrak{C}$ is a morphism such that $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism (see Definition [KS06, 5.1.4]).

14.1.1.1. Let $(V_i)_{i\in I}$ be a filtrant inductive system of $\mathfrak{C}$. Let us denote by $V := \varinjlim_i V_i$ the inductive limit computed in $\mathfrak{C}$. As K-vector space, V is the inductive limit of $(V_i)_{i\in I}$ computed in the category of K-vector spaces. The locally convex topology on V is the finest one making continuous every canonical morphisms $V_i \to V$.

Lemma 14.1.1.2. *Let $(V_i)_{i\in I}$ and $(W_i)_{i\in I}$ be two filtrant inductive systems of $\mathfrak{C}$, $f_i\colon V_i \to W_i$ be a compatible family of morphisms of $\mathfrak{C}$ and $f\colon \varinjlim_i V_i \to \varinjlim_i W_i$ the morphism of $\mathfrak{C}$ induced by passage to the inductive limit. If for any i the image of f_i is dense in W_i then the image of f is dense.*

Proof. Let F be a closed subset of $\varinjlim_i W_i$ containing the image of f. Then the inverse image of F on W_i is a closed subset containing the image of f_i which is dense in W_i. Hence, the canonical arrow $W_i \to \varinjlim_i W_i$ factorizes always via $W_i \to F$. Hence $F = \varinjlim_i W_i$. □

Lemma 14.1.1.3. *Let $(V_i)_{i\in I}$ and $(W_i)_{i\in I}$ be two filtrant inductive systems of $\mathfrak{C}$, $f_i\colon V_i \to W_i$ be a compatible family of surjective, strict morphisms of $\mathfrak{C}$ and $f\colon \varinjlim_i V_i \to \varinjlim_i W_i$ be the morphism of $\mathfrak{C}$ induced by passage to the inductive limit. Then f is a surjective strict morphism.*

Proof. The surjectivity and the continuity of f is already known. Let us denote by $W := \varinjlim_i W_i$. Let W' be an object of $\mathfrak{C}$ and $g\colon W \to W'$ be a K-linear morphism such that $g \circ f$ is continuous. Let us denote by $g_i\colon W_i \to W'$ the composition of the canonical morphism $W_i \to W$ with g. As f_i is surjective and strict, as $g_i \circ f_i\colon V_i \to W'$ is continuous, then the morphisms g_i are continuous. Following the universal property of the inductive limit, this yields g is also continuous. □

Lemma 14.1.1.4. *Let $(V_i)_{i\in I}$ be a filtrant inductive system of $\mathfrak{C}$ and J be a cofinal subpart of I. We have the canonical homeomorphism $\varinjlim_{j\in J} V_j \xrightarrow{\sim} \varinjlim_{i\in I} V_i$ whose underlying bijection is the canonical bijection.*

Proof. Straightforward. □

Definition 14.1.1.5. A "K-vector space of type LB" is a separated locally convex K-vector space V such that there exist, for any integer $m \in \mathbb{N}$, some continuous morphisms of Banach K-vector spaces $V_m \to V_{m+1}$ and a homeomorphism of the form $\varinjlim_m V_m \xrightarrow{\sim} V$. When the field K is clearly determined we simply say LB-space.

Remark 14.1.1.6. In the definition of K-vector space of type LB of 14.1.1.5 and with its notations, it is not restrictive to suppose that the morphisms $V_m \to V_{m+1}$ are injective. Indeed, denote by $j_m\colon V_m \to V$, $W_m := V_m/\ker j_m$ endowed with the quotient topology, then W_m. Since V is separated then so is W_m. Since W_m is also a quotient of a Banach K-vector space, then W_m is a Banach K-vector space. Moreover, we check by using the universal property that the canonical K-linear morphisms (which are mutually inverse to each other) $\varinjlim_m V_m \to \varinjlim_m W_m$ and $\varinjlim_m W_m \to \varinjlim_m V_m$ are continuous.

Lemma 14.1.1.7. *A separated quotient of a LB-space is an LB-space.*

Proof. For any integer $m \in \mathbb{N}$, let $V_m \to V_{m+1}$ be a continuous monomorphism of Banach K-vector spaces. We denote by $V := \varinjlim_m V_m$ and $i_m\colon V_m \hookrightarrow V$ the canonical continuous monomorphisms. Let W be a sub-K-space of V. Let $G := V/W$ be the quotient of V by W which is separated for the quotient topology. Let us denote by $G^{(m)} := V_m/i_m^{-1}(W)$ endowed with the quotient topology, i.e. such that the canonical surjection $\pi_m\colon V_m \twoheadrightarrow G^{(m)}$ is strict. Let $j_m\colon G^{(m)} \to G$ be the canonical injection. As $j_m \circ \pi_m$ is continuous, as π_m is strict, then j_m is continuous. Since G is separated and j_m is an injective continuous morphism, then $G^{(m)}$ is separated. Hence $G^{(m)}$ is a Banach K-vector space. It follows from 14.1.1.3 that the left morphism of the canonical diagram

$$\begin{array}{ccc} \varinjlim_m G^{(m)} & \xrightarrow{\sim} & G \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ \varinjlim_m V_m & \xrightarrow{\sim} & V, \end{array} \tag{14.1.1.7.1}$$

is a strict epimorphism. Since the right morphism is a strict epimorphism, since the bottom morphism is a homeomorphism, then the top isomorphism is a homeomorphism. □

14.1.2 Projective topology of a tensor product on a K-algebra

Let D be a K-algebra (non-necessarily commutative and without topology) such that K is in the center of D. Let us denote by $\mathfrak{C}_{D,l}$ (resp. $\mathfrak{C}_{D,r}$) the full subcategory of $\mathfrak{C}$ of objects of $\mathfrak{C}$ such that the underlying structure of K-vector space extends to a structure of left (resp. right) D-module.

14.1.2.1. (Continuous D-balanced map and completion). Let V be an object of $\mathfrak{C}_{D,r}$, W be an object of $\mathfrak{C}_{D,l}$ and U be an object of $\mathfrak{C}$. We endow $V \times W$ with the product topology, i.e. $V \times W$ is the product computated in $\mathfrak{C}$. Let $\beta\colon V \times W \to U$ be a D-balanced map, i.e. a K-bilinear map such that for any $v \in V$, $w \in W$ and $d \in D$ we have $\beta(vd, w) = \beta(v, dw)$. Moreover, following [Sch02, 17.1], as β is K-bilinear, the map β is continuous if and only if it is continuous at zero, i.e., for any open $\mathcal{V}$-submodule of L of U, there exist some open $\mathcal{V}$-submodules M of V and N of W such that $\beta(M \times N) \subset L$.

Let us suppose β continuous. Hence we have the K-bilinear morphism

$$\varprojlim_{M,N} V \times W/M \times N \to \varprojlim_{L} U/L,$$

where L (resp. M, resp. N) goes through the open lattices of U (resp. V, resp. W). We denote this application by $\widehat{\beta}\colon \widehat{V} \times \widehat{W} \to \widehat{U}$. As the open lattices of $\widehat{U}$ are of the form $\varprojlim_{L} L_0/L$ where L_0 is an open lattice of U and L goes through the open lattices of U included in L_0 (and similarly for $\widehat{V} \times \widehat{W}$), then we check that $\widehat{\beta}$ is continuous. As the image of $V \times W$ in $\widehat{V} \times \widehat{W}$ is dense, we check that $\widehat{\beta}$ is the unique continuous K-bilinear map inducing the commutative square :

$$\begin{array}{ccc} \widehat{V} \times \widehat{W} & \overset{\widehat{\beta}}{\dashrightarrow} & \widehat{U} \\ \uparrow & & \uparrow \\ V \times W & \xrightarrow{\beta} & U. \end{array} \tag{14.1.2.1.1}$$

But, to get the property that $\widehat{\beta}$ is D-balanced, we need further topological hypotheses on D (see 14.1.3.4).

Definition 14.1.2.2. Let V be an object of $\mathfrak{C}_{D,r}$ and W be an object of $\mathfrak{C}_{D,l}$.

(a) By endowing $V \times W$ with the product topology, the projective topology on the tensor product $V \otimes_D W$ is by definition a locally convex K-vector space topology which is the finest one such that the canonical K-bilinear morphism $\rho_{V,W}\colon V \times W \to V \otimes_D W$ is continuous. So, a lattice $L \subset V \otimes_D W$ is open if and only if $\rho_{V,W}^{-1}(L)$ is open. As we will only consider projective topologies on tensor products, it might happen that we omit indicating " projective".

(b) The object $V \otimes_D W$ satisfies the following universal property: for any D-balanced and continuous map of the form $\phi\colon V \times W \to U$, there exists a unique morphism in $\mathfrak{C}$ of the form $\theta\colon V \otimes_D W \to U$ such that $\theta \circ \rho_{V,W} = \phi$. This yields that we get in fact the canonical bifunctor

$$- \otimes_D -\colon \mathfrak{C}_{D,r} \times \mathfrak{C}_{D,l} \to \mathfrak{C}.$$

Lemma 14.1.2.3. *Let V be an object of $\mathfrak{C}_{D,r}$ and W be an object of $\mathfrak{C}_{D,l}$. We suppose that there exist a $\mathcal{V}$-submodule V_0 of V (resp. W_0 of W) such that a basis of open neighborhoods at zero of V (resp. W) is given by the family $(p^n V_0)_{n\in\mathbb{N}}$ (resp. $(p^n W_0)_{n\in\mathbb{N}}$). Let us denote by $U_0 := <\rho_{V,W}(V_0 \times W_0)>$, where $<?>$ means the "$\mathcal{V}$-submodule of $V \otimes_D W$ generated by ?". Then a basis of open neighborhoods on $V \otimes_D W$ endowed with its projective topology (see 14.1.2.2) is given by $(p^n U_0)_{n\in\mathbb{N}}$.*

Proof. Since V_0 and W_0 are respectively lattices of V and W, then U_0 is a lattice of U. For any integer $n \in \mathbb{N}$, since $\rho_{V,W}^{-1}(p^n U_0) \supset p^n(V_0 \times W_0)$, then the $p^n U_0$ are open subsets of $V \otimes_D W$. Conversely, let L be an open $\mathcal{V}$-submodule of $V \otimes_D W$. Since $\rho_{V,W}^{-1}(L)$ is open, then there exists a large enough integer n such that $\rho_{V,W}^{-1}(L) \supset p^n(V_0 \times W_0)$. Then we have $L \supset \rho_{V,W}(p^n(V_0 \times W_0)) = p^{2n}\rho_{V,W}(V_0 \times W_0)$. Since L is a $\mathcal{V}$-submodule of $V \otimes_D W$, this yields $L \supset p^{2n} U_0$. □

14.1.2.4. Let $D' \to D$ be a homomorphism of K-algebras such that K is also in the center of D'. Let V be an object of $\mathfrak{C}_{D,r}$ and W be an object of $\mathfrak{C}_{D,l}$. Since the composition morphism $V \times W \to V \otimes_{D'} W \to V \otimes_D W$ is the canonical morphism, using the definition of the topologies defined on $V \otimes_{D'} W$ and $V \otimes_D W$ (see 14.1.2.2.a), then we can check the strictness of the epimorphism $V \otimes_{D'} W \to V \otimes_D W$.

Lemma 14.1.2.5. *Let $V \to V''$ (resp. $W \to W''$) be a strict epimorphism of $\mathfrak{C}_{D,r}$ (resp. of $\mathfrak{C}_{D,l}$). Then the epimorphisms $V \otimes_D W \to V \otimes_D W''$ and $V \otimes_D W \to V'' \otimes_D W$ are strict.*

Proof. By symmetry, let us check it only for the first epimorphism. Let L be a lattice of $V \otimes_D W''$. Since $V \times W \to V \times W''$ is a strict morphism of $\mathfrak{C}$, by using the definition of the topology on $V \otimes_D W''$, we can check that L is open if and only if its inverse image on $V \times W$ is open. By using the definition of the topology on $V \otimes_D W$, this latter property is equivalent to the fact that its inverse image on $V \otimes_D W$ is an open lattice. Hence we are done.

□

14.1.3 Tensor products on locally convex K-algebras, completions

14.1.3.1. With the notations of 14.1.2, let V be an object of $\mathfrak{C}_{D,r}$ and W be an object of $\mathfrak{C}_{D,l}$. We denote by $V\widehat{\otimes}_D W$ the separated completion of $V\otimes_D W$ and $i_{V,W}\colon V\otimes_D W\to V\widehat{\otimes}_D W$ the canonical morphism. Using the universal property of the tensor product and that of the separated completion we get the following universal property : for any continuous and D-balanced map of the form $\phi\colon V\times W\to U$ with $U\in\mathfrak{D}$, there exists a unique morphism in $\mathfrak{D}$ of the form $\theta\colon V\widehat{\otimes}_D W\to U$ such that $\theta\circ i_{V,W}\circ\rho_{V,W}=\phi$.

Hence, we get the canonical bifunctor

$$-\widehat{\otimes}_D-\colon \mathfrak{C}_{D,r}\times\mathfrak{C}_{D,l}\to\mathfrak{D}.$$

Definition 14.1.3.2. Let D be a K-algebra such that K is in the center of D.

(i) We say that D is a "locally convex K-algebra", if D is endowed with a locally convex K-vector space topology such that the multiplication is a continuous K-bilinear map. A "morphism of locally convex K-algebras" is a morphism of K-algebras which is continuous for the respective topologies. We say that D is a Banach K-algebra if D is a locally convex K-algebra whose underlying topology makes D a Banach K-vector space.

When D is moreover of type LB, we say that D is a "locally convex K-algebra of type LB".

(ii) Let D be a locally convex K-algebra.

(a) A "locally convex left D-module" is a left D-module M endowed with a locally convex K-vector space topology such that the structural exterior law $D\times M\to M$ is a continuous K-bilinear map. We have a similar notion of "locally convex right D-module".

(b) A "Banach left D-module" is a locally convex left D-module which is also a Banach K-vector space (with respect to its underlying topology).

(c) A " LB left D-module" is a locally convex left D-module which is also a K-vector LB-space (with respect to its underlying topology). Even if we do not clarify K is this terminology, beware it might depend we it.

(d) A "morphism of locally convex left D-modules" (resp. "morphism of Banach left D-modules", resp "morphism of LB left D-modules") is a morphism of left D-modules which is also a morphism of locally convex K-vector spaces (resp. Banach K-vector spaces, resp. LB-spaces).

(e) We get similar definitions by replacing left by right.

(iii) Let (D,D') be two locally convex K-algebra. A "locally convex (D,D')-bimodule" is a (D,D')-bimodule M endowed with a locally convex K-vector space topology such that M is a locally convex left D-module and a locally convex right D'-module.

Lemma 14.1.3.3. *Let D be a locally convex K-algebra, $\phi\colon M\to N$ be a morphism of locally convex left (resp. right) D-modules.*

(a) The structure of separated complete locally convex K-vector space on $\widehat{D}$ extends to a canonical structure of locally convex K-algebra. Moreover, the canonical morphism $D\to\widehat{D}$ is a morphism of locally convex K-algebras.

(b) The induced morphism by separated completion $\widehat{\phi}\colon\widehat{M}\to\widehat{N}$ is a morphism of locally convex $\widehat{D}$-modules. Moreover, the canonical morphism $M\to\widehat{M}$ is a morphism of locally convex D-modules.

Proof. Let's just prove the non-respective case. Following 14.1.2.1.1, the structural continuous K-bilinear map $\mu_D\colon D\times D\to D$ induces the continuous K-bilinear map $\mu_{\widehat{D}}:=\widehat{\mu_D}\colon\widehat{D}\times\widehat{D}\to\widehat{D}$ (uniquely) fitting into the commutative diagram:

$$\begin{array}{ccc}\widehat{D}\times\widehat{D} & \xrightarrow{\mu_{\widehat{D}}} & \widehat{D}\\ \uparrow & & \uparrow\\ D\times D & \xrightarrow{\mu_D} & D.\end{array}\qquad(14.1.3.3.1)$$

As the two maps $\mu_{\widehat{D}}\circ(\mu_{\widehat{D}}\times id)$, $\mu_{\widehat{D}}\circ(id\times\mu_{\widehat{D}})\colon\widehat{D}\times\widehat{D}\times\widehat{D}\to\widehat{D}$ coincide after composition with the canonical morphism $D\times D\times D\to\widehat{D}\times\widehat{D}\times\widehat{D}$ whose image is dense, we get $\mu_{\widehat{D}}\circ(\mu_{\widehat{D}}\times id)=\mu_{\widehat{D}}\circ(id\times\mu_{\widehat{D}})$, i.e. the multiplication is associative. We check similarly the other properties making $\widehat{D}$ a locally convex K-algebra. It is clear that the continuous canonical morphism $D \to \widehat{D}$ is then a morphism of locally convex K-algebras.

Following 14.1.2.1, the structural continuous K-bilinear maps $\mu_M : D \times M \to M$ and $\mu_N : D \times N \to N$ induce the continuous K-bilinear maps $\mu_{\widehat{M}} := \widehat{\mu_M} : \widehat{D} \times \widehat{M} \to \widehat{M}$ and $\mu_{\widehat{N}} := \widehat{\mu_N} : \widehat{D} \times \widehat{N} \to \widehat{N}$. In the same way, we check that $\mu_{\widehat{M}}$ and $\mu_{\widehat{N}}$ induce respectively a canonical structure of $\widehat{D}$-module locally convex on $\widehat{M}$ and $\widehat{N}$. As the diagram

$$\begin{array}{ccc} \widehat{D} \times \widehat{M} & \xrightarrow{\mu_{\widehat{M}}} & \widehat{M} \\ \downarrow{\scriptstyle \mathrm{id} \times \widehat{\phi}} & & \downarrow{\scriptstyle \widehat{\phi}} \\ \widehat{D} \times \widehat{N} & \xrightarrow{\mu_{\widehat{N}}} & \widehat{N} \end{array}$$

is commutative after composition with $D \times M \to \widehat{D} \times \widehat{M}$ whose image is dense, this one is commutative. Hence we are done. □

Lemma 14.1.3.4. *Let D be a locally convex K-algebra, M be a locally convex right D-module, N be a locally convex left D-module and U be a locally convex K-vector space. Let $\beta : M \times N \to U$ be a D-balanced and continuous map. The application $\widehat{\beta} : \widehat{M} \times \widehat{N} \to \widehat{U}$ (see 14.1.2.1) is then a $\widehat{D}$-balanced and continuous map.*

Proof. We already know that the map $\widehat{\beta}$ is continuous. Let us consider the square :

$$\begin{array}{ccc} \widehat{M} \times \widehat{D} \times \widehat{N} & \xrightarrow{\mu_{\widehat{M}} \times id} & \widehat{M} \times \widehat{N} \\ \downarrow{\scriptstyle \mathrm{id} \times \mu_{\widehat{N}}} & & \downarrow{\scriptstyle \widehat{\beta}} \\ \widehat{M} \times \widehat{N} & \xrightarrow{\widehat{\beta}} & \widehat{U}, \end{array} \qquad (14.1.3.4.1)$$

where $\mu_{\widehat{M}} : \widehat{M} \times \widehat{D} \to \widehat{M}$ and $\mu_{\widehat{N}} : \widehat{D} \times \widehat{N} \to \widehat{N}$ are the canonical structural continuous K-bilinear maps. As the image of $M \times D \times N$ in $\widehat{M} \times \widehat{D} \times \widehat{N}$ is dense, the square 14.1.3.4.1 is then commutative because so is without the hats. □

Proposition 14.1.3.5. *Let D be a locally convex K-algebra, M be a locally convex right D-module and N be a locally convex left D-module. Hence we have the canonical isomorphism in $\mathfrak{D}$ of the form:*

$$M \widehat{\otimes}_D N \xrightarrow{\sim} \widehat{M} \widehat{\otimes}_{\widehat{D}} \widehat{N}.$$

Proof. By functoriality of the separated completion functor, we have the morphism in $\mathfrak{D}$ of the form: $M \widehat{\otimes}_D N \to \widehat{M} \widehat{\otimes}_{\widehat{D}} \widehat{N}$. To construct the inverse morphism, by using the universal property of the tensor product, it is about defining canonically a continuous map of the form $\widehat{M} \times \widehat{N} \to M \widehat{\otimes}_D N$ which is $\widehat{D}$-balanced. This can be done by using lemma 14.1.3.4 applied in the case where β is equal to the canonical map $M \times N \to M \otimes_D N$. □

14.2 LB-spaces in the theory of D-modules arithmetic

We set $\mathfrak{S} := \mathrm{Spf}\, \mathcal{V}$. Unless otherwise stated, we will use the following notations and hypotheses : let $\mathfrak{X}$ be an affine, smooth $\mathcal{V}$-formal scheme endowed with local coordinates $t_1, \dots, t_d$ and let $\partial_1, \dots, \partial_d$ be the corresponding derivations. Let $0 \le r \le s \le d$ be some integers. We denote by $\mathfrak{Z} := \cap_{i=s+1}^d V(t_i)$ and $u : \mathfrak{Z} \hookrightarrow \mathfrak{X}$ the induced closed immersion. We denote by $\mathfrak{D} = V(t_1 \cdots t_r)$ the relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor (the case $r = 0$ means $\mathfrak{D}$ is empty). This yields the relative to $\mathfrak{Z}/\mathfrak{S}$ strict normal crossing divisor $u^{-1}(\mathfrak{D})$. We set $\mathfrak{X}^\sharp := (\mathfrak{X}, M(\mathfrak{D}))$, $\mathfrak{Z}^\sharp := (\mathfrak{Z}, M(u^{-1}\mathfrak{D}))$ and $u^\sharp : \mathfrak{Z}^\sharp \hookrightarrow \mathfrak{X}^\sharp$ the exact closed immersion of $\mathfrak{S}$-log-smooth log formal schemes. Let $f \in O_{\mathfrak{X}}$, $f_0 \in O_X$ be its reduction modulo π, $T := V(f_0)$ the corresponding divisor of X. We suppose that $T \cap Z$ is a divisor of Z. Let $\lambda_0 \in L(\mathbb{N})$. To lighten the notations, we put then $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}}(T)$, $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{Z}}(T \cap Z)$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}$. We set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z)$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$, $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$ and finally $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} := \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}$. We denote by the corresponding straight letter, the global section of a sheaf on $\mathfrak{X}$ or $\mathfrak{Z}$, e.g. $D^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}} := \Gamma(\mathfrak{X}, \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}})$. We use in this section the notation 4.5.1.1.1 : for any $m \in \mathbb{N} \cup \{\infty\}$, for any $(l_1, \ldots, l_{r+s}) = \underline{l} \in \mathbb{N}^{r+s}$ and $(k_1, \ldots, k_d) = \underline{k} \in \mathbb{N}^d$, we get an element of respectively $\mathcal{D}^{(m)}_{\mathfrak{Z}^{\sharp}}$ and $\mathcal{D}^{(m)}_{\mathfrak{X}^{\sharp}}$ by setting

$$\underline{\partial}^{\langle \underline{l} \rangle_{(m)}}_{(r)} := \underline{\partial}^{\langle (l_1,\ldots,l_r,0,\ldots,0) \rangle_{(m)}}_{\sharp} \underline{\partial}^{\langle (0,\ldots,0,l_{r+1},\ldots,l_{r+s}) \rangle_{(m)}},$$
$$\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{(r)} := \underline{\partial}^{\langle (k_1,\ldots,k_r,0,\ldots,0) \rangle_{(m)}}_{\sharp} \underline{\partial}^{\langle (0,\ldots,0,k_{r+1},\ldots,k_d) \rangle_{(m)}}. \tag{14.2.0.0.1}$$

When $m = \infty$, we better write $\underline{\partial}^{[\underline{k}]}_{(r)}$.

14.2.1 Canonical topology of the global section of a log-overconvergent isocrystal

14.2.1.1. We have the canonical isomorphisms $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}} \xrightarrow{\sim} O_{\mathfrak{X}}[\frac{1}{f}]^{\dagger}_{K}$ and $\widetilde{B}^{(m)}_{\mathfrak{X}}(T) \xrightarrow{\sim} A\{T\}/(f^{p^{m+1}}T - p) = O_{\mathfrak{X}}\{\frac{p}{f^{p^{m+1}}}\}$ (see 8.7.3.11). This yields the morphism of $\mathcal{V}$-algebras $\widetilde{B}^{(m)}_{\mathfrak{X}}(T) \hookrightarrow O_{\mathfrak{X}}[\frac{1}{f}]^{\dagger}$ (for the injectivity, see 8.7.4.1.(b)). We have also the canonical inclusion $\widetilde{D}^{(m)}_{\mathfrak{X}^{\sharp},\mathbb{Q}} \hookrightarrow D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$.

Definition 14.2.1.2. (Canonical locally convex K-vector space topology of $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$). We define the following topologies.

(a) For any integer m, the *canonical Banach K-algebra topology* on $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$ is the one such that a basis of open neighborhoods at zero is given by $(p^n \widetilde{B}^{(m)}_{\mathfrak{X}}(T))_{n \in \mathbb{N}}$.

(b) We endow $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$ with a *canonical locally convex K-vector space* topology which makes the canonical isomorphism $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}} \xrightarrow{\sim} \varinjlim_{m} \widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$ a homeomorphism. We remark that following 14.1.1.4, this does not depend on the map $\lambda_0 \in L(\mathbb{N})$.

Notation 14.2.1.3. Let V be a K-vector and $(V_m)_{m \in \mathbb{N}}$ be a sequence of K-vector subspaces of V. We set $\sum_{m=0}^{\infty} V_m := \cup_{m \in \mathbb{N}} \sum_{m=0}^{n} V_m$.

Proposition 14.2.1.4. *The canonical locally convex K-vector space $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$ defined at 14.2.1.2 is a locally convex K-algebra (see definition 14.1.3.2). More precisely the following properties hold.*

(a) A basis of open neighborhoods of zero of the canonical locally convex K-vector space topology of $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$ is given by the family

$$L_{\underline{n}} := \sum_{m=0}^{\infty} p^{n_m} \widetilde{B}^{(m)}_{\mathfrak{X}^{\sharp}}(T).$$

where $\underline{n} := (n_m)_{m \in \mathbb{N}}$ goes through sequences of nonnegative integer.

(b) Moreover $L_{\underline{n}}$ is a sub K-algebra of $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$ for any sequence $\underline{n}$.

Proof. Let us check the part (a). Since tensor product commutes with filtrant inductive limits then $L_{\underline{n}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$ and then $L_{\underline{n}}$ is a lattice. The openness of $L_{\underline{n}}$ is straightforward, so is the fact that any open lattice of $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$ contains $L_{\underline{n}}$ for some large enough sequence $\underline{n}$.

Let us check the part (b). Let $P = \sum_{m \geq 0} p^{n_m} b_m$ and $Q = \sum_{m \geq 0} p^{n_m} c_m$ be two elements of $L_{\underline{n}}$, with $b_m, c_m \in \widetilde{B}^{(m)}_{\mathfrak{X}^{\sharp}}(T)$ null except for a finite number of terms. Then $PQ = \sum_{m_1, m_2 \geq 0} p^{n_{m_1} + n_{m_2}} b_{m_1} c_{m_2}$. Set

$$[PQ]_m := \sum_{\substack{0 \leq m_1, m_2 \\ \max\{m_1, m_2\} = m}} p^{n_{m_1} + n_{m_2}} b_{m_1} c_{m_2} \in p^{n_m} \widetilde{B}^{(m)}_{\mathfrak{X}^{\sharp}}(T).$$

Hence $PQ = \sum_{m \geq 0} Q_m \in L_{\underline{n}}$. □

Remark 14.2.1.5. (Canonical normed $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-algebra topology of $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$). Since $\cap_{n\in\mathbb{N}} p^n \mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^\dagger = \{0\}$, since we have the canonical isomorphism $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^\dagger_K$ (see 8.7.3.11). then $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$ can be endowed with a *canonical normed K-algebra topology structure* whose a basis of open neighborhoods at zero is given by $(p^n \mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^\dagger)_{n\in\mathbb{N}}$. Endowed with this topology, $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$ is normed but not of Banach: its separated completion is $\mathcal{O}_{\mathfrak{X}}\{\frac{1}{f}\}_K$. Let us recall finally that since $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^\dagger_K$, then $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$ is noetherian (see 17.1.1.9).

14.2.1.6. Beware that the topology on $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$ of 14.2.1.5 is different from that of 14.2.1.2. We will see that $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$ is an LB-space for the canonical locally convex topology K-algebra of 14.2.1.2 (see lemma 14.2.2.1). Since it is more convenient to work with LB-spaces, then we can forget the topology defined in 14.2.1.5 and when we refer to the canonical locally convex K-algebra topology on $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$, we mean the topology defined in 14.2.1.2.

14.2.1.7. (Canonical topology of a finite type $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module). Let E be an $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module of finite type. Following 8.4.1.11, there exist, for m_0 large enough, a $\widetilde{B}^{(m_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module of finite type $E^{(m_0)}$ and an $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-linear isomorphism of the form $\epsilon\colon \mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}} \otimes_{\widetilde{B}^{(m_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} E^{(m_0)} \xrightarrow{\sim} E$. For any integer $m \geq m_0$, we set $E^{(m)} := \widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{B}^{(m_0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} E^{(m_0)}$. We endow $E^{(m)}$ with its canonical topology of Banach $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module of finite type (equal to the quotient topology via an epimorphism of the form $(\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}})^r \to E^{(m)}$). Since, for any m, the canonical morphisms of the form $(\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}})^r \to (\widetilde{B}^{(m+1)}_{\mathfrak{X}}(T)_{\mathbb{Q}})^r$ are continuous, this yields that the canonical morphism $E^{(m)} \to E^{(m+1)}$ is continuous. By decreeing that $\epsilon\colon \varinjlim_m E^{(m)} \xrightarrow{\sim} E$ is a homeomorphism, we define a structure of locally convex $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module on E. Similarly to 14.2.1.9, we can check that this does not depend on the choice of $(m_0, E^{(m_0)}, \epsilon)$.

14.2.1.8. (Sheaves of differential operators).

(a) For any integer $m \geq 0$, $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$ is canonically endowed with a Banach $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module topology as follows: the family $(p^n \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp})_{n\in\mathbb{N}}$ forms a basis of open neighborhoods at zero. We have similarly a canonical Banach $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T\cap Z)_{\mathbb{Q}}$-module structures on respectively $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$, $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}$ and $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}$ whose basis of open neighborhoods of zero is given by respectively $(p^n \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp})_{n\in\mathbb{N}}$, $(p^n \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})_{n\in\mathbb{N}}$ and $(p^n \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp})_{n\in\mathbb{N}}$.

(b) The canonical locally convex K-algebra topology on $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ is defined as follows. First, we make $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ a locally convex K-vector space by decreeing that the canonical isomorphism of K-algebras $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}} \xrightarrow{\sim} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$ is an homeomorphism, where the right side is endowed with the inductive limit topology in $\mathfrak{C}$ and where $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$ is endowed with the topology defined above in (a). Similarly to 14.2.1.4, we can check that a basis of open neighborhoods of zero of $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ is given by the family $L_{\underline{n}} := \sum_{m=0}^{\infty} p^{n_m} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp}$, where $\underline{n} := (n_m)_{m\in\mathbb{N}}$ goes through the sequences of nonnegative integers and that $L_{\underline{n}}$ is a ring for any sequence $\underline{n}$, which implies that $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ is in fact a locally convex K-algebra.

Since $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$ is continuous, then by taking the inductive limits on the level the morphism $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}} \to D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ is continuous. Hence, $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ has two structures (the left and the right one) of locally convex $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-modules.

(c) In the same way, we define on $D^\dagger_{\mathfrak{Z}^\sharp}(^\dagger T\cap Z)_{\mathbb{Q}} = \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$, on $\widetilde{D}^\dagger_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}} = \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}$ and $\widetilde{D}^\dagger_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} = \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}$ a canonical locally convex K-vector space topology. We remark that following the lemma 14.1.1.4 we can replace the index $\mathbb{N}$ by a cofinal subset without changing the topology. Moreover, $\widetilde{D}^\dagger_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}$ is a locally convex $(D^\dagger_{\mathfrak{Z}^\sharp}(^\dagger T\cap Z)_{\mathbb{Q}}, D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}})$-bimodule and $\widetilde{D}^\dagger_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}$ is a locally convex $(D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}, D^\dagger_{\mathfrak{Z}^\sharp}(^\dagger T\cap Z)_{\mathbb{Q}})$-bimodule.

14.2.1.9. (Canonical topology of a coherent left $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module). Let E be a coherent left $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module.

We endow E with a canonical locally convex $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module topology as follows. Following 8.4.1.11, there exist for any large enough m_0 a $\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-module of finite type $E^{(m_0)}$ and a $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism of the form $\epsilon\colon D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}} E^{(m_0)} \xrightarrow{\sim} E$. For any integer $m \geq m_0$, we set $E^{(m)} := \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}} E^{(m_0)}$. Following 7.5.1.8, we endow $E^{(m)}$ with its Banach $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-module of finite type topology (equal to the quotient topology via an epimorphism of the form $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to E^{(m)}$). As for any m the canonical morphisms of the form $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to (\widetilde{D}^{(m+1)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r$ are continuous, this yields that the canonical morphism $E^{(m)} \to E^{(m+1)}$ is continuous. By decreeing that the isomorphism $\epsilon\colon \varinjlim_m E^{(m)} \xrightarrow{\sim} E$ is a homeomorphism, we get a structure of locally convex left $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module on E.

This does not depend on the choice of $(m_0, E^{(m_0)}, \epsilon)$. Indeed, for m_0' large enough let $E'^{(m_0')}$ be a $\widetilde{D}^{(m_0')}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-module of finite type and a $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear isomorphism of the form $\epsilon'\colon D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}} E'^{(m_0')} \xrightarrow{\sim} E$. For $m \geq m_0'$, we set $E'^{(m)} := \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0')}_{\mathfrak{X}^\sharp,\mathbb{Q}}} E'^{(m_0')}$. Following 8.4.1.11, for m_0'' large enough, there exist for any $m \geq m_0''$ some $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-linear isomorphisms $\epsilon_m\colon E'^{(m)} \xrightarrow{\sim} E^{(m)}$ such that $\epsilon \circ \epsilon^\dagger = \epsilon'$, with $\epsilon^\dagger := \varinjlim_m \epsilon_m$. Moreover, following [BGR84, 3.7.3] (indeed, we remark that the commutativity of rings is useless), the maps ϵ_m are homeomorphisms. Taking the inductive limit, this yields that $\epsilon^\dagger$ is a homeomorphism. Using 14.1.1.4, this implies the canonicity of the topology on E.

Proposition 14.2.1.10. *(Canonical topology of the global section of a log-overconvergent isocrystal). Let E be a coherent $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is also a finitely generated projective $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module for the induced structure. The topology on E as coherent $D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module is the same as that as finite type $O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module. Hence, we will call it the canonical topology on the (locally free) overconvergent isocrystal E.*

Proof. Let $\mathcal{E} := \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} E \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ (see notation 11.2.1.4) be the log isocrystal on $\mathfrak{X}^\sharp$ with overconvergent singularities along T associated with E. Following 11.2.1.9, increasing $\lambda_0 \in L(\mathbb{N})$ if necessary, there exists $\mathcal{E}^{(0)}$ a topologically nilpotent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-module, coherent as $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module satisfying both conditions of 11.2.1.7.(b). By taking global section functor and using theorem A, it follows from 11.2.1.9.1 that the canonical morphism

$$\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{B}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} E^{(0)} \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}} \otimes_{\widetilde{D}^{(0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}} E^{(0)}$$

is an isomorphism. Following 7.5.2.6, the canonical topology of $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \otimes_{\widetilde{B}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}} E^{(0)}$ as finite type $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module is the same as that as finite type $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-module. Hence we are done. □

14.2.2 Examples of LB-spaces

Lemma 14.2.2.1. *(a) The canonical locally convex K-algebra topology on $O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ (see 14.2.1.2.b) is finer than its normed K-algebra topology (see 14.2.1.5).*

(b) The canonical locally convex K-algebra $O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ is of type LB.

Proof. a) Since we have the monomorphism of $\mathcal{V}$-algebras $\widetilde{B}^{(m)}_{\mathfrak{X}}(T) \hookrightarrow O_{\mathfrak{X}}[\frac{1}{f}]^\dagger$, then the canonical morphism of K-algebras $\widetilde{B}^{(m)}_{\mathfrak{X}}(T)_{\mathbb{Q}} \hookrightarrow O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ is continuous, when $O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ is endowed with its topology of normed K-algebra. By taking the inductive limit, this yields the result.

b) The first part implies that the canonical locally convex K-algebra topology on $O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ is separated. Hence, this is an LB-space. □

Lemma 14.2.2.2. *Let $N \subset (O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})^r$ be a monomorphism of $O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules. Then N is closed in $(O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})^r$ for the topology induced by the canonical topology of $(O_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})^r$.*

Proof. Let us denote by $M := (O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}})^r/N$ and by M_0 the image of the composite morphism $(O_{\mathfrak{X}}[\frac{1}{f}]^\dagger)^r \subset (O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}})^r \twoheadrightarrow M$. We endow M with the quotient topology (for the canonical topology of $(O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}})^r$). Hence we have to check that M is separated for this topology. As for any integer n the $\mathcal{V}$-module $p^n(O_{\mathfrak{X}}[\frac{1}{f}]^\dagger)^r$ is open in $(O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}})^r$ (see 14.2.2.1), the $\mathcal{V}$-module $p^n M_0$ is then open in M. Moreover, as $O_{\mathfrak{X}}[\frac{1}{f}]^\dagger$ is noetherian and as $O_{\mathfrak{X}}[\frac{1}{f}]^\dagger \to O_{\mathfrak{X}}\{\frac{1}{f}\}$ is faithfully flat, as M_0 is a finite type $O_{\mathfrak{X}}[\frac{1}{f}]^\dagger$-module, then M_0 is separated for the p-adic topology (see [Mat89, Theorems 8.10 and 8.12]), i.e. $\cap_{n\in\mathbb{N}} p^n M_0 = \{0\}$. This yields that M is separated. □

Proposition 14.2.2.3. *Finitely generated $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-modules are $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-modules of type LB (for the canonical topology). Moreover, any $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-submodule of a finitely generated $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module M is closed in M for the canonical topology of M.*

Proof. Let M be a finite type $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module endowed with its canonical topology. There exists an epimorphism of $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-modules of the form $(O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}})^r \twoheadrightarrow M$. Following 14.2.3.2, this morphism is strict with respect to their canonical topologies. Using 14.2.2.2 this yields that M is separated and then M is an $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module of type LB. If M' is an $O_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-submodule of M, then the module M/M' is separated for the induced quotient topology by M. Thanks to 14.2.3.2 this yields that M' is closed in M. □

Remark 14.2.2.4. Let $I \subset (D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}})^r$ be a coherent left $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$-submodule. This is not clear that these inclusions are strict for the respective canonical topologies nor that I is closed in $(D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}})^r$. However, following a communication of Tomoyuki Abe, the coherent $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$-modules are also spaces of type LB.

Proposition 14.2.2.5. *Let $(n_m)_{m\in\mathbb{N}}$ be a strictly increasing sequence of nonnegative integers. The locally convex K-vector space $\varinjlim_m \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$ (endowed with the inductive limit topology in the category of locally convex K-vector spaces) is an LB-space.*

Proof. Following 14.1.1.4, it is sufficient to treat the case where $n_m = m$. As the K-vector spaces $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$ are Banach spaces, it only remains to check the separateness of $\varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$. With notation 14.2.0.0.1, let us denote by L the $\mathcal{V}$-submodule of $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ of elements which can be written of the form $\sum_{\underline{k}\in\mathbb{N}^d} a_{\underline{k}} \underline{\partial}^{[k]}_{(r)}$ with $a_{\underline{k}} \in \mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^\dagger$ (and the sum has to converge in $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$). Moreover, as $p^n L \supset p^n D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T) \supset \sum_{m=0}^\infty p^n \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp}$ (see notation 14.2.1.3), then $p^n L$ is an open subset of $D^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$ for any integer n. By uniqueness of the writing of the form $\sum_{\underline{k}\in\mathbb{N}^d} a_{\underline{k}} \underline{\partial}^{[k]}_{(r)}$ of elements of L, we check that $\cap_{n\in\mathbb{N}} p^n L = \{0\}$. Hence, we get the required separateness. □

Proposition 14.2.2.6. *The canonical topologies are that defined in 14.2.1.8.*

(a) For any integer $m \in \mathbb{N}$, let $V^{(m)}$ be a locally convex left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-module and $V^{(m)} \to V^{(m+1)}$ be a continuous morphism. Let $(N_m)_{m\in\mathbb{N}}$, $(N'_m)_{m\in\mathbb{N}}$ and $(N''_m)_{m\in\mathbb{N}}$ be three strictly increasing sequences of nonnegative integers such that $N''_m \leq \max\{N_m, N'_m\}$ for any $m \in \mathbb{N}$. With the notations of 14.1.3.1, the canonical isomorphism

$$\varinjlim_m \widetilde{D}^{(N'_m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(N''_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V^{(N_m)} \xrightarrow{\sim} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V^{(m)}. \tag{14.2.2.6.1}$$

is then a homeomorphism.

(b) The locally convex K-vector spaces

$$\varinjlim_m \widetilde{D}^{(N_m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}, \qquad \varinjlim_m \widetilde{D}^{(N_m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} \quad \text{and} \quad \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}$$

are LB-spaces.

Proof. We can check the first assertion similarly to 14.1.1.4. For the second assertion, we proceed similarly to 14.2.2.5. □

14.2.3 Continuity, strictness

Lemma 14.2.3.1. *Let $\phi\colon E \to E'$ be a morphism of coherent $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. of finite type $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$-modules). Endowing E and E' with their canonical topologies (see respectively 14.2.1.9 and 14.2.1.7), the morphism ϕ is continuous.*

Proof. Following 8.4.1.11, there exist for a large enough nonnegative integer m_0 a morphism of $\widetilde{D}^{(m_0)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}$-modules of finite type $\phi^{(m_0)}\colon E^{(m_0)} \to E'^{(m_0)}$ such that $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}} \phi^{(m_0)}$ is isomorphic to ϕ. As $\phi^{(m_0)}$ is continuous for the canonical topologies (see [BGR84, 3.7.3]), this yields that ϕ is continuous. The respective case is treated in a similar way. □

Lemma 14.2.3.2. *Let $\phi\colon E \twoheadrightarrow E'$ be an epimorphism of coherent $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. of finite type $O_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$-modules). Then the morphism ϕ is strict.*

Proof. Following 8.4.1.11 there exists for a large enough nonnegative integer m_0 a morphism of finite type $\widetilde{D}^{(m_0)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}$-modules $\phi^{(m_0)}\colon E^{(m_0)} \to E'^{(m_0)}$ such that $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}} \phi^{(m_0)}$ is isomorphic to ϕ. As we have $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}} \mathrm{Coker}(\phi^{(m_0)}) = \{0\}$, increasing m_0 if necessary, we can suppose that $\phi^{(m_0)}$ is surjective. Then so is $\widetilde{D}^{(m)}_{\mathfrak{X}^{\sharp},\mathbb{Q}} \otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^{\sharp},\mathbb{Q}}} \phi^{(m_0)}$, for any integer $m \geq m_0$. Moreover, these latter epimorphisms are also strict and continuous for the respective Banach topologies (see [BGR84, 3.7.3, corollary 5]). Taking the inductive limit, this yields that ϕ is strict (use 14.1.1.3). The respective case is treated in a similar way. □

Proposition 14.2.3.3. *For some integers $r, s \geq 0$, let ϕ be a left $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-linear morphism of the form $\phi\colon (D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^r \to (D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s$. Let ψ be a right $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-linear morphism of the form $\psi\colon (D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^r \to (D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s$. With notation 9.3.4.6, the morphisms ϕ, $u^*_{\mathrm{l}}(\phi)$, $u^*_{\mathrm{r}}(\psi)$ are continuous for the respective canonical topologies (see the definitions of 14.2.1.8).*

Proof. 1) The continuity of ϕ is a consequence of the lemma 14.2.3.1.

2) Let us check now that $u^*_{\mathrm{l}}(\phi)$ is continuous. For any nonnegative integer n, let us denote by ϖ^n the surjective right $D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-linear morphims of the form $\varpi^n\colon (D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^n \twoheadrightarrow (\widetilde{D}^{\dagger}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp},\mathbb{Q}})^n$. As $u^*_{\mathrm{l}}(\phi) \circ \varpi^r = \varpi^s \circ \phi$ is continuous, it is sufficient to establish that ϖ^r is a continuous and strict morphism. As the canonical surjection $\widetilde{D}^{(m)}_{\mathfrak{X}^{\sharp},\mathbb{Q}} \to \widetilde{D}^{(m)}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp},\mathbb{Q}}$ sends $\widetilde{D}^{(m)}_{\mathfrak{X}^{\sharp}}$ into $\widetilde{D}^{(m)}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp}}$, this one is then continuous. Following Banach open mapping theorem, this surjective and continuous morphism of Banach K-vector spaces is then strict. Using the lemma 14.1.1.3, this yields that so is ϖ^r by taking the inductive limit on the level.

3) To validate the continuity of $u^*_{\mathrm{r}}(\psi)$, we proceed in the same way as the step 2). □

14.3 Preservation of the coherence by local cohomological functor

Let $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ be a closed immersion of separated, quasi-compact and smooth $\mathcal{V}$-formal schemes, T be a divisor of X such that $T \cap Z$ is a divisor of Z. Let $\mathfrak{D}$ be a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor such that $u^{-1}(\mathfrak{D})$ is a relative to $\mathfrak{Z}/\mathfrak{S}$ strict normal crossing divisor. We set $\mathfrak{X}^{\sharp} := (\mathfrak{X}, M(\mathfrak{D}))$, $\mathfrak{Z}^{\sharp} := (\mathfrak{Z}, M(u^{-1}\mathfrak{D}))$ and we denote by $u^{\sharp}\colon \mathfrak{Z}^{\sharp} \hookrightarrow \mathfrak{X}^{\sharp}$ the exact closed immersion of smooth logarithmic formal schemes on $\mathcal{V}$. Let $\lambda_0\colon \mathbb{N} \to \mathbb{N}$ be an increasing map such that $\lambda_0(m) \geq m$, for any $m \in \mathbb{N}$. To lighten the notations, we put then $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}}(T)$, $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{Z}}(T \cap Z)$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}}$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}}$. In the same way, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z)$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp}} := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp}}$, $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp},\mathbb{Q}} = \varinjlim_{m} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}\to\mathfrak{X}^{\sharp},\mathbb{Q}}$ and $\widetilde{\mathcal{D}}^{\dagger}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp},\mathbb{Q}} = \varinjlim_{m} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}\leftarrow\mathfrak{Z}^{\sharp},\mathbb{Q}}$.

14.3.1 Pushforward of level m by u in local coordinates context: exactness

We suppose that $\mathfrak{X}$ is affine, $\mathfrak{X}/\mathfrak{S}$ is endowed with coordinates $t_1, \ldots, t_d$ such that for some integers $0 \le r \le s \le d$ we have $\mathfrak{D} = V(t_1 \cdots t_r)$, $\mathfrak{Z} := \cap_{s+1}^{i=d} V(t_i)$ and $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ is equal to the induced exact closed immersion. Remark, that locally such hypotheses are valid.

Notation 14.3.1.1. For any integer $m \geq 0$, we put $D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z) := \widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z) \otimes_{\mathcal{O}_{\mathfrak{Z}}} D^{(m)}_{\mathfrak{Z}^\sharp}$, $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z) := D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)$. We endow $D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}$ (resp. $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}$) with a structure of normed $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-algebra whose basis of open neighborhoods at zero is given by the family $(p^n D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z))_{n \in \mathbb{N}}$ (resp. $(p^n D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z))_{n \in \mathbb{N}}$). In other words, they are the topologies coming from the induced norms via the inclusions $D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \hookrightarrow \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}$ and $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \hookrightarrow \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}}$. Moreover, we remark that the separated completion of $D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}$ (resp. $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}$) is $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}$ (resp. $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}}$).

Lemma 14.3.1.2. *Let $V' \underset{\phi}{\longrightarrow} V \underset{\psi}{\longrightarrow} V''$ be an exact sequence of Banach $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}$-modules (see the definition 14.1.3.2 and the canonical topology of 14.2.1.8). We suppose moreover that ϕ and ψ are strict morphisms. Then the sequences*

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V' \underset{\mathrm{id} \otimes \phi}{\longrightarrow} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V \underset{\mathrm{id} \otimes \psi}{\longrightarrow} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V'', \tag{14.3.1.2.1}$$

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V' \underset{\mathrm{id} \widehat{\otimes} \phi}{\longrightarrow} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V \underset{\mathrm{id} \widehat{\otimes} \psi}{\longrightarrow} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V'' \tag{14.3.1.2.2}$$

are exact and their morphisms are strict.

Proof. 0) By breaking down the exact sequence into short exact sequences, we can suppose we have in fact the exact sequence $0 \to V' \underset{\phi}{\longrightarrow} V \underset{\psi}{\longrightarrow} V'' \to 0$.

1) First, let us check the exactness and strictness of 14.3.1.2.2. For any $\underline{k} \in \mathbb{N}^{d-s}$, let us denote by $\xi_{\underline{k},(m)}$ the image of $\underline{\partial}^{\langle (\underline{0}, \underline{k}) \rangle_{(m)}}_{(r)}$ via the canonical surjection $D^{(m)}_{\mathfrak{X}^\sharp}(T)_{\mathbb{Q}} \twoheadrightarrow D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}$. The elements of the K-vector space $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V$ can be written uniquely of the form $\sum_{\underline{k} \in \mathbb{N}^{d-s}} \xi_{\underline{k},(m)} \otimes x_{\underline{k}}$, the sum being finite and $x_{\underline{k}} \in V$. Following 14.1.2.3, if we denote by V_0 the $\mathcal{V}$-submodule of V consisting of elements of norm less or equal to 1 and by U_0 the $\mathcal{V}$-submodule of $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V$ generated by the canonical image of the arrow $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z) \times V_0 \to D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V$, then a basis of open neighborhoods of $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V$ at zero is given by the family $(p^n U_0)_{n \in \mathbb{N}}$. This yields that the canonical topology on $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V$ is induced by the norm $\| \sum_{\underline{k} \in \mathbb{N}^{d-s}} \xi_{\underline{k},(m)} \otimes x_{\underline{k}} \| = \max_{\underline{k}} \| x_{\underline{k}} \|$. We have the same description for V' or V'' instead of V. We get then the short exact sequence of normed K-vector spaces

$$0 \to D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V' \to D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V \to D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \otimes_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V'' \to 0, \tag{14.3.1.2.3}$$

whose morphisms are strict (the strictness of the injection is straightforward from the description of their norm ; the strictness of the surjection can be checked by using for instance 14.1.2.5 or by an easy computation). Moreover, following the proposition 14.1.3.5, as V is a Banach $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}$-module, the canonical morphism $D^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}} \widehat{\otimes}_{D^{(m)}_{\mathfrak{Z}^\sharp}(T \cap Z)_{\mathbb{Q}}} V \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V$ is an isomorphism. As the separated completion functor transforms short exact sequences in which the applications are strict morphisms of normed K-vector spaces in short exact sequences in which the applications are strict morphisms, we obtain the exact sequence

$$0 \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V' \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} V'' \to 0 \tag{14.3.1.2.4}$$

whose morphisms are strict.

2)Let us now check the exactness and strictness of 14.3.1.2.1. As $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}$ is a flat left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-module, we have the same exact sequence than 14.3.1.2.4 where we replace $\widehat{\otimes}$ by $\otimes$. The surjective morphism of this last exact sequence is strict thanks to 14.1.2.5. Finally, the strictness of the injective morphism is a consequence of the fact that by composing it with the strict monomorphism $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V\to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V$, we still get a strict monomorphism. □

Lemma 14.3.1.3. *Let $m'\geq m$ be two integers, V be a Banach $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-module and V' be a Banach $\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-module (see the definition 14.1.3.2). Let $\phi\colon V\hookrightarrow V'$ be a $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-linear continuous injective morphism. The continuous morphism $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V\to \widetilde{D}^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V'$ canonically induced by ϕ is then injective.*

Proof. For any $\underline{k}\in\mathbb{N}^{d-s}$, let $\xi_{\underline{k},(m')}$ be the image of $\underline{\partial}_{(r)}^{\langle(\underline{0},\underline{k})\rangle_{(m')}}$ via the canonical surjection $D^{(m')}_{\mathfrak{X}^\sharp}(T\cap Z)_{\mathbb{Q}}\twoheadrightarrow D^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}(T\cap Z)_{\mathbb{Q}}$. Since the Banach K-vector space $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}V'$ is isomorphic to $D^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}(T\cap Z)_{\mathbb{Q}}\widehat{\otimes}_{D^{(m')}_{\mathfrak{Z}^\sharp}(T\cap Z)_{\mathbb{Q}}}V'$ (see the proof of 14.3.1.2), then $\widetilde{D}^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}V'$ is the K-space consisting of elements which can be written uniquely of the form $\sum_{\underline{k}\in\mathbb{N}^{d-s}}\xi_{\underline{k},(m')}\otimes x'_{\underline{k}}$, the sum being infinite but the sequence of elements $x'_{\underline{k}}\in V'$ converging to zero when $|\underline{k}|$ converges to the infinity. As $\xi_{\underline{k},(m)}=\lambda_{\underline{k},(m,m')}\xi_{\underline{k},(m')}$, for some $\lambda_{\underline{k},(m,m')}\in\mathcal{V}\setminus\{0\}$, then the canonical morphism $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}(T\cap Z)_{\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V\to \widetilde{D}^{(m')}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}(T\cap Z)_{\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m')}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}V'$ sends the sum $\sum_{\underline{k}\in\mathbb{N}^{d-s}}\xi_{\underline{k},(m)}\otimes x_{\underline{k}}$ on $\sum_{\underline{k}\in\mathbb{N}^{d-s}}\xi_{\underline{k},(m')}\otimes\lambda_{\underline{k},(m,m')}\phi(x_{\underline{k}})$. Hence we are done. □

Proposition 14.3.1.4. *Let $0\to V'\underset{\phi}{\longrightarrow} V\underset{\psi}{\longrightarrow} V''$ be an exact sequence of Banach $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-modules (see the definition 14.1.3.2). We suppose moreover that ϕ is a strict morphism. The sequence*

$$0\to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V'\underset{\mathrm{id}\widehat{\otimes}\phi}{\longrightarrow}\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V\underset{\mathrm{id}\widehat{\otimes}\psi}{\longrightarrow}\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} V'' \qquad (14.3.1.4.1)$$

is then exact and $id\widehat{\otimes}\phi$ is strict.

Proof. By endowing $W:=V/V'$ with the quotient topology, we obtain the $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-linear, injective and continuous morphism $W\hookrightarrow V''$. This yields that W is a Banach $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-module, because it is a separated quotient of V. We conclude by applying respectively the lemmas 14.3.1.2 and 14.3.1.3 to the exact sequence $0\to V'\to V\to W\to 0$ and to the monomorphism $W\to V''$. □

Lemma 14.3.1.5. *The module $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}$ is separated for the p-adic topology.*

Proof. If E is a finite type left $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}$-submodule of $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}$, since $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}$ is the separated p-adic completion of a free right $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}$-module, then $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}E$ is separated and complete for the p-adic topology (see 7.2.1.4). By using 14.3.1.3, this yields that the canonical morphism $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}E\to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}$ is also a monomorphism. By taking the inductive limit on E, this yields that the canonical morphism $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}\to\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}$ is injective. □

14.3.1.6. We define some normed K-vector spaces as follows.

(a) We endow $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}$ with the tensor product topology defined in 14.1.2.2, i.e. following the lemma 14.1.2.3, this is the topology whose basis of open neighborhoods at zero is given by the family $p^n\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}$ where n is going through $\mathbb{N}$.

(b) We endow naturally $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})_{\mathbb{Q}}$ with the topology whose basis of open neighborhoods at zero is given by the family $p^n \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$ with n going through $\mathbb{N}$, which is in fact a Banach K-vector space.

(c) Since we have the equality

$$\left(p^n \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}\right) \cap \left(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp, \mathbb{Q}}\right) = p^n \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$$

(this is a consequence of the uniqueness of the writing as described in the proof of 14.3.1.3), then we get the strict monomorphism of normed K-vector spaces

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp, \mathbb{Q}} \hookrightarrow (\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})_{\mathbb{Q}}. \tag{14.3.1.6.1}$$

Lemma 14.3.1.7. *The morphism 14.3.1.6.1 factors through the canonical (independent of local coordinates) isomorphism of Banach K-vector spaces:*

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp, \mathbb{Q}} \xrightarrow{\sim} (\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})_{\mathbb{Q}}. \tag{14.3.1.7.1}$$

Proof. By using the universal property of separated completions of locally convex K-vector spaces (see [Sch02, 7.5]), the morphism 14.3.1.6.1 canonically factors through a morphism of the form 14.3.1.7.1. Since 14.3.1.6.1 is a strict monomorphism of normed K-vector spaces, since $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})_{\mathbb{Q}}$ is a Banach spaces, then this amount to saying that the image of the map 14.3.1.6.1 is dense. By construction of separated completions of locally convex K-vector spaces (e.g. see the proof of [Sch02, 7.5]), since $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$ is a dense open lattice of $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp, \mathbb{Q}}$, then $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$ is a dense open lattice of $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp, \mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp, \mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp, \mathbb{Q}}$. Since $\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp}$ is a dense open lattice of $(\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})_{\mathbb{Q}}$, we are done. □

14.3.2 Stability of the coherence by local cohomological functor in degree zero

Proposition 14.3.2.1. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module. The sheaf $\mathcal{H}^{\dagger 0}_Z(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module if and only if $H^0 u^{\sharp !}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module. When these two equivalent conditions are satisfied, we have the isomorphism $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}) \xrightarrow{\sim} \mathcal{H}^{\dagger 0}_Z(\mathcal{E})$ (see notation 13.1.5.3).*

Proof. 1) a) Let us suppose that $H^0 u^{\sharp !}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module. Following 9.3.2.3, we have the canonical adjoint morphism of (coherent) $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules of the form $\phi\colon u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}) \to \mathcal{E}$. Let us check that ϕ is injective.

It follows from Berthelot-Kashiwara theorem (see 9.3.5.9), that the canonical adjoint morphism $H^0 u^{\sharp !}(\mathcal{E}) \to u^{\sharp !} \circ u^\sharp_+ (H^0 u^{\sharp !}(\mathcal{E}))$ is an isomorphism and $u^\sharp_+ (H^0 u^{\sharp !}(\mathcal{E})) \xrightarrow{\sim} H^0 u^\sharp_+ (H^0 u^{\sharp !}(\mathcal{E}))$. Since the composition $H^0 u^{\sharp !}(\mathcal{E}) \to H^0 u^{\sharp !} \circ u^\sharp_+ \circ H^0 u^{\sharp !}(\mathcal{E})) \to H^0 u^{\sharp !}(\mathcal{E})$ is the identity (see 9.3.2.3), this yields that $H^0 u^{\sharp !}(\phi)$ is an isomorphism. Let $\mathcal{K} := \operatorname{Ker} \phi$. Since $H^0 u^{\sharp !}$ is left exact (on the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules), since $\operatorname{Ker} H^0 u^{\sharp !}(\phi) = 0$, then $H^0 u^{\sharp !}(\mathcal{K}) = 0$. Since $\mathcal{K}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module with support in Z, then it comes from Berthelot-Kashiwara theorem (see 9.3.5.9) that $\mathcal{K} = 0$, i.e. that ϕ is injective.

b) Since $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module with support in Z, then the canonical morphism $\mathcal{H}^{\dagger 0}_Z(u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})) \to u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})$ is an isomorphism (see 13.2.1.6.1). This yields the injection $\phi\colon u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}) \hookrightarrow \mathcal{E}$ factors through the inclusions $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}) \hookrightarrow \mathcal{H}^{\dagger 0}_Z(\mathcal{E}) \hookrightarrow \mathcal{E}$ (the second arrow do is injective thanks to 13.1.5.4).

c) Let us now check that the inclusion $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}) \hookrightarrow \mathcal{H}^{\dagger 0}_Z(\mathcal{E})$ is an isomorphism. By the absurd, suppose that this latter inclusion is not an isomorphism. In that case, there exists an affine open

$\mathfrak{U}$ of $\mathfrak{X}$, such that $\Gamma(\mathfrak{U}, u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})) \neq \Gamma(\mathfrak{U}, \mathcal{H}^{\dagger 0}_Z(\mathcal{E}))$. Let $s \in \Gamma(\mathfrak{U}, \mathcal{H}^{\dagger 0}_Z(\mathcal{E})) \setminus \Gamma(\mathfrak{U}, u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}))$. Let G be the $D^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-submodule of $\Gamma(\mathfrak{U}, \mathcal{H}^{\dagger 0}_Z(\mathcal{E}))$ generated by $\Gamma(\mathfrak{U}, u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}))$ and by the section s. Since $\mathfrak{U}$ is affine, then $\Gamma(\mathfrak{U}, \mathcal{E})$ is a coherent $D^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-module. Since the composition $G \to \Gamma(\mathfrak{U}, \mathcal{H}^{\dagger 0}_Z(\mathcal{E})) \to \Gamma(\mathfrak{U}, \mathcal{E})$ is injective, then G is a coherent $D^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-module because it is a finite type submodule of a coherent $D^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-module. We get the coherent $\mathcal{D}^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-module by setting $\mathcal{G} := \mathcal{D}^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q} \otimes_{D^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}} G$. From the inclusion $G \subset \Gamma(\mathfrak{U}, \mathcal{H}^{\dagger 0}_Z(\mathcal{E}))$ we get the morphism $\mathcal{G} \to \mathcal{H}^{\dagger 0}_Z(\mathcal{E})|\mathfrak{U}$. Since $\mathcal{G}$ and $\mathcal{E}|\mathfrak{U}$ are coherent $\mathcal{D}^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-modules, then using theorem of type A the composition $\mathcal{G} \to \mathcal{H}^{\dagger 0}_Z(\mathcal{E})|\mathfrak{U} \to \mathcal{E}|\mathfrak{U}$ is injective. Hence, so is $\mathcal{G} \to \mathcal{H}^{\dagger 0}_Z(\mathcal{E})|\mathfrak{U}$. Moreover, since $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U}$ and $\mathcal{G}$ are coherent $\mathcal{D}^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-modules, similarly we get the canonical inclusion $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U} \hookrightarrow \mathcal{G}$. Let $v^{\sharp !}\colon \mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp \hookrightarrow \mathfrak{U}^\sharp$ be the morphism induced by $u^\sharp$ by restriction. By applying the left exact functor $H^0 v^{\sharp !}$ to the inclusions $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U} \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{E}|\mathfrak{U}$, we get the monomorphisms $H^0 u^{\sharp !} u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U} \hookrightarrow H^0 v^{\sharp !}\mathcal{G} \hookrightarrow H^0 u^{\sharp !}\mathcal{E}|\mathfrak{U}$ whose composition is an isomorphism (equal to $H^0 u^{\sharp !}(\phi)|\mathfrak{U}$). Hence, we get the isomorphism $H^0 u^{\sharp !} u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U} \xrightarrow{\sim} H^0 v^{\sharp !}\mathcal{G}$. Since $\mathcal{G}$ and $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U}$ are coherent $\mathcal{D}^\dagger_{\mathfrak{U}^\sharp}({}^\dagger T\cap U)_\mathbb{Q}$-modules with support in $U \cap Z$, then by using Berthelot-Kashiwara theorem this implies that the morphism $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E})|\mathfrak{U} \to \mathcal{G}$ is an isomorphism, which is a contradiction.

2) Let us suppose that $\mathcal{H}^{\dagger 0}_Z(\mathcal{E})$ is $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}$-coherent.

a) Let us check that for any closed subscheme Z' of X containing Z we have

$$\mathcal{H}om_{\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}}(\mathcal{O}_\mathfrak{Z}({}^\dagger T\cap Z)_\mathbb{Q}, \omega_{\mathfrak{X}^\sharp/\mathfrak{S}} \otimes_{\mathcal{O}_\mathfrak{X}} H^0(\mathcal{E}({}^\dagger Z'))) \otimes_{\mathcal{O}_\mathfrak{Z}} \omega^{-1}_{\mathfrak{Z}^\sharp/\mathfrak{S}} = 0.$$

Since this is local, then we can suppose $\mathfrak{X}$ is integral and we are in the situation of 14.2. Let us denote by $n_{Z'}$ the minimal number of divisor $T_1, \ldots, T_r$ such that $Z' = T_1 \cap \cdots \cap T_r$. We check the assertion by induction on $n_{Z'}$. The case where $n_{Z'} = 0$ (i.e. is the case where $Z' = X$) is obvious and the case $n_{Z'} = 1$ (i.e. the case where Z' is a divisor containing Z) is an easy computation. Suppose now $r \geq 2$ and the proposition holds for any Z' such that $n_{Z'} < r$. Let $T_1, \ldots, T_r$ be some divisors such that $Z' = T_1 \cap \cdots \cap T_r$. Set $Z'' = T_2 \cap \cdots \cap T_r$. It follows from 13.1.5.4.(a) that the long exact sequence induced by the Mayer-Vietoris exact triangles (see 13.1.4.15.2) yields the exact sequence:

$$0 \to H^0\mathcal{E}({}^\dagger Z') \to H^0({}^\dagger T_1)(\mathcal{E}) \oplus H^0({}^\dagger Z'')(\mathcal{E}) \to H^0({}^\dagger T_1 \cup Z'')(\mathcal{E}({}^\dagger Z')) \tag{14.3.2.1.1}$$

Since the functor $\mathcal{H}om_{\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}}(\mathcal{O}_\mathfrak{Z}({}^\dagger T\cap Z)_\mathbb{Q}, \omega_{\mathfrak{X}^\sharp/\mathfrak{S}} \otimes_{\mathcal{O}_\mathfrak{X}} -) \otimes_{\mathcal{O}_\mathfrak{Z}} \omega^{-1}_{\mathfrak{Z}^\sharp/\mathfrak{S}}$ is left exact, then we can conclude the induction.

b) With notation 9.3.4.5, we get from 9.3.1.17.1 the isomorphisms

$$u^{\sharp !}(H^0\mathcal{E}({}^\dagger Z)) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}}(\mathcal{O}_\mathfrak{Z}({}^\dagger T\cap Z)_\mathbb{Q}, \omega_{\mathfrak{X}^\sharp/\mathfrak{S}} \otimes_{\mathcal{O}_\mathfrak{X}} H^0\mathcal{E}({}^\dagger Z)) \otimes_{\mathcal{O}_\mathfrak{Z}} \omega^{-1}_{\mathfrak{Z}^\sharp/\mathfrak{S}}.$$

By using the vanishing of the part (a) in the case where $Z' = Z$, this yields $H^0 u^{\sharp !} H^0(\mathcal{E}({}^\dagger Z)) = 0$. By applying the functor $H^0 u^{\sharp !}$ to the exact sequence $0 \to \mathcal{H}^{\dagger 0}_Z(\mathcal{E}) \to \mathcal{E} \to H^0(\mathcal{E}({}^\dagger Z))$, this implies that the canonical morphism $H^0 u^{\sharp !}\mathcal{H}^{\dagger 0}_Z(\mathcal{E}) \to H^0 u^{\sharp !}(\mathcal{E})$ is an isomorphism. It follows from 9.3.5.9 that $H^0 u^{\sharp !}\mathcal{H}^{\dagger 0}_Z(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T\cap Z)_\mathbb{Q}$-module. Hence, we are done.

3) If one of these two conditions is satisfied then the isomorphism $u^\sharp_+ H^0 u^{\sharp !}(\mathcal{E}) \xrightarrow{\sim} \mathcal{H}^{\dagger 0}_Z(\mathcal{E})$ has been checked at the step 1.c). □

14.3.3 Stability of the coherence by local cohomological functor in maximal degree

The purpose of this subsection is to check the corollary 14.3.3.9. This corollary completes the theorem 14.3.3.1 just below which is straightforward from 9.1.6.3 and 13.2.1.5.1.

Theorem 14.3.3.1. *We suppose that $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ is of pure codimension 1. Let $\mathcal{E}^{(\bullet)}$ be an object of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$ and $\mathcal{E} := \varinjlim (\mathcal{E}^{(\bullet)})$ the corresponding object of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q})$. The following assertions are equivalent:*

(a) $u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp})$.

(b) $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$.

(c) $({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$.

(d) $({}^\dagger Z)(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$.

(e) $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$.

When these assertions are fulfilled, we have therefore $u^{\sharp !}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T\cap Z)_{\mathbb{Q}})$ *and the isomorphism*

$$u^\sharp_+(u^{\sharp !}(\mathcal{E})) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}).$$

Proof. The equivalences $(b) \Leftrightarrow (c)$ and $(d) \Leftrightarrow (e)$ are a consequence of the distinguished triangle of localisation $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to ({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \to +1$. The equivalence $(c) \Leftrightarrow (d)$ is exactly the corollary 9.1.6.3. The equivalence $(a) \Leftrightarrow (b)$ follows from the canonical isomorphism $u^{\sharp(\bullet)}_+ \circ u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}^{(\bullet)})$ (see 13.2.1.5.1), from the canonical isomorphism $u^{\sharp(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)})$, as well as from the theorem of Berthelot-Kashiwara still valid in the context of the categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$ (see 9.3.5.13). When these assertions are fulfilled, by applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ which preserves the coherence and the isomorphisms, we get the desired consequences. □

14.3.3.2. Let $\mathfrak{B}$ be the basis of open neighborhoods of $\mathfrak{X}$ consisting of affine opens which are endowed with local coordinates satisfying the conditions of 14.2. For any $\mathfrak{U} \in \mathfrak{B}$, we denote by $\mathfrak{U}^\sharp := (\mathfrak{U}, \mathfrak{U}\cap\mathfrak{D})$. Let us describe the section on $\mathfrak{U} \in \mathfrak{B}$ of the sheaf $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})_{\mathbb{Q}}$.

Let $\mathfrak{U} \in \mathfrak{B}$. For any nonnegative integer i, we denote by X_i, $X_i^\sharp$, Z_i, $Z_i^\sharp$, U_i, $U_i^\sharp$ the reductions modulo π^{i+1} of respectively $\mathfrak{X}$, $\mathfrak{X}^\sharp$, $\mathfrak{Z}$, $\mathfrak{Z}^\sharp$, $\mathfrak{U}$, $\mathfrak{U}^\sharp$. We have the quasi-coherent $\mathcal{B}^{(m)}_{Z_i\cap U_i}(T\cap U_i)$-modules $\widetilde{\mathcal{D}}^{(m)}_{U_i^\sharp\leftarrow Z_i^\sharp\cap U_i^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}\otimes_{\mathcal{V}}\mathcal{V}/\pi^{i+1}$, $\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp\cap U_i^\sharp} = \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}\otimes_{\mathcal{V}}\mathcal{V}/\pi^{i+1}$, and finally $\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp\cap U_i^\sharp\to U_i^\sharp} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp\to\mathfrak{U}^\sharp}\otimes_{\mathcal{V}}\mathcal{V}/\pi^{i+1}$. It follows from 7.2.3.13.1 (and 7.2.3.13.(i)) that $\widetilde{D}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}$ is separated and complete for the p-adic completion and that we have the canonical isomorphism: $\widetilde{D}^{(m)}_{U_i^\sharp\leftarrow Z_i^\sharp\cap U_i^\sharp} \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}\otimes_{\mathcal{V}}\mathcal{V}/\pi^{i+1}$; similarly for $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp\to\mathfrak{U}^\sharp}$. Since the global sections functor commutes with projective limits and by quasi-coherence of our sheaves on X_i, we obtain the canonical isomorphisms

$$\begin{aligned}\Gamma(\mathfrak{U}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}) &= \Gamma(\mathfrak{U}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp\to\mathfrak{U}^\sharp})\\ \xrightarrow{\sim} \varprojlim_i \Gamma(U_i, \widetilde{\mathcal{D}}^{(m)}_{U_i^\sharp\leftarrow Z_i^\sharp\cap U_i^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp\cap U_i^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{Z_i^\sharp\cap U_i^\sharp\to U_i^\sharp}) &\xrightarrow{\sim} \varprojlim_i \widetilde{D}^{(m)}_{U_i^\sharp\leftarrow Z_i^\sharp\cap U_i^\sharp}\otimes_{\widetilde{D}^{(m)}_{Z_i^\sharp\cap U_i^\sharp}}\widetilde{D}^{(m)}_{Z_i^\sharp\cap U_i^\sharp\to U_i^\sharp}\\ &\xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp\to\mathfrak{U}^\sharp}. \qquad (14.3.3.2.1)\end{aligned}$$

Moreover, for any $\mathfrak{U} \in \mathfrak{B}$, the functor $\Gamma(\mathfrak{U}, -)$ commutes with the tensor product by $\mathbb{Q}$. This yields

$$\Gamma(\mathfrak{U}, (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})_{\mathbb{Q}}) \xrightarrow{\sim} (\widetilde{D}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp\to\mathfrak{U}^\sharp})_{\mathbb{Q}}.$$

By tensorizing by $\mathbb{Q}$ the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp}$, we get the morphism

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}} \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})_{\mathbb{Q}}. \qquad (14.3.3.2.2)$$

Notation 14.3.3.3. We keep notation 14.3.3.2.

(a) We denote by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp,\mathbb{Q}}$ the sheaf (of K-algebras) on $\mathfrak{X}$ associated to the presheaf (in fact this is a sheaf following 14.3.3.4) on $\mathfrak{B}$ defined as follows: $\mathfrak{U} \in \mathfrak{B} \mapsto \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp\leftarrow\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}}\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp\to\mathfrak{U}^\sharp,\mathbb{Q}}$, the restriction morphisms being the canonical morphisms (the separated completion is a functor).

(b) Since the canonical morphisms

$$\widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp \to \mathfrak{U}^\sharp,\mathbb{Q}} \to \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp \to \mathfrak{U}^\sharp,\mathbb{Q}}$$

are functorial in $\mathfrak{U} \in \mathfrak{B}$, we get then the canonical morphism of sheaves:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}}. \qquad (14.3.3.3.1)$$

(c) Let $\alpha^{(m)} \colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ be a left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-linear morphism. With the notation 9.3.4.1, by using the functoriality in $\mathfrak{U} \in \mathfrak{B}$ of the morphisms

$$id\widehat{\otimes}\Gamma(\mathfrak{U}\cap\mathfrak{Z}, u^*(\alpha^{(m)}))\colon \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}} (\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp \to \mathfrak{U}^\sharp,\mathbb{Q}})^r \to \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}} (\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp \to \mathfrak{U}^\sharp,\mathbb{Q}})^s,$$

we get the morphism of sheaves:

$$id\widehat{\otimes}u^*(\alpha^{(m)})\colon \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}})^r \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}})^s. \qquad (14.3.3.3.2)$$

Lemma 14.3.3.4. *We keep the notations of 14.3.3.3.*

(a) We have the canonical commutative diagram in the category of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-bimodules:

$$\begin{array}{ccc} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}} & \overset{\sim}{\dashrightarrow} & (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})_{\mathbb{Q}} \\ {\scriptstyle 14.3.3.3.1}\nwarrow & & \nearrow{\scriptstyle 14.3.3.2.2} \\ & \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}} & \end{array} \qquad (14.3.3.4.1)$$

whose top arrow is an isomorphism. For any $\mathfrak{U} \in \mathfrak{B}$, we have moreover

$$\Gamma(\mathfrak{U}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}}) \xrightarrow{\sim} \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp,\mathbb{Q}}} \widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp\cap\mathfrak{U}^\sharp \to \mathfrak{U}^\sharp,\mathbb{Q}}.$$

(b) Let $\epsilon^{(m)} \colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})^r \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})^s$ be a left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-linear morphism and $\alpha^{(m)} \colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ be the morphism induced by tensorizing by $\mathbb{Q}$. We have the commutative square

$$\begin{array}{ccc} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})^r)_{\mathbb{Q}} & \xrightarrow{(id\widehat{\otimes}u^*\epsilon^{(m)})_{\mathbb{Q}}} & (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp})^s)_{\mathbb{Q}} \\ \sim\uparrow & & \sim\uparrow \\ \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r & \xrightarrow{id\widehat{\otimes}u^*\alpha^{(m)}} & \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s, \end{array} \qquad (14.3.3.4.2)$$

whose vertical isomorphisms are induced by the factorisation of 14.3.3.4.1.

Proof. The first statement is a consequence of the canonical isomorphisms of 14.3.1.7.1. The second assertion can easily be checked. □

Remark 14.3.3.5. With the notations of 14.3.3.3, the sheaf $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp \to \mathfrak{X}^\sharp,\mathbb{Q}}$ of K-vector spaces on $\mathfrak{B}$ is also a sheaf of topological K-vector spaces on $\mathfrak{B}$ (indeed we can check that the restriction morphisms are injective and strict).

Lemma 14.3.3.6. *We suppose that $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ is of pure codimension e. Let α be a morphism of left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules which is of the form $\alpha\colon (\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^r \to (\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$. Let m_0 be a large enough integer such that there exists a left $\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}}$-linear morphism of the form $\alpha^{(m_0)}\colon (\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to (\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ inducing α by extension via $\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp,\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$. For $m \geq m_0$, we denote by $\alpha^{(m)}\colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$, the morphism induced by extension of $\alpha^{(m_0)}$.*

Hence we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^{\dagger e}_Z((\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^r) & \xrightarrow{\mathcal{H}^{\dagger e}_Z(\alpha)} & \mathcal{H}^{\dagger e}_Z((\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s) \\
\sim\uparrow & & \sim\uparrow \\
u_*\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r & \xrightarrow{\varinjlim_m (\mathrm{id}\,\widehat{\otimes} u^*\alpha^{(m)})} & u_*\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s
\end{array} \tag{14.3.3.6.1}$$

whose vertical arrows are isomorphisms.

Proof. 1) Multiplying $\alpha^{(m_0)}$ by a large enough power of p, we can suppose that $\alpha^{(m_0)}$ comes by extension from a left $\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp}$-linear morphism of the form $\epsilon^{(m_0)}\colon (\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp})^r \to (\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^\sharp})^s$. For $m \geq m_0$, let us denote by $\epsilon^{(m)}\colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})^r \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})^s$ the morphisms induced by extension. So we get the morphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$ of the form

$$\epsilon^{(\bullet+m_0)}\colon (\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^r \to (\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^s.$$

2) Let $N \in \mathbb{N}$. Since $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})^N$ is a p-torsion free quasi-coherent $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp/\mathfrak{S}}$-module (see the definition 7.2.3.5 and 7.3.1.7) for any $m \in \mathbb{N}$, then it follows from 9.3.3.1.2

$$u^{\sharp(\bullet)}_+((\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})^N) \xrightarrow{\sim} u_*\left(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}} (\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})^N\right). \tag{14.3.3.6.2}$$

Since $u^{\sharp(\bullet)!}((\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^N)[e] \xrightarrow{\sim} (\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})^N$, this yields the third (from the top) vertical isomorphisms

$$\begin{array}{ccc}
\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})[e] & \xrightarrow{\mathbb{R}\underline{\Gamma}^\dagger_Z(\alpha)[e]} & \mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^N)[e] \\
\sim\downarrow & & \sim\downarrow \\
\underrightarrow{l}^*_{\mathbb{Q}}\mathbb{R}\underline{\Gamma}^\dagger_Z((\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^N)[e] & \xrightarrow{\underrightarrow{l}^*_{\mathbb{Q}}\mathbb{R}\underline{\Gamma}^\dagger_Z\epsilon^{(\bullet+m_0)}} & \underrightarrow{l}^*_{\mathbb{Q}}\mathbb{R}\underline{\Gamma}^\dagger_Z((\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^N)[e] \\
\sim\downarrow 13.2.1.5.1 & & \sim\downarrow 13.2.1.5.1 \\
\underrightarrow{l}^*_{\mathbb{Q}}u^{\sharp(\bullet)}_+\circ u^{\sharp(\bullet)!}((\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^N)[e] & \xrightarrow{\underrightarrow{l}^*_{\mathbb{Q}}u^{\sharp(\bullet)}_+u^{\sharp(\bullet)!}\epsilon^{(\bullet+m_0)}} & \underrightarrow{l}^*_{\mathbb{Q}}u^{\sharp(\bullet)}_+\circ u^{\sharp(\bullet)!}((\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp})^N)[e] \\
\sim\downarrow 14.3.3.6.2 & & \sim\downarrow 14.3.3.6.2 \\
\underrightarrow{l}^*_{\mathbb{Q}}u_*\left(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}}(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})^N\right) & \xrightarrow{\underrightarrow{l}^*_{\mathbb{Q}}u_*(\mathrm{id}\,\widehat{\otimes}u^*\epsilon^{(\bullet+m_0)})} & \underrightarrow{l}^*_{\mathbb{Q}}u_*\left(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp/\mathfrak{S}^\sharp}}(\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{Z}^\sharp\to\mathfrak{X}^\sharp})^N\right) \\
\downarrow\sim & & \downarrow\sim \\
u_*\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}(u^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^N & \xrightarrow{\varinjlim_m(\mathrm{id}\,\widehat{\otimes}u^*\alpha^{(m)})} & u_*\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}(u^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^N.
\end{array} \tag{14.3.3.6.3}$$

This yields $\mathbb{R}\underline{\Gamma}^\dagger_Z(\alpha)[e] \xrightarrow{\sim} \mathcal{H}^{\dagger e}_Z(\alpha)$ and the commutative diagram 14.3.3.6.1 is equal to 14.3.3.6.3. □

Remark 14.3.3.7. With the notations of 14.3.3.6, let us suppose $\mathfrak{X}$ affine. Let $E^{(m_0)}$ be a coherent $\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp}$-module without p-torsion, $\mathcal{E}^{(\bullet)} := \widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}\otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp}} E^{(m_0)}$ the induced $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}$-module of finite presentation, $\mathcal{E} := \varinjlim(\mathcal{E}^\bullet)$ the associated coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module. By taking a resolution of $E^{(m_0)}$ by free finite type $\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp}$-module which induces, after applying the functor $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}\otimes_{\widetilde{D}^{(m_0)}_{\mathfrak{X}^\sharp}} -$, a resolution of $\mathcal{E}^{(\bullet)}$ by finite type free $\widetilde{\mathcal{D}}^{(\bullet+m_0)}_{\mathfrak{X}^\sharp}$-modules, we can check that the complex $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E})$ is isomorphic to a complex whose terms are of the form $\mathcal{H}^{\dagger e}_Z((\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^N)$ for some integers N.

We are now ready to prove the following theorem.

Theorem 14.3.3.8. *We suppose that* $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ *is of pure codimension* e*. Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$*-module satisfying the two following properties :*

(a) for any $i = 0, \ldots, e-1$*, locally in* $\mathfrak{Z}$*, the* $\mathcal{D}^{\dagger}_{\mathfrak{Z}^{\sharp}}({}^{\dagger}T \cap Z)_{\mathbb{Q}}$*-modules* $\mathcal{H}^i u^{\sharp !}(\mathcal{E})$ *are* $\Gamma(\mathfrak{Z}, -)$*-acyclic ;*

(b) the module $u^*(\mathcal{E})$ *is a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{Z}^{\sharp}}({}^{\dagger}T \cap Z)_{\mathbb{Q}}$*-module which is also a finitely generated locally projective* $\mathcal{O}_{\mathfrak{Z}}({}^{\dagger}T \cap Z)_{\mathbb{Q}}$*-module for the induced structure.*

The $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$*-module* $\mathcal{H}^{\dagger e}_{Z}(\mathcal{E})$ *is then coherent and is isomorphic to* $u^{\sharp}_{+} u^*(\mathcal{E})$*.*

Proof. 0) a) *Taking global sections.*

We use hypotheses 14.3.3.8.(a) as follows: Since the $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-coherence of $\mathcal{H}^{\dagger e}_{Z}(\mathcal{E})$ is local, we can suppose we are in the local situation of the subsection 14.2, and we keep its notations. In that case, we set $E := \Gamma(\mathfrak{X}, \mathcal{E})$ as usual. With the notations of 9.3.4.1, the hypotheses of 14.3.3.8.(a) yields the isomorphism $\Gamma(\mathfrak{Z}, u^*\mathcal{E}) \xrightarrow{\sim} H^0(\mathbb{R}\Gamma(\mathfrak{Z}, \mathbb{L}u^*\mathcal{E}))$ (we use the unbounded version of spectral sequence of hypercohomology of 4.6.1.6.1 for the functor of $\Gamma(\mathfrak{Z}, -)$). Then by using lemma 9.3.4.8 we get the isomorphism $\Gamma(\mathfrak{Z}, u^*\mathcal{E}) \xrightarrow{\sim} H^0(\mathbb{L}u^*(E)) = u^*(E)$.

b) *The morphisms* α*,* β*,* ϕ*,* ψ*.*

Since $\mathfrak{X}$ is affine, then via the theorem of type A of Berthelot of 8.7.5.5 which is valid for coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules, we have the exact sequence of left $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules $(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^r \underset{\alpha}{\longrightarrow} (\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s \underset{\beta}{\longrightarrow} \mathcal{E} \to 0$. Using theorems of type A and B, we get the exact sequence $(D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^r \underset{a}{\longrightarrow} (D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s \underset{b}{\longrightarrow} E \to 0$, where $a = \Gamma(\mathfrak{X}, \alpha)$ and $b = \Gamma(\mathfrak{X}, \beta)$. By setting $\phi := u^*(a)$ and $\psi := u^*(b)$, we get the commutative diagram.

$$\begin{array}{ccccccc}
\Gamma(\mathfrak{Z}, (u^*\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^r) & \xrightarrow{\Gamma(\mathfrak{Z}, u^*(\alpha))} & \Gamma(\mathfrak{Z}, (u^*\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s) & \xrightarrow{\Gamma(\mathfrak{Z}, u^*(\beta))} & \Gamma(\mathfrak{Z}, u^*(\mathcal{E})) & \longrightarrow & 0 \\
{\scriptstyle 9.3.4.3.1}\uparrow\wr & & {\scriptstyle 9.3.4.3.1}\uparrow\wr & & \uparrow & & \\
(u^* D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^r & \xrightarrow{\phi} & (u^* D^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s & \xrightarrow{\psi} & u^*(E) & \xrightarrow{\sim} & 0
\end{array} \tag{14.3.3.8.1}$$

By right exactness of the functor of the form u^* (see the definition of 9.3.4.1), we obtain the exactness of the bottom sequence. Since the canonical morphism $u^*(E) \to \Gamma(\mathfrak{Z}, u^*\mathcal{E})$ is an isomorphism, using 9.3.4.3.1, then every vertical arrows of 14.3.3.8.1 are isomorphisms. Hence, its top sequence is exact.

c) *Increasing* $\lambda_0 \in L(\mathbb{N})$ *if necessary, we define in this step* $\mathcal{M}$ *and* $\mathcal{M}^{(m)}$ *as follows.* We set $\mathcal{M} := u^*(\mathcal{E})$, $M := \Gamma(\mathfrak{Z}, \mathcal{M})$. Since $\mathcal{M} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Z}^{\sharp}, T \cap Z/\mathfrak{S})$ and is finitely generated projective $\mathcal{O}_{\mathfrak{Z}}({}^{\dagger}T \cap Z)_{\mathbb{Q}}$-module, then it follows from 11.2.1.9 (and from the remark 8.7.6.9 for the projectivity of $\mathcal{M}^{(0)}$) that, increasing $\lambda_0 \in L(\mathbb{N})$ if necessary, there exists $\mathcal{M}^{(0)}$ a topologically nilpotent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{Z}^{\sharp}, \mathbb{Q}}$-module, projective of finite type as $\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{X}}(T)_{\mathbb{Q}}$-module satisfying both conditions of 11.2.1.7.(b). For any $m \geq 0$, we set then $\mathcal{M}^{(m)} := \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}, \mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{Z}^{\sharp}, \mathbb{Q}}} \mathcal{M}^{(0)}$. Following 11.2.1.9.1, the canonical morphism

$$\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}} \otimes_{\widetilde{\mathcal{B}}^{(0)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}} \mathcal{M}^{(0)} \to \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^{\sharp}, \mathbb{Q}} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{Z}^{\sharp}, \mathbb{Q}}} \mathcal{M}^{(0)} = \mathcal{M}^{(m)}$$

is an isomorphism. Hence, $\mathcal{M}^{(m)}$ is a projective of finite type $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-module and the canonical morphisms $\mathcal{M}^{(0)} \to \mathcal{M}^{(m)}$ are injective. Moreover, we get the equality $M = \cup_{m \in \mathbb{N}} M^{(m)}$, where as usual we set $M^{(m)} := \Gamma(\mathfrak{Z}, \mathcal{M}^{(m)})$.

d) *Levels* $m \geq m_0$*, the morphisms* $\alpha^{(m)}$*,* $\beta^{(m)}$*,* $\phi^{(m)}$*,* $\psi^{(m)}$*.*

Let $m_0 \geq 0$ be a large enough integer such that there exists a left $\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}}$-linear morphism of the form $\alpha^{(m_0)}\colon (\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}})^r \to (\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}})^s$ and inducing α by extension via $\widetilde{\mathcal{D}}^{(m_0)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$. For any $m \geq m_0$, let us denote by $\alpha^{(m)}\colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}})^r \to (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}})^s$ the map induced by $\alpha^{(m_0)}$ by extension. For any $m \geq 0$, let us denote by $\beta^{(m)}\colon (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}})^s \to \mathcal{E}$ the composition of the morphism β with the canonical inclusion $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}, \mathbb{Q}})^s \hookrightarrow (\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})^s$. Let us denote by $\phi^{(m)} := u^*\Gamma(\mathfrak{Z}, \alpha^{(m)})$ and $\psi^{(m)} := u^*\Gamma(\mathfrak{Z}, \beta^{(m)})$. With the definitions and notations of 9.3.4.1, so we obtain the commutative diagram

$$\begin{array}{ccccccc}
\Gamma(\mathfrak{Z}, (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r) & \xrightarrow{\Gamma(\mathfrak{Z},u^*(\alpha^{(m)}))} & \Gamma(\mathfrak{Z}, (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s) & \xrightarrow{\Gamma(\mathfrak{Z},u^*(\beta^{(m)}))} & M & \longrightarrow & 0 \\
{\scriptstyle 9.3.4.3.1}\uparrow \wr & & {\scriptstyle 9.3.4.3.1}\uparrow \wr & & \wr\uparrow & & \\
(u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r & \xrightarrow{\phi^{(m)}} & (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s & \xrightarrow{\psi^{(m)}} & u^*(E) & \longrightarrow & 0
\end{array} \qquad (14.3.3.8.2)$$

We remark that both horizontal sequences become exact only after taking the inductive limit on m. We can identify M and u^*E.

1) *Topology of type LB on M, levels $m \geq m_1$, continuous sections θ, $\theta^{(m)}$.*

Since $\mathfrak{Z}$ is affine, then it follows from the hypotheses 14.3.3.8.(b) that the $O_{\mathfrak{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module M is projective and of finite type. We endow M with its canonical topology as overconvergent isocrystal (see 14.2.1.10). With this topology, M is in fact an $O_{\mathfrak{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module of type LB (see 14.2.2.3). Since M is a projective $O_{\mathfrak{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module, there exist therefore an $O_{\mathfrak{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-linear morphism $\theta\colon M \to (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$ such that $\psi \circ \theta = \mathrm{id}$. Since $M^{(0)}$ is a finite type $\widetilde{B}^{(0)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-module, then there exist $m_1 \geq m_0$ such that for any $m \geq m_1$, we have $\theta(M^{(0)}) \subset (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$. So, this section θ induces (uniquely) a $\widetilde{B}^{(0)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-linear morphism of the form $M^{(0)} \to (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$. For any $m \geq m_1$, we denote by $\theta^{(m)}\colon M^{(m)} \to (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ the induced $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-linear morphism. Let $\mathring{M}^{(m)}$ be a finite type $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)$-module equipped with a $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-linear isomorphism $\mathring{M}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} M^{(m)}$. A basis of open neighborhoods at zero on $(u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ (resp. $M^{(m)}$) is given by the family of $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)$-modules $\{p^n(u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp})^s\}_{n\in\mathbb{N}}$ (resp. $\{p^n(\mathring{M}^{(m)}\}_{n\in\mathbb{N}}$). For n large enough, we have the inclusion $\theta^{(m)}(p^n\mathring{M}^{(m)}) \subset (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp})^s$. This yields that the morphism $\theta^{(m)}$ is continous. By composition of continuous morphisms, then so is $M^{(m)} \to (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$. Taking the inductive limit on the level $m \geq m_1$, the section $\theta\colon M \to (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$ is then continuous. From now in the proof, $m \geq m_1$ will be an integer.

2) *The K-vector space G of type LB, the Banach K-vector spaces $G^{(m)}$, the continuous morphism ι.*

With the respective canonical topologies (see the definitions of 14.2.1.8), following the proposition 14.2.3.3, the application ϕ is a continuous morphism of K-vector spaces of LB-type (see 14.2.2.6). Let us denote by $G := (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^r/\ker(\phi)$ the locally convex K-vector space whose topology is that which makes the canonical projection

$$\pi\colon (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^r \twoheadrightarrow G$$

is a strict morphism. Let us denote by $\iota\colon G \hookrightarrow (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$ the monomorphism such that $\iota \circ \pi = \phi$. Hence, since ϕ is continuous, then ι is continuous. Since $(u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$ is separated (see 14.2.2.6), then so is G. So G is a separated quotient of a LB-space. Following 14.1.1.7, this yields that G is also an LB-space. More precisely, following its proof, denoting by $G^{(m)} := (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r/(\ker(\phi) \cap (u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r)$ endowed with the quotient topology, $G^{(m)}$ is a Banach K-vector space and the canonical isomorphism $\varinjlim_m G^{(m)} \xrightarrow{\sim} G$ is a homeomorphism. Since by definition the morphism $(u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to G^{(m)}$ is strict, it follows from 14.3.1.2, that we have the strict epimorphism

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}(u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \twoheadrightarrow \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} G^{(m)}. \qquad (14.3.3.8.3)$$

Using 14.1.1.3, this yields that we have the strict epimorphism:

$$\varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}}(u^*\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \twoheadrightarrow \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp\leftarrow\mathfrak{Z}^\sharp,\mathbb{Q}}\widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} G^{(m)}. \qquad (14.3.3.8.4)$$

3) *The maps ι, ϕ and ψ are strict morphisms.*

We have the $O_{\mathfrak{Z},\mathbb{Q}}$-linear map $(\iota, \theta)\colon G \oplus M \to (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$ which is also a continuous bijective map between two K-vector spaces of LB-type. Since Banach open mapping theorem holds for K-vector spaces of LB-type (see [Sch02, 8.8] and use the remark 14.1.1.6), then the morphism (ι, θ) is a homeomorphism. Since ι is the composition morphism

$$\iota\colon G \subset G \oplus M \underset{(\iota,\theta)}{\xrightarrow{\sim}} (u^*D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s, \qquad (14.3.3.8.5)$$

the morphism ι is then strict. Hence, ϕ is a strict morphism. We remark that as M is a separated K-space, then G is a closed subset of $G \oplus M$ (because homeomorphic to the closed subset $G \oplus \{0\}$). Finally, we obtain the canonical commutative square

$$\begin{array}{ccc} G \oplus M & \xrightarrow[\sim]{(\iota,\theta)} & (u^* D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s \\ \downarrow{\scriptstyle (0,id)} & & \downarrow{\scriptstyle \psi} \\ M & = & M, \end{array} \tag{14.3.3.8.6}$$

in which we already know that all the arrows except ψ are continuous and that the top arrow is a homeomorphism. This yields the continuity of ψ. Since ψ is a continuous and surjective morphism between two LB-spaces, this yields that ψ is a strict morphism (see [Sch02, 8.8] and use the remark 14.1.1.6).

4) *Constructions and properties of $H^{(m)}$ and $N^{(m)}$.*

Let us denote by $H^{(m)} := \iota^{-1}((u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s)$ and $\iota^{(m)} \colon H^{(m)} \hookrightarrow (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ the $\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}$-linear morphism induced by ι. We endow $H^{(m)}$ with the unique topology such that $\iota^{(m)}$ is a strict morphism (with $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ endowed with its canonical topology of Banach K-vector space). We have seen at the step 3) that $\iota(G)$ is closed. Since $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \hookrightarrow (u^* D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s$ is continuous, this yields that $\iota^{(m)}(H^{(m)})$ is a closed subset of $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$. Since $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ is a Banach $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-module, this implies that $H^{(m)}$ is also a Banach $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-module. Let us then denote by $N^{(m)} := \operatorname{Im}(\psi^{(m)})$ and $\psi'^{(m)} \colon (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \twoheadrightarrow N^{(m)}$ the canonical epimorphism which factors $\psi^{(m)}$. We get on $N^{(m)}$ a structure of locally convex $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-module by decreeing the $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-linear epimorphism $\psi'^{(m)} \colon (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \twoheadrightarrow N^{(m)}$ strict map. So we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\iota} & (u^* D^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})^s & \xrightarrow{\psi} & M & \longrightarrow & 0 \\ & & \uparrow & \square & \uparrow & \nearrow{\scriptstyle \psi^{(m)}} & \uparrow & & \\ 0 & \longrightarrow & H^{(m)} & \xrightarrow{\iota^{(m)}} & (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s & \xrightarrow{\psi'^{(m)}} & N^{(m)} & \longrightarrow & 0 \end{array} \tag{14.3.3.8.7}$$

whose horizontal morphisms are strict and form two short exact sequences (moreover the top one splits via θ). Since the composition of $H^{(m)} \subset G$ with ι is continuous, as ι is a strict monomorphism, we remark that the inclusion $H^{(m)} \subset G$ is then continuous. Finally, as $\psi^{(m)}$ is continuous, then so is the inclusion $N^{(m)} \subset M$. Since M is separated, this yields that $N^{(m)}$ is a Banach $\widetilde{B}^{(m)}_{\mathfrak{Z}}(T \cap Z)_{\mathbb{Q}}$-module

By using 14.3.1.2, the bottom short exact sequence of 14.3.3.8.7 induces the exact sequence with strict morphisms:

$$0 \to \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(m)} \underset{\mathrm{id}\,\widehat{\otimes}\iota^{(m)}}{\to} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \underset{\mathrm{id}\,\widehat{\otimes}\psi'^{(m)}}{\to} \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} N^{(m)} \to 0. \tag{14.3.3.8.8}$$

By taking the inductive limit on m, this yields the exact sequence with continuous morphisms:

$$0 \to \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(m)} \to \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \to \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} N^{(m)} \to 0. \tag{14.3.3.8.9}$$

5) *Taking the inductive limit on m for $G^{(m)}$ and $H^{(m)}$: comparaison.*

Let $j_m \colon G^{(m)} \to G$ be the canonical continuous monomorphism. For any $m \geq m_1$, since the morphism ϕ induces the continuous morphism $\phi^{(m)} \colon (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \to (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$, then we get the inclusion $\iota \circ j_m(G^{(m)}) \subset (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ and then $j_m(G^{(m)}) \subset H^{(m)}$. Hence, the continuous monomorphism j_m can be factored factorizes through a continuous monomorphism of the form $G^{(m)} \to H^{(m)}$.

Conversely, since $H^{(m)}$ is a Banach K-vector space, since we have the equality $G = \cup_{m \in \mathbb{N}} j_m(G^{(m)})$ with j_m continuous, K-linear and $G^{(m)}$ are Banach K-vector spaces, then it follows from [Sch02, 8.9] that then there exist a large enough integer $n_m \geq m$ such that the continuous homomorphism $H^{(m)} \subset G$ is the composition of a (unique) continuous morphism of the form $H^{(m)} \to G^{(n_m)}$ with j_{n_m}. It is harmless to suppose that the sequence $(n_m)_{m \geq m_1}$ is strictly increasing. With 14.3.1.3, this yields the following continuous monomorphisms:

$$\widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(m)} \hookrightarrow \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} G^{(n_m)} \hookrightarrow \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(n_m)}. \tag{14.3.3.8.10}$$

By taking the inductive limit on m of the sequence 14.3.3.8.10, we get the diagram of K-vector spaces of LB-type of the form:

$$\begin{array}{ccc} \varinjlim_m \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} G^{(n_m)} & \longrightarrow & \varinjlim_m \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(n_m)} \\ \sim\uparrow & \nwarrow & \sim\uparrow \\ \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} G^{(m)} & \longrightarrow & \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(m)}. \end{array} \tag{14.3.3.8.11}$$

Since vertical morphisms are homeomorphisms (see 14.2.2.6.1), then so are the three other ones.

6) *Taking the limit on m for $M^{(m)}$ and $N^{(m)}$, strict homeomorphism $\beta^\dagger_Z$.*

Since $\psi^{(m)} \colon (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \to M$ is continuous, since $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s$ is a Banach K-vector space, since we have the equality $M = \cup_{m\in\mathbb{N}} M^{(m)}$ where $M^{(m)}$ are Banach K-vector spaces, then following [Sch02, 8.9], there exists $n_m \geq m$ large enough such that $\psi^{(m)}$ is the composition of a (unique) continuous morphism of the form $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \to M^{(n_m)}$ with the monomorphism $M^{(n_m)} \hookrightarrow M$. Since $N^{(m)} = \mathrm{Im}(\psi^{(m)}) = \mathrm{Im}(\psi'^{(m)})$, since $N^{(m)}$ is endowed with the quotient topology, then this continuous morphism $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \to M^{(n_m)}$ uniquely decompose into continuous morphisms as follows $(u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \underset{\psi'^{(m)}}{\twoheadrightarrow} N^{(m)} \hookrightarrow M^{(n_m)}$. On the other hand, we have the continuous injective morphism $\psi'^{(m)} \circ \theta^{(m)} \colon M^{(m)} \to N^{(m)}$ whose composition with $N^{(m)} \subset M$ gives $M^{(m)} \hookrightarrow M$. This yields the diagram of K-vector spaces of LB-type of the form:

$$\begin{array}{ccc} \varinjlim_m \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} M^{(n_m)} & \longrightarrow & \varinjlim_m \widetilde{D}^{(n_m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} N^{(n_m)} \\ \sim\uparrow & \nwarrow & \sim\uparrow \\ \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} M^{(m)} & \longrightarrow & \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} N^{(m)}. \end{array} \tag{14.3.3.8.12}$$

Since vertical morphisms are homeomorphisms (see 14.2.2.6.1), then so are the three other ones.

Let us denote by $\beta^\dagger_{\mathfrak{X}}$ the following composite morphism:

$$\beta^\dagger_{\mathfrak{X}} \colon \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \underset{\varinjlim_m id \widehat{\otimes} \psi'^{(m)}}{\twoheadrightarrow} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} N^{(m)}$$

$$\xrightarrow{\sim} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} M^{(n_m)} \xleftarrow{\sim} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(n_m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} M^{(m)}. \tag{14.3.3.8.13}$$

Since the first morphism is a strict epimorphism and the other ones are homeomorphisms, then $\beta^\dagger_{\mathfrak{X}}$ is a strict epimorphism.

7) *First conclusion.*

By composing the strict epimorphism 14.3.3.8.4 with the bottom isomorphism of 14.3.3.8.11, we obtain the strict epimorphism of the sequence by composition :

$$\varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \twoheadrightarrow \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} H^{(m)} \hookrightarrow \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s, \tag{14.3.3.8.14}$$

the second arrow being the monomorphism of 14.3.3.8.9. The composition of morphisms of 14.3.3.8.14 will be denoted by $\alpha^\dagger_{\mathfrak{X}}$. We have the equality $\alpha^\dagger_{\mathfrak{X}} = \varinjlim_m id \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} \phi^{(m)}$.

By using the exact sequence 14.3.3.8.9 and the fact that the right arrow of 14.3.3.8.13 is a homeomorphism, this yields the exact sequence:

$$\varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \underset{\alpha^\dagger_{\mathfrak{X}}}{\longrightarrow} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \underset{\beta^\dagger_{\mathfrak{X}}}{\longrightarrow} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} M^{(m)} \to 0. \tag{14.3.3.8.15}$$

8) *Sheafification.*

i) Let $\mathfrak{B}$ be the basis of open neighborhoods of $\mathfrak{X}$ given by the affine opens endowed with local coordinates satisfying the hypotheses of the chapter 14.2 (same notation as 14.3.3.2). The presheaf on $\mathfrak{B}$ defined by

$$\mathfrak{U} \in \mathfrak{B} \mapsto \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}}} \Gamma(\mathfrak{U} \cap \mathfrak{Z}, \mathcal{M}^{(m)})$$

is in fact, by definition of $\mathcal{M}^{(m)}$, a sheaf whose associated sheaf on $\mathfrak{X}$ is $u^\sharp_+(\mathcal{M})$ (e.g. notice that $\widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}}} \Gamma(\mathfrak{U} \cap \mathfrak{Z}, \mathcal{M}^{(m)}) = \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}} \otimes_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}}} \Gamma(\mathfrak{U} \cap \mathfrak{Z}, \mathcal{M}^{(m)})$).

ii) With the notations of 14.3.3.3, as the functor "sheaf (of sets) associated to a presheaf (of sets)" commutes with filtrant inductive limits, then the sheaf associated to the presheaf

$$\mathfrak{U} \in \mathfrak{B} \mapsto \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{U}^\sharp \leftarrow \mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}}} \Gamma(\mathfrak{U} \cap \mathfrak{Z}, u^*((\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r))$$

is $\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r$, similarly for another integer than r.

iii) For any $\mathfrak{U} \in \mathfrak{B}$, let us denote by $u^\sharp_{\mathfrak{U}} \colon \mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp \hookrightarrow \mathfrak{U}^\sharp$ the exact closed immersion induced by $u^\sharp$. For any $m \geq m_1$, we have the (a priori non-exact) sequence: $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{U}^\sharp,\mathbb{Q}})^r \underset{\alpha^{(m)}|\mathfrak{U}}{\longrightarrow} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{U}^\sharp,\mathbb{Q}})^s \underset{\beta^{(m)}|\mathfrak{U}}{\longrightarrow} \mathcal{E}|\mathfrak{U} \to 0$. Similarly to 14.3.3.8.15, (it is sufficient to replace $u^\sharp$ by $u^\sharp|\mathfrak{U}$), we get the exact sequence

$$\varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{U} \leftarrow \mathfrak{Z} \cap \mathfrak{U},\mathbb{Q}} \widehat{\otimes} u^*_{\mathfrak{U}} (\widetilde{D}^{(m)}_{\mathfrak{U}^\sharp,\mathbb{Q}})^r \underset{\alpha^\dagger_{\mathfrak{U}}}{\longrightarrow} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{U} \leftarrow \mathfrak{Z} \cap \mathfrak{U},\mathbb{Q}} \widehat{\otimes} u^*_{\mathfrak{U}} (\widetilde{D}^{(m)}_{\mathfrak{U}^\sharp,\mathbb{Q}})^s \underset{\beta^\dagger_{\mathfrak{U}}}{\longrightarrow} \varinjlim_m \widetilde{D}^{(m)}_{\mathfrak{U} \leftarrow \mathfrak{Z} \cap \mathfrak{U},\mathbb{Q}} \widehat{\otimes} \Gamma(\mathfrak{U} \cap \mathfrak{Z}, \mathcal{M}^{(m)}) \to 0.$$

where $\widehat{\otimes} := \widehat{\otimes}_{\widetilde{D}^{(m)}_{\mathfrak{Z}^\sharp \cap \mathfrak{U}^\sharp,\mathbb{Q}}}$, $\alpha^\dagger_{\mathfrak{U}}$ and $\beta^\dagger_{\mathfrak{U}}$ are defined similarly to 14.3.3.8.13. Since these exact sequences are functorial in $\mathfrak{U}$, since the sheaf associated to a presheaf functor is exact, we get then the exact sequence:

$$\varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^r \underset{\varinjlim_m id \widehat{\otimes} u^* \alpha^{(m)}}{\longrightarrow} \varinjlim_m \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp \leftarrow \mathfrak{Z}^\sharp,\mathbb{Q}} \widehat{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{Z}^\sharp,\mathbb{Q}}} (u^* \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp,\mathbb{Q}})^s \to u^\sharp_+(\mathcal{M}) \to 0. \qquad (14.3.3.8.16)$$

9) *End of the proof.* Since the functor $\mathcal{H}^{\dagger e}_Z$ is right exact (in the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules), then the cokernel of $\mathcal{H}^{\dagger e}_Z(\alpha)$ is isomorphic to $\mathcal{H}^{\dagger e}_Z(\mathcal{E})$. Moreover, it follows from the lemma 14.3.3.6 and the exact sequence 14.3.3.8.16, that the cokernel of $\mathcal{H}^{\dagger e}_Z(\alpha)$ is isomorphic to $u^\sharp_+\mathcal{M}$. Since $u^*(\mathcal{E}) = \mathcal{M}$, so we have checked $\mathcal{H}^{\dagger e}_Z(\mathcal{E}) \xrightarrow{\sim} u^\sharp_+ u^*(\mathcal{E})$. Finally, as u is proper and $u^*(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module, then $u^\sharp_+ u^*(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module. □

Corollary 14.3.3.9. *We suppose that $u \colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ is of pure codimension 1. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module such that $H^0 u^{\sharp!}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module and that $\mathcal{H}^1 u^{\sharp!}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module which is also a locally projective of finite type $\mathcal{O}_{\mathfrak{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module for the induced structure. Then the complex $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E})$ is $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-coherent.*

Proof. Via the distinguished triangle $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E}) \to \mathcal{E} \to ({}^\dagger Z)(\mathcal{E}) \to +1$, since $({}^\dagger Z)(\mathcal{E})$ is a module, this yields that $\mathcal{H}^{\dagger i}_Z(\mathcal{E}) = 0$ for any $i \notin \{0, 1\}$. The $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-coherence of the complex $\mathbb{R}\underline{\Gamma}^\dagger_Z(\mathcal{E})$ is then a consequence of 14.3.2.1 and of 14.3.3.8. □

Corollary 14.3.3.10. *We suppose that $u \colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ is of pure codimension 1. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module such that $H^0 u^{\sharp!}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module and that $\mathcal{H}^1 u^{\sharp!}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Z}^\sharp}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module which is also an $\mathcal{O}_{\mathfrak{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module locally projective of finite type for the induced structure. Let $\mathcal{E}^{(\bullet)}$ be an object of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$ such that $\mathcal{E} \xrightarrow{\sim} \varinjlim (\mathcal{E}^{(\bullet)})$. Then $u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\sharp})$.*

Proof. This is a consequence of 14.3.3.9 and 14.3.3.1. □

Remark 14.3.3.11. We keep the notations and hypotheses of theorem 14.3.3.1.

(a) If one of the equivalent conditions of the theorem 14.3.3.1 is satisfied then we have the isomorphism $u^{\sharp}_{+} \circ u^{\sharp !}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E})$ (in the left term, the functors $u^{\sharp}_{+}$ and $u^{\sharp !}$ are computed in the respective categories of coherent $\mathcal{D}^{\dagger}$-modules).

(b) On the other hands, if we only suppose that $u^{\sharp !}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Z}^{\sharp}}({}^{\dagger}T \cap Z)_{\mathbb{Q}})$, then it is not clear that $u^{\sharp}_{+} \circ u^{\sharp !}(\mathcal{E})$ (the functors $u^{\sharp}_{+}$ and $u^{\sharp !}$ are computed in the categories of coherent $\mathcal{D}^{\dagger}$-modules) is isomorphic to $\varinjlim \circ u^{\sharp(\bullet)}_{+} \circ u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}) = \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E})$. Indeed, then it is not clear that the coherence of $u^{\sharp !}(\mathcal{E})$ implies $u^{\sharp(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^{\sharp}})$. A priori, to ensure with such implication, we need additional conditions (e.g. 14.3.3.10).

Chapter 15

Holonomicity, overcoherence

Suppose the residue field k of $\mathcal{V}$ is perfect.

15.1 Characteristic varieties

15.1.1 Cotangent space

Let X be a smooth k-variety. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$, we denote by $\mathrm{Sym}(\mathcal{E})$ the symetric algebra of $\mathcal{E}$ and by $\mathbb{V}(\mathcal{E}) := \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ endowed with its canonical projection $\mathbb{V}(\mathcal{E}) \to \mathrm{Spec}\,\mathrm{Sym}(\mathcal{O}_X) = X$. We denote by Ω^1_X the sheaf of differential form of $X/\mathrm{Spec}(k)$ (we skip k in the notation), and $\mathcal{T}_X$ the tangent space of $X/\mathrm{Spec}(k)$, i.e. the $\mathcal{O}_X$-dual of Ω^1_X. We denote by $T^*X := \mathbb{V}(\mathcal{T}_X)$ the cotangent space of X and $\pi_X \colon T^*X \to X$ the canonical projection. Recall that from [Gro61, 1.7.9], there is a canonical bijection between sections of π_X and $\Gamma(X, \Omega^1_X)$. We denote by $T^*_X X$ the section corresponding to the zero section of $\Gamma(X, \Omega^1_X)$. If $t_1, \dots, t_d$ are local coordinates of X, we get local coordinates $t_1, \dots, t_d, \xi_1, \dots, \xi_d$ of T^*X, where ξ_i is the element associated with ∂_i, the derivation with respect to t_i. Is this case, $T^*_X X = V(\xi_1, \dots, \xi_d)$ is the closed subvariety of T^*X defined by $\xi_1 = 0, \dots, \xi_d = 0$.

Let $f \colon X \to Y$ be a morphism of smooth k-varieties. Using the equality [Gro61, 1.7.11.(iv)] we get the last one $X \times_Y T^*Y = X \times_Y \mathbb{V}(\mathcal{T}_Y) = \mathbb{V}(f^*\mathcal{T}_Y)$. The morphism $f^*\Omega^1_Y \to \Omega^1_X$ induced by f yields by duality $\mathcal{T}_X \to f^*\mathcal{T}_Y$ and then by functoriality $\mathbb{V}(f^*\mathcal{T}_Y) \to \mathbb{V}(\mathcal{T}_X) = T^*X$. By composition, we get the morphism denoted by $\rho_f \colon X \times_Y T^*Y \to T^*X$. We will write by $\varpi_f \colon X \times_Y T^*Y \to T^*Y$ the base change of f under π_Y.

We denote by $\mathscr{T}_f$ the function from the set of subvarieties of T^*X to the set of subvarieties of T^*Y defined by setting, for any subvariety V of T^*X, $\mathscr{T}_f(V) := \varpi_f(\rho_f^{-1}(V))$. If f is an open immersion, then ρ_f is an isomorphism. In that case, $\mathscr{T}_f := \varpi_f \circ \rho_f^{-1} \colon T^*X \to T^*Y$ is an open immersion and this is compatible with the above definition of $\mathscr{T}_f$. The application $\mathscr{T} : f \mapsto \mathscr{T}_f$ is transitive (with respect to the composition), i.e. we have the equality $\mathscr{T}_g \circ \mathscr{T}_f = \mathscr{T}_{g \circ f}$ for any $g \colon Y \to Z$ (e.g. look at the bottom of the diagram **??** where f and u are replaced respectively by g and f).

We define the k-variety $T^*_X Y$ (recall a k-variety is a separated reduced scheme of finite type over k from our convention) by setting $T^*_X Y := \rho_f^{-1}(T^*_X X)$. When f is an immersion, $T^*_X Y$ is viewed as a subvariety of T^*Y via $T^*_X Y \subset X \times_Y T^*Y \overset{\varpi_f}{\hookrightarrow} T^*Y$, i.e. we will simply denote $\varpi_f(T^*_X Y)$ by $T^*_X Y$.

15.1.2 Cotangent space of level m

Set $S := \mathrm{Spec}\,k$, $m \in \mathbb{N}$.

Notation 15.1.2.1. We denote by $F_X \colon X \to X$ be the absolute Frobenius morphism. We denote by $X^{(m)}$ the base change of X by F^m_S the mth power of Frobenius of S, and by $F^m_{X/S} \colon X \to X^{(m)}$ the relative Frobenius morphism. We get the equality $F^m_X = \mathrm{id} \times F^m_S \circ F^m_{X/S}$

15.1.2.2 (Universal homeomorphism)**.** Let $f \colon X \to Y$ be a morphism of schemes.

1. Following Definitions [Gro60, 3.5.4] (and Remark [Gro60, 3.5.11]) or [Gro65, 2.4.2], f is by definition a universal homeomorphism (resp. is universally injective) if for any morphism of schemes $g\colon Y' \to Y$, the morphism $f_{Y'}\colon X \times_Y Y' \to Y'$ is a homeomorphism (resp. is injective).

2. Some authors use the name of "purely inseparable" (e.g. [Liu02, 5.3.13]) or "radicial" (e.g. [Gro60, 3.5.4]) instead of "universally injective". From Definition [Gro60, 3.5.4], Proposition [Gro60, 3.5.8] and Remark [Gro60, 3.5.11], the following conditions are equivalent:

   (a) f is universally injective ;

   (b) for any field K, the map $X(K) \to Y(K)$ is injective ;

   (c) f is injective and for any point x of X the monomorphism of the residue fields $k(f(x)) \to k(x)$ induced by f is purely inseparable (some authors say "radicial" instead of "purely inseparable").

3. Suppose now that $f\colon X \to Y$ is a morphism of k-varieties. Using Proposition [Gro65, 2.4.5], we check that f is a universal homeomorphism if and only if f is finite, surjectif and radicial.

Lemma 15.1.2.3. *Let X be a S-variety. Then the relative Frobenius $F^m_{X/S}\colon X \to X^{(m)}$, the morphism $F^m_S\colon X^{(m)} \to X$ (induced from F^m_S by base change) and the absolute Frobenius morphism $F^m_{X/S}\colon X \to X$ (equal to $F^m_X = F^m_S \circ F^m_{X/S}$) are universal homeomorphisms.*

Proof. From the characterization 15.1.2.2.3, $F^m_S\colon S \to S$ is a universal homeomorphism. Hence, by stability of this property by base change we get that $F^m_S\colon X^{(m)} \to X$ is a universal homeomorphism. From Lemma [Liu02, 3.2.25], we check that $F^m_{X/S}$ is finite. Hence, so is by composition F^m_X. Since F^m_X induces the identity on the underlying topological space, F^m_X is bijective. Moreover, the monomorphism of the residue fields $k(x) \to k(x)$ induced by F^m_X is the sth power of the Frobenius, hence it is radicial. From 15.1.2.2.2.(c), this yields that F^m_X is radicial. From 15.1.2.2.2.(b), this implies that $F^m_{X/S}$ is also radicial. With the characterization 15.1.2.2.3, we get that $F^m_{X/S}$ and F^m_X are universal homeomorphisms. □

Notation 15.1.2.4. Let X be a smooth S-scheme The sheaf $\mathcal{D}^{(m)}_{X/S}$ is equipped with the order filtration of differential operators. Its associated graded ring $\operatorname{gr}\mathcal{D}^{(m)}_{X/S}$ is a quasi-coherent, commutative $\mathcal{O}_X$-algebra. We will denote by $T^{(m)*}X$, and we will call *the cotangent space of level m of X*, the reduced scheme

$$T^{(m)*}X := (\operatorname{Spec}(\operatorname{gr}\mathcal{D}^{(m)}_{X/S}))_{\mathrm{red}}.$$

Example 15.1.2.5. With notation 15.1.2.4, when $m = 0$, following 2.3.3.2, we have the canonical isomorphism $\mathbb{S}(\mathcal{T}_{X/S}) \xrightarrow{\sim} \operatorname{gr}\mathcal{D}^{(0)}_{X/S}$, so that $T^{(0)*}X$ is the usual cotangent fiber $T * X$.

15.1.2.6 (Local description and notation)**.** Let X be affine S-scheme such that X/S is endowed with coordinates $t_1, \dots, t_d$. Let $i \in \{1, \dots, d\}$. Let $j < m$ be an integer. It follows from the formula 1.4.2.7.(c) that we have the equality $(\partial_i^{<p^j>_{(m)}})^p = \left(\prod_{l=2}^p \left\langle {lp^j \atop (l-1)p^j} \right\rangle_{(m)}\right) \partial_i^{<p^{j+1}>_{(m)}}$. Using the formula 1.2.1.4.1, we compute $v_p\left(\left\langle {lp^j \atop (l-1)p^j} \right\rangle_{(m)}\right) = 1$. Hence, $(\partial_i^{<p^j>_{(m)}})^p = 0$ in $\mathcal{D}^{(m)}_{X/S}$. From 1.2.1.5.(b) and 1.4.2.7.(c), for any $l \in \mathbb{N}$, we compute that there exists $u \in \mathbb{Z}_p^*$ such that $(\partial_i^{<p^m>_{(m)}})^l = u\partial_i^{<p^m l>_{(m)}}$. Let $\xi_i^{(m)}$ be the class of $\partial_i^{<p^m>_{(m)}} = \partial_i^{[p^m]}$ in $(\operatorname{gr}\mathcal{D}^{(m)}_{X/S})_{\mathrm{red}}$. Hence, with the formula 1.4.2.8.3 (the factors of this latter formula commute two by two), we get the canonical isomorphism of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X[x_1, \dots, x_d] \xrightarrow{\sim} (\operatorname{gr}\mathcal{D}^{(m)}_{X/S})_{\mathrm{red}} \qquad (15.1.2.6.1)$$

given by $x_i \mapsto \xi_i^{(m)}$.

Lemma 15.1.2.7. *Let X be a smooth S-scheme. There exists a canonical isomorphism*

$$X \times_{X^{(m)}} T^*X^{(m)} \xrightarrow{\sim} T^{(m)*}X. \qquad (15.1.2.7.1)$$

Proof. We reduce to check that the canonical morphism morphism $(\operatorname{gr}\mathcal{D}^{(m)}_{X/S})_{\mathrm{red}} \to F^{m*}_{X/S}\operatorname{gr}\mathcal{D}^{(0)}_{X^{(m)}/S}$ induced by the morphism $\mathcal{D}^{(m)}_{X/S} \to F^{m*}_{X/S}\mathcal{D}^{(0)}_{X^{(m)}/S}$ of left $\mathcal{D}^{(m)}_{X/S}$-modules is an isomorphism. Since this is local, we can suppose we are in the local situation of 15.1.2.6. Following 6.1.1.8.3, the image of $\partial_i^{<k>_{(m)}}$ via $\mathcal{D}^{(m)}_{X/S} \to F^{m*}_{X/S}\mathcal{D}^{(0)}_{X^{(m)}/S}$ is $1 \otimes \partial_i^{k/p^m}$ if p^m divides k and otherwise is 0. Hence, $\xi_i^{(m)}$ is sent to $1 \otimes \xi_i$, where ξ_i is the class of ∂_i in $\operatorname{gr}\mathcal{D}^{(0)}_{X^{(m)}/S}$, i.e. we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_X[x_1,\dots,x_d] & \xrightarrow[\text{15.1.2.6.1}]{\sim} & (\operatorname{gr}\mathcal{D}^{(m)}_{X/S})_{\mathrm{red}} \\
\downarrow{\scriptstyle\sim} & & \downarrow \\
\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(m)}}} \mathcal{O}_{X^{(m)}}[x_1,\dots,x_d] & \xrightarrow[\text{15.1.2.6.1}]{\sim} & \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(m)}}} (\operatorname{gr}\mathcal{D}^{(0)}_{X^{(m)}/S}).
\end{array}$$

□

15.1.2.8. Since $(F^m_S)^*\Omega^1_{X/S} \xrightarrow{\sim} \Omega^1_{X^{(m)}/S}$ then $(F^m_S)^*\mathcal{T}_{X/S} \xrightarrow{\sim} \mathcal{T}_{X^{(m)}/S}$ and therefore

$$: X^{(m)} \times_X T^*X \xrightarrow{\sim} S \times_{F^m_S, S} T^*X \xrightarrow{\sim} T^*X^{(m)}. \qquad (15.1.2.8.1)$$

Hence, it follows from 15.1.2.7.1 and 15.1.2.3 that we have the canonical universal homeomorphism $\iota^{(m)}_{X/S} \colon T^{(m)*}X \to T^*X^{(m)}$ and $\iota^{(m)}_S \colon T^*X^{(m)} \to T^*X$. Setting $\iota^{(m)}_X := \iota^{(m)}_S \circ \iota^{(m)}_{X/S} \colon T^*X \to T^{(m)*}X$ we get the following cartesian diagram

$$\begin{array}{ccccc}
T^{(m)*}X & \xrightarrow{\iota^{(m)}_{X/S}} & T^*X^{(m)} & \xrightarrow{\iota^{(m)}_S} & T^*X, \\
\downarrow & \square & \downarrow & \square & \downarrow \\
X & \xrightarrow{F^m_{X/S}} & X^{(m)} & \xrightarrow{F^m_S} & X
\end{array} \qquad (15.1.2.8.2)$$

where the vertical arrows are the canonical ones. In particular, we get the homeomorphism

$$\iota^{(m)}_X \colon |T^{(m)*}X| \xrightarrow{\sim} |T^*X|, \qquad (15.1.2.8.3)$$

which may allow us to identify the closed parts of $|T^{(m)*}X|$ and $|T^*X|$.

15.1.3 The characteristic variety of a coherent $\mathcal{D}^{(m)}_{X/S}$-module

Let $i, m \in \mathbb{N}$. Let X_i be a smooth variety over $S_i = \operatorname{Spec}\mathcal{V}/\pi^{i+1}\mathcal{V}$. Let X be the reduction of X_i modulo π.

15.1.3.1. Let $\mathcal{K}$ be the kernel of the epimorphism $\operatorname{gr}\mathcal{D}^{(m)}_{X_i/S_i} \twoheadrightarrow \operatorname{gr}\mathcal{D}^{(m)}_{X/S}$. Since $\mathcal{K}^{i+1} = 0$, then

$$T^{(m)*}X = (\operatorname{Spec}(\operatorname{gr}\mathcal{D}^{(m)}_{X_i/S_i}))_{\mathrm{red}}. \qquad (15.1.3.1.1)$$

Hence, $|\operatorname{Spec}(\operatorname{gr}\mathcal{D}^{(m)}_{X_i/S_i})| = |T^{(m)*}X| \xrightarrow{\sim} |T^*X|$.

Definition 15.1.3.2. Let $\mathcal{E}$ be a coherent left or right $\mathcal{D}^{(m)}_{X_i/S_i}$-module.

(a) Choose a good filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ (see the definition 4.1.3.8 or 4.1.3.27 for the right version), i.e. a filtration such that $\operatorname{gr}\mathcal{E}$ is a coherent $\operatorname{gr}\mathcal{D}^{(m)}_{X_i/S_i}$-module (see Theorem 4.1.3.25 which is still valid for the right version). The characteristic variety of level m of $\mathcal{E}$, denoted by $\operatorname{Car}^{(m)}(\mathcal{E})$ is by definition the image by the homeomorphism $\iota^{(m)}$ (see 15.1.2.8.3) of the support of $\operatorname{gr}\mathcal{E}$ in $T^{(m)*}X$ (see 15.1.3.1.1). By using the inclusions 4.1.3.10.1, we can copy the standard proof (e.g. see [HTT08, D.3.1]) of the fact that $\operatorname{Car}^{(m)}(\mathcal{E})$ does not depend on the choice of a good filtration.

(b) Suppose $\mathcal{E}$ is a left $\mathcal{D}^{(m)}_{X_i/S_i}$-module. If $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is a good filtration of $\mathcal{E}$ then $(\omega_{X_i/S_i}\otimes_{\mathcal{O}_{X_i}}\mathcal{E}_r)_{r\in\mathbb{N}}$ is a good filtration of $\omega_{X_i/S_i}\otimes_{\mathcal{O}_{X_i}}\mathcal{E}$. Hence, we get

$$\mathrm{Car}^{(m)}(\mathcal{E})=\mathrm{Car}^{(m)}(\omega_{X_i/S_i}\otimes_{\mathcal{O}_{X_i}}\mathcal{E}).$$

To simplify the statements, we will focus later on left modules case but the right versions will obviously be valid.

(c) The "dimension of level m of $\mathcal{E}$" is defined by setting $\dim^{(m)}(\mathcal{E}):=\dim\mathrm{Car}^{(m)}(\mathcal{E})$. The "codimension of level m of $\mathcal{E}$" is defined by setting $\mathrm{codim}^{(m)}(\mathcal{E}):=2\dim X-\dim^{(m)}(\mathcal{E})$.

(d) Let $\mathcal{F}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{X_i/S_i})$ By definition, we define the characteristic variety of this complex by setting $\mathrm{Car}^{(m)}(\mathcal{F}):=\cup_r\mathrm{Car}^{(m)}(H^r(\mathcal{F}))$.

Example 15.1.3.3. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{(m)}_{X_i/S_i}$-module. We have $\mathcal{E}=0$ if and only if $\mathrm{Car}^{(m)}(\mathcal{E})$ is empty.

15.1.3.4 (Conicity)**.** Let $\mathcal{E}$ be a coherent left $\mathcal{D}^{(0)}_{X_0/S_0}$-module. The characteristic variety $\mathrm{Car}^{(0)}(\mathcal{E})$ is conic, i.e. the following properties hold. Let $s\in\mathrm{Car}^{(0)}(\mathcal{E})$ and $x:=\pi_X(s)$. Then $\pi_X^{-1}(x)\cap\mathrm{Car}^{(0)}(\mathcal{E})$ is a closed subscheme of $\pi_X^{-1}(x)$ defined by an homogeneous ideal. More precisely, if X_0/S_0 is endowed with coordinates $t_1,\dots,t_d$, then the ideal defining $\pi_X^{-1}(x)\cap\mathrm{Car}^{(0)}(\mathcal{E})$ is an homegenous ideal of $k(x)[\xi_1,\dots,\xi_d]$. In particular, we get the inequality (that we will need to check 15.1.5.5):

$$\text{For any } s\in\mathrm{Car}^{(0)}(\mathcal{E})\setminus T^*_XX,\text{ we have }\dim\left(\pi_X^{-1}(\pi_X(s))\cap\mathrm{Car}^{(0)}(\mathcal{E})\right)\geq 1 \qquad (15.1.3.4.1)$$

Lemma 15.1.3.5. *Let $0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0$ be an exact sequence of coherent left $\mathcal{D}^{(m)}_{X_i/S_i}$-modules. We have the equality*

$$\mathrm{Car}^{(m)}(\mathcal{E})=\mathrm{Car}^{(m)}(\mathcal{E}')\cup\mathrm{Car}^{(m)}(\mathcal{E}''). \qquad (15.1.3.5.1)$$

In particular, we have the formula $\mathrm{codim}^{(m)}(\mathcal{E})=\min\{\mathrm{codim}^{(m)}(\mathcal{E}'),\mathrm{codim}^{(m)}(\mathcal{E}'')\}$.

Proof. Choose a good filtration $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$. Denote by $\epsilon\colon\mathcal{E}\to\mathcal{E}''$ the surjection. Following 4.1.3.26, the filtrations of $\mathcal{E}'$, $\mathcal{E}''$ defined by: $\mathcal{E}'_n:=\mathcal{E}_n\cap\mathcal{E}'$, $\mathcal{E}''_n=\epsilon(\mathcal{E}_n)$ are good. With theses filtrations, we get the exact sequence $0\to\mathrm{gr}\,\mathcal{E}'\to\mathrm{gr}\,\mathcal{E}\to\mathrm{gr}\,\mathcal{E}''\to 0$. Hence, we are done. □

Lemma 15.1.3.6. *Let $\mathcal{E}$ be coherent left $\mathcal{D}^{(m)}_{X_i/S_i}$-module. For any integer $0\leq i'\leq i$, we have the equality*

$$\mathrm{Car}^{(m)}(\mathcal{E})=\mathrm{Car}^{(m)}(\mathcal{E}/\pi^{i'+1}\mathcal{E}). \qquad (15.1.3.6.1)$$

Proof. Let $i\in\mathbb{N}$. We proceed by induction on i. When $i=0$, this is obvious. Suppose $i\geq 1$ and the Lemma holds for $i-1$. Let $0\leq i'\leq i$.

a) The multiplication by π^i induces the surjective $\mathcal{D}^{(m)}_{X_i/S_i}$-linear map $\mathcal{E}/\pi^i\mathcal{E}\to\pi^i\mathcal{E}$. From 15.1.3.5.1, this yields $\mathrm{Car}^{(m)}(\pi^i\mathcal{E})\subset\mathrm{Car}^{(m)}(\mathcal{E}/\pi^i\mathcal{E})$. By using the exact sequence $0\to\pi^i\mathcal{E}\to\mathcal{E}\to\mathcal{E}/\pi^i\mathcal{E}\to 0$, we get $\mathrm{Car}^{(m)}(\mathcal{E})=\mathrm{Car}^{(m)}(\pi^i\mathcal{E})\cup\mathrm{Car}^{(m)}(\mathcal{E}/\pi^i\mathcal{E})$. Hence, $\mathrm{Car}^{(m)}(\mathcal{E})=\mathrm{Car}^{(m)}(\mathcal{E}/\pi^i\mathcal{E})$, i.e. 15.1.3.6.1 holds when $i'=i$.

b) When $i'<i$, since $\mathcal{E}/\pi^{i'+1}\mathcal{E}\xrightarrow{\sim}(\mathcal{E}/\pi^i\mathcal{E})/\pi^{i'+1}(\mathcal{E}/\pi^i\mathcal{E})$, then from the induction hypothesis, we have $\mathrm{Car}^{(m)}(\mathcal{E}/\pi^i\mathcal{E})=\mathrm{Car}^{(m)}(\mathcal{E}/\pi^{i'+1}\mathcal{E})$. Hence, we are done. □

Lemma 15.1.3.7. *Let $F\colon X_i\to X'_i$ be a lifting of $F^s_{X/S}\colon X\to X^{(s)}$, the sth relative Frobenius of X/S. Let $\mathcal{E}'$ be a coherent left $\mathcal{D}^{(m)}_{X'_i/S_i}$-module. Then we have in $|T^*X|$ the equality:*

$$\iota^{(s)}_S(\mathrm{Car}^{(m)}(\mathcal{E}'))=\mathrm{Car}^{(m+s)}(F^*\mathcal{E}'), \qquad (15.1.3.7.1)$$

where $\iota^{(s)}_S\colon T^*X^{(s)}\to T^*X$ is the canonical universal homeomorphism induced by base change via F^s_S (see 15.1.2.8.2).

Proof. With 15.1.3.6, we can suppose $i = 0$.

1) We reduce to the case where $m = 0$ as follows. Let $F_S^s := (F_S^s \times \mathrm{id}, F_S^s)\colon X^{(s)}/S \to X/S$ be the canonical isomorphism induced by the sth power of Frobenius of S. Let $\mathcal{E}$ be a coherent left $\mathcal{D}^{(m)}_{X/S}$-module such that $\mathcal{E}' \xrightarrow{\sim} (F_S^s)^*\mathcal{E}$. Since $\iota_S^{(s)}(\mathrm{Car}^{(m)}(\mathcal{E}')) = \mathrm{Car}^{(m)}(\mathcal{E})$, then the formula 15.1.3.7.1 holds if and only if $\mathrm{Car}^{(m)}(\mathcal{E}) = \mathrm{Car}^{(m+s)}((F_X^s)^*\mathcal{E})$. Since the functor $\mathcal{E} \mapsto (F_X^s)^*\mathcal{E}$ is an equivalence of categories between the coherent left $\mathcal{D}^{(m)}_{X/S}$-modules and the coherent left $\mathcal{D}^{(m+s)}_{X/S}$-module (this follows from the Frobenius descent 6.1.1.9 and $F_S^s \circ F_{X/S}^s = F_X^s$), then by transitivity , we can suppose $m = 0$ (and s may vary).

2) We check the equality 15.1.3.7.1 in the case where $i = 0$ and $m = 0$. We set $X' := X^{(s)}$ and $F = F^s_{X/S}$. Since $\mathcal{E}'$ is a quasi-coherent $\mathcal{O}_{X'}$-module, then it is the inductive limit of its coherent $\mathcal{O}_{X'}$-submodules (see [Gro60, 9.4.9]). Hence, there exists a coherent $\mathcal{O}_{X'}$-submodule $\mathcal{G}'$ of $\mathcal{E}'$ such that the canonical $\mathcal{D}^{(m)}_{X'/S}$-linear map $\varpi\colon \mathcal{D}^{(0)}_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{G}' \twoheadrightarrow \mathcal{E}'$ is surjective. This yields the good filtration $(\mathcal{E}'_l)_{l\in\mathbb{N}}$ by setting $\mathcal{E}'_l = \varpi(\mathcal{D}^{(0)}_{X'/S,l} \otimes_{\mathcal{O}_{X'}} \mathcal{G}')$. Since $\mathcal{D}^{(s)}_{X/S} \to F^*\mathcal{D}^{(0)}_{X'/S}$ is $(\mathcal{D}^{(s)}_{X/S}, F^{-1}\mathcal{O}_{X'})$-bilinear (see 4.4.2.8), we get the surjective $\mathcal{D}^{(s)}_{X/S}$-linear maps $\varpi^{(s)}\colon \mathcal{D}^{(s)}_{X/S} \otimes_{\mathcal{O}_X} F^*\mathcal{G}' \xrightarrow{\sim} \mathcal{D}^{(s)}_{X/S} \otimes_{F^{-1}\mathcal{O}_{X'}} F^{-1}\mathcal{G}' \twoheadrightarrow F^*\mathcal{D}^{(0)}_{X/S} \otimes_{F^{-1}\mathcal{O}_{X'}} F^{-1}\mathcal{G}' \overset{F^*(\varpi)}{\twoheadrightarrow} F^*\mathcal{E}'$. Via the formula 6.1.1.8.3, we compute the induced good filtration of $F^*\mathcal{E}'$ is

$$\mathrm{Fil}_n F^*\mathcal{E}' := \varpi^{(s)}(\mathcal{D}^{(s)}_{X/S,n} \otimes_{\mathcal{O}_X} F^*\mathcal{G}') = F^*\mathcal{E}'_{q_n^{(s)}},$$

where $q_n^{(s)}$ is the quotient of the Euclidian division of n by p^s. We compute $\mathrm{gr}_{p^s n} F^*\mathcal{E}' \xrightarrow{\sim} F^* \mathrm{gr}_n \mathcal{E}'$, and $\mathrm{gr}_N F^*\mathcal{E}' = 0$ if p^s does not divides N. Recall the canonical map $\mathcal{D}^{(s)}_{X/S} \to F^*\mathcal{D}^{(0)}_{X'/S}$ induces the isomorphism $(\mathrm{gr}\,\mathcal{D}^{(s)}_{X/S})_{\mathrm{red}} \xrightarrow{\sim} F^*\,\mathrm{gr}\,\mathcal{D}^{(0)}_{X^{(s)}/S}$ (see the proof of 15.1.2.7), which is translated by the isomorphism $X \times_{X^{(s)}} T^*X^{(s)} \xrightarrow{\sim} T^{(s)*}X$ of 15.1.2.7.1. Via the above computation, the corresponding coherent $F^*\,\mathrm{gr}\,\mathcal{D}^{(0)}_{X'/S}$-module associated with the coherent $(\mathrm{gr}\,\mathcal{D}^{(s)}_{X/S})_{\mathrm{red}}$-module $\mathrm{gr}\,F^*\mathcal{E}'$ is therefore $F^*\,\mathrm{gr}\,\mathcal{D}^{(0)}_{X'/S} \otimes_{\mathrm{gr}\,\mathcal{D}^{(0)}_{X'/S}} \mathrm{gr}\,\mathcal{E}'$. Hence $\mathrm{gr}\,F^*\mathcal{E}' \xrightarrow{\sim} \iota^{(s)}_{S*}\iota^{(s)*}_S\,\mathrm{gr}\,\mathcal{E}'$. On one hand $\mathrm{Car}^{(s)}(F^*\mathcal{E}') = \iota^{(s)}_*\,\mathrm{Supp}\,\mathrm{gr}\,F^*\mathcal{E}'$. On the other hand, since $\iota^{(s)}_S$ is a flat universal homeomorphism, then $\mathrm{Supp}\,\iota^{(s)}_{S*}\iota^{(s)*}_S\,\mathrm{gr}\,\mathcal{E}' = \mathrm{Supp}\,\mathrm{gr}\,\mathcal{E}'$. □

15.1.4 The characteristic variety of a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module

Let $\mathfrak{P}$ be a smooth $\mathcal{V}$-formal scheme, P_i be the reduction of $\mathfrak{P}$ modulo π^{i+1}, P be the reduction of $\mathfrak{P}$ modulo π. Let $m \in \mathbb{N}$ be an integer.

Definition 15.1.4.1. Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module.

(a) The "characteristic variety $\mathrm{Car}^{(m)}(\mathcal{E})$ of level m" of $\mathcal{E}$ is by definition the characteristic variety of level m of $\mathcal{E}/\pi\mathcal{E}$ as coherent $\mathcal{D}^{(m)}_P$-module, i.e. $\mathrm{Car}^{(m)}(\mathcal{E}) := \mathrm{Car}^{(m)}(\mathcal{E}/\pi\mathcal{E})$.

(b) The "dimension of level m of $\mathcal{E}$" is defined by setting $\dim^{(m)}(\mathcal{E}) := \dim \mathrm{Car}^{(m)}(\mathcal{E})$. The "codimension of level m of $\mathcal{E}$" is defined by setting $\mathrm{codim}^{(m)}(\mathcal{E}) := 2\dim P - \dim^{(m)}(\mathcal{E})$.

   We have the inequalities $0 \le \dim(\mathcal{E}) \le 2\dim P$ and $0 \le \mathrm{codim}(\mathcal{E}) \le 2\dim P$.

(c) We obtain the analogous constructions and definitions for right modules. Moreover, we get

$$\mathrm{Car}^{(m)}(\mathcal{E}) = \mathrm{Car}^{(m)}(\omega_{\mathfrak{P}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{E}).$$

Proposition 15.1.4.2. *Let $\mathcal{E}$, $\mathcal{F}$ be two p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-modules, such that there exists an isogeny $\phi\colon \mathcal{E} \to \mathcal{F}$. Then $\mathrm{Car}^{(m)}(\mathcal{E}) = \mathrm{Car}^{(m)}(\mathcal{F})$.*

Proof. Since $\mathcal{E}$ is p-torsion free, then ϕ is a monomorphism and there exists $n \in \mathbb{N}$ such that $p^{n+1}(\mathcal{F}/\phi(\mathcal{E})) = 0$. Hence $\mathrm{Tor}_1^{\mathcal{O}_{\mathfrak{P}}}(\mathcal{O}_{\mathfrak{P}}/p^{n+1}\mathcal{O}_{\mathfrak{P}}, \mathcal{F}/\phi(\mathcal{E})) \xrightarrow{\sim} \mathcal{F}/\phi(\mathcal{E})$. Since $\mathcal{F}$ is p-torsion free, then $\mathrm{Tor}_1^{\mathcal{O}_{\mathfrak{P}}}(\mathcal{O}_{\mathfrak{P}}/p^{n+1}\mathcal{O}_{\mathfrak{P}}, \mathcal{F}) =$

0. Hence, by applying the functor $\mathcal{O}_{P_n} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}} -$ to the exact sequence $0 \to \mathcal{E} \xrightarrow{\phi} \mathcal{F} \to \mathcal{F}/\phi(\mathcal{E}) \to 0$ we get an exact triangle which induces the long exact sequence of coherent left $\mathcal{D}^{(m)}_{P_n/S_n}$-modules:

$$0 \to \mathcal{F}/\phi(\mathcal{E}) \to \mathcal{E}/p^{n+1}\mathcal{E} \to \mathcal{F}/p^{n+1}\mathcal{F} \to \mathcal{F}/\phi(\mathcal{E}) \to 0.$$

Hence, it follows from 15.1.3.5.1 that $\mathrm{Car}^{(m)}(\mathcal{E}/p^{n+1}\mathcal{E}) = \mathrm{Car}^{(m)}(\mathcal{F}/p^{n+1}\mathcal{F})$. We conclude thanks to 15.1.3.6.1. □

Definition 15.1.4.3. Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module.

(a) Choose a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module $\mathring{\mathcal{E}}$ such that there exists an isomorphism of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-modules of the form $\mathring{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ (see 7.5.2.8). The characteristic variety of level m of $\mathcal{E}$ denoted by $\mathrm{Car}^{(m)}(\mathcal{E})$ is by definition that of $\mathring{\mathcal{E}}$ as coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module, i.e., $\mathrm{Car}^{(m)}(\mathcal{E}) := \mathrm{Car}^{(m)}(\mathring{\mathcal{E}}/\pi\mathring{\mathcal{E}})$. It follows from 15.1.4.2 that this is well defined.

(b) The "dimension of level m of $\mathcal{E}$" is defined by setting $\dim^{(m)}(\mathcal{E}) := \dim \mathrm{Car}^{(m)}(\mathcal{E})$. The "codimension of level m of $\mathcal{E}$" is defined by setting $\mathrm{codim}^{(m)}(\mathcal{E}) := 2\dim P - \dim^{(m)}(\mathcal{E})$.

   We have the inequalities $0 \leq \dim(\mathcal{E}) \leq 2\dim P$ and $0 \leq \mathrm{codim}(\mathcal{E}) \leq 2\dim P$.

(c) We obtain the analogous constructions and definitions for right modules. Moreover, we get

$$\mathrm{Car}^{(m)}(\mathcal{E}) = \mathrm{Car}^{(m)}(\omega_{\mathfrak{P}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{E}).$$

Example 15.1.4.4. Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. The variety $\mathrm{Car}^{(m)}(\mathcal{E})$ is empty if and only if $\dim^{(m)} \mathcal{E} = 0$ if and only if $\mathcal{E} = 0$ (because, since $\mathring{\mathcal{E}}$ has no p-torsion, $\mathring{\mathcal{E}}/\pi\mathring{\mathcal{E}} = 0$ is equivalent to $\mathring{\mathcal{E}}_{\mathbb{Q}} = 0$).

Lemma 15.1.4.5. *Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of coherent left $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-modules. We have the equality*

$$\mathrm{Car}^{(m)}(\mathcal{E}) = \mathrm{Car}^{(m)}(\mathcal{E}') \cup \mathrm{Car}^{(m)}(\mathcal{E}''). \tag{15.1.4.5.1}$$

In particular, we have the formula $\mathrm{codim}^{(m)}(\mathcal{E}) = \min\{\mathrm{codim}^{(m)}(\mathcal{E}'), \mathrm{codim}^{(m)}(\mathcal{E}'')\}$.

Proof. Choose a morphism of p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-modules $g\colon \mathcal{F} \to \mathcal{F}''$ such that the induced morphism $\mathcal{F}_{\mathbb{Q}} \to \mathcal{F}''_{\mathbb{Q}}$ is isomorphic to $\mathcal{E} \to \mathcal{E}''$. Replacing $\mathcal{F}''$ by the image of g if necessary we can suppose g is surjective. Let $\mathcal{F}' := \ker g$. Then $\mathcal{F}'$ is a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module and is endowed with an isomorphism of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-modules of the form $\mathcal{F}'_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}'$. Setting $\overline{\mathcal{F}}' := \mathcal{D}^{(0)}_P \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}} \mathcal{F}'$, $\overline{\mathcal{F}} := \mathcal{D}^{(0)}_P \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}} \mathcal{F}$ and $\overline{\mathcal{F}}'' := \mathcal{D}^{(0)}_P \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}} \mathcal{F}''$ we get the exact sequence of $\mathcal{D}^{(0)}_P$-modules $0 \to \overline{\mathcal{F}}' \to \overline{\mathcal{F}} \to \overline{\mathcal{F}}'' \to 0$. Hence, this is a consequence of 15.1.3.5. □

Lemma 15.1.4.6. *Let $F\colon \mathfrak{P} \to \mathfrak{P}'$ be a morphism of smooth $\mathfrak{S}$-formal schemes which is a lifting of $F^s_{P/S}$. Let $\mathcal{E}'$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}}$-module. Then we have in $|T^*P|$ the equality:*

$$\iota^{(s)}_S(\mathrm{Car}^{(m)}(\mathcal{E}')) = \mathrm{Car}^{(m+s)}(F^*\mathcal{E}'),$$

where $\iota^{(s)}_S\colon T^*X^{(s)} \to T^*X$ is the canonical universal homeomorphism induced by base change via F^s_S (see 15.1.2.8.2).

Proof. Choose a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'/\mathfrak{S}}$-module $\mathring{\mathcal{E}}'$ such that there exists an isomorphism of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}}$-modules of the form $\mathring{\mathcal{E}}'_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}'$. By definition, $\mathrm{Car}^{(m)}(\mathcal{E}') = \mathrm{Car}^{(m)}(\mathring{\mathcal{E}}') = \mathrm{Car}^{(m)}(\mathring{\mathcal{E}}'/\pi\mathring{\mathcal{E}}')$. On the other hand, since $F^*\mathring{\mathcal{E}}'$ is a p-torsion free coherent $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{P}/\mathfrak{S}}$-module, since we have an isomorphism of $\widehat{\mathcal{D}}^{(m+s)}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}$-modules of the form $(F^*\mathring{\mathcal{E}}')_{\mathbb{Q}} \xrightarrow{\sim} F^*\mathcal{E}'$, then $\mathrm{Car}^{(m+s)}(F^*\mathcal{E}') = \mathrm{Car}^{(m+s)}(F^*\mathring{\mathcal{E}}') = \mathrm{Car}^{(m+s)}(F^*(\mathring{\mathcal{E}}'/\pi\mathring{\mathcal{E}}'))$. We conclude by using Lemma 15.1.3.7. □

15.1.5 The characteristic variety of a coherent F-$\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module

Definition 15.1.5.1. Suppose there exists an isomorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ lifting the s-power of the Frobenius of k. Let $(\mathcal{N}, \phi)$ be a coherent F-$\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module, i.e. a coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{N}$ and an isomorphism of $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules ϕ of the form $\phi\colon F^*\mathcal{N} \xrightarrow{\sim} \mathcal{N}$, where $F\colon \mathfrak{P} \to \mathfrak{P}'$ is (locally) a lifσ-linear lifting of F_P. Then there exists a (unique up to isomorphism) coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{N}^{(0)}$ and an isomorphism $\phi^{(0)}\colon \widehat{\mathcal{D}}^{(1)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{N}^{(0)} \xrightarrow{\sim} F^*\mathcal{N}^{(0)}$ which induces canonically ϕ. Then, the characteristic variety of $\mathcal{N}$ denoted by $\mathrm{Car}(\mathcal{N})$ is by definition the characteristic variety of level 0 of $\mathcal{N}^{(0)}$, i.e., $\mathrm{Car}(\mathcal{N}) := \mathrm{Car}^{(0)}(\mathcal{N}^{(0)})$.

Example 15.1.5.2. The variety $\mathrm{Car}(\mathcal{N})$ is empty if and only if $\mathcal{N} = 0$.

Proposition 15.1.5.3. *We have the following properties.*

1. *Let $\mathcal{G}$ be a coherent $\mathcal{D}^{(0)}_P$-module. Choose a good filtration $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$. Then the following assertions are equivalent*
   (a) $\mathrm{Car}^{(0)}(\mathcal{G}) \subset T^*_P P$.
   (b) $\mathrm{gr}\,\mathcal{G}$ *is $\mathcal{O}_P$-coherent (for the $\mathcal{O}_P$-module structure induced by $\mathcal{O}_P \hookrightarrow \mathrm{gr}\,\mathcal{D}^{(0)}_P$).*
   (c) $\mathcal{G}$ *is $\mathcal{O}_P$-coherent (for the $\mathcal{O}_P$-module structure induced by $\mathcal{O}_P \hookrightarrow \mathcal{D}^{(0)}_P$).*

2. *Let $\mathcal{G}$ be a coherent $\mathcal{D}^{(m)}_P$-module (resp. a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module, resp. a topologically nilpotent coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module). Then the following assertions are equivalent*
   (a) $\mathrm{Car}^{(m)}(\mathcal{G}) \subset T^*_P P$.
   (b) $\mathcal{G}$ *is $\mathcal{O}_P$-coherent (resp. $\mathcal{O}_{\mathfrak{P}}$-coherent, resp. $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent).*

Proof. 1) Let us treat the first part. Let us prove that (a) implies (b). Since this is local, we can suppose that P affine and P/S has coordinates $t_1, \ldots, t_d$. Let ξ_i be the global section of $\mathrm{gr}\,\mathcal{D}^{(0)}_P$ which is the element associated with ∂_i, the derivation with respect to t_i. Since the ideal defining the closed immersion $\mathrm{Car}^{(0)}(\mathcal{G}) \hookrightarrow T^*P$ is the radical of the annihilator of $\mathrm{gr}\,\mathcal{G}$, the inclusion $\mathrm{Car}^{(0)}(\mathcal{G}) \subset T^*_P P$ implies that $\xi_1^N, \ldots, \xi_d^N$ annihilate $\mathrm{gr}\,\mathcal{G}$ for some integer N large enough. Hence, $\mathrm{gr}\,\mathcal{G}$ is a coherent $\mathrm{gr}\,\mathcal{D}^{(0)}_P/(\xi_1, \ldots, \xi_d)^{Nd}$-module. Since $\mathrm{gr}\,\mathcal{D}^{(0)}_P/(\xi_1, \ldots, \xi_d)^{Nd}$ is a finite $\mathcal{O}_P$-algebra we conclude that $\mathrm{gr}\,\mathcal{G}$ is $\mathcal{O}_P$-coherent.

Now, suppose (b) is satisfied and let us check (c) holds. By definition of a good filtration, we get $\mathcal{G}_n = \mathcal{G}$ for n large enough. Hence, $\mathcal{G}$ is coherent $\mathcal{O}_P$-module. Finally, suppose (c). Then, the constant filtration $(\mathcal{G}_n = \mathcal{G})_{n\in\mathbb{N}}$ is a good filtration (it might be more convenient to complete the filtration by $\mathcal{G}_n = 0$ if $n < 0$). Then the action of ξ_i on $\mathrm{gr}\,\mathcal{G} = \mathcal{G}_0/\mathcal{G}_{-1} = \mathcal{G}$ is zero (because the action of ξ_i is induced by maps of the form $\mathcal{G}_n/\mathcal{G}_{n-1} \to \mathcal{G}_{n+1}/\mathcal{G}_n$, which are zero). Hence, $\mathrm{Car}^{(0)}(\mathcal{G}) \subset T^*_P P$ (recall that the construction of $\mathrm{Car}^{(0)}(\mathcal{G})$ does not depend on the choice of the good filtration).

2) i) Suppose $\mathcal{G}$ is a coherent $\mathcal{D}^{(m)}_{P/S}$-module. Let $\mathcal{G}'^{(0)}$ be a coherent $\mathcal{D}^{(0)}_{P^{(s)}/S}$-module such that $F^{m*}_{P/S}\mathcal{G}'^{(0)} \xrightarrow{\sim} \mathcal{G}$, where $F^m_{P/S}$ is the mth relative Frobenius of P/S. Since $F^m_{P/S}$ is faithfully flat, then $\mathcal{G}'^{(0)}$ is coherent over $\mathcal{O}_{P^{(s)}}$ if and only if $\mathcal{G}$ is coherent over $\mathcal{O}_P$. By using 15.1.3.7, we reduce therefore to the case where $m = 0$ which has been checked above.

ii) Suppose $\mathcal{G}$ is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module. Since $\mathcal{G}$ is coherent over $\mathcal{O}_{\mathfrak{P}}$ if and only if $\mathcal{G}/\pi\mathcal{G}$ is coherent over $\mathcal{O}_P$ (use 7.2.1.2 and theorem of type A for coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-modules), since $\mathrm{Car}^{(0)}(\mathcal{G}) = \mathrm{Car}^{(0)}(\mathcal{G}/\pi\mathcal{G})$ (see definition 15.1.4.1), then we conclude from the case i).

iii) Suppose $\mathcal{G}$ is a topologically nilpotent coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. Choose a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module $\overset{\circ}{\mathcal{G}}$ together with the isomorphism of the form $\overset{\circ}{\mathcal{G}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{G}$. By definition, $\mathrm{Car}^{(m)}(\mathcal{G}) = \mathrm{Car}^{(m)}(\overset{\circ}{\mathcal{G}})$. Since $\mathcal{G}$ is *topologically nilpotent*, then if follows from 7.5.2.9 that $\mathcal{G}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent if and only if $\overset{\circ}{\mathcal{G}}$ is $\mathcal{O}_{\mathfrak{P}}$-coherent. □

Corollary 15.1.5.4. *Suppose there exists an isomorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ lifting the s-power of the Frobenius of k. Let $(\mathcal{E}, \phi)$ be a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. The following assertions are equivalent.*

(a) $\operatorname{Car}(\mathcal{E}) \subset T^*_P P$.

(b) $\mathcal{E}$ *is* $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$*-coherent.*

Proof. i) Suppose $\operatorname{Car}(\mathcal{E}) \subset T^*_P P$. Let $m_0 \geq 0$ be an integer such that there exist a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ together with a $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)} \xrightarrow{\sim} \mathcal{E}$ (8.4.1.11). Let $\mathring{\mathcal{E}}^{(m_0)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P}}$-module together with a $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}^{(m_0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(m_0)}$ (see 7.4.5.2). Since $\operatorname{Car}(\mathcal{E}) = \operatorname{Car}^{(m_0)}(\mathring{\mathcal{E}}^{(m_0)})$, then it follows from the part 2) that $\mathring{\mathcal{E}}^{(m_0)}$ is $\mathcal{O}_{\mathfrak{P}}$-coherent. Hence $\mathcal{E}^{(m_0)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent and $\mathcal{E}$ is therefore $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent (see 15.3.1.22).

ii) Conversely if $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent then following 11.1.1.6 there exists a p-torsion free $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}/\mathfrak{S}}$-module $\mathring{\mathcal{E}}$, coherent over $\mathcal{O}_{\mathfrak{P}}$ together with a $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. Since $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ (use 11.1.1.6.(b and c)), this yields $\operatorname{Car}(\mathcal{E}) = \operatorname{Car}^{(0)}(\mathring{\mathcal{E}})$. Following the part 2) of 15.1.5.3, $\operatorname{Car}^{(0)}(\mathring{\mathcal{E}}) \subset T^*_P P$ and we are done. □

Proposition 15.1.5.5. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{(m)}_{P/S}$-module (resp. a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module, resp. a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module).*

If $\dim \operatorname{Car}^{(m)}(\mathcal{E}) \leq \dim P$, then there exists a dense open subset $\mathfrak{U} \subset \mathfrak{P}$ such that $\mathcal{E}|U$ is coherent over $\mathcal{O}_U$ (resp. $\mathcal{O}_{\mathfrak{U}}$, resp. $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$).

Proof. 1) i) Suppose $\mathcal{E}$ be a coherent $\mathcal{D}^{(0)}_{P/S}$-module. Set $S := \operatorname{Car}^{(0)}(\mathcal{E}) \setminus T^*_P P$. If $S = \emptyset$, then $\mathcal{E}$ is coherent over $\mathcal{O}_P$ by Proposition 15.1.5.3. Assume that $S \neq \emptyset$. Since S is conic (see 15.1.3.4.1), then $\dim \left(\pi_P^{-1}(\pi_P(s)) \cap \operatorname{Car}^{(0)}(\mathcal{E})\right) \geq 1$ for any $s \in S$. Hence $\dim \pi_P(S) < \dim S \leq \dim P$. Therefore, there exists a non-empty open subset $U \subset P$ such that $P \setminus \pi_P(S) \supset U$. In this case we have $\operatorname{Car}^{(0)}(\mathcal{E}|U) = \operatorname{Car}^{(0)}(\mathcal{E}) \cap T^*U \subset T^*_U U$ and hence $\mathcal{E}|U$ is coherent over $\mathcal{O}_U$ by 15.1.5.3.

ii) Suppose $\mathcal{E}$ be a coherent $\mathcal{D}^{(m)}_{P/S}$-module. Let $\mathcal{E}'^{(0)}$ be a coherent $\mathcal{D}^{(0)}_{P^{(s)}}$-module such that $F^{m*}_{P/S}\mathcal{E}'^{(0)} \xrightarrow{\sim} \mathcal{E}$. For any open U of P, since $F^m_{P/S}$ is faithfully flat, then $\mathcal{E}'^{(0)}|U^{(s)}$ is coherent over $\mathcal{O}_{U^{(s)}}$ if and only if so is $\mathcal{E}|U$. By using 15.1.3.7, we reduce therefore to the case where $m = 0$ already treated.

2) Suppose $\mathcal{E}$ a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module. Since $\mathcal{E}$ is coherent over $\mathcal{O}_{\mathfrak{P}}$ if and only if $\mathcal{E}/\pi\mathcal{E}$ is coherent over $\mathcal{O}_P$, since $\operatorname{Car}^{(m)}(\mathcal{E}) = \operatorname{Car}^{(m)}(\mathcal{E}/\pi\mathcal{E})$ (see definition 15.1.4.1), then we conclude by using the part 1).

3) Suppose $\mathcal{E}$ is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. Let $\mathring{\mathcal{E}}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module together with a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ (see 7.4.5.2). From the part 2), there exists a dense open subset $\mathfrak{U} \subset \mathfrak{P}$ such that $\mathring{\mathcal{E}}|\mathfrak{U}$ is coherent over $\mathcal{O}_{\mathfrak{U}}$. Hence $\mathcal{E}|\mathfrak{U}$ is coherent over $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$. □

Corollary 15.1.5.6. *We suppose there exists an isomorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ lifting the s-power of the Frobenius of k. Let $\mathcal{E}$ be a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module If $\dim \operatorname{Car}(\mathcal{E}) \leq \dim P$, then there exists a dense open subset $\mathfrak{U} \subset \mathfrak{P}$ such that $\mathcal{E}|\mathfrak{U}$ is coherent over $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$.*

Proof. This follows from 15.1.5.5 and 15.3.1.22 (which is checked independently of 15.1.5.6). □

15.1.6 Purity for the level 0: preliminaries on filtered modules

We use here the terminology of Laumon in [Lau85, A.1]:

(i) A filtered ring (D, D_i) is a ring D, unitary, non-necessary commutative, with an increasing filtration by additive subgroups $(D_i)_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ such that $1 \in D_0$ and $D_i \cdot D_j \subset D_{i+j}$ for any $i, j \in \mathbb{Z}$.

(ii) Let (D, D_i) be a filtered ring. We get a category of filtered (D, D_i)-modules as follows. A filtered (D, D_i)-module (M, M_i) is a D-module M endowed with a filtration $(M_i)_{i\in\mathbb{Z}}$ such that $A_i \cdot M_j \subset M_{i+j}$ for any $i, j \in \mathbb{Z}$. A morphism of $(M, M_i) \to (M', M'_i)$ of filtered (D, D_i)-modules is a morphism of D-modules $f\colon M \to M'$ such that $f(M_i) \subset M'_i$ for any $i \in \mathbb{Z}$. If (M, M_i) is a filtered

(D, D_i)-module and $n \in \mathbb{Z}$, we denote by $(M(n), M(n)_i)$ the filtered (D, D_i)-module defined as follows: $M(n) = M$ and $M(n)_i := M_{i+n}$. Following [Gro61, 2.1.2], a filtered free (D, D_i)-module (resp. filtered free (D, D_i)-module of finite type) is a direct sum (resp. a finite direct sum) in the category of filtered (D, D_i)-modules of the form $(D(n), D(n)_i)$, for some integer n. Let (M, M_i) be a filtered (D, D_i)-module. We say that the filtration M_i is good or that (M, M_i) is a good filtered (D, D_i)-module if there exists an epimorphism in the category of filtered (D, D_i)-modules of the form $\phi\colon (L, L_i) \twoheadrightarrow (M, M_i)$ such that $\phi(L_i) = M_i$ (compare with 4.1.3.8). We remark that any D-module of finite type can be endowed with a good filtration. Conversely, for any good filtered (D, D_i)-module (M, M_i), M is a D-module of finite type.

(iii) Let (D, D_i) be a filtered ring and (M, M_i) be a filtered (D, D_i)-module. The *ind-pro-complete separation* of (M, M_i), denoted by $(\widehat{M}, \widehat{M}_i)$ is a filtered (D, D_i)-module defined as follows: $\widehat{M}_i := \varprojlim_n M_i/M_{i-n}$ is the complete separation of M_i with respect to the filtration $(M_{i-n})_{n\in\mathbb{N}}$ and $\widehat{M} := \cup_{i\in\mathbb{Z}} \widehat{M}_i$, where the inclusion $\widehat{M}_i \subset \widehat{M}_{i+1}$ are that induced by complete separation from the inclusion $M_i \subset M_{i+1}$. Using the universal property of projective limits we check that $(\widehat{D}, \widehat{D}_i)$ is also a filtered ring and that $(\widehat{M}, \widehat{M}_i)$ is a filtered $(\widehat{D}, \widehat{D}_i)$-module.

We say that (M, M_i) is *ind-pro-complete separated* if the canonical morphism $(M, M_i) \to (\widehat{M}, \widehat{M}_i)$ is an isomorphism. For instance, we remark that the filtration of an ind-pro-complete separated filtered ring is exhaustive. Like Laumon, to simplify the terminology (we hope there will not be confusion with the usual notion of completion), we will simply say "complete" for "ind-pro-complete separated" and "completion" for "ind-pro-complete separation".

(iv) In this section, with our abuse of terminology, (D, D_i) will be a complete filtered ring such that $\mathrm{gr}(D, D_i)$ is a left and right noetherian ring. Hence, from Proposition [Lau85, A.1.1], a good filtered (D, D_i)-module is complete.

The following Lemmas 15.1.6.1, 15.1.6.3, 15.1.6.5, 15.1.6.7, 15.1.6.8 correspond to Laumon's Lemmas [Lau83, 3.3.2, 3.3.3, 3.3.5,3.3.8] formulated in our context.

Lemma 15.1.6.1. *Let* $0 \to (M', M'_i) \xrightarrow{f} (M, M_i) \xrightarrow{g} (M'', M''_i) \to 0$ *be a sequence of morphisms of good filtered* (D, D_i)*-modules.*

1) The following conditions are equivalent:

(a) we have $M'_i = M' \cap M_i$ *and* $M''_i = g(M'_i)$ *for any* $i \in \mathbb{Z}$*;*

(b) the sequences of abelian groups $0 \to M'_i \xrightarrow{f} M_i \xrightarrow{g} M''_i \to 0$ *are exact for any* $i \in \mathbb{Z}$*;*

(c) $g \circ f = 0$ *and the sequence of* $\mathrm{gr}\, D$*-modules* $0 \to \mathrm{gr}\, M' \to \mathrm{gr}\, M \to \mathrm{gr}\, M'' \to 0$ *is exact.*

2) When these equivalent conditions of 1) are satisfied, the sequence of D*-modules* $0 \to M' \to M \to M'' \to 0$ *is exact.*

Proof. The equivalence between (a) and (b) is obvious. Suppose the condition (b) is satisfied. The fact good filtrations are exhaustive implies that $g \circ f = 0$. We get the last condition (c) by using the nine Lemma (see the exercice [Wei94, 1.3.2]). Suppose now the condition (c) is satisfied. Using the nine Lemma (more precisely, the part 3 of the exercice [Wei94, 1.3.2]), for any $i \in \mathbb{Z}$, we check by induction on $n \geq 1$ that the sequence of abelian groups $0 \to M'_i/M'_{i-n} \to M_i/M_{i-n} \to M''_i/M''_{i-n} \to 0$ are exact. Taking the projective limits, since M'_i, M_i and M''_i are complete separated by hypothesis, since Mittag Leffler condition is satisfied, this yields that the sequence $0 \to M'_i \xrightarrow{f} M_i \xrightarrow{g} M''_i \to 0$ is exact. The part 2) of the Lemma follows from the remark that filtrations are exhaustive. □

Definition 15.1.6.2. Let $0 \to (M', M'_i) \xrightarrow{f} (M, M_i) \xrightarrow{g} (M'', M''_i) \to 0$ be a sequence of morphisms of good filtered (D, D_i)-modules satisfying the equivalent conditions of 15.1.6.1.1). We say that this sequence is an "exact" sequence of morphisms of good filtered (D, D_i)-modules.

Lemma 15.1.6.3. *Let* $u\colon (M, M_i) \to (N, N_i)$ *be a morphism of good filtered* (D, D_i)*-modules. We have the exact sequences of good filtered* (D, D_i)*-modules:*

$$0 \to \ker u \to (M, M_i) \to \mathrm{Coim} u \to 0,$$
$$0 \to \mathrm{Im} u \to (N, N_i) \to \mathrm{Coker}\, u \to 0. \qquad (15.1.6.3.1)$$

Proof. Let $\ker u$ be the kernel of u in the category of filtered (D, D_i)-modules, i.e. $\ker u = (\ker u, \ker u \cap M_i)$. Let $\operatorname{Coker} u$ be the cokernel of u in the category of filtered (D, D_i)-modules, i.e. $\operatorname{Coker} u = (\operatorname{Coker} u, N_i/N_i \cap u(M))$. From [Lau85, A.1.1.2], the filtered (D, D_i)-modules $\ker u$ and $\operatorname{Coker} u$ are good. Hence, $\ker u, \operatorname{Coker} u, \operatorname{Im} u, \operatorname{Coim} u$ exist in the category of good filtered (D, D_i)-modules (and are equal to that computed in the category of filtered (D, D_i)-modules). Hence, both sequences 15.1.6.3.1 are well defined in the category of good filtered (D, D_i)-modules. Since $\operatorname{Coim} u = (u(M), u(M_i))$ and $\operatorname{Im} u = (u(M), u(M) \cap N_i)$, then these sequence satisfy the condition (a) of Lemma 15.1.6.1 and hence they are exact.

□

Definition 15.1.6.4 (Strictness)**.** A morphism $u: (M, M_i) \to (N, N_i)$ of good filtered (D, D_i)-modules is strict if the canonical morphism $\operatorname{Coim} u \to \operatorname{Im} u$ is an isomorphism of (good) filtered (D, D_i)-modules. If $u: M \to N$ is a monomorphism (resp. epimorphism) and if $u: (M, M_i) \to (N, N_i)$ is strict, we say that u is a strict monomorphism (resp. strict epimorphism).

Lemma 15.1.6.5. *Let $u: (M, M_i) \to (N, N_i)$ be a morphism of good filtered (D, D_i)-modules.*

1. *Then u is strict if and only if $u(M_i) = u(M) \cap N_i$ for any $i \in \mathbb{Z}$.*
2. *The following conditions are equivalent*
   (a) *u is a strict monomorphism ;*
   (b) *the morphism $(M, M_i) \to \operatorname{Im} u$ is an isomorphism ;*
   (c) *the sequence of good filtered (D, D_i)-modules $0 \to (M, M_i) \to (N, N_i) \to \operatorname{Coker} u \to 0$ is exact ;*
   (d) *$\operatorname{gr} u$ is a monomorphism.*
3. *The following conditions are equivalent*
   (a) *u is a strict epimorphism ;*
   (b) *the morphism $\operatorname{Coim} u \to (N, N_i)$ is an isomorphism ;*
   (c) *the sequence of good filtered (D, D_i)-modules $0 \to \ker u \to (M, M_i) \to (N, N_i) \to 0$ is exact ;*
   (d) *$\operatorname{gr} u$ is an epimorphism.*
4. *u is an isomorphism if and only if $\operatorname{gr} u$ is an isomorphism.*

Proof. The first statement is straighforward from the description of $\operatorname{Im} u$ and $\operatorname{Coim} u$. The implication $(a) \Rightarrow (b)$ is clear from the description of $\operatorname{Im} u$ and from 1. The implication $(b) \Rightarrow (c)$ (resp. $(c) \Rightarrow (d)$) is a consequence of 15.1.6.3.1. (resp. 15.1.6.1). Finally, suppose (d) is satisfied. Let $x \in \ker u$. Suppose $x \neq 0$. There exists $i \in \mathbb{Z}$ such that $x \notin M_i$ (recall the filtration is separated). This is a contradiction with the fact that $M_{i+1}/M_i \to N_{i+1}/N_i$ is injective (because $\operatorname{gr} u$ is injective by hypothesis). Hence u is a monomorphism. The fact that $\operatorname{gr} u$ is injective implies that $M_{i+1} \cap N_i = M_i$, for any $i \in \mathbb{Z}$. By induction on $n \in \mathbb{N}$ we get that for any $i \in \mathbb{Z}$, $M_{i+n} \cap N_i = M_i$. Since the filtration is exhaustive, this yields that $M \cap N_i = M_i$. From part 1) of the Lemma, this means that u is strict. Let us check part 3). We check similarly $(a) \to (b) \to (c) \to (d)$. Now suppose that $\operatorname{gr} u$ is an epimorphism. Let $i \in \mathbb{Z}$. From part 1), it is sufficient to check $u(M_i) = N_i$ (indeed filtrations are exhausted and then u will be surjective). Let $y \in N_i$. Put $y_{-1} := y$. By induction on $n \geq 0$, we construct $y_n \in N_{i-1-n}$ and $x_n \in M_{i-n}$ such that $u(x_n) = y_{n-1} - y_n$. This is consequence of the equality $u(M_{i-n}) + N_{i-n-1} = N_{i-n}$ (because $\operatorname{gr} u$ is an epimorphism). Since N_i and M_i are separated complete for the filtrations $(N_{i-n})_{n \in \mathbb{N}}$ and $(M_{i-n})_{n \in \mathbb{N}}$, the sum $\sum_{n \geq -1}(y_n - y_{n+1})$ converges to y and $\sum_{n \in \mathbb{N}} x_n$ converges in M_i to an element, denoted by x. Hence, $u(x) = y$ and then $N_i \subset u(M_i)$. Finally, 4 is a consequence of the equivalence $(a) \Leftrightarrow (d)$ of 2 and 3. □

Remark 15.1.6.6. Let (M, M_i) be a filtered (D, D_i)-module. From 15.1.6.5.1, we remark that (M, M_i) is a good filtered (D, D_i)-module if and only if there exists a strict epimorphism of the form $u: (L, L_i) \twoheadrightarrow (M, M_i)$, where (L, L_i) is a free filtered (D, D_i)-module of finite type.

Lemma 15.1.6.7. *Let $u: (M, M_i) \to (N, N_i)$ be a morphism of good filtered (D, D_i)-modules.*

1. *The following assertions are equivalent:*

   (a) *The morphism* u *is strict ;*

   (b) *The sequence* $0 \to \operatorname{gr}\ker u \to \operatorname{gr}(M, M_i) \to \operatorname{gr}(N, N_i) \to \operatorname{gr}\operatorname{Coker} u \to 0$ *is exact.*

   (c) $\ker \operatorname{gr}(u) = \operatorname{gr}\ker(u)$ *and* $\operatorname{Coker}\operatorname{gr}(u) = \operatorname{gr}\operatorname{Coker}(u)$.

2. *If* u *is strict then we have also* $\operatorname{im}\operatorname{gr}(u) = \operatorname{gr}\operatorname{im}(u)$.

Proof. By applying the functor gr to the exact sequences 15.1.6.3.1, we get that $(a) \to (b)$. Conversely, suppose (b) satisfied. First, remark the following fact available in an abelian category $\mathfrak{A}$: let $\alpha\colon M_1 \twoheadrightarrow M_2$ (resp. $\beta\colon M_2 \to M_3$, resp. $\gamma\colon M_3 \hookrightarrow M_4$) be a epimorphism (resp. a morphism, resp. a monomorphism) of $\mathfrak{A}$. Then if $\ker\alpha = \ker \gamma\circ\beta\circ\alpha$ then β is a monomorphism. Moreover, if $\operatorname{Im}\gamma = \operatorname{Im}\gamma\circ\beta\circ\alpha$, then β is surjective. By applying the functor gr to the exact sequences 15.1.6.3.1, with this remark, the condition (b) implies that the morphism $\operatorname{gr}\operatorname{Coim}(u) \to \operatorname{gr}\operatorname{Im}(u)$ is an isomorphism of the abelian category of $\operatorname{gr} D$-modules. With Lemma 15.1.6.5.4, this implies that $\operatorname{Coim}(u) \to \operatorname{Im}(u)$ is an isomorphism.

The equivalence $(b) \Leftrightarrow (c)$ is straightforward. We check the statement 2) by applying gr to the exact sequences 15.1.6.3.1. □

Lemma 15.1.6.8. *Let* $u\colon (M, M_i) \to (N, N_i)$ *and* $v\colon (N, N_i) \to (O, O_i)$ *be two morphisms of good filtered* (D, D_i)*-modules.*

1. *If* v *is a strict monomorphism and* u *is strict then* $v \circ u$ *is strict.*

2. *If* u *is a strict epimorphism and* v *is strict then* $v \circ u$ *is strict.*

3. *If* $v \circ u$ *is strict epimorphism then* v *is a strict epimorphism.*

4. *If* $v \circ u$ *is strict monomorphism then* u *is a strict monomorphism.*

Proof. This can be checked elementarily from the characterization 15.1.6.5.1 For instance, let us check 1. Suppose v is a strict monomorphism and u is strict. We have $v(u(M)) \cap O_i \subset v(N) \cap O_i = v(N_i)$ (because v is strict). Hence, we get $v(u(M)) \cap O_i \subset v(u(M)) \cap v(N_i) = v(u(M) \cap N_i) = v(u(M_i))$ (to check the equalities, use respectively that v is a monomorphism and u is strict). This implies $v(u(M_i)) = v(u(M)) \cap O_i$. Let us check 4. If $v\circ u$ is strict monomorphism then u is a monomorphism and we have $M_i \subset M \cap u^{-1}(N_i) \subset M \cap (v\circ u)^{-1}(O_i) = M_i$. Hence, $M_i = M \cap u^{-1}(N_i)$, i.e. $u(M_i) = u(M) \cap N_i$. We leave the other statements to the reader. □

Remark 15.1.6.9. With the notation 15.1.6.8, this is not true in general that if u and v are strict then $v \circ u$ is also strict. However, using 15.1.6.8 and 15.1.6.3, we remark that a morphism of good filtered (D, D_i)-modules is strict if and only if it is the composition (in the category of good filtered (D, D_i)-modules) of a strict epimorphism with a strict monomorphism. This last characterization of strictness was Laumon's definition of strictness given in [Lau83, 1.0.1].

Proposition 15.1.6.10. With the definition 15.1.6.4 of strictness, the category of good filtered (D, D_i)-modules is exact (see the definition in [Lau83, 1.0.2]).

Proof. This is straighforward from previous Lemmas. For instance, the condition [Lau83, 1.0.2.(vi)] are the last two statements of 15.1.6.8. □

Notation 15.1.6.11 (Localisation)**.** Let f be a homogeneous element of $\operatorname{gr} D$. We denote by $(D_{[f]}, D_{[f],i})$ the complete filtered ring of (D, D_i) relatively to $S_1(f) := \{f^n\ ,\ n \in \mathbb{N}\} \subset \operatorname{gr} D$ (see the definition after [Lau85, Corollaire A.2.3.4]).

Let (M, M_i) be a good filtered (D, D_i)-module. We put

$$(M_{[f]}, M_{[f],i}) := (D_{[f]}, D_{[f],i}) \otimes_{(D,D_i)} (M, M_i), \tag{15.1.6.11.1}$$

the localized filtered module of (M, M_i) with respect to $S_1(f)$. We remind that $(M_{[f]}, M_{[f],i})$ is also a good filtered $(D_{[f]}, D_{[f],i})$-module (see [Lau85, A.2.3.6]) and $\operatorname{gr} M_{[f]} \xrightarrow{\sim} \operatorname{gr} D_{[f]} \otimes_{\operatorname{gr} D} \operatorname{gr} M$ (see [Lau85, A.1.1.3]).

The results and proofs of Malgrange in [Mal76, IV.4.2.3] (we can also find the proof in the book [HTT08, D.2.2]) can be extended without further problem in the context of complete filtered rings:

Lemma 15.1.6.12. *Let (M, M_i) be a good filtered (D, D_i)-module. Then there exists some free filtered (D, D_i)-modules of finite type $(L_n, L_{n,i})$ with $n \in \mathbb{N}$ and strict morphisms of good filtered (D, D_i)-modules $(L_{n+1}, L_{n+1,i}) \to (L_n, L_{n,i})$ and $(L_0, L_{0,i}) \to (M, M_i)$ such that $L_\bullet \to M$ is a resolution of M (in the category of D-modules).*

We call such a resolution $(L_\bullet, L_{\bullet,i})$ a "good resolution" of (M, M_i).

Proof. This is almost the same as [Mal76, IV.4.2.3.2]. For the reader, we remind the construction: with the remark 15.1.6.6, there exists a strict epimorphism of good filtered (D, D_i)-modules of the form $\phi_0 \colon (L_0, L_{0,i}) \to (M, M_i)$, with $(L_0, L_{0,i})$ a free filtered (D, D_i)-module of finite type. Let $(M_1, M_{1,i})$ be the kernel of ϕ_0 (in the category of good filtered (D, D_i)-modules: see 15.1.6.10). Since $(M_1, M_{1,i})$ is good, there exists a strict epimorphism of the form $\phi_1 \colon (L_1, L_{1,i}) \to (M_1, M_{1,i})$, with $(L_1, L_{1,i})$ a free filtered (D, D_i)-module of finite type. Hence, the morphism $(L_1, L_{1,i}) \to (L_0, L_{0,i})$ is strict. We go on similarly.

□

Remark 15.1.6.13. Let $(L_\bullet, L_{\bullet,i})$ be a good resolution of (M, M_i). Then $\mathrm{gr}(L_\bullet, L_{\bullet,i})$ is a resolution of $\mathrm{gr}(M, M_i)$ by free $\mathrm{gr}(D, D_i)$-modules of finite type (use the equivalence between 15.1.6.7.1.(a) and 15.1.6.7.1.(c)).

Lemma 15.1.6.14. *Let $K^\bullet$ be a complex of abelian groups. Let $(F_i K^\bullet)_{i \in \mathbb{Z}}$ be an increasing filtration of $K^\bullet$. We put*

$$F_i H^r(K^\bullet) := \mathrm{Im}(H^r(F_i K^\bullet) \to H^r(K^\bullet)). \tag{15.1.6.14.1}$$

Then $\mathrm{gr}_i^F(H^r(K^\bullet))$ is a subquotient of $H^r(\mathrm{gr}_i^F K^\bullet)$.

Proof. For instance, we can follow the last seven lines of the proof of [HTT08, D.2.4] (or also at Malgrange's description of the corresponding spectral sequence in [Mal76, IV.4.2.3.2]): denote by $d^r \colon K^r \to K^{r+1}$ the morphism in $K^\bullet$, $d_i^r \colon F_i K^r \to F_i K^{r+1}$ the morphism in $F_i K^\bullet$, $\overline{d}_i^r \colon \mathrm{gr}_i^F K^r = F_i K^r / F_{i-1} K^r \to F_i K^{r+1} / F_{i-1} K^{r+1} = \mathrm{gr}_i^F K^{r+1}$. Since $\ker d_i^r = \ker d^r \cap F_i K^r$, we get $F_i H^r(K^\bullet) = \ker d^r \cap F_i K^r + \mathrm{Im}\, d^{r-1} / \mathrm{Im}\, d^{r-1}$. By definition we obtain:

$$\mathrm{gr}_i^F(H^r(K^\bullet)) := F_i H^r(K^\bullet)/F_{i-1} H^r(K^\bullet) = \ker d^r \cap F_i K^r + \mathrm{Im}\, d^{r-1} / \ker d^r \cap F_{i-1} K^r + \mathrm{Im}\, d^{r-1}. \tag{15.1.6.14.2}$$

We have $\ker \overline{d}_i^r = \ker(F_i K^r \to \mathrm{gr}_i^F K^{r+1})/F_{i-1} K^r$ and $\mathrm{Im}\, \overline{d}_{i-1}^r = d_i^{r-1}(F_i K^{r-1}) + F_{i-1} K^r / F_{i-1} K^r$. Hence

$$H^r(\mathrm{gr}_i^F K^\bullet) := \ker \overline{d}_i^r / \mathrm{Im}\, \overline{d}_{i-1}^r = \ker(F_i K^r \to \mathrm{gr}_i^F K^{r+1})/d_i^{r-1}(F_i K^{r-1}) + F_{i-1} K^r. \tag{15.1.6.14.3}$$

Set $L = \ker d^r \cap F_i K^r / d_i^{r-1}(F_i K^{r-1}) + \ker d^r \cap F_{i-1} K^r$. The inclusion $\ker d^r \cap F_i K^r \subset \ker(F_i K^r \to \mathrm{gr}_i^F K^{r+1})$ induces the map $\phi \colon \ker d^r \cap F_i K^r \to \ker(F_i K^r \to \mathrm{gr}_i^F K^{r+1})/d_i^{r-1}(F_i K^{r-1}) + F_{i-1} K^r$. Let $x \in \ker d^r \cap F_i K^r$ be an element in the kernel of ϕ. Then there exist $y \in F_i K^{r-1}$ and $z \in F_{i-1} K^r$ such that $x = d_i^{r-1}(y) + z$. Since $d_i^{r-1}(y) \in \ker d^r$, we get $z \in \ker d^r$ and then $z \in \ker d^r \cap F_{i-1} K^r$. Hence $\ker \phi \subset d_i^{r-1}(F_i K^{r-1}) + \ker d^r \cap F_{i-1} K^r$. Since the converse is obvious, we get $\ker \phi = d_i^{r-1}(F_i K^{r-1}) + \ker d^r \cap F_{i-1} K^r$. From 15.1.6.14.3, this yields that L is a subobject of $H^r(\mathrm{gr}_i^F K^\bullet)$. Moreover, we have the epimorphism $\ker d^r \cap F_i K^r \to \ker d^r \cap F_i K^r + \mathrm{Im}\, d^{r-1} / \ker d^r \cap F_{i-1} K^r + \mathrm{Im}\, d^{r-1}$. Since $d_i^{r-1}(F_i K^{r-1}) + \ker d^r \cap F_{i-1} K^r$ is in the kernel of this map, we get the factorisation $L \to \ker d^r \cap F_i K^r + \mathrm{Im}\, d^{r-1} / \ker d^r \cap F_{i-1} K^r + \mathrm{Im}\, d^{r-1}$, which is still an epimorphism. From 15.1.6.14.2, this implies that L is a quotient of $\mathrm{gr}_i^F(H^r(K^\bullet))$. Hence, $\mathrm{gr}_i^F(H^r(K^\bullet))$ is a quotient of a submodule of $H^r(\mathrm{gr}_i^F K^\bullet)$.
□

15.1.6.15. Let (M, M_i) and (N, N_i) be two filtered (D, D_i)-modules. For any integer $i \in \mathbb{Z}$, let $F_i \mathrm{Hom}_D(M, N)$ be the subgroup of $Hom_D(M, N)$ of the elements ϕ such that, for any integer $j \in \mathbb{Z}$, $\phi(M_j) \subset N_{i+j}$. For any integer $j \in \mathbb{Z}$, for any $\phi \in F_i \mathrm{Hom}_D(M, N)$, we get a morphism $\mathrm{gr}_i M \to \mathrm{gr}_{i+j} N$ defined by sending the class of a element $x \in M_i$ to the class of $\phi(x)$. Hence ϕ induces a map $\mathrm{gr}\, M \to \mathrm{gr}\, N$, which is in fact gr_D-linear. Hence we get the canonical morphism $F_i \mathrm{Hom}_D(M, N) \to \mathrm{Hom}_{\mathrm{gr}\, D}(\mathrm{gr}\, M, \mathrm{gr}\, N)$ and then

$$\mathrm{gr}^F \mathrm{Hom}_D(M, N) \to \mathrm{Hom}_{\mathrm{gr}\, D}(\mathrm{gr}\, M, \mathrm{gr}\, N). \tag{15.1.6.15.1}$$

Lemma 15.1.6.16. *Let (L, L_i) be a free filtered (D, D_i)-module of finite type. Let (N, N_i) be filtered (D, D_i)-module. The canonical morphism*

$$\operatorname{gr}\operatorname{Hom}_D(L, N) \to \operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} L, \operatorname{gr} N)$$

of 15.1.6.15.1 is an isomorphism of abelian groups.

Proof. Let (M, M_i), (M', M'_i) be two filtered (D, D_i)-modules and $(M'', M''_i) := (M, M_i) \oplus (M', M'_i)$. Then the morphism of 15.1.6.15.1 $\operatorname{gr}^F \operatorname{Hom}_D(M'', N) \to \operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} M'', \operatorname{gr} N)$ is an isomorphism if and only if so are $\operatorname{gr}^F \operatorname{Hom}_D(M, N) \to \operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} M, \operatorname{gr} N)$ and $\operatorname{gr}^F \operatorname{Hom}_D(M', N) \to \operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} M', \operatorname{gr} N)$. Hence, we can suppose that $(L, L_i) = (D(n), D(n)_i)$, for some integer n. Twisting the filtrations, we can suppose $n = 0$. Finally, we compute that the morphism of 15.1.6.15.1 $\operatorname{gr}\operatorname{Hom}_D(D, N) \to \operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} D, \operatorname{gr} N)$ is, modulo the identifications $N = \operatorname{Hom}_D(D, N)$ and $\operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} D, \operatorname{gr} N) = \operatorname{gr} N$, the identity, which is an isomorphism. □

Proposition 15.1.6.17. Let (M, M_i) be a good filtered (D, D_i)-module. Let (N, N_i) be a filtered (D, D_i)-module. For any integer r, there exists a filtration F of $\operatorname{Ext}^r_D(M, N)$ satisfying the following properties

1. $\operatorname{Ext}^r_D(M, N) = \cup_{i \in \mathbb{Z}} F_i \operatorname{Ext}^r_D(M, N)$,
2. $\operatorname{gr}^F \operatorname{Ext}^r_D(M, N)$ is a subquotient of $\operatorname{Ext}^r_{\operatorname{gr} D}(\operatorname{gr} M, \operatorname{gr} N)$.
3. Suppose $(N, N_i) = (D, D_i)$. The filtration $(F_i \operatorname{Ext}^r_D(M, D))_{i \in \mathbb{Z}}$ of the right D-module $\operatorname{Ext}^r_D(M, D)$ is a good filtration. In particular, $0 = \cap_{i \in \mathbb{Z}} F_i \operatorname{Ext}^r_D(M, D)$. Moreover, we have the implication

$$\operatorname{Ext}^r_{\operatorname{gr} D}(\operatorname{gr} M, \operatorname{gr} D) = 0 \Rightarrow \operatorname{Ext}^r_D(M, D) = 0.$$

Proof. From 15.1.6.12 and with its definition, there exists a good resolution $(L_\bullet, L_{\bullet,i})$ of (M, M_i). We put $K^\bullet := \operatorname{Hom}_D(L_\bullet, N)$. Since $L_\bullet$ is a resolution of M by projective D-modules, we get $H^r(K^\bullet) = \operatorname{Ext}^r_D(M, N)$.

Let $F_i K^n$ be the subset of the elements ϕ of $Hom_D(L_n, N)$ such that, for any integer $j \in \mathbb{Z}$, $\phi(L_{n,j}) \subset N_{i+j}$. Since L_n is a D-module of finite type, we get $\cup_{i \in \mathbb{Z}} F_i K^n = K^n$. With the induced filtration on $H^r(K^\bullet) = \operatorname{Ext}^r_D(M, N)$ (see 15.1.6.14), this yields the first property. Since L_n is a free filtered (D, D_i)-modules of finite type, from Lemma 15.1.6.16, the canonical morphism $\operatorname{gr} K^n \to \operatorname{Hom}_{\operatorname{gr} D}(\operatorname{gr} L_n, \operatorname{gr} N)$ is an isomorphism. Since $\operatorname{gr} L_\bullet$ is a resolution of $\operatorname{gr} M$ by projective $\operatorname{gr} D$-modules, we get $H^r(\operatorname{gr} K^\bullet) = \operatorname{Ext}^r_{\operatorname{gr} D}(\operatorname{gr} M, \operatorname{gr} N)$. This implies the second point by using Lemma 15.1.6.14.

When $(N, N_i) = (D, D_i)$, the filtration $F_i K^n$ of the right D-module K^n is a good filtration. We denote by $d_n \colon K^n \to K^{n+1}$ the canonical morphisms. From [Lau85, A.1.1.2], the induced filtrations on $\ker d_n$ and next on $\ker d_n / Imd_{n-1}$ (induced from the surjection $\ker d_n \to \ker d_n / Imd_{n-1}$) are good. We notice that this filtration on $\ker d_n / \operatorname{Im} d_{n-1} = H^n(K^\bullet)$ is the same as that defined at 15.1.6.14.1, which is the first assertion of the third point. With the second point, this yields the rest of the third point. □

15.1.7 A criterium on the purity of the characteristic variety of a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-module

In the subsection, we prove that when $\mathcal{E}$ is holonomic, its characteristic variety $\operatorname{Car}(\mathcal{E})$ is of pure dimension $\dim X$. One main ingredient of the proof is to use the homological characterization of the holonomicity (see 15.2.4.8) and another one is to use the sheaf of microdifferential operators (for instance, see [Abe14b]). Both ideas comes from the original proof of Kashiwara of the analogous property in the theory of analytic $\mathcal{D}$-modules (see [Kas77]).

Let $\mathcal{V}$ be a complete discrete valued ring of mixed characteristic $(0, p)$, π be a uniformizer, K its field of fractions, k its residue field which is supposed to be perfect. A k-variety is a separated reduced scheme of finite type over k.

Lemme 15.1.7.1. Let X be an affine smooth variety over k, $\overline{D} := \Gamma(X, \mathcal{D}^{(0)}_{X/k})$, $(\overline{D}_i)_{i \in \mathbb{N}}$ be its order filtration of the operators, $\overline{f}$ be an homogeneous element of $\operatorname{gr} \overline{D}$. Let $(\overline{M}, \overline{M}_i)$ and $(\overline{N}, \overline{N}_i)$ be two good filtered $(\overline{D}_{[\overline{f}]}, \overline{D}_{[\overline{f}],i})$-modules and r be an integer.

1. We have $\operatorname{codim} \operatorname{Ext}^r_{\operatorname{gr} \overline{D}_{[\overline{f}]}}(\operatorname{gr} \overline{M}, \operatorname{gr} \overline{N}) \geq r$.

2. If $r < \operatorname{codim} \operatorname{gr} \overline{M}$ then $\operatorname{Ext}^r_{\operatorname{gr} \overline{D}_{[\overline{f}]}}(\operatorname{gr} \overline{N}, \operatorname{gr} \overline{D}_{[\overline{f}]}) = 0$.

Proof. By construction (see [Lau85, A.2]), we get $\operatorname{gr} \overline{D}_{[\overline{f}]} \xrightarrow{\sim} (\operatorname{gr} \overline{D})_{\overline{f}}$. Then, this is well known (e.g. see [HTT08, D.4.4]). □

15.1.7.2 (Localisation and π-adic completion)**.** Let $\mathfrak{P}$ be an affine smooth $\mathcal{V}$-formal scheme, X_n be the reduction of $\mathfrak{P}$ modulo π^{n+1}. We put $D := \Gamma(X, \mathcal{D}^{(0)}_{\mathfrak{P}/\mathfrak{S}})$ and $D_n := \Gamma(X, \mathcal{D}^{(0)}_{X_n/S_n})$. These rings are canonically filtered by the order of the differential operators ; we denote by (D, D_i) and $(D_n, D_{n,i})$ the (ind-pro) complete filtered rings. Let f be an homogeneous element of $\operatorname{gr} D$ and f_n be its image in $\operatorname{gr} D_n$. With the notation of 15.1.6.11, by using the same arguments as in the proof of [Abe14b, 2.3], we get the canonical isomorphism of (ind-pro) complete filtered ring

$$(D_{[f]}, D_{[f],i}) \otimes_{\mathcal{V}} \mathcal{V}/\pi^{n+1} \xrightarrow{\sim} (D_{n[f_n]}, D_{n[f_n],i}). \tag{15.1.7.2.1}$$

We put $\widehat{D}_{[f]}$ (be careful that this notation is slightly different from that of 15.1.6.11) as the π-adic completion of $D_{[f]}$, i.e. $\widehat{D}_{[f]} := \varprojlim_n D_{[f]}/\pi^{n+1} D_{[f]} \xrightarrow{\sim} \varprojlim_n D_{n,[f_n]}$. Using Corollary [Lau85, A.1.1.1] and 7.2.1.2.(c), we get from the isomorphism 15.1.7.2.1 the noetherianity of $\widehat{D}_{[f]}$.

Finally, when there is no confusion with the notation 15.1.6.11, for any coherent $\widehat{D}$-module (resp. coherent $\widehat{D}_{\mathbb{Q}}$-module) M (resp. N), we set (by default in this new context) $M_{[f]} := \widehat{D}_{[f]} \otimes_{\widehat{D}} M$ (resp. $N_{[f]} := \widehat{D}_{[f]} \otimes_{\widehat{D}} N$).

Lemme 15.1.7.3. With the notation of 15.1.7.2, the homomorphism $\widehat{D} \to \widehat{D}_{[f]}$ is flat.

Proof. This is a consequence of 7.2.1.3.(g), [Lau85, A.2.3.4.(ii)] and 15.1.7.2.1. □

Remark 15.1.7.4. With the notation of 15.1.7.2, let M be a coherent $\widehat{D}$-module. We put $M_n := M/\pi^{n+1}M$. From 4.1.3.17, there exists a good filtration $(M_{n,i})_{i \in \mathbb{N}}$ of M_n indexed by $\mathbb{N}$. We recall (see notation 15.1.6.11.1) that we get a good filtered $(D_{n,[f_n]}, D_{n,[f_n],i})$-module by putting $(M_{n,[f_n]}, M_{n,[f_n],i}) := (D_{n,[f_n]}, D_{n,[f_n],i}) \otimes_{(D_n, D_{n,i})} (M_n, M_{n,i})$. Moreover, since $\widehat{D}_{[f]}/\pi^{n+1}\widehat{D}_{[f]} \xrightarrow{\sim} D_{n,[f_n]}$ (use 15.1.7.2.1), then

$$M_{[f]}/\pi^{n+1} M_{[f]} \xrightarrow{\sim} M_{n,[f_n]}. \tag{15.1.7.4.1}$$

From 7.2.1.4, since $M_{[f]}$ is a $\widehat{D}_{[f]}$-module of finite type, then $M_{[f]}$ is complete for the π-adic topology. Hence, using 15.1.7.4.1 we get the canonical isomorphism of $\widehat{D}_{[f]}$-modules$M_{[f]} \xrightarrow{\sim} \varprojlim_n M_{n,[f_n]}$.

Lemme 15.1.7.5. We keep notation 15.1.7.2. Let $\mathcal{N}$ be a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-module, $\operatorname{Car}^{(0)}(\mathcal{N})$ its characteristic variety of level 0 (see the definition in 15.1.4.3). We put $N := \Gamma(X, \mathcal{N})$. The following assertions are equivalent

1. $D(f_0) \cap \operatorname{Car}^{(0)}(\mathcal{N}) = \emptyset$.

2. $N_{[f]} = 0$.

Proof. From 7.4.5.1 and 7.4.5.2, there exits a p-torsion free coherent $\widehat{D}$-module M such that $M_{\mathbb{Q}} \xrightarrow{\sim} N$. Since the extension $\widehat{D} \to \widehat{D}_{[f]}$ is flat (see 15.1.7.3), we get that $M_{[f]}$ is also p-torsion free (p is in the center of $\widehat{D}$ and $\widehat{D}_{[f]}$). This yields that $N_{[f]} = 0$ if and only if $M_{[f]} = 0$. Let $\overline{M} := M/\pi M$. From 15.1.7.4.1, we have $M_{[f]}/\pi M_{[f]} \xrightarrow{\sim} \overline{M}_{[f_0]}$. Hence, $M_{[f]} = 0$ if and only if $\overline{M}_{[f_0]} = 0$ (7.2.1.2.(c)). Following 4.1.3.17, there exists a good filtration $(\overline{M}_i)_{i \in \mathbb{N}}$ of $\overline{M}$ indexed by $\mathbb{N}$. From the remark 15.1.7.4, this induces canonically the (ind-pro) complete $(\overline{D}_{[f_0]}, \overline{D}_{[f_0],i})$-module $(\overline{M}_{[f_0]}, \overline{M}_{[f_0],i})$. Since $\overline{M}_{[f_0]}$ is (ind-pro) complete, then the equalities $\overline{M}_{[f_0]} = 0$ and $\operatorname{gr}(\overline{M}_{[f_0]}, \overline{M}_{[f_0],i})) = 0$ are equivalent. Also, $(\operatorname{gr} \overline{M})_{f_0} = 0$ if and only if $D(f_0) \cap \operatorname{Supp}(\operatorname{gr} \overline{M}) = \emptyset$. Since $(\operatorname{gr} \overline{M})_{f_0} \xrightarrow{\sim} \operatorname{gr}(\overline{M}_{[f_0]})$ (see [Lau85, A.1.1.3]) and since by definition $\operatorname{Car}^{(0)}(\mathcal{N}) = \operatorname{Supp}(\operatorname{gr}(\overline{M}, \overline{M}_i))$, we conclude the proof. □

Remark 15.1.7.6. Let $A = \oplus_{i\in\mathbb{N}} A_i$ be a graded ring. Let I be a graded ideal. Let $a_1, \ldots, a_r$ be some homogeneous generators of I. We notice that $|\operatorname{Spec} A| \setminus V(I) = \cup_{i=1}^r D(a_i)$.

The following proposition is the analogue of [Kas77, 2.11]:

Proposition 15.1.7.7. Let $\mathfrak{P}$ be a smooth $\mathcal{V}$-formal scheme. Let $\mathcal{N}$ be a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-module and V be an irreducible component of codimension r of $\operatorname{Car}^{(0)}(\mathcal{N})$, the characteristic variety of level 0 of $\mathcal{N}$ (see 15.1.4.3). Then, $\operatorname{Car}^{(0)}(\mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}))$ contains V.

Proof. We follow the proof of [Kas77, 2.11]: first, we can suppose $\mathfrak{P}$ affine with local coordinates. We set $D := \Gamma(\mathfrak{P}, \mathcal{D}^{(0)}_{\mathfrak{P}/\mathfrak{S}})$, $N := \Gamma(X, \mathcal{N})$, $\overline{D} := \Gamma(X, \mathcal{D}^{(0)}_{X/S})$. Let M be a p-torsion free coherent $\widehat{D}$-module such that $M_{\mathbb{Q}} \xrightarrow{\sim} N$. Let $\overline{M} := M/\pi M$. Following 4.1.3.17, there exists a good filtration $(\overline{M}_i)_{i\in\mathbb{N}}$ of $\overline{M}$ indexed by $\mathbb{N}$. By definition, we have $\operatorname{Car}^{(0)}(\mathcal{N}) = \operatorname{Supp}(\operatorname{gr}(\overline{M}, \overline{M}_i))$ (we recall that this is independent on the choice of the good filtration). Let η be the generic point of V. From [Mat80, 7.D and 10.B.i)], the irreducible components of $\operatorname{Supp}(\operatorname{gr} \overline{M})$ are of the form $V(J)$ with J a homogeneous ideal. Let Z be the union of the irreducible components of $\operatorname{Supp}(\operatorname{gr} \overline{M})$ which do not contain η. Then, we get from the remark 15.1.7.6 that there exists a homegeneous element $f \in \operatorname{gr} D$ such that $\eta \in D(f_0)$ and $D(f_0) \cap Z = \emptyset$ (in other words, $D(f_0) \cap \operatorname{Car}^{(0)}(\mathcal{N}) = D(f_0) \cap V \neq \emptyset$).

Now, suppose absurdly that $\eta \notin \operatorname{Car}^{(0)}(\mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}))$. Using the same arguments as above, there exists a homogeneous element $g \in \operatorname{gr} D$ such that $\eta \in D(g_0)$ and $D(g_0) \cap \operatorname{Car}^{(0)}(\mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})) = \emptyset$. We put $h = fg$. Hence, we have $\eta \in D(h_0)$ and $D(h_0) \cap \operatorname{Car}^{(0)}(\mathcal{N}) = D(h_0) \cap V$ and $D(h_0) \cap \operatorname{Car}^{(0)}(\mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})) = \emptyset$.

1) Since $\mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})$ is a coherent (right) $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-module, from Theorem A and B of 7.2.3.16, we get the equality $\Gamma(\mathfrak{P}, \mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})) = \operatorname{Ext}^r_{\widehat{D}_{\mathbb{Q}}}(N, \widehat{D}_{\mathbb{Q}})$. From 15.1.7.5, this implies $\operatorname{Ext}^r_{\widehat{D}_{\mathbb{Q}}}(N, \widehat{D}_{\mathbb{Q}}) \otimes_{\widehat{D}_{\mathbb{Q}}} \widehat{D}_{[h],\mathbb{Q}} = 0$. Since the extension $\widehat{D}_{\mathbb{Q}} \to \widehat{D}_{[h],\mathbb{Q}}$ is flat (see 15.1.7.3), we get $\operatorname{Ext}^r_{\widehat{D}_{[h],\mathbb{Q}}}(N_{[h]}, \widehat{D}_{[h],\mathbb{Q}}) \xrightarrow{\sim} \operatorname{Ext}^r_{\widehat{D}_{\mathbb{Q}}}(N, \widehat{D}_{\mathbb{Q}}) \otimes_{\widehat{D}_{\mathbb{Q}}} \widehat{D}_{[h],\mathbb{Q}} = 0$.

2) a) Since $\operatorname{Car}^{(0)}(\mathcal{N}) = \operatorname{Supp}(\operatorname{gr} \overline{M})$, then $D(h_0) \cap \operatorname{Car}^{(0)}(\mathcal{N}) = \operatorname{Supp}(\operatorname{gr} \overline{M})_{h_0})$. Since we have also $D(h_0) \cap \operatorname{Car}^{(0)}(\mathcal{N}) = D(h_0) \cap V$, then in particular we get $\operatorname{Codim}(\operatorname{gr} \overline{M})_{h_0} = r$. Since $(\operatorname{gr} \overline{M})_{h_0} = \operatorname{gr}(\overline{M}_{[h_0]})$, then from 15.1.7.1 for any $i < r$ we obtain $\operatorname{Ext}^i_{\operatorname{gr} \overline{D}_{[h_0]}}(\operatorname{gr}(\overline{M}_{[h_0]}), \operatorname{gr} \overline{D}_{[h_0]}) = 0$. From 15.1.6.17.3, this yields that for $i < r$, $\operatorname{Ext}^i_{\overline{D}_{[h_0]}}(\overline{M}_{[h_0]}, \overline{D}_{[h_0]}) = 0$. On the other hand, from 15.1.7.1 we get for any $i > r$ the inequality $\operatorname{Codim}(\operatorname{Ext}^i_{\operatorname{gr} \overline{D}_{[h_0]}}(\operatorname{gr}(\overline{M}_{[h_0]}), \operatorname{gr} \overline{D}_{[h_0]})) > r$. Hence, by reducing $D(h_0)$ if necessary (use again the remark 15.1.7.6), for any $i > r$ we get $\operatorname{Ext}^i_{\operatorname{gr} \overline{D}_{[h_0]}}(\operatorname{gr}(\overline{M}_{[h_0]}), \operatorname{gr} \overline{D}_{[h_0]}) = 0$ and then $\operatorname{Ext}^i_{\overline{D}_{[h_0]}}(\overline{M}_{[h_0]}, \overline{D}_{[h_0]}) = 0$. To sum up, we have found an homogeneous element $h \in \operatorname{gr} D$ such that $\eta \in D(h_0)$ and for $i \neq r$, $\operatorname{Ext}^i_{\overline{D}_{[h_0]}}(\overline{M}_{[h_0]}, \overline{D}_{[h_0]}) = 0$.

2) b) Now, since $M_{[h]}$ is p-torsion free, $\mathbb{R}\operatorname{Hom}_{\widehat{D}_{[h]}}(M_{[h]}, \widehat{D}_{[h]}) \otimes^{\mathbb{L}}_{\widehat{D}_{[h]}} \overline{D}_{[h_0]} \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\overline{D}_{[h_0]}}(\overline{M}_{[h_0]}, \overline{D}_{[h_0]})$. From the exact sequence of universal coefficients (e.g. see the proof of 15.2.1.1), we get the inclusion $\operatorname{Ext}^i_{\widehat{D}_{[h]}}(M_{[h]}, \widehat{D}_{[h]}) \otimes_{\widehat{D}_{[h]}} \overline{D}_{[h_0]} \hookrightarrow \operatorname{Ext}^i_{\overline{D}_{[h_0]}}(\overline{M}_{[h_0]}, \overline{D}_{[h_0]})$. Hence, for any $i \neq r$, from the step 2) a) of the proof, we obtain the vanishing $\operatorname{Ext}^i_{\widehat{D}_{[h]}}(M_{[h]}, \widehat{D}_{[h]}) \otimes_{\widehat{D}_{[h]}} \overline{D}_{[h_0]} = 0$. By using 7.2.1.2.(b), since $\operatorname{Ext}^i_{\widehat{D}_{[h]}}(M_{[h]}, \widehat{D}_{[h]})$ is a coherent $\widehat{D}_{[h]}$-module, for $i \neq r$ we get $\operatorname{Ext}^i_{\widehat{D}_{[h]}}(M_{[h]}, \widehat{D}_{[h]}) = 0$ and then $\operatorname{Ext}^i_{\widehat{D}_{[h],\mathbb{Q}}}(N_{[h]}, \widehat{D}_{[h],\mathbb{Q}}) = 0$ (because $\widehat{D}_{[h]} \to \widehat{D}_{[h],\mathbb{Q}}$ is flat).

3) From steps 1) and 2), we have checked that $\mathbb{R}\operatorname{Hom}_{\widehat{D}_{[h],\mathbb{Q}}}(N_{[h]}, \widehat{D}_{[h],\mathbb{Q}}) = 0$. By using the biduality isomorphism (see 4.6.4.6 and notice that $N_{[h]}$ is a perfect complex because so is N and because the extension $\widehat{D}_{\mathbb{Q}} \to \widehat{D}_{[h],\mathbb{Q}}$ is flat), we get $N_{[h]} = 0$, which is absurd following Lemma 15.1.7.5 because $\eta \in D(h_0)$. □

Theorem 15.1.7.8. Let $\mathfrak{P}$ be a smooth $\mathcal{V}$-formal scheme. Let r be an integer, $\mathcal{N}$ be a coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-module such that $\mathcal{E}xt^s_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}) = 0$ for any $s \neq r$. Then, the characteristic variety $\mathrm{Car}^{(0)}(\mathcal{N})$ of $\mathcal{N}$ is purely of codimension r.

Proof. If V is an irreducible component of $\mathrm{Car}^{(0)}(\mathcal{N})$ of codimension s, then from 15.1.7.7 we get $\mathcal{E}xt^s_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{N}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}) \neq 0$ since it contains V. Hence $s = r$. □

15.1.7.9. When $\mathcal{E}$ is a log extendable overconvergent F-isocrystal, we establish the inclusion of $\mathrm{Car}(\mathcal{E})$ into a explicit lagrangian subvariety of the cotangent space of X. With the above purity theorem, this inclusion implies the Lagrangianity of $\mathrm{Car}(\mathcal{E})$. Moreover, one another application of this inclusion in a further work will be to get some "relative generic $\mathcal{O}$-coherence" (see precisely the proof of Theorem [Car19a, 1.4.3]). This implies some Betti number estimates (see [Car19a]). In the theory of arithmetic $\mathcal{D}$-modules, we recall that to check some property we are often able to reduce to the case of log extendable overconvergent F-isocrystals (e.g. in the proof of Theorem [Car19a, 1.4.3]). Indeed, overholonomic F-complexes of arithmetic $\mathcal{D}$-modules are devissable in overconvergent F-isocrystals (see later 18.3.2.3) and thanks to Kedlaya's semistable reduction theorem any overconvergent F-isocrystal becomes log-extendable after the pullback by some generically etale alteration (see [Ked11]).

15.2 Holonomic $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module

Let $\mathfrak{P}$ be a smooth $\mathfrak{S}$-formal scheme.

15.2.1 Dimension of level m and vanishing of $\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(-,-)$

Lemma 15.2.1.1 (Universal coefficient exact sequence)**.** *Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}})$ and $\mathcal{E}_0 := \mathcal{D}^{(m)}_P \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} \mathcal{E}$. For any $n \in \mathbb{N}$, we have the exact sequence:*

$$0 \to \mathcal{E}xt^n_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} \mathcal{D}^{(m)}_P \to \mathcal{E}xt^n_{\mathcal{D}^{(m)}_P}(\mathcal{E}_0, \mathcal{D}^{(m)}_P) \to \mathcal{T}or^1_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(\mathcal{E}xt^{n+1}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}), \mathcal{D}^{(m)}_P) \to 0. \tag{15.2.1.1.1}$$

Proof. The second spectral sequence of the functor $-\otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} \mathcal{D}^{(m)}_P$ applied to the complex $\mathcal{F} := \mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}})$ is:

$$E_2^{r,s} = \mathcal{T}or^{-r}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(H^s\mathcal{F}, \mathcal{D}^{(m)}_P) \Rightarrow H^{r+s}(\mathcal{F} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} \mathcal{D}^{(m)}_P).$$

Since dual functors commute with extensions (see 4.6.4.7.1), then we get the isomorphism $\mathcal{F} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} \mathcal{D}^{(m)}_P \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_P}(\mathcal{E}_0, \mathcal{D}^{(m)}_P)$. This yields the spectral sequence:

$$E_2^{r,s} = \mathcal{T}or^{-r}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(\mathcal{E}xt^s_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}), \mathcal{D}^{(m)}_P) \Rightarrow \mathcal{E}xt^n_{\mathcal{D}^{(m)}_P}(\mathcal{E}_0, \mathcal{D}^{(m)}_P).$$

Since $E_2^{r,s} = 0$ except for $r = 0$ and $r = -1$, then we are done.

□

Lemma 15.2.1.2. *Suppose $\mathfrak{P}$ affine. Let $\mathcal{E}$ be a coherent left $\mathcal{D}^{(0)}_{P/S}$-module. We endow the left $\mathcal{D}^{(0)}_{P/S}$-module $\mathcal{D}^{(0)}_{P/S}$ with its canonical order filtration which is a good filtration. Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a good filtration of $\mathcal{E}$. Let $i \in \mathbb{N}$. There is a good filtration of $\mathcal{E}xt^i_{\mathcal{D}^{(0)}_{P/S}}(\mathcal{E}, \mathcal{D}^{(0)}_{P/S})$, such that $\mathrm{gr}\,\mathcal{E}xt^i_{\mathcal{D}^{(0)}_{P/S}}(\mathcal{E}, \mathcal{D}^{(0)}_{P/S})$ is a $\mathrm{gr}\,\mathcal{D}^{(0)}_{P/S}$-subquotient of $\mathcal{E}xt^i_{\mathrm{gr}\,\mathcal{D}^{(0)}_{P/S}}(\mathrm{gr}\,\mathcal{E}, \mathrm{gr}\,\mathcal{D}^{(0)}_{P/S})$.*

Proof. By using the theorem of type A for quasi-coherent $\mathcal{D}^{(0)}_{P/S}$-modules, this is a consequence of 15.1.6.17. □

Lemma 15.2.1.3. *Let $\mathcal{E}, \mathcal{F}$ be two coherent left $\mathcal{D}^{(0)}_{P/S}$-modules. Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ (resp. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a good filtration of $\mathcal{E}$ (resp. $\mathcal{F}$). Hence:*

(a) $\operatorname{codim}(\mathcal{E}xt^i_{\operatorname{gr}\mathcal{D}^{(0)}_{P/S}}(\operatorname{gr}\mathcal{E}, \operatorname{gr}\mathcal{D}^{(0)}_{P/S})) \geq i$*, for any* $i \geq 0$*.*

(b) $\mathcal{E}xt^i_{\operatorname{gr}\mathcal{D}^{(0)}_{P/S}}(\operatorname{gr}\mathcal{E}, \operatorname{gr}\mathcal{D}^{(0)}_{P/S}) = 0$*, for any* $i < \operatorname{codim}^{(0)}(\mathcal{E})$*.*

Proof. Since this is local in $\mathfrak{P}$, we can suppose $\mathfrak{P}$ is affine. Via the theorem of type A for quasi-coherent $\mathcal{D}^{(0)}_{P/S}$-modules, this is a consequence of 15.1.7.1. □

Corollary 15.2.1.4. *Let $\mathcal{E}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}$-module. Hence:*

(a) $\operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})) \geq i$*, for any* $i \geq 0$*.*

(b) $\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}) = 0$*, for any* $i < \operatorname{codim}^{(0)}(\mathcal{E})$*.*

Proof. Let $i \in \mathbb{N}$. Since $\mathcal{E}$ is p-torsion free, we have

$$\mathcal{E}_0 := \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}} \mathcal{E} \xrightarrow{\sim} \mathcal{D}^{(0)}_{P} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}} \mathcal{E}.$$

By definition (see 15.1.4.1),

$$\operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})) = \operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}) \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}} \mathcal{D}^{(0)}_{P}).$$

By using the injective morphism of the exact sequence 15.2.1.1.1 and the formula 15.1.3.5, this yields

$$\operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})) \geq \operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\mathcal{D}^{(0)}_{P}}(\mathcal{E}_0, \mathcal{D}^{(0)}_{P})).$$

It follows from 15.2.1.2 (resp. 15.2.1.3.(a)) the first (resp. second) inequality:

$$\operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\mathcal{D}^{(0)}_{P}}(\mathcal{E}_0, \mathcal{D}^{(0)}_{P})) \geq \operatorname{codim}(\mathcal{E}xt^i_{\operatorname{gr}\mathcal{D}^{(0)}_{P/S}}(\operatorname{gr}\mathcal{E}_0, \operatorname{gr}\mathcal{D}^{(0)}_{P/S})) \geq i,$$

which yields (a).

Suppose $i < \operatorname{codim}^{(0)}(\mathcal{E})$. Following 15.2.1.2 and 15.2.1.3.(b), we have: $\mathcal{E}xt^i_{\mathcal{D}^{(0)}_{P}}(\mathcal{E}_0, \mathcal{D}^{(0)}_{P}) = 0$. By using the injective morphism of 15.2.1.1.1, this yields $\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})) = 0$. □

Corollary 15.2.1.5. *Let $\mathcal{E}^{(m)}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. Hence:*

(a) $\operatorname{codim}^{(m)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})) \geq i$*, for any* $i \geq 0$*.*

(b) $\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) = 0$*, for any* $i < \operatorname{codim}^{(m)}(\mathcal{E}^{(m)})$*.*

Proof. 1) Suppose $m = 0$. Choose a p-torsion free coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}$-module $\mathcal{F}$ such that there exists an isomorphism of $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}$-modules of the form $\mathcal{F}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(0)}$ (see 7.5.2.8). Let $\mathcal{N}$ be the quotient of $\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{F}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})$ by its p-torsion part. Since $\mathcal{N}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})$ then we get the first equality by definition (see 15.1.4.3):

$$\operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})) = \operatorname{codim}^{(0)}(\mathcal{N}).$$

Since $\mathcal{N}$ is a quotient of $\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{F}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})$, since the functor $\mathcal{F} \mapsto \mathcal{F}/\pi\mathcal{F}$ is right exact then we get from 15.1.3.5:

$$\operatorname{codim}^{(0)}(\mathcal{N}) \geq \operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}})),$$

Hence, $\operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P},\mathbb{Q}})) \geq \operatorname{codim}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}}(\mathcal{E}^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}}))$. By using the example 15.1.4.4 and 15.2.1.4, we are done.

2) Let us go back to the general case. Since this is local in $\mathfrak{P}$, we can suppose there exists $F\colon \mathfrak{P} \to \mathfrak{P}'$ a morphism of smooth $\mathfrak{S}$-schemes lifting $F^m_{P_0/S_0}$. Let $\mathcal{E}'^{(0)} := F^\flat \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}$ be the corresponding left $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}}$-module. Following 8.8.1.3, we have the isomorphism: $F^*\mathcal{E}'^{(0)} \xrightarrow{\sim} \mathcal{E}^{(m)}$. Hence, it follows from 11.3.2.1.1 that we have the canonical isomorphism

$$F^\flat \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}}}(\mathcal{E}'^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}). \qquad (15.2.1.5.1)$$

Following (the right version of) 15.1.4.6, we get the first (second) equality:

$$\iota^{(m)}_S(\operatorname{Car}^{(0)}(\mathcal{E}'^{(0)}) = \operatorname{Car}^{(m)}(\mathcal{E}^{(m)}),$$
$$\operatorname{Car}^{(0)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}}}(\mathcal{E}'^{(0)}, \widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}})) = \operatorname{Car}^{(m)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}}(\mathcal{E}'^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'/\mathfrak{S},\mathbb{Q}})).$$

Hence, we can reduce to the case $m = 0$ of the first part and we are done. □

15.2.2 The filtration by the codimension of a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module

Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module.

Notation 15.2.2.1. Since we have the biduality isomorphism $\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathbb{R}\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}), \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}$ (see 4.6.4.6), then we get the converging biregular spectral sequence (see [Sta22, 0BDU])

$$E_2^{r,s} = \mathcal{E}xt^r_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}xt^{-s}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}), \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \Rightarrow H^n(\mathcal{E}) = E^n. \qquad (15.2.2.1.1)$$

Denote by $(F^i_{(m)}(\mathcal{E}))_{i\geq 0}$ the decreasing filtration of $\mathcal{E}$ induced by the spectral sequence 15.2.2.1.1 for $n = 0$. One gets the explicit construction at [Sta22, 012P] in the case where the filtered complex is given by the total complex associated to the bicomplex $\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{H}om_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{I}^\bullet), \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is an injective resolution of the bimodule $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$, this total complex being filtrated by the first filtration F_I according to notation [Sta22, 012Z].

One important property, is the functoriality of this filtration: for any morphism $\alpha\colon \mathcal{E}' \to \mathcal{E}$ of coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-modules, α is in fact a morphism of filtered coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-modules, i.e. $\alpha(F^i_{(m)}(\mathcal{E}')) \subset F^i_{(m)}(\mathcal{E})$.

15.2.2.2. Set $k := \operatorname{codim}^{(m)}(\mathcal{E})$. We have $E_2^{i,j} = 0$ (and therefore $E_\infty^{i,j} = 0$) in the following case:

(a) for any $i < 0$ or $-j < 0$ (obvious),

(b) for any $i > 2d$ or $-j > 2d$ (use 8.7.7.6),

(c) for any $-j < k$ (use 15.2.1.5(b)),

(d) for any $i < -j$ (use both properties of 15.2.1.5).

Proposition 15.2.2.3. *With the notations of 15.2.2.1, for any $0 \leq i \leq 2d$, $F^i_{(m)}(\mathcal{E})$ is the greatest coherent sub-$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module of codimension greater or equal to i of $\mathcal{E}$.*

Proof. Let $0 \leq i \leq 2d$. Since the spectral sequence 15.2.2.1.1 is converging, then we have by definition of the convergence $F^j_{(m)}(\mathcal{E})/F^{j+1}_{(m)}(\mathcal{E}) = E_\infty^{j,-j}$ for any $j \in \mathbb{N}$.

1) Suppose $0 \leq i \leq k$. Since for any $0 \leq j < i$, we have $E_\infty^{j,-j} = 0$ (see 15.2.2.2.(c)), then we get the equality $F^i_{(m)}(\mathcal{E}) = \mathcal{E}$.

2) Following 15.2.1.5, for any $i \leq j \leq 2d$, $\operatorname{codim}^{(m)}(E_2^{j,-j}) \geq j \geq i$. With the lemma 15.1.4.5, this yields we get $\operatorname{codim}^{(m)}(E_\infty^{j,-j}) \geq i$, for any $i \leq j \leq 2d$. Since $F^j_{(m)}(\mathcal{E})/F^{j+1}_{(m)}(\mathcal{E}) = E_\infty^{j,-j}$, this yields that $F^i_{(m)}(\mathcal{E})$ has codimension greater or equal to i.

3) Let $\mathcal{E}'$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-submodule of $\mathcal{E}$ of codimension greater or equal to i. By functoriality of the filtration (see 15.2.2.1.1), we get the commutative diagram:

$$\begin{array}{ccc} F^i_{(m)}(\mathcal{E}')[r]^- & \hookrightarrow & \mathcal{E}' \\ \downarrow & & \downarrow \\ F^i_{(m)}(\mathcal{E}) & \hookrightarrow & \mathcal{E}. \end{array}$$

Since $\mathcal{E}'$ has codimension greater or equal to i, then the top arrow is an equality following the step 1) (applied to $\mathcal{E}'$). This implies the inclusion $\mathcal{E}' \hookrightarrow F^i_{(m)}(\mathcal{E})$ factoring this commutative diagram. Hence we are done. □

Proposition 15.2.2.4. *Let $\mathcal{E}^{(m)}$ be a nonzero coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. We have the equality:*

$$\operatorname{codim}^{(m)}(\mathcal{E}^{(m)}) = \min \left\{ i | \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \neq 0 \right\}. \tag{15.2.2.4.1}$$

Proof. Set $k := \operatorname{codim}^{(m)}(\mathcal{E}^{(m)})$ and $l := \min \left\{ i | \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \neq 0 \right\}$. It follows from 15.2.1.5.(b) that $l \geq k$. Hence, it remains to check that $\mathcal{E}xt^k_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \neq 0$. With notation 15.2.2.1.1, it is therefore sufficient to prove that $E_2^{k,-k} \neq 0$. By the absurd, let us suppose $E_2^{k,-k} = 0$. Following 15.2.1.5.(b), $E_2^{i,-i} = 0$ for $i < k$. Moreover, via 15.2.1.5.(a), $\operatorname{codim}(E_2^{i,-i}) \geq k+1$ for any $i \geq k+1$. By using 15.1.4.5, this yields $\operatorname{codim}(E_\infty^{i,-i}) \geq k+1$ for any $i \geq 0$ (in fact $E_\infty^{i,-i} = 0$ for any $i \leq k$). As the filtration of $\mathcal{E}^{(m)}$ induced by the spectral sequence is finite (use 8.7.7.6), then it follows from 15.1.4.5 that we obtain: $\operatorname{codim}(\mathcal{E}^{(m)}) \geq k+1$, which is absurd. □

Let us finish by results concerning the variation of the level.

Proposition 15.2.2.5. *Let $m' \geq m \geq$ be two integers, $\mathcal{E}^{(m)}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. We denote by $\mathcal{E}^{(m')} := \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}$, $(F^i_{(m)}(\mathcal{E}^{(m)}))_{i\geq 0}$ and $(F^i_{(m')}(\mathcal{E}^{(m')}))_{i\geq 0}$ the decreasing filtrations of 15.2.2.1. We have the canonical isomorphism*

$$\widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} F^i_{(m)}(\mathcal{E}^{(m)}) \xrightarrow{\sim} F^i_{(m')}(\mathcal{E}^{(m')}).$$

Proof. It comes from the fact that the extension $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}$ is flat (see 7.5.3.1) and that the dual functor commutes with extensions (see 4.6.4.4.1). □

Corollary 15.2.2.6. *Let $\mathcal{E}^{(m)}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. For $m' \geq m$, we set $\mathcal{E}^{(m')} := \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}$. We have the inequality* $\operatorname{codim}^{(m')}(\mathcal{E}^{(m')}) \geq \operatorname{codim}^{(m)}(\mathcal{E}^{(m)})$.

Proof. If $\mathcal{E}^{(m')} = 0$ then the assertion is straightforward. Let us suppose then $\mathcal{E}^{(m')} \neq 0$. As the extension $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}$ is flat, for any $i < \operatorname{codim}^{(m)}(\mathcal{E}^{(m)})$, we obtain:

$$0 = \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m')}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$$

Hence we are done via 15.2.2.4. □

Remark 15.2.2.7. With the notations of 15.2.2.6, following a counterexample of Berthelot and of Abe, there is no inclusion between the varieties characteristics of respectively $\mathcal{E}^{(m)}$ and $\mathcal{E}^{(m')}$.

15.2.3 The (co)dimension of a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module

Let T be a divisor of P and $\mathfrak{U} := \mathfrak{P} \setminus T$. In this subsection, we define the notion of the (co)dimension of a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. We also check here that the (co)dimension of the level m properties of the subsections 15.2.1 and 15.2.2 still hold for coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules.

Definition 15.2.3.1. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. Let m_0 be the smaller integer $m \geq 0$ such that there exists a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U},\mathbb{Q}}$-module $\mathcal{E}^{(m)}$ together with a $\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U},\mathbb{Q}}} \mathcal{E}^{(m)} \xrightarrow{\sim} \mathcal{E}|\mathfrak{U}$. Following 8.4.1.11, this integer is well defined. Let us choose $\mathcal{E}^{(m_0)}$ such a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{U},\mathbb{Q}}$-module. For any integer $m \geq m_0$, we denote by then $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{U}\mathbb{Q}}} \mathcal{E}^{(m_0)}$. Following 15.2.2.6, for any $m' \geq m \geq m_0$, we have $\dim^{(m)}(\mathcal{E}^{(m)}) \geq \dim^{(m')}(\mathcal{E}^{(m')})$.

The *dimension of* $\mathcal{E}$, denoted by $\dim(\mathcal{E})$, is by definition the minimum of $\{\dim^{(m)}(\mathcal{E}^{(m)}), m \geq m_0\}$, i.e., the limit of the stationary sequence sequence $(\dim^{(m)}(\mathcal{E}^{(m)}))_{m \geq m_0}$. The *codimension of* $\mathcal{E}$, denoted by $\operatorname{codim}(\mathcal{E})$, is the integer $\operatorname{codim}(\mathcal{E}) := 2\dim P - \dim(\mathcal{E})$. Remark that following 8.4.1.11 the integers $\dim(\mathcal{E})$ and $\operatorname{codim}(\mathcal{E})$ do not depend on the choice of $\mathcal{E}^{(m_0)}$.

Lemma 15.2.3.2. *Let* $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ *be an exact sequence of coherent left* $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$*-modules. We have the formula* $\operatorname{codim}(\mathcal{E}) = \min\{\operatorname{codim}(\mathcal{E}'), \operatorname{codim}(\mathcal{E}'')\}$.

Proof. By definition of the codimension, we can suppose the divisor T is empty. This is therefore a consequence of 15.1.4.5 and 8.4.1.11. □

Let us extend via the following two propositions the results of the preceding section:

Proposition 15.2.3.3. *Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$*-module.*

(a) $\operatorname{codim}(\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})) \geq i$*, for any* $i \geq 0$.

(b) $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) = 0$*, for any* $i < \operatorname{codim}(\mathcal{E})$.

Proof. By definition of the codimension or by using 8.7.6.11, we reduce to case where the divisor T is empty. With the notations of 15.2.3.1, for any $m \geq m_0$, as the extension $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$ is flat (see 7.5.3.1), as dual functors commute with extensions (see 4.6.4.4.1), then we obtain the second isomorphism:

$$\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}. \tag{15.2.3.3.1}$$

For m large enough, $\operatorname{codim}^{(m)}(\mathcal{E}^{(m)}) = \operatorname{codim}(\mathcal{E})$. Following 15.2.1.5, this yields that $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) = 0$ for any $i < \operatorname{codim}(\mathcal{E})$, i.e. we have checked the part (b) of the proposition. By definition of the codimension of $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$, the isomorphism 15.2.3.3.1 can be translated for m large enough by

$$\operatorname{codim}(\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})) = \operatorname{codim}^{(m)}(\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})).$$

From 15.2.1.5.(a), this yields therefore (a). □

Notation 15.2.3.4. Since we have the biduality isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}), \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}$$

(see 4.6.4.6), then similarly to 15.2.2.1 we get the converging biregular spectral sequence

$$E_2^{r,s} = \mathcal{E}xt^r_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}xt^{-s}_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}), \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \Rightarrow H^n(\mathcal{E}) = E^n. \tag{15.2.3.4.1}$$

Denote by $(F^i(\mathcal{E}))_{i \geq 0}$ the decreasing filtration of $\mathcal{E}$ induced by the spectral sequence 15.2.3.4.1 for $n = 0$. This filtration is functorial: for any morphism $\alpha \colon \mathcal{E}' \to \mathcal{E}$ of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules, α is in fact a morphism of filtered coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules, i.e. $\alpha(F^i(\mathcal{E}')) \subset F^i(\mathcal{E})$.

15.2.3.5. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. Set $k := \operatorname{codim}(\mathcal{E}) \leq d$. With the notation 15.2.3.4.1, we have $E_2^{i,j} = 0$ (and therefore $E_\infty^{i,j} = 0$) in the following case:

(i) for any $i < 0$ or $-j < 0$ (obvious),

(ii) for any $i > 2d + 2$ or $-j > 2d + 2$ (use 8.7.7.9),

(iii) for any $-j < k$ (use 15.2.3.3(b)),

(iv) for any $i < -j$ (use both properties of 15.2.3.3).

Proposition 15.2.3.6. *With the notations of 15.2.3.4, for any $0 \leq i \leq 2d$, $F^i(\mathcal{E})$ is the greatest coherent sub-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module of codimension greater or equal to i of $\mathcal{E}$.*

Proof. Copy the check of 15.2.2.3. ☐

Proposition 15.2.3.7. *Let $\mathcal{E}^{(m)}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. We denote by $\mathcal{E} := \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}$, $(F^i_{(m)}(\mathcal{E}^{(m)}))_{i\geq 0}$ and $(F^i(\mathcal{E})_{i\geq 0}$ the decreasing filtrations of 15.2.2.1 and 15.2.3.4. We have the canonical isomorphism*

$$\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} F^i_{(m)}(\mathcal{E}^{(m)}) \xrightarrow{\sim} F^i(\mathcal{E}). \tag{15.2.3.7.1}$$

Proof. The isomorphism 15.2.3.7.1 comes from the fact that the extension $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$ is flat (this follows from 7.5.3.1) and that the dual functor commutes with extensions (see 4.6.4.4.1). ☐

Proposition 15.2.3.8. *Let $\mathcal{E}$ be a nonzero coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. Hence:*

$$\operatorname{codim}(\mathcal{E}) = \min \left\{ i | \mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \neq 0 \right\}. \tag{15.2.3.8.1}$$

Proof. By definition of the codimension and by using 8.7.6.11, we reduce to case where the divisor T is empty. Set $k := \operatorname{codim}(\mathcal{E})$ and $l := \min \left\{ i | \mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \neq 0 \right\}$. It follows from 15.2.3.3 that $k \leq l$. It remains to check $\mathcal{E}xt^k_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \neq 0$. With notation 15.2.3.1, for m large enough, $\mathcal{E}^{(m)} \neq 0$ and $k = \operatorname{codim}(\mathcal{E}^{(m)})$. We get from 15.2.2.4 that $\mathcal{E}xt^k_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \neq 0$ for any m large enough. For any $m' \geq m$ large enough, as the extension $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}$ is flat (see 7.5.3.1), as dual functors commute with extensions (see 4.6.4.4.1), we obtain the second isomorphism:

$$\mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m')}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}}(\widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{E}xt^i_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}^{(m)}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}) \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}}.$$

Via the isomorphisms 15.2.3.3.1, this yields $\mathcal{E}xt^k_{\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \neq 0$. ☐

15.2.4 Inequality of Bernstein, holonomicity, homological criterion, Berthelot-Kashiwara theorem

Let T be a divisor of P. Unless otherwise stated, we suppose P integral.

Theorem 15.2.4.1 (Inequality of Bernstein)**.** *For any nonzero coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module $\mathcal{E}$, we have*

$$\dim(\mathcal{E}) \geq \dim P. \tag{15.2.4.1.1}$$

Proof. By definition of the dimension, we reduce to case where the divisor T is empty. We proceed by induction on the dimension of P.

1) Let us suppose that the support of $\mathcal{E}$ is P. With notation 15.2.3.1, let $m \geq m_0$ and $\mathring{\mathcal{E}}^{(m)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module such that $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} \mathring{\mathcal{E}}^{(m)} \xrightarrow{\sim} \mathcal{E}^{(m)}$. Since the support of $\mathcal{E}^{(m)}$ is P, then the support of $\operatorname{gr}(\mathring{\mathcal{E}}^{(m)}/\pi\mathring{\mathcal{E}}^{(m)})$ in T^*P contains $T^*_P P$ and the inequality holds.

2) Suppose the support Z of $\mathcal{E}$ is a closed subset of dimension smaller than that of P. Replacing if necessary P by an affine open U such that $Z \cap U$ is smooth and dense in an irreducible component of Z of maximal dimension, we can suppose Z smooth and that $Z \subset P$ lifts to a closed immersion $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{P}$ of smooth $\mathfrak{S}$-formal schemes. As $\mathcal{E}^{(m)}$ has its support in Z, following 9.3.5.8, increasing m if necessary, there exists a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z},\mathbb{Q}}$-module $\mathcal{F}^{(m)}$ such that $u^{(m)}_{+}(\mathcal{F}^{(m)}) \xrightarrow{\sim} \mathcal{E}^{(m)}$ and $\dim(\mathcal{E}) = \dim^{(m)}(\mathcal{E}^{(m)})$. Since this is local, we can suppose $\mathfrak{P}$ is affine and there exists a morphism of smooth $\mathfrak{S}$-formal schemes $F\colon \mathfrak{P} \to \mathfrak{P}'$ (resp. $F\colon \mathfrak{Z} \to \mathfrak{Z}'$) which is a lifting of the relative Frobenius $F^{m}_{P/S}\colon P \to P^{(m)}$ (resp. $F^{m}_{Z/S}\colon Z \to Z^{(m)}$). Let $u'\colon \mathfrak{Z}' \hookrightarrow \mathfrak{X}'$ be a lifting of the closed immersion $u'_0 := u^{(m)}_0$. We set $\mathcal{E}'^{(0)} := F^{\flat}\widehat{\mathcal{D}}^{(0)}_{\mathfrak{P}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)}$ and $\mathcal{F}'^{(0)} := F^{\flat}\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Z}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z},\mathbb{Q}}} \mathcal{F}^{(m)}$. Following 8.8.1.3, $F^{*}\mathcal{E}'^{(0)} \xrightarrow{\sim} \mathcal{E}^{(m)}$ and $F^{*}\mathcal{F}'^{(0)} \xrightarrow{\sim} \mathcal{F}^{(m)}$. Hence, by using 15.1.4.6, we reduce to check $\mathrm{Car}^{(0)}(\mathcal{E}'^{(0)}) \geq \dim P$.

Let $\mathring{\mathcal{F}}'^{0)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Z}'}$-module such that $\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Z}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Z}'}} \mathring{\mathcal{F}}'^{0)} \xrightarrow{\sim} \mathcal{F}'^{(0)}$. We set $\overline{\mathcal{F}}'^{(0)} := \mathcal{D}^{(0)}_{Z^{(m)}} \otimes_{\widehat{\mathcal{D}}^{(0)}_{\mathfrak{Z}'}} \mathring{\mathcal{F}}'^{0)}$. By definition, $\mathrm{Car}^{(0)}(\overline{\mathcal{F}}'^{(0)}) = \mathrm{Car}^{(0)}(\mathring{\mathcal{F}}'^{0)}) = \mathrm{Car}^{(0)}(\mathcal{F}'^{(0)})$. Since $\mathcal{F}'^{(0)}$ is not null, then by induction hypothesis (and by 15.2.2.6) we get $\dim \mathrm{Car}^{(0)}(\overline{\mathcal{F}}'^{(0)}) \geq \dim Z$.

Via an easy computation in local coordinates, we can check the formula

$$\mathrm{Car}^{(0)}(u'^{(0)}_{0+}(\overline{\mathcal{F}}'^{(0)})) = \varpi_{u'_0}(\rho^{-1}_{u'_0}\mathrm{Car}^{(0)}(\overline{\mathcal{F}}'^{(0)})),$$

where $T^*Z \xleftarrow{\rho_{u'_0}} Z\times_P T^*P \xrightarrow{\varpi_{u'_0}} T^*P$ are the canonical maps (see 15.1.1). With notation 15.1.1, since $\rho_{u'_0}$ is flat of relative dimension $\dim P - \dim Z$ and $\varpi_{u'_0}$ is a closed immersion, then $\dim \varpi_{u'_0}(\rho^{-1}_{u'_0}\mathrm{Car}^{(0)}(\overline{\mathcal{F}}'^{(0)})) = \dim \mathrm{Car}^{(0)}(\overline{\mathcal{F}}'^{(0)}) + \dim P - \dim Z \geq \dim P$. Hence, $\dim \mathrm{Car}^{(0)}(u'^{(0)}_{0+}(\overline{\mathcal{F}}'^{(0)})) \geq \dim P$. As $u'^{(0)}_{0+}(\mathring{\mathcal{F}}'^{0)})$ gives a p-torsion free model of $\mathcal{E}'^{(0)}$, then $\mathrm{Car}^{(0)}(\mathcal{E}'^{(0)}) = \mathrm{Car}^{(0)}(u^{(0)}_{0+}(\overline{\mathcal{F}}'^{(0)})) \geq \dim P$ and we are done.

□

Remark 15.2.4.2. Let $\mathcal{E}^{(m)}$ be a nonzero coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module. We have $\dim^{(m)}(\mathcal{E}^{(m)}) \geq \dim(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)})$. However, we cannot deduce $\dim^{(m)}(\mathcal{E}^{(m)}) \geq \dim P$ from 15.2.4.1.1 because it might happen that $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)} = 0$.

The Bernstein inequality 15.2.4.1.1 is false when P is not connected. For instance is the connected composantes of $\mathfrak{P}$ are $\widehat{\mathbb{A}}^1_{\mathfrak{S}}$ and $\mathfrak{S}$, take $\mathcal{E}$ be a nonzero coherent module with support in $\mathfrak{S}$.

Theorem 15.2.4.3. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module. For any $i > \dim P$, we have*

$$\mathcal{E}xt^{i}_{\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) = 0.$$

Proof. By the absurd, let us suppose that there exist $i > \dim P$ such that $\mathcal{E}xt^{i}_{\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \neq 0$. The inequality of Bernstein can be translated by $\mathrm{codim}(\mathcal{E}xt^{i}_{\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})) \leq \dim P$, which is in contradiction with 15.2.3.3.a. □

The following corollary improves 8.7.7.6 in our specific context.

Corollary 15.2.4.4. *We have the following properties.*

(a) For any $i > \dim P$, for any coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module $\mathcal{E}$ and any $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module$\mathcal{F}$,

$$\mathcal{E}xt^{i}_{\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{F}) = 0.$$

(b) For any affine open subset $\mathfrak{U}$ of $\mathfrak{P}$, we have the equality $\mathrm{tor.dim}\,\Gamma(\mathfrak{U}, \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) = \dim U$ *and the inequalities* $\dim U \leq \mathrm{gl.dim}\,\Gamma(\mathfrak{U}, \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \leq \dim U + 1$.

Proof. a) Since $\mathcal{E}$ is a perfect complex, then it follows from 4.6.3.6.1 the isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}} \mathcal{F} \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{F}),$$

and we conclude by using 15.2.4.3.

b) We can suppose $\mathfrak{P}$ affine. Since the functor $\Gamma(\mathfrak{P}, -)$ is exact on the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-modules, then we get for any $r \in \mathbb{N}$:

$$\Gamma(\mathfrak{P}, -) \circ \mathcal{E}xt^r_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Ext}^r_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}).$$

With 15.2.4.3 and 1.4.3.30.(g), this yields $\mathrm{tor}\,.\dim \Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) \leq \dim U$. Since the converse inequality is already knows (see 8.7.7.4), then we get $\mathrm{tor}\,.\dim \Gamma(\mathfrak{U}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) = \dim U$. The last inequalities follows from 2.3.4.3. □

Definition 15.2.4.5 (Holonomicity)**.** We do not suppose $\mathfrak{P}$ is integral. As $\mathfrak{P}$ is smooth, $\mathfrak{P}$ is the sum of its connected components denoted by $(\mathfrak{P}_r)_r$.

(a) Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-module. We say that $\mathcal{E}$ is "$\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-holonomic" or "holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-module" if, for any r, we have $\dim(\mathcal{E}|\mathfrak{P}_r) \leq \dim P_r$.

(b) Moreover, a complex $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ is by definition *holonomic* if so are its cohomogical sheaves. It will be denoted by $D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ the full subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ consisting in holonomic $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-complexes.

(c) An F-complex $(\mathcal{E}, \Phi) \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ is by definition *holonomic* is $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$. We denote by $F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ the full subcategory of $F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ consisting in holonomic F-$\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-complexes.

15.2.4.6. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$. Let $\mathfrak{U} := \mathfrak{P} \setminus T$. It is straightforward from the definition of the holonomicity and of the dimension that $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ if and only if $\mathcal{E}|\mathfrak{U} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}})$.

Lemma 15.2.4.7. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-module. Hence $\mathcal{E}^* := H^0(\mathbb{D}_T(\mathcal{E}))$ is $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-holonomic.*

Proof. This follows from the proposition 15.2.3.3. □

Proposition 15.2.4.8 (Homological criterion of holonomicity)**.** *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-module. Hence $\mathcal{E}$ is holonomic if and only if, for any $i \neq \dim P$, $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) = 0$.*

Proof. Since the case $\mathcal{E} = 0$ is obvious, then we can suppose $\mathcal{E} \neq 0$.

1) If $\mathcal{E}$ is holonomic that by definition $\dim(\mathcal{E}) = \mathrm{codim}(\mathcal{E}) = \dim P$. By using 15.2.3.3, we get $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) = 0$ for any $i < \dim P$ and via 15.2.4.3 we obtain $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) = 0$ for any $i > \dim P$.

2) Conversely, suppose $\mathcal{E}xt^i_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) = 0$ for any $i \neq \dim P$. Set $k = \mathrm{codim}(\mathcal{E})$. By using the proposition 15.2.3.8, we get $\mathcal{E}xt^k_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}) \neq 0$. Hence, $k = \mathrm{codim}(\mathcal{E})$. □

Example 15.2.4.9. We have the following properties.

(a) The coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-modules which are $\mathcal{O}_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-coherent are $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-holonomic. Indeed, this is a consequence of the isomorphism $\mathbb{D}_T(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}}(\mathcal{E}, \mathcal{O}_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}})$ (see 11.2.6.3.4).

(b) We will give more example at 18.2.3.13.

Corollary 15.2.4.10. *Let $\mathfrak{U} := \mathfrak{P} \setminus T$. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-module. Hence $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T)_{\mathbb{Q}}$-holonomic if and only if $\mathcal{E}|\mathfrak{U}$ is $\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}}$-holonomic.*

Proof. This is a consequence of 15.2.4.8 and 8.7.6.11. ☐

Remark 15.2.4.11. The $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomicity does not imply the $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-holonomicity. Indeed, there exist some coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules $\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-coherent which are not $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-coherent (nor a fortiori $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-holonomic): when $\mathfrak{P}$ is proper, the de Rham cohomology of a coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module is of finite type on K. Now, if $\mathcal{E} = \mathrm{sp}_* E$, the de Rham cohomology of $\mathcal{E}$ is identified with the rigid cohomology of E. If we take $\mathcal{V} = \mathbb{Z}_{(p)}$, $\mathfrak{P} = \widehat{\mathbb{P}}^1_{\mathcal{V}}$, $Y = \mathbb{G}_{m,k}$, and if E is the overconvergent isocrystal on Y defined by $j^\dagger \mathcal{O}_{\mathfrak{P}_K}$, equipped with the connection ∇ such that $\nabla(1) = (\alpha/t)dt$, where $\alpha \in \mathbb{Z}_{(p)}$ is a Liouville number, the rigid cohomology of E is given by the action of ∇ on the space of analytical functions on annulus of the form $1-\epsilon < |t| < 1+\epsilon$, with $\epsilon > 0$ variable, and it is classic that its H^1 is of infinite dimension. But, when the module are endowed with Frobenius structures, Berthelot conjecture in [Ber02, 5.3.6] that this is the case:

15.2.4.12. The properties of coherence and of holonomicity are stable by base change. More precisely, let $\mathcal{V} \to \mathcal{V}'$ be a morphism of complete discrete valuation rings of unequal characteristic $(0,p)$ and perfect residue fields denoting by $\mathfrak{S} := \mathrm{Spf}(\mathcal{V})$, $\mathfrak{S}' := \mathrm{Spf}(\mathcal{V}')$, let $f\colon \mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}' \to \mathfrak{P}$ be the canonical morphism, $T' := f^{-1}(T)$. If $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}})$, then $f^*(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'/\mathfrak{S}'}({}^\dagger T')_{\mathbb{Q}})$. Moreover, if $\mathcal{E}$ is a holonomic $\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module (see 15.2.4.5) then $f^*(\mathcal{E})$ is a holonomic $\mathcal{D}^\dagger_{\mathfrak{P}'/\mathfrak{S}'}({}^\dagger T')_{\mathbb{Q}}$-module. Indeed, this is a consequence of the fact that the extension $f^{-1}\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T')_{\mathbb{Q}}$ is flat (see 9.2.7.1), of the fact that the dual functor commutes with extensions (see 4.6.4.4.1) and of the homological criterion of holonomicity] (see 15.2.4.8).

Conjecture 15.2.4.13 (Berthelot)**.** We suppose there exists an isomorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ lifting the s-power of the Frobenius of k. Let $\mathcal{E}$ be a coherent F-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. If $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic then $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-holonomic.

Proposition 15.2.4.14. *We do not suppose $\mathfrak{P}$ is integral. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules. Hence $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic if and only if $\mathcal{E}'$ and $\mathcal{E}''$ are $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic.*

Proof. We can suppose $\mathfrak{P}$ is integral. Suppose $\mathcal{E}'$ and $\mathcal{E}''$ are $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic. By considering the long exact sequence induced by the exact triangle $\mathbb{D}_T(\mathcal{E}'') \to \mathbb{D}_T(\mathcal{E}) \to \mathbb{D}_T(\mathcal{E}') \to +1$, we get the $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomicity of $\mathcal{E}$. Conversely, suppose $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic. By using 15.2.4.10, we reduce to the case where the divisor T is empty. The inequality of Bernstein (see 15.2.4.1.1) can be translated by $\mathrm{codim}(\mathcal{E}') \le \dim P$ and $\mathrm{codim}(\mathcal{E}'') \le \dim P$. Since we have $\mathrm{codim}(\mathcal{E}) = \min\{\mathrm{codim}(\mathcal{E}'), \mathrm{codim}(\mathcal{E}'')\}$ (see 15.2.3.2), since $\mathrm{codim}(\mathcal{E}) = \dim P$ then $\mathrm{codim}(\mathcal{E}') = \dim P$ and $\mathrm{codim}(\mathcal{E}'') = \dim P$ and we are done. ☐

Proposition 15.2.4.15. *The functor $\mathbb{D}_T$ induces an exact auto-equivalence of the categories of holonomic $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules (resp. $D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$).*

Proof. This is a consequence of the isomorphism of biduality (see 5.1.4.4) and of Proposition 15.2.4.8. ☐

Notation 15.2.4.16. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module.

(a) Thanks to 15.2.4.3, we can improve the vanishing properties of the biduality spectral sequence of 15.2.3.5.(ii) by replacing $2d$ by d. In particular, with its notation we get we have $E_2^{i,j} = 0$ in the following case:

  (i) for any $i \notin [0,d]$ or $j \notin [-d,0]$ (use 15.2.4.3),

  (ii) for any $i < -j$ (use both properties of 15.2.3.3).

  This implies the equalities

$$F^d(\mathcal{E}) = E_\infty^{d,-d} = E_2^{d,-d} = (\mathcal{E}^*)^*. \tag{15.2.4.16.1}$$

(b) We set $\mathcal{E}^{\mathrm{hol}} := (\mathcal{E}^*)^*$. Following 15.2.3.6 and 15.2.4.16.1, $\mathcal{E}^{\mathrm{hol}}$ is the greatest holonomic submodule of $\mathcal{E}$. We get the functor $(-)^{\mathrm{hol}}\colon \mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \to \mathrm{Hol}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ defined by $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{hol}}$.

  We remark moreover that the forgetful functor $\mathrm{Hol}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \to \mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ is left adjoint of the functor $(-)^{\mathrm{hol}}$. In particular, the functor $(-)^{\mathrm{hol}}$ is left exact and commutes with projective limits.

Lemma 15.2.4.17. *Let $\mathcal{E}' \subset \mathcal{E}$ be a monomorphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules. The morphism $\mathcal{E}'^{\mathrm{hol}} \to \mathcal{E}^{\mathrm{hol}}$ induced by the functor $(-)^{\mathrm{hol}}$ (see 15.2.4.16) is then a monomorphism. Moreover, $\mathcal{E}'^{\mathrm{hol}} = \mathcal{E}' \cap \mathcal{E}^{\mathrm{hol}}$.*

Proof. The first assertion results from the left exactness of the functor $(-)^{\mathrm{hol}}$ (see 15.2.4.16). Since $\mathcal{E}' \cap \mathcal{E}^{\mathrm{hol}}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module which is included into the holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module $\mathcal{E}'$, then it follows from 15.2.4.14 that it is holonomic. Since $\mathcal{E}'^{\mathrm{hol}}$ is the greatest holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-submodule of $\mathcal{E}'$, this yields the equality $\mathcal{E}'^{\mathrm{hol}} = \mathcal{E}' \cap \mathcal{E}^{\mathrm{hol}}$. □

Lemma 15.2.4.18. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module. The canonical morphism $\mathcal{E}^{*} \to (\mathcal{E}^{\mathrm{hol}})^{*}$ is an isomorphism, i.e., $(\mathcal{E}/\mathcal{E}^{\mathrm{hol}})^{*} = 0$. In particular, $(\mathcal{E}/\mathcal{E}^{\mathrm{hol}})^{\mathrm{hol}} = 0$.*

Proof. Set $\mathcal{E}_{\text{n-hol}} := \mathcal{E}/\mathcal{E}^{\mathrm{hol}}$. Since $\mathcal{E}^{\mathrm{hol}}$ is holonomic, then by using 15.2.4.8 we get $H^{-1}\mathbb{D}_T(\mathcal{E}^{\mathrm{hol}}) = 0$. By applying the functor $H^0\mathbb{D}_T$ to the exact sequence $0 \to \mathcal{E}^{\mathrm{hol}} \underset{\mathrm{adj}}{\longrightarrow} \mathcal{E} \longrightarrow \mathcal{E}_{\text{n-hol}} \to 0$, this yields the exact sequence $0 \to \mathcal{E}^{*}_{\text{n-hol}} \to \mathcal{E}^{*} \to (\mathcal{E}^{\mathrm{hol}})^{*} \to 0$. Since this a morphism of holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules, by using the biduality isomorphism it follows that it is sufficient to check that by applying the functor $H^0\mathbb{D}_T$ to the morphism $\mathcal{E}^{*} \to (\mathcal{E}^{\mathrm{hol}})^{*}$, we get an isomorphism. However, this latter is the morphism $(-)^{\mathrm{hol}}(\mathrm{adj})\colon (\mathcal{E}^{\mathrm{hol}})^{\mathrm{hol}} \to \mathcal{E}^{\mathrm{hol}}$. Since the adjunction morphism of the form $\mathrm{adj}\colon \mathcal{F}^{\mathrm{hol}} \subset \mathcal{F}$ is functorial in $\mathcal{F}$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{E}^{\mathrm{hol}} & \xhookrightarrow{\ \mathrm{adj}\ } & \mathcal{E} \\ {\scriptstyle \mathrm{adj}}\uparrow & & \uparrow{\scriptstyle \mathrm{adj}} \\ (\mathcal{E}^{\mathrm{hol}})^{\mathrm{hol}} & \xhookrightarrow{\ (-)^{\mathrm{hol}}(\mathrm{adj})\ } & \mathcal{E}^{\mathrm{hol}}. \end{array}$$

Since $\mathcal{E}^{\mathrm{hol}}$ is holonomic, the left arrow is bijective. Moreover, since the right and top arrows are identical then this implies that the bottom arrow is an isomorphism.

□

Theorem 15.2.4.19 (Holonomic version of Berthelot-Kashiwara Theorem)**.** *Let $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ be a closed immersion of smooth $\mathfrak{S}$-formal schemes. Let T be a divisor of X such that $D := X \cap T$ is a divisor of X.*

(a) For any holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module $\mathcal{E}$ supported in X, for any holonomic $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module $\mathcal{F}$, we have $H^r u_{+}(\mathcal{F}) = 0$ and $H^r u^{!}(\mathcal{E}) = 0$ for any $r \neq 0$.

(b) The functors u_{+} and $u^{!}$ are exact quasi-inverse equivalences between the category of holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules supported in X and that of holonomic $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-modules.

Proof. This follows from Berthelot-Kashiwara theorem 9.3.5.9, from the criterion of holonomicity (see 15.2.4.8) and from the relative duality isomorphism (see 13.2.4.1). □

Corollary 15.2.4.20. *Let $(Y, X, \mathfrak{P}, T)/\mathcal{V}$ be a completely smooth d-frame. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ (see definition 12.2.1.4). Then $\mathcal{E}$ is a holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module.*

Proof. Since this is local, we can suppose there exists a closed immersion $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ of smooth $\mathfrak{S}$-formal schemes which lifts $X \hookrightarrow \mathfrak{P}$. This yields $u^{!}(\mathcal{E}) \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{X}, T \cap X/\mathcal{V})$ (see 12.2.1.9). Following 15.2.4.9.(a), $u^{!}\mathcal{E}$ is a holonomic $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T \cap X)_{\mathbb{Q}}$-module. Following the coherent version of Berthelot-Kashiwara theorem (see 9.3.5.9), we have $u_{+}u^{!}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. We conclude thanks to 15.2.4.19. □

15.2.5 Purity of the characteristic variety of a holonomic F-$\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module

Corollaire 15.2.5.1. We suppose there exists an isomorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ lifting the s-power of the Frobenius of k. Let $\mathfrak{P}$ be a smooth integral $\mathcal{V}$-formal scheme of dimension d. Let $\mathcal{N} \neq 0$ be a holonomic F-$\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. Then, the characteristic variety $\mathrm{Car}(\mathcal{N})$ of $\mathcal{N}$ is purely of codimension d.

Proof. The is a consequence of holonomicity characterization (see 15.2.4.8) and of Theorem 15.1.7.8. □

15.3 Overcoherence

Let $\mathfrak{P}$ be a smooth $\mathfrak{S}$-formal scheme and T a divisor of its special fiber P.

15.3.1 Generical $\mathcal{O}$-coherence of a coherent $\mathcal{D}$-module with finite fibers or an holonomic $\mathcal{D}$-module

We check in this subsection that a coherent module whose extraordinary fibers are finite becomes $\mathcal{O}$-coherent on a dense open subset of its support (more precisely, see the theorem 15.3.1.19). This corresponds to a well known property of the holonomicity in characteristic zero. This will be a fundamental ingredient of the proof of the overconvergent holonomicity criterion (see 15.3.2.8).

Notation 15.3.1.1. For any closed point x of P, we will denote by $i_x\colon \operatorname{Spf}\mathcal{V}_x \hookrightarrow \mathfrak{P}$ a morphism of smooth $\mathfrak{S}$-formal schemes which is a lifting of the canonical closed immersion induced by x. Since k is perfect then $\mathfrak{S}_x \to \mathfrak{S}$ is finite and etale. Hence, $\mathcal{D}^\dagger_{\mathfrak{S}_x/\mathfrak{S}} = \mathcal{O}_{\mathfrak{S}_x} = \mathcal{V}_x$ and $\mathcal{D}^\dagger_{\mathfrak{S}_x/\mathfrak{S},\mathbb{Q}} = \mathcal{O}_{\mathfrak{S}_x,\mathbb{Q}} = K_x$. We remark that $\mathcal{V}_x$ is a complete discrete valuation ring of unequal characteristics $(0,p)$. Its field of fractions will be designated by K_x. We set $\mathfrak{S}_x := \operatorname{Spf}\mathcal{V}_x$. According to notation 9.2.1.21, we have the functor $\mathbb{L}i_x^* := i_x^![d_{i_x}]\colon D^-(\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_\mathbb{Q} \to D^-(\mathcal{O}_{\mathfrak{S}_x,\mathbb{Q}})$. We set $i_x^* := H^0\mathbb{L}i_x^*$.

Lemma 15.3.1.2. *Let $f\colon \mathfrak{P}' \hookrightarrow \mathfrak{P}$ be a closed immersion of $\mathcal{V}$-smooth formal schemes, $\mathcal{I} \subset \mathcal{O}_\mathfrak{P}$ the ideal defining f. Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_\mathfrak{P}$-module (resp. a coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module) which is flat as $\mathcal{O}_\mathfrak{P}$-module (resp. $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module). With notation 7.5.5.6 (resp. 9.2.1.21), we have therefore the natural isomorphism: $\mathbb{L}f^{(m)*}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}/\mathcal{I}\mathcal{E}$ (resp. $\mathbb{L}f^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}/\mathcal{I}\mathcal{E}$).*

Proof. Since the respective case is treated similarly, then let us treat the non-respective one. Since $\widehat{\mathcal{D}}^{(m)}_\mathfrak{P}$ is $\mathcal{O}_\mathfrak{P}$-flat, this therefore implies the isomorphism of $\mathcal{O}_\mathfrak{P}$-modules:

$$\mathcal{E}/\mathcal{I}\mathcal{E} \xrightarrow{\sim} \mathcal{O}_\mathfrak{P}/\mathcal{I}\mathcal{O}_\mathfrak{P} \otimes^\mathbb{L}_{\mathcal{O}_\mathfrak{P}} \mathcal{E} \overset{9.3.1.8.2}{\xrightarrow{\sim}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'\hookrightarrow\mathfrak{P}/\mathfrak{S}} \otimes^\mathbb{L}_{\widehat{\mathcal{D}}^{(m)}_\mathfrak{P}} \mathcal{E} = f^{(m)*}(\mathcal{E})$$

□

Lemma 15.3.1.3. *Let A be a p-adically separated complete $\mathcal{V}$-algebra, M be a π-torsion free separate complete A-module. We assume that $M/\pi M$ is a free $A/\pi A$-module. There then exists a free A-module M' and an A-linear isomorphism $\widehat{M'} \xrightarrow{\sim} M$.*

Proof. By hypothesis, there exists an isomorphism of the form $\overline{\phi}\colon (A/\pi A)^{(J)} \xrightarrow{\sim} M/\pi M$. Choose a lifting $\phi\colon A^{(J)} \to M$ of $\overline{\phi}$. Since this is an isomorphism modulo π, lemma 3.3.2.5 allows us to conclude. □

Remark 15.3.1.4. It follows from 7.2.1.4 that if $\mathcal{E}$ is an $\mathcal{O}_\mathfrak{P}$-module of the form $(\mathcal{O}_\mathfrak{P}^{(J)})^\wedge$, i.e., is isomorphic to the p-adic completion of a free $\mathcal{O}_\mathfrak{P}$-module, then $\mathcal{E}$ is $\mathcal{O}_\mathfrak{P}$-flat.

Proposition 15.3.1.5. Assume that $\mathfrak{P}$ is affine and fix a closed point x of $\mathfrak{P}$. Let $\mathcal{I} \subset \mathcal{O}_\mathfrak{P}$ be the ideal defining i_x. Set $A := \Gamma(\mathfrak{P}, \mathcal{O}_\mathfrak{P})$ and $I := \Gamma(\mathfrak{P}, \mathcal{I})$. Let $\mathcal{E}$ be any coherent $\widehat{\mathcal{D}}^{(m)}_\mathfrak{P}$-module (resp. coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module, resp. coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module). Put $E := \Gamma(\mathfrak{P}, \mathcal{E})$.

(a) The canonical morphism $E/IE \to \mathcal{E}/\mathcal{I}\mathcal{E}$ is an isomorphism.

(b) In the non-respective case, if $\mathcal{E}$ is $\mathcal{O}_\mathfrak{P}$-flat, then E/IE is p-torsion free, p-adically separated and complete. Moreover, the subset IE is closed in E, when E is endowed with its p-adic topology. Furthermore, if $\mathcal{E}$ is of the form $(\mathcal{O}_\mathfrak{P}^{(J)})^\wedge$ for some set J, then E/IE is isomorphic to $(\mathcal{V}_x^{(J)})^\wedge$.

Proof. a) Let's start by proving the first assertion. The other cases being similar, we reduce to deal with the non-respected case. Let $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_\mathfrak{P}$-module. Setting $\widehat{D}^{(m)}_\mathfrak{P} := \Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_\mathfrak{P})$, the morphism $\widehat{\mathcal{D}}^{(m)}_\mathfrak{P} \otimes_{\widehat{D}^{(m)}_\mathfrak{P}} E \to \mathcal{E}$ is an isomorphism (see 7.2.3.16.(ib)) and therefore we get the last isomorphism:

$$\mathcal{E}/\mathcal{I}\mathcal{E} \xrightarrow{\sim} \mathcal{O}_\mathfrak{P}/\mathcal{I}\mathcal{O}_\mathfrak{P} \otimes_{\mathcal{O}_\mathfrak{P}} \mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_\mathfrak{P}/\mathcal{I}\widehat{\mathcal{D}}^{(m)}_\mathfrak{P} \otimes_{\widehat{\mathcal{D}}^{(m)}_\mathfrak{P}} \mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_\mathfrak{P}/\mathcal{I}\widehat{\mathcal{D}}^{(m)}_\mathfrak{P} \otimes_{\widehat{D}^{(m)}_\mathfrak{P}} E$$

As $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}/\mathcal{I}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$ has its support in $\{x\}$, then we get the first isomorphism:

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}/\mathcal{I}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}} \otimes_{\widehat{D}^{(m)}_{\mathfrak{P}}} E \xrightarrow{\sim} \Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}/\mathcal{I}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}) \otimes_{\widehat{D}^{(m)}_{\mathfrak{P}}} E$$
$$\overset{9.3.1.3.5}{\xrightarrow{\sim}} \widehat{D}^{(m)}_{\mathfrak{P}}/I\widehat{D}^{(m)}_{\mathfrak{P}} \otimes_{\widehat{D}^{(m)}_{\mathfrak{P}}} E \xrightarrow{\sim} E/IE$$

By composition the above isomorphisms, we get the assertion (a) of the Lemma.

By applying the lemmas 15.3.1.2, 7.5.5.12.(b) in the case of the closed immersion $i_x\colon \operatorname{Spf}\mathcal{V}_x \hookrightarrow \mathfrak{P}$, it follows from the part a) that E/IE is separated and complete for the p-adic topology. Hence, subset IE is closed in E (e.g. see [Mat89, 23.B]). □

Remark 15.3.1.6. With notation 15.3.1.5, if M is a flat separated complete $\mathcal{V}_x$-module, then M is the p-adic separated completion of a free $\mathcal{V}_x$-module. (Indeed, this follows from 15.3.1.3 and the fact that "p-torsion free" is equivalent to "$\mathcal{V}$-flat".) Hence, in the part (b), if $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{P}}$-flat then E/IE is always isomorphic to $(\mathcal{V}_x^{(J)})^\wedge$ for some set J.

In order to prove Proposition 15.3.1.8, we need the following lemma.

Lemma 15.3.1.7. *We assume that $\mathfrak{P}$ is affine and $\mathfrak{P}/\mathfrak{S}$ is endowed with coordinates. Let $m \geq m_0 \geq 0$ be two integers, $\mathcal{E}^{(m_0)}$ be a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module. We set $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$ and $E^{(m)} := \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$.*

The image of $E^{(m_0)}$ in $E^{(m)}$ is dense in $E^{(m)}$, where $E^{(m)}$ is equipped with the topology induced by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type (see 7.5.1.8).

Proof. Let $P \in \Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$. We can (uniquely) write $P = \sum_{\underline{k}} a_{\underline{k}} \underline{\partial}^{<\underline{k}>_{(m)}}$, where the $a_{\underline{k}}$ form a sequence of $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}})$ converging to 0 when $|\underline{k}|$ tends to infinity. For any integer $N \geq 0$, we set $P_N := \sum_{|\underline{k}| \leq N} a_{\underline{k}} \underline{\partial}^{<\underline{k}>_{(m)}}$. The sequence $(P_N)_{N\in\mathbb{N}}$ converges to P, for the topology of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$ defined in 7.5.1.8. Let $x \in E^{(m_0)}$ and $1 \otimes x$ be the image of x in $E^{(m)}$. The action of P (resp. P_N) on $1 \otimes x$ is equal to $P \otimes x$ (resp. $P_N \otimes x$). Since the action of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$ on $E^{(m)}$ is continuous, then we get the second equality:

$$P \otimes x = (\lim_{N\to\infty} P_N) \otimes x = \lim_{N\to\infty} (P_N \otimes x) = \lim_{N\to\infty} (1 \otimes P_N \cdot x),$$

the last equality resulting from the remark that $P_N \in \Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}})$. Hence the result. □

Proposition 15.3.1.8. We keep the notations and hypotheses of 15.3.1.7. If the image of the canonical morphism $E^{(m_0)} \to E^{(m)}/IE^{(m)}$ is a K_x-vector space of finite dimension then this map is surjective. In particular, $E^{(m)}/IE^{(m)}$ has finite dimension on K.

Proof. We endow $E^{(m)}/IE^{(m)}$ with the quotient topology induced by the topology of $E^{(m)}$ given by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type (see 7.5.1.8). Let us denote by $G^{(m)}$, the image of $E^{(m_0)} \to E^{(m)}/IE^{(m)}$. As $G^{(m)}$ is of finite dimension on K, then it is a K-Banach space. Hence, $G^{(m)}$ is closed in $E^{(m)}/IE^{(m)}$. Following 15.3.1.7, $G^{(m)}$ is dense in $E^{(m)}/IE^{(m)}$. Hence $G^{(m)} = E^{(m)}/IE^{(m)}$ and we are done. □

Lemma 15.3.1.9. *Let $(M^{(m)})_{m\in\mathbb{N}}$ be a inductive system of Banach spaces such that $M := \varinjlim_m M^{(m)}$ is a K-vector space of finite dimension. Then, for any integer m_0, there exists $m_1 \geq m_0$ such that, for any $m \geq m_1$, the canonical arrow $\operatorname{Im}(M^{(m_0)} \to M^{(m)}) \to M$ is injective.*

Proof. Let $m_0 \in \mathbb{N}$. First, suppose $M = 0$. Let K_m be the kernel of the arrow $M^{(m_0)} \to M^{(m)}$. Since $M^{(m)}$ is separated, then K_m is closed. 1) Since $M = 0$, then we have $\cup_{m\geq m_0} K_m = M^{(m_0)}$. Since $M^{(m_0)}$ is a Banach K-space, it satisfies the Baire property. This implies that for m_1 large enough $K_{m_1} = M^{(m_0)}$, which allows us to conclude.

2) Let us now consider the general case. Increasing if necessary m_0, we can suppose that $M^{(m_0)} \to M$ is surjective. Let $V^{(m_0)}$ be a K-vector subspace of $M^{(m_0)}$ of finite dimension such that $V^{(m_0)} \to M$ is bijective. Denoting by $V^{(m)}$ the image of $V^{(m_0)}$ in $M^{(m)}$, it is sufficient to prove that $\operatorname{Im}(M^{(m_0)} \to M^{(m)}) \subset V^{(m)}$ for m large enough. This results from the first case used for $(M^{(m)}/V^{(m)})_{m\geq m_0}$. □

Proposition 15.3.1.10. *Suppose $\mathfrak{P}$ affine and $\mathfrak{P}/\mathfrak{S}$ is endowed with coordinates. Fix a closed point x of $\mathfrak{P}$. Furthermore, let $\mathcal{I} \subset \mathcal{O}_{\mathfrak{P}}$ be the ideal defining i_x (see notation 15.3.1.1) and $I := \Gamma(\mathfrak{P}, \mathcal{I})$.*

Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. Let $m_0 \geq 0$ be an integer such that there exist a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ and a $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)} \xrightarrow{\sim} \mathcal{E}$ (see 8.4.1.11). For any integer $m \geq m_0$, we set $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$, $E^{(m)} := \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ and $E := \Gamma(\mathfrak{P}, \mathcal{E})$.

Suppose the following assertions are satisfied:

(i) For any $m \geq m_0$, we can choose a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module $\overset{\circ}{\mathcal{E}}{}^{(m)}$ together with a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\overset{\circ}{\mathcal{E}}{}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(m_0)}$ such that the $\mathcal{V}_x$-module $\overset{\circ}{\mathcal{E}}{}^{(m)}/\mathcal{I}\overset{\circ}{\mathcal{E}}{}^{(m)}$ is p-torsion free.

(ii) E/IE is a K_x-vector space of finite dimension.

Then there exists an integer $m \geq m_0$ such that $E^{(m)}/IE^{(m)}$ is a K_x-vector space of finite dimension.

Proof. Following 15.3.1.5, $\overset{\circ}{E}{}^{(m)}/I\overset{\circ}{E}{}^{(m)}$ is p-torsion free, p-adically separated and complete. and the subset $I\overset{\circ}{E}{}^{(m)}$ is closed in $\overset{\circ}{E}{}^{(m)}$. Hence the subset $IE^{(m)}$ is closed in $E^{(m)}$, for the linear topology given by the p-adic topology of $\overset{\circ}{E}{}^{(m)}$. This topology is equal to topology induced by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type (see 7.5.1.8). Hence, $E^{(m)}/IE^{(m)}$ is a Banach K-vector space for the quotient topology. Since the inductive system $(E^{(m)})_{m \in \mathbb{N}}$ can be considered as an inductive system in the category of A-modules whose inductive limit is isomorphic to E, then by applying the functor $A/I \otimes_A -$, which commutes to inductive limits, to this isomorphism we get the following one $E/IE \xrightarrow{\sim} \varinjlim_m E^{(m)}/IE^{(m)}$. By using Lemma 15.3.1.9 for the sequence $(E^{(m)}/IE^{(m)})_{m \geq m_0}$, since E/IE has finite dimension, then so is the image of $E^{(m_0)}/IE^{(m_0)} \to E^{(m)}/IE^{(m)}$ for m large enough. Hence, using 15.3.1.8, we get that $E^{(m)}/IE^{(m)}$ has finite dimension for m large enough.

□

Lemma 15.3.1.11. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module, $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent. Then there exists a dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{E}|\mathfrak{U}$ is a free $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-module of finite type.*

Proof. We can suppose $\mathfrak{P}$ affine and endowed with local coordinates. Let $m \in \mathbb{N}$. Following 11.1.1.6.(a), there exists a p-torsion free $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module $\overset{\circ}{\mathcal{E}}$, coherent over $\mathcal{O}_{\mathfrak{P}}$ together with a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\overset{\circ}{\mathcal{E}}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. We get the coherent $\mathcal{D}^{(m}_{P}$-module $\mathcal{F} := \mathcal{O}_P \otimes_{\mathcal{O}_{\mathfrak{P}}} \overset{\circ}{\mathcal{E}}$, which is also $\mathcal{O}_P$-coherent. It follows from 4.1.3.28, that there exist $\mathfrak{U} \subset \mathfrak{P}$ an affine dense open such that $\mathcal{F}|U$ is a free $\mathcal{O}_U$-module of finite type. With 15.3.1.3, since $\overset{\circ}{\mathcal{E}}$ is p-torsion free, this yields that $\overset{\circ}{\mathcal{E}}|\mathfrak{U}$ is a free $\mathcal{O}_{\mathfrak{U}}$-module of finite type and we are done. □

Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module such that for any closed point x of P, the K-vector space $i_x^*(\mathcal{E})$ is of finite dimension. A technical additional problem is that when the level m increases, the open subset above which the coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module associated to $\mathcal{E}$ becomes $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent a priori narrows. The phenomenon of contagiosity of the lemma 15.3.1.12 below allow us to solve this obstacle.

Lemma 15.3.1.12. *We suppose $\mathfrak{P}$ affine and endowed with local coordinates. Let $m' \geq m$ be two integers, $\mathfrak{U}$ be an affine dense open subset of $\mathfrak{P}$, $\mathcal{E}$ be a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-module such that $\mathcal{E}$ is a projective $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module of finite type.*

We suppose that $\Gamma(\mathfrak{U}, \mathcal{E})$ is endowed with a structure of $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-module extending its structure of $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module. Then $\Gamma(\mathfrak{P}, \mathcal{E})$ is endowed with a unique of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-module extending its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module.

Proof. The uniqueness comes from the fact that $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$ is dense in $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$ (e.g. see 15.3.1.7). Let us check the existence. Let $e \in \Gamma(\mathfrak{P}, \mathcal{E})$ and $1 \otimes e$ the induced element of $\Gamma(\mathfrak{U}, \mathcal{E})$. As the elements $\underline{\partial}^{\langle \underline{k} \rangle_{(m')}}$ are of norm 1 in $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$ (for the Gauss norm), the family $\underline{\partial}^{\langle \underline{k} \rangle_{(m')}} \cdot (1 \otimes e)$ with $\underline{k}$ going through $\mathbb{N}^d$ is bounded for the topology of $\Gamma(\mathfrak{U}, \mathcal{E})$ given by its structure of $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-module of finite type.

Let us check now that the family $\{\underline{\partial}^{\langle \underline{k}\rangle_{(m')}} \cdot e | \underline{k} \in \mathbb{N}^d\}$ is a bounded subset of $\Gamma(\mathfrak{P}, \mathcal{E})$. Following 7.5.2.6, the topology of $\Gamma(\mathfrak{P}, \mathcal{E})$ (resp. $\Gamma(\mathfrak{U}, \mathcal{E})$) given by its structure of $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$-module of finite type or $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type (resp. $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$-module of finite type or $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-module of finite type) coincide. Let N be an integer and $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module such that $\mathcal{E} \oplus \mathcal{F} \xrightarrow{\sim} \mathcal{O}^N_{\mathfrak{P},\mathbb{Q}}$. We get the commutative diagram:

$$\begin{array}{ccccc} \Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})^N & \xrightarrow{\sim} & \Gamma(\mathfrak{P}, \mathcal{E}) \oplus \Gamma(\mathfrak{P}, \mathcal{F}) & \twoheadrightarrow & \Gamma(\mathfrak{P}, \mathcal{E}) \\ \downarrow & & & & \downarrow \\ \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})^N & \xrightarrow{\sim} & \Gamma(\mathfrak{U}, \mathcal{E}) \oplus \Gamma(\mathfrak{U}, \mathcal{F}) & \twoheadrightarrow & \Gamma(\mathfrak{U}, \mathcal{E}) \end{array}$$

whose right injective arrow sends $\underline{\partial}^{\langle \underline{k}\rangle_{(m')}}.e$ on $\underline{\partial}^{\langle \underline{k}\rangle_{(m')}}.(1 \otimes e)$ (that does make sense because $\underline{\partial}^{\langle \underline{k}\rangle_{(m')}} \in \Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$). Via the top (resp. bottom) morphisms, the topology of $\Gamma(\mathfrak{P}, \mathcal{E})$ (resp. $\Gamma(\mathfrak{U}, \mathcal{E})$) is the quotient topology induced by that of $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})^N$ (resp. $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})^N$). As $\mathfrak{P}/\mathfrak{S}$ is smooth and U is dense in P, then the morphism $\Gamma(P, \mathcal{O}_P) \to \Gamma(U, \mathcal{O}_P)$ is injective. Since $\mathfrak{P}$ is p-torsion free, this yields $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P}}) \cap \Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}}) = \Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}})$. Hence the family $\{\underline{\partial}^{\langle \underline{k}\rangle_{(m')}} \cdot e | \underline{k} \in \mathbb{N}^d\}$ is a bounded subset of $\Gamma(\mathfrak{P}, \mathcal{E})$.

Let $P = \sum_{\underline{k} \in \mathbb{N}^d} a_{\underline{k}} \underline{\partial}^{\langle \underline{k}\rangle_{(m')}}$ be an element of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$. This yields the convergence in $\Gamma(\mathfrak{P}, \mathcal{E})$ of the sum $\sum_{\underline{k} \in \mathbb{N}^d} a_{\underline{k}}(\underline{\partial}^{\langle \underline{k}\rangle_{(m')}}.e)$. We set then $P.e := \sum_{\underline{k} \in \mathbb{N}^d} a_{\underline{k}}(\underline{\partial}^{\langle \underline{k}\rangle_{(m')}}.e)$. It remains to check that this endows $\Gamma(\mathfrak{P}, \mathcal{E})$ with a structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-module. By multiplying by $a_{\underline{k}}$ the equalities $\underline{\partial}^{\langle \underline{k}\rangle_{(m')}}.e = \underline{\partial}^{\langle \underline{k}\rangle_{(m')}}.(1 \otimes e)$, and next by summing it, we obtain $P.e = P.(1 \otimes e)$. So, $\Gamma(\mathfrak{P}, \mathcal{E})$ is a $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-submodule of $\Gamma(\mathfrak{U}, \mathcal{E})$. □

Lemma 15.3.1.13. *Let $m_0 \geq 0$ be an integer, $\mathcal{E}^{(m_0)}$ be a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module. Let $m \geq m_0$ be an integer, $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$.*

(a) If $\mathcal{E}^{(m_0)}$ and $\mathcal{E}^{(m)}$ are $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent, then the canonical morphism $\mathcal{E}^{(m_0)} \to \mathcal{E}^{(m)}$ is surjective.

(b) Let $r \in \mathbb{N}$ and $\mathfrak{U}$ be a dense open of $\mathfrak{P}$. We suppose that $\mathcal{E}^{(m_0)}$ is a free $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module of rank r and $\mathcal{E}^{(m)}|\mathfrak{U}$ is a free $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-module of rank r. Then the canonical morphism $\mathcal{E}^{(m_0)} \to \mathcal{E}^{(m)}$ is an isomorphism.

Proof. Since the lemma is local, we can assume that $\mathfrak{P}$ and $\mathfrak{U}$ are affine and $\mathfrak{P}/\mathfrak{S}$ is endowed with coordinates.

a) Let us prove the first statement. Following [BGR84, 3.7.3.1], by endowing $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)})$ and $\Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ with the topology given by their structure of $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$-module of finite type, the image of $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)})$ in $\Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ is closed. Moreover, following 15.3.1.7, the image of $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)})$ in $\Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ is dense, when $\Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ is endowed with the topology given by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type. But, thanks to 7.5.2.6, both topologies on $\Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ coincide. This yields that the canonical morphism $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)}) \to \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ is surjective. Hence, using the theorem of type A for coherent $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-modules, we get the canonical morphism $\mathcal{E}^{(m_0)} \to \mathcal{E}^{(m)}$ is an isomorphism.

b) i) Let us check the canonical morphism $\mathcal{E}^{(m_0)}|\mathfrak{U} \to \mathcal{E}^{(m)}|\mathfrak{U}$ is an isomorphism. Following the part a), this morphism is surjective. Let $K(\mathfrak{U})$ be the field of fraction of the integral ring $B(\mathfrak{U}) := \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$. By applying the functor $K(\mathfrak{U}) \otimes_{B(\mathfrak{U})} -$ to the surjective morphism $\Gamma(\mathfrak{U}, \mathcal{E}^{(m_0)}) \to \Gamma(\mathfrak{U}, \mathcal{E}^{(m)})$ of free $B(\mathfrak{U})$-module of rank r, we get a surjective morphism of $K(\mathfrak{U})$-vector spaces of dimension r which is therefore injective. Hence, the morphism $\Gamma(\mathfrak{U}, \mathcal{E}^{(m_0)}) \to \Gamma(\mathfrak{U}, \mathcal{E}^{(m)})$ is injective and therefore bijective. Via theorem of type A for of coherent $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-modules, this yields the canonical morphism $\mathcal{E}^{(m_0)}|\mathfrak{U} \to \mathcal{E}^{(m)}|\mathfrak{U}$ is an isomorphism.

b)ii) We end the proof. It follows from i) that $\Gamma(\mathfrak{U}, \mathcal{E}^{(m_0)})$ is endowed with a structure of $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module extending its structure of $\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}})$-module. With 15.3.1.12, this yields that $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)})$ is endowed with a (unique) structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module extending its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}})$. This implies that the canonical morphism $\rho_m \colon \Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)}) \to \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ admits a canonical retraction. The morphism ρ_m is therefore injective. Let us check now the surjectivity ρ_m. The canonical morphism $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)}) \to \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$, where for $m' = m$ or $m' = m_0$ $\Gamma(\mathfrak{P}, \mathcal{E}^{(m')})$ is endowed with the topology given by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m')}_{\mathfrak{P},\mathbb{Q}})$-module of finite type, is continuous. Moreover, it follows from 7.5.2.6

that the topology of $\Gamma(\mathfrak{P}, \mathcal{E}^{(m_0)})$ induced by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type and the one induced by its structure of $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-module of finite type are identical. As $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}})$ is dense in $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$, since ρ_m is continuous and $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}})$-linear, this yields that ρ_m is $\Gamma(\mathfrak{P}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}})$-linear. Hence, its image is therefore closed (see the last paragraph of 7.5.1.8). As it is also dense (see 15.3.1.7), the morphism ρ_m is therefore surjective. Hence, ρ_m is a bijection. Similarly, for any affine open $\mathfrak{P}'$ of $\mathfrak{P}$ the canonical homomorphism $\rho_m \colon \Gamma(\mathfrak{P}', \mathcal{E}^{(m_0)}) \to \Gamma(\mathfrak{P}', \mathcal{E}^{(m)})$ is an isomorphism. Hence, we get the canonical morphism $\mathcal{E}^{(m_0)} \to \mathcal{E}^{(m)}$ is an isomorphism. □

Lemma 15.3.1.14. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. Let $m_0 \geq 0$ be an integer such that there exists a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ inducing $\mathcal{E}$ by extension (8.4.1.11). For any integer $m \geq m_0$, set $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$. Suppose there exist a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ and a sequence of dense opens $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of $\mathfrak{P}$ such that $\mathcal{E}^{(m_n)}|\mathfrak{U}_n$ is a free $\mathcal{O}_{\mathfrak{U}_n,\mathbb{Q}}$-module of finite rank. Then, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{E}|\mathfrak{U}_{n_0}$ is $\mathcal{O}_{\mathfrak{U}_{n_0},\mathbb{Q}}$-coherent.*

Proof. We can suppose the sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ is decreasing. It follows from Lemma 15.3.1.13.(a) that the sequence $(\operatorname{rank} \mathcal{E}^{(m_n)}|\mathfrak{U}_n)_{n\in\mathbb{N}}$ is decreasing. Let r be the minimal rank. Hence, there exists n_0 such that for any $n \geq n_0$ the $\mathcal{O}_{\mathfrak{U}_n,\mathbb{Q}}$-module $\mathcal{E}^{(m_n)}|\mathfrak{U}_n$ is free of rank r. Using Lemma 15.3.1.13.(b), this yields that for any $n \geq n_0$, the canonical morphism $\mathcal{E}^{(m_{n_0})}|\mathfrak{U}_{n_0} \to \mathcal{E}^{(m_n)}|\mathfrak{U}_{n_0}$ is an isomorphism. By taking the inductive limit on n, this yields that $\mathcal{E}^{(m_{n_0})}|\mathfrak{U}_{n_0} \to \mathcal{E}|\mathfrak{U}_{n_0}$ is an isomorphism (of free $\mathcal{O}_{\mathfrak{U}_{n_0},\mathbb{Q}}$-module of rank r). □

Proposition 15.3.1.15. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. Let $m_0 \geq 0$ be an integer such that there exists a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ inducing $\mathcal{E}$ by extension (8.4.1.11). For any integer $m \geq m_0$, let us denote by $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$. The following conditions are equivalent.*

(a) The sheaf $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent.

(b) There exists $m_1 \geq m_0$ such that for any $m \geq m_1$ the sheaf $\mathcal{E}^{(m)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent and the canonical homomorphism $\mathcal{E}^{(m)} \to \mathcal{E}$ is an isomorphism.

(c) There exists a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ such that $\mathcal{E}^{(m_n)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent.

Proof. The assertion is local and we can suppose that $\mathfrak{P}$ is affine. If $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent then $\mathcal{E}$ is associated to a convergent isocrystal on $\mathfrak{P}/\mathfrak{S}$ (11.1.1.3). Hence, it follows from 11.1.1.10 that we get the implication $(a) \Rightarrow (b)$. Since $(b) \to (c)$ is obvious, let us check $(c) \to (a)$. Let us suppose that there exists a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ such that $\mathcal{E}^{(m_n)}$ is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent. Then, it follows from Lemma 15.3.1.13.(a) that the canonical morphism $\Gamma(\mathfrak{X}, \mathcal{E}^{(m_n)}) \to \Gamma(\mathfrak{X}, \mathcal{E}^{(m_{n+1})})$ is surjective. By taking the limit on n, this yields the surjectivity of $\Gamma(\mathfrak{X}, \mathcal{E}^{(m_n)}) \to \Gamma(\mathfrak{X}, \mathcal{E})$. Following 11.1.1.8, $\mathcal{E}$ is then $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent. □

Lemma 15.3.1.16. *Let $\mathcal{F}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module. Then there exists an affine dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{F}|\mathfrak{U}$ is isomorphic to the p-adic completion of a free $\mathcal{O}_{\mathfrak{U}}$-module.*

Proof. Since the sheaf $\mathcal{O}_P \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{F}$ is a coherent $\mathcal{D}^{(m)}_P$-module, then it follows from 4.1.3.28 that there exists an affine dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{O}_P \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{F}|U$ is a free $\mathcal{O}_U$-module. Using 15.3.1.3, this yields that $\Gamma(\mathfrak{U}, \mathcal{F})$ is isomorphic to the p-adic completion of a free $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})$-module. Moreover, taking the projective limit of 4.3.4.6.1, we obtain the canonical isomorphism

$$\mathcal{O}_{\mathfrak{U}} \widehat{\otimes}_{\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P}})} \Gamma(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}} \widehat{\otimes}_{\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}})} \Gamma(\mathfrak{U}, \mathcal{F}).$$

As $\mathcal{F}|\mathfrak{U}$ is a coherent (and therefore pseudo quasi-coherent) $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}}$-module, via the theorem of type A of 7.2.3.16.(i) and 7.2.3.13.(i), we get therefore the isomorphisms:

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}} \widehat{\otimes}_{\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}})} \Gamma(\mathfrak{U}, \mathcal{F}) \xleftarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}} \otimes_{\Gamma(\mathfrak{U}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}})} \Gamma(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}|\mathfrak{U}.$$

This yields the isomorphism $\mathcal{O}_{\mathfrak{U}} \widehat{\otimes}_{\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{P}})} \Gamma(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}|\mathfrak{U}$ and we are done. □

Before considering the general case (see 15.3.1.19), let us first check the case where the residue field k is uncountable.

Lemma 15.3.1.17. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module such that for any closed point x of P, the K-vector space $i_x^*(\mathcal{E})$ is of finite dimension (see notation 15.3.1.1). Suppose k is uncountable. Then there exists an affine dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{E}|\mathfrak{U}$ is a free $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-module of finite rank.*

Proof. 1) Since the theorem is local, let us suppose $\mathfrak{P}$ affine, integral and $\mathfrak{P}/\mathfrak{S}$ is endowed with coordinates. Let $m_0 \geq 0$ be an integer such that there exist a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ and a $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)} \xrightarrow{\sim} \mathcal{E}$ (see 8.4.1.11). For any integer $m \geq m_0$, we set $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$, $E^{(m)} := \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ and $E := \Gamma(\mathfrak{P}, \mathcal{E})$. Let $\mathring{\mathcal{E}}^{(m_0)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P}}$-module together with a $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}^{(m_0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(m_0)}$ (see 7.4.5.2). We denote by $\mathring{\mathcal{E}}^{(m)}$ the quotient of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P}}} \mathring{\mathcal{E}}^{(m_0)}$ by its π-torsion part. Following 7.4.5.1, $\mathring{\mathcal{E}}^{(m)}$ is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module. We set $\mathring{E}^{(m)} := \Gamma(\mathfrak{P}, \mathring{\mathcal{E}}^{(m)})$. It follows from 15.3.1.16 that for any integer $m \geq m_0$, there exists an affine dense open $\mathfrak{U}_m$ of $\mathfrak{P}$ such that $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m$ is isomorphic to the p-adic completion of a free $\mathcal{O}_{\mathfrak{U}_m}$-module. As k is uncountable, then $\cap_{m \geq m0} \mathfrak{U}_m$ is not empty. Let x be a closed point of $\cap_{m \geq m0} \mathfrak{U}_m$. Shrinking the sequence $(\mathfrak{U}_m)_{m \geq m_0}$ if necessary, we can suppose $\mathfrak{U}_m \supset \mathfrak{U}_{m+1}$. Let us denote by $\mathcal{I} \subset \mathcal{O}_{\mathfrak{P}}$ the ideal induced by i_x, $I := \Gamma(\mathfrak{P}, \mathcal{I})$ and $A := \Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}})$.

2) Let's prove that there exists $m_1 \geq m_0$ such that for any $m \geq m_1$, the sheaf $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m$ is a free $\mathcal{O}_{\mathfrak{U}_m}$-module of finite type.

From the propositions 15.3.1.2 and 15.3.1.5.(a), we get $E/IE \xrightarrow{\sim} i_x^*(\mathcal{E})$, $E^{(m)}/IE^{(m)} \xrightarrow{\sim} i_x^*(\mathcal{E}^{(m)})$ and $\mathring{E}^{(m)}/I\mathring{E}^{(m)} \xrightarrow{\sim} i_x^*(\mathring{\mathcal{E}}^{(m)})$. Hence E/IE is K_x-vector spaces of finite dimension. Let us denote by $i_x^{(m)}$ the closed immersion $\mathrm{Spf}\, \mathcal{V}_x \hookrightarrow \mathfrak{U}_m$ induced by i_x. As $i_x^*(\mathring{\mathcal{E}}^{(m)}) \xrightarrow{\sim} i_x^{(m)*}(\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m)$, since following the step 1) the $\mathcal{O}_{\mathfrak{U}_m}$-module $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m$ is flat, then $\mathring{E}^{(m)}/I\mathring{E}^{(m)}$ is p-torsion free. Hence, using 15.3.1.10, there exists an integer $m_1 \geq m_0$ such that for any $m \geq m_1$ the K_x-vector space $E^{(m)}/IE^{(m)}$ is of finite dimension. Let $m \geq m_1$ and J_m be a set such that $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m$ is of the form $(\mathcal{O}_{\mathfrak{P}}^{(J_m)})^{\wedge}$. Then $\mathring{E}^{(m)}/I\mathring{E}^{(m)}$ is isomorphic to $(\mathcal{V}_x^{(J_m)})^{\wedge}$ (see 15.3.1.5.(b)). Since $(\mathring{E}^{(m)}/I\mathring{E}^{(m)})_{\mathbb{Q}} \xrightarrow{\sim} E^{(m)}/IE^{(m)}$, since $E^{(m)}/IE^{(m)}$ is a K_x-vector space of finite dimension, then J_m much be finite and we are done.

3) It follows from the step 2) that $\mathcal{E}^{(m)}|\mathfrak{U}_m$ is a free $\mathcal{O}_{\mathfrak{U}_m,\mathbb{Q}}$-module of finite rank for any $m \geq m_1$. Hence, using 15.3.1.14 we are done. □

In order to extend to Lemma 15.3.1.17 to the general case, we proceed by descent via the Lemma:

Lemma 15.3.1.18. *Let $\mathcal{V} \to \mathcal{V}'$ be a morphism of $\mathrm{DVR}(\mathcal{V})$ (see notation 9.2.6.12), $\mathfrak{S}' := \mathrm{Spf}(\mathcal{V}')$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f\colon \mathfrak{P}' \to \mathfrak{P}$ the canonical projection. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}$-module and $\mathcal{E}' := f^!(\mathcal{E}) := \mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}',\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}} f^{-1}\mathcal{E}$, i.e. $\mathcal{E}'$ is the base change of $\mathcal{E}$ via $\mathcal{V} \to \mathcal{V}'$ (see 9.2.7.1). The following assertions are then equivalent:*

(a) There exists a dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{E}|\mathfrak{U}$ is a free $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-module of finite type.

(b) There exists a dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{E}|\mathfrak{U}$ is $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-coherent.

(c) There exists a dense open $\mathfrak{U}'$ of $\mathfrak{P}'$ such that $\mathcal{E}'|\mathfrak{U}'$ is a free $\mathcal{O}_{\mathfrak{U}',\mathbb{Q}}$-module of finite type.

(d) There exists a dense open $\mathfrak{U}'$ of $\mathfrak{P}'$ such that $\mathcal{E}'|\mathfrak{U}'$ is $\mathcal{O}_{\mathfrak{U}',\mathbb{Q}}$-coherent.

Proof. The equivalence between (a) and (b) (resp. between (c) and (d)) follows from 15.3.1.11. The implication $(a) \Rightarrow (c)$ is straightforward. Conversely, let us suppose that there exist a dense open $\mathfrak{U}'$ of $\mathfrak{P}'$ such that $\mathcal{E}'|\mathfrak{U}'$ is a free $\mathcal{O}_{\mathfrak{U}',\mathbb{Q}}$-module of finite rank r. Since the property (a) is local on $\mathfrak{P}$, then we can suppose $\mathfrak{P}$ affine, integral and $\mathfrak{P}/\mathfrak{S}$ is endowed with coordinates. For any integer $m \geq m_0$, we set $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$, $E^{(m)} := \Gamma(\mathfrak{P}, \mathcal{E}^{(m)})$ and $E := \Gamma(\mathfrak{P}, \mathcal{E})$. Let $\mathring{\mathcal{E}}^{(m_0)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P}}$-module together with a $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}^{(m_0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(m_0)}$ (see 7.4.5.2). We denote by $\mathring{\mathcal{E}}^{(m)}$ the quotient of $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P}}} \mathring{\mathcal{E}}^{(m_0)}$ by its π-torsion part. Following 7.4.5.1, $\mathring{\mathcal{E}}^{(m)}$ is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module. Hence, following 15.3.1.16, for any $m \geq m_0$, there exist an affine dense open $\mathfrak{U}_m$

of $\mathfrak{P}$, a set A_m and an $\mathcal{O}_{\mathfrak{U}_m}$-linear isomorphism of the form $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m \xrightarrow{\sim} (\mathcal{O}_{\mathfrak{U}_m}^{(A_m)})^{\wedge}$. Shrinking $\mathfrak{U}_m$ if necessary, we can suppose $\mathfrak{U}_m \supset \mathfrak{U}_{m+1}$. The rank (an integer if it is finite, if not we define it equal to $+\infty$) of $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m$ as $\mathcal{O}_{\mathfrak{U}_m}$-module is then the same one than that of $\mathcal{E}^{(m)}|\mathfrak{U}_m$ as $\mathcal{O}_{\mathfrak{U}_m,\mathbb{Q}}$-module.

Let $g\colon \mathfrak{U}' \to \mathfrak{P}$ be the morphism induce by f. Set $g^{(m)!}(\mathcal{E}^{(m)}) := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'/\mathfrak{S}',\mathbb{Q}} \otimes_{g^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}/\mathfrak{S},\mathbb{Q}}} g^{-1}\mathcal{E}^{(m)}$, which is a coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'/\mathfrak{S}',\mathbb{Q}}$-module. We deduce from 9.2.6.5.2 the isomorphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}}$-modules: $\mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}',\mathbb{Q}}} (g^{(m)!}(\mathcal{E}^{(m)})) \xrightarrow{\sim} \mathcal{E}'|\mathfrak{U}'$ (use also 9.2.7.1 and the fact that $g^{(m)!}$ is the composition of the base change $f^{(m)!}$ by $\mathcal{V} \to \mathcal{V}'$ with the restriction $|\mathfrak{U}'$). By 15.3.1.15, as $\mathcal{E}'|\mathfrak{U}'$ is moreover $\mathcal{O}_{\mathfrak{U}',\mathbb{Q}}$-coherent, this yields there exists m_0 such that for any $m \geq m_0$ we have the canonical isomorphism $g^{(m)!}(\mathcal{E}^{(m)}) \xrightarrow{\sim} \mathcal{E}'|\mathfrak{U}'$.

Let $m \geq m_0$, $\mathfrak{U}'_m := g^{-1}(\mathfrak{U}_m)$ and $g_m\colon \mathfrak{U}'_m \to \mathfrak{U}_m$ be the morphism induced by g. We get the isomorphism:

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'_m/\mathfrak{S}'} \otimes_{g_m^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}_m/\mathfrak{S}}} g_m^{-1}(\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m) \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'_m/\mathfrak{S}'} \widehat{\otimes}^{\mathbb{L}}_{g_m^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}_m/\mathfrak{S}}} g_m^{-1}(\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m)$$

$$\xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'_m/\mathfrak{S}'} \widehat{\otimes}^{\mathbb{L}}_{g_m^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}_m/\mathfrak{S}}} g_m^{-1}((\mathcal{O}^{(A_m)}_{\mathfrak{U}_m})^{\wedge}) \overset{7.5.5.12.(c)}{\xrightarrow{\sim}} (\mathcal{O}^{(A_m)}_{\mathfrak{U}'_m})^{\wedge} \qquad (15.3.1.18.1)$$

the last one follows from the coherence of $\mathring{\mathcal{E}}^{(m)}$ and the flatness of $g^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}/\mathfrak{S},\mathbb{Q}} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'/\mathfrak{S}'}$. By tensoring with $\mathbb{Q}$ the composition of 15.3.1.18.1, we get the last $\mathcal{O}_{\mathfrak{U}'_m,\mathbb{Q}}$-linear isomorphism:

$$\mathcal{E}'|\mathfrak{U}'_m \xrightarrow{\sim} g^{(m)!}(\mathcal{E}^{(m)})|\mathfrak{U}'_m \xrightarrow{\sim} ((\mathcal{O}^{(A_m)}_{\mathfrak{U}'_m})^{\wedge})_{\mathbb{Q}}.$$

This yields that for any $m \geq m_0$, the set A_m has r elements and then $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}_m$ is $\mathcal{O}_{\mathfrak{U}_m}$-free of rank r. Hence $\mathcal{E}^{(m)}|\mathfrak{U}_m$ is a free $\mathcal{O}_{\mathfrak{U}_m,\mathbb{Q}}$-module of rank r. We conclude thanks to 15.3.1.14. □

Theorem 15.3.1.19. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module such that for any closed point x of P, $i_x^*(\mathcal{E})$ is a K-vector space of finite dimension. Then there exists a divisor $T' \supset T$ of P such that $({}^{\dagger}T')(\mathcal{E})$ is an isocrystal on $\mathfrak{P}/\mathfrak{S}$ overconvergent along T'.*

Proof. Let $\mathcal{V} \to \mathcal{V}'$ be a morphism of DVR($\mathcal{V}$) (see notation 9.2.6.12), $\mathfrak{S}' := \mathrm{Spf}(\mathcal{V}')$, such that the residue field of $\mathcal{V}'$ is uncountable. Let $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f\colon \mathfrak{P}' \to \mathfrak{P}$ be the canonical projection and $\mathcal{E}' := f^*(\mathcal{E})$ be the coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}',\mathbb{Q}}$-module induced by base change. Then for any closed point x' of P', $i^*_{x'}(\mathcal{E})$ is a K'-space vector of finite dimension. Hence, following 15.3.1.17, there exists a dense open $\mathfrak{U}'$ of $\mathfrak{P}'$ such that $\mathcal{E}'|\mathfrak{U}'$ is a free $\mathcal{O}_{\mathfrak{U}',\mathbb{Q}}$-module of finite type. We deduce from 15.3.1.18 that there exists a dense open $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{E}|\mathfrak{U}$ is a free $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-module of finite type. Shrinking $\mathfrak{U}$if necessary, we can suppose $\mathfrak{U}$ is the open complementary to the support of a divisor. We end the proof thanks to theorem 11.2.1.14.(e). □

Theorem 15.3.1.20. *Let $\mathcal{E}$ be a holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module. There exits a divisor T' containing T such that $({}^{\dagger}T')(\mathcal{E})$ is $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T')_{\mathbb{Q}}$-coherent.*

Proof. We can suppose $\mathfrak{P}$ integral. Moreover, it follows from 11.2.1.14.(e) that we reduce to check there exists a dense open subset $\mathfrak{U}$ of $\mathfrak{P} \setminus T$ such that $\mathcal{E}|\mathfrak{U}$ is coherent over $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$. In particular, we can suppose $T = \emptyset$. By definition of the holonomicity, we have $\dim(\mathcal{E}) \leq \dim P$. Let $m_0 \geq 0$ be an integer such that there exist a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ together with a $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)} \xrightarrow{\sim} \mathcal{E}$ (8.4.1.11). For any integer $m \geq m_0$, let us denote by $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$. Increasing $m_0$if necessary, we have $\dim(\mathcal{E}) = \dim^{(m)}(\mathcal{E}^{(m)})$ for any $m \geq m_0$. Following 15.1.5.5, there exists a dense open subset $\mathfrak{U}_m$ of $\mathfrak{P}$ such that $\mathcal{E}^{(m)}|\mathfrak{U}_m$ is coherent over $\mathcal{O}_{\mathfrak{U}_m,\mathbb{Q}}$ for any $m \geq m_0$. We conclude thanks to 15.3.1.14. □

15.3.1.21. With the notations of the theorem 15.3.1.15, the fact that $\mathcal{E}^{(m_0)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent does not a priori imply that $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent. However, with a Frobenius structure this is the case:

Proposition 15.3.1.22. *Let $\mathcal{E}$ be a coherent F-$\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module, $m_0 \geq 0$ an integer and $\mathcal{E}^{(m_0)}$, the coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module associated with $\mathcal{E}$ (8.4.1.13). Then $\mathcal{E}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent if and only if $\mathcal{E}^{(m_0)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent.*

Proof. Since $F^*\mathcal{E}^{(m_0)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent is and only if $\mathcal{E}^{(m_0)}$ is $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent, then this follows from 15.3.1.15. □

15.3.2 Finite extraordinary fibers and a holonomicity criterion

Definition 15.3.2.1. A coherent $\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}}$-module $\mathcal{E}$ (resp. a complex $\mathcal{E}$ of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}})$) is said to have "finite extraordinary fibers" if, for any closed point x of P, with notation 15.3.1.1 we have $i^!_x(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{S}_x/\mathfrak{S},\mathbb{Q}})$. The property "having finite extraordinary fibers" is closed under devissage, i.e. the complexes having finite extraordinary fibers is a triangulated subcategories of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}}({}^\dagger T)_{\mathbb{Q}})$.

Example 15.3.2.2. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. If $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ then $\mathcal{E}$ has finite extraordinary fibers.

Definition 15.3.2.3. A complex $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ is said to have "finite extraordinary fibers" if, for any closed point x of P, we have $i^{(\bullet)!}_x(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{O}^{(\bullet)}_{\mathfrak{S}_x})$ (see notation 15.3.1.1). The property "having finite extraordinary fibers" is closed under devissage.

15.3.2.4. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ and $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. If $\mathcal{E}^{(\bullet)}$ has finite extraordinary fibers then so is $\mathcal{E}$. Beware that the converse is not clear (but at least with the extra hypotheses of 14.3.3.10 for modules such converse property can be checked). In the case of curves, such an equivalence will be established below (thanks to 14.3.3.10).

We get by devissage the complex analogue of Theorem 15.3.1.19:

Theorem 15.3.2.5. *Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. If $\mathcal{E}$ has finite extraordinary fibers then there exists a divisor $T' \supset T$ of P such that $({}^\dagger T')(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$.*

Proof. We can suppose P integral of dimension d. Thanks to theorem 11.2.1.14.(e), we reduce to check there exists a dense open $\mathfrak{U}'$ of $\mathfrak{P} \setminus T$ such that $\mathcal{E}|\mathfrak{U}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{U}',\mathbb{Q}})$. Consider the set $A := \{n \in \mathbb{Z}$ such that $H^n(\mathcal{E}) \neq 0\}$. Let N be the maximum of A. Let x be a closed point of P. By using the spectral sequence $H^r i^!_x H^s(\mathcal{E}) \to H^n i^!_x(\mathcal{E})$, since $\mathcal{E}$ has finite extraordinary fibers then we get $i^*_x(H^N(\mathcal{E})) \xrightarrow{\sim} H^{N+d} i^!_x(\mathcal{E})$ is a K-vector space of finite dimension. Hence, following 15.3.1.19 there exists a dense open $\mathfrak{U}''$ of $\mathfrak{P} \setminus T$ such that $H^N(\mathcal{E})|\mathfrak{U}'$ is $\mathcal{O}_{\mathfrak{U}',\mathbb{Q}}$-coherent. In particular, $H^N(\mathcal{E})|\mathfrak{U}'$ has finite extraordinary fibers (see 15.3.2.2). Proceeding by induction on the cardinal of A, we conclude therefore the proof by devissage. □

We will need the following result Lemmas ot check Theorem 15.3.2.8:

Lemma 15.3.2.6. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module having finite extraordinary fibers. Let Z be the support of $\mathcal{E}$. There exists an open set $\mathfrak{U}$ of $\mathfrak{P}$ such that $Y := Z \cap U$ is affine, smooth and dense in Z and such that $\mathcal{E}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$ (see notation 12.2.1.4) In particular, $\mathcal{E}|\mathfrak{U}$ is holonomic and nonzero.*

Proof. Let $\mathfrak{U}'$ be an open set of $\mathfrak{P}$ such that $Y' := Z \cap U'$ is affine, smooth and dense in Z. We can lift Y' to an affine and smooth $\mathfrak{S}$-formal scheme $\mathfrak{Y}'$. Since $\mathfrak{U}'$ is smooth and $\mathfrak{Y}'$ is affine, then there exists a morphism of smooth $\mathfrak{S}$-formal schemes of the form $v \colon \mathfrak{Y}' \hookrightarrow \mathfrak{U}'$ lifting $Y' \hookrightarrow \mathfrak{U}'$. Since $\mathcal{E}|\mathfrak{U}'$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}}$-module with support in Y' and having finite extraordinary fibers, it comes from the theorem of Berthelot-Kashiwara (see 9.3.5.9)) that $v^!(\mathcal{E}|\mathfrak{U}')$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Y}',\mathbb{Q}}$-module and having finite extraordinary fibers. Following 15.3.1.19, then there exists an open set $\mathfrak{U}$ of $\mathfrak{U}'$ such that $\mathfrak{Y} := \mathfrak{Y}' \cap \mathfrak{U}$ is affine and dense in $\mathfrak{Y}'$ and such that $v^!(\mathcal{E}|\mathfrak{U}')|\mathfrak{Y}$ is moreover $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Denote by $u \colon \mathfrak{Y} \hookrightarrow \mathfrak{U}$ the closed immersion induced by v. Then $u^!(\mathcal{E}|\mathfrak{U})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}}$-module which is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Since $\mathcal{E}|\mathfrak{U}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}}$-module supported in Y, then this yields $\mathcal{E}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$. The fact that $\mathcal{E}|\mathfrak{U}$ is holonomic follows from 15.2.4.20. □

Lemma 15.3.2.7. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be some coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules having finite extraordinary fibers. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are not all zero, then there exists an open set $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{E}_1|\mathfrak{U}, \ldots, \mathcal{E}_n|\mathfrak{U}$ are holonomic and not all zero.*

Proof. We proceed by induction on the number N of element $i \in \{1,\dots,n\}$ such that $\mathcal{E}_i \neq 0$. When $N=1$, this comes from 15.3.2.6. Suppose now $N \geq 2$. Reindexing if necessary, we can suppose $\mathcal{E}_1 \neq 0$. Following 15.3.2.6, there exists an open set $\mathfrak{U}'$ of $\mathfrak{P}$ such that $\mathcal{E}_1|\mathfrak{U}'$ is holonomic and nonzero. The case where, for any $i \geq 2$, $\mathcal{E}_i|\mathfrak{U}' = 0$ is straightforward. Otherwise, suppose that the $\mathcal{E}_2|\mathfrak{U}',\dots,\mathcal{E}_n|\mathfrak{U}'$ are not all zero. By induction hypothesis, there exists an open set $\mathfrak{U}$ of $\mathfrak{U}'$ such that $\mathcal{E}_2|\mathfrak{U},\dots,\mathcal{E}_n|\mathfrak{U}$ are holonomic and not all zero. The fact that $\mathcal{E}_1|\mathfrak{U}$ is also holonomic allows us to conclude. □

Theorem 15.3.2.8 (Holonomicity criterion)**.** *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. If the cohomological spaces of $\mathbb{D}_T(\mathcal{E})$ have finite extraordinary fibers, then $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic.*

Proof. By definition of the $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic, we can suppose T is empty. Let d the dimension of P. We can suppose P integral. For any integer $d \geq i \geq 0$, set $\mathcal{F}_i := H^{-i}\mathbb{D}(\mathcal{E})$. For $d \geq i \geq 1$, let Z_i be the support of $\mathcal{F}_i$ and $Z := \cup_{i=1,\dots,d} Z_i$ their union. Following the homological criterion of holonomicity of 15.2.4.8, $\mathcal{E}$ is holonomic if and only if $\mathcal{F}_1,\dots,\mathcal{F}_d$ are all zero. By the absurd, suppose that $\mathcal{F}_1,\dots,\mathcal{F}_d$ are not all zero. It comes from 15.3.2.7 that there exists an open set $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathcal{F}_1|\mathfrak{U},\dots,\mathcal{F}_{\mathrm{r}}|\mathfrak{U}$ are holonomic and not all zero. By 15.2.4.7, $\mathcal{F}_0$ is holonomic too. So, the cohomological spaces of $\mathbb{D}(\mathcal{E}|\mathfrak{U})$ are all holonomic, i.e., the complex $\mathbb{D}(\mathcal{E}|\mathfrak{U})$ is holonomic. Hence, thanks to theorem of biduality (see 8.7.7.3) and to the preservation of the holonomicity by the functor $\mathbb{D}$ (see 15.2.4.15), $\mathcal{E}|\mathfrak{U}$ is holonomic. However, following the homological criterion of holonomicity this implies that $\mathcal{F}_1|\mathfrak{U},\dots,\mathcal{F}_d|\mathfrak{U}$ are all zero. Hence we get a contradiction. □

15.3.3 The case of curves

We suppose in this subsection that P has dimension 1.

Proposition 15.3.3.1. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. The following assertions are equivalent:*

(a) The sheaf $\mathcal{E}$ is a holonomic $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module.

(b) There exits a dense open $\mathfrak{U}$ of $\mathfrak{P} \setminus T$ such that $\mathcal{E}|\mathfrak{U}$ is $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-coherent.

(c) There exits a divisor T' containing T such that $({}^\dagger T')(\mathcal{E})$ is $\mathcal{O}_{\mathfrak{P}}({}^\dagger T')_{\mathbb{Q}}$-coherent.

Proof. Following 11.2.1.14.(e), we get the equivalence between (b) and (c). The implication $(a) \Rightarrow (c)$ is always true (see 15.3.1.20). Let's prove now the implication $(b) \Rightarrow (a)$. By definition of the holonomicity, we can suppose $\mathfrak{P}$ integral and T empty. Let $m_0 \geq 0$ be an integer such that there exist a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}^{(m_0)}$ together with a $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m)} \xrightarrow{\sim} \mathcal{E}$ (8.4.1.11). For any integer $m \geq m_0$, let us denote by $\mathcal{E}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{P},\mathbb{Q}}} \mathcal{E}^{(m_0)}$. Let $\mathring{\mathcal{E}}^{(m)}$ be a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}$-module together with a $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{E}}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}^{(m)}$ (see 7.4.5.2). Since by hypothesis $\mathcal{E}|\mathfrak{U}$ is $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-coherent, then following 15.3.1.15 there exists $m_1 \geq m_0$ such that for any $m \geq m_1$ the sheaf $\mathcal{E}^{(m)}|\mathfrak{U}$ is $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-coherent and the canonical homomorphism $\mathcal{E}^{(m)}|\mathfrak{U} \to \mathcal{E}|\mathfrak{U}$ is an isomorphism. Hence, it follows from 11.1.1.11 that for any $m \geq m_1$ the sheaf $\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}$ is $\mathcal{O}_{\mathfrak{U}}$-coherent. It follows from 15.1.5.3 that $\mathrm{Car}^{(m)}(\mathring{\mathcal{E}}^{(m)}|\mathfrak{U}) \subset T^*_U U$. Since U is dense, this implies that $\dim \mathrm{Car}^{(m)}(\mathring{\mathcal{E}}^{(m)}) \leq 1$. Hence, for any $m \geq m_1$, we have $\dim \mathrm{Car}^{(m)}(\mathcal{E}^{(m)}) \leq 1$. We conclude by passing to the limit. □

Remark 15.3.3.2. When $\mathfrak{X}$ is not a formal curve, the implication $(b) \Rightarrow (a)$ of 15.3.3.1 is false in general. For example, let $\mathfrak{Z}$ (resp. $\mathfrak{P}$) be the p-adic completion of $\mathbb{A}^1_{\mathcal{V}}$ (resp. $\mathbb{A}^2_{\mathcal{V}}$) and $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{P}$ be the closed immersion given by $t_2 = 0$ if t_1, t_2 are coordinates of $\mathfrak{X}/\mathfrak{S}$. Then $\mathcal{E} := u_+(\mathcal{D}^\dagger_{\mathfrak{Z},\mathbb{Q}})$ satisfies (b) but it is not holonomic since it has dimension 3.

Lemma 15.3.3.3. *Let k be a perfect field and Z be a reduced k-scheme of finite type and dimension* 0*. So Z is a finite and etale k-scheme.*

Proof. As Z is a Noetherian scheme of dimension 0, according to [Gro65, 0.14.1.9] and [Gro60, 2.8.2], Z is an Artinian scheme. If A is the ring corresponding to Z, A is therefore equal to the (finite) product of local (Artinian) rings of Z. As we further assume that Z is reduced, and as a reduced local Artinian ring is a field, then the local rings of A are all fields. Now, as k is perfect, the fields of Z are separable extensions of k. Thus, A is isomorphic to a finite product of finite extensions separable from k. Hence the result. □

Proposition 15.3.3.4. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ and $\mathcal{E} := \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. The following assertions are equivalent.*

(a) The complex $\mathcal{E}$ has finite extraordinary fibers (in the sense of definition 15.3.2.1).

(b) For any $n \in \mathbb{Z}$, the module $H^n\mathcal{E}$ has finite extraordinary fibers (in the sense of definition 15.3.2.1).

(c) The complex $\mathcal{E}^{(\bullet)}$ has finite extraordinary fibers (in the sense of definition 15.3.2.3).

(d) For any $n \in \mathbb{Z}$, the module $H^n\mathcal{E}^{(\bullet)}$ has finite extraordinary fibers (in the sense of definition 15.3.2.3).

(e) For any divisor D de P, $({}^{\dagger}D)(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}({}^{\dagger}T))$.

(f) There exists a divisor T' containing T such that $({}^{\dagger}T')(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T')_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

Proof. The implications $(b) \Rightarrow (a)$ is obvious. Conversely, suppose for any $n \in \mathbb{Z}$, the module $H^n\mathcal{E}$ has finite extraordinary fibers. Using the spectral sequence $E_2^{r,s} = H^r i^!_x(H^s(\mathcal{E})) \Rightarrow H^{r+s} i^!_x(\mathcal{E})$, since $E_2^{r,s} = 0$ is $r \notin \{0,1\}$, then we get the exact sequence for any $n \in \mathbb{Z}$:

$$0 \to H^1 i^!_x(H^n(\mathcal{E})) \to H^{n+1} i^!_x(\mathcal{E}) \to H^0 i^!_x(H^{n+1}(\mathcal{E})) \to 0.$$

Since $H^n i^!_x(\mathcal{E})$ is a K-vector space of finite dimension for any $n \in \mathbb{Z}$, then so are $H^0 i^!_x(H^n(\mathcal{E}))$ an $H^1 i^!_x(H^n(\mathcal{E}))$ which means that $H^n(\mathcal{E})$ has finite extraordinary fibers. The equivalence between (b) and (d) is a consequence of 14.3.3.10. The implications $(d) \Rightarrow (c) \to (a)$ are straightforward.

It follows from 14.3.3.1 that $\mathcal{E}$ has finite extraordinary fibers if and only if for any divisor D_x of P whose support is a closed point x, $\mathbb{R}\underline{\Gamma}^{\dagger}_{D_x}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Since a reduced divisor of P has dimension 0 then it is finite and etale over k and then D is sum of its connected components which are closed points. Hence, using 13.1.4.16.1, we get the equivalence between (c) and (e).

It remains to check $(a) \Leftrightarrow (f)$. Suppose $\mathcal{E}$ has finite extraordinary fibers. Then following 15.3.2.5 there exists a divisor $T' \supset T$ of P such that $({}^{\dagger}T')(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Since $\mathcal{E}$ has finite extraordinary fibers then we have also $\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ (see above). Hence, we are done. Conversely, suppose $\mathcal{E}$ satisfies (f). Suppose there exists a divisor T' such that $({}^{\dagger}T')(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T')_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Then $({}^{\dagger}T')(\mathcal{E})$ has finite extraordinary fibers (indeed, when x is a closed point of P, then $i^!_x(({}^{\dagger}T')(\mathcal{E})) = 0$ if $x \in T'$ otherwise $i^!_x(({}^{\dagger}T')(\mathcal{E})) = i^!_x(\mathcal{E})$). By using the triangle of localisation of $\mathcal{E}$ with respect to T', we get $\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Using Berthelot-Kashiwara theorem this implies that $\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E})$ has finite extraordinary fibers. Hence, so is $\mathcal{E}$ by devissage. □

Remark 15.3.3.5. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. If T' is a divisor containing T such that $({}^{\dagger}T')(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T')_{\mathbb{Q}}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$, then we get the exact triangle of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E}) \to \mathcal{E} \to ({}^{\dagger}T')(\mathcal{E}) \to +1 \tag{15.3.3.5.1}$$

such that for any $n \in \mathbb{Z}$ we have $H^n(\mathbb{R}\underline{\Gamma}^{\dagger}_{T'}(\mathcal{E})) \in \mathrm{MIC}^{\dagger\dagger}(T', \mathfrak{P}/\mathcal{V})$ and $H^n(({}^{\dagger}T')(\mathcal{E})) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T'/\mathcal{V})$ (see notation 11.2.1.4). Hence, if $\mathcal{E}$ has finite extraordinary fibers then it admits a devissage in overconvergent isocrystals. We will study later such devissages in a wider context.

Corollary 15.3.3.6. *Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$. If $\mathcal{E}$ has finite extraordinary fibers, then it is holonomic.*

Proof. This is a consequence of 15.3.3.1 and 15.3.3.4. □

Proposition 15.3.3.7. *Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$. Then $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-coherent.*

Proof. This be proved later 18.3.2.2 in a wider context. For the case of curves, this can be checked using the finite monodromy theorem. □

Remark 15.3.3.8. Thanks to the proposition 15.3.3.7, we show below that for curves having finite extraordinary fibers is equivalent to be holonomic. In higher dimension this is still an open question. To get a more suitable notion, we will introduce that of overcoherence in the next subsection.

Proposition 15.3.3.9. *Let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$. The following assertions are equivalent:*

(a) The complex $\mathcal{E}$ has finite extraordinary fibers.

(b) $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$.

(c) For any divisor T of P, $({}^{\dagger}T)\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$.

Proof. We already know $(a) \to (b)$ (see 15.3.3.6). Conversely, suppose $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$. By devissage, we can suppose $\mathcal{E}$ is an holonomic $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. Thanks to 15.3.1.20, there exists a divisor T of P such that $({}^{\dagger}T)\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, T/\mathcal{V})$. Hence, $({}^{\dagger}T)\mathcal{E}$ has finite extraordinary fibers. Moreover, following 15.3.3.7, $({}^{\dagger}T)\mathcal{E}$ is therefore $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-coherent. This yields by devissage that $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E})$ has its support in T (which is finite and etale over S), then using Berthelot-Kashiwara theorem we get that $\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{E})$ has extraordinary finite fibers.

Since the property "having finite extraordinary fibers" is stable under the localisation functor by a divisor $({}^{\dagger}T)$ (see 15.3.3.4), then we get from the equivalence between (a) and (b), but we get the equivalence between (b) and (c).

□

15.3.4 Overcoherence (after any base change) in a smooth $\mathfrak{S}$-formal scheme

Definition 15.3.4.1. Let $\mathcal{E}^{(\bullet)} \in (F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

(a) We say that $\mathcal{E}^{(\bullet)}$ is "$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-overcoherent in $\mathfrak{P}$" if, for any divisor D of P, we have $({}^{\dagger}D)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ the full subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ consisting of overcoherent in $\mathfrak{P}$ complexes. We denote by $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ the full subcategory of $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ consisting of objects $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-module belonging to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

(b) We say that $\mathcal{E}^{(\bullet)}$ is "$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-overcoherent in $\mathfrak{P}$ after any base change" if for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$ (see notation 9.2.6.12), denoting by $\mathfrak{S}' := \mathrm{Spf}\,\mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f \colon \mathfrak{P}' \to \mathfrak{P}$ the projection, $T' := f^{-1}(T)$, we have $\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}'}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'))$ (see notation 9.2.6.13). We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ the full subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ consisting of overcoherent after any base in $\mathfrak{P}$ complexes. We denote by $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{oc},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ the full subcategory of $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ consisting of objects $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-module belonging to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

Definition 15.3.4.2. Let $\mathcal{E}$ be an object of $(F\text{-})D(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

(a) The complex $\mathcal{E}$ is "$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent in $\mathfrak{P}$" if $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ and if, for any divisor D of P, we have $({}^{\dagger}D)(\mathcal{E}) \in (F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ (see 9.1.6.10 for a definition of the functor $({}^{\dagger}D)$ on $(F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$). We denote by $(F\text{-})D^{\mathrm{b}}_{\mathrm{ovcoh},\mathfrak{P}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ the full subcategory of $(F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ consisting of $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent in $\mathfrak{P}$ complexes. A $(F\text{-})\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module is overcoherent in $\mathfrak{P}$ if so is as an object of $(F\text{-})D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

(b) We say that $\mathcal{E}$ is " $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent in $\mathfrak{P}$ after any base change" if, for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$, denoting by $\mathfrak{S}' := \mathrm{Spf}\,\mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f \colon \mathfrak{P}' \to \mathfrak{P}$ the projection, $T' := f^{-1}(T)$, we have $\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}'}({}^{\dagger}T')_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}} f^{-1}\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh},\mathfrak{P}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$, i.e. the base change $f^{!}_{T}(\mathcal{E})$ of $\mathcal{E}$ via $\mathcal{V} \to \mathcal{V}'$ is overcoherent in $\mathfrak{P}'$ (see 9.2.7.1). A $(F\text{-})\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module is overcoherent in $\mathfrak{P}$ after any base change if so is as an object of $(F\text{-})D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. We denote by $(F\text{-})D^{\mathrm{b}}_{\mathrm{oc},\mathfrak{P}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ the full subcategory of $(F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ consisting of $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent in $\mathfrak{P}$ after any base change complexes.

Notation 15.3.4.3. In this subsection, let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$.

Example 15.3.4.4. The $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$ is overcoherent in $\mathfrak{P}$. Indeed, this is a consequence of 12.2.7.1.

Proposition 15.3.4.5. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. The property $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ is equivalent to the property $\underrightarrow{l}^{*}_{\mathbb{Q}}\, \mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\natural,\mathfrak{P}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$, where $\underrightarrow{l}^{*}_{\mathbb{Q}}$ is the equivalence of categories of 12.2.1.6.1). The equivalence of categories $\underrightarrow{l}^{*}_{\mathbb{Q}}$ of 8.7.5.4.1 preserves overcoherence and induces the equivalence of categories of the form $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)) \cong D^{\mathrm{b}}_{\natural,\mathfrak{P}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

Proof. Let T' a divisor of P. Since $\underrightarrow{l}^{*}_{\mathbb{Q}} \circ ({}^{\dagger}T')(\mathcal{E}^{(\bullet)}) \underset{9.1.3.2}{\xrightarrow{\sim}} \underrightarrow{l}^{*}_{\mathbb{Q}} \circ ({}^{\dagger}T' \cup T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^{\dagger}T' \cup T, T) \circ \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$, it is sufficient to apply the corollary 9.1.6.3 to the complex $\mathcal{E}'^{(\bullet)} := ({}^{\dagger}T')(\mathcal{E}^{(\bullet)})$. □

15.3.4.6. It follows from 15.3.4.5 that the properties concerning the overcoherence are still valid replacing the categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}$ by $D^{\mathrm{b}}_{\natural,\mathfrak{P}}$ and vice versa. We will therefore in the following simply write and check one of the two contexts.

Like coherence, we verify that the notion of overcoherent (after any base change) in a smooth $\mathfrak{S}$-formal scheme is local in $\mathfrak{P}$. Likewise, we extend the standard properties from coherent modules as follows:

Proposition 15.3.4.7. Let $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \mathcal{E}_4 \to \mathcal{E}_5$ an exact sequence of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$). If $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$ are overcoherent (after any base change) in $\mathfrak{P}$ then so is $\mathcal{E}_3$.

Proof. Since the respective case is checked similarly, let us treat the non-respective one. Let D be a divisor of P. Since the extension $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T \cup D)_{\mathbb{Q}}$ is flat, then the functor $({}^{\dagger}D)$ from the category of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules in that of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T \cup D)_{\mathbb{Q}}$-modules is exact. We get therefore the exact sequence $({}^{\dagger}D)(\mathcal{E}_1) \to ({}^{\dagger}D)(\mathcal{E}_2) \to ({}^{\dagger}D)(\mathcal{E}_3) \to ({}^{\dagger}D)(\mathcal{E}_4) \to ({}^{\dagger}D)(\mathcal{E}_5)$. Since by hypothesis all terms except the middle one are coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules, then so is the middle one. So, $\mathcal{E}_3$ is an overcoherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module. Similarly, since the base change given by a morphism $\mathcal{V} \to \mathcal{V}'$ of DVR($\mathcal{V}$) is exact (see 9.2.7.1), then we are done. □

Proposition 15.3.4.8. Let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$. Let $\Phi : \mathcal{E} \to \mathcal{F}$ be a morphism of overcoherent (after any base change) $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. of $\underrightarrow{LM}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$). Then $\mathrm{Ker}\,\Phi$, $\mathrm{Coker}\,\Phi$ and $\mathrm{Im}\Phi$ are overcoherent (after any base change) $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. belong to $\underrightarrow{LM}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$).

Proof. Let D be a divisor of P. Since the functor $({}^{\dagger}D)\colon \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T \cup D)_{\mathbb{Q}})$ is exact, then this functor $({}^{\dagger}D)$ commutes with kernels, cokernels and images. Since the kernels, cokernels, images of a morphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules are coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules, then we get the overcoherent case. Similarly, since the base change given by a morphism $\mathcal{V} \to \mathcal{V}'$ of DVR($\mathcal{V}$) is exact (see 9.2.7.1), then we are done. □

Lemma 15.3.4.9. *Let* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

(a) Let $(\mathfrak{P}_i)_i$ *be an open covering of* $\mathfrak{P}$. *We have* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ *if and only if* $\mathcal{E}^{(\bullet)}|\mathfrak{P}_i \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}_i}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}_i}(T \cap P_i))$ *for any* i.

(b) If two terms of a distinguished triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ *belong to* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$*, then so is the third.*

(c) We have $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ *if and only if* $H^j(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{oc},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ *for any integer* $j \in \mathbb{Z}$.

Proof. The two first assertions are obvious. The third results from the fact that the functor $({}^{\dagger}D)$ with D a divisor of X and base change via a morphism of DVR are exact on the category of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules. □

Lemma 15.3.4.10. *Let* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. *The following conditions are equivalent:*

(a) $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ *;*

(b) *For any subscheme (resp. open subscheme, resp. closed subscheme) Y of P, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.*

Proof. By using the case where Y is equal to P or to the complementary of a divisor (or a divisor and use 15.3.4.9.(b)), we get $(b) \to (a)$. Conversely, let us check the implication $(a) \Rightarrow (b)$. By construction of the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}$ (see 13.1.5.1), using 15.3.4.9.(b) we reduce by devissage to the case where Y is an open subscheme of P and we denote by Z the complementary closed subscheme. Let us denote by n_Z the minimal number of divisor $T_1, \ldots, T_r$ such that $Z = T_1 \cap \cdots \cap T_r$. We check the assertion by induction on n_Z. Suppose $n_Z \leq 1$, i.e. Z is a divisor. For any divisor D of X, we have $({}^{\dagger}D) \circ ({}^{\dagger}Z)(\mathcal{E}) \xrightarrow{\sim} ({}^{\dagger}Z \cup D)(\mathcal{E})$ (see 9.1.3.2) and we are done. Suppose now $r \geq 2$ and the proposition holds for any Z such that $n_Z < r$. Set $Z' = T_2 \cap \cdots \cap T_r$. We conclude by using the induction hypothesis via the following Mayer-Vietoris exact triangles (see 13.1.4.15.2):

$$\mathcal{E}^{(\bullet)} \to ({}^{\dagger}T_1)(\mathcal{E}^{(\bullet)}) \oplus ({}^{\dagger}Z')(\mathcal{E}^{(\bullet)}) \to ({}^{\dagger}T_1 \cup Z')(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}[1]. \tag{15.3.4.10.1}$$

□

Remark 15.3.4.11. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$, $\mathcal{E} := \underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$ The complex $\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ if and only if $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh},\mathfrak{P}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ (see 15.3.4.5) and thanks to 15.3.4.10 and 14.3.3.1 these properties are equivalent to one of theses properties:

(a) For any subscheme (resp. open subscheme, resp. closed subscheme) Y of P, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

(b) For any subscheme (resp. open subscheme, resp. closed subscheme) Y of P, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

However, when Y do not vary this is not clear. More precisely, if X be a closed subscheme of P, if $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ then $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. When X is a divisor of P the converse is true (see 14.3.3.1) but otherwise this is an open question. Because of this remark, if X and X' are closed subschemes P such that $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ and next $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E})) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$, then this is not clear that we have the isomorphism

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X\cap X'}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E})).$$

Lemma 15.3.4.12. *For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$, for any subscheme Y of P, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.*

Proof. Since base changes commute with localisation functors (see 13.1.5.7), then we reduce to the case where $\natural = \mathrm{ovcoh}$. Hence, this follows from 15.3.4.10 and 13.1.5.6.1. □

Proposition 15.3.4.13. *Let $f\colon \mathfrak{P} \to \mathfrak{Q}$ be a proper morphism of smooth $\mathfrak{S}$-formal schemes, U be a divisor of Q such that $T = f^{-1}(U)$. X be a closed subscheme Q. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ with support in $f^{-1}(X)$. Then $f_{+}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathbb{Q}}(U))$ with support in X.*

Proof. Since base changes commute to localisation functors (see 13.1.5.7) and to pushforwards (see 9.2.6.9), then we reduce to the case where $\natural = \mathrm{ovcoh}$. This follows from the commutation of the localisation functor to direct images (see 9.4.3.3) and the stability of the coherence by pushforward by a proper morphism (see 9.4.2.4) and 15.3.4.10. □

Remark 15.3.4.14. Let $f\colon \mathfrak{P} \to \mathfrak{Q}$ be a smooth morphism of smooth $\mathfrak{S}$-formal schemes, U be a divisor of Q. It is not clear if $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}}(U))$ then $f^{(\bullet)!}(\mathcal{E}^{(\bullet)})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. We will introduce the notion of overcoherence in order to get this stability by definition (see 15.3.6.1 and 15.3.6.2).

Proposition 15.3.4.15. *Let $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ be a closed immersion of smooth $\mathfrak{S}$-formal schemes. Let $\mathfrak{U}$ the open of $\mathfrak{P}$ complementary to $u(X)$, T a divisor of P such that $T \cap X$ is a divisor of X.*

(a) *For any complex $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$, we have therefore $u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{X}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T \cap X))$.*

(b) *The functors $u^{(\bullet)}_{+}$ and $u^{(\bullet)!}$ induce quasi-inverse equivalences between the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{X}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T \cap X))$ and the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}|\mathfrak{U} \xrightarrow{\sim} 0$.*

Proof. Let us check the first assertion. Since base changes commute to localisation functors (see 13.1.5.7) and pullbacks 9.2.6.6, then we reduce to the case where $\natural$ = ovcoh. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ and Y be a subvariety of X. It follows from 15.3.4.10 that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. Since this latter is in particular coherent with support in X, then it follows from Theorem 9.3.5.13 that $u^{(\bullet)!}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)})) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T\cap X))$. Moreover, using 13.2.1.4.1, we get the canonical isomorphism $u^{(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Y} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)})$. Using 15.3.4.10, this implies $u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{X}}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T\cap X))$. Now let's deal with the assertion (b). Since u is proper, then the functor $u^{(\bullet)}_{+}$ preserves the overcoherence in a smooth $\mathfrak{S}$-formal by 15.3.4.13. Since the overcoherence is a stronger condition than the coherence and is preserved by $u^{(\bullet)}_{+}$ and $u_{(\bullet)+}$, then we conclude thanks to the coherent version of Berthelot-Kashiwara Theorem (see 9.3.5.13.(c)). □

Remark 15.3.4.16. It follows from 15.3.4.15 that overcoherence in $\mathfrak{P}$ is a stronger condition than having finite extraordinary fibers. Hence, an overcoherent in $\mathfrak{P}$ module is generically $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent (see 15.3.1.20).

Proposition 15.3.4.17. *Let $a\colon \mathfrak{Z} \hookrightarrow \mathfrak{P}$ be a closed immersion of smooth affine $\mathfrak{S}$-formal schemes. We suppose $\mathfrak{P}$ endowed with local coordinates $t_1, \ldots, t_n$ such that $\mathfrak{Z} = V(t_1, \ldots, t_r)$. Let $\mathcal{E}$ be a $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module which overcoherent in $\mathfrak{P}$. With notation 9.3.4.1, we have the canonical isomorphism:*

$$a^{*}(E) \xrightarrow{\sim} \Gamma(\mathfrak{Z}, a^{*}(\mathcal{E})). \tag{15.3.4.17.1}$$

Proof. We proceed by induction on r. When $r = 1$, this comes from the lemma 9.3.4.9. Denote by $\mathfrak{P}' := V(t_1)$. Using 15.3.4.9.(c) and 15.3.4.15.(a), denoting by $b\colon \mathfrak{P}' \hookrightarrow \mathfrak{P}$ the canonical closed immersion, we can check that $\mathcal{E}' := b^{*}(\mathcal{E})$ is a $\mathcal{D}^{\dagger}_{\mathfrak{P}',\mathbb{Q}}$-module which is overcoherent in $\mathfrak{P}'$. By noting $c\colon \mathfrak{Z} \hookrightarrow \mathfrak{P}'$ the canonical closed immersion, by induction hypothesis, we get the isomorphisms $b^{*}(E) \xrightarrow{\sim} \Gamma(\mathfrak{P}', b^{*}(\mathcal{E})) = E'$ and $c^{*}(E') \xrightarrow{\sim} \Gamma(\mathfrak{Z}, c^{*}(\mathcal{E}'))$. Since $a^{*}(E) \xrightarrow{\sim} c^{*} \circ b^{*}(E)$ and $a^{*}(\mathcal{E}) \xrightarrow{\sim} c^{*} \circ b^{*}(\mathcal{E})$, we can deduce the result. □

Remark 15.3.4.18. The isomorphism 15.3.4.17.1 is used in the proof of the lemma 15.3.5.3 (see the step II.4)). It is an open question to know if the isomorphism 15.3.4.17.1 is still valid when $\mathcal{E}$ is only a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module.

15.3.5 The overcoherence in $\mathfrak{P}$ after any base change implies the holonomicity

Lemma 15.3.5.1. *Let $f\colon \mathfrak{P} \to \mathfrak{Q}$ be a finite etale morphism of smooth $\mathfrak{S}$-formal schemes, $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module.*

(a) With notation 15.2.4.16, we have the isomorphisms $f_{+}(\mathcal{E}^{\mathrm{hol}}) \xrightarrow{\sim} f_{+}(\mathcal{E})^{\mathrm{hol}}$ and $f_{+}(\mathcal{E}/\mathcal{E}^{\mathrm{hol}}) \xrightarrow{\sim} f_{+}(\mathcal{E})/f_{+}(\mathcal{E})^{\mathrm{hol}}$.

(b) The module $\mathcal{E}$ is holonomic if and only if $f_{+}(\mathcal{E})$ is holonomic.

Proof. Since f is finite, then f is projective. Hence, it follows from the relative duality theorem in the case of a projective morphism (see 9.4.4.5) and of the exactness of the functor f_{+} when f is finite and etale (see 9.2.4.15) that $f_{+}(\mathcal{E}^{*}) \xrightarrow{\sim} (f_{+}(\mathcal{E}))^{*}$. Hence, $f_{+}(\mathcal{E}^{\mathrm{hol}}) \xrightarrow{\sim} (f_{+}(\mathcal{E}))^{\mathrm{hol}}$. By exactness of f_{+}, this yields $f_{+}(\mathcal{E}/\mathcal{E}^{\mathrm{hol}}) \xrightarrow{\sim} f_{+}(\mathcal{E})/f_{+}(\mathcal{E})^{\mathrm{hol}}$. The statement (b) comes from the first isomorphism of (a) via the fact that a monomorphism $\mathcal{E}' \hookrightarrow \mathcal{E}$ of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules is an isomorphism if and only if $f_{+}(\mathcal{E}') \hookrightarrow f_{+}(\mathcal{E})$ is an isomorphism. □

Lemma 15.3.5.2. *Let M be a p-torsion free p-adically separated complete $\mathcal{V}$-module. Let $x_1, \ldots, x_r$ be some elements of $M_K := M \otimes_{\mathcal{V}} K$. Then there exists an integer $N_0 \geq 0$ such that, for every integer n, we have the inclusion:*

$$(Kx_1 + \cdots + Kx_r) \cap p^{N_0+n}M \subset p^{n}(\mathcal{V}x_1 + \cdots + \mathcal{V}P_r). \tag{15.3.5.2.1}$$

Proof. According to the terminology of 7.5.1.7, we have the p-adic norm on M_K given by M. This induces a norm on $Kx_1 + \cdots + Kx_r$. On the other hand, we have the p-adic norm on $Kx_1 + \cdots + Kx_r$ given by $\mathcal{V}x_1 + \cdots + \mathcal{V}x_r$. Since the K-vector space $Kx_1 + \cdots + Kx_r$ is of finite dimension, following [Sch02, 4.13], both norms are equivalent. Hence, the induced topology are the same. A basis of neighborhood of zero of the induced norm is given by $((Kx_1 + \cdots + Kx_r) \cap p^n M)_{n\in\mathbb{N}}$ and the second one is given by $(p^n(\mathcal{V}P_1 + \cdots + \mathcal{V}P_r))_{n\in\mathbb{N}}$. Hence, there exists N_0 such that $(Kx_1 + \cdots + Kx_r) \cap p^{N_0} M \subset \mathcal{V}x_1 + \cdots + \mathcal{V}P_r$ and we are done. □

Lemma 15.3.5.3. *Let $\mathcal{E}$ be a $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-overcoherent after any base change in $\mathfrak{P}$ module. Denote by Z the support of $\mathcal{E}/\mathcal{E}^{\mathrm{hol}}$. By the absurd, we suppose that Z is non-empty. For any irreducible component Z' of Z, then there exists an affine open set $\mathfrak{U}$ of $\mathfrak{P}$ such that*

(a) the open set $U \cap Z'$ is smooth and dense in Z' ;

(b) the module $(\mathcal{E}/\mathcal{E}^{\mathrm{hol}})|\mathfrak{U}$ is holonomic.

Proof. We remark that the finiteness hypothesis on $\mathcal{E}$ is only used at the step $II.4$).

Step I)*: Preliminary, reduction of the problem and notations.*

Step 1). The lemma is Zariski local on $\mathfrak{P}$. Replacing if necessary $\mathfrak{P}$ by an affine open set $\mathfrak{U}$ of $\mathfrak{P}$ such that $U \cap Z' = U \cap Z$ and such that $U \cap Z'$ is a smooth dense open of Z', we can then suppose $\mathfrak{P}$ affine, $Z = Z'$ and Z smooth. Following [SGA1, Exp. III], then there exist a closed immersion of smooth affine $\mathcal{V}$-schemes of the form $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{P}$ which lifts $Z \hookrightarrow P$. Following the theorem 15.3.1.19 and the proposition 15.3.4.15.(a), there exists an open set $\mathfrak{U}$ of $\mathfrak{P}$ such that $\mathfrak{V} := \mathfrak{U} \cap \mathfrak{Z}$ is dense in $\mathfrak{Z}$ and $u^!(\mathcal{E})|\mathfrak{V}$ is $\mathcal{O}_{\mathfrak{V},\mathbb{Q}}$-coherent and therefore holonomic. Hence, following the holonomic version of Berthelot-Kashiwara (see 15.2.4.19), we get that $\mathcal{E}|\mathfrak{U}$ is holonomic. This implies that the dimension of the support of $\mathcal{E}_{\text{n-hol}} := \mathcal{E}/\mathcal{E}^{\mathrm{hol}}$ is smaller than that of P. Denote by $r \geq 1$ the codimension of Z in P.

Shrinking if necessary $\mathfrak{P}$, there exists a finite and etale morphism of the form $h\colon \mathfrak{P} \to \widehat{\mathbb{A}}^d_{\mathcal{V}}$ such that $Z = h^{-1}(\mathbb{A}^{d-r}_k)$ where $\mathbb{A}^{d-r}_k$ is the closed subvariety of $\mathbb{A}^d_k = \operatorname{Spec} k[t_1, \ldots, t_d]$ equal to $V(t_1, \ldots, t_r)$. (see the main theorem of Kedlaya of [Ked05]). Thanks to Proposition 15.3.4.13 and Lemma 15.3.5.1, we reduce then to the case where $\mathfrak{Z} \hookrightarrow \mathfrak{P}$ is the closed immersion $\widehat{\mathbb{A}}^{d-r}_{\mathcal{V}} \hookrightarrow \widehat{\mathbb{A}}^d_{\mathcal{V}}$. In particular, we are in the local situation of the subsection 9.3.3 and 9.3.4 and we will use their notations. Since $\mathcal{E}_{\text{n-hol}}$ is supported in Z, following the theorem of Berthelot-Kashiwara (see 9.3.5.9), there exists a coherent $\mathcal{D}^{\dagger}_{\mathfrak{Z},\mathbb{Q}}$-module $\mathcal{F}$ such that $\mathcal{E}_{\text{n-hol}} \xrightarrow{\sim} u_+(\mathcal{F})$.

Step 2). *It is sufficient to check that there exists a dense open set $\mathfrak{V}$ of $\mathfrak{Z}$ such that $\mathcal{F}|\mathfrak{V}$ is a free $\mathcal{O}_{\mathfrak{V},\mathbb{Q}}$-module of finite type, which will be established in the step* II).

Indeed, suppose $\mathcal{F}|\mathfrak{V}$ is a free $\mathcal{O}_{\mathfrak{V},\mathbb{Q}}$-module of finite type. Let $\mathfrak{U}$ be an open set of $\mathfrak{P}$ such that $U \cap Z = V$. Then $\mathcal{E}_{\text{n-hol}}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(V, \mathfrak{U}/\mathcal{V})$. Hence, $\mathcal{E}_{\text{n-hol}}|\mathfrak{U}$ is a holonomic $\mathcal{D}^{\dagger}_{\mathfrak{V},\mathbb{Q}}$-module (see 15.2.4.20) and we are done.

Step 3). Following 15.3.1.18, thanks to the step 2), we reduce by descent to the case where k is algebraically closed and uncountable.

Step 4). There exist $m_0 \in \mathbb{N}$, a coherent $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Z},\mathbb{Q}}$-module $\mathcal{F}^{(m_0)}$ together with a $\mathcal{D}^{\dagger}_{\mathfrak{Z},\mathbb{Q}}$-linear isomorphism of the form $\mathcal{D}^{\dagger}_{\mathfrak{Z},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Z},\mathbb{Q}}} \mathcal{F}^{(m_0)} \xrightarrow{\sim} \mathcal{F}$. From now, m will always be an integer greater or equal than m_0. Denote by $\mathcal{F}^{(m)} := \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Z},\mathbb{Q}}} \mathcal{F}^{(m_0)}$. We have the canonical $\widehat{D}^{(m)}_{\mathfrak{Z},\mathbb{Q}}$-linear morphisms $\rho_m\colon F^{(m)} \to F^{(m+1)}$ (recall according to notation 9.3.3 that sheaves are denoted with curly letters and the corresponding straight letters mean their global sections). Let $\mathring{\mathcal{F}}^{(m_0)}$ be a coherent p-torsion free $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Z}}$-module together with a $\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Z},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{F}}^{(m_0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{F}^{(m_0)}$. For any $m \geq m_0 + 1$, let $\mathring{\mathcal{F}}^{(m)}$ be the quotient of the coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}}$-module $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}} \otimes_{\widehat{\mathcal{D}}^{(m_0)}_{\mathfrak{Z}}} \mathring{\mathcal{F}}^{(m_0)}$ by its p-torsion part. Then $\mathring{\mathcal{F}}^{(m)}$ is a p-torsion free coherent $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}}$-module $\mathring{\mathcal{F}}^{(m)}$ and we have the $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z},\mathbb{Q}}$-linear isomorphism $\mathring{\mathcal{F}}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{F}^{(m)}$ and such that $\rho_{m-1}(\mathring{F}^{(m-1)}) \subset \mathring{F}^{(m)}$. Since $\mathring{\mathcal{F}}^{(m)}$ is p-torsion free, then we get the isomorphism: $\mathring{\mathcal{F}}^{(m)}_i := \mathcal{V}/\pi^{i+1}\mathcal{V} \otimes^{\mathbb{L}}_{\mathcal{V}} \mathring{\mathcal{F}}^{(m)} \xrightarrow{\sim} \mathcal{V}/\pi^{i+1}\mathcal{V} \otimes_{\mathcal{V}} \mathring{\mathcal{F}}^{(m)}$.

Next, we set $\mathring{\mathcal{E}}^{(m)}_{\text{n-hol}} := u^{(m)}_+(\mathring{\mathcal{F}}^{(m)})$ and $\mathcal{E}^{(m)}_{\text{n-hol}} := \mathring{\mathcal{E}}^{(m)}_{\text{n-hol},\mathbb{Q}}$. Thanks to 15.3.1.16, there exists a dense open set $\mathfrak{Z}_m$ of $\mathfrak{Z}$ such that $\mathring{\mathcal{F}}^{(m)}|\mathfrak{Z}_m$ is isomorphic to the p-adic completion of a free $\mathcal{O}_{\mathfrak{Z}_m}$-module. Hence, we have an $\mathcal{O}_{\mathfrak{Z}_m}$-linear isomorphism of the form $\mathring{\mathcal{F}}^{(m)}|\mathfrak{Z}_m \xrightarrow{\sim} (\mathcal{O}^{(A_m)}_{\mathfrak{Z}_m})^{\wedge}$ where A_m is a set.

Let $t_1, \dots, t_d$ be such that $\mathfrak{P} = \operatorname{Spf} \mathcal{V}\{t_1, \dots, t_d\}$ and $\mathfrak{Z} = V(t_1, \dots, t_r)$. We obtain some closed $\mathfrak{S}$-formal subschemes of $\mathfrak{P}$ by setting $\mathfrak{P}' := V(t_{r+1}, \dots, t_d)$ and $\mathfrak{Z}' := V(t_1, \dots, t_d)$. We obtain then the canonical diagram cartesien:

$$\begin{array}{ccc} \mathfrak{Z} & \overset{u}{\hookrightarrow} & \mathfrak{P} \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle a} \\ \mathfrak{Z}' & \overset{u'}{\hookrightarrow} & \mathfrak{P}'. \end{array} \tag{15.3.5.3.1}$$

Since k is algebraically closed and uncountable (see the step 3)), by changing the choice of the coordinates $t_{r+1}, \dots, t_d$ if necessary, we can suppose that $|Z'| \in \cap_{m \in \mathbb{N}} \mathfrak{Z}_m$.

Step 5). Since $|Z'| \in \mathfrak{Z}_m$, via moreover 7.5.5.12.(c), we get the isomorphism $b^*(\mathring{\mathcal{F}}^{(m)}) \xrightarrow{\sim} (\mathcal{V}^{(E_m)})^{\wedge}$. So, $b^*(\mathring{\mathcal{F}}^{(m)})$ is p-torsion free, separated and complete for the p-adic topology. With notation 9.3.4.1, following 9.3.4.10.2 and 9.3.4.10.3 we have therefore the last isomorphism

$$b^*(\mathring{\mathcal{F}}^{(m)}) \overset{15.3.1.5}{\xrightarrow{\sim}} b^*(\mathring{F}^{(m)}) \xrightarrow{\sim} \varprojlim_i b_i^*(\mathring{F}_i{}^{(m)}).$$

Step II)*: We prove in this step that there exists an integer $m_1 \geq m_0$ such that $\mathcal{F}|\mathfrak{Z}_{m_1}$ is a free $\mathcal{O}_{\mathfrak{Z}_{m_1},\mathbb{Q}}$-module of finite type.*

Step 1). *Acyclicity.* Following 5.2.4.5.1, we have the isomorphisms $\mathbb{L}a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \xrightarrow{\sim} u_{i+}'^{(m)} \circ \mathbb{L}b_i^*(\mathring{\mathcal{F}}_i{}^{(m)})$. As $\mathring{\mathcal{F}}^{(m)}|\mathfrak{Z}_m$ is flat, we get $\mathbb{L}b_i^*(\mathring{\mathcal{F}}_i{}^{(m)}) \xleftarrow{\sim} b_i^*(\mathring{\mathcal{F}}_i{}^{(m)})$. Since the functors $u_{i+}^{(m)}$ and $u_{i+}'^{(m)}$ are exact (see 5.2.3.1), this yields the isomorphisms

$$\mathbb{L}a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \xrightarrow{\sim} a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \xrightarrow{\sim} u_{i+}'^{(m)} \circ b_i^*(\mathring{\mathcal{F}}_i{}^{(m)}). \tag{15.3.5.3.2}$$

Step 2). *The module $a^*(\mathring{\mathcal{E}}^{(m)}_{\text{n-hol}})$ is pseudo-quasi-coherent (see the definition 7.2.3.5). More precisely, we have the canonical isomorphisms*

$$\mathcal{D}^{(m)}_{P_i'} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) \xrightarrow{\sim} a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}),$$
$$a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)}) \xrightarrow{\sim} \varprojlim_i a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}). \tag{15.3.5.3.3}$$

Proof: Since $u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})$ is coherent, then it follows from 7.5.5.13.(c) that $\mathbb{L}a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})$ is quasi-coherent. This yields

$$\mathbb{L}a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)}) \xrightarrow{\sim} \mathbb{R}\varprojlim_i \mathcal{D}^{(m)}_{P_i'} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(\mathbb{L}a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right). \tag{15.3.5.3.4}$$

$$u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \overset{7.5.8.8.2}{\xrightarrow{\sim}} \mathcal{D}^{(m)}_{P_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} u_+^{(m)}(\mathring{\mathcal{F}}^{(m)}) \xrightarrow{\sim} \mathcal{D}^{(m)}_{P_i} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} u_+^{(m)}(\mathring{\mathcal{F}}^{(m)}). \tag{15.3.5.3.5}$$

We have the isomorphism:

$$\begin{aligned} \mathcal{D}^{(m)}_{P_i'} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(\mathbb{L}a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) &\overset{9.2.1.11.2}{\xrightarrow{\sim}} \mathbb{L}a_i^* \left(\mathcal{D}^{(m)}_{P_i} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}} u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) \\ &\overset{7.5.8.8.2}{\xrightarrow{\sim}} \mathbb{L}a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \overset{15.3.5.3.2}{\xrightarrow{\sim}} a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \end{aligned} \tag{15.3.5.3.6}$$

This yields the isomorphisms:

$$\mathcal{D}^{(m)}_{P_i'} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(\mathbb{L}a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) \xrightarrow{\sim} \mathcal{D}^{(m)}_{P_i'} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) \xrightarrow{\sim} a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}). \tag{15.3.5.3.7}$$

Then it comes from 15.3.5.3.7 and 15.3.5.3.4 the top isomorphisms of the diagram below:

$$\begin{array}{ccccc} \mathbb{L}a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)}) & \xrightarrow{\sim} & \mathbb{R}\varprojlim_i \mathcal{D}^{(m)}_{P_i'} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) & \xrightarrow{\sim} & \mathbb{R}\varprojlim_i a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}) \\ \downarrow\wr & & \wr\uparrow & & \wr\uparrow \\ a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)}) & \xrightarrow{\sim} & \varprojlim_i \mathcal{D}^{(m)}_{P_i'} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'}} \left(a^* \circ u_+^{(m)}(\mathring{\mathcal{F}}^{(m)})\right) & \xrightarrow{\sim} & \varprojlim_i a_i^* \circ u_{i+}^{(m)}(\mathring{\mathcal{F}}_i{}^{(m)}). \end{array}$$

(15.3.5.3.8)

Since the top left horizontal arrow is an isomorphism from a complex of $D^{\leq 0}$ to a complex of $D^{\geq 0}$, then every morphisms of the left square are isomorphisms. Hence so are that of the right square. Hence we are done.

Step 3) *We have the canonical isomorphism* $\theta^{(m)}\colon a^*(\mathring{E}^{(m)}_{\text{n-hol}}) \xrightarrow{\sim} u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)})$ *which is compatible with the level increasing.*

II) The isomorphism $\theta^{(m)}$ is the composition of the canonical isomorphisms

$$a^*(\mathring{E}^{(m)}_{\text{n-hol}}) \xrightarrow{\sim} \varprojlim_i a_i^* \circ u^{(m)}_{i+}(\mathring{F}_i{}^{(m)}) \xrightarrow{\sim} \varprojlim_i u'^{(m)}_{i+} \circ b_i^*(\mathring{F}_i{}^{(m)}) \xrightarrow{\sim} u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)}), \qquad (15.3.5.3.9)$$

where the functor $u'^{(m)}_{+}$ is defined at 9.3.3.6.1 (this has a meaning since $b^*(\mathring{F}^{(m)}$ is a p-torsion free separated complete modules).

i) The first isomorphism of 15.3.5.3.2 is the composition of the following isomorphisms:

$$a^*(\mathring{E}^{(m)}_{\text{n-hol}}) = a^*(\Gamma(\mathfrak{P}, u^{(m)}_{+}(\mathring{\mathcal{F}}^{(m)}))) \overset{9.3.4.10.3}{\xrightarrow{\sim}} \Gamma(\mathfrak{P}', a^* \circ u^{(m)}_{+}(\mathring{\mathcal{F}}^{(m)}))$$
$$\xrightarrow{\sim} \Gamma(\mathfrak{P}', \varprojlim_i a_i^* \circ u^{(m)}_{i+}(\mathring{\mathcal{F}}_i{}^{(m)})) \xrightarrow{\sim} \varprojlim_i \Gamma(\mathfrak{P}', a_i^* \circ u^{(m)}_{i+}(\mathring{\mathcal{F}}_i{}^{(m)})) \xrightarrow{\sim} \varprojlim_i a_i^* \circ u^{(m)}_{i+}(\mathring{F}_i{}^{(m)})).$$

These isomorphisms are built as follows: since $u^{(m)}_{+}(\mathring{\mathcal{F}}^{(m)})$ is pseudo quasi-coherent, then we get the first isomorphism. By applying the functor $\Gamma(\mathfrak{P}', -)$ to the composite isomorphism of the bottom horizontal maps of the diagram 15.3.5.3.8 we get the second isomorphism. We obtain therefore the first isomorphism: Since the functor $\Gamma(\mathfrak{P}', -)$ commutes with projective limits, then we get the third isomorphism. By using the theorem of type A for quasi-coherent modules on schemes, we obtain the forth.

Likewise, we obtain the second isomorphism of 15.3.5.3.9 by taking the projective limit of the global section of 15.3.5.3.2. Finally, since $b_i^*(\mathring{F}_i{}^{(m)}) \xrightarrow{\sim} \mathcal{V}/\pi^{i+1}\mathcal{V} \otimes_{\mathcal{V}} b^*(\mathring{F}^{(m)})$, then we get the last isomorphism of 15.3.5.3.9 (recall also the notation 9.3.3.6.1). Hence we are done.

ii) Taking projective limits and global section of the right square of 5.2.4.3.2, we get that the middle isomorphism of 15.3.5.3.9 commutes with the level increasing. Since this is clear for the other isomorphism, hence so is $\theta^{(m}$, i.e. we get the commutative square:

$$\begin{array}{ccc} a^*(\mathring{E}^{(m)}_{\text{n-hol}}) & \underset{\theta^{(m_0)}}{\xrightarrow{\sim}} & u'^{(m_0)}_{+} \circ b^*(\mathring{F}^{(m_0)}) \\ \downarrow & & \downarrow \\ a^*(\mathring{E}^{(m_0+s)}_{\text{n-hol}}) & \underset{\theta^{(m_0+s)}}{\xrightarrow{\sim}} & u'^{(m_0+s)}_{+} \circ b^*(\mathring{F}^{(m_0+s)}). \end{array} \qquad (15.3.5.3.10)$$

Step 4)*: Construction of* $G^{(m)}$.

Via the theorems of type A and B for coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules, we have the $D^\dagger_{\mathfrak{P},\mathbb{Q}}$-linear surjection $E \twoheadrightarrow E_{\text{n-hol}}$. By right exactness of the functor a^*, we get the $D^\dagger_{\mathfrak{P}',\mathbb{Q}}$-linear surjection $a^*(E) \twoheadrightarrow a^*(E_{\text{n-hol}})$. Since by hypothesis $\mathcal{E}$ is overcoherent in $\mathfrak{P}$, thanks to 15.3.4.17.1, then we have the isomorphism $\Gamma(\mathfrak{P}', a^*(\mathcal{E})) \xrightarrow{\sim} a^*(E)$. Since $a^*(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{P}',\mathbb{Q}}$-module, via the theorem of type A for coherent $\mathcal{D}^\dagger_{\mathfrak{P}',\mathbb{Q}}$-modules (see 8.7.5.5), we can deduce that $a^*(E_{\text{n-hol}})$ is a $D^\dagger_{\mathfrak{P}',\mathbb{Q}}$-module of finite type. Let $x_1, \ldots, x_N \in a^*(E_{\text{n-hol}})$ be some elements such that $a^*(E_{\text{n-hol}}) = \sum_{l=1}^N D^\dagger_{\mathfrak{P}',\mathbb{Q}} \cdot x_l$.

Since $a^*(E_{\text{n-hol}}) \xrightarrow{\sim} \varinjlim_m a^*(E^{(m)}_{\text{n-hol}})$, by increasing if necessary m_0, there exists $x_1^{(0)}, \ldots, x_N^{(0)} \in a^*(E^{(m_0)}_{\text{n-hol}})$ such that, for any $l = 1, \ldots, N$, $x_l^{(0)}$ is sent on x_l via the morphism $a^*(E^{(m_0)}_{\text{n-hol}}) \to a^*(E_{\text{n-hol}})$. For any $m \geq m_0$, put $G^{(m)} := \sum_{l=1}^N \widehat{D}^{(m)}_{\mathfrak{P}',\mathbb{Q}} \cdot x_l^{(m-m_0)} \subset a^*(E^{(m)}_{\text{n-hol}})$, where $x_l^{(m-m_0)}$ is the image of $x_l^{(0)}$ via the canonical morphism $a^*(E^{(m_0)}_{\text{n-hol}}) \to a^*(E^{(m)}_{\text{n-hol}})$.

Step 5)*: Construction of* $\mathring{G}^{(m)}$.

Since $G^{(m)}$ is $\widehat{D}^{(m)}_{\mathfrak{P}',\mathbb{Q}}$-coherent, then there exists a p-torsion free coherent $\widehat{D}^{(m)}_{\mathfrak{P}'}$-module $\mathring{G}^{(m)}$ such that $\mathring{G}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} G^{(m)}$. Since we have the $\widehat{D}^{(m)}_{\mathfrak{P}',\mathbb{Q}}$-linear inclusion $G^{(m)} \subset a^*(E^{(m)}_{\text{n-hol}})$, since $\left(a^*(\mathring{E}^{(m)}_{\text{n-hol}})\right)_{\mathbb{Q}} \xrightarrow{\sim} a^*(E^{(m)}_{\text{n-hol}})$, by multiplying if necessary $\mathring{G}^{(m)}$ by a convenient power of p, we can moreover suppose $\mathring{G}^{(m)} \subset a^*(\mathring{E}^{(m)}_{\text{n-hol}})$.

Step 6)*: Construction of* $\mathring{H}^{(m)}$ *and* $H^{(m)}$.

We set $\mathring{H}^{(m)} := \theta^{(m)}(\mathring{G}^{(m)})$, and $H^{(m)} := \theta^{(m)}_{\mathbb{Q}}(G^{(m)})$, where $\theta^{(m)}$ is the isomorphism 15.3.5.3.9 and

$$\theta^{(m)}_{\mathbb{Q}} \colon a^*(E^{(m)}_{\text{n-hol}}) \xrightarrow{\sim} (a^*(\mathring{E}^{(m)}_{\text{n-hol}}))_{\mathbb{Q}} \overset{15.3.5.3.9}{\xrightarrow{\sim}} (u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)}))_{\mathbb{Q}}. \tag{15.3.5.3.11}$$

We get $(\mathring{H}^{(m)})_{\mathbb{Q}} \xrightarrow{\sim} H^{(m)}$.

Step 7)*: With notation 9.3.3.10.1, the* $\mathcal{V}$*-module* $H^0u'^{(m)!}(\mathring{H}^{(m)})$ *is free of finite rank.*

Since $H^0u'^{(m)!}(\mathring{H}^{(m)})$ is p-torsion free, separated and complete for the p-adic topology (see notation and explanation of 9.3.3.10), this one is the p-adic completion of a free $\mathcal{V}$-module (e.g. use 15.3.1.3). It remains to check that it is of finite type over $\mathcal{V}$. Following the step $I.5$), the module $b^*(\mathring{F}^{(m)})$ is p-torsion free, separated and complete for the p-adic topology. Then we have the canonical square

$$\begin{array}{ccc} \mathring{H}^{(m)} & \hookrightarrow & u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)}) \\ \uparrow & & \uparrow\sim \\ u'^{(m)}_{+} H^0u'^{(m)!}(\mathring{H}^{(m)}) & \hookrightarrow & u'^{(m)}_{+} H^0u'^{(m)!}(u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)})) \end{array}$$

whose bottom horizontal morphisms is induced by functoriality of the tautological inclusion $\mathring{H}^{(m)} \subset u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)})$, whose vertical arrows are induced by the adjunction morphisms of 9.3.3.14.1. The square is commutative by functoriality. Following 9.3.3.15.3, the right vertical arrow is an isomorphism. Since the functor $H^0u'^{(m)!}$ is left exact, we get the canonical inclusion $H^0u'^{(m)!}(\mathring{H}^{(m)}) \subset H^0u'^{(m)!}(u'^{(m)}_{+} \circ b^*(\mathring{F}^{(m)}))$. Hence, by applying to it the functor $u'^{(m)}_{+}$, it follows from 9.3.3.7 that the bottom horizontal arrow is injective. This implies that the canonical arrow $u'_{+}H^0u'^{(m)!}(\mathring{H}^{(m)}) \to \mathring{H}^{(m)}$ is injective. Since $\mathring{H}^{(m)}$ is $\widehat{D}^{(m)}_{\mathfrak{P}'}$-coherent, this yields by noetherianity of $\widehat{D}^{(m)}_{\mathfrak{P}'}$ that $u'_{+}H^0u'^{(m)!}(\mathring{H}^{(m)})$ is $\widehat{D}^{(m)}_{\mathfrak{P}'}$-coherent. It comes from the lemma 9.3.3.8 that $H^0u'^{(m)!}(\mathring{H}^{(m)})$ is a free $\mathcal{V}$-module of finite rank.

Step 8)*: By the absurd, we suppose that, for any* $m \geq m_0$, $b^*(\mathring{F}^{(m)})$ *is not of finite type over* $\mathcal{V}$. *We get then a contradiction.*

i) We construct by induction on $s \in \mathbb{N}$ some elements $y^{(0)}_0, \dots, y^{(0)}_s \in b^*(\mathring{F}^{(m_0)})$ and some integers $n_0, \dots, n_s$ as follows.

It comes from 9.3.3.13.1 that the canonical morphism $b^*(\mathring{F}^{(m)}) \to H^0u'^{(m)!}(u'^{(m)}_{+}(b^*(\mathring{F}^{(m)})))$ is an isomorphism. Via this isomorphism, we will identify $H^0u'^{(m)!}(\mathring{H}^{(m)})$ as a subset of $b^*(\mathring{F}^{(m)})$. Setting $H^0u'^{(m)!}(H^{(m)}) := (H^0u'^{(m)!}(\mathring{H}^{(m)}))\otimes_{\mathcal{V}} K$, we will also identify $H^0u'^{(m)!}(H^{(m)})$ as a subset of $b^*(F^{(m)}) = \left(b^*(\mathring{F}^{(m_0)})\right)_{\mathbb{Q}}$. Since $b^*(\mathring{F}^{(m_0)})$ is p-torsion free, separated and complete for the p-adic topology (see the step $I.5$)), since we suppose that $b^*(\mathring{F}^{(m_0)})$ is not a $\mathcal{V}$-module of finite type, then the K-vector space $b^*(F^{(m_0)})$ is not of finite dimension. Since $H^0u'^{(m)!}(\mathring{H}^{(m)})$ is a free $\mathcal{V}$-module of finite type (see the step II.7), then $H^0u'^{(m)!}(H^{(m)})$ then is a K-vector space is of finite dimension. Hence, there exists an element $y^{(0)}_0 \in b^*(\mathring{F}^{(m_0)})$ such that $y^{(0)}_0 \notin H^0u'^{(m)!}(H^{(m_0)})$. Moreover, since the K-vector space $Ky^{(0)}_0 + H^0u'^{(m)!}(H^{(m)})$ is of finite dimension, then following 15.3.5.2 there exists an integer $n_0 \geq 0$ such that, for any integer n, we have the inclusion:

$$\left(Ky^{(0)}_0 \oplus H^0u'^{(m)!}(H^{(m_0)})\right) \cap p^{n_0+n} b^*(\mathring{F}^{(m_0)}) \subset p^n \left(\mathcal{V}y^{(0)}_0 \oplus H^0u'^{(m)!}(\mathring{H}^{(m_0)})\right). \tag{15.3.5.3.12}$$

Suppose now constructed $y^{(0)}_0, \dots, y^{(0)}_s \in b^*(\mathring{F}^{(m_0)})$ as well as the integers $n_0, \dots, n_s$. For any element $y^{(0)}$ of $b^*(F^{(m_0)})$, we denote by $y^{(j)}$ its image via the morphism $b^*(F^{(m_0)}) \to b^*(F^{(m_0+j)})$ for any $j \in \mathbb{N}$. (Remark that if $y^{(0)} \in b^*(\mathring{F}^{(m_0)})$ then $y^{(j)} \in b^*(\mathring{F}^{(m_0+j)})$.) Since for any $m \geq m_0$ the K-vector space $b^*(F^{(m)})$ is not of finite dimension, it comes from 15.3.1.8 that the image of $b^*(F^{(m_0)}) \to b^*(F^{(m)})$ is a K-vector space of infinite dimension. Set $M^{(s+1)} := H^0u'^{(m)!}(H^{(m_0+s+1)}) + \sum_{j=0}^{s} Ky^{(s+1)}_j$. Let $\mathring{M}^{(s+1)}$ be a free $\mathcal{V}$-module such that $\mathring{M}^{(s+1)} \otimes \mathbb{Q} \xrightarrow{\sim} M^{(s+1)}$. Since $M^{(s+1)}$ is of finite dimension on K, then there exists $y^{(0)}_{s+1} \in p^{s+1}b^*(\mathring{F}^{(m_0)})$ such that $y^{(s+1)}_{s+1} \notin M^{(s+1)}$. Multiplying if necessary by a power of p, we can moreover suppose that for any $j = 0, \dots, s$, we have $y^{(j)}_{s+1} \in p^{1+n_j}b^*(\mathring{F}^{(m_0+j)})$. Following 15.3.5.2, then there exists an integer $n_{s+1} \geq n_s + 1$ such that for any integer n we have

$$\left(M^{(s+1)} \oplus Ky^{(s+1)}_{s+1}\right) \cap p^{n_{s+1}+n} b^*(\mathring{F}^{(m_0+s+1)}) \subset p^n \left(\mathring{M}^{(s+1)} \oplus \mathcal{V}y^{(s+1)}_{s+1}\right). \tag{15.3.5.3.13}$$

ii) Set $y^{(0)} := \sum_{j=0}^{\infty} y_j^{(0)} \in b^*(F^{(m_0)})$. Let $x^{(0)} \in a^*(E_{\text{n-hol}}^{(m_0)})$ be the element such that $\theta_{\mathbb{Q}}^{(m_0)}(x^{(0)}) = 1 \otimes y^{(0)}$ (see the description 9.3.3.6.1 of the pushforward to understand the notation $1 \otimes y^{(0)}$). By definition of $G^{(m)}$ (see the step II.4), since $a^*(E_{\text{n-hol}}) \xrightarrow{\sim} \varinjlim_m a^*(E_{\text{n-hol}}^{(m)})$, then there exists $s \geq 1$ large enough such that the image $x^{(s)}$ of $x^{(0)}$ via the canonical morphism $a^*(E_{\text{n-hol}}^{(m_0)}) \to a^*(E_{\text{n-hol}}^{(m_0+s)})$ belongs to $G^{(m_0+s)}$. By tensoring with $\mathbb{Q}$ 15.3.5.3.10 and by functoriality of the first isomorphism of 15.3.5.3.11, we get the commutative diagram

$$\begin{array}{ccc} a^*(E_{\text{n-hol}}^{(m_0)}) & \xrightarrow[\theta_{\mathbb{Q}}^{(m_0)}]{\sim} & (u'^{(m_0)}_{+} \circ b^*(\mathring{F}^{(m_0)}))_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ a^*(E_{\text{n-hol}}^{(m_0+s)}) & \xrightarrow[\theta_{\mathbb{Q}}^{(m_0+s)}]{\sim} & (u'^{(m_0+s)}_{+} \circ b^*(\mathring{F}^{(m_0+s)}))_{\mathbb{Q}}. \end{array} \qquad (15.3.5.3.14)$$

The image of $1 \otimes y^{(0)} = \sum_{j=0}^{\infty} 1 \otimes y_j^{(0)}$ via the right arrow of 15.3.5.3.14 is $1 \otimes y^{(s)}$, where $y^{(s)} := \sum_{j=0}^{\infty} y_j^{(s)} \in b^*(\mathring{F}^{(m_0+s)})$. This implies the equality $\theta_{\mathbb{Q}}^{(m_0+s)}(x^{(s)}) = 1 \otimes y^{(s)}$ and then$1 \otimes y^{(s)} \in H^{(m_0+s)}$. Since $1 \otimes y^{(s)}$ is killed by $t_1, \ldots, t_r$, we have in fact $1 \otimes y^{(s)} \in H^0 u'^{(m)!}(H^{(m_0+s)})$. Modulo the identification of $H^0 u'^{(m)!} H^{(m_0+s)})$ as a subset of $b^*(F^{(m_0+s)})$, we get then $y^{(s)} \in H^0 u'^{(m)!}(H^{(m_0+s)})$. So, $y^{(s)} - \sum_{j=0}^{s} y_j^{(s)} \in (H^0 u'^{(m)!}(H^{(m_0+s)}) + \sum_{j=0}^{s-1} K y_j^{(s)}) + K y_s^{(s)} = M^{(s)} \oplus K y_s^{(s)}$. However, for any $j \geq s+1$, we have $y_j^{(s)} \in p^{1+n_s} b^*(\mathring{F}^{(m_0+s)})$. Since $y^{(s)} - \sum_{j=0}^{s} y_j^{(s)} = \sum_{j=s+1}^{\infty} y_j^{(s)}$, we get $y^{(s)} - \sum_{j=0}^{s} y_j^{(s)} \in \left(M^{(s)} \oplus K y_s^{(s)}\right) \cap p^{1+n_s} b^*(\mathring{F}^{(m_0+s)}) \subset p \left(\mathring{M}^{(s)} \oplus \mathcal{V} y_s^{(s)}\right)$. Since $y^{(s)} - \sum_{j=0}^{s} y_j^{(s)}$ splits in $M^{(s)} \oplus K y_s^{(s)}$ of the form $(y^{(s)} - \sum_{j=0}^{s-1} y_j^{(s)}) - y_s^{(s)}$, we get a contradiction.

Step 9)*: Conclusion*

Following the step 8), there exists an integer m_1 such that $b^*(F^{(m_1)})$ is a K-vector space of finite dimension. By 15.3.1.8, this yields that for any $m \geq m_1$, $b^*(F^{(m)})$ is a K-vector space of finite dimension. Let $m \geq m_1$. Since $b^*(\mathring{\mathcal{F}}^{(m)}) \xrightarrow{\sim} (\mathcal{V}^{(A_m)})^{\wedge}$ (see step I.4), as $(b^*(\mathring{\mathcal{F}}^{(m)})_{\mathbb{Q}} \xrightarrow{\sim} b^*(F^{(m)})$, this yields that A_m is a finite set and therefore $\mathring{\mathcal{F}}^{(m)}|\mathfrak{Z}_m$ is a free $\mathcal{O}_{\mathfrak{Z}_m}$-module of finite rank. Hence, $\mathcal{F}^{(m)}|\mathfrak{Z}_m$ is a free $\mathcal{O}_{\mathfrak{Z}_m,\mathbb{Q}}$-module of rank r. Using Lemma 15.3.1.14, for m large enough $\mathcal{F}|\mathfrak{Z}_m$ is a free $\mathcal{O}_{\mathfrak{Z}_m,\mathbb{Q}}$-module of finite rank. Hence, we are done. □

Theorem 15.3.5.4. *Let $\mathcal{E}$ be an overcoherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module in $\mathfrak{P}$ after any base change. Then $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-holonomic.*

Proof. Following 15.2.4.6, we reduce to the case where the divisor T is empty. Set $\mathcal{E}_{\text{n-hol}} := \mathcal{E}/\mathcal{E}^{\text{hol}}$. Denote by Z the support of $\mathcal{E}_{\text{n-hol}}$. By the absurd, suppose Z is non-empty and let Z' be an irreducible component. Following 15.3.5.3, then there exists an open set $\mathfrak{U}$ of $\mathfrak{P}$ such that the open subset $U \cap Z'$ is smooth and dense in Z' and such that the module $\mathcal{E}_{\text{n-hol}}|\mathfrak{U}$ is holonomic. However, $(\mathcal{E}_{\text{n-hol}}|\mathfrak{U})^{\text{hol}} = (\mathcal{E}_{\text{n-hol}})^{\text{hol}}|\mathfrak{U} = 0$, the last equality coming from 15.2.4.18. Moreover, since the module $\mathcal{E}_{\text{n-hol}}|\mathfrak{U}$ is holonomic, we get the equality $(\mathcal{E}_{\text{n-hol}}|\mathfrak{U})^{\text{hol}} = (\mathcal{E}_{\text{n-hol}}|\mathfrak{U})$ (see 15.2.4.16). Hence we get a contradiction. □

15.3.6 $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherence (after any base change)

Definition 15.3.6.1. Let $\mathcal{E}^{(\bullet)} \in (F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

(a) We say that $\mathcal{E}^{(\bullet)}$ is "$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-overcoherent" if, for any smooth morphism of the form $f\colon \mathfrak{P}' \to \mathfrak{P}$, we have $f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}'}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(f^{-1}(T)))$. We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$) the full subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. of $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$) consisting of overcoherent complexes.

(b) We say that $\mathcal{E}^{(\bullet)}$ is "$\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-overcoherent after any base change" if for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$ (see notation 9.2.6.12), denoting by $\mathfrak{S}' := \operatorname{Spf} \mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f\colon \mathfrak{P}' \to \mathfrak{P}$ the projection,

$T' := f^{-1}(T)$, we have $\mathcal{V}'\widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}}\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'))$ (see notation 9.2.6.13). We denote by $(F$-$)\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $(F$-$)\underrightarrow{LM}_{\mathbb{Q},\mathrm{oc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$) the full subcategory of $(F$-$)\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$) consisting of $\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)$-overcoherent after any base change complexes.

Definition 15.3.6.2. Let $\mathcal{E}$ be an object of $(F$-$)D(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

(a) The complex $\mathcal{E}$ is "$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent" if for any smooth morphism of the form $f\colon \mathfrak{P}' \to \mathfrak{P}$, we have $f^{!}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh},\mathfrak{P}'}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}f^{-1}(T))_{\mathbb{Q}})$. We denote by $(F$-$)D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ the full subcategory of $(F$-$)D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ consisting of $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent complexes.

(b) We say that $\mathcal{E}$ is " $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent after any base change" if, for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$, denoting by $\mathfrak{S}' := \mathrm{Spf}\,\mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P}\times_{\mathfrak{S}}\mathfrak{S}'$, $f\colon \mathfrak{P}' \to \mathfrak{P}$ the projection, $\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}'}({}^{\dagger}T')_{\mathbb{Q}}\otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S}}({}^{\dagger}T)_{\mathbb{Q}}} f^{-1}\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$, i.e. the base change $f^{!}_{T}(\mathcal{E})$ of $\mathcal{E}$ via $\mathcal{V} \to \mathcal{V}'$ is overcoherent (see 9.2.7.1). We denote by $(F$-$)D^{\mathrm{b}}_{\mathrm{oc}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ the full subcategory of $(F$-$)D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ consisting of $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent after any base change complexes.

Example 15.3.6.3. Since the constant sheaf is stable under base change and inverse images, then it follows from 15.3.4.4 that the $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$ is overcoherent after any base change.

Like coherence, we verify that the notion of overcoherent is local in $\mathfrak{P}$. Likewise, we extend the standard properties from coherent modules to overcoherent modules:

Proposition 15.3.6.4. We have the following properties.

(a) Let $\Phi : \mathcal{E} \to \mathcal{F}$ be a morphism of overcoherent (after any base change) $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$). Then $\ker\Phi$ and $\mathrm{Im}\Phi$ are overcoherent (after any base change).

(b) Let $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \mathcal{E}_4 \to \mathcal{E}_5$ an exact sequence of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules (resp. of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$). If $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$ are overcoherent (after any base change) then so is $\mathcal{E}_3$.

Proof. Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism of $\mathfrak{S}$-formal schemes. Denote by $f^{*} := H^{0}f^{!}[-d_{X'/X}]$ and $T' := f^{-1}(T)$. Since f is smooth, then the functor f^{*} is exact and the image by f^{*} of a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}}$-module. Hence, this functor f^{*} commutes with kernels, cokernels and images, which implies (a) thanks to 15.3.4.8. Moreover, we get the exact sequence $f^{*}(\mathcal{E}_1) \to f^{*}(\mathcal{E}_2) \to f^{*}(\mathcal{E}_3) \to f^{*}(\mathcal{E}_4) \to f^{*}(\mathcal{E}_5)$. Since by hypothesis all terms except the middle one are overcoherent (after any base change) in $\mathfrak{P}$ then thanks to 15.3.4.7 so is $\mathcal{E}_3$. □

The next proposition is obvious.

Proposition 15.3.6.5. *Let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$.*

(a) A direct summand of a complex of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $D^{\mathrm{b}}_{\natural}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$) is a complex of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $D^{\mathrm{b}}_{\natural}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$).

(b) Let $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}'[1]$ be a distinguished triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},qc}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $D(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$). If two of the three complexes belong to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $D^{\mathrm{b}}_{\natural}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$) then so does the third. In particular, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $D^{\mathrm{b}}_{\natural}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$) $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ (resp. $D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$)

Proposition 15.3.6.6. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. Let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$. The property $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ is equivalent to the property $\underrightarrow{l}^{*}_{\mathbb{Q}}\ \mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\natural}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Both properties are local in $\mathfrak{P}$. We have the equivalence of categories $\underrightarrow{l}^{*}_{\mathbb{Q}}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T)) \cong D^{\mathrm{b}}_{\natural}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

Proof. It follows from the proposition 15.3.4.5 and from the fact that, for any smooth morphism f of smooth $\mathfrak{S}$-formal schemes, the functor $f^{(\bullet)!}$ preserves the coherence. □

15.3.6.7. It follows from 15.3.6.6 that the properties concerning the overcoherence are still valid replacing the categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural,\mathfrak{P}}$ by $D^{\mathrm{b}}_{\natural,\mathfrak{P}}$ and vice versa. We will therefore in the following simply write and check one of the two contexts.

Let us now move on to the stability properties of overcoherent by cohomological operations.

Lemma 15.3.6.8. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. The following conditions are equivalent:*

(a) $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$;

(b) For any smooth morphism of the form $f\colon \mathfrak{P}' \to \mathfrak{P}$, for any subscheme (resp. open subscheme, resp. closed subscheme) Y' of P', we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} \circ f^{(\bullet)!}(\mathcal{E}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(f^{-1}(T)))$.

(c) For any smooth morphism of the form $f\colon \mathfrak{P}' \to \mathfrak{P}$, for any subscheme Y' of P', we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} \circ f^{(\bullet)!}(\mathcal{E}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(f^{-1}(T)))$.

Proof. We proceed therefore similarly to 15.3.4.10. □

Lemma 15.3.6.9. *For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$, for any subscheme Y of P, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.*

Proof. Since base changes commute to localisation functors (see 13.1.5.7) and pullbacks 9.2.6.6, then we reduce to the case where $\natural = \mathrm{ovcoh}$. Hence, this follows from 15.3.6.8 and 13.1.5.6.1. □

15.3.6.10. If D is any divisor of P, then we have defined the functor $({}^{\dagger}D)$ of 9.1.1.12.1 factors through $({}^{\dagger}D)\colon D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Then copying the proof of 13.1.1.4 where we replace $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$ by $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$, we get the *canonical* functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{D}\colon D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Then, thanks to the stability of the overcoherence, we can copy the proof of 13.1.3.4 which remains valid for $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ instead of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$. Let X be a reduced closed subscheme of P. We can therefore copy the construction of respectively 13.1.3.8 and 13.1.4.4 in order to define the functors $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}$ and $({}^{\dagger}X)$. Let Y be a subscheme of P. Similarly to 13.1.5.1, we construct the functor:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\colon D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}).$$

Notation 15.3.6.11. Let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$. Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of smooth $\mathcal{V}$-formal schemes, T and T' some divisors of respectively P and P' such that $T' = f^{-1}(T)$.

Proposition 15.3.6.12. *With the notations 15.3.6.11, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$, we have $f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'))$.*

Proof. Since base changes commute to localisation functors (see 13.1.5.7) and pullbacks 9.2.6.6, then we reduce to the case where $\natural = \mathrm{ovcoh}$. Since f decomposes into a closed immersion followed by a smooth morphism and since the case where f is a smooth morphism is immediate, we reduce to the case where f is a closed immersion. Let $g : \mathfrak{Q}' \to \mathfrak{P}'$ be a smooth morphism and Z' a closed subscheme of Q'. It is enough to prove that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} g^{(\bullet)!}(f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}'}({}^{\dagger}g^{-1}(T'))_{\mathbb{Q}})$, which is local in $\mathfrak{Q}'$. We can therefore assume that g decomposes into a closed immersion $\mathfrak{Q}' \hookrightarrow \widehat{\mathbb{A}}^{n}_{\mathfrak{P}'}$ followed by the projection $\widehat{\mathbb{A}}^{n}_{\mathfrak{P}'} \to \mathfrak{P}'$. By denoting p the projection $\widehat{\mathbb{A}}^{n}_{\mathfrak{P}} \to \mathfrak{P}$ and i the closed immersion $\mathfrak{Q}' \hookrightarrow \widehat{\mathbb{A}}^{n}_{\mathfrak{P}}$, we obtain $f \circ g = p \circ i$. Hence the isomorphism:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} g^{(\bullet)!}(f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} i^{(\bullet)!} p^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \overset{13.2.1.4.1}{\xrightarrow{\sim}} i^{(\bullet)!}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} p^{(\bullet)!}(\mathcal{E}^{(\bullet)}).$$

As $\mathcal{E}^{(\bullet)}$ is overcoherent, since p is smooth, then $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} p^{(\bullet)!}(\mathcal{E}^{(\bullet)})) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\widehat{\mathbb{A}}^{n}_{\mathfrak{P}}}({}^{\dagger}p^{-1}(T))_{\mathbb{Q}})$ (see 15.3.6.8) with support in $Z' \subset P'$. The theorem 9.3.5.13.(c) allows us to conclude. □

Remark 15.3.6.13. The proposition 15.3.6.12 implies that an overcoherent complex has finite extraordinary fibers. Hence, we can apply for instance 15.3.1.19 and 15.3.2.8 for overcoherent complexes.

Proposition 15.3.6.14. *With the notations 15.3.6.11, suppose f is realizable with respect to T in the sense of 13.2.3.1. For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'))$ with proper support over P, we have therefore $f'^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.*

Proof. Since base changes commute to localisation functors (see 13.1.5.7), pullbacks 9.2.6.6 and to push-forwards (see 9.2.6.9), then we reduce to the case where $\natural$ = ovcoh. Let $g\colon \mathfrak{Q} \to \mathfrak{P}$ be a smooth morphism and Z a closed subscheme of Q. Following 15.3.6.8, it is therefore sufficient to check that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} g^{(\bullet)!} f'^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}}({}^{\dagger}g^{-1}(T))_{\mathbb{Q}})$. We denote by $\mathfrak{Q}' := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{Q}$, by $f'\colon \mathfrak{Q}' \to \mathfrak{Q}$ and $g'\colon \mathfrak{Q}' \to \mathfrak{P}'$ the two canonical projections. We get the isomorphism:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} g^{(\bullet)!} f'^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow[\sim]{13.2.3.7} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} f'^{(\bullet)}_{+} \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \xrightarrow[\sim]{13.2.1.4.2} f'^{(\bullet)}_{+} \mathbb{R}\underline{\Gamma}^{\dagger}_{f'^{-1}(Z)} g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$$

However, it results from the characterisation of the overcoherence of 15.3.6.8.(b) that $\mathbb{R}\underline{\Gamma}^{\dagger}_{f'^{-1}(Z)} g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}'}({}^{\dagger}p^{-1}(T'))_{\mathbb{Q}})$. Since f' is realizable with respect to $g^{-1}(T)$, then it follows from 13.2.3.4 that $f'^{(\bullet)}_{+} \mathbb{R}\underline{\Gamma}^{\dagger}_{f'^{-1}(Z)} g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}}({}^{\dagger}g^{-1}(T))_{\mathbb{Q}})$. □

15.3.6.15. By using some devissage in overconvergent isocrystals, we will improve later the stability property of Proposition 15.3.6.14 by removing the assumption of realisability of the morphism (see 16.3.3.1).

In the next section we will need the following lemma.

Lemme 15.3.6.16. Let $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ be a closed immersion of $\mathcal{V}$-smooth formal schemes and $\mathcal{E} \in F\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. We have a canonical compatible to Frobenius isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ of the form $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)})$ fitting into the canonical diagram

$$\begin{array}{ccc} u^{(\bullet)}_{+} \circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)}) & \xrightarrow{\text{adj}} & \mathcal{E}^{(\bullet)} \\ \downarrow{\scriptstyle\sim} & & \| \\ \mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) & \longrightarrow & \mathcal{E}^{(\bullet)}. \end{array} \tag{15.3.6.16.1}$$

where the top (resp. bottom) horizontal arrow is the adjunction morphism of 9.5.4.5 (resp. is induced by functoriality in X of the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}$).

Proof. This follows from 13.2.1.7. □

15.3.7 $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}Z)_{\mathbb{Q}}$-overcoherence (after any base change)

Let $\mathfrak{P}$ be a smooth formal scheme over $\mathfrak{S}$. Let Z be a closed subscheme of P. Set $\mathfrak{U} := \mathfrak{P} \setminus Z$.

Definition 15.3.7.1. We introduce the following categories.

(a) We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ the strictly full subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$ consisting of objects $\mathcal{E}^{(\bullet)}$ satisfying the following properties:

  (i) The canonical morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$
  $$\mathcal{E}^{(\bullet)} \to ({}^{\dagger}Z)(\mathcal{E}^{(\bullet)})$$
  is an isomorphism ;

  (ii) For any smooth morphism $f\colon \mathfrak{P}' \to \mathfrak{P}$, for any divisor T' of P' containing $f^{-1}(Z)$ with notation 9.2.1.15 and 15.3.4.1 we have
  $$({}^{\dagger}T')f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}'}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T')).$$

(b) We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ the strictly full subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$, denoting by $\mathfrak{S}' := \mathrm{Spf}\,\mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f\colon \mathfrak{P}' \to \mathfrak{P}$ the projection, we have $\mathcal{V}'\widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}}\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}'}(Z'))$.

(c) The objects of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ (resp. $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$) are said to be $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)$-overcoherent $(F\text{-})$complexes (resp. after any base change).

Example 15.3.7.2. When Z is the support of a divisor of P, then both definition 15.3.6.1 and 15.3.7.1 are the same.

From now, let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$. Let us give the following straightforward properties.

Lemma 15.3.7.3. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$.*

(a) Let $(\mathfrak{P}_i)_{i\in I}$ be an open covering of $\mathfrak{P}$. Set $Z_i := Z \cap P_i$. Then $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ if and only if for any $i \in I$, $\mathcal{E}^{(\bullet)}|\mathfrak{P}_i \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(Z_i))$.

(b) Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism. Then the functor $f^{*(\bullet)}$ (see notation 9.2.1.15) induces

$$f^{*(\bullet)}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(f^{-1}(Z))). \tag{15.3.7.3.1}$$

Proof. Straightforward. □

Lemma 15.3.7.4. *For any closed subscheme Z' of P, the functor $({}^\dagger Z')$ induces*

$$({}^\dagger Z')\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z\cup Z')) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)). \tag{15.3.7.4.1}$$

Proof. Since base changes commute with localisation functors (see 13.1.5.7), then we reduce to the case where $\natural = \mathrm{ovcoh}$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism, T' be a divisor of P' containing $f^{-1}(Z)$. ince the canonical morphism $({}^\dagger Z')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger Z'')(({}^\dagger Z')(\mathcal{E}^{(\bullet)}))$ is an isomorphism for $Z'' = Z'$ or $Z'' = Z' \cup Z$, then we reduce to prove $({}^\dagger T')f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}'}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$. This is check by using 15.3.4.10 via the isomorphisms:

$$\begin{aligned}({}^\dagger T')f^{*(\bullet)}\left(({}^\dagger Z')(\mathcal{E}^{(\bullet)})\right) &\underset{13.2.1.4.1}{\xrightarrow{\sim}} ({}^\dagger T')({}^\dagger f^{-1}(Z'))f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \\ &\xrightarrow{\sim} ({}^\dagger f^{-1}(Z'))\left(({}^\dagger T')f^{*(\bullet)}(\mathcal{E}^{(\bullet)})\right) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}'}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T')).\end{aligned}$$

□

Lemma 15.3.7.5. *For any subscheme Y' of P, the functor $\mathbb{R}\underline{\Gamma}^\dagger_{Y'}$ induces*

$$\mathbb{R}\underline{\Gamma}^\dagger_{Y'}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\mathcal{D}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)). \tag{15.3.7.5.1}$$

Proof. By construction of the local cohomological functor, since $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ is a full triangulated subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$, then we reduce to the case where Y' is an open, which is a consequence of 15.3.7.4.1. □

The following proposition is an extension of 8.7.6.11.

Proposition 15.3.7.6. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. The following properties are equivalent.*

(a) $\mathcal{E}^{(\bullet)} \xrightarrow{\sim} 0$.

(b) $\mathcal{E}^{(\bullet)}|\mathfrak{U} \xrightarrow{\sim} 0$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Let us treat the converse. We can suppose $\mathfrak{P}$ is integral. Let us denote by n_Z the minimal number of divisor $T_1, \dots, T_r$ such that $Z = T_1 \cap \dots \cap T_r$. We check the proposition by induction on n_Z. The case where $n_Z = 0$ (i.e. is the case where $Z = P$) is obvious and the case $n_Z = 1$ (i.e. the case where Z is a divisor) is a consequence of 8.7.6.11. Suppose now $r \geq 2$ and the proposition holds for any closed subschemes $\widetilde{Z}$ such that $n_{\widetilde{Z}} < r$. Set $Z' = T_2 \cap \dots \cap T_r$. By using Mayer-Vietoris exact triangles (see 13.1.4.15.2), we get the exact triangle

$$\mathcal{E}^{(\bullet)} \to ({}^\dagger T_1)(\mathcal{E}^{(\bullet)}) \oplus ({}^\dagger Z')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger T_1 \cup Z')(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}[1]. \tag{15.3.7.6.1}$$

For $D \in \{T_1, Z', T_1 \cup Z'\}$, set $\mathfrak{U}' := \mathfrak{P} \setminus D \subset \mathfrak{U}$. Since $({}^\dagger D)(\mathcal{E}^{(\bullet)})|\mathfrak{U}' = 0$, then by using the induction hypothesis, $({}^\dagger Z')(\mathcal{E}^{(\bullet)}) = 0$. Hence, we get from 15.3.7.6.1 that $\mathcal{E}^{(\bullet)} = 0$. □

Corollary 15.3.7.7. *Let $\phi\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a homomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. The morphism ϕ is an isomorphism if and only if so is $\phi|\mathfrak{U}$.*

Proof. Suppose $\phi|_{\mathfrak{U}}$ is an isomorphism. Let $\mathcal{G}^{(\bullet)}$ be a mapping cone of ϕ, which is an object of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. Since $\phi|_{\mathfrak{U}}$ is an isomorphism, then $\mathcal{G}^{(\bullet)}|\mathfrak{U} \xrightarrow{\sim} 0$. It follows from 15.3.7.6 that $\mathcal{G}^{(\bullet)} \xrightarrow{\sim} 0$, i.e. ϕ is an isomorphism. □

Proposition 15.3.7.8. *Let $\natural \in \{\mathrm{ovcoh}, \mathrm{oc}\}$. Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ a proper morphism of smooth $\mathcal{V}$-formal schemes, Z be a closed subscheme of P and $Z' := f^{-1}(Z)$. For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$ with proper support over P, we have therefore $f'^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$.*

Proof. Since base changes commute to localisation functors (see 13.1.5.7), pullbacks 9.2.6.6 and to push-forwards (see 9.2.6.9), then we reduce to the case where $\natural = \mathrm{ovcoh}$. Let $g\colon \mathfrak{Q} \to \mathfrak{P}$ be a smooth morphism and T be a divisor of Q containing $f^{-1}(Z)$. It is sufficient to check that $({}^\dagger T) \circ g^{(\bullet)!} f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}}({}^\dagger g^{-1}(T))_{\mathbb{Q}})$. We denote by $\mathfrak{Q}' := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{Q}$, by $f'\colon \mathfrak{Q}' \to \mathfrak{Q}$ and $g'\colon \mathfrak{Q}' \to \mathfrak{P}'$ the two canonical projections. We get the isomorphism:

$$({}^\dagger T) \circ g^{(\bullet)!} \circ f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \overset{13.2.3.7}{\xrightarrow{\sim}} ({}^\dagger T) \circ f'^{(\bullet)}_{+} \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \overset{13.2.1.4.2}{\xrightarrow{\sim}} f'^{(\bullet)}_{+} \circ ({}^\dagger f'^{-1}(T)) \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$$

However, it results from 15.3.8.6.1 and 15.3.7.4.1 $({}^\dagger f'^{-1}(T)) \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}'}({}^\dagger p^{-1}(T'))_{\mathbb{Q}})$. Hence, it remains to prove $f'^{(\bullet)}_{+} \circ ({}^\dagger f'^{-1}(T)) \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}}({}^\dagger g^{-1}(T))_{\mathbb{Q}})$. Since $\mathfrak{Q}'$ is suppose, then we can suppose Q' is integral. So either $f'^{-1}(T)$ is a divisor of Q' or $f'^{-1}(T) = Q'$. The second case is obvious since we get therefore $({}^\dagger f'^{-1}(T)) \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) = 0$. Since f' is proper, then the first case is a consequence of 15.3.6.14 and we are done.

□

15.3.7.9. The stability under extraordinary pullbacks will be checked in the next subsection (see 15.3.8.27) because its proof uses an overcoherent version of Berthelot-Kashiwara theorem.

15.3.8 Partial overcoherence and t-structure over a c-frame $(Y, X, \mathfrak{P}, Z/\mathfrak{S})$

Let Z and X be two closed subschemes of P. Set $\mathfrak{U} := \mathfrak{P} \setminus Z$, $Y := X \setminus Z$ and $\mathfrak{V} := \mathfrak{P} \setminus X$.

Definition 15.3.8.1. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. Since $\mathcal{E}^{(\bullet)}|\mathfrak{U} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$, then its support $\mathrm{Supp}(\mathcal{E}^{(\bullet)}|\mathfrak{U})$ is well defined (see definition 13.1.4.17). We define the support of $\mathcal{E}^{(\bullet)}$ as the closure in P of $\mathrm{Supp}(\mathcal{E}^{(\bullet)}|\mathfrak{U})$. We denote by $\mathrm{Supp}(\mathcal{E}^{(\bullet)})$ the support of $\mathcal{E}^{(\bullet)}$.

We extend the proposition 13.1.4.8 as follows.

Proposition 15.3.8.2. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. The following assertions are equivalent:*

(a) $\mathrm{Supp}(\mathcal{E}^{(\bullet)}) \subset X$.

(b) We have in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{V}/\mathfrak{S}})$ the isomorphism $\mathcal{E}^{(\bullet)}|\mathfrak{V} \xrightarrow{\sim} 0$.

(c) *The canonical morphism* $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ *is an isomorphism in* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$.

(d) *We have in* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$ *the isomorphism* $({}^{\dagger}X)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$.

Proof. i) First, let us check the equivalence between (a) and (b). By definition, we have the equivalences $\mathrm{Supp}(\mathcal{E}^{(\bullet)}) \subset X \Leftrightarrow \mathrm{Supp}(\mathcal{E}^{(\bullet)}|\mathfrak{U}) \subset X \cap U \Leftrightarrow \mathcal{E}^{(\bullet)}|\mathfrak{U}\cap\mathfrak{V} \xrightarrow{\sim} 0$. Since $\mathcal{E}^{(\bullet)}|\mathfrak{V} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{V}/\mathfrak{S}}(Z\cap V))$ (see 15.3.7.3), then it follows from 15.3.7.6 the equivalence $\mathcal{E}^{(\bullet)}|\mathfrak{V}\cap\mathfrak{U} \xrightarrow{\sim} 0 \Leftrightarrow \mathcal{E}^{(\bullet)}|\mathfrak{V} \xrightarrow{\sim} 0$. Hence, we are done.

ii) The equivalence between (c) and (d) is a consequence of the exact triangle

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to ({}^{\dagger}X)(\mathcal{E}^{(\bullet)}) \to +1.$$

iii) The implication (d) ⇒ (b) is obvious. Let us now check the converse implication. We can suppose $\mathfrak{P}$ is integral. Let us denote by n_Z the minimal number of divisor $T_1, \ldots, T_r$ such that $Z = T_1 \cap \cdots \cap T_r$. We check the proposition by induction on n_Z. The case where $n_Z = 0$ (i.e. is the case where $Z = X$) is obvious and the case $n_Z = 1$ (i.e. the case where Z is a divisor) follows from the proposition 13.1.4.8. Suppose now $r \geq 2$ and the proposition holds for any closed subscheme $\widetilde{Z}$ such that $n_{\widetilde{Z}} < r$. Set $Z' = T_2 \cap \cdots \cap T_r$. By using Mayer-Vietoris exact triangles (see 13.1.4.15.2), we get the exact triangle

$$\mathcal{E}^{(\bullet)} \to ({}^{\dagger}T_1)(\mathcal{E}^{(\bullet)}) \oplus ({}^{\dagger}Z')(\mathcal{E}^{(\bullet)}) \to ({}^{\dagger}T_1 \cup Z')(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}[1].$$

For $D \in \{T_1, Z', T_1 \cup Z'\}$, we have $({}^{\dagger}D)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(D))$ and $({}^{\dagger}D)(\mathcal{E}^{(\bullet)})|\mathfrak{V} \xrightarrow{\sim} 0$. Hence, by using the induction hypothesis we get $({}^{\dagger}X)(\mathcal{E}^{(\bullet)}({}^{\dagger}D)) \xrightarrow{\sim} 0$ and we are done. □

Definition 15.3.8.3. We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{povcoh}}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$ (resp. $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$) the subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ (resp. $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$) consisting of ($F$-)complexes $\mathcal{E}^{(\bullet)}$ with support in X (see definition 15.3.8.1). The objects of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$ are called "partially overcoherent over $(X, \mathfrak{P}, Z/\mathfrak{S})$ (F-)complexes (resp. after any base change)" when $\natural$ = povcoh (resp. $\natural$ = poc).

From now, in this subsection, let $\natural \in \{\mathrm{povcoh}, \mathrm{poc}\}$. We can simply write $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$ instead of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$ and when Z is empty we can remove it in the notation.

Example 15.3.8.4. When $X = P$, we have $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{povcoh}}(P, \mathfrak{P}, Z/\mathfrak{S}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}(P, \mathfrak{P}, Z/\mathfrak{S}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$.

Remark 15.3.8.5. We have added the word "partially" to avoid confusion between $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{povcoh}}(P, \mathfrak{P}, Z/\mathfrak{S})$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ that we will denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathfrak{S})$ where $Y = P \setminus Z$ (see 19.1.2.2 and 19.2.1.5).

Lemma 15.3.8.6. *Let* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$.

(a) *Let* $(\mathfrak{P}_i)_{i\in I}$ *be an open covering of* $\mathfrak{P}$. *Set* $Z_i := Z \cap P_i$ *and* $X_i := X \cap P_i$. *Then* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$ *if and only if for any* $i \in I$, $\mathcal{E}^{(\bullet)}|\mathfrak{P}_i \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X_i, \mathfrak{P}_i, Z_i/\mathfrak{S})$.

(b) *Let* $f\colon \mathfrak{P}' \to \mathfrak{P}$ *be a smooth morphism. Then the functor* $f^{*(\bullet)}$ *induces*

$$f^{*(\bullet)}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(f^{-1}(X), \mathfrak{P}', f^{-1}(Z)/\mathfrak{S}). \tag{15.3.8.6.1}$$

Proof. Straightforward. □

Lemma 15.3.8.7. *Let* X' *be a closed subscheme of* X. *We have the inclusion* $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X', \mathfrak{P}, Z/\mathfrak{S}) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. *This inclusion has the left adjoint* $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X', \mathfrak{P}, Z/\mathfrak{S})$.

Proof. This is a consequence of 15.3.7.5.1. □

Remark 15.3.8.8. Beware that if Z' is a closed subscheme of P containing Z, then we can not compare a priori $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z'/\mathfrak{S})$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$. Moreover, it follows from 15.3.7.4.1 that the functor $({}^\dagger Z')$ of is a right adjoint to the inclusion of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z'/\mathfrak{S})\cap\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$.

Notation 15.3.8.9. Let $n\in\mathbb{Z}$. Following 13.1.4.18, we have a canonical t-structure on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$ and the truncated categories are denoted by $\underrightarrow{LD}^{\leq n}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$ and $\mathcal{E}^{(\bullet)}|\mathfrak{U}\in\underrightarrow{LD}^{\geq n}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$.

We denote by $\underrightarrow{LD}^{\leq n}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$ (resp. $\underrightarrow{LD}^{\geq n}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$) the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$ of complexes $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}|\mathfrak{U}\in\underrightarrow{LD}^{\leq n}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$ (resp. $\mathcal{E}^{(\bullet)}|\mathfrak{U}\in\underrightarrow{LD}^{\geq n}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$). We have the obvious equalities $\underrightarrow{LD}^{\leq n}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})=\underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})[-n]$, $\underrightarrow{LD}^{\geq n}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})=\underrightarrow{LD}^{\geq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})[-n]$ and the inclusions $\underrightarrow{LD}^{\geq 1}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})\subset\underrightarrow{LD}^{\geq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$, $\underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})\subset\underrightarrow{LD}^{\leq 1}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$.

Example 15.3.8.10. When Z is the support of a divisor T of P, the definitions of 15.3.8.9 is related to that of 13.1.4.18 via the equalities

$$\underrightarrow{LD}^{\star}_{\mathbb{Q},\natural}(X,\mathfrak{P},T/\mathfrak{S})=\underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))\cap\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},T/\mathfrak{S}),\qquad(15.3.8.10.1)$$

where $\star$ is either "$\leq n$" or "$\geq n$". Hence, the natural t-structure on $\underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ induces a natural t-structure on $\underrightarrow{LD}^{\star}_{\mathbb{Q},\natural}(X,\mathfrak{P},T/\mathfrak{S})$.

Lemma 15.3.8.11. *Let Z' be a closed subscheme of P.*

(a) For any $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\star}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$, we have $({}^\dagger Z')(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\star}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z\cup Z'/\mathfrak{S})$, where $\star$ is either "b" or "$\leq n$" or "$\geq n$".

(b) Let $f\colon\mathcal{E}^{(\bullet)}\to\mathcal{F}^{(\bullet)}$ be an homomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$ such that $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$, $\mathcal{F}^{(\bullet)}\in\underrightarrow{LD}^{\geq 1}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z\cup Z'/\mathfrak{S})$. Then we have $f=0$.

Proof. The assertion (a) is obvious. Let us prove the part (b). The morphism f factors through a morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z\cup Z'/\mathfrak{S})$ of the form $({}^\dagger Z')(\mathcal{E}^{(\bullet)})\to\mathcal{F}^{(\bullet)}$. Since $({}^\dagger Z')(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z\cup Z'/\mathfrak{S})$, then we reduce to the case where Z' is empty. We can suppose that P is integral. We proceed by induction on n_Z, the minimal number r of divisors $T_1,\dots,T_r$ of P such that $Z=T_1\cap\dots\cap T_r$. The case $n_Z=0$ and $n_Z=1$ follows from 15.3.8.10.1. Suppose now $n_Z\geq 2$ and the proposition holds for $n_Z\leq r-1$. Set $Z_2=T_2\cap\dots\cap T_r$. We have the commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{E}^{(\bullet)} & \longrightarrow & ({}^\dagger T_1)(\mathcal{E}^{(\bullet)})\oplus({}^\dagger Z_2)(\mathcal{E}^{(\bullet)}) & \longrightarrow & ({}^\dagger T_1\cup Z_2)(\mathcal{E}^{(\bullet)}) & \longrightarrow & \mathcal{E}^{(\bullet)}[1]\ . \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle ({}^\dagger T_1)(f)\oplus({}^\dagger Z_2)(f)} & & \downarrow{\scriptstyle ({}^\dagger T_1\cup Z_2)(f)} & & \downarrow{\scriptstyle f[1]} \\
\mathcal{F}^{(\bullet)} & \longrightarrow & ({}^\dagger T_1)(\mathcal{F}^{(\bullet)})\oplus({}^\dagger Z_2)(\mathcal{F}^{(\bullet)}) & \longrightarrow & ({}^\dagger T_1\cup Z_2)(\mathcal{F}^{(\bullet)}) & \longrightarrow & \mathcal{F}^{(\bullet)}[1],
\end{array}\qquad(15.3.8.11.1)$$

where lines are Mayer-Vietoris exact triangles (see 13.1.4.15.2). By induction hypotheses and using the part (a) of the Lemma, we get $({}^\dagger T_1)(f)\oplus({}^\dagger Z_2)(f)=0$ and $({}^\dagger T_1\cup Z_2)(f)=0$. Let $g\colon\mathcal{E}^{(\bullet)}\to({}^\dagger T_1\cup Z_2)(\mathcal{F}^{(\bullet)})[-1]$ be a morphism. The morphism g factors through a morphism $h\colon({}^\dagger T_1\cup Z_2)(\mathcal{E}^{(\bullet)})\to({}^\dagger T_1\cup Z_2)(\mathcal{F}^{(\bullet)})[-1]$. Since $({}^\dagger T_1\cup Z_2)(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},T_1\cup Z_2/\mathfrak{S})$ and $({}^\dagger T_1\cup Z_2)(\mathcal{F}^{(\bullet)})[-1]\in\underrightarrow{LD}^{\geq 2}_{\mathbb{Q},\natural}(X,\mathfrak{P},T_1\cup Z_2/\mathfrak{S})\subset\underrightarrow{LD}^{\geq 1}_{\mathbb{Q},\natural}(X,\mathfrak{P},T_1\cup Z_2/\mathfrak{S})$, then by induction hypotheses, we get $h=0$ and then $g=0$. This yields f is the unique morphism making commutative the (left square of the) diagram 15.3.8.11.1 whose vertical morphisms of the middle square are zero. Hence, $f=0$. □

Proposition 15.3.8.12 (existence of t-structure)**.** We keep notation 15.3.8.9. Let $n\in\mathbb{N}$ and $\star$ means either $\leq n$ or $\geq n$.

(a) The paire $(\underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S}),\underrightarrow{LD}^{\geq 0}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S}))$ defines a t-structure, called the *canonical t-structure*, on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$. We denote by

$$\tau^{\star}_Z\colon\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})\to\underrightarrow{LD}^{\star}_{\mathbb{Q},\natural}(X,\mathfrak{P},Z/\mathfrak{S})$$

the truncation functors given by the canonical t-structure.

(b) For any morphism $\mathcal{V} \to \mathcal{V}'$ of DVR($\mathcal{V}$), denoting by $\mathfrak{S}' := \mathrm{Spf}\, \mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f\colon \mathfrak{P}' \to \mathfrak{P}$ the projection, $X' := f^{-1}(X)$, the functor

$$\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} - \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}(X', \mathfrak{P}', Z'/\mathfrak{S}) \tag{15.3.8.12.1}$$

is t-exact, i.e. preserves the bounded categories of the form $\underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{poc}}$ (see definition [HTT08, 8.1.13]).

(c) For any smooth morphism $f\colon \mathfrak{P}' \to \mathfrak{P}$, the functor

$$f^{*(\bullet)}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(f^{-1}(X), \mathfrak{P}', f^{-1}(Z)/\mathfrak{S}).$$

of 15.3.8.6.1 is t-exact.

(d) For any closed subscheme Z' of P, the functor

$$({}^{\dagger}Z')\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z \cup Z'/\mathfrak{S}) \tag{15.3.8.12.2}$$

is t-exact.

(e) For any divisor T of P, the functor

$$({}^{\dagger}T)\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \tag{15.3.8.12.3}$$

is t-exact

Proof. 1) We prove now the t-exactness of 15.3.8.12.1. We have to check the isomorphism:

$$\tau^{\star}_{Z'}(\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \tau^{\star}_{Z}(\mathcal{E}^{(\bullet)}).$$

i) We check that the objects $\tau^{\star}_{Z'}(\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)})$ and $\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}}(\tau^{\star}_{Z}(\mathcal{E}^{(\bullet)}))$ belong to $\underrightarrow{LD}^{\star}_{\mathbb{Q},poc}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}'}(Z'))$ as follows. We reduce by definition to check it outside Z and Z', i.e. we can suppose Z is empty. The base change functor $\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} - \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}'})$ corresponds via the equivalence of categories $\underrightarrow{l}^{*}_{\mathbb{Q}}$ of 15.3.6.6 to the functor $\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}',\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}} f^{-1} - \colon D^{\mathrm{b}}_{\mathrm{poc}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'/\mathfrak{S}',\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{poc}}(\mathcal{D}^{\dagger}_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}})$ which is exact. Hence, we are done.

ii) We conclude by using the distinguished triangles:

$$\tau^{\leq 0}_{Z'}(\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)}) \to \mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)} \to \tau^{\geq 1}_{Z'}(\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)}) \to +1,$$
$$\mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}}(\tau^{\leq 0}_{Z}(\mathcal{E}^{(\bullet)})) \to \mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}} \mathcal{E}^{(\bullet)} \to \mathcal{V}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{V}}(\tau^{\geq 1}_{Z}(\mathcal{E}^{(\bullet)})) \to +1.$$

2) Since the case $\natural = \mathrm{ovcoh}$ implies the other one (because of 15.3.8.12.1), let us prove the proposition for $\natural = \mathrm{ovcoh}$. We can suppose $\mathfrak{P}$ is integral. Let us denote by n_Z the minimal number of divisor $T_1, \dots, T_r$ such that $Z = T_1 \cap \cdots \cap T_r$. We check the proposition by induction on n_Z. The case where $n_Z = 0$ (i.e. is the case where $Z = X$) or $n_Z = 1$ (i.e. the case where Z is a divisor) follows from 15.3.8.10.1. Suppose now $r \geq 2$ and the proposition holds for any closed subscheme Z' such that $n_{Z'} < n_Z = r$. Let $\star$ equal to "≤ 0" or "≥ 1". Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$.

1) We prove in this step the part (a).

i) Thanks to Lemma 15.3.8.11 (used in the case where Z' is empty), it remains to check that there exists a distinguished triangle of the form

$$(\mathcal{E}^{(\bullet)})^{\leq 0} \to \mathcal{E}^{(\bullet)} \to (\mathcal{E}^{(\bullet)})^{\geq 1} \xrightarrow{+1},$$

where $(\mathcal{E}^{(\bullet)})^{\leq 0} \in \underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\mathrm{ovcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$ and $(\mathcal{E}^{(\bullet)})^{\geq 1} \in \underrightarrow{LD}^{\geq 1}_{\mathbb{Q},\mathrm{ovcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$.

We set $Z_2 := T_2 \cap \cdots \cap T_r$, $\mathcal{E}^{(\bullet)}_1 := \mathcal{E}^{(\bullet)}({}^{\dagger}T_1)$, $\mathcal{E}^{(\bullet)}_2 := \mathcal{E}^{(\bullet)}({}^{\dagger}Z_2)$, $\mathcal{E}^{(\bullet)}_{12} := \mathcal{E}^{(\bullet)}({}^{\dagger}T_1 \cup Z_2)$. We have the canonical morphism $\alpha_1\colon \mathcal{E}^{(\bullet)} \to \mathcal{E}^{(\bullet)}_1$, $\alpha_2\colon \mathcal{E}^{(\bullet)} \to \mathcal{E}^{(\bullet)}_2$, $\beta_1\colon \mathcal{E}^{(\bullet)}_1 \to \mathcal{E}^{(\bullet)}_{12}$, $\beta_2\colon \mathcal{E}^{(\bullet)}_2 \to \mathcal{E}^{(\bullet)}_{12}$. We get the Mayer-Vietoris exact triangle

$$\mathcal{E}^{(\bullet)} \xrightarrow{\alpha_1 + \alpha_2} \mathcal{E}^{(\bullet)}_1 \oplus \mathcal{E}^{(\bullet)}_2 \xrightarrow{(\beta_1, -\beta_2)} \mathcal{E}^{(\bullet)}_{12} \to \mathcal{E}^{(\bullet)}[1].$$

By using the induction hypothesis, since $\mathcal{E}_1^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},T_1/\mathfrak{S})$, then we can set $\mathcal{E}_1^{(\bullet)\star} := \tau^{\star}_{T_1}\mathcal{E}_1^{(\bullet)}$, where $\star$ is either "≤ 0" or "≥ 1". Similarly, as $\mathcal{E}_2^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z_2/\mathfrak{S})$, then we can put $\mathcal{E}_2^{(\bullet)\star} := \tau^{\star}_{Z_2}\mathcal{E}_2^{(\bullet)}$; since $\mathcal{E}_{12}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},T_1\cup Z_2/\mathfrak{S})$, then we can set $\mathcal{E}_{12}^{(\bullet)\star} := \tau^{\star}_{T_1\cup Z_2}\mathcal{E}_{12}^{(\bullet)}$.

ii) Let us check that $\mathcal{E}_2^{(\bullet)\star} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$. Let $f\colon \mathfrak{P}'\to\mathfrak{P}$ be a smooth morphism, $T'_i := f^{-1}(T_i)$ for any i, $Z'_2 := T'_2\cap\cdots\cap T'_r$, $Z' := f^{-1}(Z)$, $X' := f^{-1}(X)$. We have to prove $({}^\dagger T')\circ f^{*(\bullet)}\circ \tau^{\star}_{Z_2}\circ({}^\dagger Z_2)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}'}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$. By using the induction hypothesis for the part (c) of the proof (and use [HTT08, 8.1.15]), we get the first isomorphism:

$$f^{*(\bullet)}\circ\tau^{\star}_{Z_2}\circ({}^\dagger Z_2)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^{\star}_{Z'_2}\circ f^{*(\bullet)}\circ({}^\dagger Z_2)(\mathcal{E}^{(\bullet)}) \underset{13.2.1.4.1}{\xrightarrow{\sim}} \tau^{\star}_{Z'_2}\circ({}^\dagger Z'_2)\circ f^{*(\bullet)}(\mathcal{E}^{(\bullet)}).$$

Since $f^{*(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X',\mathfrak{P}',Z'/\mathfrak{S})$ (see 15.3.8.6.1), then we reduce to the case $f=\mathrm{id}$, i.e. we only need to prove that for any divisor T of P containing Z, we have $({}^\dagger T)\circ\tau^{\star}_{Z_2}\circ({}^\dagger Z_2)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$.

We have the distinguished triangles:

$$({}^\dagger T)\circ\tau^{\leq 0}_{Z_2}(\mathcal{E}_2^{(\bullet)}) \to ({}^\dagger T)(\mathcal{E}_2^{(\bullet)}) \to ({}^\dagger T)\circ\tau^{\geq 1}_{Z_2}(\mathcal{E}^{(\bullet)}) \to +1,$$
$$({}^\dagger Z_2)\circ\tau^{\leq 0}_{T}\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \to ({}^\dagger Z_2)\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \to ({}^\dagger Z_2)\circ\tau^{\geq 1}_{T}\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \to +1.$$

It follows from 15.3.8.11.(a) that $({}^\dagger Z_2)\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger T)(\mathcal{E}_2^{(\bullet)})$ are in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},T\cup Z_2/\mathfrak{S})$, $({}^\dagger T)\circ\tau^{\star}_{Z_2}(\mathcal{E}_2^{(\bullet)}) \in \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},T\cup Z_2/\mathfrak{S})$ and $({}^\dagger Z_2)\circ\tau^{\star}_{T}\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},T\cup Z_2/\mathfrak{S})$. Hence, it follows from the standard property of a t-structure that we get the isomorphism of $\underrightarrow{LD}^{\star}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},T\cup Z_2/\mathfrak{S})$:

$$({}^\dagger T)\circ\tau^{\star}_{Z_2}(\mathcal{E}_2^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger Z_2)\circ\tau^{\star}_{T}\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}).$$

Since $\tau^{\star}_{T}\circ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh},\mathfrak{P}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$, then so is $({}^\dagger Z_2)\circ\tau^{\star}_{T}\circ({}^\dagger T)(\mathcal{E}^{(\bullet)})$ (see 15.3.4.10) and we are done.

ii') Similarly, we check that $\mathcal{E}_1^{(\bullet)\star}, \mathcal{E}_{12}^{(\bullet)\star} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$.

iii) By using the induction hypothesis of 15.3.8.12.2 we get the t-exactness of the functor $({}^\dagger T_1\cup Z_2)$ and then we have the isomorphisms $\mathcal{E}_{12}^{(\bullet)\star} \xrightarrow{\sim} ({}^\dagger T_1\cup Z_2)(\mathcal{E}_1^{(\bullet)\star})$ and $\mathcal{E}_{12}^{(\bullet)\star} \xrightarrow{\sim} ({}^\dagger T_1\cup Z_2)(\mathcal{E}_2^{(\bullet)\star})$. Hence, we get a unique morphism $(\beta_1^{\leq 0}, -\beta_2^{\leq 0})\colon \mathcal{E}_1^{(\bullet)\leq 0}\oplus\mathcal{E}_2^{(\bullet)\leq 0} \to \mathcal{E}_{12}^{(\bullet)\leq 0}$, making commutative the diagram of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$:

$$\begin{array}{ccc}
\mathcal{E}_1^{(\bullet)\leq 0}\oplus\mathcal{E}_2^{(\bullet)\leq 0} & \xrightarrow{(\beta_1^{\leq 0},-\beta_2^{\leq 0})} & \mathcal{E}_{12}^{(\bullet)\leq 0} \\
\downarrow & & \downarrow \\
\mathcal{E}_1^{(\bullet)}\oplus\mathcal{E}_2^{(\bullet)} & \xrightarrow{(\beta_1,-\beta_2)} & \mathcal{E}_{12}^{(\bullet)}.
\end{array}$$

Choose a distinguished triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$ of the form

$$\mathcal{F}^{(\bullet)} \longrightarrow \mathcal{E}_1^{(\bullet)\leq 0}\oplus\mathcal{E}_2^{(\bullet)\leq 0} \xrightarrow{(\beta_1^{\leq 0},-\beta_2^{\leq 0})} \mathcal{E}_{12}^{(\bullet)\leq 0} \longrightarrow \mathcal{F}^{(\bullet)}[1]. \tag{15.3.8.12.4}$$

Set $\mathfrak{U}_1 := \mathfrak{P}\setminus T_1$, $\mathfrak{U}_2 := \mathfrak{P}\setminus Z_2$. We have $\mathcal{E}_1^{(\bullet)\leq 0}|\mathfrak{U}_1 = \mathcal{E}^{(\bullet)\leq 0}|\mathfrak{U}_1$, $\mathcal{E}_2^{(\bullet)\leq 0}|\mathfrak{U}_1 = \mathcal{E}_{12}^{(\bullet)\leq 0}|\mathfrak{U}_1$, and $\beta_2^{\leq 0}|\mathfrak{U}_1 = \mathrm{id}$. Hence, $\mathcal{F}^{(\bullet)}|\mathfrak{U}_1 \xrightarrow{\sim} \mathcal{E}^{(\bullet)\leq 0}|\mathfrak{U}_1$. Similarly, we get $\mathcal{F}^{(\bullet)}|\mathfrak{U}_2 \xrightarrow{\sim} \mathcal{E}^{(\bullet)\leq 0}|\mathfrak{U}_2$. This yields that $\mathcal{F}^{(\bullet)}|\mathfrak{U} \in \underrightarrow{LD}^{\leq n}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$. Hence, we have checked $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\leq 0}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$.

Following [Sta22, 05R0] (more precisely this is checked in the proof, up to a shift), there exist $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$ and morphisms in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X,\mathfrak{P},Z/\mathfrak{S})$ denoted by dotted arrows making commutative, except for the lower right square which is anticommutative, the diagram:

$$\begin{array}{ccccccc}
\mathcal{F}^{(\bullet)} & \longrightarrow & \mathcal{E}_1^{(\bullet)\leq 0} \oplus \mathcal{E}_2^{(\bullet)\leq 0} & \longrightarrow & \mathcal{E}_{12}^{(\bullet)\leq 0} & \longrightarrow & \mathcal{F}^{(\bullet)}[1] \\
\vdots & & \downarrow & & \downarrow & & \vdots \\
\mathcal{E}^{(\bullet)} & \xrightarrow{\alpha_1+\alpha_2} & \mathcal{E}_1^{(\bullet)} \oplus \mathcal{E}_2^{(\bullet)} & \xrightarrow{(\beta_1,-\beta_2)} & \mathcal{E}_{12}^{(\bullet)} & \longrightarrow & \mathcal{E}^{(\bullet)}[1] \\
\vdots & & \downarrow & & \downarrow & & \vdots \\
\mathcal{G}^{(\bullet)} & \cdots\to & \mathcal{E}_1^{(\bullet)\geq 1} \oplus \mathcal{E}_2^{(\bullet)\geq 1} & \cdots\to & \mathcal{E}_{12}^{(\bullet)\geq 1} & \longrightarrow & \mathcal{G}^{(\bullet)}[1] \\
\vdots & & \downarrow & & \downarrow & & \vdots \\
\mathcal{F}^{(\bullet)}[1] & \longrightarrow & \mathcal{E}_1^{(\bullet)\leq 0} \oplus \mathcal{E}_2^{(\bullet)\leq 0}[1] & \longrightarrow & \mathcal{E}_{12}^{(\bullet)\leq 0}[1] & \longrightarrow & \mathcal{F}^{(\bullet)}[2]
\end{array}$$

where lines and columns are distinguished triangles. From the third line, we get $\mathcal{G}^{(\bullet)} \in \underrightarrow{LD}^{\geq 1}_{\mathbb{Q},\mathrm{ovcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$. Hence, the distinguished triangle of the left column is the desired one.

3) To check the t-exactness of the assertions c),d),e), we reduce by definition to check it outside Z or $Z \cup Z'$, which is already well known. $\square$

Definition 15.3.8.13. We denote by $\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$ the heart of the canonical t-structure on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. When $X = P$, we can simply write $\underrightarrow{LM}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. We define for any integer n, the n-th cohomology functor $H^n_Z \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$ by putting $H^n_Z(\mathcal{E}^{(\bullet)}) := \tau^{\leq 0}_Z \tau^{\geq 0}_Z(\mathcal{E}^{(\bullet)}[n])$ for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. Beware that when Z is not a divisor of P, we do have to distinguish H^n_Z with the standard cohomological functor H^n of $D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}))$ (see the counter-example 15.3.8.22 or 15.3.8.20.1 for a comparison), Moreover, beware that $\mathcal{H}^{\dagger,n}_Z$ has another meaning (see notation 13.1.5.3).

Example 15.3.8.14. When Z is the support of a divisor of P, the category $\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$ is simply the category of overcoherent $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)$-modules (after any base change) with support in X. In particular, when $X = P$, we retrieve the (usual) heart of the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$.

15.3.8.15. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$, $\widetilde{Z}$ be a closed subscheme of P containing Z, T be a divisor of P. Let $\star$ be either "$\leq n$" or "$\geq n$". Let $f \colon \mathfrak{P}' \to \mathfrak{P}$ be smooth morphism, $Z' := f^{-1}(Z)$ (e.g. f is the open immersion $\mathfrak{U} \subset \mathfrak{P}$ and therefore Z' is empty). It follows from the t-exactness properties of 15.3.8.12.c-e and of [HTT08, 8.1.15] that we have the isomorphisms

$$f^{*(\bullet)} \circ \tau^{\star}_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^{\star}_{Z'} \circ f^{*(\bullet)}(\mathcal{E}^{(\bullet)}), \qquad f^{*(\bullet)} \circ H^n_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} H^n_{Z'} \circ f^{*(\bullet)}(\mathcal{E}^{(\bullet)}), \tag{15.3.8.15.1}$$

$$({}^\dagger \widetilde{Z}) \circ \tau^{\star}_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^{\star}_{\widetilde{Z}} \circ ({}^\dagger \widetilde{Z})(\mathcal{E}^{(\bullet)}), \quad ({}^\dagger \widetilde{Z}) \circ H^n_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} H^n_{\widetilde{Z}} \circ ({}^\dagger \widetilde{Z})(\mathcal{E}^{(\bullet)}), \tag{15.3.8.15.2}$$

$$({}^\dagger T) \circ \tau^{\star}_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^{\star}_Z \circ ({}^\dagger T)(\mathcal{E}^{(\bullet)}), \quad ({}^\dagger T) \circ H^n_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} H^n_Z \circ ({}^\dagger T)(\mathcal{E}^{(\bullet)}). \tag{15.3.8.15.3}$$

15.3.8.16. The following properties are straightforward:

(a) For any closed subscheme Z' of P, the t-exact functor 15.3.8.12.2 induces the exact functor

$$({}^\dagger Z') \colon \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z \cup Z'/\mathfrak{S}). \tag{15.3.8.16.1}$$

(b) For any divisor T of P, the t-exact functor 15.3.8.12.3 induces the exact functor

$$({}^\dagger T) \colon \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}). \tag{15.3.8.16.2}$$

Lemma 15.3.8.17. *Let $\phi \colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a homomorphism of $\underrightarrow{LM}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. The morphism ϕ is an isomorphism (resp. a monomorphism, an epimorphism) if and only if so is $\phi|\mathfrak{U}$.*

Proof. Let $\mathcal{K}^{(\bullet)}$ be the kernel or the cokernel of ϕ. Then following 15.3.7.6 the object $\mathcal{K}^{(\bullet)}$ is null if and only if so is $\mathcal{K}^{(\bullet)}|\mathfrak{U}$. $\square$

Lemma 15.3.8.18. *The restriction functor*

$$|\mathfrak{U}\colon \underrightarrow{LM}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z)) \to \underrightarrow{LM}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{U}/\mathfrak{S}})$$

is faithful.

Proof. Let $\phi\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a homomorphism of $\underrightarrow{LM}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. The property $\phi = 0$ is equivalent to saying that the morphism $\ker\phi \to \mathcal{E}^{(\bullet)}$ is an isomorphism. Hence, the lemma 15.3.8.17 implies that $\phi = 0 \Leftrightarrow \phi|\mathfrak{U} = 0$. □

15.3.8.19. Let $T_1, \dots, T_r$ be divisors of P such that $Z = T_1 \cap \dots \cap T_r$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/K)$. Set $\mathcal{E}^{(\bullet)}_{ij} := ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)})$ and $\mathcal{E}^{(\bullet)}_{i} := ({}^\dagger T_i)\mathcal{E}^{(\bullet)}$ for any $i, j \in \{1, \dots, r\}$. We get the morphism $\theta^1_{ij}\colon \mathcal{E}^{(\bullet)}_i \to ({}^\dagger T_i \cup T_j)\mathcal{E}^{(\bullet)}_i = \mathcal{E}^{(\bullet)}_{ij}$. Following 15.3.8.16.2, this is a map of the abelian category $\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. This yields the maps $\theta^1\colon \oplus_{i=1}^r \mathcal{E}^{(\bullet)}_i \to \oplus_{i,j=1}^r \mathcal{E}^{(\bullet)}_{ij}$. Similarly, for any i, j we have the maps $\theta^2_{ij}\colon \mathcal{E}^{(\bullet)}_j \to ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}_j) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}_{ij}$, which yields the map $\theta^2\colon \oplus_{i=1}^r \mathcal{E}^{(\bullet)}_i \to \oplus_{i,j=1}^r \mathcal{E}^{(\bullet)}_{ij}$ of $\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. We get the exact sequence of $\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$:

$$0 \to \mathcal{E}^{(\bullet)} \to \oplus_{i=1}^r ({}^\dagger T_i)(\mathcal{E}^{(\bullet)}) \xrightarrow{\theta^2 - \theta^1} \oplus_{i,j=1}^r ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}). \tag{15.3.8.19.1}$$

Indeed, to check the exactness, by using 15.3.8.17 we reduce to the case where Z is empty. Set $\mathfrak{U}_i = \mathfrak{P} \backslash T_i$ for any $i = 1, \dots, r$. Let $i \in \{1, \dots, r\}$. We reduce to check that the sequence 15.3.8.19.1 is exact over $\mathfrak{U}_i$, i.e. to the case where the divisor T_i is empty, which is a straighforward computation.

Remark 15.3.8.20. (a) Let $\mathcal{E} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$. Let $\star$ be either "$\leq n$" or "$\geq n$". Then for any divisor T containing Z, we have $({}^\dagger T) \circ \tau^\star(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^\star \circ ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^\star_T \circ ({}^\dagger T)(\mathcal{E}^{(\bullet)})$, where $\tau^\star$ is the usual truncation for coherent complexes (see 13.1.4.18). This yields that $({}^\dagger Z) \circ \tau^\star(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^\star_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. Hence, we get the morphisms

$$({}^\dagger Z) \circ \tau^{\leq n}(\mathcal{E}^{(\bullet)}) \to \tau^{\leq n}_Z(\mathcal{E}^{(\bullet)}), \qquad \tau^{\geq n}_Z(\mathcal{E}^{(\bullet)}) \to ({}^\dagger Z) \circ \tau^{\geq n}(\mathcal{E}^{(\bullet)}) \tag{15.3.8.20.1}$$

which are isomorphisms since this is the case outside Z. This yields the isomorphism

$$({}^\dagger Z) \circ H^n(\mathcal{E}) \xrightarrow{\sim} H^n_Z(\mathcal{E}).$$

(b) For any $n \in \mathbb{Z}$, $\mathcal{E} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$, it follows from 15.3.8.15.1 that we have $H^n_Z(\mathcal{E})|\mathfrak{U} \xrightarrow{\sim} H^n(\mathcal{E}|\mathfrak{U})$. Hence, from Lemma 15.3.7.6, we get $H^n_Z(\mathcal{E}) = 0$ if and only if $H^n(\mathcal{E}|\mathfrak{U}) = 0$.

(c) We have the inclusion $\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) \subset \underrightarrow{LD}^{\geq 0}_{\mathbb{Q}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$ (see 13.1.4.19 for the t-structure on the right side). Indeed, by using Mayer-Vietoris exact triangles and by induction on the minimal number of divisors whose intersection is Z, we reduce to the case where Z is a divisor.

Proposition 15.3.8.21. *Let D be a divisor of P and Let $\widetilde{Z} := Z \cup D$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, \widetilde{Z}/\mathfrak{S}) \cap \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. Then we have the canonical isomorphisms*

$$\tau^\star_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^\star_{\widetilde{Z}}(\mathcal{E}^{(\bullet)}),\ H^n_{\widetilde{Z}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} H^n_Z(\mathcal{E}^{(\bullet)}),$$

where $\star$ means either $\leq n$ or $\geq n$ with $n \in \mathbb{Z}$.

Proof. Since the morphism $\mathcal{E}^{(\bullet)} \to ({}^\dagger Z')(\mathcal{E}^{(\bullet)})$ is an isomorphism for any closed subscheme Z' included in $\widetilde{Z}$, then we get the first and the last isomorphisms:

$$\tau^\star_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^\star_Z \circ ({}^\dagger D)(\mathcal{E}^{(\bullet)}) \underset{15.3.8.15.3}{\xrightarrow{\sim}} ({}^\dagger D) \circ \tau^\star_Z(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger \widetilde{Z}) \circ \tau^\star_Z(\mathcal{E}^{(\bullet)})$$
$$\underset{15.3.8.15.2}{\xrightarrow{\sim}} \tau^\star_{\widetilde{Z}} \circ ({}^\dagger \widetilde{Z})(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \tau^\star_{\widetilde{Z}}(\mathcal{E}^{(\bullet)}).$$

□

Remark 15.3.8.22. Let us give a counter-example of the proposition 15.3.8.21 when D is not a divisor of P and Z is empty. Suppose D is a rational closed point of $\mathbb{P}^2_k$, $\mathfrak{P} = \widehat{\mathbb{P}}^2_{\mathcal{V}}$, $X = \mathbb{P}^2_k$, and $Y := X \setminus D$. Since $({}^\dagger D)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})|\mathfrak{U} \xrightarrow{\sim} \mathcal{O}^{(\bullet)}_{\mathfrak{U}}$, we get $({}^\dagger D)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \in \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, D/\mathfrak{S})$. However, by using the localization triangle with respect to D of $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$, we have $\tau^{\leq 0}(({}^\dagger D)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})) \xrightarrow{\sim} \mathcal{O}^{(\bullet)}_{\mathfrak{P}} \notin \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, D/\mathfrak{S})$, where $\tau^{\leq 0}$ is the standard truncation functor of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$. Now, let $u\colon \mathfrak{D} \to \mathfrak{P}$ be a lifting of the closed immersion $D \hookrightarrow P$. We have $\mathbb{R}\underline{\Gamma}^\dagger_D(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \cong u^{(\bullet)}_+ u^{(\bullet)!}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \cong u^{(\bullet)}_+(\mathcal{O}^{(\bullet)}_{\mathfrak{D}})[-2]$. Using again the localization triangle, we have $\tau_{\geq 1}(({}^\dagger D)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})) \xrightarrow{\sim} u^{(\bullet)}_+(\mathcal{O}^{(\bullet)}_{\mathfrak{D}})[-1]$. Hence, the object $({}^\dagger D)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$ is not an element of $\underrightarrow{LM}_{\mathbb{Q}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$, i.e. a module in the usual sense.

Proposition 15.3.8.23. *Let T be a divisor of P and let $\widetilde{Z} := Z \cup T$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, \widetilde{Z}/\mathfrak{S})$. Then both assertions are equivalent.*

(a) $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$;

(b) for any $n \in \mathbb{Z}$, we have $H^n_{\widetilde{Z}}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$.

Proof. We easily check by devissage the implication $(b) \Rightarrow (a)$. The implication $(a) \Rightarrow (b)$ is a consequence of 15.3.8.21. ☐

Proposition 15.3.8.24. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. We have the equality*

$$\operatorname{Supp} \mathcal{E}^{(\bullet)} = \cup_{j \in \mathbb{Z}} \operatorname{Supp} H^j_Z(\mathcal{E}^{(\bullet)}). \tag{15.3.8.24.1}$$

Proof. For any closed subscheme X' of P, it follows from 13.1.4.20.1 the last equivalence: $\operatorname{Supp} \mathcal{E}^{(\bullet)} \subset X' \Leftrightarrow \operatorname{Supp}(\mathcal{E}^{(\bullet)}|\mathfrak{U}) \subset X' \cap U \Leftrightarrow \forall j \in \mathbb{Z}$, $\operatorname{Supp} H^j(\mathcal{E}^{(\bullet)}|\mathfrak{U}) \subset X' \cap U$. It follows from 15.3.8.15.1 that we have $H^j_Z(\mathcal{E})|\mathfrak{U} \xrightarrow{\sim} H^j(\mathcal{E}|\mathfrak{U})$. Hence, $\forall j \in \mathbb{Z}$, $\operatorname{Supp} H^j(\mathcal{E}^{(\bullet)}|\mathfrak{U}) \subset X' \cap U \Leftrightarrow \forall j \in \mathbb{Z}$, $\operatorname{Supp} H^j_Z(\mathcal{E}^{(\bullet)}) \subset X'$. ☐

Lemma 15.3.8.25. *Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, X, X' be two smooth closed subschemes of P, Z, Z' be two closed subschemes of P such that $Z \cap X = Z' \cap X'$. We get then the equality: $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X', \mathfrak{P}, Z'/\mathfrak{S})$.*

Proof. 1) Suppose $Z = Z'$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/\mathfrak{S})$. We have to prove that $\mathcal{E}^{(\bullet)}$ has support in X'. We have to check that the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is an isomorphism (see 15.3.8.2. Since $\mathcal{E}^{(\bullet)} \to ({}^\dagger Z)\mathcal{E}^{(\bullet)}$ is an isomorphism, then we get the isomorphisms

$$\mathbb{R}\underline{\Gamma}^\dagger_X(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_X({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \underset{13.1.5.1.1}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^\dagger_{X'}({}^\dagger Z)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{X'}(\mathcal{E}^{(\bullet)}). \tag{15.3.8.25.1}$$

2) Suppose $X = X'$ and $Z \subset Z'$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/K)$. Since $X \setminus Z = X \setminus Z'$, then the canonical morphism $\mathcal{E}^{(\bullet)} \to ({}^\dagger Z')(\mathcal{E}^{(\bullet)})$ is an isomorphism. Since $({}^\dagger Z')(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/K)$, this yields the inclusion $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z/K) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z'/K)$. Conversely, let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, Z'/K)$. We have to check that $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$.

a) Suppose Z and Z' are the support of some divisors T and T'. Since this is of local nature in $\mathfrak{P}$, we can suppose $\mathfrak{P}$ affine and X is integral. Let us choose then $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ is a closed immersion of smooth formal $\mathfrak{S}$-schemes which is a lifting of $X \hookrightarrow P$. Either $T \cap X$ is a divisor of X or T contains X. When T contains X, the considered categories are reduced to the zero object. Suppose now that $Z := T \cap X$ is a divisor of X. Following the Berthelot-Kashiwara theorem (see 9.3.5.13) and from the stability of the overcoherence in the divisor case (see 15.3.6.11 and 15.3.6.14) that the functor $u^{(\bullet)}_{T+}$ (resp. $u^{(\bullet)}_{T'+}$) induces an equivalence of categories between $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{X}, T \cap X)$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T)$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T')$). Following 9.2.4.19, the functors $u^{(\bullet)}_{T+}$ and $\mathrm{forg}_{T,T'} \circ u^{(\bullet)}_{T'+}$ are isomorphic. Hence, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T')$.

b) Suppose now Z' (and not Z) is the support of some divisors T'. Let T divisor of P containing Z. It remains to prove $({}^\dagger T)\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. We have $({}^\dagger T)(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger T \cup T')(\mathcal{E}^{(\bullet)}) \in$

$\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},T\cup T'/K)$. Since $X\setminus T=X\setminus(T\cup T')$, then it follows from the case a) that $({}^\dagger T)(\mathcal{E}'^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},T/K)$ and we are done.

c) We have to check that for any divisor T of P containing Z we have $({}^\dagger T)\mathcal{E}'^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. Since $({}^\dagger T)(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}({}^\dagger T\cup Z')(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},T\cup Z'/K)$, then it follows from the case b) that $({}^\dagger T)(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}({}^\dagger T\cup Z')(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(X,\mathfrak{P},T/K)$, and we are done.

3) Since $X\setminus Z=X\setminus(Z\cup Z')=X'\setminus(Z\cup Z')=X'\setminus Z'$, then we get the general case from the two preceding cases. □

Theorem 15.3.8.26 (Overcoherent version of Berthelot-Kashiwara Theorem)**.** *Let $\natural\in\{\mathrm{ovcoh},\mathrm{oc}\}$. Let $u\colon\mathfrak{P}'\hookrightarrow\mathfrak{P}$ be a closed immersion of log smooth formal log $\mathcal{V}$-schemes. Let $\mathfrak{U}$ be the open set of $\mathfrak{P}$ complementary to $u(P')$, Z be a closed subscheme of P and $Z':=Z\cap P'$. Let $\mathcal{F}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$, $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$ with support in Z'.*

(a) We have the canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$:

$$\mathcal{F}^{(\bullet)}\xrightarrow{\sim}u^{(\bullet)!}\circ u^{(\bullet)}_{+}(\mathcal{F}^{(\bullet)}).\qquad(15.3.8.26.1)$$

(b) We have $u^{(\bullet)!}(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$ and we benefit from the canonical isomorphism

$$u^{(\bullet)}_{+}\circ u^{(\bullet)!}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim}\mathcal{E}^{(\bullet)}.\qquad(15.3.8.26.2)$$

(c) The functors $u^{(\bullet)}_{+}$ and $u^{(\bullet)!}$ induce t-exact quasi-inverse equivalences between the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$ (resp. $\underrightarrow{LD}^{0}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$) and the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$ (resp. $\underrightarrow{LD}^{0}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$) consisting of complexes $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}|\mathfrak{U}\xrightarrow{\sim}0$.

Proof. 1) Let us prove $u^{(\bullet)!}(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$. Following 15.3.8.25, we have $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z'))$. Let T' be a divisor of P' containing P. Let us check $({}^\dagger T')u^{(\bullet)!}(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$. Since this is of local nature in $\mathfrak{P}$, we can suppose $\mathfrak{P}$ affine and there exists a global section $f\in\Gamma(\mathfrak{P},\mathcal{O}_{\mathfrak{P}})$ whose image $\overline{f}'$ via $\Gamma(\mathfrak{P},\mathcal{O}_{\mathfrak{P}})\twoheadrightarrow\Gamma(P',\mathcal{O}_{P'})$ gives a equation of the divisor T'. Let T be the divisor of P given by $\overline{f}$, the image of f via $\Gamma(\mathfrak{P},\mathcal{O}_{\mathfrak{P}})\to\Gamma(P,\mathcal{O}_{P})$. Since T contains Z' then $({}^\dagger T)(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. since $T'=T\cap X$, then it follows from 15.3.6.12 that $u^{(\bullet)!}(({}^\dagger T)(\mathcal{E}^{(\bullet)}))\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'))$. Since $u^{(\bullet)!}(({}^\dagger T)(\mathcal{E}^{(\bullet)}))\xrightarrow{\sim}({}^\dagger T')u^{(\bullet)!}(\mathcal{E}^{(\bullet)})$, then we are done.

2) The stability by $u^{(\bullet)}_{+}$ comes from the stability of overcoherence by a proper morphism (see 15.3.7.8

3) It follows from 5.2.6.3 that we have the canonical adjunction morphism $\mathcal{F}^{(\bullet)}\to u^{(\bullet)!}\circ u^{(\bullet)}_{+}(\mathcal{F}^{(\bullet)})$ of $D({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'\sharp})$. Using the part 1) and 2) of the proof, this induces a morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}({}^{1}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$. To check that this morphism is an isomorphism, using 15.3.8.17, we reduce to the case where Z is empty, which is already known (see 9.3.5.13).

4) We get the isomorphism 15.3.8.26.2 from 15.3.8.2 and 13.2.1.5.1

5) To check the t-exactness of the functors induced by $u^{(\bullet)}_{+}$ and $u^{(\bullet)!}$, by definition (see 15.3.8.12) we reduce to the case where Z is empty, which is already known (see 9.3.5.13). □

Corollary 15.3.8.27. *Let $\natural\in\{\mathrm{ovcoh},\mathrm{oc}\}$. Let $f\colon\mathfrak{P}'\to\mathfrak{P}$ be a morphism of smooth $\mathcal{V}$-formal schemes, Z a closed subscheme of P, and $Z':=f^{-1}(Z)$. For any $\mathcal{E}^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$, we have $f^{(\bullet)!}(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(Z'))$.*

Proof. Since base changes commute to localisation functors (see 13.1.5.7) and pullbacks 9.2.6.6, then we reduce to the case where $\natural=\mathrm{ovcoh}$. Since f decomposes into a closed immersion followed by a smooth morphism and since the case where f is a smooth morphism is immediate, we reduce to the case where f is a closed immersion. Then this follows from $\mathbb{R}\underline{\Gamma}^{\dagger}_{P'}(\mathcal{E}^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$ (see 15.3.7.5.1), from 15.3.8.26 and the isomorphism

$$f^{(\bullet)!}(\mathcal{E}^{(\bullet)})\xrightarrow{\sim}f^{(\bullet)!}\circ\mathbb{R}\underline{\Gamma}^{\dagger}_{P'}(\mathcal{E}^{(\bullet)}).$$

□

Chapter 16

Arithmetic $\mathcal{D}$-modules associated with overconvergent isocrystals

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. When we work with F-complex, we suppose there exists an automorphism $\sigma \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining.

16.1 Partially overcoherent isocrystals: divisorial case

16.1.1 1-holonomicity of overconvergent isocrystals on completely smooth d-frames

Let $\mathfrak{P}$ be a smooth $\mathfrak{S}$-formal scheme, T a divisor of P.

16.1.1.1 (1-overholonomicity)**.** Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. We say that $\mathcal{E}$ is 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomic if $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ (see notation 15.3.6.2) and if for all smooth morphism $\alpha \colon \mathfrak{Q} \to \mathfrak{P}$, for all divisor D of Q, putting $U := \alpha^{-1}(T)$, we get $\mathbb{D}_{\mathfrak{Q},U} \circ ({}^{\dagger}D) \circ \alpha^{!}_{T}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}}({}^{\dagger}U)_{\mathbb{Q}})$.

We denote by $D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ the strictly full subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ consisting of 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomic complexes.

Remark 16.1.1.2. It is unclear whether the 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomicity is preserved by dual functor. We will define later the notion of n-$\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-overholonomicity for any integer n (see 18.1.2.1), this might be extend to the notion of n-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomicity but we will not need it.

Lemma 16.1.1.3. *As the fact that an object of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ belongs to $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ is local on $\mathfrak{P}$, then the 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-holonomicity is local on $\mathfrak{P}$, i.e., if $(\mathfrak{P}_{\alpha})_{\alpha\in\Lambda}$ is an open covering of $\mathfrak{P}$, then $\mathcal{E}$ is 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-holonomic if and only if $\mathcal{E}|_{\mathfrak{P}_{\alpha}}$ is 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}_{\alpha}}({}^{\dagger}T \cap P_{\alpha})_{\mathbb{Q}}$-holonomic for all $\alpha \in \Lambda$.*

Lemma 16.1.1.4. *Let $\mathcal{E} \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Then for all morphism $\alpha \colon \mathfrak{Q} \to \mathfrak{P}$ such that $U := \alpha^{-1}(T)$ is a divisor of Q, for any subscheme $Y \subset Q$, we have $\mathbb{D}_{\mathfrak{Q},U}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\alpha^{!}_{T}(\mathcal{E})) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}}({}^{\dagger}U)_{\mathbb{Q}})$*

Proof. We have a splitting of the form $\alpha = p\circ\gamma$, where $p \colon \mathfrak{P}' \to \mathfrak{P}$ is a smooth morphism and $\gamma \colon \mathfrak{Q} \hookrightarrow \mathfrak{P}'$ is a closed immersion ($\mathfrak{P}'$ is an open of $\mathfrak{Q} \times \mathfrak{P}$ and γ is induced by the graph of α). Since $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$, then $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\alpha^{!}_{T}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}}({}^{\dagger}U)_{\mathbb{Q}})$. Hence, $\mathbb{D}_{\mathfrak{Q},U}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\alpha^{!}_{T}(\mathcal{E})) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}}({}^{\dagger}U)_{\mathbb{Q}})$. By Kashiwara's theorem, we get therefore the first isomorphism: theorem of relative duality and the commutation of local cohomology with direct image, putting $T' = p^{-1}(T)$, we have

$$\begin{aligned}
\mathbb{D}_{\mathfrak{Q},U}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\alpha^{!}_{T}(\mathcal{E})) &\underset{9.3.5.9}{\xrightarrow{\sim}} \gamma^{!}_{T'}\gamma_{T'+}\mathbb{D}_{\mathfrak{Q},U}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\alpha^{!}_{T}(\mathcal{E})) \underset{9.4.5.2.1}{\xrightarrow{\sim}} \gamma^{!}_{T'}\mathbb{D}_{\mathfrak{P}',T'}\gamma_{T'+}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\alpha^{!}_{T}(\mathcal{E})) \\
&\underset{13.2.1.4.2}{\xrightarrow{\sim}} \gamma^{!}_{T'}\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\gamma_{T'+}(\alpha^{!}_{T}(\mathcal{E})) \xrightarrow{\sim} \gamma^{!}_{T'}\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\gamma_{T'+}\gamma^{!}_{T'}p^{!}_{T}(\mathcal{E}) \\
&\xrightarrow{\sim} \gamma^{!}_{T'}\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\mathbb{R}\underline{\Gamma}^{\dagger}_{Q}p^{!}_{T}(\mathcal{E}) \xrightarrow{\sim} \gamma^{!}_{T'}\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}p^{!}_{T}(\mathcal{E}).
\end{aligned}$$

Using localization exact triangles, since p is smooth then we get by devissage that $\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}p^{!}_{T}(\mathcal{E})$ is overcoherent. We conclude by stability of the overcoherence by extraordinary inverse image of Proposition 15.3.6.12. □

Proposition 16.1.1.5. Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of smooth $\mathfrak{S}$-formal schemes such that $T' := f^{-1}(T)$ is a divisor of Q.

(a) For any subscheme Y' of P', for any $\mathcal{E} \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$, we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'}f^{!}_{T}(\mathcal{E}) \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$.

(b) If f is proper then, for any $\mathcal{E}' \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$, $f_{T,+}(\mathcal{E}') \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$.

Proof. The first assertion results from 16.1.1.2.(iii) and from the commutation of extraordinary inverse image with local cohomology (see 13.2.1.4.1) and with composition. Next we prove the second assertion. Let $\alpha\colon \mathfrak{Q} \to \mathfrak{P}$ be a smooth morphism and Z a closed subscheme of Q. Write $\mathfrak{Q}' := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{Q}$, $\alpha'\colon \mathfrak{Q}' \to \mathfrak{P}'$ and $f'\colon \mathfrak{Q}' \to \mathfrak{Q}$ the projections, $U := \alpha^{-1}(T)$, $U' := \alpha'^{-1}(T')$ and $Z' := f'^{-1}(Z)$. We get: $\mathbb{D}_{\mathfrak{Q},U}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}\alpha^{!}_{T}(f_{T,+}(\mathcal{E}')) \xrightarrow{\sim} f'_{T,+}\mathbb{D}_{\mathfrak{Q}',U'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}\alpha'^{!}_{T}(\mathcal{E}')$ because of 9.4.5.1, 13.2.1.4 and 13.2.3.7.

Since $\mathbb{D}_{\mathfrak{Q}',U'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}\alpha'^{!}_{T}(\mathcal{E}')$ is by definition $\mathcal{D}^{\dagger}_{\mathfrak{Q}'}({}^{\dagger}U')_{\mathbb{Q}}$-overcoherent, then it follows from 15.3.6.14 From this we derive the $\mathcal{D}^{\dagger}_{\mathfrak{Q}}({}^{\dagger}U)_{\mathbb{Q}}$-overcoherence of $f'_{T,+}\mathbb{D}_{\mathfrak{Q}',U'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}\alpha'^{!}_{T}(\mathcal{E}')$. □

16.1.1.6. Let $\theta = (b, a, f)\colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ be a morphism of smooth d-frames.

(a) We have therefore the factorisations

$$\theta^{!} := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{!}_{T',T}\colon D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}}) \tag{16.1.1.6.1}$$

$$\theta^{+} := \mathbb{D}_{T'} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{!}_{T',T} \circ \mathbb{D}_{T}\colon D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}) \to D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}}). \tag{16.1.1.6.2}$$

(b) Suppose f is proper and $T' = f^{-1}(T)$ and set $\theta_{+} := f_{T+}$. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ and $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$. Since f is proper, then we get by adjunction (see 9.4.5.5) the morphisms: $f_{T+} \circ f^{!}_{T}(\mathcal{E}) \to \mathcal{E}$ and $\mathcal{E}' \to f^{!}_{T} \circ f_{T+}(\mathcal{E}')$. This yields $(\theta_{+}, \theta^{!})$ is an adjoint paire (on overcoherent categories) and we get the morphisms

$$\theta_{+} \circ \theta^{!}(\mathcal{E}) \to \mathcal{E} \text{ and } \mathcal{E}' \to \theta^{!} \circ \theta_{+}(\mathcal{E}'). \tag{16.1.1.6.3}$$

Let $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ and $\mathcal{F}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$. Suppose $\mathbb{D}_{T}(\mathcal{F}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ and $\mathbb{D}_{T'}(\mathcal{F}') \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$, e.g. $\mathcal{F} \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$ and $\mathcal{F}' \in D^{\mathrm{b}}_{1\text{-ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$. Since f is proper, using the relative duality theorem (see 9.4.5.2), we get by duality from 16.1.1.6.3 the morphisms:

$$\mathcal{F} \to \theta_{+} \circ \theta^{+}(\mathcal{F}) \text{ and } \theta^{+} \circ \theta_{+}(\mathcal{F}') \to \mathcal{F}'. \tag{16.1.1.6.4}$$

Theorem 16.1.1.7. *Let X be a smooth closed subscheme of P such that $T_X := T \cap X$ is a divisor of X. Let $\mathfrak{U}$ (resp. Y) denote the open complement of T (resp. T_X) in $\mathfrak{P}$ (resp. X). Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ (see notation 12.2.1.4). Then $\mathcal{E}$ is 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomic (see definition 16.1.1.1).*

Proof. 0) As the theorem is local on $\mathfrak{P}$, we can suppose that P and X are affine and irreducible. The closed immersion $X \hookrightarrow P$ lifts to a closed immersion of smooth $\mathfrak{S}$-formal schemes. By 16.1.1.5, we are therefore reduced to treat the case where $X = P$. We write T for T_X and $\mathfrak{X}$ for $\mathfrak{P}$.

Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{X}, T/\mathcal{V})$ (see 12.2.1.6) be an object $\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}$. Let Z be any closed subscheme of X. Firstly, since the categories of the form $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ are preserved by pullbacks (more precisely, see 12.2.1.14.1), then it suffices to prove that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(T))$ and next that $\mathbb{D}_{\mathfrak{X},T}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$. Remark that the property a) is equivalent to saying $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent but according to the remark 15.3.4.11, then this is wiser to work with $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}(T))$ before knowing the overcoherence.

According to the desingularisation theorem of de Jong ([dJ96]), there exists a projective, surjective, generically finite and étale morphism $a\colon X' \to X$ such that X' is irreducible and k-smooth, and such that $a^{-1}(Z)$ is a strict normal crossing divisor of X'. As a is projective, there exists thus a smooth $\mathfrak{S}$-formal scheme $\mathfrak{P}'$, a closed immersion $u'\colon X' \hookrightarrow \mathfrak{P}'$, a smooth proper morphism $f\colon \mathfrak{P}' \to \mathfrak{X}$ such that $f \circ u' = u \circ a$. Write $T' = f^{-1}(T)$ and $Y' := a^{-1}(Y)$.

1) i) Since f is proper then we get by adjunction (see 9.4.5.5) the last morphism:

$$f_{T+}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}(\mathcal{E}) \to f_{T+}f^{!}_{T}(\mathcal{E}) \to \mathcal{E}.$$

ii) By applying the functor $\mathbb{D}_{\mathfrak{X},T}$ to the similar morphism $f_{T+}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}(\mathbb{D}_{\mathfrak{X},T}(\mathcal{E})) \to \mathbb{D}_{\mathfrak{X},T}(\mathcal{E})$, via the theorem of relative duality and of biduality (see 9.4.5.2 and 8.7.7.3) we obtain:

$$\mathcal{E} \to f_{T+}\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}\mathbb{D}_{\mathfrak{X},T}(\mathcal{E}).$$

iii) Now, as X' is irreducible and T' is a divisor of P', $X' \cap T'$ is then either a divisor of X' or is equal to X'. Since a is generically finite and étale, we cannot have $X' \cap T' = X'$. From this we conclude that $X' \cap T'$ is a divisor of X'. Thus it follows from 12.2.4.1.2 that we have the natural isomorphisms of $\mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/\mathcal{V})$ of the form: $\mathcal{E}' := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}(\mathcal{E}) \xrightarrow{\sim} \mathrm{sp}_{X'\hookrightarrow\mathfrak{P}',T',+}(a^{*}(E))$, $\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}\mathbb{D}_{\mathfrak{X},T}(\mathcal{E}) \xrightarrow{\sim} \mathrm{sp}_{X'\hookrightarrow\mathfrak{P}',T'+}(a^{*}(E^{\vee})^{\vee}) \xrightarrow{\sim} \mathrm{sp}_{X'\hookrightarrow\mathfrak{P}',T'+}(a^{*}(E))$ (12.2.5.6).

iv) From i) (resp. ii), resp. iii)), we get the left (resp. right, resp. middle isomorphism):

$$\mathcal{E} \to f_{T+}\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}\mathbb{D}_{\mathfrak{X},T}(\mathcal{E}) \xrightarrow{\sim} f_{T+}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}(\mathcal{E}) \to \mathcal{E}$$

is an isomorphism. Indeed, using the third part of Proposition 11.2.1.14.c, since this composition is a morphism of the abelian category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathcal{V})$, we reduce to check that its restriction to a dense open subset is an isomorphism. Hence, we can suppose that T is empty and $a\colon X' \to X$ is finite and étale, which is easy.

v) Hence, $\mathcal{E}$ is a direct factor of $f_{T+}(\mathcal{E}')$. Hence, $\mathcal{E}$ is a direct factor of $f_{T+}\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}(\mathcal{E})$. Since $\mathcal{E}'^{(\bullet)} := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}f^{!}_{T}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V})$ (see 12.2.1.11.2), then it is in particular coherent and since f is proper we get therefore $f^{(\bullet)}_{T,+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T))$ together with the isomorphism $\underrightarrow{l}^{*}_{\mathbb{Q}}(f^{(\bullet)}_{T,+}(\mathcal{E}'^{(\bullet)})) \xrightarrow{\sim} (\mathcal{E}')$. Since $\underrightarrow{l}^{*}_{\mathbb{Q}}$ is fully faithful on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T))$, then $\mathcal{E}^{(\bullet)}$ is a direct factor of $f^{(\bullet)}_{T,+}(\mathcal{E}'^{(\bullet)})$. By 16.1.1.5 and by 13.2.1.4, it is therefore sufficient to prove $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$ and next that $\mathbb{D}_{\mathfrak{P}',T'}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T')_{\mathbb{Q}})$. In other word, we reduce to the case where Z is strict normal crossing divisor of X.

2) By means of Mayer-Vietoris exact triangles (see 13.1.4.15.2), proceeding by induction on the number of irreducible components of Z, it remains to treat the case where Z is a smooth integral closed subscheme of X. Thus we have two cases: either $Z \subset T$, or $Z \cap T$ is a divisor of Z. The first case gives the equalities $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) = 0$ and $\mathbb{D}_{\mathfrak{X},T}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) = 0$. Now consider the second case. The two properties being local on $\mathfrak{X}$, we can suppose that the closed immersion $Z \hookrightarrow X$ lifts to a closed immersion $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ of smooth $\mathfrak{S}$-formal schemes.

i) Since we have the isomorphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u^{(\bullet)}_{+}u^{(\bullet)!}(\mathcal{E}^{(\bullet)})$ and $u^{(\bullet)!}(\mathcal{E})[-d_{Z/X}] \in \mathrm{MIC}^{(\bullet)}(Z, \mathfrak{Z}, T\cap Z/\mathcal{V})$ (see 12.2.1.9), then $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(T))$. Hence, we have proved that $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$.

ii) Since $\mathbb{D}_{\mathfrak{Z},T\cap Z}u^{!}(\mathcal{E})[-d_{Z/X}] \in \mathrm{MIC}^{\dagger\dagger}(Z, \mathfrak{Z}, T\cap Z/\mathcal{V})$, then from the part 2.i), we get $\mathbb{D}_{\mathfrak{Z},T\cap Z}u^{!}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{Z}}({}^{\dagger}T\cap Z)_{\mathbb{Q}})$. Hence, following 15.3.6.14, we obtain $\mathbb{D}_{\mathfrak{X},T}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) \xrightarrow{\sim} u_{+}\mathbb{D}_{\mathfrak{Z},T\cap Z}u^{!}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$. □

16.1.2 Overcoherent isocrystals on completely smooth d-frames, (extraordinary) pullbacks

Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, T be a divisor of P, X be a closed subscheme of P, $\mathfrak{U}$ the open set of $\mathfrak{P}$ complementary to T. We suppose that $Y := X \setminus T$ is a smooth k-scheme.

Definition 16.1.2.1. We define the category $(F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ to be the full subcategory of that of coherent $(F\text{-})\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules with support in X consisting of objects $\mathcal{E}$ such that

(i) $\mathcal{E}|\mathfrak{U}$ is in the essential image of the functor $\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}$.

(ii) $\mathcal{E}$ and $\mathbb{D}_{\mathfrak{P},T}(\mathcal{E})$ are $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-surcoherent.

When the divisor T is empty, we omit T from the notation. The objects of $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ are "the partially overcoherent isocrystals over $(X, \mathfrak{P}, T/\mathcal{V})$".

Example 16.1.2.2. It follows from Theorem 16.1.1.7 that when X/S is smooth this notation coincides with that of 12.2.1.4. Remark that the condition (i) of 16.1.2.1 is equivalent to the property $\mathcal{E}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$.

Notation 16.1.2.3. We will denote by $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$ the strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$ such that the equivalence of categories of 8.4.5.6 induces the equivalence of categories

$$\underrightarrow{l}^{*}_{\mathbb{Q}} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \cong \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}). \qquad (16.1.2.3.1)$$

16.1.2.4. We keep notation 16.1.2.3.

(a) Following 11.2.2.5.2 we have the equality

$$\mathrm{MIC}^{(\bullet)}(\mathcal{P}, T, P/\mathcal{V}) = \mathrm{MIC}^{(\bullet)}(\mathcal{P}, T/\mathcal{V}) := \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathcal{P}}(T)) \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\mathcal{B}^{(\bullet)}_{\mathcal{P}}(T)). \qquad (16.1.2.4.1)$$

(b) It follows from 15.3.6.6 that we have $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

The stability by inverse image of the categories of the form $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ is not straightforward. To go around this technical difficulty, let us introduce the temporary notation of the categories of 16.1.2.5 (this is temporary because they will turn out to be equal to $\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ following 16.1.6.11). Their stability by (extraordinary) inverse images is almost tautological (see the proposition 16.1.2.8), which will allows us by descent to the completely smooth case. But, on the other hand until the end of this section (more precisely until 16.1.6.11) beware that we will have to do without the stability by dual functor when we work with the categories of 16.1.2.5.

Notation 16.1.2.5. (a) We denote by $\mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V})$ the strictly full subcategory of $\mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V})$ (see notation 9.3.7.4) consisting of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules $\mathcal{E}$ such that $\mathbb{D}_T(\mathcal{E})$ is $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent and $\mathcal{E}|\mathfrak{U}$ is in the essential image of the functor $\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$.

(b) We denote by $\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$ the strictly full subcategory of $\mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V})$ of overcoherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules $\mathcal{E}$ such that $\mathcal{E}|\mathfrak{U}$ is in the essential image of the functor $\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$.

When the divisor T is empty, we do not indicate it in the notations 16.1.2.5 above.

16.1.2.6 (Completely smooth case)**.** When X is smooth, it follows from 16.1.1.7 that we have the equalities $\mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$.

Remark 16.1.2.7. With the notations 16.1.2.5, we have the two properties straightforward below.

(a) Let $\mathcal{E} \in \mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V})$. Since the functor $\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$ commutes with duality (see 11.2.7.5.1), then using 12.2.5.6 for any integer $j \neq 0$ we have the vanishing $H^j(\mathbb{D}(\mathcal{E}|\mathfrak{U})) = 0$. By 8.7.6.11, this yields, for any integer $j \neq 0$, $H^j(\mathbb{D}_T(\mathcal{E})) = 0$. Finally, following 16.1.2.6, we have the equalities $\mathrm{MIC}^{*}(Y, \mathfrak{U}/\mathcal{V}) = \mathrm{MIC}^{**}(Y, \mathfrak{U}/\mathcal{V}) = \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$.

(b) Since the functor $\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$ commutes with duality (see 11.2.7.5.1) and via the biduality theorem (see 8.7.7.3), then we check the functor $\mathbb{D}_T$ induced an equivalence between the categories $\mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V})$ and $\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$. In fact, we will establish that these two categories are equal (see 16.1.6.11) but before we have to distinguish them.

Proposition 16.1.2.8. *Let $\theta = (b, a, f) \colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ be a morphism of smooth d-frames. Let $\mathcal{E}$ be an object of $(F\text{-})\mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V})$ and $\mathcal{F}$ an object of $(F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$.*

(a) For any integer $j \in \mathbb{Z} \setminus \{0\}$, the following equalities are satisfied: $H^j(({}^{\dagger}T')\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{!}(\mathcal{F})[-d_{X'/X}]) = 0$, $H^j(\mathbb{D}_{T'}({}^{\dagger}T')\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{!}\mathbb{D}_T(\mathcal{E})[-d_{X'/X}]) = 0$.

(b) We have therefore the factorisations [1]

$$\theta^{!} := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{!}_{T',T}[-d_{X'/X}] \colon (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{**}(X', \mathfrak{P}', T'/\mathcal{V}) \quad (16.1.2.8.1)$$

$$\theta^{+} := \mathbb{D}_{T'} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{!}_{T',T}[-d_{X'/X}] \circ \mathbb{D}_T \colon (F\text{-})\mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{*}(X', \mathfrak{P}', T'/\mathcal{V})). \quad (16.1.2.8.2)$$

[1] To stay in the abelian categories of the form MIC and to simplify notation we added some shift $[-d_{X'/X}]$ which should not appear for complexes

These functors are transitive with respect to the composition: if $\theta' = (b', a', f') \colon (Y'', X'', \mathfrak{P}'', T'') \to (Y', X', \mathfrak{P}', T')$ is a morphism of smooth d-frames, we have the canonical isomorphisms $\theta'^! \circ \theta^! \xrightarrow{\sim} (\theta \circ \theta')^!$ and $\theta'^+ \circ \theta^+ \xrightarrow{\sim} (\theta \circ \theta')^+$.

Proof. To establish the two vanishing formulas, by using 8.7.6.11, we can reduce to the case where the divisor T is empty, which is already checked (see 12.2.1.14). By stability of the overcoherence by extraordinary inverse images and local cohomological functor with proper support (see 15.3.6.12 and 15.3.6.9), to check the factorisations 16.1.2.8.1 and 16.1.2.8.2 we reduce therefore to check $\theta^!(\mathcal{F})|\mathfrak{U}' \in \mathrm{MIC}^{\dagger\dagger}(Y', \mathfrak{U}'/\mathcal{V}))$ and $\theta^+(\mathcal{F})|\mathfrak{U}' \in \mathrm{MIC}^{\dagger\dagger}(Y', \mathfrak{U}'/\mathcal{V}))$, where $\mathfrak{U}' := \mathfrak{P}' \setminus T'$. Hence, we come back to the completely smooth case, which was already proved (see 12.2.1.14). The transitivity with respect to the composition of morphisms is a consequence of that of the extraordinary inverse image functors and of the commutation isomorphisms of the extraordinary inverse images to local cohomological functors. □

Lemma 16.1.2.9. *With the notations of 16.1.2.5 denote by "?" one of the symbol "$*$", "$**$" or "$\dagger\dagger$". Let $\overline{Y}$ the closure of Y in X. Let $Y_1, \ldots, Y_N$ the irreducible components of Y and $\overline{Y}_1, \ldots, \overline{Y}_N$ their closure in X.*

(a) We have the equality $(F\text{-})\mathrm{MIC}^?(\overline{Y}, \mathfrak{P}, T/\mathcal{V}) = (F\text{-})\mathrm{MIC}^?(X, \mathfrak{P}, T/\mathcal{V})$.

(b) We have a canonical equivalence of categories:
$(F\text{-})\mathrm{MIC}^?(X, \mathfrak{P}, T/\mathcal{V}) \cong \prod_{r=1}^N (F\text{-})\mathrm{MIC}^?(\overline{Y}_r, \mathfrak{P}, T/\mathcal{V})$.

Proof. a) Let us start with the case $? = **$. The inclusion of the first equality is obvious. Conversely, is $\mathcal{E} \in (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$. By stability of the overcoherence by local cohomological functor (see 15.3.6.12), then we have therefore the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}}(\mathcal{E}) \to \mathcal{E}$ of $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. Since this is the case outside T (because $\overline{Y} \setminus T = Y$), with 8.7.6.11, this morphism is then an isomorphism. The sheaf $\mathcal{E}$ is then with support in $\overline{Y}$. Hence the converse inclusion. With the second remark of 16.1.2.7, this yields by duality the first equality when $? = *$. By using the first two first cases, this yields the case where $? = \dagger\dagger$.

b) Let $\mathcal{E} \in (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$. The canonical morphism $\oplus_{r=1}^N \mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r} \mathcal{E} \to \mathcal{E}$ of $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ is an isomorphism (because so is outside T and use theorem 8.7.6.11). We know $\mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r} \mathcal{E} \in (F\text{-})\mathrm{MIC}^{**}(\overline{Y}_r, \mathfrak{P}, T/\mathcal{V})$ (see 16.1.2.8.1). If moreover $\mathcal{E} \in (F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$, this yields $\mathbb{D}_T(\mathcal{E}) \xrightarrow{\sim} \oplus_{r=1}^N \mathbb{D}_T(\mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r} \mathcal{E})$. Hence, $\mathbb{D}_T(\mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r} \mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. This yields $\mathbb{D}_T(\mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r} \mathcal{E}) \in (F\text{-})\mathrm{MIC}^{\dagger\dagger}(\overline{Y}_r, \mathfrak{P}, T/\mathcal{V})$. Hence the functor $\oplus_{r=1}^N \mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r}$ induces the equivalence of categories: $(F\text{-})\mathrm{MIC}^?(X, \mathfrak{P}, T/\mathcal{V}) \cong \prod_{r=1}^N (F\text{-})\mathrm{MIC}^?(\overline{Y}_r, \mathfrak{P}, T/\mathcal{V})$ for $? = **$ or for $? = \dagger\dagger$. From the case $? = **$, we get the case $? = *$ by duality (more precisely, the equivalence of categories is given by $\mathbb{D}_T \circ (\oplus_{r=1}^N \mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}_r}) \circ \mathbb{D}_T$). □

Remark 16.1.2.10. The rigid version of the lemma 16.1.2.9 is well known (see [LS07] or [Ber96b]). More precisely, with the notations of 16.1.2.9, we have the equality $(F\text{-})\mathrm{MIC}^\dagger(Y, \overline{Y}/\mathcal{V}) = (F\text{-})\mathrm{MIC}^\dagger(Y, X/\mathcal{V})$ and the canonical equivalence of categories: $(F\text{-})\mathrm{MIC}^\dagger(Y, X/\mathcal{V}) \cong \prod_{r=1}^N (F\text{-})\mathrm{MIC}^\dagger(Y_r, \overline{Y}_r/\mathcal{V})$

Lemma 16.1.2.11. *Let $f \colon \mathfrak{P}' \to \mathfrak{P}$ be a realizable with respect to T (in the sense of 13.2.3.1) morphism of separated and smooth $\mathfrak{S}$-formal schemes, X be a closed subscheme of P' such that the induced morphism $X \to P$ is a closed immersion, Y an open set of X, T be a divisor of P such that $T' := f^{-1}(T)$ is a divisor of P' and such that $Y = X \setminus T$ (and then $Y = X \setminus T'$). We suppose Y smooth.*

(a) Let $\mathcal{E} \in (F\text{-})\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$, $\mathcal{F} \in (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$. For $? = *$ or $? = **$, let $\mathcal{E}' \in (F\text{-})\mathrm{MIC}^?(X, \mathfrak{P}', T'/\mathcal{V})$). For any integer $j \in \mathbb{Z} \setminus \{0\}$, the following equalities are then satisfied:

$$H^j(\mathbb{D}_{T'} \mathbb{R}\underline{\Gamma}^\dagger_X f^!_T \mathbb{D}_T(\mathcal{E})) = 0, \qquad H^j(\mathbb{R}\underline{\Gamma}^\dagger_X f^!_T(\mathcal{F})) = 0, \qquad H^j(f_{T+}(\mathcal{E}')) = 0.$$

(b) The functors $\mathbb{R}\underline{\Gamma}^\dagger_X f^!_T$ (resp. $\mathbb{D}_{T'} \mathbb{R}\underline{\Gamma}^\dagger_X f^!_T \mathbb{D}_T$) and f_{T+} induce quasi-inverse equivalences between the categories $(F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$ and $(F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}', T'/\mathcal{V})$ (resp. between $(F\text{-})\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$ and $(F\text{-})\mathrm{MIC}^*(X, \mathfrak{P}', T'/\mathcal{V})$).

Proof. The other vanishing formulas of (a) being already well known (see 16.1.2.8), then let us treat the last. It follows from the stability of the overcoherence of 15.3.6.14), since $\mathcal{E}'$ has proper support over P, then $f_{T+}(\mathcal{E}')$ is an overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T')_{\mathbb{Q}}$-module. By 8.7.6.11, it is therefore sufficient to check it outside T, which reduce to the case where X is smooth and T is empty. Since this is local on $\mathfrak{P}$, suppose P affine. Since X is an affine and smooth k-scheme then there exists an affine and smooth $\mathfrak{S}$-formal scheme $\mathfrak{X}$ which is a lifting of X. Since $\mathfrak{P}'$ is smooth and since $\mathfrak{P}$ is affine, then we have a lifting of $X \to P'$ of the form $u'\colon \mathfrak{X} \to \mathfrak{P}'$. By setting $u := f \circ u'$, we get then a lifting $u\colon \mathfrak{X} \to \mathfrak{P}$ of $X \to P$. By hypothesis, there exists $\mathcal{G}$ a coherent $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module, $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent such that $\mathcal{E}' \xrightarrow{\sim} u'_+(\mathcal{G})$. Hence $f_+(\mathcal{E}') \xrightarrow{\sim} u_+(\mathcal{G})$. Since u is a closed immersion, then the functor u_+ is exact. This yields then that for any $j \in \mathbb{Z} \setminus \{0\}$, $H^j(f_+(\mathcal{E}')) = 0$.

Let us consider now (b). Suppose $? = **$. Let $\mathcal{F}' \in (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}', T'/\mathcal{V})$. Since f is realisable with respect to T, then we have the adjunction morphisms $f_{T+} \circ \mathbb{R}\underline{\Gamma}^\dagger_X f^!_T(\mathcal{F}) \to \mathcal{F}$ and $\mathcal{F}' \to \mathbb{R}\underline{\Gamma}^\dagger_X f^!_T \circ f_{T+}(\mathcal{F}')$ (see 13.2.4.3). In order the check that these morphisms are isomorphisms, by 8.7.6.11 it is sufficient to prove it outside respectively the divisor T and T', i.e. we reduce to the case where T and T' are empty. Since this is local, then we can suppose as above that there exists a lifting $u'\colon \mathfrak{X} \to \mathfrak{P}'$ of $X \to P'$. Set $u = f \circ u'$. Then $f_+ \circ \mathbb{R}\underline{\Gamma}^\dagger_X f^!(\mathcal{F}) \to \mathcal{F}$ is canonically isomorphic to the natural map $u_+ \circ u^!(\mathcal{F}) \to \mathcal{F}$ which is an isomorphism thanks to Kashiwara-Berthelot theorem. Moreover, the map $\mathcal{F}' \to \mathbb{R}\underline{\Gamma}^\dagger_X f^! \circ f_+(\mathcal{F}')$ is canonical isomorphic to the image under u'_+ of $\mathcal{G}' \to u^! \circ u_+(\mathcal{G}')$, where $\mathcal{G}' = u'^!(\mathcal{F}')$. Using again Kashiwara-Berthelot theorem, the map is therefore an isomorphism. Hence we are done. □

Remark 16.1.2.12. The lemma 16.1.2.11 will be extended (the morphism f is not necessarily realizable and the divisor T' is independent from T) in Lemma 16.2.7.6. However, before establishing this more general case, Lemma 16.1.2.11 will be used to build the adjunction morphisms in the proof of Lemma 16.1.10.4 (more precisely in fact, in the proof of the lemma 16.1.10.3 that allow to establish 16.1.10.4). This will imply that the category $(F\text{-})\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$ does only depend on the pair (Y, X).

Lemma 16.1.2.13. *The category* $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$ *is stable under kernels, images, cokernels.*

Proof. Let ϕ be a morphism of $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$. Since the category $\mathrm{MIC}^\dagger(Y, Y/K)$ is stable under kernels, images and cokernels, since we have the equivalence of categories $\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}\colon \mathrm{MIC}^\dagger(Y, Y/K) \cong \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$ of 12.2.2.6.1, then so is $\mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$. Let $\mathcal{E}$ be the kernel or the image or the cokernel of ϕ. This yields, for any integer $j \neq 0$, the equality $H^j(\mathbb{D}(\mathcal{E}|\mathfrak{U})) = 0$ (use 12.2.5.6). Using 8.7.6.11, this implies, for any integer $j \neq 0$, the vanishing $H^j\mathbb{D}_T(\mathcal{E}) = 0$. This yields that the dual of the kernel (resp. the image, resp. the cokernel) of ϕ is the cokernel (resp. the image, resp. the kernel) of $\mathbb{D}_T(\phi)$. Moreover, since $\mathbb{D}_T(\phi)$ is a morphism of overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules, its kernel, its image, its cokernel are $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overcoherent. □

Notation 16.1.2.14 (Finite and etale outside singularities case)**.** Let the commutative diagram

$$\begin{array}{ccccc} Y^{(0)} & \overset{j^{(0)}}{\hookrightarrow} & X^{(0)} & \overset{u^{(0)}}{\hookrightarrow} & \mathfrak{P}^{(0)} \\ \downarrow b & \square & \downarrow a & & \downarrow f \\ Y & \overset{j}{\hookrightarrow} & X & \overset{u}{\hookrightarrow} & \mathfrak{P}, \end{array} \tag{16.1.2.14.1}$$

where the left square is cartesian, f is a proper smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes, a is a proper surjective morphism of k-varieties, b is a finite and etale morphism of smooth k-varieties, j and $j^{(0)}$ are open immersions, u and $u^{(0)}$ are closed immersions. We suppose moreover there exists a divisor T of P such that $Y = X \setminus T$. We denote by $\mathfrak{U} := \mathfrak{P} \setminus T$, $T^{(0)} := f^{-1}(T)$, $\mathfrak{U}^{(0)} := \mathfrak{P}^{(0)} \setminus T^{(0)}$ and $g\colon \mathfrak{U}^{(0)} \to \mathfrak{U}$ the morphism induced by f. We suppose $Y^{(0)} = X^{(0)} \setminus T^{(0)}$. We get the morphism $\theta = (b, a, f)$ of smooth d-frames over $\mathfrak{S}$ (see definition 12.2.1.1). By abuse of notation, it might happen that we denote by $a := (b, a, f)$ and $b := (b, b, g)\colon (Y^{(0)}, Y^{(0)}, \mathfrak{U}^{(0)}/\mathfrak{S}) \to (Y, Y, \mathfrak{U}/\mathfrak{S})$ the morphism of smooth d-frames over $\mathfrak{S}$.

16.1.2.15 (b_+ is a left and right adjoint functor of b^+)**.** We keep the notations of 16.1.2.14.

(a) Following 12.2.2.6.1, we have the equivalence of categories of the form $\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}\colon \mathrm{MIC}^\dagger(Y, Y/K) \cong \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$. The functors of the form $\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}$ commute with duality and with (extraordinary) inverse images (see 12.2.5.6 and 12.2.4.1). Both functors $b^! := \mathbb{R}\underline{\Gamma}^\dagger_{Y^{(0)}} \circ g^!$ and $b^+ := \mathbb{D} \circ \mathbb{R}\underline{\Gamma}^\dagger_{Y^{(0)}} \circ g^! \circ \mathbb{D}$ of $\mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V})$ are then isomorphic. Hence, we get the commutative diagram (up to canonical isomorphism):

$$\begin{array}{ccc} \mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K) & \xrightarrow[\cong]{\mathrm{sp}_{Y^{(0)} \hookrightarrow \mathfrak{U}^{(0)},+}} & \mathrm{MIC}^{\dagger\dagger}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V}) \\ {\scriptstyle b^*}\uparrow & & \uparrow{\scriptstyle b^+} \\ \mathrm{MIC}^\dagger(Y, Y/K) & \xrightarrow[\cong]{\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}} & \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V}). \end{array} \qquad (16.1.2.15.1)$$

(b) Moreover, since b is finite and etale, the functor g_+ factors as follows $g_+ \colon \mathrm{MIC}^{\dagger\dagger}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$. Indeed, since this is local on $\mathfrak{U}$, we can suppose $\mathfrak{U}$ affine. Since in this case Y is affine and smooth, then there exists a smooth $\mathfrak{S}$-formal scheme $\mathfrak{Y}$ which is a lifting of Y. By smoothness of $\mathfrak{U}$ and affinity of $\mathfrak{Y}$, this yields a lifting $\mathfrak{Y} \hookrightarrow \mathfrak{U}$ of $Y \hookrightarrow U$. Similarly, since $Y^{(0)}$ is affine and smooth, since $\mathfrak{Y} \times_{\mathfrak{U}} \mathfrak{U}^{(0)}$ is smooth, we get a lifting $\mathfrak{Y}^{(0)} \hookrightarrow \mathfrak{Y} \times_{\mathfrak{U}} \mathfrak{U}^{(0)}$ of $Y^{(0)} \hookrightarrow Y \times_U U^{(0)}$. This yields of liftings $\mathfrak{Y}^{(0)} \to \mathfrak{Y}$ and $\mathfrak{Y}^{(0)} \to \mathfrak{U}^{(0)}$ inducing the same morphism over $\mathfrak{U}$. Via the theorem of Berthelot-Kashiwara (see 9.3.5.9), we reduce therefore to the case where $Y = U$ and $Y^{(0)} = U^{(0)}$. In this case, since g is finite and etale, then the functor g_+ preserves both the $\mathcal{D}^\dagger$-coherence and the $\mathcal{O}$-coherence, and we are done. We denote by b_+ this factorisation.

(c) Since g is proper then $g^!$ is right adjoint to g_+. This yields that the functor $b^!$ is right adjoint to b_+. Since g is proper, using the relative duality theorem (see 9.4.5.2), this yields that the functor b^+ is left adjoint to b_+.

(d) Since $b^! \xrightarrow{\sim} b^+$, this yields therefore that $b^!$ is left adjoint to b_+ and that b^+ is a right adjoint to b_+.

(e) Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$. We get by adjunction the maps:

$$\mathcal{E} \to b_+ b^+(\mathcal{E}) \to \mathcal{E}, \qquad (16.1.2.15.2)$$

whose composition is an isomorphism. Indeed, using Berthelot-Kashiwara theorem we reduce to the case where $Y = U$. In that case $b_+ = b_*$ and $b^+ = b^*$ and we are done.

Remark 16.1.2.16. With the notations of 16.1.2.15, since b is finite and etale, we have the direct image functor $b_* \colon \mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K) \to \mathrm{MIC}^\dagger(Y, Y/K)$ (see [Tsu02, 5]). The functor b^* is left adjoint to the functor b_*. By uniqueness of the adjoint functors, we get the commutative diagram (up to canonical isomorphism):

$$\begin{array}{ccc} \mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}/K) & \xrightarrow[\cong]{\mathrm{sp}_{Y^{(0)} \hookrightarrow \mathfrak{U}^{(0)},+}} & \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}^{(0)}/\mathcal{V}) \\ \downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle b_+} \\ \mathrm{MIC}^\dagger(Y, Y/K) & \xrightarrow[\cong]{\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}} & \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V}), \end{array} \qquad (16.1.2.16.1)$$

so that the adjunction morphisms are compatible.

In order to define 16.1.3.1, we will need the notion of the direct image by a morphism which is finite and etale outside overconvergent singularities as defined in the proposition 16.1.2.17 below.

Lemma 16.1.2.17. *We keep the notations of 16.1.2.14.*

(a) We have the factorisation $\theta_+ := f_{T,T^{(0)}+} \colon \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \to \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$.

(b) The functor θ_+ *is right adjoint to* θ^+ *(defined at 16.1.2.8.2). We denote by* adj_a *the adjunction morphisms* $\mathrm{id} \to \theta_+ \circ \theta^+$ *and* $\theta^+ \circ \theta_+ \to \mathrm{id}$. *These adjunction morphisms are transitive with respect to the composition of diagram of the form 16.1.2.14.1 and satisfying the required conditions.*

Proof. First let us prove (a). Let $\mathcal{E}^{(0)}$ an object of $\mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V})$. Since f is proper, then f_+ commutes with duality (see 9.4.5.2) and the (over)coherence is closed by f_+ (see 15.3.6.14). This yields that $f_+(\mathcal{E}^{(0)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ is coherent and $\mathbb{D}_T \circ f_+(\mathcal{E}^{(0)}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. Moreover, $f_+(\mathcal{E}^{(0)})|\mathfrak{U} \xrightarrow{\sim} g_+(\mathcal{E}^{(0)}|\mathfrak{U}^{(0)})$. Following the second point of 16.1.2.15, this yields $f_+(\mathcal{E}^{(0)})|\mathfrak{U} \in$

$\mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$, i.e., $f_+(\mathcal{E}^{(0)})|\mathfrak{U}$ is in the essential image of $\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}$. By 8.7.6.11, this yields that $f_+(\mathcal{E}^{(0)}) \in \mathrm{Coh}(X, \mathfrak{P}, T/\mathcal{V})$. We get then the required factorisation of f_+.

We construct canonically the adjunction morphisms between θ_+ and θ^+ as follows: let $\mathcal{E}^{(0)} \in \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V})$ and $\mathcal{E} \in \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$. Since the functor f_+ is left adjoint to $f^!$ (for the overcoherent complexes) we get the morphism $f_+ \circ f^! \circ \mathbb{D}_T(\mathcal{E}) \to \mathbb{D}_T(\mathcal{E})$. Via the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \to \mathrm{id}$, we get by functoriality $f_+ \circ \mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \circ f^! \circ \mathbb{D}_T(\mathcal{E}) \to \mathbb{D}_T(\mathcal{E})$. By applying the functor $\mathbb{D}_T$ this yields the morphism

$$\mathcal{E} \xrightarrow[\sim]{8.7.7.3} \mathbb{D}_T \circ \mathbb{D}_T(\mathcal{E}) \to \mathbb{D}_T \circ f_+ \circ \mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \circ f^! \circ \mathbb{D}_T(\mathcal{E}) \xrightarrow[\sim]{9.4.5.2} f_+ \circ \mathbb{D}_{T^{(0)}} \circ \mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \circ f^! \circ \mathbb{D}_T(\mathcal{E}),$$

i.e. we get then the morphism $\mathcal{E} \to \theta_+ \circ \theta^+(\mathcal{E})$. We build similarly the adjunction morphism $\theta^+ \circ \theta_+(\mathcal{E}^{(0)}) \to \mathcal{E}^{(0)}$. □

16.1.2.18. We keep notation of 16.1.2.17. Let $\widetilde{T}$ be a divisor of $\mathfrak{P}$ containing T. Set $\widetilde{T}^{(0)} := f^{-1}(\widetilde{T})$, $\widetilde{b}\colon (X^{(0)} \setminus \widetilde{T}^{(0)}) \to (X \setminus \widetilde{T})$ the morphism induced by a and $\widetilde{\theta} = (f, a, \widetilde{b})$ the associated morphism of smooth d-frames. Since the dual functors commute with localisation outside a divisor (see 9.2.4.22.3), then we get the functors $({}^\dagger \widetilde{T})\colon \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T}/\mathcal{V})$ and $({}^\dagger \widetilde{T}^{(0)})\colon \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \to \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})$. Moreover, the extraordinary inverse image and direct image functors commute with localisation functors (see 13.2.1.4). This yields that the functors θ^+ and θ_+ of 16.1.2.8 and 16.1.2.17 commute with localisation, i.e., we have some canonical isomorphisms $\widetilde{\theta}^+ \circ ({}^\dagger \widetilde{T}) \xrightarrow{\sim} ({}^\dagger \widetilde{T}^{(0)}) \circ \theta^+$ and $\widetilde{\theta}_+ \circ ({}^\dagger \widetilde{T}^{(0)}) \xrightarrow{\sim} ({}^\dagger \widetilde{T}) \circ \theta_+$.

16.1.3 Finite and etale (outside overconvergent singularities) descent

We keep the notations and hypotheses of 16.1.2.14. Moreover, denote by $\mathfrak{P}^{(1)} := \mathfrak{P}^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$, $\mathfrak{P}^{(2)} := \mathfrak{P}^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$, $f_1, f_2\colon \mathfrak{P}^{(1)} \to \mathfrak{P}^{(0)}$ the left and right projections, $f_{ij}\colon \mathfrak{P}^{(2)} \to \mathfrak{P}^{(1)}$ the projections on the factors of the indices i, j for $i < j$. Similarly, we denote by $X^{(1)} := X^{(0)} \times_X X^{(0)}$, $X^{(2)} := X^{(0)} \times_X X^{(0)} \times_X X^{(0)}$, $a_i\colon X^{(1)} \to X^{(0)}$, $a_{ij}\colon X^{(2)} \to X^{(1)}$ the canonical projections ; similarly by replacing respectively a by b (resp. by g) and X by Y (resp. $\mathfrak{U}$). We denote by $T^{(1)} := f_1^{-1}(T^{(0)}) = f_2^{-1}(T^{(0)})$, $T^{(2)} := f_{12}^{-1}(T^{(1)}) = f_{23}^{-1}(T^{(1)}) = f_{13}^{-1}(T^{(1)})$. By abuse of notation, $a := (b, a, f)$, $a_i := (b_i, a_i, f_i)$ and $a_{ij} := (b_{ij}, a_{ij}, f_{ij})$ can be viewed as morphisms of smooth d-frames over $\mathfrak{S}$ which satisfy the same properties as in 16.1.2.14, i.e. they are finite and etale outside overconvergent singularities. Hence, we can use 16.1.2.17 and its notation, e.g. $a_{ij}^+ := \mathbb{D}_{T^{(2)}} \circ \mathbb{R}\underline{\Gamma}^\dagger_{X^{(2)}} \circ f^!_{ij, T^{(2)}, T^{(1)}} \circ \mathbb{D}_{T^{(1)}}$ and $a_{ij+} := f_{ij, T^{(2)}, T^{(1)}+}$. By abuse of notation, $b_i := (b_i, b_i, g_i)$ and $b_{ij} := (b_{ij}, b_{ij}, g_{ij})$ can be viewed as morphisms of completely smooth d-frames over $\mathfrak{S}$ (with empty divisors). We can use 16.1.2.17 and its notation, e.g. $b_{ij}^+ := \mathbb{D} \circ \mathbb{R}\underline{\Gamma}^\dagger_{Y^{(2)}} \circ g^!_{ij} \circ \mathbb{D}$ and $b_{ij+} := g_{ij+}$.

Definition 16.1.3.1. We define the category $\mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$ as follows:

(i) The objects are the sheaves $\mathcal{E}^{(0)} \in \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V})$ together with a glueing datum, i.e., an isomorphism in $\mathrm{MIC}^*(X^{(1)}, \mathfrak{P}^{(1)}, T^{(1)}/\mathcal{V})$ of the form $\epsilon\colon a_2^+(\mathcal{E}^{(0)}) \xrightarrow{\sim} a_1^+(\mathcal{E}^{(0)})$ satisfying the cocycle condition: $a_{13}^+(\epsilon) = a_{12}^+(\epsilon) \circ a_{23}^+(\epsilon)$.

(ii) The morphisms $(\mathcal{E}^{(0)}, \epsilon) \to (\mathcal{F}^{(0)}, \tau)$ are the $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-linear morphisms $\phi\colon \mathcal{E}^{(0)} \to \mathcal{F}^{(0)}$ commuting to glueing data, i.e. such that $\tau \circ a_2^+(\phi) = a_1^+(\phi) \circ \epsilon$.

When the divisors are empty, as usual we remove them in the notation.

16.1.3.2. We have the functor $\mathcal{L}oc\colon \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$ defined by $\mathcal{E} \mapsto (a^+(\mathcal{E}), \epsilon)$, where ϵ is the isomorphism induced by the transitivity with respect to the composition of the functors of the form a^+ (see 16.1.2.17).

16.1.3.3. We build the functor $\mathcal{G}lue\colon \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V}) \to \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$ as follows: let $(\mathcal{E}^{(0)}, \epsilon) \in \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$. Set $\widetilde{a} := a \circ a_1 = a \circ a_2$, $\widetilde{b} = b \circ b_1 = b \circ b_2$ and $\widetilde{f} = f \circ f_1 = f \circ f_2$. By abuse of notation, we write $\widetilde{a} := (\widetilde{b}, \widetilde{a}, \widetilde{f})$. Remark the equality $\widetilde{a} = a \circ a_1 = a \circ a_2$ still holds as morphism of smooth d-frames over $\mathfrak{S}$. We have the composition morphism

$$\phi_1\colon a_+(\mathcal{E}^{(0)}) \underset{16.1.2.17.b}{\xrightarrow{\mathrm{adj}_{a_1}}} a_+ \circ a_{1+} \circ a_1^+(\mathcal{E}^{(0)}) \xrightarrow{\sim} \tilde{a}_+ \circ a_1^+(\mathcal{E}^{(0)}).$$

On the other hand, we get a second composition morphism:

$$\phi_2 \colon a_+(\mathcal{E}^{(0)}) \underset{16.1.2.17.b}{\overset{\mathrm{adj}_{a_2}}{\longrightarrow}} a_+ \circ a_{2+} \circ a_2^+(\mathcal{E}^{(0)}) \xrightarrow{\sim} \tilde{a}_+ \circ a_2^+(\mathcal{E}^{(0)}) \underset{\epsilon}{\xrightarrow{\sim}} \tilde{a}_+ \circ a_1^+(\mathcal{E}^{(0)}).$$

We set $\mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) = \ker \left(a_+(\mathcal{E}^{(0)}) \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \tilde{a}_+ \circ a_1^+(\mathcal{E}^{(0)}) \right)$. With 16.1.2.13 and 16.1.2.17, we check $\mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) \in \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$.

16.1.3.4. Let $\widetilde{T}$ be a divisor of P containing T and $\widetilde{T}^{(0)} := f^{-1}(\widetilde{T})$. Via the remark of 16.1.2.18, we easily check that the functors $\mathcal{L}oc$ of 16.1.3.2 and $\mathcal{G}lue$ of 16.1.3.3 commute with localisation outside a divisor, i.e., we have some canonical isomorphisms $\mathcal{G}lue \circ ({}^\dagger\widetilde{T}^{(0)}) \xrightarrow{\sim} ({}^\dagger\widetilde{T}) \circ \mathcal{G}lue$ and $\mathcal{L}oc \circ ({}^\dagger\widetilde{T}) \xrightarrow{\sim} ({}^\dagger\widetilde{T}^{(0)}) \circ \mathcal{L}oc$ (the functors $\mathcal{L}oc$ and $\mathcal{G}lue$ indicate the functors of 16.1.3.2 and 16.1.3.3, are similarly replacing T by $\widetilde{T}$).

Before considering the general case of this subsection, first let us consider the finite and etale descent of convergent isocrystals over smooth k-schemes, which corresponds to the case where the overconvergent singularities are empty:

Lemma 16.1.3.5 (Outside overconvergent singularities)**.** *The canonical functors* $\mathcal{L}oc \colon \mathrm{MIC}^*(Y, \mathfrak{U}/\mathcal{V}) \to \mathrm{MIC}^*(Y^{(\bullet)}, \mathfrak{U}^{(\bullet)}/\mathcal{V})$ *and* $\mathcal{G}lue \colon \mathrm{MIC}^*(Y^{(\bullet)}, \mathfrak{U}^{(\bullet)}/\mathcal{V}) \to \mathrm{MIC}^*(Y, \mathfrak{U}/\mathcal{V})$ *defined respectively in 16.1.3.2 and 16.1.3.3 are quasi-inverse.*

Proof. Let $\mathcal{E} \in \mathrm{MIC}^*(Y, \mathfrak{U}/\mathcal{V})$ and $(\mathcal{E}^{(0)}, \epsilon) \in \mathrm{MIC}^*(Y^{(\bullet)}, \mathfrak{U}^{(\bullet)}/\mathcal{V})$. Following the steps I.1) and II.1) of the proof of 16.1.3.6 (these two steps do not use 16.1.3.5), we have the morphisms $\mathcal{E} \to \mathcal{G}lue \circ \mathcal{L}oc(\mathcal{E})$ and $b^+ \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) \to \mathcal{E}^{(0)}$. The check that the first (resp. second) morphism is an isomorphism (resp. commutes with glueing data and is an isomorphism) is local. Via the theorem of Berthelot-Kashiwara (see 9.3.5.9), we reduce therefore to suppose $Y = U$ and $Y^{(0)} = U^{(0)}$. In this case $g = b$ is a finite and etale morphism and b_+ (resp. b^+) is canonically isomorphic to b_* (resp. b^*), and similarly by adding indices i or ij at the bottom.

In the case where $g = b$, via the classic theory of the faithfully flat descent, we check the following assertions:

(a) The category $\mathrm{MIC}^{\dagger\dagger}(Y, Y/\mathcal{V})$ is equivalent to that of objects $\mathcal{E}^{(0)} \in \mathrm{MIC}^{\dagger\dagger}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V})$ endowed with a glueing datum, i.e., an isomorphism $\epsilon \colon b_2^*(\mathcal{E}^{(0)}) \xrightarrow{\sim} b_1^*(\mathcal{E}^{(0)})$ satisfying the usual cocycle condition.

(b) The quasi-inverse functor of this equivalence of categories is given by the glueing functor defined by setting $\mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) = \ker \left(b_*(\mathcal{E}^{(0)}) \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \widetilde{b}_* \circ b_1^*(\mathcal{E}^{(0)}) \right)$, where $\phi_1 \colon b_*(\mathcal{E}^{(0)}) \xrightarrow{\mathrm{adj}_{b_1}} b_* \circ b_{1*} \circ b_1^*(\mathcal{E}^{(0)}) \xrightarrow{\sim} \widetilde{b}_* \circ b_1^*(\mathcal{E}^{(0)})$ and where $\phi_2 \colon b_*(\mathcal{E}^{(0)}) \xrightarrow{\mathrm{adj}_{b_2}} b_* \circ b_{2*} \circ b_2^*(\mathcal{E}^{(0)}) \xrightarrow{\sim} \widetilde{b}_* \circ b_2^*(\mathcal{E}^{(0)}) \underset{\epsilon}{\xrightarrow{\sim}} \widetilde{b}_* \circ b_1^*(\mathcal{E}^{(0)})$.

(c) The morphism induced by adjunction $b^*\mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) \to (\mathcal{E}^{(0)}, \epsilon)$ is an isomorphism which commutes with glueing data (see for example, [Mil80, beginning of the page 19]).

□

Proposition 16.1.3.6. *The functors $\mathcal{L}oc$ and $\mathcal{G}lue$ induce quasi-inverse equivalences between the categories* $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$ *and* $\mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$.

Proof. I) Let $\mathcal{E} \in \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$. Let us establish the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{G}lue \circ \mathcal{L}oc(\mathcal{E})$.

1) Construction of this morphism: we put $(\mathcal{E}^{(0)}, \epsilon) := \mathcal{L}oc(\mathcal{E})$. Consider the diagram below:

$$\begin{array}{ccccccc}
\mathcal{E} & \xrightarrow{\mathrm{adj}_a} & a_+(\mathcal{E}^{(0)}) & \xrightarrow{\mathrm{adj}_{a_2}} & a_+ \circ a_{2+} \circ a_2^+(\mathcal{E}^{(0)}) & \xrightarrow{\sim} & \widetilde{a}_+ \circ a_2^+(\mathcal{E}^{(0)}) \\
\| & & \downarrow \mathrm{adj}_{a_1} & & & \swarrow_{\epsilon}^{\sim} & \downarrow \sim \\
\| & & a_+ \circ a_{1+} \circ a_1^+(\mathcal{E}^{(0)}) & \xrightarrow{\sim} & \widetilde{a}_+ \circ a_1^+(\mathcal{E}^{(0)}) & & \\
\| & & \downarrow \sim & \swarrow^{\sim} & & & \\
\mathcal{E} & \xrightarrow{\mathrm{adj}_{\mathbf{a}}} & \widetilde{a}_+ \circ \widetilde{a}^+(\mathcal{E}) & = & = & = & \widetilde{a}_+ \circ \widetilde{a}^+(\mathcal{E}).
\end{array} \tag{16.1.3.6.1}$$

The middle triangle is commutative by definition of the isomorphism $a_+ \circ a_{1+} \circ a_1^+(\mathcal{E}^{(0)}) \xrightarrow{\sim} \widetilde{a}_+ \circ \widetilde{a}^+(\mathcal{E})$. The right triangle is commutative by definition of the isomorphism ϵ. By transitivity of the adjunction morphism, the left rectangle of the diagram is commutative. For the same reason, the outer of 16.1.3.6.1 is commutative. In the trapeze of the top, we notice that the composition $a_+(\mathcal{E}^{(0)}) \to \widetilde{a}_+ \circ a_1^+(\mathcal{E}^{(0)})$ going through the bottom (resp. the top) is ϕ_1 (resp. ϕ_2). This yields the factorisation: $\mathcal{E} \to \mathcal{G}lue \circ \mathcal{L}oc(\mathcal{E})$.

2) Let us now check that this is an isomorphism. Following 8.7.6.11, it is sufficient to establish it outside T. We reduce then to the situation where T is empty, i.e. to the case where $\mathfrak{U} = \mathfrak{P}$, i.e., to the situation of 16.1.3.5. Hence we are done.

II) Conversely, is $(\mathcal{E}^{(0)}, \epsilon) \in \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$ and let us check the isomorphism $\mathcal{L}oc \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) \xrightarrow{\sim} (\mathcal{E}^{(0)}, \epsilon)$.

1) Let us construct canonically this morphism. By definition, we have the inclusion $\mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) \subset a_+(\mathcal{E}^{(0)})$. By adjunction (see 16.1.2.17), this yields the morphism $\phi\colon a^+ \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon) \to \mathcal{E}^{(0)}$.

2) Denote by τ the glueing datum of $a^+ \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon)$ satisfying the equality $(a^+ \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon), \tau) = \mathcal{L}oc \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \epsilon)$. We have to check that ϕ is an isomorphism commuting to respective glueing data, i.e., such that $\tau \circ a_2^+(\phi) = a_1^+(\phi) \circ \epsilon$. By 8.7.6.11, it is sufficient to establish it outside T, which reduces to the situation of 16.1.3.5. □

Corollary 16.1.3.7. *We suppose moreover $X^{(0)}$ smooth over k. We have the equalities:* $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$.

Proof. With the remark 16.1.2.7.(b), it is sufficient to prove the equality $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$. We have the inclusion $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \supset \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$. Conversely, let $\mathcal{E} \in \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$. Set

$$a^! := \mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \circ f^!_{T^{(0)},T} \colon D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}) \xrightarrow{f^!_{T^{(0)},T}} D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}}) \xrightarrow{\mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}}} D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}}).$$

Since $\mathbb{D}_T(\mathcal{E})$ is an overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module, by stability of the overcoherence by extraordinary inverse images and local cohomological functor (see 15.3.6.12 and 15.3.6.9), we get $a^!(\mathbb{D}_T(\mathcal{E})) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}})$. Moreover, since $\mathcal{E}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$, then following the smooth case treated in 12.2.1.11.2, we have

$$a^!(\mathbb{D}_T(\mathcal{E}))|\mathfrak{U}^{(0)} \xrightarrow{\sim} b^!(\mathcal{E}\mathfrak{U}) \in \mathrm{MIC}^{\dagger\dagger}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V}).$$

With 8.7.6.11, this yields that $a^!(\mathbb{D}_T(\mathcal{E}))$ is (isomorphic to) an (over)coherent $\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}}$-module. Since $X^{(0)}$ is smooth, via the characterization of 12.2.1.5, this yields $a^!(\mathbb{D}_T(\mathcal{E})) \in \mathrm{MIC}^{\dagger\dagger}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V})$. Hence, $a^!(\mathbb{D}_T(\mathcal{E}))$ is 1-$\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}}$-overholonomic (see 16.1.1.7). This implies in particular that $a^+(\mathcal{E})$ is an overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}}$-module and (moreover by using the isomorphism of biduality of 8.7.7.3) that $a_1^+(a^+(\mathcal{E}))$ is an overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}^{(1)}}({}^\dagger(T^{(1)}))_{\mathbb{Q}}$-module. By preservation of the overcoherence by kernel and by direct image by a proper smooth morphism, this yields $\mathcal{G}lue \circ \mathcal{L}oc(\mathcal{E})$ is an overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. Moreover, following 16.1.3.6, $\mathcal{E} \xrightarrow{\sim} \mathcal{G}lue \circ \mathcal{L}oc(\mathcal{E})$. Hence we are done. □

16.1.4 Full faithfulness of the localisation outside a divisor functor

With the notations 16.1.2.5, the theorem below is the analogue of the theorem of Tsuzuki [Tsu02, 4.1.1] or of the extended version of Kedlaya of [Ked07, 5.2.1]:

Theorem 16.1.4.1. *Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, $T \subset T'$ two divisors of P, X be a closed subscheme of P. By setting $Y := X \setminus T$, $Y' := X \setminus T'$, we suppose moreover Y smooth and Y' dense in Y. The functor $({}^\dagger T')$ induced the fully faithful functors:*

$$({}^\dagger T')\colon (F\text{-})\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^*(X, \mathfrak{P}, T'/\mathcal{V}), \tag{16.1.4.1.1}$$

$$({}^\dagger T')\colon (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{**}(X, \mathfrak{P}, T'/\mathcal{V}), \tag{16.1.4.1.2}$$

$$({}^\dagger T')\colon (F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T'/\mathcal{V}). \tag{16.1.4.1.3}$$

Proof. To lighten the notation, we omit indicating "$(F\text{-})$". Since the functor $\mathbb{D}_T$ (resp. $\mathbb{D}_{T'}$) induced an equivalence between $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$ and $\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$ (resp. between $\mathrm{MIC}^*(X, \mathfrak{P}, T'/\mathcal{V})$ and

$\mathrm{MIC}^{**}(X, \mathfrak{P}, T'/\mathcal{V})$), since we have the canonical isomorphism $\mathbb{D}_{T'} \circ ({}^\dagger T') \xrightarrow{\sim} ({}^\dagger T') \circ \mathbb{D}_T$, then it is sufficient to check 16.1.4.1.2.

Let $\mathcal{E}_1$, $\mathcal{E}_2$ be two objects of $\mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V})$. Set $\mathfrak{U} := \mathfrak{P} \setminus T$, $\mathfrak{U}' := \mathfrak{P} \setminus T'$. Via the lemma 16.1.2.9 and the remark 16.1.2.10, we reduce to the case where Y is integral and dense in X.

1) *For any $i = 1, 2$, the canonical morphism $\mathcal{E}_i \to \mathcal{E}_i({}^\dagger T')$ is injective.*

Proof: let $\mathcal{E}'_i$ be the kernel of $\mathcal{E}_i \to \mathcal{E}_i({}^\dagger T')$. By 8.7.6.11, since $\mathcal{E}'_i$ is an (over)coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module, it is sufficient to establish that $\mathcal{E}'_i|\mathfrak{U} = 0$. We reduce then to the case where T is empty. In this case since $X = Y$ is smooth, as the assertion is local in $\mathfrak{P}$, then we can suppose there exists a morphism $\mathfrak{X} \hookrightarrow \mathfrak{P}$ of affine and smooth $\mathfrak{S}$-formal schemes which is a lifting of $X \hookrightarrow P$. Since Y' is dense in $Y = X$, then T' does not contains X. Hence, $T' \cap X$ is a divisor of X (because X is integral and smooth). By using the theorem of Berthelot-Kashiwara (see 9.3.5.9), it follows that the functors $u_{T',+}$ and $u^!_{T'}$ (resp. u_+ and $u^!$) induce quasi-inverse equivalences of categories between $\mathrm{MIC}^{**}(X, \mathfrak{X}, T' \cap X/\mathcal{V})$ and $\mathrm{MIC}^{**}(X, \mathfrak{P}, T'/\mathcal{V})$ (resp. between $\mathrm{MIC}^{**}(X, \mathfrak{X}/\mathcal{V})$ and $\mathrm{MIC}^{**}(X, \mathfrak{P}/\mathcal{V})$). We reduce then to treat the case where $X = P$ (P is still affine and T empty). Following 11.2.1.14, the category $\mathrm{MIC}^{**}(P, \mathfrak{P}/\mathcal{V})$ is equal to the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules which are locally projective $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-module of finite type. In particular, $\mathcal{E}_1$ and $\mathcal{E}_2$ are projective $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-modules. Since the canonical morphism $\mathcal{O}_{\mathfrak{P},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{P}}({}^\dagger T')_{\mathbb{Q}}$ is injective, then so is of $\mathcal{E}_i \to \mathcal{E}_i({}^\dagger T')$.

2) This is a consequence of the step 1) that the functor 16.1.4.1.2 is faithful.

3) Let $\phi\colon \mathcal{E}_1({}^\dagger T') \to \mathcal{E}_2({}^\dagger T')$ be a morphism of $\mathrm{MIC}^{**}(X, \mathfrak{P}, T'/\mathcal{V})$. It remains to prove that ϕ comes by extension from a morphism of the form $\mathcal{E}_1 \to \mathcal{E}_2$. Following the step 2), this is local. We can therefore suppose $\mathfrak{P}$ affine and $\mathfrak{P}/\mathfrak{S}$ is endowed with coordinates.

4) Denote by $\mathcal{G}$ the image of the composition: $\mathcal{E}_1 \hookrightarrow \mathcal{E}_1({}^\dagger T') \xrightarrow{\phi} \mathcal{E}_2({}^\dagger T')$. Since $\mathcal{G}$, $\mathcal{E}_2$ are (over) coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-submodules of the (over)coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module $\mathcal{E}_2({}^\dagger T')$, the sheaf $\mathcal{G} \cap \mathcal{E}_2$ (equal by definition to the kernel of $\mathcal{G} \to \mathcal{E}_2({}^\dagger T')/\mathcal{E}_2$) is also an (over)coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-submodule of $\mathcal{E}_2({}^\dagger T')$.

5) *The inclusion $\mathcal{G} \cap \mathcal{E}_2 \subset \mathcal{G}$ is an isomorphism.*

Proof: via 8.7.6.11, we reduce to the case where T is empty. Moreover, via the theorem of Berthelot-Kashiwara (see 9.3.5.9), we can suppose $X = P$. By using the theorem of type A for the coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules, since $\mathcal{G}$ is the image of a morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules of the form $\mathcal{E}_1 \to \mathcal{E}_2({}^\dagger T')$, we get a $\Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$-linear surjective map $\Gamma(\mathfrak{P}, \mathcal{E}_1) \to \Gamma(\mathfrak{P}, \mathcal{G})$. Moreover, via the theorem of type A for coherent $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-modules, $\Gamma(\mathfrak{P}, \mathcal{E}_1)$ is $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$-coherent. By notherianite of $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$, this yields $\Gamma(\mathfrak{P}, \mathcal{G})$ and $\Gamma(\mathfrak{P}, \mathcal{G} \cap \mathcal{E}_2)$ are $\Gamma(\mathfrak{P}, \mathcal{O}_{\mathfrak{P},\mathbb{Q}})$-coherent. Since $\mathcal{G}$ and $\mathcal{G} \cap \mathcal{E}_2$ are moreover $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-coherent, using 11.1.1.8 this implies that $\mathcal{G}$ and $\mathcal{G} \cap \mathcal{E}_2$ are $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$-coherent. The morphism $\mathcal{G} \cap \mathcal{E}_2 \subset \mathcal{G}$ is then a morphism of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}/\mathcal{V})$. Hence, $\mathcal{G}/\mathcal{G} \cap \mathcal{E}_2 \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}/\mathcal{V})$. Let $E \in \mathrm{MIC}^\dagger(P, P/K)$ be a convergent isocrystal over P such that $\mathrm{sp}_* E \xrightarrow{\sim} \mathcal{G}/\mathcal{G} \cap \mathcal{E}_2$ (see 11.1.1.2). Since $\mathcal{G}/\mathcal{G} \cap \mathcal{E}_2 | \mathfrak{U}' = 0$, the isocrystal convergent over U' induced by E is null. Since $\mathfrak{U}'$ is dense in $\mathfrak{P}$, this yields $E = 0$, i.e. $\mathcal{G}/\mathcal{G} \cap \mathcal{E}_2 = 0$.

6) From 5), we get the morphisms $\theta\colon \mathcal{E}_1 \twoheadrightarrow \mathcal{G} \xleftarrow{\sim} \mathcal{G} \cap \mathcal{E}_2 \hookrightarrow \mathcal{E}_2$ whose composition with $\mathcal{E}_2 \to \mathcal{E}_2({}^\dagger T')$ is equal to the composition morphism $\mathcal{E}_1 \hookrightarrow \mathcal{E}_1({}^\dagger T') \xrightarrow{\phi} \mathcal{E}_2({}^\dagger T')$. Since $({}^\dagger T')(\epsilon)|\mathfrak{U}' = \phi|\mathfrak{U}'$, this yields by faithfulness of $|\mathfrak{U}'$ the equality $({}^\dagger T')(\theta) = \phi$. □

Remark 16.1.4.2. The theorem 16.1.4.1 is wrong if Y' is not dense in Y. For example, if we take $\mathfrak{P}$ a curve, Y a point and Y' the empty set.

16.1.5 Full faithfulness of the "restriction-inverse image" functor

We keep in the rest of this subsection 16.1.5 the following notations: let the commutative diagram

$$\begin{array}{ccccccc} \widetilde{Y}^{(0)} & \overset{l^{(0)}}{\hookrightarrow} & Y^{(0)} & \overset{j^{(0)}}{\hookrightarrow} & X^{(0)} & \overset{u^{(0)}}{\hookrightarrow} & \mathfrak{P}^{(0)} \\ \downarrow c & \square & \downarrow b & \square & \downarrow a & & \downarrow f \\ \widetilde{Y} & \overset{l}{\hookrightarrow} & Y & \overset{j}{\hookrightarrow} & X & \overset{u}{\hookrightarrow} & \mathfrak{P}, \end{array} \qquad (16.1.5.0.1)$$

where the left and middle squares are cartesian, f is a proper smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes, a is a proper surjective morphism of k-varieties, b is a morphism of smooth k-varieties, c is a finite and etale morphism, l, $l^{(0)}$, j and $j^{(0)}$ are open immersions, u and $u^{(0)}$ are closed immersions, $\widetilde{Y}$ is dense in Y and $\widetilde{Y}^{(0)}$ is dense in $Y^{(0)}$. We denote by $\widetilde{j}\colon \widetilde{Y} \hookrightarrow X$ and $\widetilde{j}^{(0)}\colon \widetilde{Y}^{(0)} \hookrightarrow X^{(0)}$ the induced open immersions. We suppose moreover there exists a divisor T (resp. $\widetilde{T}$) of P such that $Y = X \setminus T$ (resp. $\widetilde{Y} = X \setminus \widetilde{T}$). We denote by $\mathfrak{U} := \mathfrak{P} \setminus T$, $T^{(0)} := f^{-1}(T)$, $\mathfrak{U}^{(0)} := \mathfrak{P}^{(0)} \setminus T^{(0)}$ and $g\colon \mathfrak{U}^{(0)} \to \mathfrak{U}$ the morphism induced by f ; the same by adding tildes (e.g. we set $\widetilde{\mathfrak{U}} := \mathfrak{P} \setminus \widetilde{T}$ etc.).

Denote by $\mathfrak{P}^{(1)} := \mathfrak{P}^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$, $\mathfrak{P}^{(2)} := \mathfrak{P}^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$, $f_1, f_2\colon \mathfrak{P}^{(1)} \to \mathfrak{P}^{(0)}$ the left and right projections, $f_{ij}\colon \mathfrak{P}^{(2)} \to \mathfrak{P}^{(1)}$ the projections on the factors of the indices i, j for $i < j$. Similarly, we denote by $X^{(1)} := X^{(0)} \times_X X^{(0)}$, $X^{(2)} := X^{(0)} \times_X X^{(0)} \times_X X^{(0)}$, $a_i\colon X^{(1)} \to X^{(0)}$, $a_{ij}\colon X^{(2)} \to X^{(1)}$ the canonical projections ; similarly by replacing respectively a by b, c or g and X by Y, $\widetilde{Y}$ or $\mathfrak{U}$. We denote by $T^{(1)} := f_1^{-1}(T^{(0)}) \cup f_2^{-1}(T^{(0)})$, $T^{(2)} := f_{12}^{-1}(T^{(1)}) \cup f_{23}^{-1}(T^{(1)}) \cup f_{13}^{-1}(T^{(1)})$; similarly with tildes.

By abuse of notation, $a := (c, a, f)$, $a_i := (c_i, a_i, f_i)$ and $a_{ij} := (c_{ij}, a_{ij}, f_{ij})$ can be viewed as morphisms of smooth d-frames over $\mathfrak{S}$ which satisfy the same properties as in 16.1.2.14, i.e. they are finite and etale outside overconvergent singularities. Hence, we can use 16.1.2.17 and its notation, e.g. $a_{ij}^{+} := \mathbb{D}_{T^{(2)}} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X^{(2)}} \circ f^{!}_{ij,T^{(2)},T^{(1)}} \circ \mathbb{D}_{T^{(1)}}$ and $a_{ij+} := f_{ij,T^{(2)},T^{(1)}+}$. If we denote by $\mathfrak{j} := (\mathrm{id}, j, |\mathfrak{U})$ and $\widetilde{\mathfrak{j}} := (\mathrm{id}, \widetilde{j}, |\widetilde{\mathfrak{U}})$, then with notation 16.1.2.8.2 we get the functors $\mathfrak{j}^{+}\colon (F\text{-})\mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{*}(Y, \mathfrak{U}/\mathcal{V})$ and $\widetilde{\mathfrak{j}}^{+}\colon (F\text{-})\mathrm{MIC}^{*}(X, \mathfrak{P}, \widetilde{T}/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{*}(\widetilde{Y}, \widetilde{\mathfrak{U}}/\mathcal{V})$ which are respectively canonically isomorphic to the restriction functors $|\mathfrak{U}$ and $|\widetilde{\mathfrak{U}}$.

By abuse of notation, we denote by $a = (b, a, f)$ the morphism of smooth d-frames. Following 16.1.2.8.2, we get the functor $a^{+}\colon \mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^{*}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V})$. Beware that we have two distinguished functors a^{+} but it will be obvious which one is used following the context.

Lemma 16.1.5.1. *The functor*

$$(a^{+}, |\widetilde{\mathfrak{U}})\colon \mathrm{MIC}^{*}(X, \mathfrak{P}, \widetilde{T}/\mathcal{V}) \to \mathrm{MIC}^{*}(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{*}(\widetilde{\mathfrak{U}}^{(0)}, \widetilde{Y}^{(0)}/\mathcal{V})} \mathrm{MIC}^{*}(\widetilde{Y}, \widetilde{\mathfrak{U}}/\mathcal{V}) \quad (16.1.5.1.1)$$

is fully faithful.

Proof. Since the functor $|\widetilde{\mathfrak{U}}$ is faithful (see 8.7.6.8), the functor $(a^{+}, |\widetilde{\mathfrak{U}})$ is faithful. It remains to prove that this faithfulness is full. Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{MIC}^{*}(X, \mathfrak{P}, \widetilde{T}/\mathcal{V})$. By setting $\mathcal{E}_1^{(0)} := a^{+}(\mathcal{E}_1)$, $\mathcal{E}_2^{(0)} := a^{+}(\mathcal{E}_2)$ let $\phi^{(0)}\colon \mathcal{E}_1^{(0)} \to \mathcal{E}_2^{(0)}$ and $\psi\colon \mathcal{E}_1|\widetilde{\mathfrak{U}} \to \mathcal{E}_2|\widetilde{\mathfrak{U}}$ be two morphisms inducing canonically the same morphism in $\mathrm{MIC}^{*}(\widetilde{\mathfrak{U}}^{(0)}, \widetilde{Y}^{(0)}/\mathcal{V})$. Consider the square:

$$\begin{array}{ccc} a_2^{+}(\mathcal{E}_1^{(0)}) & \xrightarrow[\epsilon_1]{\sim} & a_1^{+}(\mathcal{E}_1^{(0)}) \\ \downarrow a_1^{+}(\phi^{(0)}) & & \downarrow a_2^{+}(\phi^{(0)}) \\ a_2^{+}(\mathcal{E}_2^{(0)}) & \xrightarrow[\epsilon_2]{\sim} & a_1^{+}(\mathcal{E}_2^{(0)}) \end{array} \quad (16.1.5.1.2)$$

of $\mathrm{MIC}^{*}(X^{(1)}, \mathfrak{P}^{(1)}, \widetilde{T}^{(1)}/\mathcal{V})$, where ϵ_1 and ϵ_2 are the canonical isomorphisms induced by transitivity (see 16.1.2.8). Since the morphisms $\phi^{(0)}$ and ψ are compatible, the diagram 16.1.5.1.2 becomes commutative after applying the functor $|\widetilde{\mathfrak{U}}^{(1)}$. Thanks to the proposition 8.7.6.11, this yields the commutativity of 16.1.5.1.2. Hence, we have the morphism $\phi^{(0)}\colon (\mathcal{E}_1^{(0)}, \epsilon_1) \to (\mathcal{E}_2^{(0)}, \epsilon_2)$ of $\mathrm{MIC}^{*}(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, \widetilde{T}^{(\bullet)}/\mathcal{V})$. Hence the morphism $\phi^{(0)}|\widetilde{\mathfrak{U}}$ commutes with glueing data and we still denote by $\phi^{(0)}|\widetilde{\mathfrak{U}}\colon (\mathcal{E}_1^{(0)}|\widetilde{\mathfrak{U}}, \epsilon_1|\widetilde{\mathfrak{U}}^{(1)}) \to (\mathcal{E}_2^{(0)}|\widetilde{\mathfrak{U}}, \epsilon_2|\widetilde{\mathfrak{U}}^{(1)})$ the induced morphism of $\mathrm{MIC}^{*}(\widetilde{Y}^{(\bullet)}, \widetilde{\mathfrak{U}}^{(\bullet)}/\mathcal{V})$. Moreover, for $i = 1, 2$, $\mathcal{L}oc(\mathcal{E}_i) = (\mathcal{E}_i^{(0)}, \epsilon_i)$. Since the functor $\mathcal{L}oc$ is fully faithful (see 16.1.3.6), then there exists a morphism $\phi\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that $\mathcal{L}oc(\phi) = \phi^{(0)}$, and in particular we get $a^{+}(\phi) = \phi^{(0)}$. Since $\mathcal{L}oc(\phi|\widetilde{\mathfrak{U}}) = \phi^{(0)}|\widetilde{\mathfrak{U}}^{(0)} = \mathcal{L}oc(\psi)$, by faithfulness of $\mathcal{L}oc$, this yields $\phi|\widetilde{\mathfrak{U}} = \psi$. Hence, we have checked $(a^{+}, |\widetilde{\mathfrak{U}})(\phi) = (\phi^{(0)}, \psi)$. □

Proposition 16.1.5.2. *The functor*

$$(a^{+}, |\mathfrak{U})\colon \mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^{*}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{*}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V})} \mathrm{MIC}^{*}(Y, \mathfrak{U}/\mathcal{V}) \quad (16.1.5.2.1)$$

is fully faithful.

Proof. Consider the following diagram

$$\begin{array}{ccc} \mathrm{MIC}^{*}(X, \mathfrak{P}, T/\mathcal{V}) & \xrightarrow{(a^{+}, |\mathfrak{U})} & \mathrm{MIC}^{*}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{*}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V})} \mathrm{MIC}^{*}(Y, \mathfrak{U}/\mathcal{V}) \\ \downarrow ({}^{\dagger}\widetilde{T}) & & \downarrow (({}^{\dagger}\widetilde{T}^{(0)}), |\widetilde{\mathfrak{U}}) \\ \mathrm{MIC}^{*}(X, \mathfrak{P}, \widetilde{T}/\mathcal{V}) & \xrightarrow{(a^{+}, |\widetilde{\mathfrak{U}})} & \mathrm{MIC}^{*}(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{*}(\widetilde{Y}^{(0)}, \widetilde{\mathfrak{U}}^{(0)}/\mathcal{V})} \mathrm{MIC}^{*}(\widetilde{Y}, \widetilde{\mathfrak{U}}/\mathcal{V}). \end{array} \quad (16.1.5.2.2)$$

Following 13.2.1.4, the local functors commute with extraordinary inverse images. It follows from 4.6.4.7.1, the localisation outside a divisor commutes with duality. This yields the commutativity up to canonical isomorphism of the diagram 16.1.5.2.2. Since following 16.1.4.1 (resp. 16.1.5.1) the left functor (resp. bottom) is fully faithful, as the right one is faithful, the upper functor is then fully faithful. □

16.1.6 An equivalence of categories induced by the "localisation-inverse image" functor

In this section, we keep the notations and hypotheses of the section 16.1.5.

Lemma 16.1.6.1. *The canonical functor below induced by* $(a^+,\, ({}^\dagger \widetilde{T}))$*:*

$$\mathrm{MIC}^*(X,\mathfrak{P},T/\mathcal{V}) \to \mathrm{MIC}^*(X^{(0)},\mathfrak{P}^{(0)},T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^*(X^{(0)},\mathfrak{P}^{(0)},\widetilde{T}^{(0)}/\mathcal{V})} \mathrm{MIC}^*(X,\mathfrak{P},\widetilde{T}/\mathcal{V}) \quad (16.1.6.1.1)$$

is fully faithful.

Proof. Since the functor $({}^\dagger \widetilde{T})$ is faithful, then $(a^+,\, ({}^\dagger \widetilde{T}))$ is faithful. Since $({}^\dagger \widetilde{T})$ is fully faithful (see the theorem 16.1.4.1) and since $({}^\dagger \widetilde{T}^{(0)})$ is faithful, we check the faithfulness of $(a^+,\, ({}^\dagger \widetilde{T}))$ is full. □

Lemma 16.1.6.2. *Let* $\widetilde{\mathcal{E}} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger \widetilde{T})_{\mathbb{Q}})$.

(a) If $\widetilde{\mathcal{E}} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$*, then the canonical morphism of* $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger \widetilde{T})_{\mathbb{Q}})$

$$\widetilde{\mathcal{E}} \to ({}^\dagger \widetilde{T}, T)(\widetilde{\mathcal{E}}) \quad (16.1.6.2.1)$$

is a isomorphism. We have the isomorphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger \widetilde{T})_{\mathbb{Q}})$*:*

$$({}^\dagger \widetilde{T}, T) \circ \mathbb{D}_T(\widetilde{\mathcal{E}}) \xrightarrow{\sim} \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}). \quad (16.1.6.2.2)$$

(b) If $\mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$*, then there exists a canonical morphism of* $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ *of the form:*

$$\iota\colon \mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}}. \quad (16.1.6.2.3)$$

Proof. a) Since the map 16.1.6.2.1 is a morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger \widetilde{T})_{\mathbb{Q}})$ which an isomorphism outside $\widetilde{T}$, then it follows from 8.7.6.11 that 16.1.6.2.1 is an isomorphism. Hence, we get:

$$({}^\dagger \widetilde{T}, T) \circ \mathbb{D}_T(\widetilde{\mathcal{E}}) \underset{4.6.4.4.1}{\xrightarrow{\sim}} \mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T}, T)(\widetilde{\mathcal{E}}) \overset{16.1.6.2.1}{\xleftarrow{\sim}} \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}).$$

b) It follows from a) that the canonical morphism $\mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \to ({}^\dagger \widetilde{T}, T)(\mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}))$ is an isomorphism. Hence, we build the morphism 16.1.6.2.3 by composition as follows:

$$\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \to ({}^\dagger \widetilde{T}, T) \circ \mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \overset{4.6.4.4.1}{\xrightarrow{\sim}} \mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T}, T) \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \underset{16.1.6.2.1}{\xleftarrow{\sim}} \mathbb{D}_{\widetilde{T}} \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \overset{8.7.7.3}{\xrightarrow{\sim}} \widetilde{\mathcal{E}}.$$

□

Lemma 16.1.6.3. *We suppose* X *smooth. Let* $\mathcal{E} \in \mathrm{MIC}^*(X,\mathfrak{P},T/\mathcal{V})$*. Then we have the following properties hold:*

(a) $({}^\dagger \widetilde{T})(\mathcal{E})$ *and* $\mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T})(\mathcal{E})$ *are both overcoherent and holonomic as* $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$*-module.*

(b) We have a canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathrm{Im}\left(\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T})(\mathcal{E}) \overset{16.1.6.2.3}{\longrightarrow} ({}^\dagger \widetilde{T})(\mathcal{E})\right)$.

Proof. Since X is smooth, following 16.1.1.7 $\mathcal{E}$ is therefore 1-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overholonomic. With 15.3.2.8, this yields $\mathcal{E}({}^\dagger \widetilde{T})$ is both overcoherent and holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. Since the category $\mathrm{MIC}^*(X,\mathfrak{P},T/\mathcal{V})$ is stable under the functor $\mathbb{D}_T$ (see 16.1.2.6), this yields similarly that $\mathbb{D}_{\widetilde{T}}(\mathcal{E}({}^\dagger \widetilde{T})) \overset{4.6.4.4.1}{\xrightarrow{\sim}} ({}^\dagger \widetilde{T})\mathbb{D}_T(\mathcal{E})$ is both overcoherent and holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module.

We denote by α the morphisms induced by functoriality by $\mathrm{id} \to ({}^\dagger \widetilde{T})$. Let $\mathcal{F}$ be the image of the morphism $\iota\colon \mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T})(\mathcal{E}) \to ({}^\dagger \widetilde{T})(\mathcal{E})$ constructed at 16.1.6.2.3. Since ι is an isomorphism outside $\widetilde{T}$, then so is the canonical morphism $\rho\colon ({}^\dagger \widetilde{T})(\mathcal{F}) \to \mathcal{E}({}^\dagger \widetilde{T})$ of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger \widetilde{T})_{\mathbb{Q}}$-modules (which factors by definition through the canonical inclusion $\mathcal{F} \subset \mathcal{E}({}^\dagger \widetilde{T})$). By 8.7.6.11, this morphism ρ is an isomorphism.

Following the step 1) of the proof of 16.1.4.1, since $\mathcal{E}, \mathbb{D}_T(\mathcal{E}) \in \mathrm{MIC}^{**}(X,\mathfrak{P},T/\mathcal{V})$, the canonical morphisms $\alpha\colon \mathcal{E} \to ({}^\dagger \widetilde{T})(\mathcal{E})$ and $\alpha\colon \mathbb{D}_T(\mathcal{E}) \to ({}^\dagger \widetilde{T})(\mathbb{D}_T(\mathcal{E}))$ are injective. Thanks to **??**, by applying to it $\mathbb{D}_T$, we get the surjection $\alpha^*\colon \mathbb{D}_T({}^\dagger \widetilde{T})(\mathbb{D}_T(\mathcal{E})) \twoheadrightarrow \mathbb{D}_T\mathbb{D}_T(\mathcal{E})$. Since the morphism $({}^\dagger \widetilde{T})(\alpha^*)$ is an isomorphism outside $\widetilde{T}$, this is an isomorphism. We get by functoriality the commutativity of squares of the bottom of the following diagram:

$$\begin{array}{ccccccc}
\mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T}) \circ \mathbb{D}_{\widetilde{T}}(({}^\dagger \widetilde{T})(\mathcal{E})) & \xrightarrow{\sim} & \mathbb{D}_{\widetilde{T}} \circ \mathbb{D}_{\widetilde{T}}(({}^\dagger \widetilde{T})(\mathcal{E})) & \xleftarrow{\sim} & \mathbb{D}_{\widetilde{T}} \circ \mathbb{D}_{\widetilde{T}}(({}^\dagger \widetilde{T})(\mathcal{E})) & \xrightarrow[8.7.7.3]{\sim} & ({}^\dagger \widetilde{T})(\mathcal{E}) \\
\sim \uparrow {\scriptstyle 4.6.4.4.1} & & & & & & \| \\
({}^\dagger \widetilde{T}) \circ \mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T})(\mathcal{E}) & \xrightarrow[4.6.4.4.1]{\sim} & ({}^\dagger \widetilde{T}) \circ \mathbb{D}_T \circ ({}^\dagger \widetilde{T})(\mathbb{D}_T(\mathcal{E})) & \xrightarrow[({}^\dagger \widetilde{T})(\alpha^*)]{\sim} & ({}^\dagger \widetilde{T}) \circ \mathbb{D}_T(\mathbb{D}_T(\mathcal{E})) & \xrightarrow[8.7.7.3]{\sim} & ({}^\dagger \widetilde{T})(\mathcal{E}) \\
\alpha \uparrow & & \alpha \uparrow & & \alpha \uparrow & & \alpha \uparrow \\
\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}} \circ ({}^\dagger \widetilde{T})(\mathcal{E}) & \xrightarrow[4.6.4.4.1]{\sim} & \mathbb{D}_T \circ ({}^\dagger \widetilde{T})(\mathbb{D}_T(\mathcal{E})) & \underset{\alpha^*}{\twoheadrightarrow} & \mathbb{D}_T(\mathbb{D}_T(\mathcal{E})) & \xrightarrow[8.7.7.3]{\sim} & \mathcal{E}.
\end{array} \tag{16.1.6.3.1}$$

To validate the commutativity of the (top) rectangle of the diagram 16.1.6.3.1, since all terms are coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger \widetilde{T})_{\mathbb{Q}}$-modules, then it is sufficient via 8.7.6.11 to establish it outside $\widetilde{T}$, which is straightforward.

We see that the morphism ι is equal to the composition of the morphisms from the left and next from the top of the diagram 16.1.6.3.1. We get therefore the commutative diagram below:

$$\begin{array}{ccccc}
\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\mathcal{E}({}^\dagger \widetilde{T})) & \twoheadrightarrow & \mathcal{F} & \hookrightarrow & \mathcal{E}({}^\dagger \widetilde{T}) \\
\| & & & & \| \\
\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(({}^\dagger \widetilde{T})(\mathcal{E})) & \twoheadrightarrow & \mathcal{E} & \overset{\alpha}{\hookrightarrow} & ({}^\dagger \widetilde{T})(\mathcal{E}),
\end{array} \tag{16.1.6.3.2}$$

whose bottom surjective arrow is the composition of the bottom morphisms of the diagram 16.1.6.3.1. This yields the canonical isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{F}$. □

Lemma 16.1.6.4. *Let $\mathcal{E}_1, \mathcal{E}_2$ two overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules such that $\mathbb{D}_T(\mathcal{E}_1), \mathbb{D}_T(\mathcal{E}_2)$ are $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overcoherent. Let $\phi\colon \mathcal{E}_1 \to \mathcal{E}_2$ be a $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-linear morphism. Let $\mathcal{K}$ be either the kernel of ϕ, or its image or its cokernel. Then $\mathcal{K}$, $H^0\mathbb{D}_T(\mathcal{K})$ are overcoherent and holonomic as a $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules.*

Proof. Using Theorem 15.3.2.8 implies that $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-holonomic. With 15.2.4.14 and 15.2.4.15, this yields $\mathcal{K}$, $H^0\mathbb{D}_T(\mathcal{K})$, are holonomic as a$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules. Moreover, since the overcoherence is stable under kernel, image and cokernel, then $\mathcal{K}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overcoherent. Since the functor $H^0\mathbb{D}_T$ is an exact functor over the category of holonomic $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules, this yields $H^0\mathbb{D}_T(\mathcal{K})$ is either the cokernel, or the image, or the kernel of $\mathbb{D}_T(\phi)$. Using the $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overcoherence of $\mathbb{D}_T(\mathcal{E}_1), \mathbb{D}_T(\mathcal{E}_2)$, we are done. □

Lemma 16.1.6.5. *We suppose $X^{(0)}$ smooth. Let $\widetilde{\mathcal{E}}^{(0)} \in \mathrm{MIC}^*(X^{(0)},\mathfrak{P}^{(0)},\widetilde{T}^{(0)}/\mathcal{V})$. Then we have the canonical isomorphism $a_{1+} \circ a_1^+(\widetilde{\mathcal{E}}^{(0)}) \xrightarrow{\sim} a_{1+} \circ a_1^!(\widetilde{\mathcal{E}}^{(0)})$.*

Proof. We remark first that since $X^{(0)}$ is smooth, $\widetilde{\mathcal{E}}^{(0)} \in \mathrm{MIC}^*(X^{(0)},\mathfrak{P}^{(0)},\widetilde{T}^{(0)}/\mathcal{V}) = \mathrm{MIC}^{**}(X^{(0)},\mathfrak{P}^{(0)},\widetilde{T}^{(0)}/\mathcal{V})$. Following the de Jong 's desingularisation theorem (of the form 10.4.1.2), we have a surjective, projective, generically finite and etale morphism $\alpha\colon X'' \to X^{(1)}$ such that X'' is smooth and $\alpha^{-1}(T^{(1)} \cap X^{(1)})$ is a divisor of X''. Then there exists a projective and smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes of the form $f''\colon \mathfrak{P}'' \to \mathfrak{P}^{(1)}$, a closed immersion $X'' \hookrightarrow \mathfrak{P}''$ such that their composition gives the composite morphism $X'' \to X^{(1)} \hookrightarrow \mathfrak{P}^{(1)}$. Set $T'' := (f'')^{-1}(\widetilde{T}^{(0)})$, $Y'' := X'' \setminus T''$, $\beta\colon Y'' \to Y^{(1)}$ the morphism induced by α and $\alpha := (\beta, \alpha, f'')\colon (Y'', X'', \mathfrak{P}'', \widetilde{T}''/\mathcal{V}) \to (Y^{(1)}, X^{(1)}, \mathfrak{P}^{(1)}, \widetilde{T}^{(1)}/\mathcal{V})$. Moreover, since $a_1 \circ \alpha\colon (X'', \mathfrak{P}'', \widetilde{T}''/\mathcal{V}) \to (X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})$ is a morphism of completely smooth frames, then we can use the commutation isomorphisms 12.2.1.14.2 and 12.2.1.14.3. Hence, we get the middle isomorphism:

$$\alpha^+ \circ a_1^+(\widetilde{\mathcal{E}}^{(0)}) \xrightarrow{\sim} (a_1 \circ \alpha)^+(\widetilde{\mathcal{E}}^{(0)}) \xrightarrow{\sim} (a_1 \circ \alpha)^!(\widetilde{\mathcal{E}}^{(0)}) \xrightarrow{\sim} \alpha^! \circ a_1^!(\widetilde{\mathcal{E}}^{(0)}).$$

This yields the middle morphism:

$$a_{1+} \circ a_1^+(\widetilde{\mathcal{E}}^{(0)}) \overset{16.1.1.6.4}{\longrightarrow} a_{1+} \circ \alpha_+ \circ \alpha^+ \circ a_1^+(\widetilde{\mathcal{E}}^{(0)}) \xrightarrow{\sim} a_{1+} \circ \alpha_+ \circ \alpha^! \circ a_1^!(\widetilde{\mathcal{E}}^{(0)}) \overset{16.1.1.6.3}{\longrightarrow} a_{1+} \circ a_1^!(\widetilde{\mathcal{E}}^{(0)}).$$

To establish that this morphism is an isomorphism, it is sufficient to check it outside $\widetilde{T}^{(0)}$, which reduces the case where the divisor $\widetilde{T}^{(0)}$ is empty and in particular the case where $X^{(1)}$ is also smooth. In this case, we have the canonical isomorphism $a_1^+(\widetilde{\mathcal{E}}^{(0)}) \xrightarrow{\sim} a_1^!(\widetilde{\mathcal{E}}^{(0)})$ and the composition of the canonical morphisms $\mathrm{id} \to \alpha_+ \circ \alpha^+ \xrightarrow{\sim} \alpha_+\alpha^! \to \mathrm{id}$ of functors defined over $\mathrm{MIC}^{\dagger\dagger}(X^{(1)}, \mathfrak{P}^{(1)}/\mathcal{V}) \cap D^{\mathrm{b}}_{\text{1-ovhol}}(\mathcal{D}^\dagger_{\mathfrak{P}^{(1)}}({}^\dagger T^{(1)})_{\mathbb{Q}})$ is an isomorphism (indeed, since this is a morphism of overconvergent isocrystals, it is sufficient to check it over a dense open subset of $X^{(1)}$ where the induced by α morphism is finite and etale). □

Lemma 16.1.6.6. *We suppose that the k-variety $X^{(0)}$ is smooth. Let $(\mathcal{E}^{(0)}, \widetilde{\mathcal{E}}, \rho)$ be an object of the category* $\mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})} \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T}/\mathcal{V})$. *Then $\widetilde{\mathcal{E}}$, $\mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}})$, $\mathbb{D}_T(\widetilde{\mathcal{E}})$, $\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}})$ are overcoherent and holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module.*

Proof. By definition of the $\mathcal{L}oc\colon \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T}/\mathcal{V}) \to \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, \widetilde{T}^{(\bullet)}/\mathcal{V})$, we have $\mathcal{L}oc(\widetilde{\mathcal{E}}) = (a^+(\widetilde{\mathcal{E}}), \theta)$ where θ is the isomorphism induced by the transitivity with respect to the composition of pullbacks functors (see 16.1.3.2).

Since c is a finite and etale morphism (recall notation 16.1.5.0.1), then following Proposition 16.1.3.6, the functors $\mathcal{L}oc$ and $\mathcal{G}lue$ are quasi-inverse equivalence of categories. Hence, $\widetilde{\mathcal{E}} \xrightarrow{\sim} \mathcal{G}lue(a^+(\widetilde{\mathcal{E}}), \theta)$. By definition of 16.1.3.3 of the functor $\mathcal{G}lue$, $\widetilde{\mathcal{E}}$ is then isomorphic to the kernel of a morphism of the form $a_+(\widetilde{\mathcal{E}}) \to a_+ \circ a_{1+} \circ a_1^+(\widetilde{\mathcal{E}})$. By hypothesis, we have the isomorphism $\rho\colon a^+(\widetilde{\mathcal{E}}) \xrightarrow{\sim} \mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})$. Hence, it follows from the Proposition 16.1.6.5 that $\widetilde{\mathcal{E}}$ is isomorphic to the kernel of a morphism of the form:

$$\phi\colon a_+(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})) \to \tilde{a}_+ \circ a_1^!(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})).$$

Moreover, since $X^{(0)}$ is smooth, then following Theorem 16.1.1.7 $\mathcal{E}^{(0)}$ is 1-$\mathcal{D}^\dagger_{\mathfrak{P}^{(0)}}({}^\dagger T^{(0)})_{\mathbb{Q}}$-overholonomic. By stability of this property by extraordinary inverse images, functor of localisation and direct image by a proper morphism (see 16.1.1.5), this yields the two terms of the morphism ϕ are 1-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overholonomic. Since $\widetilde{\mathcal{E}}$ is isomorphic to the kernel of ϕ, then it follows from Lemma 16.1.6.4 that $\widetilde{\mathcal{E}}$ and $\mathbb{D}_T(\widetilde{\mathcal{E}})$ are overcoherent and holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules.

Since $H^0\mathbb{D}_T$ is an exact functor on the category of holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules, then $\mathbb{D}_T(\widetilde{\mathcal{E}})$ is the cokernel of $\mathbb{D}_T(\phi)$. By using 16.1.6.2.2, we get $({}^\dagger \widetilde{T}, T) \circ \mathbb{D}_T(\phi) \xrightarrow{\sim} \mathbb{D}_{\widetilde{T}}(\phi)$. Since the functor $({}^\dagger \widetilde{T}, T)$ is exact over the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules, this yields that $\mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}})$ is the cokernel $\mathbb{D}_{\widetilde{T}}(\phi)$. Using again 16.1.6.4, to conclude the proof, it is therefore sufficient to prove that both terms of the morphism $\mathbb{D}_{\widetilde{T}}(\phi)$ are 1-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overholonomic. On one hand, we have

$$\mathbb{D}_{\widetilde{T}} \circ a_+(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})) \overset{9.4.5.2.1}{\xrightarrow{\sim}} a_+ \circ \mathbb{D}_{\widetilde{T}^{(0)}}(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})) \overset{4.6.4.4.1}{\xrightarrow{\sim}} a_+ \circ ({}^\dagger \widetilde{T}^{(0)})(\mathbb{D}_{T^{(0)}}(\mathcal{E}^{(0)})). \tag{16.1.6.6.1}$$

On the other hand, we get

$$\begin{aligned} \mathbb{D}_{\widetilde{T}} \circ \tilde{a}_+ \circ a_1^!(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})) &\overset{9.4.5.2.1}{\xrightarrow{\sim}} \tilde{a}_+ \circ \mathbb{D}_{\widetilde{T}^{(1)}} \circ a_1^!(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})) \overset{8.7.7.3}{\xrightarrow{\sim}} \tilde{a}_+ \circ a_1^+ \circ \mathbb{D}_{\widetilde{T}^{(0)}}(\mathcal{E}^{(0)}({}^\dagger \widetilde{T}^{(0)})) \\ &\overset{4.6.4.4.1}{\xrightarrow{\sim}} \tilde{a}_+ \circ a_1^+ \circ ({}^\dagger \widetilde{T}^{(0)})(\mathbb{D}_{T^{(0)}}(\mathcal{E}^{(0)})) \overset{16.1.6.5}{\xrightarrow{\sim}} \tilde{a}_+ \circ a_1^! \circ ({}^\dagger \widetilde{T}^{(0)})(\mathbb{D}_{T^{(0)}}(\mathcal{E}^{(0)})) \end{aligned} \tag{16.1.6.6.2}$$

Since $\mathbb{D}_{T^{(0)}}(\mathcal{E}^{(0)}) \in \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})$, then the last term of respectively 16.1.6.6.1 and 16.1.6.6.2 is 1-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overholonomic. This yields that both terms of $\mathbb{D}_{\widetilde{T}}(\phi)$ are 1-$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overholonomic, and we are done. □

Notation 16.1.6.7. We denote by $\mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T} \supset T/\mathcal{V})$ the full subcategory of $\mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T}/\mathcal{V})$ consisting of objects $\widetilde{\mathcal{E}}$ such that there exists $\mathcal{G} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$ together with an isomorphism of the form $\widetilde{\mathcal{E}}|\mathfrak{U} \xrightarrow{\sim} ({}^\dagger \widetilde{T} \cap U)(\mathcal{G})$.

We have the factorisation $({}^\dagger \widetilde{T})\colon \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T} \supset T/\mathcal{V})$.

Theorem 16.1.6.8. *Suppose $X^{(0)}$ smooth. With the notations of 16.1.6.7, the canonical functor*

$$(a^+, ({}^\dagger \widetilde{T}))\colon \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})} \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T} \supset T/\mathcal{V}) \tag{16.1.6.8.1}$$

is an equivalence of categories. We have moreover a canonical quasi-inverse functor denoted by $\mathcal{G}lue$.

Proof. I) Following Lemma 16.1.6.1, we already know that the functor $(a^+, ({}^\dagger \widetilde{T}))$ is fully faithful.

II) 1) Let us construct the quasi-inverse to $(a^+, ({}^\dagger \widetilde{T}))$ functor $\mathcal{G}lue$.

Let $(\mathcal{E}^{(0)}, \widetilde{\mathcal{E}}, \rho) \in \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})} \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T} \supset T/\mathcal{V})$. By using 16.1.6.6), the canonical morphism of overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules $\phi\colon \mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}}$ of 16.1.6.2.3 is such that $\mathbb{D}_T(\phi)$ is a morphism of overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules. We can therefore define a $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module $\mathcal{E}$ by setting

$$\mathcal{E} := \mathcal{G}lue(\mathcal{E}^{(0)}, \widetilde{\mathcal{E}}, \rho) := \mathrm{Im}(\mathbb{D}_T \circ \mathbb{D}_{\widetilde{T}}(\widetilde{\mathcal{E}}) \xrightarrow{\phi} \widetilde{\mathcal{E}}). \tag{16.1.6.8.2}$$

Following the Lemma 16.1.6.4, $\mathcal{E}$ and $\mathbb{D}_T(\mathcal{E})$ are overcoherent and holonomic as $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module. By definition of 16.1.6.7, there exists $\mathcal{G} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/\mathcal{V})$ together with and isomorphism of the form $\widetilde{\mathcal{E}}|\mathfrak{U} \xrightarrow{\sim} ({}^\dagger \widetilde{T} \cap U)(\mathcal{G})$. Hence,

$$\mathcal{E}|\mathfrak{U} \xrightarrow{\sim} \mathrm{Im}(\mathbb{D} \circ \mathbb{D}_{\widetilde{T} \cap U}(({}^\dagger \widetilde{T} \cap U)(\mathcal{G})) \xrightarrow{\phi|\mathfrak{U}} ({}^\dagger \widetilde{T} \cap U)(\mathcal{G})).$$

Since Y/S is smooth, then it follows from 16.1.6.3 that $\mathcal{E}|\mathfrak{U} \xrightarrow{\sim} \mathcal{G} \in \mathrm{MIC}^*(Y, \mathfrak{U}/\mathcal{V})$. Hence, $\mathcal{E}|\mathfrak{U}$ is in the essential image of the functor $\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$. Hence, we have checked that $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$ and therefore $\mathcal{E} \in \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V})$.

2) Since $({}^\dagger \widetilde{T})(\phi)$ is an isomorphism, then so is the canonical arrow $\mathcal{E}({}^\dagger \widetilde{T}) \to \widetilde{\mathcal{E}}$. By full faithfulness of $({}^\dagger \widetilde{T}^{(0)})$ (see 16.1.4.1), this yields the isomorphism: $(a^+, ({}^\dagger \widetilde{T})) \circ \mathcal{G}lue(\mathcal{E}^{(0)}, \widetilde{\mathcal{E}}, \rho) = (a^+(\mathcal{E}), \mathcal{E}({}^\dagger \widetilde{T})) \xrightarrow{\sim} (\mathcal{E}^{(0)}, \widetilde{\mathcal{E}}, \rho)$. Since the functor $(a^+, ({}^\dagger \widetilde{T}))$ is fully faithful, this yields the functors $(a^+, ({}^\dagger \widetilde{T}))$ and $\mathcal{G}lue$ are quasi-inverse. □

Remark 16.1.6.9. Following the proof of 16.1.6.8, the functor $\mathcal{G}lue$ of 16.1.6.8.2 factorizes as follows:

$$\mathcal{G}lue\colon \mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^*(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V})} \mathrm{MIC}^*(X, \mathfrak{P}, \widetilde{T} \supset T/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V}).$$

Lemma 16.1.6.10. *Let P be a smooth k-variety, T be a divisor of P, Y be an integral closed subscheme of $P \setminus T$, Y' be a non-empty open subset of Y. Then there exists a divisor $T' \supset T$ such that $Y \setminus T'$ is included in Y' and is non-empty.*

Proof. We can suppose that P is integral. Set $U := P \setminus T$ and let $(U_i)_{i \in I}$ be a finite covering of U with affine open subschemes. Put $Y_i := Y \cap U_i$, $Y'_i := Y' \cap U_i$. Since Y' is dense in Y, then Y'_i is empty if and only if Y_i is empty. Let I' be the set of the elements $i \in I$ such that Y_i is non-empty. Let $i \in I'$. Then there exists a divisor T'_i of U_i such that $Y_i \setminus T'_i$ is included in Y'_i and is non-empty. By setting $Z' := \cup_{i \in I'} T'_i$, this yields that Z' does not contain the generic point of Y. Moreover, for any $i \in I'$, $Y_i \setminus Z' \subset Y_i \setminus T'_i \subset Y'_i$. For any $i \in I \setminus I'$, we have $Y_i \setminus Z' \subset Y'_i$ because they are both empty. Hence, we have checked $Y \setminus Z'$ is not empty and $Y \setminus Z' \subset Y'$. So, it is sufficient to choose T' equal to the union of T and of the closure of Z' in P.

□

Corollary 16.1.6.11. *Let $(Y, X, \mathfrak{P}, T)$ be a smooth d-frame over $\mathfrak{S}$ (see the definition 12.2.1.1). We have the equalities $\mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{**}(X, \mathfrak{P}, T/\mathcal{V}) = \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/\mathcal{V})$. Moreover, the functor $\mathbb{D}_T$ preserves these categories.*

Proof. With the lemma 16.1.2.9 (and the remark 16.1.2.10), we can suppose Y integral and dense in X. By using de Jong 's desingularisation theorem, there exists a diagram of the form 16.1.5.0.1 with $X^{(0)}$ smooth and satisfying the conditions which are required. Indeed, the existence of the diagram 16.1.5.0.1 is straightforward (thanks to de Jong). Moreover, since Y is integral, then it follows from 16.1.6.10 that shrinking $\widetilde{Y}$ if necessary, we can suppose there exists a divisor $\widetilde{T}$ of P such that $\widetilde{Y} = X \setminus \widetilde{T}$. We can therefore use any result of this section. Hence, the equalities follow from the theorem 16.1.6.8 and of the remark 16.1.6.9. Thanks to the second remark of 16.1.2.7, this yields the last assertion. □

16.1.6.12. Following 16.1.2.8 and 16.1.6.11, and since these functors commutes with Frobenius, then we get the functor exact:

$$\theta^* := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{!}_{T',T}[-d_{Y'/Y}] \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V}). \qquad (16.1.6.12.1)$$

16.1.7 The equivalence sp_+: finite and etale (outside overconvergent singularities) case

By using the notations and hypotheses of 16.1.3, suppose moreover $X^{(0)}$ smooth over k.

Definition 16.1.7.1. We define the category $\mathrm{MIC}^{\dagger}(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/\mathcal{V})$ as follows:

(i) The objects are the sheaves $E^{(0)} \in \mathrm{MIC}^{\dagger}(Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}/K)$ together with a glueing datum, i.e., an isomorphism in $\mathrm{MIC}^{*}(Y^{(1)}, X^{(1)}, \mathfrak{P}^{(1)}/K)$ of the form $\epsilon \colon a^{*}_{2K}(E^{(0)}) \xrightarrow{\sim} a^{*}_{1K}(E^{(0)})$ satisfying the cocycle condition: $a^{*}_{13K}(\epsilon) = a^{*}_{12K}(\epsilon) \circ a^{*}_{23K}(\epsilon)$ (see notation 10.2.2.8).

(ii) The morphisms $(E^{(0)}, \epsilon) \to (F^{(0)}, \tau)$ are morphisms $\phi \colon E^{(0)} \to F^{(0)}$ of $\mathrm{MIC}^{\dagger}(Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}/K)$ commuting to glueing data, i.e. such that $\tau \circ a^{*}_{2K}(\phi) = a^{*}_{1K}(\phi) \circ \epsilon$.

Proposition 16.1.7.2. *With the notations of 16.1.7.1 and 16.1.3.1, we have the canonical equivalence of categories of the form* $\mathrm{sp}_{X^{(\bullet)} \hookrightarrow \mathfrak{P}^{(\bullet)}, T^{(\bullet)},+} \colon \mathrm{MIC}^{\dagger}(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K) \cong \mathrm{MIC}^{*}(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$.

Proof. Via the lemma 16.1.2.9 and the remark 16.1.2.10, by noticing that the closure of a irreducible component of $Y^{(0)}$ is still smooth, we reduce to the case where Y (resp. $Y^{(0)}$) is integral and dense in X (resp. $X^{(0)}$). Following the theorem of desingularization of de Jong (of the form 10.4.1.2), we have a surjective, projective, generically finite and etale morphism, $\alpha \colon X'' \to X^{(1)}$ such that X'' is smooth and $\alpha^{-1}(T^{(1)} \cap X^{(1)})$ is a divisor of X''. Then there exists a projective and smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes of the form $f'' \colon \mathfrak{P}'' \to \mathfrak{P}^{(1)}$, a closed immersion $X'' \hookrightarrow \mathfrak{P}''$ such that their composition gives the composite morphism $X'' \to X^{(1)} \hookrightarrow \mathfrak{P}^{(1)}$. Set $T'' := (f'')^{-1}(\widetilde{T}^{(0)})$, $Y'' := X'' \setminus T''$, $\mathfrak{U}'' := \mathfrak{P}'' \setminus T''$, $\beta \colon Y'' \to Y^{(1)}$ the morphism induced by α and $\alpha := (\beta, \alpha, f'') \colon (Y'', X'', \mathfrak{P}'', \widetilde{T}''/\mathcal{V}) \to (Y^{(1)}, X^{(1)}, \mathfrak{P}^{(1)}, \widetilde{T}^{(1)}/\mathcal{V})$.

I) Construction of $\mathrm{sp}_{X^{(\bullet)} \hookrightarrow \mathfrak{P}^{(\bullet)}, T^{(\bullet)},+} \colon \mathrm{MIC}^{\dagger}(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K) \to \mathrm{MIC}^{*}(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$. Let $(E^{(0)}, \epsilon) \in \mathrm{MIC}^{\dagger}(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K)$. Since $X^{(0)}$ is smooth, we can use the functor of 12.2.2.6.1 and we set $\mathcal{E}^{(0)} := \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)},+}(E^{(0)})$. Set $\mathcal{F}_1 := a_1^{+}(\mathcal{E}^{(0)})$ and $\mathcal{F}_2 := a_2^{+}(\mathcal{E}^{(0)})$.

i) Construction of the isomorphism $\varepsilon'' \colon \alpha^{+}(\mathcal{F}_2) \xrightarrow{\sim} \alpha^{+}(\mathcal{F}_1)$.

a) For $i = 1, 2$, since X'' and $X^{(0)}$ are smooth, by commutativity of functors of the form $\mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)},+}$ with inverse images (see 12.2.4.1) and dual functors (see 12.2.5.6), we get the canonical isomorphism $(\alpha \circ a_i)^{+} \circ \mathcal{E}^{(0)} \xrightarrow{\sim} \mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'',+} \circ {}^{\vee} \circ (\alpha \circ a_i)^{*}_{K} \circ {}^{\vee}(E^{(0)})$, where $\vee$ is the dual functor in the categories of overconvergent isocrystals. Moreover, since dual functors and inverse images commute over the category of overconvergent isocrystals, then we get the isomorphism: $(\alpha \circ a_i)^{*}_{K}(E^{(0)}) \xrightarrow{\sim} {}^{\vee} \circ (\alpha \circ a_i)^{*}_{K} \circ {}^{\vee}(E^{(0)})$. Moreover, by transitivity of the inverse image, $\alpha^{+}(\mathcal{F}_i) \xrightarrow{\sim} (\alpha \circ a_i)^{+} \circ \mathcal{E}^{(0)}$. Hence, $\alpha^{+}(\mathcal{F}_i) \xrightarrow{\sim} \mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'',+} \circ (\alpha \circ a_i)^{*}_{K}(E^{(0)})$.

b) By applying α^{*}_{K} to the glueing structural isomorphism $\epsilon \colon a^{*}_{2K}(E^{(0)}) \xrightarrow{\sim} a^{*}_{1K}(E^{(0)})$, we get the isomorphism $\epsilon'' \colon (\alpha \circ a_2)^{*}_{K}(E^{(0)}) \xrightarrow{\sim} (\alpha \circ a_1)^{*}_{K}(E^{(0)})$. This yields the canonical isomorphism $\varepsilon'' \colon \alpha^{+}(\mathcal{F}_2) \xrightarrow{\sim} \alpha^{+}(\mathcal{F}_1)$ making commutative the following diagram:

$$\begin{array}{ccccc} \alpha^{+}(\mathcal{F}_2) & = & \alpha^{+} \circ a_2^{+}(\mathcal{E}^{(0)}) & \xrightarrow{\sim} & \mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'',+} \circ (\alpha \circ a_2)^{*}(E^{(0)}) \\ {\scriptstyle \varepsilon''} \downarrow {\scriptstyle \sim} & & {\scriptstyle \varepsilon''} \downarrow {\scriptstyle \sim} & & {\scriptstyle \sim} \downarrow {\scriptstyle \mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'',+}(\epsilon'')} \\ \alpha^{+}(\mathcal{F}_1) & = & \alpha^{+} \circ a_1^{+}(\mathcal{E}^{(0)}) & \xrightarrow{\sim} & \mathrm{sp}_{X'' \hookrightarrow \mathfrak{P}'', T'',+} \circ (\alpha \circ a_1)^{*}(E^{(0)}) \end{array} \qquad (16.1.7.2.1)$$

whose horizontal isomorphisms are built just above at I.i)a).

ii) Denote by $\widehat{E}^{(0)} := E^{(0)}|]Y^{(0)}[_{\mathfrak{U}^{(0)}}$ and by $\widehat{\epsilon}\colon b_2^*(\widehat{E}^{(0)}) \xrightarrow{\sim} b_1^*(\widehat{E}^{(0)})$ the canonically induced by ϵ isomorphism of $\mathrm{MIC}^\dagger(Y^{(1)}, Y^{(1)}, \mathfrak{U}^{(1)}/K)$. We get the canonical isomorphism $\widehat{\varepsilon}\colon \mathcal{F}_2|\mathfrak{U}^{(1)} \xrightarrow{\sim} \mathcal{F}_1|\mathfrak{U}^{(1)}$ making commutative the diagram below:

$$\begin{array}{ccccc} \mathcal{F}_2|\mathfrak{U}^{(1)} & \xrightarrow{\sim} & b_2^+(\mathcal{E}^{(0)}|\mathfrak{U}^{(0)}) & \xrightarrow{\sim} & \mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}\circ b_2^*(\widehat{E}^{(0)}) \\ {\scriptstyle\widehat{\varepsilon}}\downarrow\wr & & & & \wr\downarrow{\scriptstyle\mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}(\widehat{\epsilon})} \\ \mathcal{F}_1|\mathfrak{U}^{(1)} & \xrightarrow{\sim} & b_1^+(\mathcal{E}^{(0)}|\mathfrak{U}^{(0)}) & \xrightarrow{\sim} & \mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}\circ b_1^*(\widehat{E}^{(0)}) \end{array} \tag{16.1.7.2.2}$$

whose horizontal isomorphisms are built similarly to that of 16.1.7.2.1 (because b_1 and b_2 are morphisms of completely smooth frames and so we can use 12.2.4.1 and 12.2.5.6).

iii) The morphisms ε'', $\widehat{\varepsilon}$ induce canonically the same morphism of the form $\alpha^+(\mathcal{F}_2)|\mathfrak{U}'' \xrightarrow{\sim} \alpha^+(\mathcal{F}_1)|\mathfrak{U}''$. This means that the pair $(\varepsilon'', \widehat{\varepsilon})$ is a morphism of $\mathrm{MIC}^*(X'', \mathfrak{P}'', T''/\mathcal{V}) \times_{\mathrm{MIC}^*(Y'',\mathfrak{U}''/\mathcal{V})} \mathrm{MIC}^*(Y^{(1)}, \mathfrak{U}^{(1)}/\mathcal{V})$. Moreover, following 16.1.5.2 the canonical functor

$$(\alpha^+, |\mathfrak{U}^{(1)})\colon \mathrm{MIC}^*(X^{(1)}, \mathfrak{P}^{(1)}, T^{(1)}/\mathcal{V}) \to \mathrm{MIC}^*(X'', \mathfrak{P}'', T''/\mathcal{V}) \times_{\mathrm{MIC}^*(Y'',\mathfrak{U}''/\mathcal{V})} \mathrm{MIC}^*(Y^{(1)}, \mathfrak{U}^{(1)}/\mathcal{V})$$

is fully faithful. Let $\varepsilon\colon \mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_1$ be the morphism such that $(\alpha^+, |\mathfrak{U}^{(1)})(\varepsilon) = (\varepsilon'', \widehat{\varepsilon})$. By faithfulness of the restriction functor $|\mathfrak{U}^{(2)}$, this isomorphism $\varepsilon\colon a_2^+(\mathcal{E}^{(0)}) \xrightarrow{\sim} a_1^+(\mathcal{E}^{(0)})$ satisfies the cocycle condition. Hence, we have built the functor $\mathrm{sp}_+\colon \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K) \to \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$ by setting $\mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}(E^{(0)}, \epsilon) := (\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(E^{(0)}), \varepsilon)$.

II) Let check that the functor $\mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}$ induces an equivalence of categories.

i) Since $\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}$ is faithful, then so is of $\mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}$. We prove the fullness as follow: Let $(E^{(0)}, \epsilon), (\widetilde{E}^{(0)}, \varepsilon) \in \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K)$. Set $(\mathcal{E}^{(0)}, \varepsilon) := \mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}(E^{(0)}, \epsilon)$, $(\widetilde{\mathcal{E}}^{(0)}, \widetilde{\varepsilon}) := \mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}(\widetilde{E}^{(0)}, \widetilde{\epsilon})$. Let

$$\psi\colon (\mathcal{E}^{(0)}, \varepsilon) \to (\widetilde{\mathcal{E}}^{(0)}, \widetilde{\varepsilon})$$

be a morphism of $\mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$. We keep notation of the step I) of the proof concerning the morphisms ε'', $\widehat{\varepsilon}$, ε, ϵ'', $\widehat{\epsilon}$ induced by ϵ ; similarly by adding tildes. Since $X^{(0)}$ is smooth, then the functor $\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}$ of 12.2.2.6.1 is fully faithful. Let $\phi\colon E^{(0)} \to \widetilde{E}^{(0)}$ be morphism such that $\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(\phi) = \psi$. It remains to prove that ϕ commutes with the respective glueing data. Consider the diagram below:

$$\begin{array}{ccccccc} & & \alpha^+\circ a_2^+(\widetilde{\mathcal{E}}^{(0)}) & \longrightarrow & & & \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}\circ(\alpha\circ a_2)^*(\widetilde{E}^{(0)}) \\ & \nearrow^{\psi} & & & & \nearrow^{\phi} & \\ \alpha^+\circ a_2^+(\mathcal{E}^{(0)}) & \longrightarrow & & & \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}\circ(\alpha\circ a_2)^*(E^{(0)}) & & \downarrow{\scriptstyle\mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}(\widetilde{\epsilon}'')} \\ & & \downarrow{\scriptstyle\widetilde{\varepsilon}''} & & & & \\ {\scriptstyle\varepsilon''}\downarrow & & \alpha^+\circ a_1^+(\widetilde{\mathcal{E}}^{(0)}) & \longrightarrow & \downarrow{\scriptstyle\mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}(\epsilon'')} & \longrightarrow & \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}\circ(\alpha\circ a_1)^*(\widetilde{E}^{(0)}) \\ & \nearrow_{\psi} & & & & \nearrow_{\phi} & \\ \alpha^+\circ a_1^+(\mathcal{E}^{(0)}) & \longrightarrow & & & \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}\circ(\alpha\circ a_1)^*(E^{(0)}), & & \end{array} \tag{16.1.7.2.3}$$

where the horizontal arrows toward the back are the ones canonically induced by ϕ or ψ. Since ψ commutes with glueing data, the left square is commutative. The horizontal squares are commutative by functoriality, while the commutativity of the front and back squares are by definition (see 16.1.7.2.1). Since the five other squares of the cube 16.1.7.2.3 are commutative, then so is the right square. Since the functor $\mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}$ is faithful, this square remains commutative without $\mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}$.

Similarly, using the cube deduced from 16.1.7.2.2 by functoriality, we can check the morphism $\phi|]Y^{(0)}[_{\mathfrak{U}^{(0)}}\colon E^{(0)}|]Y^{(0)}[_{\mathfrak{U}^{(0)}} \to \widetilde{E}^{(0)}|]Y^{(0)}[_{\mathfrak{U}^{(0)}}$ of $\mathrm{MIC}^\dagger(Y^{(0)}, Y^{(0)}, \mathfrak{U}^{(0)}/K)$ commutes with glueing data. By faithfulness of the functor $(\alpha^*, j^{(0)*})$ (see the theorem 10.4.1.1), this yields ϕ commutes to glueing data and we are done.

ii) Let $(\mathcal{E}^{(0)}, \varepsilon) \in \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V})$. Since the functor $\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}$ is essentially surjective, then there exists $E^{(0)} \in \mathrm{MIC}^\dagger(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/\mathcal{V})$ together with an isomorphism

$$\iota\colon \mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(E^{(0)}) \xrightarrow{\sim} \mathcal{E}^{(0)}.$$

Since the functor $\mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}$ is fully faithful, then there exists an unique isomorphism $\epsilon''\colon \alpha_K^*\circ a_{2K}^*(E^{(0)})\xrightarrow{\sim}\alpha_K^*\circ a_{1K}^*(E^{(0)})$ making commutative the following diagram:

$$\begin{array}{ccccc}\alpha^+\circ a_2^+(\mathcal{E}^{(0)}) & \xrightarrow[\iota]{\sim} & \alpha^+\circ a_2^+(\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(E^{(0)})) & \xrightarrow{\sim} & \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}\circ\alpha_K^*\circ a_{2K}^*(E^{(0)}) \\ {\scriptstyle \alpha^+(\varepsilon)}\downarrow\wr & & & & \wr\downarrow{\scriptstyle \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}(\epsilon'')} \\ \alpha^+\circ a_1^+(\mathcal{E}^{(0)}) & \xrightarrow[\iota]{\sim} & \alpha^+\circ a_1^+(\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(E^{(0)})) & \xrightarrow{\sim} & \mathrm{sp}_{X''\hookrightarrow\mathfrak{P}'',T'',+}\circ\alpha_K^*\circ a_{1K}^*(E^{(0)})\end{array} \quad (16.1.7.2.4)$$

whose horizontal right isomorphisms are built as above at I.i)a).

Set $\widehat{E}^{(0)}:=E^{(0)}|]Y^{(0)}[_{\mathfrak{U}^{(0)}}$. Since the functor $\mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}$ is fully faithful, then there exists a unique isomorphism $\widehat{\epsilon}\colon b_2^*(\widehat{E}^{(0)})\xrightarrow{\sim} b_1^*(\widehat{E}^{(0)})$ making commutative the diagram below:

$$\begin{array}{ccccc}b_2^+(\mathcal{E}^{(0)}|\mathfrak{U}^{(0)}) & \xrightarrow[\iota]{\sim} & b_2^+(\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(E^{(0)})|\mathfrak{U}^{(0)}) & \xrightarrow{\sim} & \mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}\circ b_2^*(\widehat{E}^{(0)}) \\ {\scriptstyle \varepsilon|\mathfrak{U}^{(0)}}\downarrow\wr & & & & \wr\downarrow{\scriptstyle \mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}(\widehat{\epsilon})} \\ b_1^+(\mathcal{E}^{(0)}|\mathfrak{U}^{(0)}) & \xrightarrow[\iota]{\sim} & b_1^+(\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}(E^{(0)})|\mathfrak{U}^{(0)}) & \xrightarrow{\sim} & \mathrm{sp}_{Y^{(1)}\hookrightarrow\mathfrak{U}^{(1)},+}\circ b_1^*(\widehat{E}^{(0)})\end{array} \quad (16.1.7.2.5)$$

Following 10.4.1.1, the canonical functor

$$(\alpha_K^*,\,|]Y^{(1)}[_{\mathfrak{U}^{(1)}})\colon \mathrm{MIC}^\dagger(X^{(1)},\mathfrak{P}^{(1)},T^{(1)}/\mathcal{V})\to\mathrm{MIC}^\dagger(Y'',X'',\mathfrak{P}''/\mathcal{V})\times_{\mathrm{MIC}^\dagger(Y'',\mathfrak{U}''/\mathcal{V})}\mathrm{MIC}^\dagger(Y^{(1)},\mathfrak{U}^{(1)}/\mathcal{V})$$

is fully faithful. Let $\epsilon\colon a_{2K}^*(E^{(0)})\xrightarrow{\sim} a_{1K}^*(E^{(0)})$ be the morphism such that $(\alpha_K^*,\,|]Y^{(1)}[_{\mathfrak{U}^{(1)}})(\epsilon)=(\epsilon'',\widehat{\epsilon})$. By faithfulness of the restriction functor $|]Y^{(2)}[_{\mathfrak{U}^{(2)}}$, this isomorphism ϵ satisfies the cocycle condition. By construction of the functor $\mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}$ (see the part I), we can check that ι induces the isomorphism: $\iota\colon \mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}(E^{(0)},\epsilon)\xrightarrow{\sim}(\mathcal{E}^{(0)},\varepsilon)$. Hence, we are done. □

Corollary 16.1.7.3. *With the notations and hypotheses of the section, we have the canonical equivalence of categories* $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}\colon\mathrm{MIC}^\dagger(Y,X,\mathfrak{P}/K)\cong\mathrm{MIC}^*(X,\mathfrak{P},T/\mathcal{V})$.

Proof. By using the lemma 16.1.2.9 and the remark 16.1.2.10, we reduce to the case where Y (resp. $Y^{(0)}$) is integral and dense in X (resp. $X^{(0)}$).

I) Construction of $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$.

Denote by $\mathcal{L}oc\colon\mathrm{MIC}^\dagger(Y,X,\mathfrak{P}/K)\to\mathrm{MIC}^\dagger(Y^{(\bullet)},X^{(\bullet)},\mathfrak{P}^{(\bullet)}/K)$ the canonical functor (built similarly to 16.1.3.2). Following the descent theorem of Shiho (see [Shi07, 7.3]), this functor $\mathcal{L}oc$ is an equivalence of categories. We get the functor $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$ by setting

$$\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}:=\mathcal{G}lue\circ\mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}\circ\mathcal{L}oc.$$

Moreover, following 16.1.7.2 (resp. 16.1.3.6) the functor $\mathrm{sp}_{X^{(\bullet)}\hookrightarrow\mathfrak{P}^{(\bullet)},T^{(\bullet)},+}$ (resp. $\mathcal{G}lue$) is also an equivalence of categories. Then so is the functor $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$.

II) Let's check now that this equivalence is canonical i.e. does not depend up to canonical isomorphism on the morphism $(Y^{(0)},X^{(0)},\mathfrak{P}^{(0)},T^{(0)})\to(Y,X,\mathfrak{P},T)$ of smooth d-frames (see the definition 12.2.1.1) such that $X^{(0)}$ is integral, smooth and whose corresponding conditions of the paragraph 16.1.2.14 are satisfied.

Let $(b',a',f')\colon(Y'^{(0)},X'^{(0)},\mathfrak{P}'^{(0)},T'^{(0)})\to(Y,X,\mathfrak{P},T)$ be a second morphism of smooth d-frames such that $X'^{(0)}$ is integral and smooth, $Y'^{(0)}$ is dense in $X'^{(0)}$ and whose conditions of the paragraph 16.1.2.14 are fulfilled. Concerning this second choice, we keep the similar to 16.1.3 notations by adding primes. We denote by $\mathrm{sp}_{X'^{(\bullet)}\hookrightarrow\mathfrak{P}'^{(\bullet)},T'^{(\bullet)},+}$ the functor defined at 16.1.7.2 by replacing the glueing data over $X^{(0)}$ by that over $X'^{(0)}$. Moreover, in order to distinguish between the two choices, denote by $\mathcal{L}oc'\colon\mathrm{MIC}^*(X,\mathfrak{P},T/\mathcal{V})\to\mathrm{MIC}^*(X'^{(\bullet)},\mathfrak{P}'^{(\bullet)},T'^{(\bullet)}/\mathcal{V})$ and $\mathcal{L}oc'\colon\mathrm{MIC}^\dagger(Y,X,\mathfrak{P}/K)\to\mathrm{MIC}^\dagger(Y'^{(\bullet)},X'^{(\bullet)},\mathfrak{P}'^{(\bullet)}/K)$ the usually denoted by $\mathcal{L}oc$ functors. Similarly, the glueing functor of the second choice is denoted by $\mathcal{G}lue'\colon\mathrm{MIC}^*(X'^{(\bullet)},\mathfrak{P}'^{(\bullet)},T'^{(\bullet)}/\mathcal{V})\to\mathrm{MIC}^*(X,\mathfrak{P},T/\mathcal{V})$.

By setting $\mathrm{sp}'_{X\hookrightarrow\mathfrak{P},T,+}:=\mathcal{G}lue'\circ\mathrm{sp}_{X'^{(\bullet)}\hookrightarrow\mathfrak{P}'^{(\bullet)},T'^{(\bullet)},+}\circ\mathcal{L}oc'$, it is a question of establishing that the two functors $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$ and $\mathrm{sp}'_{X\hookrightarrow\mathfrak{P},T,+}$ are canonically isomorphic.

i) We reduce to the case where $(b',a',f')\colon(Y'^{(0)},X'^{(0)},\mathfrak{P}'^{(0)},T'^{(0)})\to(Y,X,\mathfrak{P},T)$ factors through $(Y'^{(0)},X'^{(0)},\mathfrak{P}'^{(0)},T'^{(0)})\to(Y^{(0)},X^{(0)},\mathfrak{P}^{(0)},T^{(0)})$.

Following de Jong 's desingularisation theorem (of the form 10.4.1.2), there exists a projective, surjective, generically finite and etale morphism $\alpha\colon X''^{(0)}\to X'^{(0)}\times_X X^{(0)}$ with $X''^{(0)}$ smooth. As this latter morphism is projective, then there exists a projective and smooth morphism of the form $q\colon \mathfrak{P}''^{(0)} \to \mathfrak{P}'^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$, a closed immersion $X''^{(0)} \hookrightarrow \mathfrak{P}''^{(0)}$ such that their composition gives the composite morphism $X''^{(0)} \to X'^{(0)} \times_X X^{(0)} \hookrightarrow \mathfrak{P}'^{(0)} \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$. Then we get a morphism $(Y''^{(0)}, X''^{(0)}, \mathfrak{P}''^{(0)}, T''^{(0)}) \to (Y, X, \mathfrak{P}, T)$ which factors through $(Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}) \to (Y, X, \mathfrak{P}, T)$ and $(Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y, X, \mathfrak{P}, T)$.

Since α is generically finite and etale, there exists then a divisor $\widetilde{T}$ containing T such that $\widetilde{Y} := X \setminus \widetilde{T}$ is dense in X and such that the morphism $X''^{(0)} \to X$ is finite and etale outside $\widetilde{T}$. Following 16.1.3.4, the functor $\mathcal{G}lue$ of 16.1.3.3 commute with the functors of the form $({}^\dagger \widetilde{T})$. Moreover, so is the functors of the form $\mathcal{L}oc\colon \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) \to \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K)$ or of the form $\mathrm{sp}_{X^{(\bullet)} \hookrightarrow \mathfrak{P}^{(\bullet)}, T^{(\bullet)},+}$. Since $X \setminus \widetilde{T}$ is dense in Y, then according to 16.1.4.1, the functor extension $({}^\dagger \widetilde{T})$ is fully faithful. Hence, we can suppose the morphism $Y''^{(0)} \to Y$ finite and etale. Hence, we are done.

ii) a) Denote the factorization by $\alpha^{(0)}\colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$. We remark that the morphism $\alpha^{(0)}$ induced the factorisation $\alpha^{(1)}\colon (Y'^{(1)}, X'^{(1)}, \mathfrak{P}'^{(1)}, T'^{(1)}) \to (Y^{(1)}, X^{(1)}, \mathfrak{P}^{(1)}, T^{(1)})$ such that, for $i := 1, 2$, $\alpha^{(0)} \circ a'_i = a_i \circ \alpha^{(1)}$. Let $(E^{(0)}, \epsilon) \in \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K)$. The glueing isomorphism ϵ induces then canonically a glueing isomorphism over $\alpha_K^{(0)*}(E^{(0)})$ that we denote by ϵ'. We get the functor $\mathcal{L}oc^{(0)}\colon \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K) \to \mathrm{MIC}^\dagger(Y'^{(\bullet)}, X'^{(\bullet)}/K)$ by setting $\mathcal{L}oc^{(0)}(E^{(0)}, \epsilon) := (\alpha_K^{(0)*}(E^{(0)}), \epsilon')$. We have moreover the canonical isomorphism $\mathcal{L}oc^{(0)} \circ \mathcal{L}oc \xrightarrow{\sim} \mathcal{L}oc'$. Similarly, we build the functor $\mathcal{L}oc^{(0)}\colon \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V}) \to \mathrm{MIC}^*(X'^{(\bullet)}, \mathfrak{P}'^{(\bullet)}, T'^{(\bullet)}/\mathcal{V})$ and we get the canonical isomorphism $\mathcal{L}oc^{(0)} \circ \mathcal{L}oc \xrightarrow{\sim} \mathcal{L}oc'$.

b) Let $E \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ and $(E^{(0)}, \epsilon) := \mathcal{L}oc(E) \in \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K)$. Since $\alpha^{(0)}$ is a morphism of completely smooth frames, then we have the canonical isomorphism

$$\alpha^{(0)+} \circ \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)},+}(E^{(0)}) \xrightarrow{\sim} \mathrm{sp}_{X'^{(0)} \hookrightarrow \mathfrak{P}'^{(0)}, T'^{(0)},+} \circ \alpha_K^{(0)*}(E^{(0)}).$$

In order to check that this latter isomorphism commutes with glueing data induced by ϵ, by faithfulness of the functor $|\mathfrak{U}'^{(1)}$, we reduce to the case where the divisor T is empty (i.e. $Y = X$), which is straightforward. Hence, we get the isomorphism

$$\mathcal{L}oc^{(0)} \circ \mathrm{sp}_{X^{(\bullet)} \hookrightarrow \mathfrak{P}^{(\bullet)}, T^{(\bullet)},+}(\mathcal{L}oc(E)) \xrightarrow{\sim} \mathrm{sp}_{X'^{(\bullet)} \hookrightarrow \mathfrak{P}'^{(\bullet)}, T'^{(\bullet)},+} \circ \mathcal{L}oc^{(0)}(\mathcal{L}oc(E)). \qquad (16.1.7.3.1)$$

To check that $\mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}(E)$ and $\mathrm{sp}'_{X \hookrightarrow \mathfrak{P}, T,+}(E)$ are canonically isomorphic, it is sufficient to construct a canonical functorial isomorphism of the form $\mathcal{L}oc' \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}(E) \xrightarrow{\sim} \mathcal{L}oc' \circ \mathrm{sp}'_{X \hookrightarrow \mathfrak{P}, T,+}(E)$. Moreover, since $\mathcal{L}oc^{(0)} \circ \mathcal{L}oc \xrightarrow{\sim} \mathcal{L}oc'$ and $\mathcal{L}oc \circ \mathcal{G}lue \xrightarrow{\sim} \mathrm{id}$ (see the proof of 16.1.3.6), the term $\mathcal{L}oc' \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}(E)$ is canonically isomorphic to the left term of 16.1.7.3.1. Similarly, $\mathcal{L}oc' \circ \mathrm{sp}'_{X \hookrightarrow \mathfrak{P}, T,+}(E)$ is canonically isomorphic to the right one.

□

Remark 16.1.7.4. With the notations of 16.1.7.3, when X is smooth, the functor $\mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}$ is by construction (e.g. see the step II) of the proof of 16.1.7.3) canonically isomorphic to that built in the smooth case (see 12.2.2.6.1).

16.1.7.5. With the notations of 16.1.7.3, for any $E \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$, by construction of the functor $\mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}$ (see the step I) of the proof of 16.1.7.3) we have the canonical isomorphism:

$$a^+ \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}(E) \xrightarrow{\sim} \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)},+} \circ a^*(E). \qquad (16.1.7.5.1)$$

Remark 16.1.7.6. With the notations of 16.1.7.3, we will see later that $a^+ = a^!$ (see Theorem 16.1.10.5. But beware that we do not have *by construction* and isomorphism of the form 16.1.7.5.1 where a^+ replaced by $a^!$.

16.1.7.7. Let T' be a divisor containing T. Denote by $Y' := X \setminus T'$, $Y'^{(0)} := X^{(0)} \setminus T'^{(0)}$, $j'\colon Y' \hookrightarrow X$, $j'^{(0)}\colon Y'^{(0)} \hookrightarrow X^{(0)}$ the induced open immersions etc (i.e. we add primes). Consider the diagram

$$\begin{array}{ccccccc}
\mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) & \xrightarrow{\mathcal{L}oc} & \mathrm{MIC}^\dagger(Y^{(\bullet)}, X^{(\bullet)}, \mathfrak{P}^{(\bullet)}/K) & \xrightarrow{\mathrm{sp}_{X^{(\bullet)} \hookrightarrow \mathfrak{P}^{(\bullet)}, T^{(\bullet)},+}} & \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T^{(\bullet)}/\mathcal{V}) & \xrightarrow{\mathcal{G}lue} & \mathrm{MIC}^*(X, \mathfrak{P}, T/\mathcal{V}) \\
\downarrow j'^\dagger & & \downarrow j'^{(0)\dagger} & & \downarrow ({}^\dagger T'^{(0)}) & & \downarrow ({}^\dagger T') \\
\mathrm{MIC}^\dagger(Y', X, \mathfrak{P}/K) & \xrightarrow{\mathcal{L}oc} & ({}^\dagger T'^{(0)})\mathrm{MIC}^\dagger(Y'^{(\bullet)}, X^{(\bullet)}/K) & \xrightarrow{\mathrm{sp}_{X^{(\bullet)} \hookrightarrow \mathfrak{P}^{(\bullet)}, T'^{(\bullet)},+}} & \mathrm{MIC}^*(X^{(\bullet)}, \mathfrak{P}^{(\bullet)}, T'^{(\bullet)}/\mathcal{V}) & \xrightarrow{\mathcal{G}lue} & \mathrm{MIC}^*(X, \mathfrak{P}, T'/\mathcal{V})
\end{array} \qquad (16.1.7.7.1)$$

where $({}^{\dagger}T'^{(0)})$ (resp. $({}^{\dagger}T'^{(0)})$) is the canonical functor induced by $({}^{\dagger}T'^{(0)})$ (resp. $({}^{\dagger}T'^{(0)})$). Since the functors of the form $\mathrm{sp}_{X'^{(0)}\hookrightarrow\mathfrak{P}'^{(0)},T'^{(0)},+}$ (i.e., the functors sp_+ built at 12.2.2.6.1 in the case of a completely smooth d-frame) canonically commute to the functors of the form $j'^{\dagger}$ or $({}^{\dagger}T')$ (this is a special case of 12.2.4.1), then we get the commutativity up to canonical isomorphism of the central square of the diagram 16.1.7.7.1. Since so is two other squares (straightforward for the left square and see 16.1.3.4 for the right one), then we have the canonical isomorphism:

$$({}^{\dagger}T')\circ \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+} \xrightarrow{\sim} \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T',+}\circ j'^{\dagger}. \tag{16.1.7.7.2}$$

16.1.8 The equivalence sp_+

16.1.8.1 (Kedlaya contagiosity phenomena)**.** Let X be a k-variety. Let $j\colon Y\subset X$ and $j'\colon Y'\subset X$ be two open immersions such that Y is k-smooth and Y' is included and dense in Y. The contagiosity theorem of Kedlaya [Ked07, 5.3.7] means that the functor

$$(j'^{\dagger}, j^{*})\colon \mathrm{MIC}^{\dagger}(Y,X/K)\to \mathrm{MIC}^{\dagger}(Y',X/K)\times_{\mathrm{MIC}^{\dagger}(Y',Y/K)}\mathrm{MIC}^{\dagger}(Y,Y/K) \tag{16.1.8.1.1}$$

is an equivalence of categories.

Lemma 16.1.8.2. *With the notations of the section 16.1.5, the functor* $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}$ *built in 16.1.7.3 induces the equivalence of categories:* $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}\colon \mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K)\cong \mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V})$*, where* $\mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K)$ *the essential image of the fully faithful functor* $\widetilde{j}^{\dagger}\colon \mathrm{MIC}^{\dagger}(Y,X/K)\to\mathrm{MIC}^{\dagger}(\widetilde{Y},X/K)$ *(see 10.4.3.1) and the second category is defined at 16.1.6.7.*

Proof. Let $E\in\mathrm{MIC}^{\dagger}(Y,X/K)$, $\widetilde{E}:=\widetilde{j}^{\dagger}E\in\mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K)$, and $\widetilde{\mathcal{E}}:=\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}\widetilde{E}$. Let $G:=E|(Y,Y)\in\mathrm{MIC}^{\dagger}(Y,Y/K)$ be the convergent isocrystal on Y induced by restriction from E, and $\widetilde{G}:=\widetilde{E}|(\widetilde{Y},Y)\in\mathrm{MIC}^{\dagger}(\widetilde{Y},Y/K)$ be the induced restriction by an open immersion. We have the canonical isomorphism $\widetilde{\mathcal{E}}|\mathfrak{U}\xrightarrow{\sim}\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},\widetilde{T}\cap U,+}(\widetilde{G})\overset{16.1.7.7.2}{\xrightarrow{\sim}}({}^{\dagger}\widetilde{T}\cap U)\circ\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}(G)$. Hence, $\widetilde{\mathcal{E}}\in\mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V})$. In other words, we get the commutative (up to canonical isomorphism) diagram below

$$\begin{array}{ccc}\mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K) & \hookrightarrow & \mathrm{MIC}^{\dagger}(\widetilde{Y},X/K)\\ \downarrow{\scriptstyle \mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}} & & {\scriptstyle 16.1.7.3}\downarrow{\scriptstyle \mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}}\\ \mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V}) & \hookrightarrow & \mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}/\mathcal{V}).\end{array} \tag{16.1.8.2.1}$$

Since the bottom, top and right functor of the diagram 16.1.8.2.1 are fully faithful then so is the left one. It remains to prove its essential surjectivity. Let $\widetilde{\mathcal{E}}\in\mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V})$. Since the right functor $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}$ of the diagram 16.1.8.2.1 is essentially surjective (see 16.1.7.3), then there exists an isocrystal $\widetilde{E}$ of $\mathrm{MIC}^{\dagger}(\widetilde{Y},X/K)$ such that $\widetilde{\mathcal{E}}\xrightarrow{\sim}\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}(\widetilde{E})$. Denote by $\widetilde{G}:=\widetilde{E}|(\widetilde{Y},Y)\in\mathrm{MIC}^{\dagger}(\widetilde{Y},Y/K)$. We get $\widetilde{\mathcal{E}}|\mathfrak{U}\xrightarrow{\sim}\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},\widetilde{T}\cap U,+}(\widetilde{G})$. By hypothesis, there exists $\mathcal{G}\in\mathrm{MIC}^{\dagger\dagger}(Y,\mathfrak{U}/\mathcal{V})$ together with an isomorphism of the form $\widetilde{\mathcal{E}}|\mathfrak{U}\xrightarrow{\sim}({}^{\dagger}\widetilde{T}\cap U)(\mathcal{G})$. Following 12.2.2.6.1, there exists $G\in\mathrm{MIC}^{\dagger}(Y,Y,\mathfrak{U}/K)$ together with an isomorphism $\mathcal{G}\xrightarrow{\sim}\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}(G)$. Hence,

$$\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},\widetilde{T}\cap U,+}(\widetilde{G})\xrightarrow{\sim}({}^{\dagger}\widetilde{T}\cap U)(\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},+}(G))\overset{16.1.7.7.2}{\xrightarrow{\sim}}\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},\widetilde{T}\cap U,+}(G|(\widetilde{Y},Y)).$$

Since the functor $\mathrm{sp}_{Y\hookrightarrow\mathfrak{U},\widetilde{T}\cap U,+}$ is fully faithful, we get the isomorphism $\widetilde{E}|(\widetilde{Y},Y)\xrightarrow{\sim}G|(\widetilde{Y},Y)$, i.e. $(\widetilde{E},G)$ is an object of $\mathrm{MIC}^{\dagger}(\widetilde{Y},X/K)\times_{\mathrm{MIC}^{\dagger}(\widetilde{Y},Y/K)}\mathrm{MIC}^{\dagger}(Y,Y/K)$. Hence, by using the theorem of contagiosity of Kedlaya (see 16.1.8.1), $\widetilde{E}$ comes from an isocrystal of $\mathrm{MIC}^{\dagger}(Y,X/K)$, i.e. $\widetilde{E}\in\mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K)$. □

Remark 16.1.8.3. The fully faithful functor $({}^{\dagger}\widetilde{T})$ of 16.1.4.1 factors through:

$$({}^{\dagger}\widetilde{T})\colon \mathrm{MIC}^{*}(X,\mathfrak{P},T/\mathcal{V})\to\mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V}). \tag{16.1.8.3.1}$$

However, this is not obvious that it is essentially surjective (but this is a consequence of the theorem 16.1.8.4 below). The category $\mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V})$ (and not the essential image of 16.1.8.3.1) was defined so that we get almost for free the equivalence of categories of 16.1.8.2.

Theorem 16.1.8.4. *Let $(Y,X,\mathfrak{P},T)$ be a smooth d-frame (see the definition 12.2.1.1). We have a canonical equivalence of categories denoted by* $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}\colon \mathrm{MIC}^{\dagger}(Y,X/K)\cong \mathrm{MIC}^{*}(X,\mathfrak{P},T/K)$.

Proof. With the lemma 16.1.2.9 (and the remark 16.1.2.10), we can suppose Y integral and dense in X. By using de Jong 's desingularisation theorem, there exists a divisor $\widetilde{T}$ containing T and a diagram of the form 16.1.5.0.1 satisfying the required conditions of 16.1.5 and such that moreover $X^{(0)}$ is smooth.

With the notations of the subsection 16.1.5, we define the functor $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$ as the one making commutative the diagram below:

$$\begin{array}{ccc}
\mathrm{MIC}^{\dagger}(Y,X/K) & \xrightarrow{(a^*,\widetilde{j}^{\dagger})} & \mathrm{MIC}^{\dagger}(Y^{(0)},X^{(0)}/K)\times_{\mathrm{MIC}^{\dagger}(\widetilde{Y}^{(0)},X^{(0)}/K)}\mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K)\\
\Big\downarrow{\scriptstyle \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}} & & \Big\downarrow{\scriptstyle \mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}\times\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}}\\
\mathrm{MIC}^{*}(X,\mathfrak{P},T/\mathcal{V}) & \xleftarrow[\mathcal{G}lue]{} & \mathrm{MIC}^{*}(X^{(0)},\mathfrak{P}^{(0)},T^{(0)}/\mathcal{V})\times_{\mathrm{MIC}^{*}(X^{(0)},\mathfrak{P}^{(0)},\widetilde{T}^{(0)}/\mathcal{V})}\mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V}),
\end{array}\tag{16.1.8.4.1}$$

whose bottom functor is the equivalence of categories built at 16.1.6.8.2 in the proof of Theorem 16.1.6.8. Following 10.4.3.2 (resp. 16.1.8.2, resp. 12.2.2.6.1), so is the top functor $(a^*,\widetilde{j}^{\dagger})$ (resp. the right functor $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}$, resp. the right functor $\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}$). This yields the left functor of 16.1.8.4.1 is an equivalence of categories.

It remains to prove that this functor does not depend on the choice of the a diagram of the form 16.1.5.0.1 and satisfying the required conditions. First, lLet us consider the independance in $\widetilde{T}$. Let $\widetilde{T}'$ be a second divisor such that $a^{-1}(X\setminus\widetilde{T}')\to(X\setminus\widetilde{T}')$ is finite and etale. By considering $\widetilde{T}\cup\widetilde{T}'$ if necessary, we can suppose $\widetilde{T}'\supset\widetilde{T}$. Denote by $\widetilde{j}'\colon X\setminus\widetilde{T}'\subset X$ the canonical inclusion. Following 16.1.7.7.2, we have the canonical isomorphisms $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T}',+}\circ\widetilde{j}'^{\dagger}\xrightarrow{\sim}({}^{\dagger}\widetilde{T}')\circ\mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}$. Moreover, following 16.1.6.8 (and its proof), we have the canonical isomorphisms $\mathcal{L}oc\circ\mathcal{G}lue\xrightarrow{\sim}\mathrm{id}$, with is $\mathcal{L}oc=(a^+,({}^{\dagger}\widetilde{T}))$ or $\mathcal{L}oc=(a^+,({}^{\dagger}\widetilde{T}'))$ and the quasi-inverse functor $\mathcal{G}lue$ (see 16.1.6.8.2). We get the canonical isomorphism $\mathcal{G}lue\circ(\mathrm{id}\times({}^{\dagger}\widetilde{T}'))\xrightarrow{\sim}\mathcal{G}lue$. Using the canonical isomorphisms of 16.1.7.7.2 and 12.2.4.1.2, this yields the independance from $\widetilde{T}$.

Denote by $\eta=(b,a,f)\colon(Y^{(0)},X^{(0)},\mathfrak{P}^{(0)})\to(Y,X,\mathfrak{P},T)$ the morphism of d-frames that we have chosen and make a second choice: Let $\eta'=(b',a',f')\colon(Y'^{(0)},X'^{(0)},\mathfrak{P}'^{(0)})\to(Y,X,\mathfrak{P},T)$ be a morphism of d-frames such that denoting by $\widetilde{Y}'^{(0)}:=Y'^{(0)}\setminus f'^{-1}(\widetilde{T})$, we get a diagram of the form 16.1.5.0.1 diagram where we replace "(0)" by "$\prime$(0)" and which satisfies the required conditions of 16.1.5. Suppose moreover $X'^{(0)}$ is smooth. Similarly to in the step II.i) of the proof of 16.1.7.3, desingularising $X'^{(0)}\times_X X^{(0)}$ (by using de Jong theorem) and increasing $\widetilde{T}$ if necessary, we reduce to suppose that we have the strict morphisms of smooth d-frames $\tau\colon(Y'^{(0)},X'^{(0)},\mathfrak{P}'^{(0)},T'^{(0)})\to(Y^{(0)},X^{(0)},\mathfrak{P}^{(0)},T^{(0)})$ and $\widetilde{\tau}\colon(\widetilde{Y}'^{(0)},X'^{(0)},\mathfrak{P}'^{(0)},\widetilde{T}'^{(0)})\to(\widetilde{Y}^{(0)},X^{(0)},\mathfrak{P}^{(0)},\widetilde{T}^{(0)})$ (which factors the diagram 16.1.5.0.1 where we replace "(0)" by "$\prime$(0)"). We have the canonical isomorphism $\eta^+\circ\mathrm{sp}_{X^{(0)}\hookrightarrow\mathfrak{P}^{(0)},T^{(0)},+}\xrightarrow{\sim}\mathrm{sp}_{X'^{(0)}\hookrightarrow\mathfrak{P}'^{(0)},T'^{(0)},+}\circ\tau^*$ (use the commutation with extraordinary inverse image and dual functors of 12.2.4.1 12.2.5.6). Since $\mathcal{L}oc\circ\mathcal{G}lue\xrightarrow{\sim}\mathrm{id}$ with $\mathcal{L}oc=(\eta^+,({}^{\dagger}\widetilde{T}))$ or $\mathcal{L}oc=(\eta'^+,({}^{\dagger}\widetilde{T}))$, we check the canonical isomorphism $\mathcal{G}lue\circ(\eta^+\times\mathrm{id})\xrightarrow{\sim}\mathcal{G}lue$. By using 12.2.4.1.2, we are done. □

16.1.8.5. It follows from the commutative up to canonical isomorphism diagram 16.1.8.4.1 that we get the commutative up to canonical isomorphism diagram:

$$\begin{array}{ccc}
\mathrm{MIC}^{\dagger}(Y,X/K) & \xrightarrow[\cong]{\widetilde{j}^{\dagger}} & \mathrm{MIC}^{\dagger}(\widetilde{Y}\subset Y,X/K)\\
\cong\Big\downarrow{\scriptstyle \mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}} & & \cong\Big\downarrow{\scriptstyle \mathrm{sp}_{X\hookrightarrow\mathfrak{P},\widetilde{T},+}}\\
\mathrm{MIC}^{*}(X,\mathfrak{P},T/\mathcal{V}) & \xrightarrow{({}^{\dagger}\widetilde{T})} & \mathrm{MIC}^{*}(X,\mathfrak{P},\widetilde{T}\supset T/\mathcal{V}).
\end{array}\tag{16.1.8.5.1}$$

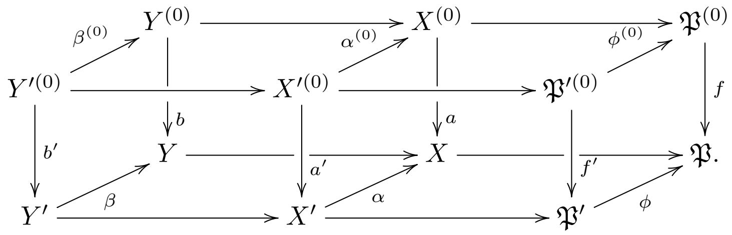
This implies that the bottom functor $({}^{\dagger}\widetilde{T})$ is an equivalence of categories.

Before establishing via the proposition 16.1.11.6 that this equivalence of categories commutes with the duality, Let us check first its commutation to inverse images via the proposition 16.1.8.6:

Proposition 16.1.8.6. *Let $\theta\colon(Y',X',\mathfrak{P}',T')\to(Y,X,\mathfrak{P},T)$ be a morphism of smooth d-frames. We have the canonical isomorphism:*

$$\mathrm{sp}_{X'\hookrightarrow\mathfrak{P}',T',+}\circ\theta^*\xrightarrow{\sim}\theta^+\circ\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}.\tag{16.1.8.6.1}$$

Proof. Set $\theta = (\beta, \alpha, \phi)$. With the lemma 16.1.2.9 (and the remark 16.1.2.10), we reduce to the case where Y (resp. Y') is integral and dense in X (resp. X'). When X and X' are smooth, this theorem of commutation is already well known (see 12.2.4.1.2). To obtain the general case, the idea is to come down to the smooth case thanks to theorem of full faithfulness 16.1.5.2 as follows: by using de Jong 's desingularisation theorem, there exists a morphism of smooth d-frames of the form $\eta = (b, a, f) \colon (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}) \to (Y, X, \mathfrak{P}, T)$ such that $X^{(0)}$ is smooth, f is projective and smooth, a is projective, surjective, generically finite and etale, $f^{-1}(T) = T^{(0)}$ and $T^{(0)} \cap X^{(0)}$ is a (strict normal crossing) divisor of $X^{(0)}$. Similarly by using the de Jong 's desingularisation theorem of the form 10.4.1.2 (applied to $X' \times_X X^{(0)}$ element of $(Y', X', \mathfrak{P}', T') \times_{(Y,X,\mathfrak{P},T)} (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$ and to its divisor complementary to $Y' \times_Y Y^{(0)}$), we build a frame $(Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)})$ smooth with $X'^{(0)}$ smooth and two morphisms of the form $\eta' = (b', a', f') \colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y', X', \mathfrak{P}', T')$ and $\theta^{(0)} = (\beta^{(0)}, \alpha^{(0)}, \phi^{(0)}) \colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$ such that $\theta \circ \eta' = \eta \circ \theta^{(0)}$ and such that f' is projective and smooth, a' is projective, surjective, generically finite and etale, $(f')^{-1}(T') = T'^{(0)}$ and $T'^{(0)} \cap X'^{(0)}$ is a (strict normal crossing) divisor of $X'^{(0)}$. The diagram illustrates our notation:



(16.1.8.6.2)

By full faithfulness of the localisation outside a divisor functor (see the theorem 16.1.4.1), increasing T and T' if necessary, it follows from 16.1.8.5 that we reduce to the case where b and b' are finite and etale. Set $\mathfrak{U} := \mathfrak{P} \setminus T$ and $\mathfrak{U}' := \mathfrak{P}' \setminus T'$ and let $\widehat{\theta} \colon (Y', Y', \mathfrak{U}') \to (Y, Y, \mathfrak{U})$ be the morphism of smooth d-frames induced by θ. Following the case of the smooth partial compactification (use 12.2.4.1 and 12.2.5.6), we have the canonical isomorphism $\mathrm{sp}_{Y' \hookrightarrow \mathfrak{U}',+} \circ \widehat{\theta}^* \xrightarrow{\sim} \widehat{\theta}^+ \circ \mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$. Hence:

$$|\mathfrak{U}' \circ \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}',T',+} \circ \theta^* \xrightarrow{\sim} \mathrm{sp}_{Y' \hookrightarrow \mathfrak{U}',+} \circ \widehat{\theta}^* \circ |\mathfrak{U} \xrightarrow{\sim} \widehat{\theta}^+ \circ \mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+} \circ |\mathfrak{U} \xrightarrow{\sim} |\mathfrak{U}' \circ \theta^+ \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P},T,+}. \quad (16.1.8.6.3)$$

Moreover, following 16.1.5.2, the functor $(\eta'^+, |\mathfrak{U}')$ is fully faithful. Hence, it remains then to build a compatible with 16.1.8.6.3 isomorphism of the form $\eta'^+ \circ \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}',T',+} \circ \theta^* \xrightarrow{\sim} \eta'^+ \circ \theta^+ \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P},T,+}$. Since $X^{(0)}$ and $X'^{(0)}$ are smooth, then $\mathrm{sp}_{X'^{(0)} \hookrightarrow \mathfrak{P}'^{(0)},T'^{(0)},+} \circ (\theta^{(0)})^* \xrightarrow{\sim} (\theta^{(0)})^+ \circ \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)},T^{(0)},+}$ (use 12.2.4.1 and 12.2.5.6). Moreover, following 16.1.7.5, since b and b' are finite and etale, then we have the isomorphisms $\eta^+ \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P},T,+} \xrightarrow{\sim} \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)},T^{(0)},+} \circ \eta^*$ and $\eta'^+ \circ \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}',T',+} \xrightarrow{\sim} \mathrm{sp}_{X'^{(0)} \hookrightarrow \mathfrak{P}'^{(0)},T'^{(0)},+} \circ \eta'^*$. This yields the canonical isomorphisms

$$\begin{aligned} \eta'^+ \circ \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}',T',+} \circ \theta^* &\xrightarrow{\sim} \mathrm{sp}_{X'^{(0)} \hookrightarrow \mathfrak{P}'^{(0)},T'^{(0)},+} \circ \eta'^* \circ \theta^* \\ &\xrightarrow{\sim} \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)},T^{(0)},+} \circ \theta^{(0)*} \circ \eta^* \xrightarrow{\sim} \theta^{(0)+} \circ \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)},T^{(0)},+} \circ \eta^* \\ &\xrightarrow{\sim} \theta^{(0)+} \circ \eta^{(0)+} \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P},T,+} \xrightarrow{\sim} \eta^{(0)+} \circ \theta^+ \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P},T,+}. \end{aligned} \quad (16.1.8.6.4)$$

Hence we are done. □

16.1.9 Canonical independance in completely smooth d-frame

In this section, when X is smooth we study the independance in $\mathfrak{P}$ and T of the category $(F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$.

Lemma 16.1.9.1. *Let $\mathfrak{P}$ be a smooth $\mathfrak{S}$-formal scheme, $T \subset T'$ two divisors of P and $\mathcal{E}$ a coherent $(F\text{-})\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module.*

(a) There exists a structure of coherent $(F\text{-})\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T')_{\mathbb{Q}}$-module on $\mathcal{E}$ extending its structure of $(F\text{-})\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module.

(b) The canonical morphism $\mathcal{E} \to \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T')_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}$ is an isomorphism.

In particular, the existence of such a structure is local on $\mathfrak{P}$ and such coherent (F-)$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T')_{\mathbb{Q}}$-module is unique if it exists.

Proof. This follows from 8.7.6.11. □

Lemma 16.1.9.2. *Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, X, X' be two smooth closed subschemes of P, T, T' be two divisors of P such that $T \cap X = T' \cap X'$. Let $\natural \in \{\mathrm{coh}, \mathrm{ovcoh}, \mathrm{oc}\}$. We get then the equalities: $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T) = (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X', \mathfrak{P}, T')$ and $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/K) = (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}, T'/K)$*

Proof. Since the other equality is checked similarly, let us only prove the equality $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T) = (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X', \mathfrak{P}, T')$. 1) Suppose $T = T'$. Let $\mathcal{E}^{(\bullet)} \in (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T)$. We have to prove that $\mathcal{E}^{(\bullet)}$ has support in X'. We have to check that the canonical morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is an isomorphism (see 13.1.4.8). Since $\mathcal{E}^{(\bullet)} \to ({}^{\dagger}T)\mathcal{E}^{(\bullet)}$ is an isomorphism, then we get the isomorphisms

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X}({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \underset{13.1.5.1.1}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X'}(\mathcal{E}^{(\bullet)}). \tag{16.1.9.2.1}$$

2) Suppose $X = X'$ and $T \subset T'$. Since this is of local nature in $\mathfrak{P}$, we can suppose $\mathfrak{P}$ affine and X is integral. Let us choose then $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ is a closed immersion of smooth formal $\mathfrak{S}$-schemes which is a lifting of $X \hookrightarrow P$. Either $T \cap X$ is a divisor of X or T contains X. When T contains X, the considered categories are reduced to the zero object (this is a consequence of 8.7.6.11). Suppose now that $Z := T \cap X$ is a divisor of X.

Let $\mathcal{E}^{(\bullet)}$ be an object of $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T)$. Following the Berthelot-Kashiwara theorem (see 9.3.5.13 or respectively 15.3.8.26), the functor $u^{(\bullet)}_{T+}$ (resp. $u^{(\bullet)}_{T'+}$) induces an equivalence of categories between $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{X}, Z)$ and $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T)$ (resp. $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T')$). Following 9.2.4.19, the functors $u^{(\bullet)}_{T+}$ and $\mathrm{forg}_{T,T'} \circ u^{(\bullet)}_{T'+}$ are isomorphic. Hence, $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T) = (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T')$.

3) Since $X \setminus T = X \setminus (T \cup T') = X' \setminus (T \cup T') = X' \setminus T'$, then to check the general case we deduce to the two preceding cases.

□

Lemma 16.1.9.3. *Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of separated and smooth $\mathfrak{S}$-formal schemes, X be a smooth closed subscheme of P' such that the morphism induced $X \to P$ is a closed immersion, Y an open set of X, T be a divisor of P (resp. T' be a divisor of P') such that $Y = X \setminus T$ (resp. $Y = X \setminus T'$). Let $\natural \in \{\mathrm{coh}, \mathrm{ovcoh}, \mathrm{oc}\}$.*

(a) For any $\mathcal{E}^{(\bullet)} \in (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T)$, for any $\mathcal{E}'^{(\bullet)} \in (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}', T')$, for any $j \in \mathbb{Z} \setminus \{0\}$,

$$H^j(\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) = 0, H^j(f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)})) = 0.$$

(b) The functors $\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}$ and $f^{(\bullet)}_{+}$ induce then quasi-inverse equivalences between the categories $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T)$ and $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}', T')$ (resp. between $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/K)$ and $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}', T'/K)$).

Proof. In order to lighten notations, the case with Frobenius structure being similar, we avoid indicating "(F-)" in all categories. It is harmless to suppose P and P' integral. Similarly, since X is smooth and since modules have support in X, then we reduce to the case where X integral. We distinguish then two case: either Y is empty or $T \cap X = T' \cap X$ is a divisor of X. The first case is straightforward since the categories are in this case null. Let us consider now the second case. In this case $f^{-1}(T)$ is a divisor of P'. By 16.1.9.2, since $f^{-1}(T) \cap X = T' \cap X$, then we can suppose $T' = f^{-1}(T)$.

Let us fix $(\mathfrak{P}_{\alpha})_{\alpha \in \Lambda}$ an affine open covering of $\mathfrak{P}$. We denote by $X_{\alpha} := X \cap P_{\alpha}$, $\mathfrak{P}'_{\alpha} := f^{-1}(\mathfrak{P}_{\alpha})$, For any $\alpha \in \Lambda$, let us choose $\mathfrak{X}_{\alpha}$ some smooth formal $\mathfrak{S}$-schemes which is a lifting of X_{α} Let $u'_{\alpha}\colon \mathfrak{X}_{\alpha} \to \mathfrak{P}'_{\alpha}$ some liftings of $X_{\alpha} \to P'_{\alpha}$ We denote by $f_{\alpha}\colon \mathfrak{P}'_{\alpha} \to \mathfrak{P}_{\alpha}$ the morphism induced by f. We set $u_{\alpha} := f_{\alpha} \circ u'_{\alpha}\colon \mathfrak{X}_{\alpha} \to \mathfrak{P}_{\alpha}$. Remark when we will have to check the commutations to glueing data, we have to fix other liftings (e.g. of $X_{\alpha} \cap X_{\beta}$, $X_{\alpha} \cap X_{\beta} \cap X_{\gamma}$) similarly to 9.3.7, but we leave the details to the reader as an exercise. Since the overcoherence (after any base change) is stable under proper pushforward and extraordinary pullbacks, then Theorem 9.3.7.12 still holds replacing the coherence by the overcoherence (after any base change).

1) i) We have the canonical isomorphisms:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})|\mathfrak{P}'_{\alpha} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{X_\alpha} f^{(\bullet)!}_{\alpha}(\mathcal{E}^{(\bullet)}|\mathfrak{P}_{\alpha}) \underset{\sim}{\overset{13.2.1.5.1}{\longrightarrow}} u'^{(\bullet)}_{\alpha+} \circ u'^{(\bullet)!}_{\alpha} \circ f^{(\bullet)!}_{\alpha}(\mathcal{E}^{(\bullet)}|\mathfrak{P}_{\alpha}) \xrightarrow{\sim} u'^{(\bullet)}_{\alpha+} \circ u^{(\bullet)!}_{\alpha}(\mathcal{E}^{(\bullet)}|\mathfrak{P}_{\alpha}). \tag{16.1.9.3.1}$$

Since this is local on $\mathfrak{P}'$, since $\mathcal{E}^{(\bullet)}$ has in support in X, then by using Berthelot-Kashiwara theorem (see 9.3.5.13 or respectively 15.3.8.26) this yields, for any integer $j \neq 0$, we have $H^j(\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) = 0$ and $\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}', T')$.

ii) The equivalence of categories $u_0^{(\bullet)!} \colon \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T) \cong \underrightarrow{LM}_{\mathbb{Q},\natural}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T \cap X)$ of 9.3.7.12.(a) is defined by setting $u_0^{(\bullet)!}(\mathcal{E}^{(\bullet)}) = (u_\alpha^{(\bullet)!}(\mathcal{E}^{(\bullet)}|\mathfrak{P}_\alpha))_\alpha$, this latter is endowed with its canonical glueing data. Similarly with of primes. Moreover, by applying $u'^{(\bullet)!}_{\alpha} \circ \underrightarrow{l}^{*}_{\mathbb{Q}}$ to 16.1.9.3.1, via the theorem Berthelot-Kashiwara (see 9.3.5.13 or respectively 15.3.8.26), we get the canonical isomorphism $u'^{(\bullet)!}_{\alpha}(\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})|\mathfrak{P}'_\alpha) \xrightarrow{\sim} u^{(\bullet)!}_{\alpha}(\mathcal{E}^{(\bullet)}|\mathfrak{P}_\alpha)$, this isomorphism commuting to the respective glueing data. Then we have the commutative diagram (up to canonical isomorphism):

$$\begin{array}{ccc} \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}, T) & \underset{\cong}{\xrightarrow{u_0^{(\bullet)!}}} & \underrightarrow{LM}_{\mathbb{Q},\natural}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T \cap X) \\ \downarrow{\scriptstyle \mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}} & & \| \\ \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}', T') & \underset{\cong}{\xrightarrow{u'^{(\bullet)!}_0}} & \underrightarrow{LM}_{\mathbb{Q},\natural}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T' \cap X). \end{array} \tag{16.1.9.3.2}$$

2) i) Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\natural}(X, \mathfrak{P}', T')$. Following the theorem of Berthelot-Kashiwara (see 9.3.5.13 or respectively 15.3.8.26), since $\mathcal{E}'^{(\bullet)}|\mathfrak{P}'_\alpha$ has its support in X_α, then $u'^{!}_{\alpha}(\mathcal{E}'^{(\bullet)}|\mathfrak{P}'_\alpha) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}_\alpha}(T' \cap X_\alpha))$ and we have the isomorphism: $\mathcal{E}'^{(\bullet)}|\mathfrak{P}'_\alpha \xrightarrow{\sim} u'^{(\bullet)}_{\alpha+} \circ u'^{(\bullet)!}_{\alpha}(\mathcal{E}'^{(\bullet)}|\mathfrak{P}'_\alpha)$. Hence we get the isomorphism $f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)})|\mathfrak{P}_\alpha \xrightarrow{\sim} f^{(\bullet)}_{\alpha+}(\mathcal{E}'^{(\bullet)}|\mathfrak{P}'_\alpha) \xrightarrow{\sim} u^{(\bullet)}_{\alpha+} \circ u'^{!}_{\alpha}(\mathcal{E}'^{(\bullet)}|\mathfrak{P}'_\alpha)$. Since this is local on $\mathfrak{P}$, this yields, for $j \neq 0$, $H^j(f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)})) = 0$ and $f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}, T)$.

ii) The equivalence of categories $u'^{(\bullet)}_{0+} \colon \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{P}_\alpha)_{\alpha\in\Lambda}, T' \cap X) \cong \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}', T')$ of b is defined by $(\mathcal{E}^{(\bullet)}_{\alpha})_\alpha \mapsto (u'^{(\bullet)}_{\alpha+}(\mathcal{E}^{(\bullet)}_{\alpha}))_\alpha$, where $(u'^{(\bullet)}_{\alpha+}(\mathcal{E}^{(\bullet)}_{\alpha}))_\alpha$ is endowed with a canonical glueing data. Similarly without primes. This yields the commutative canonical diagram (up to canonical isomorphism):

$$\begin{array}{ccc} \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T \cap X) & \underset{\cong}{\xrightarrow{u_{0+}^{(\bullet)}}} & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}, T) \\ \| & & \uparrow{\scriptstyle f_+^{(\bullet)}} \\ \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T' \cap X) & \underset{\cong}{\xrightarrow{u'^{(\bullet)}_{0+}}} & \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}', T'). \end{array} \tag{16.1.9.3.3}$$

3) Since the functors $u_0^{(\bullet)!}$ and $u_{0+}^{(\bullet)}$ are quasi-inverse (see 9.3.7.12), via the commutative diagrams (up to canonical isomorphism) 16.1.9.3.2 and 16.1.9.3.3, this yields the functors $\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}$ and $f_+^{(\bullet)}$ induce quasi-inverse equivalences between the categories $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}, T)$ and $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}', T')$.

4) Concerning the equivalence of categories between $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/K)$ and $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}', T'/K)$, we proceed in the same way: We replace respectively in the proof the category $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T \cap X)$ by $\mathrm{MIC}^{(\bullet)}((\mathfrak{X}_\alpha)_{\alpha\in\Lambda}, T \cap X/\mathcal{V})$ and the category $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(X, \mathfrak{P}, T)$ by $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$ and similarly with primes (and replace the use of 9.3.7.12 by that of 12.2.2.5). □

Notation 16.1.9.4. Let $f \colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of separated and smooth $\mathfrak{S}$-formal schemes, X be a smooth closed subscheme of P' such that the morphism induced $X \to P$ is a closed immersion, Y an open set of X, T be a divisor of P (resp. T' be a divisor of P') such that $Y = X \setminus T$ (resp. $Y = X \setminus T'$).

Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ such that $\underrightarrow{l}^{*}_{\mathbb{Q}}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$ (see notation 9.1.6.1). Then we write

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X} f^{(\bullet)!}(\mathcal{E}) := \underrightarrow{l}^{*}_{\mathbb{Q}} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}).$$

Beware that since $f^!(\mathcal{E})$ is not in general coherent (if f is not smooth and $\mathcal{E}$ is not overcoherent), then $\mathbb{R}\underline{\Gamma}^{\dagger}_{X}(f^!(\mathcal{E}))$ has a priori no meaning.

Corollary 16.1.9.5. *We keep notation 16.1.9.4.*

(a) For any $\mathcal{E} \in (F\text{-})\mathrm{Coh}(X, \mathfrak{P}, T)$, for any $\mathcal{E}' \in (F\text{-})\mathrm{Coh}(X, \mathfrak{P}', T')$, for any $j \in \mathbb{Z} \setminus \{0\}$,

$$H^j(\mathbb{R}\underline{\Gamma}^\dagger_X f^!(\mathcal{E})) = 0, H^j(f_+(\mathcal{E}') = 0.$$

(b) The functors $\mathbb{R}\underline{\Gamma}^\dagger_X f^!$ and f_+ induce then quasi-inverse equivalences between the categories $(F\text{-})\mathrm{Coh}(X, \mathfrak{P}, T)$ and $(F\text{-})\mathrm{Coh}(X, \mathfrak{P}', T')$ (resp. between $(F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$ and $(F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}', T'/K)$).

Remark 16.1.9.6. The lemma 16.1.9.3 will have a variation at 16.2.7.6 in the case of isocrystals.

16.1.10 Isomorphism between inverse images and extraordinary inverse images of partially overcoherent isocrystals

Lemma 16.1.10.1. *Consider the commutative diagram below*

$$\begin{array}{ccc} \mathfrak{P}'^{(0)} & \xrightarrow{\phi^{(0)}} & \mathfrak{P}^{(0)} \\ \downarrow f' & & \downarrow f \\ \mathfrak{P}' & \xrightarrow{\phi} & \mathfrak{P}, \end{array} \qquad (16.1.10.1.1)$$

where ϕ and $\phi^{(0)}$ are proper morphisms of smooth formal $\mathfrak{S}$-schemes, f and f' are smooth morphisms of smooth formal $\mathfrak{S}$-schemes. Let T be a divisor of P such that $T' := \phi^{-1}(T)$ is a divisor of P'. We set $T^{(0)} := f^{-1}(T)$ and $T'^{(0)} := f'^{-1}(T')$. Let $\mathcal{E}'^{(\bullet)} \in D^{\mathrm b}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T')_{\mathbb{Q}})$. Hence, we get the canonical adjunction morphisms: $\phi^{(0)}_+ \circ f'^!(\mathcal{E}'^{(\bullet)}) \to f^! \circ \phi_+(\mathcal{E}'^{(\bullet)})$ and $f^+ \circ \phi^{(0)}_+(\mathcal{E}'^{(\bullet)}) \to \phi_+ \circ f'^+(\mathcal{E}'^{(\bullet)})$. Moreover, when the diagram 16.1.10.1.1 is cartesian, these two morphisms are isomorphisms.

Proof. By using the isomorphism of relative duality of 9.4.5.2.1, we construct the second adjunction morphism from the first one by duality. Hence, we reduce to treat the first adjunction morphism. Let $\mathfrak{P}'' := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$, $\iota \colon \mathfrak{P}'^{(0)} \to \mathfrak{P}''$, $f'' \colon \mathfrak{P}'' \to \mathfrak{P}'$ and $\phi'' \colon \mathfrak{P}'' \to \mathfrak{P}^{(0)}$ the canonical morphisms. Since ϕ'' and $\phi^{(0)}$ are proper, ι then so is. Hence, we have the adjunction morphism $\iota_+ \circ \iota^! \to \mathrm{id}$ (see 9.4.5.5).

Since f'' is smooth, then the functor $f''^{(\bullet)!}$ preserves the coherence (see 9.4.1.7). Hence, via the equivalence of categories $\underrightarrow{l}^*_{\mathbb{Q}}$ of 8.7.5.4 (i.e. of the form $\underrightarrow{LD}^{\mathrm b}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T)) \cong D^{\mathrm b}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S}^\sharp}(T)_{\mathbb{Q}})$) we get the transitivity isomorphism: $f'^!(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \iota^! \circ f''^!(\mathcal{E}'^{(\bullet)})$ (see 9.2.1.14 and 9.2.1.24). Since f' is smooth, this yields that $\iota^! \circ f''^!(\mathcal{E}'^{(\bullet)})$ is coherent. Since ι is proper, following the proposition 13.2.3.7 this yields we have the canonical isomorphism $\iota_+ \circ \iota^! \circ f''^!(\mathcal{E}'^{(\bullet)}) \to f''^!(\mathcal{E}'^{(\bullet)})$. Hence, we get:

$$\phi^{(0)}_+ \circ f'^!(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \phi''_+ \circ \iota_+ \circ \iota^! \circ f''^!(\mathcal{E}'^{(\bullet)}) \to \phi''_+ \circ f''^!(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f^! \circ \phi_+(\mathcal{E}'^{(\bullet)}).$$

When the diagram 16.1.10.1.1 is cartesian, ι is an isomorphism and the adjunction morphism $\iota_+ \circ \iota^! \to \mathrm{id}$ is an isomorphism. Hence, we get the last assertion. □

Notation 16.1.10.2 (Finite and etale outside singularities case)**.** Let $\theta = (\beta, \alpha, \phi) \colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$, $\eta = (b, a, f) \colon (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}) \to (Y, X, \mathfrak{P}, T)$, $\theta^{(0)} = (\beta^{(0)}, \alpha^{(0)}, \phi^{(0)}) \colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$ and $\eta' = (b', a', f') \colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y', X', \mathfrak{P}', T')$ four morphisms of smooth d-frames such that $\theta \circ \eta' = \eta \circ \theta^{(0)}$. We get then the rectangular parallelepiped of 16.1.8.6.2.

In order to give a meaning to the notion of the direct image (on the formal side), we suppose that the four rectangle of 16.1.8.6.2 (i.e., the faces of the front side, of the back side, of the top, bottom) satisfy the hypotheses of 16.1.2.14.1, i.e. they are "finite and etale outside singularities morphisms of d-frames"

Lemma 16.1.10.3. *With the notations and hypotheses of 16.1.10.2, for any $\mathcal{E}' \in \mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/K)$, we have the functorial in $\mathcal{E}'$ canonical morphisms:*

$$\theta^{(0)}_+ \circ \eta'^!(\mathcal{E}') \to \eta^! \circ \theta_+(\mathcal{E}'^{(\bullet)}), \qquad \eta^+ \circ \theta_+(\mathcal{E}'^{(\bullet)}) \to \theta^{(0)}_+ \circ \eta'^+(\mathcal{E}'). \qquad (16.1.10.3.1)$$

When the left face of 16.1.8.6.2 is cartesian, both morphisms are isomorphisms.

Proof. Via the isomorphism of relative duality (see 9.4.5.2.1) and of biduality (see 8.7.7.3), the second morphism is build by applying the dual functor to the first. We define the first morphism as equal to the composition:

$$\theta^{(0)}_+ \circ \eta'^!(\mathcal{E}') = \phi^{(0)}_+ \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'^{(0)}} \circ f'^!(\mathcal{E}') \to \phi^{(0)}_+ \circ \mathbb{R}\underline{\Gamma}^\dagger_{(\phi^{(0)})^{-1}(X^{(0)})} \circ f'^!(\mathcal{E}') \underset{13.2.1.4.2}{\xrightarrow{\sim}}$$
$$\mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \circ \phi^{(0)}_+ \circ f'^!(\mathcal{E}') \underset{16.1.10.1}{\longrightarrow} \mathbb{R}\underline{\Gamma}^\dagger_{X^{(0)}} \circ f^! \circ \phi_+(\mathcal{E}') = \eta^! \circ \theta_+(\mathcal{E}'^{(\bullet)}). \qquad (16.1.10.3.2)$$

Suppose now that the left face of 16.1.8.6.2 is cartesian and let us check that the composite morphism 16.1.10.3.2 becomes an isomorphism. Thanks to 8.7.6.11, we reduce to the case where T is empty, i.e., the face of the middle of 16.1.8.6.2 is equal to the left one and is in particular cartesian. We get a similar to 16.1.8.6.2 commutative diagram where $\mathfrak{P}'^{(0)}$ is replaced by $\mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$ (indeed, we have a canonically induced closed immersion: $X' \times_X X^{(0)} \hookrightarrow \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$). Then denote by $\iota\colon \mathfrak{P}'^{(0)} \to \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$ the canonical morphism.

Since ι is proper (because the morphisms f, f', ϕ, $\phi^{(0)}$ are proper), following the lemma 16.1.2.11, ι_+ and $\iota^!$ induce quasi-inverse equivalences between $\mathrm{MIC}^{\dagger\dagger}(X'^{(0)}, \mathfrak{P}'^{(0)}/K)$ and $\mathrm{MIC}^{\dagger\dagger}(X'^{(0)}, \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}/K)$. We reduce therefore to the case where $\mathfrak{P}'^{(0)} = \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$. In this case, following 16.1.10.1, the last morphism of 16.1.10.3.2 is an isomorphism. Moreover, $(\phi^{(0)})^{-1}(X^{(0)}) = P' \times_P X^{(0)}$. Moreover, since $\mathcal{E}'$ has its support in X', $f'^!(\mathcal{E}')$ has its support in $(f')^{-1}(X') = X' \times_P P^{(0)}$. Since $(P' \times_P X^{(0)}) \cap (X' \times_P P^{(0)}) = X' \times_X X^{(0)} = X'^{(0)}$, this yields $\mathbb{R}\underline{\Gamma}^\dagger_{(\phi^{(0)})^{-1}(X^{(0)})} \circ f'^!(\mathcal{E}')$ has its support in $X'^{(0)}$. The first morphism of 16.1.10.3.2 is then an isomorphism. Hence we are done. □

Lemma 16.1.10.4. *Let $\theta = (\beta, \alpha, \phi)\colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ be a strict morphism of smooth d-frames such that ϕ is proper and smooth, α is proper, surjective and β is finite and etale. The functor $\theta_+\colon \mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/K) \to \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$ is a left adjoint functor of θ^+.*

Proof. 0) With the lemma 16.1.2.9 (and the remark 16.1.2.10), we reduce to the case where Y (resp. Y') is integral and dense in X (resp. X'). By using de Jong 's desingularisation theorem, then there exists a completely smooth d-frame $(Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$ together with a strict morphism $\eta = (b, a, f)\colon (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}) \to (Y, X, \mathfrak{P}, T)$ such that f is projective and smooth, a is projective, surjective, generically finite and etale, and $T^{(0)} \cap X^{(0)}$ is a (strict normal crossing) divisor of $X^{(0)}$.

Moreover, following 16.1.4.1, for any divisor $\widetilde{T}'$ containing T' and such that $X' \setminus \widetilde{T}'$ is dense in Y', the functor $({}^\dagger\widetilde{T}')$ is fully faithful. Hence, increasing T if necessary (and using 13.2.1.4), we can suppose that b is finite and etale. Set $Y'^{(0)} := Y' \times_Y Y^{(0)}$, $X''^{(0)} := X' \times_X X^{(0)}$ $\mathfrak{P}''^{(0)} := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}^{(0)}$. Denote by $\varpi\colon X'^{(0)} \to X''^{(0)}$ the normalisation map (which is the identity over $Y'^{(0)}$). Since $X''^{(0)}$ is an integral k-variety, then ϖ is finite. Hence, there exists a strict morphism of d-frame of the form $(Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y'^{(0)}, X''^{(0)}, \mathfrak{P}''^{(0)}, T''^{(0)})$ such that $\mathfrak{P}'^{(0)} \to \mathfrak{P}''^{(0)}$ is projective and smooth. Let $\theta^{(0)} = (\beta^{(0)}, \alpha^{(0)}, \phi^{(0)})\colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$ and $\eta' = (b', a', f')\colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y', X', \mathfrak{P}', T')$ be the strict morphisms of frames induced by the projections. Then we are in the context of 16.1.10.2 with moreover $X^{(0)}$ smooth, $X'^{(0)}$ normal and the left face of the diagram of 16.1.8.6.2 cartesian.

1) Construction of the morphism $\theta^{(0)}_+ \circ \theta^{(0)+} \to \mathrm{id}$.

Since $\alpha^{(0)}$ is proper and surjective with $X^{(0)}$ smooth and $X'^{(0)}$ normal, since $\beta^{(0)}$ is finite and etale, Tsuzuki has built in this context (see the section [Tsu02, 5]), the pushforward functor of the form $\theta^{(0)}_*\colon \mathrm{MIC}^\dagger(Y'^{(0)}, X'^{(0)}/K) \to \mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K)$ which is right and left adjoint to the pullback functor $(\theta^{(0)})^*$. To lighten the notations, we set $\mathrm{sp}_+ := \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +}$ and $\mathrm{sp}'_+ := \mathrm{sp}_{X'^{(0)} \hookrightarrow \mathfrak{P}'^{(0)}, T'^{(0)}, +}$. Let $E^{(0)} \in \mathrm{MIC}^\dagger(Y^{(0)}, X^{(0)}/K)$, $E'^{(0)} \in \mathrm{MIC}^\dagger(Y'^{(0)}, X'^{(0)}/K)$.

i) Since sp_+ is fully faithful and since $\theta^{(0)}_*$ is right adjoint of $(\theta^{(0)})^*$, we get the canonical functorial isomorphism in $E^{(0)}$ and $E'^{(0)}$:

$$\mathrm{Hom}_{\mathrm{MIC}^\dagger(Y'^{(0)}, X'^{(0)}/K)}((\theta^{(0)})^*(E^{(0)}), E'^{(0)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{MIC}^{\dagger\dagger}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/K)}(\mathrm{sp}_+(E^{(0)}), \mathrm{sp}_+ \circ \theta^{(0)}_*(E'^{(0)}))$$

ii) Following 16.1.8.6, we have the isomorphism $\mathrm{sp}'_+ \circ (\theta^{(0)})^* \xrightarrow{\sim} (\theta^{(0)})^+ \circ \mathrm{sp}_+$. Moreover, following 16.1.2.17.b, the functor $\theta^{(0)}_+$ is right adjoint to $(\theta^{(0)})^+$. Since the functor sp'_+ is fully faithful, using these two facts the functorial in $E^{(0)}$ and $E'^{(0)}$ canonical isomorphism:

$$\mathrm{Hom}_{\mathrm{MIC}^\dagger(Y'^{(0)}, X'^{(0)}/K)}((\theta^{(0)})^*(E^{(0)}), E'^{(0)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{MIC}^{\dagger\dagger}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/K)}(\mathrm{sp}_+(E^{(0)}), \theta^{(0)}_+ \circ \mathrm{sp}'_+(E'^{(0)}))$$

iii) Using both bijections of i) and ii), we get the functorial in $E'^{(0)}$ canonical isomorphism:

$$\mathrm{sp}_+ \circ \theta_*^{(0)}(E'^{(0)}) \xrightarrow{\sim} \theta_+^{(0)} \circ \mathrm{sp}'_+(E'^{(0)}). \qquad (16.1.10.4.1)$$

iv) Let $\mathcal{E}^{(0)} \in \mathrm{MIC}^{\dagger\dagger}(X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}/K)$, $\mathcal{E}'^{(0)} \in \mathrm{MIC}^{\dagger\dagger}(X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}/K)$. Since the functor sp_+ is essentially surjective, there exists $E_1^{(0)}$, $E_2^{(0)}$ (we make two choices in order to check the canonicity) such that $\mathcal{E}^{(0)} \xleftarrow{\sim} \mathrm{sp}_+(E_1^{(0)})$ and $\mathcal{E}^{(0)} \xrightarrow{\sim} \mathrm{sp}_+(E_2^{(0)})$. Since $\theta_*^{(0)}$ is left adjoint to $(\theta^{(0)})^*$ (resp. using 16.1.10.4.1 and 16.1.8.6), we get the bottom (resp. top) vertical isomorphisms of the diagram:

$$\begin{array}{ccccc} \theta_+^{(0)} \circ \theta^{(0)+} \circ \mathrm{sp}_+(E_1^{(0)}) & \xrightarrow{\sim} & \theta_+^{(0)} \circ \theta^{(0)+}(\mathcal{E}^{(0)}) & \xrightarrow{\sim} & \theta_+^{(0)} \circ \theta^{(0)+} \circ \mathrm{sp}_+(E_2^{(0)}) \\ \downarrow\sim & & & & \downarrow\sim \\ \mathrm{sp}_+ \circ \theta_*^{(0)} \circ \theta^{(0)*}(E_1^{(0)}) & & & & \mathrm{sp}_+ \circ \theta_*^{(0)} \circ \theta^{(0)*}(E_2^{(0)}) \\ \downarrow \mathrm{adj} & & & & \downarrow \mathrm{adj} \\ \mathrm{sp}_+(E_1^{(0)}) & \xrightarrow{\sim} & \mathcal{E}^{(0)} & \xrightarrow{\sim} & \mathrm{sp}_+(E_2^{(0)}) \end{array} \qquad (16.1.10.4.2)$$

Since sp_+ is fully faithful, the bottom composite isomorphism $\mathrm{sp}_+(E_1^{(0)}) \xrightarrow{\sim} \mathrm{sp}_+(E_2^{(0)})$ comes from an isomorphism $E_1^{(0)} \xrightarrow{\sim} E_2^{(0)}$. By functoriality of the vertical isomorphisms, this yields that the diagram 16.1.10.4.2 is commutative. Hence, we get a canonical morphism $\theta_+^{(0)} \circ \theta^{(0)+}(\mathcal{E}^{(0)}) \to \mathcal{E}^{(0)}$.

1 bis) Similarly to the step 1)iv), since $\theta_*^{(0)}$ is left adjoint to $(\theta^{(0)})^*$, then using 16.1.10.4.1 and 16.1.8.6, we build the canonical morphism $\mathcal{E}'^{(0)} \to \theta^{(0)+} \circ \theta_+^{(0)}(\mathcal{E}'^{(0)})$.

2) Construction of the canonical morphism $\theta_+ \circ \theta^+ \to \mathrm{id}$.

Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$, $\mathcal{E}' \in \mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/K)$. Following 16.1.10.3, we have the change base isomorphism $\eta^+ \circ \theta_+(\mathcal{E}'^{(\bullet)}) \to \theta_+^{(0)} \circ \eta'^+(\mathcal{E}')$. With the morphism $\theta_+^{(0)} \circ \theta^{(0)+} \to \mathrm{id}$ constructed at the step 1), we get the canonical morphism:

$$\eta^+ \circ \theta_+ \circ \theta^+(\mathcal{E}) \xrightarrow{\sim} \theta_+^{(0)} \circ \eta'^+ \circ \theta^+(\mathcal{E}) \xrightarrow{\sim} \theta_+^{(0)} \circ \theta^{(0)+} \circ \eta^+(\mathcal{E}) \xrightarrow{\mathrm{adj}} \eta^+(\mathcal{E}). \qquad (16.1.10.4.3)$$

On the other hand, denoting by $\mathfrak{U} := \mathfrak{P} \setminus T$, $\mathfrak{U}' := \mathfrak{P}' \setminus T'$, $\varphi \colon \mathfrak{U}' \to \mathfrak{U}$ the induced map by ϕ and $\vartheta = (\beta, \beta, \phi) \colon (Y', Y', \mathfrak{U}', T') \to (Y, Y, \mathfrak{U})$, following 16.1.2.15, we have the adjunction morphism

$$|\mathfrak{U} \circ \theta_+ \circ \theta^+(\mathcal{E}) \xrightarrow{\sim} \vartheta_+ \circ \vartheta^+(\mathcal{E}|\mathfrak{U}) \xrightarrow{\mathrm{adj}} \mathcal{E}|\mathfrak{U}. \qquad (16.1.10.4.4)$$

Since the functor $(\eta^+, |\mathfrak{U})$ is fully faithful (see proposition 16.1.5.2), then via the canonical morphisms of 16.1.10.4.3 and 16.1.10.4.4, we get the canonical morphism $\theta_+ \circ \theta^+(\mathcal{E}) \to \mathcal{E}$.

2 bis) Similarly, we build the morphism $\mathcal{E}'^{(\bullet)} \to \theta^+ \circ \theta_+(\mathcal{E}')$.

3) To deduce that θ_+ is left adjoint to θ^+, then it is sufficient (thanks to the proposition 8.7.6.11) to check it outside the divisor. We reduce then to the situation of 16.1.2.15. □

Theorem 16.1.10.5. *Let* $\theta = (\beta, \alpha, \phi) \colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ *be a morphism of smooth d-frames. The functors* $\theta^+, \theta^! \colon \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K) \to \mathrm{MIC}^{\dagger\dagger}(X', \mathfrak{P}', T'/K)$ *are canonically isomorphic.*

Proof. 0) With the lemma 16.1.2.9, we reduce to the case where Y (resp. Y') is integral and dense in X (resp. X').

1) Let us first check the theorem when ϕ is proper and smooth, $T' = \phi^{-1}(T)$, α is proper, surjective, generically finite and etale.

Thanks to 16.1.4.1, for any divisor $\widetilde{T}'$ containing T' and such that $X' \setminus \widetilde{T}'$ is dense in Y', the localisation functor $({}^\dagger \widetilde{T}')$ is fully faithful. Increasing T if necessary, we reduce therefore to the case where β is finite and etale. In this case, following 16.1.10.4, the functor θ^+ is then right adjoint to θ_+ Moreover, then so is $\theta^!$ (see 16.1.1.6.3). Hence we are done.

2) General case. Via the de Jong 's desingularisation theorem, we check (as in the proof of 16.1.8.6) there exists $\eta = (b, a, f) \colon (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)}) \to (Y, X, \mathfrak{P}, T)$, $\theta^{(0)} = (\beta^{(0)}, \alpha^{(0)}, \phi^{(0)}) \colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$ and $\eta' = (b', a', f') \colon (Y'^{(0)}, X'^{(0)}, \mathfrak{P}'^{(0)}, T'^{(0)}) \to (Y', X', \mathfrak{P}', T')$ three strict morphisms of smooth d-frames such that $\theta \circ \eta' = \eta \circ \theta^{(0)}$, f and f' are projective and smooth, a and a' are projective, generically finite and etale, $X'^{(0)}$ and $X^{(0)}$ are smooth, $T^{(0)} \cap X^{(0)}$ is a (strict normal crossing) divisor of $X^{(0)}$, $T'^{(0)} \cap X'^{(0)}$ is a (strict normal crossing) divisor of $X'^{(0)}$. It follows from the step 1) that the functors η^+ and $\eta^!$ (resp. η'^+ and $\eta'^!$) are canonically isomorphic. Moreover, following the proposition 16.1.5.2, the functor $(\eta'^+, |\mathfrak{P}'^{(0)} \setminus T'^{(0)})$ is fully faithful. Hence, we reduce the case where X and X' are smooth, which is already well known (this follows from 12.2.4.1 and 12.2.5.6). $\square$

Remark 16.1.10.6. The main goal of this section was to establish 16.1.10.5. Beware, following [Abe14a, 5.6], to get a compatible with Frobenius isomorphism between θ^+ and $\theta^!$, we have to add a twist. However, when we have a Frobenius structure, this isomorphism can be checked more easily since we then have the theorem of full faithfulness of Kedlaya of [Ked04a]. Indeed, thanks to this theorem of full faithfulness, we reduce to the situation of the smooth partially compactification (i.e., that of 12.2).

This isomorphism of 16.1.10.5 is a fundamental ingredient in the proof of 16.1.11.2.

16.1.11 1-overholonomicity of partially overcoherent isocrystals, duality

Lemma 16.1.11.1. *Let $\mathfrak{P}$ be a separated and smooth $\mathfrak{S}$-formal scheme, $X \hookrightarrow P$ be a closed immersion with X integral, Z be a closed subscheme of P not containing X and such that $Y := X \setminus Z$ is smooth. Then there exists a commutative diagram of the form*

$$\begin{array}{ccccc} X' & \xrightarrow{u'} & \mathbb{P}^N_P & \longrightarrow & \widehat{\mathbb{P}}^N_{\mathfrak{P}} \\ {\scriptstyle a}\downarrow & & \downarrow & \square & \downarrow{\scriptstyle q} \\ X & \xrightarrow{u} & P & \longrightarrow & \mathfrak{P}, \end{array} \tag{16.1.11.1.1}$$

where X' is smooth over k, q is the canonical projection, u' is a closed immersion, $a^{-1}(Z \cap X)$ is a strict normal crossing divisor of X', a is proper, surjective, generically finite and etale.

Proof. By using de Jong 's desingularisation theorem, there exists a quasi-projective smooth variety X', a projective generically finite and etale morphism $a\colon X' \to X$ such that $a^{-1}(Z \cap X)$ is a strict normal crossing divisor of X'. Then there exists an immersion of the form $X' \hookrightarrow \mathbb{P}^N_k$. Since a is proper, the induced immersion $X' \hookrightarrow \mathbb{P}^N_X$ is closed. Hence we get a closed immersion: $X' \hookrightarrow \widehat{\mathbb{P}}^N_{\mathfrak{P}}$ which gives the existence of the commutative diagram 16.1.11.1.1 satisfying the required properties. $\square$

Lemma 16.1.11.2. *Let $\mathfrak{P}$ be a separated and smooth $\mathfrak{S}$-formal scheme, $X \hookrightarrow P$ be a closed immersion with X integral, T be a divisor of P not containing X and such that $Y := X \setminus T$ is dense and smooth. Suppose given a commutative diagram of the form*

$$\begin{array}{ccccc} X' & \xrightarrow{u'} & P' & \longrightarrow & \mathfrak{P}' \\ {\scriptstyle a}\downarrow & & \downarrow & \square & \downarrow{\scriptstyle q} \\ X & \xrightarrow{u} & P & \longrightarrow & \mathfrak{P}, \end{array} \tag{16.1.11.2.1}$$

where X' is smooth over k, q is a proper morphism of smooth $\mathfrak{S}$-formal schemes such $T' := q^{-1}(T)$ is a divisor of P', u' is a closed immersion, $a^{-1}(T \cap X)$ is a divisor of X', a is proper, surjective, generically finite and etale. Let and $\eta := (b, a, q)\colon (Y', X', \mathfrak{P}', T') \to (Y, X, \mathfrak{P}, T)$ be the morphism of smooth d-frames given by the diagram 16.1.11.2.1. Let $\mathcal{E} \in (F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$ (resp. $\mathcal{E}^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/K)$). Then $\mathcal{E}$ (resp. $\mathcal{E}^{(\bullet)}$) is a direct summand of $\eta_+ \circ \eta^!(\mathcal{E})$ (resp. $\eta_+^{(\bullet)} \circ \eta^{(\bullet)!}(\mathcal{E}^{(\bullet)})$), where $\eta_+ := q_+$ (resp. $\eta_+^{(\bullet)} := q_+^{(\bullet)}$).

Proof. Since the respective case is treated similarly, then let's only prove the non-respective one. Since q is proper, since $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ and $\mathbb{D}_T(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$ then it follows from 16.1.1.6.(b) that we have the morphisms $\mathcal{E} \to \eta_+ \circ \eta^+(\mathcal{E})$ and $\eta_+ \circ \eta^!(\mathcal{E}) \to \mathcal{E}$. Moreover, following 16.1.10.5, $\eta^!(\mathcal{E}) \xrightarrow{\sim} \eta^+(\mathcal{E})$. Hence, we get the sequence $\mathcal{E} \to \eta_+ \circ \eta^!(\mathcal{E}) \to \mathcal{E}$. Let $T \subset T'$ be a divisor such that $Y' := X \setminus T'$ is dense in Y. Since the functor $|\mathfrak{U}'$, where $\mathfrak{U}' := \mathfrak{P} \setminus T'$, is faithful (use 8.7.6.11 and 16.1.4.1), then to check that the composition of $\mathcal{E} \to \eta_+ \circ \eta^!(\mathcal{E}) \to \mathcal{E}$ is an isomorphism we reduce to the case where a is finite and etale and T is empty, which is already known (see 16.1.2.15.2). $\square$

Proposition 16.1.11.3. *Let $(Y, X, \mathfrak{P}, T)$ be a smooth d-frame over $\mathfrak{S}$.*

(a) *For any object* $\mathcal{E} \in (F\text{-})\mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$*,* $\mathcal{E}$ *is* 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$*-overholonomic (see the definition 16.1.1.1).*

(b) *We have the inclusion:*

$$\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T)). \tag{16.1.11.3.1}$$

(c) *The functor* $\mathbb{D}^{(\bullet)}_{T}$ *of 9.2.4.20 induces the equivalence of categories:*

$$\mathbb{D}^{(\bullet)}_{T} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}). \tag{16.1.11.3.2}$$

Proof. a) By using the lemma 16.1.11.1, we get a diagram of the form 16.1.11.1.1 and satisfying the required conditions. It follows from 16.1.11.2 that $\mathcal{E}$ is therefore a direct summand of $\eta_{+} \circ \eta^{!}(\mathcal{E})$. Since $\eta^{!}(\mathcal{E}) \in \mathrm{MIC}^{\dagger\dagger}(X', \widehat{\mathbb{P}}^{N}_{\mathfrak{P}}, q^{-1}(T)/K)$, as X' is smooth, it follows from 16.1.1.7 that $\eta'^{!}(\mathcal{E})$ is 1-$\mathcal{D}^{\dagger}_{\widehat{\mathbb{P}}^{N}_{\mathfrak{P}}}({}^{\dagger}q^{-1}(T))_{\mathbb{Q}}$-overholonomic By stability 1-overholonomicity by direct image by the a proper morphism (see 16.1.1.5), $\eta_{+} \circ \eta^{!}(\mathcal{E})$ is 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomic. Hence we are done.

b) and c) When X is smooth, this is already known (see respectively 12.2.1.7.1 and 12.2.1.13.1). Hence, we proceed similarly to the part a) by using the stability of perfectness by pushforward of a proper morphism (see 9.4.2.6) and the relative duality isomorphism 9.4.5.2.1. □

Let us give some applications of 16.1.11.3.(a):

Proposition 16.1.11.4. *Let* $\mathfrak{P}$ *be a smooth separated* $\mathfrak{S}$*-formal scheme,* $T \subset T'$ *two divisors of* P*,* X *be a closed subscheme of* P*. We set* $\mathfrak{U} := \mathfrak{P} \setminus T$*,* $\mathfrak{U}' := \mathfrak{P} \setminus T'$*,* $Y := X \setminus T$*,* $Y' := X \setminus T'$*. We suppose moreover* Y *smooth and* Y' *dense in* Y*. Let* $\mathcal{E}' \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T'/K)$*. If there exists* $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$ *such that* $\mathcal{E}' \xrightarrow{\sim} ({}^{\dagger}T')(\mathcal{E})$*, then* $\mathcal{E} \xrightarrow{\sim} \mathrm{Im}\,(\mathbb{D}_{T} \circ \mathbb{D}_{T'}(\mathcal{E}') \to \mathcal{E}')$*.*

Proof. By applying the functor $\mathbb{D}_{T}$ to the isomorphism $\mathcal{E}' \xrightarrow{\sim} ({}^{\dagger}T')(\mathcal{E})$ we get the first isomorphism:

$$\mathbb{D}_{T} \circ \mathbb{D}_{T'}(\mathcal{E}') \xleftarrow{\sim} \mathbb{D}_{T} \circ \mathbb{D}_{T'} \circ ({}^{\dagger}T')(\mathcal{E}) \overset{9.2.4.22.3}{\xrightarrow{\sim}} \mathbb{D}_{T} \circ ({}^{\dagger}T') \circ \mathbb{D}_{T}(\mathcal{E}). \tag{16.1.11.4.1}$$

Since $\mathbb{D}_{T}(\mathcal{E}) \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$, then following 16.1.11.3 $\mathbb{D}_{T}(\mathcal{E})$ is 1-$\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overholonomic. Hence, it follows from the isomorphisms 16.1.11.4.1 that $\mathbb{D}_{T} \circ \mathbb{D}_{T'}(\mathcal{E}')$ is $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent. Since $\mathcal{E}' \xrightarrow{\sim} ({}^{\dagger}T')(\mathcal{E})$, as $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$, then $\mathcal{E}'$ is $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent. Hence we get the morphism of overcoherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules $\mathbb{D}_{T} \circ \mathbb{D}_{T'}(\mathcal{E}') \to \mathcal{E}'$ and we denote by $\mathcal{F}$ its image, which is also an overcoherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-module. Moreover, following the smooth case 16.1.6.3, $\mathcal{F}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y, \mathfrak{U}/K)$. In of the other terms, $\mathcal{F}|\mathfrak{U}$ is in the essential image of the functor $\mathrm{sp}_{Y \hookrightarrow \mathfrak{U},+}$. Hence $\mathcal{F} \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{P}, T/K)$.

The canonical inclusion $\mathcal{F} \subset \mathcal{E}'$ induced the canonical isomorphism $({}^{\dagger}T')(\mathcal{F}) \xrightarrow{\sim} \mathcal{E}'$ (Indeed, via 8.7.6.11, it is sufficient to check it outside T'). By full faithfulness of the functor $({}^{\dagger}T')$ (see the theorem 16.1.4.1), this yields $\mathcal{F} \xrightarrow{\sim} \mathcal{E}$. □

Lemma 16.1.11.5. *Let* $\mathfrak{P}$ *be a smooth formal scheme over* $\mathfrak{S}$*. Let* Z *and* X *be two closed subschemes of* P*. Set* $\mathfrak{U} := \mathfrak{P} \setminus Z$*,* $Y := X \setminus Z$ *and* $\mathfrak{V} := \mathfrak{P} \setminus X$*. We suppose* Y *smooth. For any divisor* T *of* P *containing* Z*, we have the factorisation:*

$$\mathbb{D}^{(\bullet)}_{T} \circ ({}^{\dagger}T) \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S}) \cap \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathfrak{S}), \tag{16.1.11.5.1}$$

where $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ *is defined at 16.2.1.1.*

Proof. Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S})$. It follows from 16.2.1.5.2 and 16.1.11.3.2, that $\mathbb{D}^{(\bullet)}_{T} \circ ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathfrak{S})$. Let us now check that $\mathbb{D}^{(\bullet)}_{T} \circ ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$. Let T' be a divisor of P containing Z. We get the isomorphisms:

$$({}^{\dagger}T') \circ \mathbb{D}^{(\bullet)}_{T} \circ ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^{\dagger}T \cup T') \circ \mathbb{D}^{(\bullet)}_{T} \circ ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \underset{9.2.4.20.4}{\xrightarrow{\sim}} \mathbb{D}^{(\bullet)}_{T \cup T'} \circ ({}^{\dagger}T \cup T')(\mathcal{E}^{(\bullet)}).$$

By symmetry in T and T', this yields the canonical isomorphism

$$({}^{\dagger}T') \circ \mathbb{D}^{(\bullet)}_{T} \circ ({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^{\dagger}T) \circ \mathbb{D}^{(\bullet)}_{T'} \circ ({}^{\dagger}T')(\mathcal{E}^{(\bullet)}).$$

It follows from 16.1.11.3.2 that $\mathbb{D}^{(\bullet)}_{T'} \circ ({}^{\dagger}T')(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, T'/\mathfrak{S})$. Hence, $({}^{\dagger}T) \circ \mathbb{D}^{(\bullet)}_{T'} \circ ({}^{\dagger}T')(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, T'/\mathfrak{S})$ and we are done. □

Here is a corollary of the theorem 16.1.10.5:

Proposition 16.1.11.6. *Let $(Y, X, \mathfrak{P}, T)$ be a smooth d-frame over $\mathfrak{S}$ (see the definition 12.2.1.1). Let $E \in \mathrm{MIC}^{\dagger}(Y, X, \mathfrak{P}/K)$. We have the canonical isomorphism:*

$$\mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E^{\vee}) \xrightarrow{\sim} \mathbb{D}_T \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E),$$

where $\vee$ is the dual in $\mathrm{MIC}^{\dagger}(Y, X/K)$ and $\mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}$ was defined in theorem 16.1.8.4.

Proof. With the lemma 16.1.2.9 (and the remark 16.1.2.10), we reduce to the case where X is integral and Y is dense in X. When X is smooth, this proposition was established at 12.2.5.6. To reduce to the case where X is smooth, we use the theorem of full faithfulness 16.1.5.2 as follows: Following the de Jong 's desingularisation theorem, there exists a divisor $\widetilde{T}$ containing T and a diagram of the form 16.1.5.0.1 satisfying the required conditions of 16.1.5 and such that moreover $X^{(0)}$ is smooth. We keep the corresponding notations. We have the following isomorphisms:

$$\begin{aligned} a^{+} \circ \mathbb{D}_T \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E) &\overset{8.7.7.3}{\xrightarrow{\sim}} \mathbb{D}_{T^{(0)}} \circ a^{!} \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E) \\ \overset{16.1.10.5}{\xrightarrow{\sim}} \mathbb{D}_{T^{(0)}} \circ a^{+} \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E) &\overset{16.1.7.5.1}{\xrightarrow{\sim}} \mathbb{D}_{T^{(0)}} \circ \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +} \circ a^{*}(E). \end{aligned} \tag{16.1.11.6.1}$$

On the other hand, we have:

$$\begin{aligned} a^{+} \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E^{\vee}) &\overset{16.1.7.5.1}{\xrightarrow{\sim}} \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +} \circ a^{*}(E^{\vee}) \xrightarrow{\sim} \\ \xrightarrow{\sim} \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +}((a^{*}(E))^{\vee}) &\overset{12.2.5.6}{\xrightarrow{\sim}} \mathbb{D}_{T^{(0)}} \circ \mathrm{sp}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +} \circ a^{*}(E). \end{aligned} \tag{16.1.11.6.2}$$

By 16.2.6.3.2 and 16.1.11.6.2, we get the canonical isomorphism $a^{+} \circ \mathbb{D}_T \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E) \xrightarrow{\sim} a^{+} \circ \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}(E^{\vee})$. Set $\mathfrak{U} := \mathfrak{P} \setminus T$, $\mathfrak{U}^{(0)} := \mathfrak{P}^{(0)} \setminus T^{(0)}$ and $\widehat{E} \in \mathrm{MIC}^{\dagger}(Y, Y, \mathfrak{U}/K)$ be the induced convergent isocrystal on $(Y, Y, \mathfrak{U}/K)$. By applying the functor $\mathfrak{U}^{(0)}$ to this latter isomorphism, we get an arrow isomorphic to the image by b^{+} of the canonical isomorphism $\mathbb{D} \circ \mathrm{sp}_{Y \hookrightarrow \mathfrak{U}, +}(\widehat{E}) \xrightarrow{\sim} \mathrm{sp}_{Y \hookrightarrow \mathfrak{U}, +}(\widehat{E}^{\vee})$. Hence we are done by full faithfulness of $(a^{+}, |\mathfrak{P} \setminus T)$ (see 16.1.5.2).

□

16.2 Partially overcoherent isocrystals

16.2.1 Definition and the equivalence sp_{+}

Let $\mathfrak{P}$ be a smooth separated formal scheme over $\mathfrak{S}$. Let Z and X be two closed subschemes of P. Set $\mathfrak{U} := \mathfrak{P} \setminus Z$, $Y := X \setminus Z$ and $\mathfrak{V} := \mathfrak{P} \setminus X$. We suppose Y/S smooth.

Notation 16.2.1.1. We denote by $(F\text{-})\mathrm{MIC}^{(\bullet)}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$ or by $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S})$ the strictly full abelian subcategory of $(F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$ (see definition 15.3.8.13) consisting of $(F\text{-})$complexes $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}|\mathfrak{U} \in \mathrm{MIC}^{(\bullet)}(Y, \mathcal{U}, \emptyset/\mathcal{V})$, where the latter category is defined in 16.1.2.3.

Example 16.2.1.2. When Z is a divisor, the category $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S})$ defined at 16.2.1.1 is the same as that defined at 16.1.2.3.

Remark 16.2.1.3. It is not clear if the following functor

$$\underrightarrow{l}^{*}_{\mathbb{Q}} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \to D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}) \tag{16.2.1.3.1}$$

is fully faithful. So the analogue of $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ in term of $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module is not clear. With the equivalence of categories 16.2.1.10.1, this is another argument to prefer to work with categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$.

Lemma 16.2.1.4. *Let $\overline{Y}$ the closure of Y in X. Let $Y_1, \ldots, Y_N$ the irreducible components of Y and $\overline{Y}_1, \ldots, \overline{Y}_N$ their closure in X.*

(a) We have the equality $(F\text{-})\mathrm{MIC}^{(\bullet)}(\overline{Y}, \mathfrak{P}, Z/\mathcal{V}) = (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$.

(b) We have a canonical equivalence of categories:

$$\oplus_{r=1}^{N} \mathbb{R}\underline{\Gamma}^{\dagger}_{\overline{Y}_r} \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \cong \prod_{r=1}^{N} (F\text{-})\mathrm{MIC}^{(\bullet)}(\overline{Y}_r, \mathfrak{P}, Z/\mathcal{V}). \tag{16.2.1.4.1}$$

Proof. The inclusion of the equality of (a) is obvious. Conversely, is $\mathcal{E} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. According to 15.3.7.5.1, we have the canonical morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{\overline{Y}}(\mathcal{E}) \to \mathcal{E}$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. Since this is the case outside Z (because $\overline{Y} \setminus Z = Y$), then it follows from 15.3.7.6 that this morphism is an isomorphism. Hence, the sheaf $\mathcal{E}$ has its support in $\overline{Y}$ and we get the inclusion inverse.

Similarly, it follows from 15.3.7.6 that the functor $\oplus_{r=1}^{N} \mathbb{R}\underline{\Gamma}^{\dagger}_{\overline{Y}_r}$ induces the equivalence of categories 16.2.1.4.1. □

16.2.1.5. The following properties are straightforward:

(a) For any closed subscheme $\widetilde{Z} \supset Z$ of P, the t-exact functor 15.3.8.12.2 induces the functor

$$({}^{\dagger}\widetilde{Z}) \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, \widetilde{Z}/\mathfrak{S}). \tag{16.2.1.5.1}$$

(b) For any divisor T of P, the t-exact functor 15.3.8.12.3 induces the functor

$$({}^{\dagger}T) \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S}) \cap \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathfrak{S}). \tag{16.2.1.5.2}$$

Notation 16.2.1.6. Let Z_1 and Z_2 be two closed subschemes of P such that $Z_1 \cap Z_2 = Z$. We write $Y_i := X \setminus Z_i$, for $i = 1, 2$.

(a) We define the category $\mathrm{MIC}^{\dagger}((Y_1, Y_2), X, \mathfrak{P}/\mathfrak{S})$ as follows. An object (E_1, E_2, ϵ) of the category $\mathrm{MIC}^{\dagger}((Y_1, Y_2), X, \mathfrak{P}/\mathfrak{S})$ consists of the data of an object $E_i \in \mathrm{MIC}^{\dagger}(Y_i, X, \mathfrak{P}/\mathfrak{S})$ for $i = 1, 2$ and of an isomorphism $\epsilon \colon j^{\dagger}_{Y_1 \cap Y_2} E_2 \xrightarrow{\sim} j^{\dagger}_{Y_1 \cap Y_2} E_1$.

(b) We denote by $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, (Z_1, Z_2)/\mathfrak{S})$ the following category. An object $(\mathcal{E}^{(\bullet)}_1, \mathcal{E}^{(\bullet)}_2, \theta)$ of the category $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, (Z_1, Z_2)/\mathfrak{S})$ consists of the data of an object $\mathcal{E}^{(\bullet)}_1 \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S}) \cap \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z_1/\mathfrak{S})$, an object $\mathcal{E}^{(\bullet)}_2 \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S}) \cap \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z_2/\mathfrak{S})$ and of an isomorphism $\theta \colon ({}^{\dagger}Z_1 \cup Z_2)(\mathcal{E}^{(\bullet)}_2) \xrightarrow{\sim} ({}^{\dagger}Z_1 \cup Z_2)(\mathcal{E}^{(\bullet)}_1)$.

Proposition 16.2.1.7. *We keep the notation 16.2.1.6.*

(a) The canonical morphism $\mathrm{Loc} \colon \mathrm{MIC}^{\dagger}(Y, X, \mathfrak{P}/\mathfrak{S}) \to \mathrm{MIC}^{\dagger}((Y_1, Y_2), X, \mathfrak{P}/\mathfrak{S})$ *given by* $E \mapsto (j^{\dagger}_{Y_1} E, j^{\dagger}_{Y_2} E, \epsilon_E)$, *where* ϵ_E *is the canonical isomorphism, is an equivalence of categories and has a canonical quasi-inverse functor.*

(b) The canonical morphism $\mathrm{Loc} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, (Z_1, Z_2)/\mathfrak{S})$ *given by* $\mathcal{E}^{(\bullet)} \mapsto (({}^{\dagger}Z_1)(\mathcal{E}^{(\bullet)}), ({}^{\dagger}Z_2)(\mathcal{E}^{(\bullet)}), \theta_{\mathcal{E}^{(\bullet)}})$, *where* $\theta_{\mathcal{E}^{(\bullet)}}$ *is the canonical isomorphism, is an equivalence of categories and has a canonical quasi-inverse functor.*

Proof. I) The first statement is well known.

II) 1) We construct a canonical quasi-inverse functor as follows. Let $(\mathcal{E}^{(\bullet)}_1, \mathcal{E}^{(\bullet)}_2, \theta) \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, (Z_1, Z_2)/\mathfrak{S})$. We set $\mathcal{E}^{(\bullet)}_{12} := ({}^{\dagger}Z_1 \cup Z_2)(\mathcal{E}^{(\bullet)}_1)$. We get the canonical morphism $\alpha \colon \mathcal{E}^{(\bullet)}_1 \to \mathcal{E}^{(\bullet)}_{12}$ and $\beta \colon \mathcal{E}^{(\bullet)}_2 \to ({}^{\dagger}Z_1 \cup Z_2)(\mathcal{E}^{(\bullet)}_2) \overset{\theta}{\xrightarrow{\sim}} \mathcal{E}^{(\bullet)}_{12}$. Let $\mathrm{Glue}(\mathcal{E}^{(\bullet)}_1, \mathcal{E}^{(\bullet)}_2, \theta)$ be the kernel of the morphism $(\alpha, -\beta) \colon \mathcal{E}^{(\bullet)}_1 \oplus \mathcal{E}^{(\bullet)}_2 \to \mathcal{E}^{(\bullet)}_{12}$ of the abelian category $\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$.

2) We check in this step that $\mathcal{F}^{(\bullet)} := \mathrm{Glue}(\mathcal{E}^{(\bullet)}_1, \mathcal{E}^{(\bullet)}_2, \theta) \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S})$. Since $\mathcal{F}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$, it remains to prove that $\mathcal{F}^{(\bullet)}|\mathfrak{U} \in \mathrm{MIC}^{(\bullet)}(Y, \mathfrak{U}, \emptyset/\mathfrak{S})$.

Set $\mathfrak{U}_i := \mathfrak{P} \setminus Z_i$, for $i = 1, 2$. Since $\alpha|\mathfrak{U}_2 = \mathrm{id}$, then $\mathcal{F}^{(\bullet)}|\mathfrak{U}_2 \xrightarrow{\sim} \mathcal{E}^{(\bullet)}_2|\mathfrak{U}_2 \in \mathrm{MIC}^{(\bullet)}(X \cap U_2, \mathfrak{U}_2, \emptyset/\mathfrak{S})$. Similarly, since $\beta|\mathfrak{U}_1 = \theta|\mathfrak{U}_1$ is an isomorphism then we get the isomorphism $\mathcal{F}^{(\bullet)}|\mathfrak{U}_1 \xrightarrow{\sim} \mathcal{E}^{(\bullet)}_1|\mathfrak{U}_1 \in \mathrm{MIC}^{(\bullet)}(X \cap U_1, \mathfrak{U}_1, \emptyset/\mathfrak{S})$. Since $\mathfrak{U} = \mathfrak{U}_1 \cup \mathfrak{U}_2$, then we are done.

3) By setting $:= (\alpha, -\beta) \colon \mathcal{E}_1^{(\bullet)} \oplus \mathcal{E}_2^{(\bullet)} \to \mathcal{E}_{12}^{(\bullet)}$, we get the funcor Glue: $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, (Z_1, Z_2)/\mathfrak{S}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathfrak{S})$.

4) The fact that Loc and Glue are quasi-inverse is easy. □

Definition 16.2.1.8. We define the category of c-frames (over $\mathfrak{S}$) which extends that of d-frames (see 12.2.1.1) as follows:

(a) A "c-frame $(Y, X, \mathfrak{P}, Z)/\mathfrak{S}$" over $\mathfrak{S}$ is the data of a separated and smooth $\mathfrak{S}$-formal scheme $\mathfrak{P}$, of two closed subschemes Z and X of P such that $Y = X \setminus Z$. A c-frame $(Y, X, \mathfrak{P}, Z)/\mathfrak{S}$ is "smooth" if Y is smooth. A c-frame $(Y, X, \mathfrak{P}, Z)/\mathfrak{S}$ can simply be written $(X, \mathfrak{P}, Z)/\mathfrak{S}$ or even $(X, \mathfrak{P}, Z)$.

(b) A morphism $\theta \colon (Y', X', \mathfrak{P}', Z') \to (Y, X, \mathfrak{P}, Z)$ of c-frames is the data of a morphism $f \colon \mathfrak{P}' \to \mathfrak{P}$ of formal schemes such that $X' \subset f^{-1}X$ and $Z' \supset f^{-1}(Z)$.

Remark we have a forgetful functor from the category of c-frames to that of frames (see definition 10.1.1.4) given by $(Y, X, \mathfrak{P}, Z)/\mathfrak{S} \mapsto (Y, X, \mathfrak{P})/\mathfrak{S}$. The letter c stands for "closed subscheme" instead of "d" for "divisor".

16.2.1.9. When Z is a divisor that we prefer to denote by T, it follows from 16.1.8.4 and 16.1.2.3.1 that we have the canonical equivalence of categories

$$\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, T, +} \colon \mathrm{MIC}^{\dagger}(X, \mathfrak{P}, T/K) \cong \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}), \tag{16.2.1.9.1}$$

so that $\underrightarrow{l}^*_{\mathbb{Q}} \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, T, +} = \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T, +}$.

Theorem 16.2.1.10. *We have the canonical equivalence of categories*

$$\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +} \colon \mathrm{MIC}^{\dagger}(X, \mathfrak{P}, Z/K) \cong \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}), \tag{16.2.1.10.1}$$

which is 16.2.1.9.1 when Z is a divisor.

Proof. Following 16.2.1.9.1, the equivalence of categories of 16.2.1.10.1 are already known when Z is a divisor. To check the general case, we can suppose P is integral. We proceed therefore by induction on the number of divisors $T_1, \dots, T_r$ such that $Z = T_1 \cap \dots \cap T_r$. To do so, we use 16.2.1.7. It remains to check the independence of the construction of 16.2.1.10.1 from the choice of the writing of Z as an intersection of divisors of P. Make a second choice $Z = T_{r+1} \cap \dots \cap T_{r+s}$. Then we show that both constructions of the isomorphism 16.2.1.10.1 are canonically isomorphic to the one in the case of $Z = T_1 \cap \dots \cap T_{r+s}$. □

16.2.1.11. Let T be a divisor containing Z and $j_T \colon (X \setminus T, X, \mathfrak{P}) \to (X \setminus T, X, \mathfrak{P})$ the induced frame morphism. By construction of equivalence 16.2.1.10.1

$$({}^{\dagger}T) \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +} \xrightarrow{\sim} \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, T, +} \circ j_T^{\dagger}. \tag{16.2.1.11.1}$$

16.2.1.12. Let $T_1, \dots, T_r$ be divisors of P such that $Z = T_1 \cap \dots \cap T_r$. Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. Since $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ is an abelian subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathcal{V})$, then it follows from 15.3.8.19.1 that we have the exact sequence

$$0 \to \mathcal{E}^{(\bullet)} \to \oplus_{i=1}^r ({}^{\dagger}T_i)(\mathcal{E}^{(\bullet)}) \xrightarrow{\theta^2 - \theta^1} \oplus_{i,j=1}^r ({}^{\dagger}T_i \cup T_j)(\mathcal{E}^{(\bullet)}). \tag{16.2.1.12.1}$$

where $\theta^n \colon \oplus_{i=1}^r ({}^{\dagger}T_i)(\mathcal{E}^{(\bullet)}) \to \oplus_{i,j=1}^r ({}^{\dagger}T_i \cup T_j)(\mathcal{E}^{(\bullet)})$ for $n = 0, 1$ are the canonical maps defined at 15.3.8.19.

16.2.2 Full faithfulness of the localisation functor

Theorem 16.2.2.1. *Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, $Z \subset Z'$ two closed subschemes of P, X be a closed subscheme of P. By setting $Y := X \setminus Z$, $Y' := X \setminus Z'$, we suppose moreover Y smooth and Y' dense in Y. The functor $({}^{\dagger}Z')$ induced the fully faithful functors:*

$$({}^{\dagger}Z') \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z'/\mathcal{V}). \tag{16.2.2.1.1}$$

Proof. 0) Let us check the faithfulness of the functor $({}^\dagger Z')$. Let $\mathfrak{U} := \mathfrak{P} \setminus Z$, $\mathfrak{U}' := \mathfrak{P} \setminus Z'$. It follows from 15.3.8.18 that we reduce to check the faithfulness of the functor $\mathfrak{U}' \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(Y, \mathfrak{U}/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{(\bullet)}(Y', \mathfrak{U}'/\mathcal{V})$. We conclude using 16.1.4.1.1. Let us now check that faithfulness is full. Since this is local in $\mathfrak{P}$ and we can therefore suppose Y is integral. Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ together with a morphism $\psi \colon \mathcal{E}^{(\bullet)}({}^\dagger Z') \to \mathcal{F}^{(\bullet)}({}^\dagger Z')$.

1) First suppose $Z = T$ is a divisor of P. Let $T_1, \ldots, T_r$ be divisors of P such that $Z' = T_1 \cap \cdots \cap T_r$. Since Y' is dense in Y, then there exists at least on divisor T_i which does not contain Y. Let Z'' be the intersection of such divisors. Since $Z' \subset Z''$, we reduce to the case where all divisors T_i do not contain Y. Following 16.1.4.1.1, since $Y \setminus T_i$ is dense in Y, then the functor $({}^\dagger T_i) \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V}) \to (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T_i/\mathcal{V})$ is fully faithful. Hence, there exists $\phi_i \colon \mathcal{E}^{(\bullet)}({}^\dagger T_i) \to \mathcal{F}^{(\bullet)}({}^\dagger T_i)$ making commutative the left diagram

$$\begin{array}{ccccc}
\mathcal{E}^{(\bullet)} & \longrightarrow & ({}^\dagger T_i)(\mathcal{E}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i)(\mathcal{E}^{(\bullet)}({}^\dagger Z')) \\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle ({}^\dagger T_i)(\phi_i)} & & \downarrow{\scriptstyle ({}^\dagger T_i)\psi} \\
\mathcal{F}^{(\bullet)} & \longrightarrow & ({}^\dagger T_i)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i)(\mathcal{F}^{(\bullet)}({}^\dagger Z')),
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{E}^{(\bullet)} & \longrightarrow & ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}({}^\dagger Z')) \\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle ({}^\dagger T_i \cup T_j)(\phi_i)} & & \downarrow{\scriptstyle ({}^\dagger T_i \cup T_j)\psi} \\
\mathcal{F}^{(\bullet)} & \longrightarrow & ({}^\dagger T_i \cup T_j)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i \cup T_j)(\mathcal{F}^{(\bullet)}({}^\dagger Z'))
\end{array}
\tag{16.2.2.1.2}$$

This yields the commutative right diagram of 16.2.2.1.2. Since $({}^\dagger T_i \cup T_j)(\phi_i) = ({}^\dagger T_i \cup T_j)(\phi_j)$, then by faithfulness of the functor $({}^\dagger T_i \cup T_j)$ we get $\phi_i = \phi_j$ that we denote by ϕ. $({}^\dagger Z')(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ then following 16.2.2.1.1 we get the exact sequence:

$$\begin{array}{ccccc}
\mathcal{E}^{(\bullet)} & \longrightarrow & \oplus_{i=1}^r ({}^\dagger T_i)(\mathcal{E}^{(\bullet)}) & \xrightarrow{\theta^2_{\mathcal{E}} - \theta^1_{\mathcal{E}}} & \oplus_{i,j=1}^r ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \oplus_i ({}^\dagger T_i)\phi} & & \downarrow{\scriptstyle \oplus_{ij} ({}^\dagger T_i \cup T_j)\phi} \\
\mathcal{F}^{(\bullet)} & \longrightarrow & \oplus_{i=1}^r ({}^\dagger T_i)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\theta^2_{\mathcal{F}} - \theta^1_{\mathcal{F}}} & \oplus_{i,j=1}^r ({}^\dagger T_i \cup T_j)(\mathcal{F}^{(\bullet)})
\end{array}
\tag{16.2.2.1.3}$$

where $\theta^n_{\mathcal{E}} \colon \oplus_{i=1}^r \mathcal{E}_i^{(\bullet)} \to \oplus_{i,j=1}^r \mathcal{E}_{ij}^{(\bullet)}$ for $n = 0, 1$ are the canonical maps defined at 15.3.8.19. Since $({}^\dagger Z')(\mathcal{E}^{(\bullet)}) = \ker(\theta^2_{\mathcal{E}} - \theta^1_{\mathcal{E}})$ and $({}^\dagger Z')(\mathcal{F}^{(\bullet)}) = \ker(\theta^2_{\mathcal{F}} - \theta^1_{\mathcal{F}})$ (see 16.2.7.4.1), then we are done.

2) Let $T_1, \ldots, T_r$ be divisors of P such that $Z = T_1 \cap \cdots \cap T_r$. By using the step 1), there exists $\phi_i \colon \mathcal{E}^{(\bullet)}({}^\dagger T_i) \to \mathcal{F}^{(\bullet)}({}^\dagger T_i)$ making commutative the diagram

$$\begin{array}{ccccc}
({}^\dagger T_i)(\mathcal{E}^{(\bullet)}) & \longrightarrow & ({}^\dagger Z')(({}^\dagger T_i)\mathcal{E}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i)(\mathcal{E}^{(\bullet)}({}^\dagger Z')) \\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle ({}^\dagger Z')\phi_i} & & \downarrow{\scriptstyle ({}^\dagger T_i)\psi} \\
({}^\dagger T_i)(\mathcal{F}^{(\bullet)}) & \longrightarrow & ({}^\dagger Z')(({}^\dagger T_i)\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i)(\mathcal{F}^{(\bullet)}({}^\dagger Z')),
\end{array}
\tag{16.2.2.1.4}$$

where the horizontal isomorphisms are the canonical ones. Since we have the commutative diagram

$$\begin{array}{ccccc}
({}^\dagger T_i)(\mathcal{E}^{(\bullet)}) & \longrightarrow & ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}({}^\dagger Z')) \\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle ({}^\dagger T_i \cup T_j)(\phi_i)} & & \downarrow{\scriptstyle ({}^\dagger T_i \cup T_j)\psi} \\
({}^\dagger T_i)(\mathcal{F}^{(\bullet)}) & \longrightarrow & ({}^\dagger T_i \cup T_j)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\sim} & ({}^\dagger T_i \cup T_j)(\mathcal{F}^{(\bullet)}({}^\dagger Z')),
\end{array}
\tag{16.2.2.1.5}$$

then $({}^\dagger T_i \cup T_j)(\phi_i) = ({}^\dagger T_i \cup T_j)(\phi_j)$. Hence, we get the unique map ϕ making commutative the diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{E}^{(\bullet)} & \longrightarrow & \oplus_{i=1}^r ({}^\dagger T_i)(\mathcal{E}^{(\bullet)}) & \xrightarrow{\theta^2_{\mathcal{E}} - \theta^1_{\mathcal{E}}} & \oplus_{i,j=1}^r ({}^\dagger T_i \cup T_j)(\mathcal{E}^{(\bullet)}) \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \oplus_i ({}^\dagger T_i)\phi_i} & & \downarrow{\scriptstyle \oplus_{ij} ({}^\dagger T_i \cup T_j)\phi_i} \\
0 & \longrightarrow & \mathcal{F}^{(\bullet)} & \longrightarrow & \oplus_{i=1}^r ({}^\dagger T_i)(\mathcal{F}^{(\bullet)}) & \xrightarrow{\theta^2_{\mathcal{F}} - \theta^1_{\mathcal{F}}} & \oplus_{i,j=1}^r ({}^\dagger T_i \cup T_j)(\mathcal{F}^{(\bullet)})
\end{array}
\tag{16.2.2.1.6}$$

□

16.2.3 Duality, commutation with sp_+

Proposition 16.2.3.1. *Let $\mathfrak{P}$ be a smooth formal scheme over $\mathfrak{S}$. Let Z and X be two closed subschemes of P. Set $\mathfrak{U} := \mathfrak{P} \setminus Z$, $Y := X \setminus Z$ and $\mathfrak{V} := \mathfrak{P} \setminus X$. We have the equivalence of categories*

$$\mathbb{D}_Z^{(\bullet)} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \tag{16.2.3.1.1}$$

which is characterized by the following property: for any $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$, for any divisor T containing Z, we have the canonical isomorphism of $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$:

$$({}^\dagger T) \circ \mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{D}_T^{(\bullet)} \circ ({}^\dagger T)(\mathcal{E}^{(\bullet)}). \tag{16.2.3.1.2}$$

Proof. Let $T_1, \ldots, T_r$ be divisors of P such that $Z = T_1 \cap \cdots \cap T_r$. Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/K)$. Set $\mathcal{E}_{ij}^{(\bullet)} := ({}^\dagger T_i \cup T_j)\mathcal{E}^{(\bullet)}$ and $\mathcal{E}_i^{(\bullet)} := ({}^\dagger T_i)\mathcal{E}^{(\bullet)}$ for any $i, j \in \{1, \ldots, r\}$. We get the morphism

$$\theta_{ij}^1 \colon \mathbb{D}_{T_i}^{(\bullet)} \mathcal{E}_i^{(\bullet)} \to ({}^\dagger T_i \cup T_j) \circ \mathbb{D}_{T_i}^{(\bullet)} \mathcal{E}_i^{(\bullet)} \underset{9.2.4.20.4}{\xrightarrow{\sim}} \mathbb{D}_{T_i \cup T_j}^{(\bullet)} \mathcal{E}_{ij}^{(\bullet)}.$$

Following 16.1.11.5.1, this is a map of the abelian category $\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$. This yields the maps

$$\theta^1 \colon \oplus_{i=1}^r \mathbb{D}_{T_i}^{(\bullet)} \mathcal{E}_i^{(\bullet)} \to \oplus_{i,j=1}^r \mathbb{D}_{T_i \cup T_j}^{(\bullet)} \mathcal{E}_{ij}^{(\bullet)}.$$

Similarly, for any i, j we have the maps

$$\theta_{ij}^2 \colon \mathbb{D}_{T_j}^{(\bullet)} \mathcal{E}_j^{(\bullet)} \to ({}^\dagger T_i \cup T_j) \circ \mathbb{D}_{T_j}^{(\bullet)} \mathcal{E}_j^{(\bullet)} \underset{9.2.4.20.4}{\xrightarrow{\sim}} \mathbb{D}_{T_i \cup T_j}^{(\bullet)} \mathcal{E}_{ij}^{(\bullet)},$$

which yields the map $\theta^2 \colon \oplus_{i=1}^r \mathbb{D}_{T_i}^{(\bullet)} \mathcal{E}_i^{(\bullet)} \to \oplus_{i,j=1}^r \mathbb{D}_{T_i \cup T_j}^{(\bullet)} \mathcal{E}_{ij}^{(\bullet)}$ of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$.

We remark that if T_{r+1} is a divisor containing Z, then denoting by for $k = 1, 2$ the map $\theta'^k \colon \oplus_{i=1}^{r+1} \mathbb{D}_{T_i}^{(\bullet)} \mathcal{E}_i^{(\bullet)} \to \oplus_{i,j=1}^{r+1} \mathbb{D}_{T_i \cup T_j}^{(\bullet)} \mathcal{E}_{ij}^{(\bullet)}$ constructed as above, we compute $\ker(\theta^2 - \theta^1) = \ker(\theta'^2 - \theta'^1)$. Hence, we define canonically an object of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, Z/\mathfrak{S})$ by setting $\mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)}) := \ker(\theta^2 - \theta^1)$. It remains to prove that $\mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)})|\mathfrak{U} \in \mathrm{MIC}^{(\bullet)}(Y, \mathfrak{U}/\mathcal{V})$, i.e. we reduce to the case where Z is a (empty) divisor. In that case $\mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)}) = \mathbb{D}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}/\mathcal{V})$ (choose $r = 1$ and $T_1 = Z$). □

Proposition 16.2.3.2 (Biduality)**.** *Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. We have the canonical isomorphism of $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$:*

$$\mathbb{D}_Z^{(\bullet)} \circ \mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}. \tag{16.2.3.2.1}$$

Proof. With the lemma 16.2.1.4, we reduce to the case where Y is integral and dense in X. Since Z is the intersection of some divisors of P, then there exists a divisor T of P containing Z such that $X \setminus T$ is dense in Y. We have the isomorphisms of $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$:

$$({}^\dagger T) \circ \mathbb{D}_Z^{(\bullet)} \circ \mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)}) \overset{16.2.3.1.2}{\xrightarrow{\sim}} \mathbb{D}_T^{(\bullet)} \circ \mathbb{D}_T^{(\bullet)} \circ ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \overset{8.7.7.3}{\xrightarrow{\sim}} ({}^\dagger T)(\mathcal{E}^{(\bullet)}).$$

We conclude by using the full faithfulness of the functor $({}^\dagger T)$ of Theorem 16.2.2.1. □

Proposition 16.2.3.3. *Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. For any closed subset Z' of P containing Z, we have the canonical isomorphism of $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z'/\mathcal{V})$:*

$$({}^\dagger Z') \circ \mathbb{D}_Z^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{D}_{Z'}^{(\bullet)} \circ ({}^\dagger Z')(\mathcal{E}^{(\bullet)}). \tag{16.2.3.3.1}$$

Proof. With the lemma 16.2.1.4, we reduce to the case where Y is integral and dense in X. If Z' contains X, then the isomorphism 16.2.3.3.1 is $0 \xrightarrow{\sim} 0$. Suppose now $Y' := X \setminus Z'$ is dense in Y. Since Z' is the intersection of some divisors of P, then there exists a divisor T' of P containing Z' such that $X \setminus T'$ is dense in Y' (and in Y). By using the full faithfulness of the functor $({}^\dagger T')$ (see 16.2.2.1) and the commutation of the duality with localisation outside a divisor (see 16.2.3.1.2), we are done. □

Proposition 16.2.3.4. *Let $(Y, X, \mathfrak{P}, Z)$ be a smooth c-frame over $\mathfrak{S}$. Let $E \in \mathrm{MIC}^{\dagger}(X, \mathfrak{P}, Z/K)$. We have the canonical isomorphism:*

$$\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}(E^{\vee}) \xrightarrow{\sim} \mathbb{D}^{(\bullet)}_{Z} \circ \mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}(E),$$

where $\vee$ is the dual in $\mathrm{MIC}^{\dagger}(X, \mathfrak{P}, Z/K)$ and $\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}$ was defined at theorem 16.2.1.10.1.

Proof. With the lemma 16.2.1.4, we reduce to the case where Y is integral and dense in X. Since Z is the intersection of some divisors of P, then there exists a divisor T of P containing Z such that $X \setminus T$ is dense in Y. By using the full faithfulness of $({}^{\dagger}T)$ (see 16.2.2.1) and the commutation of the duality with localisation (see 16.2.3.1.2), we reduce to the case where Z is a divisor, which was already checked (see 16.1.11.6). □

16.2.4 (Extraordinary) pullbacks, commutation with sp_{+}

Proposition 16.2.4.1. *Let $\theta = (b, a, f) \colon (Y', X', \mathfrak{P}', Z') \to (Y, X, \mathfrak{P}, Z)$ be a morphism of smooth c-frames. Let $\mathcal{E}^{(\bullet)}$ be an object of $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/K)$. Hence, for any integer $j \in \mathbb{Z}\setminus\{0\}$, the following equality is satisfied: $H^{j}_{Z}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})[-d_{X'/X}]) = 0$ (see notation 15.3.8.13). Moreover, we have the factorisation*

$$\theta^{(\bullet)!} := \mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} \circ f^{(\bullet)!}[-d_{X'/X}] \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/K) \to (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/K). \tag{16.2.4.1.1}$$

These functors are transitive with respect to the composition: if $\theta' = (b', a', f') \colon (Y'', X'', \mathfrak{P}'', Z'') \to (Y', X', \mathfrak{P}', Z')$ is a morphism of smooth c-frames, we have the canonical isomorphism $\theta'^{(\bullet)!} \circ \theta^{(\bullet)!} \xrightarrow{\sim} (\theta \circ \theta')^{(\bullet)!}$.

Proof. By definition of the t-structure, we reduce to check $H^{j}(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})[-d_{X'/X}])|\mathfrak{U}' = 0$, where $\mathfrak{U}' := \mathfrak{P}' \setminus Z'$, i.e. we reduce to the case where Z is empty. This case was proved at 12.2. □

Notation 16.2.4.2. Let $\theta = (b, a, f) \colon (Y', X', \mathfrak{P}', Z') \to (Y, X, \mathfrak{P}, Z)$ be a morphism of smooth c-frames. Let $\mathcal{E}^{(\bullet)}$ be an object of $(F\text{-})\mathrm{MIC}^{*}(X, \mathfrak{P}, Z/K)$. With 16.2.4.1.1 and 16.2.3.1.1, we set

$$\theta^{(\bullet)+} := \mathbb{D}^{(\bullet)}_{Z'} \circ \theta^{(\bullet)!} \circ \mathbb{D}^{(\bullet)}_{Z} \colon (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/K) \to (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/K). \tag{16.2.4.2.1}$$

It follows from 16.2.3.2 and 16.2.4.1 that these functors are transitive with respect to the composition: if $\theta' = (b', a', f') \colon (Y'', X'', \mathfrak{P}'', Z'') \to (Y', X', \mathfrak{P}', Z')$ is a morphism of smooth c-frames, we have the canonical isomorphism $\theta'^{(\bullet)+} \circ \theta^{(\bullet)+} \xrightarrow{\sim} (\theta \circ \theta')^{(\bullet)+}$.

Corollary 16.2.4.3. *Let $\theta = (\beta, \alpha, \phi) \colon (Y', X', \mathfrak{P}', Z') \to (Y, X, \mathfrak{P}, Z)$ be a morphism of smooth c-frames.*

(a) The functors $\theta^{(\bullet)+}, \theta^{(\bullet)!} \colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/K) \to \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/K)$ are canonically isomorphic. [2] *We can denote them by $\theta^{(\bullet)*}$.*

(b) We have the canonical isomorphism:

$$\mathrm{sp}^{(\bullet)}_{X'\hookrightarrow\mathfrak{P}',Z',+} \circ \theta^{*} \xrightarrow{\sim} \theta^{(\bullet)+} \circ \mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}. \tag{16.2.4.3.1}$$

Proof. With the lemma 16.2.1.4, we reduce to the case where Y (resp. Y') is integral and dense in X (resp. X'). Let $T_1, \ldots, T_r$ be some of divisors of P such that $Z = T_1 \cap \cdots \cap T_r$. Since $Z' \supset \phi^{-1}(T_1) \cap \cdots \cap \phi^{-1}(T_r)$ and Z' does not contains X' then there exists i_0 such that T_{i_0} (resp. $\phi^{-1}(T_{i_0})$) does not contains X (resp. X'). In particular $\widetilde{Y} := X \setminus T_{i_0}$ is dense in Y. Set $\widetilde{T} := T_{i_0}$. Similarly, since $Z' \cup \phi^{-1}(T_{i_0})$ is an intersection of some divisor $T'_1, \ldots T'_{r'}$, there exists i'_0 such that $T'_{i'_0}$ does not contains X', i.e. $\widetilde{Y}' := X' \setminus T'_{i'_0}$ is dense in Y'. Set $\widetilde{T}' := T'_{i'_0}$. Let $\widetilde{\theta} = (\widetilde{\beta}, \alpha, \phi) \colon (\widetilde{Y}', X', \mathfrak{P}', \widetilde{T}') \to (\widetilde{Y}, X, \mathfrak{P}, \widetilde{T})$, $\iota \colon (\widetilde{Y}, X, \mathfrak{P}, \widetilde{T}) \to (Y, X, \mathfrak{P}, Z)$ and $\iota' \colon (\widetilde{Y}', X', \mathfrak{P}', \widetilde{T}') \to (Y', X', \mathfrak{P}', Z')$ be the induced morphism of smooth c-frames. Let $\widetilde{j} \colon (\widetilde{Y}, X, \mathfrak{P}) \to (Y, X, \mathfrak{P})$ and $\widetilde{j}' \colon (\widetilde{Y}', X', \mathfrak{P}') \to (Y', X', \mathfrak{P}')$ be the frame morphism.

[2] Beware this isomorphism is not compatible with Frobenius, so for the Frobenius structure we need to distinguish both inverse image functors: for instance, see 11.3.5.1.1

(a) Since $\iota^! = ({}^\dagger \widetilde{T})$ and $\iota^+ \xrightarrow{\sim} ({}^\dagger \widetilde{T})$ (use 16.2.3.3.1 and 16.2.3.3.1), then by transitivity we get the isomorphism:

$$\widetilde{\theta}^{(\bullet)+} \circ ({}^\dagger \widetilde{T}) \xrightarrow{\sim} ({}^\dagger \widetilde{T}') \circ \theta^{(\bullet)+}, \qquad \widetilde{\theta}^{(\bullet)!} \circ ({}^\dagger \widetilde{T}) \xrightarrow{\sim} ({}^\dagger \widetilde{T}') \circ \theta^{(\bullet)!}.$$

Hence, by full faithfulness of $({}^\dagger \widetilde{T}')$ (see 16.2.2.1), we reduce to check the canonical isomorphism $\widetilde{\theta}^{(\bullet)+} \xrightarrow{\sim} \widetilde{\theta}^{(\bullet)!}$, which is Theorem 16.1.10.5.

(b) We get the isomorphisms:

$$({}^\dagger \widetilde{T}') \circ \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}', Z', +} \circ \theta^* \overset{16.2.1.11.1}{\xrightarrow{\sim}} \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}', \widetilde{T}', +} \circ \widetilde{j}'^\dagger \circ \theta^* \xrightarrow{\sim} \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}', \widetilde{T}', +} \circ \widetilde{\theta}^* \circ \widetilde{j}^\dagger$$
$$\overset{16.1.8.6.1}{\xrightarrow{\sim}} \widetilde{\theta}^{(\bullet)+} \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, \widetilde{T}, +} \circ \widetilde{j}^\dagger \overset{16.2.1.11.1}{\xrightarrow{\sim}} \widetilde{\theta}^{(\bullet)+} \circ ({}^\dagger \widetilde{T}) \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +} \xrightarrow{\sim} ({}^\dagger \widetilde{T}') \circ \theta^{(\bullet)+} \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}.$$

Hence, we conclude by using the full faithfulness of $({}^\dagger \widetilde{T}')$ (see 16.2.2.1). □

16.2.5 Full faithfulness of the "restriction-inverse image" functor

Notation 16.2.5.1. Consider the commutative diagram

$$\begin{array}{ccccc} Y^{(0)} & \overset{j^{(0)}}{\hookrightarrow} & X^{(0)} & \overset{u^{(0)}}{\hookrightarrow} & \mathfrak{P}^{(0)} \\ \downarrow b & \square & \downarrow a & & \downarrow f \\ Y & \underset{j}{\hookrightarrow} & X & \underset{u}{\hookrightarrow} & \mathfrak{P}, \end{array} \tag{16.2.5.1.1}$$

where the left and middle squares are cartesian, f is a proper smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes, a is a proper surjective morphism of k-varieties, b is a generically finite and etale morphism of smooth k-varieties, j and $j^{(0)}$ are open immersions, u and $u^{(0)}$ are closed immersions. Let Z be a closed subscheme of P such that $Y = X \setminus Z$. We denote by $\mathfrak{U} := \mathfrak{P} \setminus Z$, $Z^{(0)} := f^{-1}(Z)$, $\mathfrak{U}^{(0)} := \mathfrak{P}^{(0)} \setminus Z^{(0)}$ and $g\colon \mathfrak{U}^{(0)} \to \mathfrak{U}$ the morphism induced by f. We denote by $a = (b, a, f)$ the induced morphism of smooth c-frames.

Proposition 16.2.5.2. *With notation 16.2.5.1, the functor*

$$(a^{(\bullet)+}, |\mathfrak{U})\colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X^{(0)}, \mathfrak{P}^{(0)}, Z^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{(\bullet)}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V})} \mathrm{MIC}^{(\bullet)}(Y, \mathfrak{U}/\mathcal{V}) \tag{16.2.5.2.1}$$

is fully faithful.

Proof. We can suppose $\mathfrak{P}$ is integral. Since Z is a finite intersection of divisors of P, then there exists a divisor T of P containing Z such that $\widetilde{Y} := X \setminus T$ is dense in Y. We denote by $\widetilde{\mathfrak{U}} := \mathfrak{P} \setminus \widetilde{T}$, $\widetilde{T}^{(0)} := f^{-1}(\widetilde{T})$, $\widetilde{\mathfrak{U}}^{(0)} := \mathfrak{P}^{(0)} \setminus \widetilde{T}^{(0)}$, $\widetilde{Y}^{(0)} := X^{(0)} \setminus \widetilde{T}^{(0)}$, $c\colon \widetilde{Y}^{(0)} \to \widetilde{Y}$ the morphism induced by a. Then $\widetilde{Y}^{(0)}$ is dense in $Y^{(0)}$. We denote by $\widetilde{a} = (c, a, f)$ the induced morphism of smooth d-frames. We denote by $\widetilde{j}\colon \widetilde{Y} \hookrightarrow X$ and $\widetilde{j}^{(0)}\colon \widetilde{Y}^{(0)} \hookrightarrow X^{(0)}$ the induced open immersions. Consider the following diagram

$$\begin{array}{ccc} \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) & \xrightarrow{(a^{(\bullet)+}, |\mathfrak{U})} & \mathrm{MIC}^{(\bullet)}(X^{(0)}, \mathfrak{P}^{(0)}, Z^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{(\bullet)}(Y^{(0)}, \mathfrak{U}^{(0)}/\mathcal{V})} \mathrm{MIC}^{(\bullet)}(Y, \mathfrak{U}/\mathcal{V}) \\ \downarrow ({}^\dagger \widetilde{T}) & & \downarrow (({}^\dagger \widetilde{T}^{(0)}), |\widetilde{\mathfrak{U}}) \\ \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, \widetilde{T}/\mathcal{V}) & \xrightarrow{(\widetilde{a}^{(\bullet)+}, |\widetilde{\mathfrak{U}})} & \mathrm{MIC}^{(\bullet)}(X^{(0)}, \mathfrak{P}^{(0)}, \widetilde{T}^{(0)}/\mathcal{V}) \times_{\mathrm{MIC}^{(\bullet)}(\widetilde{Y}^{(0)}, \widetilde{\mathfrak{U}}^{(0)}/\mathcal{V})} \mathrm{MIC}^{(\bullet)}(\widetilde{Y}, \widetilde{\mathfrak{U}}/\mathcal{V}). \end{array} \tag{16.2.5.2.2}$$

Following 13.2.1.4, the local functors commute with extraordinary inverse images. It follows from 16.2.3.3.1, the localisation outside a divisor commutes with duality. This yields the commutativity up to canonical isomorphism of the diagram 16.2.5.2.2. Since following 16.2.2.1 (resp. 16.1.5.2) the left functor (resp. bottom) is fully faithful, as the right one is faithful, the upper functor is then fully faithful. □

Remark 16.2.5.3. The extension from the case of divisors to that of closed subschemes in 16.1.11.4 is not clear since we need first to define the dual functor $\mathbb{D}^{(\bullet}_Z$ in this context.

16.2.5.4. Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, $Z \subset Z'$ be two closed subschemes of P, X be a closed subscheme of P. We set $\mathfrak{U} := \mathfrak{P} \setminus Z$, $\mathfrak{U}' := \mathfrak{P} \setminus Z'$, $Y := X \setminus Z$, $Y' := X \setminus Z'$, $j \colon Y \subset X$ and $j' \colon Y' \subset X$ the canonical open immersions. We suppose moreover Y smooth and Y' dense in Y.

(a) Following the contagiosity theorem of Kedlaya (see 16.1.8.1), the bottom functor of the square

$$\begin{array}{ccc} \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/K) & \xrightarrow{((^\dagger Z'),|\mathfrak{U})} & \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z'/K) \times_{\mathrm{MIC}^{(\bullet)}(Y,\mathfrak{U},Z'\cap U/K)} \mathrm{MIC}^{(\bullet)}(Y,\mathfrak{U}/K) \\ \uparrow \mathrm{sp}_{X\hookrightarrow \mathfrak{P},Z,+} & & \uparrow (\mathrm{sp}_{X\hookrightarrow \mathfrak{P},Z',+},\mathrm{sp}_{Y\hookrightarrow \mathfrak{U},+}) \\ \mathrm{MIC}^{\dagger}(Y,X/K) & \xrightarrow{(j'^\dagger,j^*)} & \mathrm{MIC}^{\dagger}(Y',X/K) \times_{\mathrm{MIC}^{\dagger}(Y',Y/K)} \mathrm{MIC}^{\dagger}(Y,Y/K) \end{array} \tag{16.2.5.4.1}$$

is an equivalence of categories. Since this square is commutative up to canonical isomorphism, since the vertical functors are equivalences of categories (see 16.2.1.10.1), then so is the top functor $((^\dagger Z'),|\mathfrak{U})$.

(b) When Z and Z' are in fact the support of some divisors T and T', then by using to 16.1.11.4, we can explicitly build the quasi-inverse functor $\mathcal{G}lue$ of $((^\dagger T'),|\mathfrak{U})$ by setting, for any object $(\mathcal{E}',\mathcal{F}_{\mathfrak{U}},\rho) \in \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T'/K) \times_{\mathrm{MIC}^{\dagger\dagger}(Y,\mathfrak{U},T'/K)} \mathrm{MIC}^{\dagger\dagger}(Y,\mathfrak{U}/K)$,

$$\mathcal{G}lue(\mathcal{E}',\mathcal{F}_{\mathfrak{U}},\rho) := \mathrm{Im}\left(\mathbb{D}_T \circ \mathbb{D}_{T'}(\mathcal{E}') \to \mathcal{E}'\right).$$

16.2.6 Tensor products, commutation with sp_+

Let $(Y,X,\mathfrak{P},Z)$ and $(Y',X',\mathfrak{P}',Z')$ be two smooth c-frames (see the definition 16.2.1.8). We set $\mathfrak{P}'' := \mathfrak{P}\times\mathfrak{P}'$, $X'' := X \times X'$, $Y'' := Y \times Y'$, $j \colon Y \subset X$, $j' \colon Y' \subset X'$ and $j'' \colon Y'' \subset X''$ the canonical inclusions. We denote by $\theta = (b,a,p) \colon (Y'',X'',\mathfrak{P}'',Z'') \to (Y,X,\mathfrak{P},Z)$ and $\theta' = (b',a',p') \colon (Y'',X'',\mathfrak{P}'',Z'') \to (Y',X',\mathfrak{P}',Z')$ the morphisms of d-frames induced by the canonical projections, where $Z'' = p^{-1}(Z) \cup p'^{-1}(Z')$.

Divisorial case: When Z (resp. Z') are the support of a divisor T (resp. T'), then we get smooth d-frames (see the definition 12.2.1.1) and θ and θ' becomes morphisms of d-frames.

Lemma 16.2.6.1. *Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/\mathcal{V})$, $\mathcal{E}'^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X',\mathfrak{P}',Z'/\mathcal{V})$.*

(a) For any integer $j \neq 0$, we have $H^j(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} 0$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}(Z''))$. Moreover, $H^0(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X'',\mathfrak{P}'',Z''/\mathcal{V})$.

(b) We have $H^0(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}[d_{Y/P}]) \in \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/\mathcal{V})$ and the isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$:

$$\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}[d_{Y/P}] \xrightarrow{\sim} H^0(\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}[d_{Y/P}]).$$

Proof. 1) Suppose Z (resp. Z') are the support of a divisor T (resp. T').

a) Let us check (a). Let us denote by $\mathfrak{U} := \mathfrak{P}\setminus T$, $\mathfrak{U}' := \mathfrak{P}'\setminus T'$, $\mathfrak{U}'' := \mathfrak{P}''\setminus T''$. As $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}|\mathfrak{U}'' = \mathcal{E}^{(\bullet)}|\mathfrak{U}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}|\mathfrak{U}'$, following the already known case of completely smooth d-frames (see 12.2.1.8), then we have $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}|\mathfrak{U}'' \in \mathrm{MIC}^{(\bullet)}(Y'',\mathfrak{U}''/\mathcal{V})$. It is then sufficient to prove that $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}(T''))$. As $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}(T''))$ the idea is as usual to proceed by descent: with the lemma 16.1.2.9 (and the remark 16.1.2.10), we reduce to the case where Y is dense in X with X integral and where Y' is dense in X' with X' integral. Following 16.1.11.1 there exist a morphism of d-frames of the form $\alpha = (h,g,f) \colon (\widetilde{Y},\widetilde{X},\widetilde{\mathfrak{P}},\widetilde{T}) \to (Y,X,\mathfrak{P},T)$ where $\widetilde{X}$ is smooth, $\widetilde{T} = f^{-1}(T)$ and $\widetilde{T}\cap\widetilde{X}$ is a strict normal crossing divisor of $\widetilde{X}$, f is a proper and smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes, g is a proper, surjective, generically finite etale morphism of k-varieties. We set $\widetilde{\mathfrak{P}}'' := \widetilde{\mathfrak{P}}\times\mathfrak{P}'$, $\widetilde{X}'' := \widetilde{X}\times X'$, $\widetilde{Y}'' := \widetilde{Y}\times Y'$, $\alpha'' = (h'',g'',f'') \colon (\widetilde{Y}'',\widetilde{X}'',\widetilde{\mathfrak{P}}'',\widetilde{T}'') \to (Y'',X'',\mathfrak{P}'',T'')$ the morphism of d-frames induced by α, where $\widetilde{T}'' = f''^{-1}(T'')$. We denote by $\widetilde{\theta} = (\widetilde{b},\widetilde{a},\widetilde{p}) \colon (\widetilde{Y}'',\widetilde{X}'',\widetilde{\mathfrak{P}}'',\widetilde{T}'') \to (\widetilde{\mathfrak{P}},\widetilde{T},\widetilde{X},\widetilde{Y})$ and $\widetilde{\theta}' = (\widetilde{p}',\widetilde{a}',\widetilde{b}') \colon (\widetilde{Y}'',\widetilde{X}'',\widetilde{\mathfrak{P}}'',\widetilde{T}'') \to (Y',X',\mathfrak{P}',T')$ the morphisms of d-frames induced by the canonical projections. Let us denote by $\widetilde{\mathcal{E}}^{(\bullet)} := \alpha^{(\bullet)}(\mathcal{E}^{(\bullet)}) = \mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{X}} f_T^{(\bullet)!}(\mathcal{E}^{(\bullet)})$. By stability of the overcoherence by extraordinary inverse image (see 15.3.6.12), we get $\widetilde{\mathcal{E}}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\widetilde{\mathfrak{P}}}(\widetilde{T}))$.

Following 16.1.11.2, $\mathcal{E}^{(\bullet)}$ is a direct summand of $f_{T+}^{(\bullet)}(\widetilde{\mathcal{E}}^{(\bullet)})$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. This implies that $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}$ is a direct summand of $f_{T+}^{(\bullet)}(\widetilde{\mathcal{E}}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}$ in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}(T''))$. Moreover, following 9.4.4.2.2, we have the isomorphism $f_{T,+}^{(\bullet)}(\widetilde{\mathcal{E}}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)} \xrightarrow{\sim} f''^{(\bullet)}_{T'',+}\left(\widetilde{\mathcal{E}}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)}\right)$. By stability of the overcoherence by the direct image of a proper morphism, we reduce then to the case where X is smooth and $T\cap X$ is strict normal crossing divisor of X. In the same way, we reduce then to the case where X' is smooth and $T'\cap X'$ is a strict normal crossing divisor of X'. In that case, following 12.2.1.8, we obtain $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X'',\mathfrak{P}'',T''/\mathcal{V})$.

b) Let us check (b). Let $\delta\colon \mathfrak{P}\hookrightarrow\mathfrak{P}\times\mathfrak{P}$ be the diagonal immersion. By using 9.2.5.15.1 (recall also 9.1.2.8.1 and 9.2.5.3) in the case where $(Y,X,\mathfrak{P},T) = (Y',X',\mathfrak{P}',T')$, we get $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} \delta^{(\bullet)!}(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)})[-d_P]$. According to (a), we have $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}(T''))$. By stability of the overcoherence by extraordinary inverse image (see 15.3.6.12, this yields that $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T))$. Hence, it is sufficient to check the lemma apart from T (e.g. we use 8.7.6.11, the characterization of categories of the form $\mathrm{MIC}^{(\bullet)}$ (see 16.2.1.1)

Let us denote by $\mathfrak{U} := \mathfrak{P}\backslash T$. Using the case treated in 12.2.1.13, this yields that $\mathcal{E}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathcal{F}^{(\bullet)}[d_{Y/P}]|_{\mathfrak{U}} = \mathcal{E}^{(\bullet)}|_{\mathfrak{U}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{U}}}\mathcal{F}^{(\bullet)}|_{\mathfrak{U}}[d_{Y/U}] \xrightarrow{\sim} H^0(\mathcal{E}^{(\bullet)}|_{\mathfrak{U}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{U}}}\mathcal{F}^{(\bullet)}|_{\mathfrak{U}}[d_{Y/U}]) \in \mathrm{MIC}^{(\bullet)}(Y,\mathfrak{U}/\mathcal{V})$.

2) Let us go back to the general case. Since tensor products commute with extraordinary pullbacks and with localisation functors (see 13.1.2.1), then the part (b) follows from the divisorial case. Since the categories of the form $\mathrm{MIC}^{(\bullet)}$ commutes with inverse images, the part (a) is a consequence of (b) (recall definition 9.2.5.1.5). ☐

Corollary 16.2.6.2 (Divisorial case)**.** *Suppose Z (resp. Z') are the support of a divisor T (resp. T'). Let $\mathcal{E},\mathcal{F} \in \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V})$, $\mathcal{E}' \in \mathrm{MIC}^{\dagger\dagger}(X',\mathfrak{P}',T'/\mathcal{V})$.*

(a) We have $H^0(\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{F}[d_{Y/P}]) \in \mathrm{MIC}^{\dagger\dagger}(X,\mathfrak{P},T/\mathcal{V})$ and the isomorphism

$$\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{F}[d_{Y/P}] \xrightarrow{\sim} H^0(\mathcal{E}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{F}[d_{Y/P}]).$$

(b) For any integer $j\neq 0$, we have the isomorphism $H^j(\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},T,T'}\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} 0$ and $H^0(\mathcal{E}\overset{\mathbb{L}}{\boxtimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{S}},T,T'}\mathcal{E}'^{(\bullet)}) \in \mathrm{MIC}^{\dagger\dagger}(X'',\mathfrak{P}'',T''/\mathcal{V})$.

Lemma 16.2.6.3. *Let $E,F \in \mathrm{MIC}^{\dagger}(Y,X,\mathfrak{P}/K)$. We have the canonical isomorphism in* $\mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/\mathcal{V})$*:*

$$\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}(E\otimes_{j^{\dagger}\mathcal{O}_{]X[_{\mathfrak{P}}}}F) \xrightarrow{\sim} \mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}(E)\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}(F)[d_{Y/P}].$$

Proof. Following 16.2.1.10.1, we have the equivalence of categories $\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}\colon \mathrm{MIC}^{\dagger}(Y,X,\mathfrak{P}/K) \cong \mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/\mathcal{V})$. Following the lemma 16.2.6.1.(b), this yields that the two terms are in $\mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/\mathcal{V})$. With the lemma 16.2.1.4 (and the remark 16.1.2.10), we reduce to the case where X is integral and Y is dense in X. To come down to the case where X is smooth (already known case in 12.2.6.5), we use the fully faithfulness theorem 16.2.5.2 as follows: following the theorem of desingularisation of de Jong (see [dJ96, 4.1]), there exist a diagram of the form

$$\begin{array}{ccccc} Y^{(0)} & \xhookrightarrow{j^{(0)}} & X^{(0)} & \xhookrightarrow{u^{(0)}} & \mathfrak{P}^{(0)} \\ \downarrow b & \square & \downarrow a & & \downarrow f \\ Y & \xhookrightarrow{j} & X & \xhookrightarrow{u} & \mathfrak{P}, \end{array} \qquad (16.2.6.3.1)$$

where the left square is cartesian, $X^{(0)}$ is smooth, f is a proper and smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes, a is a proper, surjective, generically finite etale morphism of k-varieties,

$T^{(0)} := f^{-1}(Z)$ is a divisor of $P^{(0)}$, j and $j^{(0)}$ are open immersions, u and $u^{(0)}$ are closed immersions. Let us denote by $\theta := (b, a, f)$ the morphism of c-frames. We have the isomorphisms:

$$\begin{aligned}
\theta^* \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(E \otimes_{j^\dagger \mathcal{O}_{]X[_\mathfrak{P}}} F) &\xrightarrow[16.2.4.3.1]{\sim} \mathrm{sp}^{(\bullet)}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +} \circ \theta^*(E \otimes_{j^\dagger \mathcal{O}_{]X[_\mathfrak{P}}} F) \\
&\xrightarrow{\sim} \mathrm{sp}^{(\bullet)}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +} \left(\theta^*(E) \otimes_{j^{(0)\dagger} \mathcal{O}_{]X^{(0)}[_{\mathfrak{P}^{(0)}}}} \theta^*(F) \right) \\
&\xrightarrow[12.2.6.5.1]{\sim} \mathrm{sp}^{(\bullet)}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +}(\theta^*(E)) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}^{(0)}}} \mathrm{sp}^{(\bullet)}_{X^{(0)} \hookrightarrow \mathfrak{P}^{(0)}, T^{(0)}, +}(\theta^*(F))[d_{Y^{(0)}/P^{(0)}}] \\
&\xrightarrow[16.2.4.3.1]{\sim} \theta^* \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(E) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}^{(0)}}} \theta^* \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(F)[d_{Y^{(0)}/P^{(0)}}] \\
&\xrightarrow{\sim} \theta^* \left(\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(E) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(F)[d_{Y/P}] \right). \qquad (16.2.6.3.2)
\end{aligned}$$

Outside the divisor $T^{(0)}$, this isomorphism is 12.2.6.5.1. As the functor $(\theta^*, |\mathfrak{U})$ is fully faithful (see 12.2.6.5, this yields the proposition. □

Notation 16.2.6.4. Let $E \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ and $E' \in \mathrm{MIC}^\dagger(Y', X', \mathfrak{P}'/K)$. With the notations of 16.1.2.8 (see also 16.1.6.11), we define the bifunctor $- \boxtimes -: \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K) \times \mathrm{MIC}^\dagger(Y', X', \mathfrak{P}'/K) \to \mathrm{MIC}^\dagger(Y'', X'', \mathfrak{P}''/K)$ by setting

$$E \boxtimes E' := \theta^*(E) \otimes_{j''^\dagger \mathcal{O}_{]X''[_{\mathfrak{P}''}}} \theta'^*(E').$$

Proposition 16.2.6.5. *With the notations 16.2.6.4, we have the canonical isomorphism in* $\mathrm{MIC}^{(\bullet)}(X'', \mathfrak{P}'', Z''/\mathcal{V})$*:*

$$\mathrm{sp}^{(\bullet)}_{X'' \hookrightarrow \mathfrak{P}'', Z'', +}(E \boxtimes E') \xrightarrow{\sim} \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(E) \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_\mathfrak{S}} \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}', Z', +}(E'). \qquad (16.2.6.5.1)$$

Proof. By using 16.2.6.1.(a), the right term of 16.2.6.5.1 belongs to $\mathrm{MIC}^{(\bullet)}(X'', \mathfrak{P}'', Z''/\mathcal{V})$. Since so is the the left one, then using 15.3.8.2 and replacing the use of Proposition 12.2.6.5 by that of Lemma 16.2.6.3, 12.2.4.1.2 by 16.2.4.3, we can copy the proof of 12.2.6.7. □

Theorem 16.2.6.6. *We suppose $\mathfrak{P}' = \mathfrak{P}$ and that $(Y \cap Y', X \cap X', \mathfrak{P}, Z \cup Z')$ is a smooth c-frame. Let us denote by $i: (Y \cap Y', X \cap X', \mathfrak{P}, Z \cup Z') \to (Y, X, \mathfrak{P}, Z)$, $i': (Y \cap Y', X \cap X', \mathfrak{P}, Z \cup Z') \to (Y', X', \mathfrak{P}, Z')$ the canonical morphisms of c-frames and $\widetilde{j}: Y \cap Y' \subset X \cap X'$ be the canonical inclusion.*

(a) For any $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$, $\mathcal{E}'^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}, Z'/\mathcal{V})$, we have the canonical isomorphism in $\mathrm{MIC}^{(\bullet)}(X \cap X', \mathfrak{P}, Z \cup Z'/\mathcal{V})$ of the form:

$$\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_\mathfrak{P}} \mathcal{E}'^{(\bullet)}[d_Y + d_{Y'} - d_{Y \cap Y'} - d_P] \xrightarrow{\sim} i^{(\bullet)*}(\mathcal{E}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_\mathfrak{P}} i'^{(\bullet)*}(\mathcal{E}'^{(\bullet)})[d_{Y \cap Y'/P}]. \qquad (16.2.6.6.1)$$

(b) For any $E \in \mathrm{MIC}^\dagger(Y, X, \mathfrak{P}/K)$ and $E' \in \mathrm{MIC}^\dagger(Y', X', \mathfrak{P}'/K)$, we have the canonical isomorphism in $\mathrm{MIC}^{(\bullet)}(X \cap X', \mathfrak{P}, Z \cup Z'/\mathcal{V})$:

$$\begin{aligned}
&\mathrm{sp}^{(\bullet)}_{X \cap X' \hookrightarrow \mathfrak{P}, Z \cup Z', +}(i^*(E) \otimes_{\widetilde{j}^\dagger \mathcal{O}_{]X \cap X'[_\mathfrak{P}}} i'^*(E')) \\
&\xrightarrow{\sim} \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z, +}(E) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_\mathfrak{P}} \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}, Z', +}(E')[d_Y + d_{Y'} - d_{Y \cap Y'} - d_P]. \qquad (16.2.6.6.2)
\end{aligned}$$

Proof. Let us treat first 16.2.6.6.1. Since we have the isomorphisms $\mathbb{R}\underline{\Gamma}^\dagger_Y \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$, $\mathbb{R}\underline{\Gamma}^\dagger_{Y'} \mathcal{E}'^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'^{(\bullet)}$ and since $\mathbb{R}\underline{\Gamma}^\dagger_Y \circ \mathbb{R}\underline{\Gamma}^\dagger_{Y'} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Y \cap Y'}$ (see 13.1.5.6.1), then we get from 13.1.5.6.2 the isomorphism:

$$\begin{aligned}
\mathcal{E}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_\mathfrak{P}} \mathcal{E}'^{(\bullet)} &\xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Y \cap Y'}(\mathcal{E}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_\mathfrak{P}} \mathbb{R}\underline{\Gamma}^\dagger_{Y \cap Y'}(\mathcal{E}'^{(\bullet)}) \\
&= i^{(\bullet)*}(\mathcal{E}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_\mathfrak{P}} i'^{(\bullet)*}(\mathcal{E}'^{(\bullet)})[2d_{Y \cap Y'} - (d_Y + d_{Y'})]. \qquad (16.2.6.6.3)
\end{aligned}$$

It follows from 16.2.4.3 that $i^{(\bullet)*}(\mathcal{E}^{(\bullet)})$, $i'^{(\bullet)*}(\mathcal{E}'^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X \cap X', \mathfrak{P}, Z \cup Z'/\mathcal{V})$. The lemma 16.2.6.1.(b) allow us to conclude the check of 16.2.6.6.1. Finally, the isomorphism 16.2.6.6.2 can be built by composing the isomorphisms below:

$$\begin{aligned}
&\mathrm{sp}^{(\bullet)}_{X\cap X' \hookrightarrow \mathfrak{P}, Z\cup Z',+}(i^*(E) \otimes_{\widetilde{j}^\dagger \mathcal{O}_{]X\cap X'[_{\mathfrak{P}}}} i'^*(E')) \\
&\overset{16.2.6.3}{\underset{\sim}{\longrightarrow}} \mathrm{sp}^{(\bullet)}_{X\cap X' \hookrightarrow \mathfrak{P}, Z\cup Z',+}(i^*(E)) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathrm{sp}^{(\bullet)}_{X\cap X' \hookrightarrow \mathfrak{P}, Z\cup Z',+}(i'^*(E'))[d_{Y\cap Y'/P}] \\
&\underset{16.2.4.3.1}{\overset{\sim}{\longrightarrow}} i^{(\bullet)*} \circ \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z,+}(E) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} i'^{(\bullet)*} \circ \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}, Z',+}(E')[d_{Y\cap Y'/P}] \\
&= \mathbb{R}\underline{\Gamma}^\dagger_{Y\cap Y'} \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z,+}(E) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathbb{R}\underline{\Gamma}^\dagger_{Y\cap Y'} \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}, Z',+}(E')[d_Y + d_{Y'} - d_{Y\cap Y'} - d_P] \\
&\overset{16.2.6.6.3}{\underset{\sim}{\longrightarrow}} \mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z,+}(E) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} \mathrm{sp}^{(\bullet)}_{X' \hookrightarrow \mathfrak{P}, Z',+}(E')[d_Y + d_{Y'} - d_{Y\cap Y'} - d_P].
\end{aligned}$$

□

16.2.7 Canonical independence in smooth c-frame

Lemma 16.2.7.1. *Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of separated and smooth $\mathfrak{S}$-formal schemes, $u'\colon X' \hookrightarrow P'$ be a closed immersion with X' integral, Z be a closed subscheme of P such that $Z' := f^{-1}(Z)$ is a closed subscheme of P' not containing X' and such that $Y' := X' \setminus Z'$ is smooth. We suppose $f \circ u'$ proper. Let $\mathcal{E}'^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$.*

(a) We have $f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$.

(b) If Z is the support of some divisor T and $\mathcal{E}' := \underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$ then $\mathbb{D}_T \circ f_{T,+}(\mathcal{E}'^{(\bullet)!}) \in (F\text{-})D^{\mathrm{b}}_{\mathrm{oc}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$.

(c) Suppose the induced morphism $Y' \to P$ is an immersion. Let X be the closure of Y' in P. Then $f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$.

Proof. Let us prove a): by using the stability of our category with respect to (extraordinary) inverse images (see 16.2.4.3), by using the commutation of the pushforward by a proper morphism to the extraordinary pullback (see 13.2.3.7), we reduce to check that for any divisor T of P containing Z we have $({}^\dagger T) f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. We get

$$({}^\dagger T) \circ f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \overset{13.2.1.4.2}{\underset{\sim}{\longrightarrow}} f^{(\bullet)}_+ \circ ({}^\dagger f^{-1}(T))(\mathcal{E}'^{(\bullet)}).$$

We can suppose P' is integral. So either $f^{-1}(T)$ is a divisor of P' or $f^{-1}(T) = P'$. Since the second case is obvious (because we get therefore $({}^\dagger f^{-1}(T))(\mathcal{E}'^{(\bullet)}) = 0$, then we can suppose $T' := f^{-1}(T)$ is a divisor of P'. Set $\widetilde{Y}' := X' \setminus T'$. Set $\widetilde{\mathcal{E}}'^{(\bullet)} := ({}^\dagger T')(\mathcal{E}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', T'/\mathcal{V})$. It follows from de Jong desingularization theorem that there exists a diagram of the form:

$$\begin{array}{ccccccccc}
\widetilde{Y}'' & \longrightarrow & Y'' & \xrightarrow{j''} & X'' & \xrightarrow{u''} & \widehat{\mathbb{P}}^N_{\mathfrak{P}'} & \xrightarrow{\widehat{\mathbb{P}}^N_f} & \widehat{\mathbb{P}}^N_{\mathfrak{P}} \\
\downarrow{\scriptstyle c'} & \Box & \downarrow{\scriptstyle b'} & \Box & \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle q'} & \Box & \downarrow{\scriptstyle q} \\
\widetilde{Y}' & \longrightarrow & Y' & \xrightarrow{j'} & X' & \xrightarrow{u'} & \mathfrak{P}' & \xrightarrow{f} & \mathfrak{P},
\end{array} \qquad (16.2.7.1.1)$$

where X'' is quasi-projective and smooth over k, q and q' are the canonical projections, u'' is a closed immersion, $a'^{-1}(T' \cap X')$ is a strict normal crossing divisor of X'', a' is proper, surjective, generically finite and etale. More precisely, there exists an immersion $\iota\colon X'' \hookrightarrow \widehat{\mathbb{P}}^N_{\mathfrak{S}}$ such that u'' is the composition of the graph of $u' \circ a'$ with $\iota \times \mathrm{id}\colon X' \times \mathfrak{P}' \hookrightarrow \widehat{\mathbb{P}}^N_{\mathfrak{P}'}$. Since $u' \circ a'$ and q are proper, then so is u'' which is therefore a closed immersion. Similarly, we can check that the morphism $\tilde{f} \circ u''$ is an immersion which is closed because of the properness of $f \circ u' \circ a$ and q. Denote by $\tilde{f} := \widehat{\mathbb{P}}^N_f$, $\widetilde{\mathfrak{P}} := \widehat{\mathbb{P}}^N_{\mathfrak{P}}$, $\widetilde{T} := q^{-1}(T)$,$\eta' := (c', a', q')$ and $\widetilde{\mathcal{E}}''^{(\bullet)} := \eta'^{!}(\widetilde{\mathcal{E}}'^{(\bullet)})$.

Since $T'' \cap X'' = \widetilde{T} \cap X''$, via 16.1.9.3.(b), then we get $\tilde{f}^{(\bullet)}_+(\widetilde{\mathcal{E}}''^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X'', \widetilde{\mathfrak{P}}, \widetilde{T}/\mathcal{V})$. Since q is proper, $q^{(\bullet)}_+$ preserves the overcoherence (see 15.3.6.14). Moreover, by transitivity of the direct image $f^{(\bullet)}_+ \circ q'^{(\bullet)}_+(\widetilde{\mathcal{E}}''^{(\bullet)}) \xrightarrow{\sim} q^{(\bullet)}_+ \circ \tilde{f}^{(\bullet)}_+(\widetilde{\mathcal{E}}''^{(\bullet)})$. This yields that $f^{(\bullet)}_+ \circ q'^{(\bullet)}_+(\widetilde{\mathcal{E}}''^{(\bullet)}) \in (F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^*\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. Since $\widetilde{\mathcal{E}}'^{(\bullet)}$ is a direct summand of $q'^{(\bullet)}_+(\widetilde{\mathcal{E}}''^{(\bullet)})$, then $f^{(\bullet)}_+(\widetilde{\mathcal{E}}'^{(\bullet)})$ is a direct summand of $f^{(\bullet)}_+ \circ q'^{(\bullet)}_+(\widetilde{\mathcal{E}}''^{(\bullet)})$ Hence $f^{(\bullet)}_+(\widetilde{\mathcal{E}}'^{(\bullet)}) \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$.

Let us check (b). In the case where $Z = T$ and $Z' = T'$, set $\mathcal{E}''^{(\bullet)} := \eta'^{!}(\widetilde{\mathcal{E}}'^{(\bullet)})$ and $\mathcal{E}'' := \mathcal{E}''^{(\bullet)}$. Since $f_+ \circ q'_+(\mathcal{E}'') \xrightarrow{\sim} q_+ \circ \tilde{f}_+(\mathcal{E}'')$. Since $\tilde{f}^{(\bullet)}_+(\mathcal{E}''^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X'', \widetilde{\mathfrak{P}}, \widetilde{T}/\mathcal{V})$, then it follows from 16.1.11.3 that $\mathbb{D}_T \circ f_+ \circ q'_+(\mathcal{E}'') \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$. Hence, we get $\mathbb{D}_T \circ f_+(\mathcal{E}'^{(\bullet)})$ is a direct summand of $\mathbb{D}_T \circ f_+ \circ q'_+(\mathcal{E}'')$. Hence $\mathbb{D}_T \circ f_+(\mathcal{E}') \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}})$.

Let us consider now c). Following a), it is already known that $f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$. Hence, it remains more to prove that $f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})|\mathfrak{U} \in \mathrm{MIC}^{(\bullet)}(Y', \mathfrak{U}/\mathcal{V})$, where $\mathfrak{U} := \mathfrak{P} \setminus T$. This is a consequence of the completely smooth d-frame case which was already treated at 16.1.9.3. Hence we are done. □

Corollary 16.2.7.2. *With notation 16.2.7.1, let $g\colon \mathfrak{Q} \to \mathfrak{P}$ be a smooth morphism of $\mathfrak{S}$-smooth formal schemes. We denote by $\mathfrak{Q}' := \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{Q}$, by $f'\colon \mathfrak{Q}' \to \mathfrak{Q}$ and $g'\colon \mathfrak{Q}' \to \mathfrak{P}'$ the two canonical projections, $U := g^{-1}(Z)$, $U' := g'^{-1}(Z')$. There then exists a canonical isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Q}/\mathfrak{S}}(U))$:*

$$g^{(\bullet)!} \circ f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} f'^{(\bullet)}_+ \circ g'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}). \tag{16.2.7.2.1}$$

Proof. Using Lemma 16.2.7.1 and replacing the coherent version by the overconvergent version of Berthelot-Kashiwara theorem (see 15.3.8.26), then we can indeed copy the proof of 13.2.3.7.

□

Lemma 16.2.7.3. *Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, X, X' be two smooth closed subschemes of P, Z, Z' be two closed subschemes of P such that $Z \cap X = Z' \cap X'$. We get then the equality: $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) = (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}, Z'/\mathcal{V})$.*

Proof. This is checked similarly to 15.3.8.25 □

Proposition 16.2.7.4. Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, X, Z, X' and Z' be some closed subschemes of P such that $Y := X \setminus Z$ is smooth and $X \setminus Z = X' \setminus Z'$. Then we have the equality $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) = (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}, Z'/\mathcal{V})$. In particular, writing the schematic closure of Y in P by $\overline{Y}$, we get $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) = (F\text{-})\mathrm{MIC}^{(\bullet)}(\overline{Y}, \mathfrak{P}, \overline{Y} \setminus Y/\mathcal{V})$.

Proof. 1) Suppose $Z = Z'$. This case is checked similarly to the part 1) of the proof of 16.1.9.2. we are reduced to the case where $X = X' = \overline{Y}$ and $X \setminus Z$ is not empty.

2) Suppose $X = X'$ and $Z \subset Z'$. Similarly to the part 2) of the proof of 16.1.9.2, we get $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \subset (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z'/\mathcal{V})$. Conversely, let $\mathcal{E}'^{(\bullet)}$ be an object of $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z'/\mathcal{V})$. By 16.2.1.4, we can suppose that X is irreducible. It remains to show that $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$ (indeed, the second condition of the definition 16.2.1.1 holds since we are in the completely smooth context of 16.2.7.3). By using the stability of our category with respect to (extraordinary) inverse images (see 16.2.4.3), we reduce to check that for any divisor T of P containing Z we have $({}^\dagger T)\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. Since X is integral, then either $T \cap X$ is a divisor of X or $T \cap X = X$. Since the second case is obvious (because we get therefore $({}^\dagger T)(\mathcal{E}'^{(\bullet)}) = 0$, then we can suppose $D := T \cap X$ is a divisor of X. Set $\widetilde{Y} := X \setminus T$. Set $\widetilde{\mathcal{E}}'^{(\bullet)} := ({}^\dagger T)(\mathcal{E}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T \cup Z'/\mathcal{V})$. We have to check that $\widetilde{\mathcal{E}}'^{(\bullet)} := ({}^\dagger T)(\mathcal{E}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$.

Let $T'_1, \ldots, T'_r$ be some divisors of P such that $T \cup Z' = T'_1 \cap \cdots \cap T'_r$. Since $\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T \cup Z'/\mathcal{V})$ is an abelian subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{poc}}(X, \mathfrak{P}, T \cup Z'/\mathcal{V})$, then it follows from 15.3.8.19.1 that we have the exact sequence

$$0 \to \widetilde{\mathcal{E}}'^{(\bullet)} \to \oplus_{i=1}^r ({}^\dagger T'_i)(\widetilde{\mathcal{E}}'^{(\bullet)}) \xrightarrow{\theta^2 - \theta^1} \oplus_{i,j=1}^r ({}^\dagger T'_i \cup T'_j)(\widetilde{\mathcal{E}}'^{(\bullet)}). \tag{16.2.7.4.1}$$

where $\theta^n\colon \widetilde{\mathcal{E}}'^{(\bullet)} \to \oplus_{i=1}^{r} ({}^\dagger T'_i)(\widetilde{\mathcal{E}}'^{(\bullet)}) \to \oplus_{i,j=1}^{r} ({}^\dagger T'_i \cup T'_j)(\widetilde{\mathcal{E}}'^{(\bullet)})$ for $n = 0, 1$ are the canonical maps defined at 15.3.8.19. Hence, we reduce to check that for any divisor T' containing $T \cup Z'$, we have $({}^\dagger T')(\widetilde{\mathcal{E}}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T/\mathcal{V})$.

Set $\widetilde{Y} := X \setminus T$ and $\widetilde{Y}' := X \setminus T'$. It follows from de Jong desingularization theorem that there exists a diagram of the form:

$$\begin{array}{ccccccccc} \widetilde{Y}'^{(0)} & \xrightarrow{l^{(0)}} & \widetilde{Y}^{(0)} & \xrightarrow{l^{(0)}} & Y^{(0)} & \xrightarrow{j^{(0)}} & X^{(0)} & \xrightarrow{u^{(0)}} & \mathfrak{P}^{(0)} \\ \downarrow d & \square & \downarrow c & \square & \downarrow b & \square & \downarrow a & & \downarrow f \\ \widetilde{Y}' & \xrightarrow{l} & \widetilde{Y} & \xrightarrow{l} & Y & \xrightarrow{j} & X & \xrightarrow{u} & \mathfrak{P}, \end{array} \qquad (16.2.7.4.2)$$

where the left and middle squares are cartesian, f is a proper smooth morphism of separated and smooth $\mathfrak{S}$-formal schemes, a is a proper surjective morphism of k-varieties, b is a morphism of smooth k-varieties, c is a generically finite and etale morphism, l, $l^{(0)}$, j and $j^{(0)}$ are open immersions, u and $u^{(0)}$ are closed immersions, $\widetilde{Y}$ is dense in Y and $\widetilde{Y}^{(0)}$ is dense in $Y^{(0)}$. We denote by $\widetilde{j}\colon \widetilde{Y} \hookrightarrow X$ and $\widetilde{j}^{(0)}\colon \widetilde{Y}^{(0)} \hookrightarrow X^{(0)}$ the induced open immersions. We set $\mathfrak{U} := \mathfrak{P} \setminus Z$, $Z^{(0)} := f^{-1}(Z)$, $Z'^{(0)} := f^{-1}(Z')$, $T^{(0)} := f^{-1}(T)$,$T'^{(0)} := f^{-1}(T')$, $\mathfrak{U}^{(0)} := \mathfrak{P}^{(0)} \setminus Z^{(0)}$ and $g\colon \mathfrak{U}^{(0)} \to \mathfrak{U}$ the morphism induced by f.

We denote by $\eta = (c, a, f)$ and $\eta' = (d, a, f)$ the induced morphism of smooth d-frames. Since $\widetilde{\mathcal{E}}^{(\bullet)}({}^\dagger T') \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, T'/\mathcal{V})$, then it follows from 16.1.11.2 that $\widetilde{\mathcal{E}}^{(\bullet)}({}^\dagger T')$ is a direct factor of $\eta'^{(\bullet)}_{+} \circ \eta'^{(\bullet)!}(\widetilde{\mathcal{E}}^{(\bullet)}({}^\dagger T'))$, where $\eta'^{(\bullet)}_{+} := f^{(\bullet)}_{+}$. According to 16.2.4.1.1, we have $\eta^{(\bullet)!}(\widetilde{\mathcal{E}}'^{(\bullet)}) := \mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{Y}^{(0)}} \circ f^{(\bullet)!}(\widetilde{\mathcal{E}}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(\widetilde{Y}^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, Z'^{(0)} \cup T^{(0)})$. With 16.2.7.3, since $(Z'^{(0)} \cup T^{(0)}) \cap X^{(0)} = T^{(0)} \cap X^{(0)}$, this implies $\eta^{(\bullet)!}(\widetilde{\mathcal{E}}'^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(\widetilde{Y}^{(0)}, X^{(0)}, \mathfrak{P}^{(0)}, T^{(0)})$. By stability of the overcoherence, via the isomorphism

$$\eta'^{(\bullet)}_{+} \circ \eta'^{(\bullet)!}(\widetilde{\mathcal{E}}^{(\bullet)}({}^\dagger T')) \xrightarrow{\sim} ({}^\dagger T') \circ \eta^{(\bullet)}_{+}\left(\eta^{(\bullet)!}(\widetilde{\mathcal{E}}^{(\bullet)})\right),$$

(where $\eta^{(\bullet)}_{+} := f^{(\bullet)}_{+}$) we get $\eta'^{(\bullet)}_{+} \circ \eta'^{(\bullet)!}(\widetilde{\mathcal{E}}^{(\bullet)}({}^\dagger T')) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathcal{P}}(T))$. Hence, so is $\widetilde{\mathcal{E}}^{(\bullet)}({}^\dagger T')$ and we are done.

3) Since $X \setminus Z = X \setminus (Z \cup Z') = X' \setminus (Z \cup Z') = X' \setminus Z'$, then to check the general case we deduce to the two preceding cases.

□

Notation 16.2.7.5. Let $\mathfrak{P}$ be a smooth separated $\mathfrak{S}$-formal scheme, Y be some subscheme of P. We write $(F\text{-})\mathrm{MIC}^{(\bullet)}(Y, \mathfrak{P}/\mathcal{V}) := (F\text{-})\mathrm{MIC}^{(\bullet)}(\overline{Y}, \mathfrak{P}, \overline{Y} \setminus Y/\mathcal{V})$.

Proposition 16.2.7.6. *Let $\theta = (\mathrm{id}, a, f)\colon (Y, X', \mathfrak{P}', Z') \to (Y, X, \mathfrak{P}, Z)$ be a morphism of smooth c-frames (see definition 16.2.1.8) such that a is proper.*

(a) Let $\mathcal{E}^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$, $\mathcal{E}'^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$. For any $l \in \mathbb{Z} \setminus \{0\}$,

$$H^l(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) = 0, \qquad\qquad H^l(f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)})) = 0.$$

(b) The functors $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} f^{(\bullet)!}$ and $f^{(\bullet)}_{+}$ induce canonically quasi-inverse equivalences between the categories $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ and $(F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$.

Proof. Let $\mathcal{E}^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$, $\mathcal{E}'^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$. Denote by $\mathfrak{U} := \mathfrak{P} \setminus Z$ and $\mathfrak{U}' := \mathfrak{P}' \setminus Z'$. Using 16.2.7.4, we can suppose Y is dense in X'. Since a is proper, then the open immersion $Y \hookrightarrow X' \setminus f^{-1}(Z)$ is also closed. Hence, $f^{-1}(Z)$ is a closed subscheme of P' such that $X' \setminus f^{-1}(Z) = Y$. Following 16.2.7.4, we can suppose $Z' = f^{-1}(Z)$. Moreover, using lemma 16.2.1.4, we reduce to the case where Y integral.

Since a is proper, then it follows from lemma 16.2.7.1.(c) (resp. thanks to the proposition 16.2.4.1), we obtain the assertion (a) for the direct image (resp. the other functor) and $f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)!}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ (resp. $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$).

Let us now check that we have the canonical isomorphism: $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{(\bullet)!} \circ f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}'^{(\bullet)!}$. It follows from the lemma 16.1.11.1), that there exists a diagram of the form 16.2.7.1.1 and satisfying the required conditions. With these notations, since the functor $(a'^{(\bullet)!}, |\mathfrak{U}')$ is fully faithful (see 16.2.5.2), it is sufficient to build two compatible isomorphisms $a'^{(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{(\bullet)!} \circ f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} a'^{(\bullet)!}(\mathcal{E}'^{(\bullet)!})$

and $|\mathfrak{U}' \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'} \circ f^{(\bullet)!} \circ f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}'^{(\bullet)!}|\mathfrak{U}'$. The second isomorphism is a consequence of the case of the smooth partial compactification (see 16.1.9.5). Let us establish now the first. We recall that $a'^{(\bullet)!} = \mathbb{R}\underline{\Gamma}^\dagger_{X''} \circ q'^!$. Denote by $\tilde{f} := \widehat{\mathbb{P}}^N_f : \widehat{\mathbb{P}}^N_{\mathfrak{P}'} \to \widehat{\mathbb{P}}^N_{\mathfrak{P}}$. Following 16.2.7.1.a, $f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)})$ is coherent. We have the isomorphisms:

$$a'^{(\bullet)!} \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'} \circ f^{(\bullet)!} \circ f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{X''} \circ \widetilde{f}^{(\bullet)!} \circ q^{(\bullet)!} \circ f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)}) \overset{16.2.7.2}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^\dagger_{X''} \circ \widetilde{f}^{(\bullet)!} \circ \widetilde{f}_+^{(\bullet)} \circ q'^!(\mathcal{E}'^{(\bullet)})$$
$$\xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{X''} \circ \widetilde{f}^{(\bullet)!} \circ \widetilde{f}_+^{(\bullet)} \circ a'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} a'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}),$$

the last isomorphism, since X'' is smooth, coming from the completely smooth case of 16.1.9.5.

Similarly, we established the canonical isomorphism and $f_+^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'} \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. Hence we are done. □

Corollary 16.2.7.7. *Let $(Y, X, \mathfrak{P}, Z)$ be a smooth c-frame over $\mathfrak{S}$. Denote by $p_1 : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ and $p_2 : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ the left and right projections. For any $\mathcal{E}^{(\bullet)} \in (F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$, we have the canonical isomorphisms:*

$$\mathbb{R}\underline{\Gamma}^\dagger_X p_1^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xleftarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_X p_1^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_X p_2^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Y p_2^{(\bullet)!}(\mathcal{E}^{(\bullet)}), \tag{16.2.7.7.1}$$

where by abuse of notation X is $\delta(X)$ and Y is $\delta(Y)$.

Proof. Since $\delta(X) \cap (P \times Y) = \delta(X) \cap (Y \times P) = \delta(Y)$, then we get the first and the last isomorphisms. Let $\delta : \mathfrak{P} \hookrightarrow \mathfrak{P} \times \mathfrak{P}$ be the diagonal immersion. Pour $i = 1, 2$, set $\mathcal{F}_i^{(\bullet)} := \mathbb{R}\underline{\Gamma}^\dagger_X p_i^{(\bullet)!}(\mathcal{E}^{(\bullet)})$. We have therefore the canonical isomorphisms:

$$\mathcal{F}_i^{(\bullet)} \overset{16.2.7.6}{\xrightarrow{\sim}} \delta_+^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^\dagger_X \delta^{(\bullet)!}(\mathcal{F}_i^{(\bullet)}) \xrightarrow{\sim} \delta_+^{(\bullet)}(\mathcal{E}^{(\bullet)}),$$

the last one coming from $p_i \circ \delta = \mathrm{id}$. Hence, we are done. □

Definition 16.2.7.8. A "realizable pair $(Y, X)/\mathcal{V}$(of k-varieties)" is the data of a k-variety X, of an open set Y of X such that there exists a c-frame of the form $(Y, X, \mathfrak{P}, Z)$. In that case we say that $(Y, X, \mathfrak{P}, Z)$ is a c-frame enclosing $(Y, X)/\mathcal{V}$. A morphism $(Y', X') \to (Y, X)$ of realizable pairs of k-varieties is a morphism of varieties $a : X' \to X$ such that $a(Y') \subset Y$.

Proposition 16.2.7.9 (and Definition)**.** *Let $(Y, X, \mathfrak{P}, Z)$ be a c-frame.*

(a) The category $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ does not depend, up to canonical equivalence of categories, on the choice of the c-frame enclosing $(Y, X)/\mathcal{V}$.

(b) If there is no ambiguity, we denote therefore by $(F\text{-})\mathrm{MIC}^{(\bullet)}(Y, X/\mathcal{V})$ instead of $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. Its objects are the "$(F\text{-})$partially overcoherent isocrystals over $(Y, X)/\mathcal{V}$" or simply "$(F\text{-})$overcoherent isocrystals over $(Y, X)/\mathcal{V}$".

Proof. Let $(Y, X, \mathfrak{P}', Z')$ be another c-frame enclosing (Y, X). Set $\mathfrak{P}'' := \mathfrak{P} \times \mathfrak{P}'$. Let $q : \mathfrak{P}'' \to \mathfrak{P}$, $q' : \mathfrak{P}'' \to \mathfrak{P}'$ the canonical projections and $Z'' := q^{-1}(Z)$. Following 16.2.7.6, the functors $q_+^{(\bullet)}$ and $\mathbb{R}\underline{\Gamma}^\dagger_X q^{(\bullet)!}$ (resp. $q'^{(\bullet)}_+$ and $\mathbb{R}\underline{\Gamma}^\dagger_X q'^!$) induce then quasi-inverse equivalences between the categories $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ and $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}'', Z''/\mathcal{V})$ (resp. $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}', Z'/\mathcal{V})$ and $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}'', Z''/\mathcal{V})$). Hence we are done. □

16.2.7.10. Suppose there exists an automorphism $\sigma : \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. Let $(Y, X, \mathfrak{P}, Z)$ be a smooth c-frames. We denote by $\mathfrak{P}' := \mathfrak{P}^\sigma$ the $\mathcal{V}$-formal scheme deduced from $\mathfrak{P}$ by the base change defined by σ. We denote by $Z' := Z^\sigma$, $X' := X^\sigma$, $Y' := Y^\sigma$. Let $F^s_{Y/S} : Y \to Y'$ be the relative Frobenius of Y. The functor $F^{*(\bullet)}_{\mathfrak{X}} : \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}) \to \underrightarrow{LD}^-_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ defined at 9.5.1.2.2 (with empty divisors) induces the functor:

$$F^{*(\bullet)}_{\mathfrak{X}} : \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}). \tag{16.2.7.10.1}$$

Composing the graph $\gamma_{F^s_{Y/S}} : Y \hookrightarrow Y \times Y'$ of $F^s_{Y/S}$ with the immersion $Y \times Y' \hookrightarrow \mathfrak{P} \times \mathfrak{P}'$, we get the immersion $\gamma : Y \hookrightarrow \mathfrak{P} \times \mathfrak{P}'$. Let X'' be the closure Y in $P \times P'$ and $Z'' := (Z \times P') \cup (P \times Z')$, we get the c-frame $(Y, X'', \mathfrak{P}\times\mathfrak{P}', Z'')$. We get the morphisms of c-frames $\theta_2 = (F^s_{Y/S}, a, p_2)\colon (Y, X'', \mathfrak{P}\times\mathfrak{P}', Z'') \to (Y', X', \mathfrak{P}', Z')$, $\theta_1 = (\mathrm{id}, a, p_1)\colon (Y, X'', \mathfrak{P}\times\mathfrak{P}', Z'') \to (Y, X, \mathfrak{P}, Z)$ where p_1 and p_2 are respectively the left and the right projection. It follows from 16.2.7.6 that p_{1+} induces canonically the equivalence of categories $\theta^{(\bullet)}_{1+} = p^{(\bullet)}_{1+}\colon F\text{-MIC}^{(\bullet)}(X'', \mathfrak{P}\times\mathfrak{P}', Z''/\mathcal{V}) \cong F\text{-MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. We get the functor

$$\theta^{(\bullet)}_{1+}\circ\theta^{(\bullet)!}_{2}\colon F\text{-MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V}) \to F\text{-MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}). \qquad (16.2.7.10.2)$$

which is canonically isomorphic to the functor $F^{*(\bullet)}_{\mathfrak{X}}$ defined of 16.2.7.10.1.

It follows from 16.2.7.6 and 16.2.4.1.1, that for any $\mathcal{E}^{(\bullet)} \in F\text{-MIC}^{(\bullet)}(X'', \mathfrak{P}\times\mathfrak{P}', Z''/\mathcal{V})$, we get the isomorphism:

$$\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}\circ\theta^{(\bullet)}_{1+}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \theta^{(\bullet)}_{1+}\circ\mathrm{sp}^{(\bullet)}_{X''\hookrightarrow\mathfrak{P}\times\mathfrak{P}',Z'',+}(\mathcal{E}^{(\bullet)}).$$

Using again 16.2.4.1.1, since both functors 16.2.7.10.2 and 16.2.7.10.2 are isomorphic, this yields for any $\mathcal{E}'^{(\bullet)} \in F\text{-MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$

$$\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}\circ F^{*(\bullet)}_{\mathfrak{X}}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} F^{*(\bullet)}_{\mathfrak{X}}\mathrm{sp}^{(\bullet)}_{X'\hookrightarrow\mathfrak{P}',Z',+}(\mathcal{E}'^{(\bullet)}). \qquad (16.2.7.10.3)$$

16.2.7.11. Let (Y, X) be a pair of k-varieties. Following lemma 16.2.1.4, if $\overline{Y}$ is the closure of Y in X then $(F\text{-})\mathrm{MIC}^{(\bullet)}(Y, \overline{Y}/\mathcal{V}) = (F\text{-})\mathrm{MIC}^{(\bullet)}(Y, X/\mathcal{V})$.

Let $(Y, X, \mathfrak{P}, Z)$ be a c-frame enclosing (Y, X). Since the equivalence of categories of 16.2.1.10.1 commutes with base change, using 16.2.7.10.3, we obtain the functor:

$$\mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+}\ :\ (F\text{-})\mathrm{MIC}^{\dagger}(Y, X/\mathcal{V}) \cong (F\text{-})\mathrm{MIC}^{(\bullet)}(Y, X/\mathcal{V}), \qquad (16.2.7.11.1)$$

which does not depend canonically on the choice of the c-frame $(Y, X, \mathfrak{P}, Z)$ enclosing (Y, X) and will be simply denoted by $\mathrm{sp}^{(\bullet)}_{(Y,X),+}$. When X/S is proper, $\mathrm{sp}^{(\bullet)}_{(Y,X),+}$ only depends on Y and will simply be denoted by $\mathrm{sp}^{(\bullet)}_{Y+}$.

Proposition 16.2.7.12 (and Definition)**.** *Let Y be a smooth k-variety. We suppose there exists a c-frame of the form $(Y, X, \mathfrak{P}, Z)$, with X a proper k-variety.*

(a) The category $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$ does not depend, up to canonical equivalence of categories, on the choice of the c-frame of the form $(Y, X, \mathfrak{P}, Z)$, with X a proper k-variety.

(b) We denote then by $(F\text{-})\mathrm{MIC}^{(\bullet)}(Y/\mathcal{V})$ instead of $(F\text{-})\mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V})$. Its objects are called "$(F\text{-})$overcoherent isocrystals over Y".

Proof. Let $(Y, X', \mathfrak{P}', Z')$ be another c-frame with X' proper. Let X'' be the closure of Y in $X\times X'$. Since the projection $X''\to X$ and $X''\to X'$ are proper, we conclude the proof by using 16.2.7.6. □

Notation 16.2.7.13. Let $\theta = (b, a, f)\colon (Y', X', \mathfrak{P}', Z') \to (Y, X, \mathfrak{P}, Z)$ be a morphism of smooth c-frames. The functors $\theta^{(\bullet)+}, \theta^{(\bullet)!}\colon \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(X', \mathfrak{P}', Z'/\mathcal{V})$ of 16.2.4.3 only depends on the morphism of realizable pairs $(b, a)\colon (X', Y') \to (Y, X)$. Following 16.2.4.3, those are isomorphic. Hence we will denote it without ambiguity $(b, a)^*$.

16.3 Devissability in overconvergent isocrystals

16.3.1 Definition and first properties

Let $\mathfrak{P}$ be a separated and smooth $\mathfrak{S}$-formal scheme.

Definition 16.3.1.1. Let Y be a subvariety of P. We say that Y is "d-embedabble in P" if there exist a divisor T of P such that Y is closed in $P\setminus T$. We remark that this is equivalent to suppose that there exist a d-frame of the form $(Y, X, \mathfrak{P}, T)$. Moreover, if Y', Y are d-embedabble in P two subvarieties, then $Y\cap Y'$ is d-embedabble in P.

Definition 16.3.1.2. Let Y be a subvariety of P.

(a) A "stratification of Y" is the data (for a certain integer $r \geq 1$) of r subvarieties $(Y_1, Y_2, \ldots, Y_r)$ such that, by setting $Y_0 := \emptyset$, for any integer i satisfying $1 \leq i \leq r-1$, the variety Y_i is an open of $Y \setminus (\cup_{0 \leq j \leq i-1} Y_j)$ and such that $Y_r = Y \setminus (\cup_{0 \leq j \leq r-1} Y_j)$. In other words, we have the direct sum $Y = \sqcup_{i=1,\ldots,r} Y_i$ such that, for any $1 \leq i \leq r-1$, the variety Y_i is an open subset of $\sqcup_{j=i,\ldots,r} Y_j$. Taking care of the order, we also say such a split $Y = \sqcup_{i=1,\ldots,r} Y_i$ is a stratification of Y.

(b) Let $Y = \sqcup_{i=1,\ldots,r} Y_i$ be a stratification. We say that $Y = \sqcup_{i=1,\ldots,r} Y_i$ or $(Y_1, Y_2, \ldots, Y_r)$ is a smooth stratification (resp. a d-stratification in P, resp. is a smooth d-stratification) if, for any $1 \leq i \leq r$, the variety Y_i is smooth (resp. is d-embeddable, resp. is smooth and d-embeddable).

Remark 16.3.1.3. Let Y be a subvariety of P and $Y = \sqcup_{i=1,\ldots,r} Y_i$ be a stratification. For any, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$, for any $1 \leq i \leq r-1$, we have distinguished triangle of localisation

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{\sqcup_{j=i+1,\ldots,r} Y_j}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{\sqcup_{j=i,\ldots,r} Y_j}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Y_i}(\mathcal{E}^{(\bullet)}) \to +1$$

Notation 16.3.1.4. Let Y be a smooth subvariety of P. Let us choose X and Z be two closed subschemes of P, such that $Y = X \setminus Z$. We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},X}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$ the strictly full triangulated subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{poc}}(Y, X, \mathfrak{P}, Z/\mathfrak{S})$ (see notation 15.3.8.3) of complexes $\mathcal{E}^{(\bullet)}$ such that for any $j \in \mathbb{Z}$, with notation 15.3.8.13 we have

$$H^j_Z(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(X, \mathfrak{P}, Z/\mathcal{V}).$$

This category do not depend on the choice of the closed subschemes X an Z such that $Y = X \setminus Z$. Indeed, let X', Z' be a closed subschemes of P such that $X' \setminus Z' = Y$. It follows from 16.2.7.4 by devissage in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},X}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$ the inclusion $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},X}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z)) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},X'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z'))$. By symmetry, we get this inclusion is an equality.

Hence, we can then simply denote $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},X}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(Z))$ by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$.

The following notion of devissability into overconvergent isocrystals extends that of [Car06a] since we do not bother with divisors (but both notions are equal).

Definition 16.3.1.5. Let Y be a subvariety of P. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$.

(a) The complex $\mathcal{E}^{(\bullet)}$ "splits into overconvergent isocrystals on Y" if there exists a smooth stratification of Y (see Definition 16.3.1.2) of the form $Y = \sqcup_{i=1,\ldots,r} Y_i$ such that, for any $i = 1, \ldots, r$, we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y_i}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y_i}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. We also say that the complex "$\mathcal{E}^{(\bullet)}$ splits (or is devissable) into overconvergent isocrystals on the smooth stratification $Y = \sqcup_{i=1,\ldots,r} Y_i$".

(b) When $Y = P$, we say that $\mathcal{E}^{(\bullet)}$ splits (or is devissable) into overconvergent isocrystals. We will denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ of devissable into overconvergent isocrystals complexes.

Lemma 16.3.1.6. *Let Y, Y' be two smooth subvarieties of P such that $Y' \subset Y$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. If $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ then $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$.*

Proof. Following 16.2.4.3 and with notation 16.2.7.5 we have the exact functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} \colon \mathrm{MIC}^{(\bullet)}(Y, \mathfrak{P}/\mathcal{V}) \to \mathrm{MIC}^{(\bullet)}(Y', \mathfrak{P}/\mathcal{V})$. Hence we are done. □

Lemma 16.3.1.7. *Let Y be a subvariety of P, $Y = \sqcup_{i=1,\ldots,r} Y_i$ be a stratification. For any $i = 1, \ldots, r$, let $Y_i = \sqcup_{j=1,\ldots,j_i} Y_{i,j}$ a smooth stratification (see definition 16.3.1.2). Hence $Y = (\sqcup_{j=1,\ldots,j_1} Y_{1,j}) \sqcup \cdots \sqcup (\sqcup_{j=1,\ldots,j_r} Y_{r,j})$ is a smooth stratification. We say that such a stratification is a "smooth substratification" of $Y = \sqcup_{i=1,\ldots,r} Y_i$.*

Proof. Let $1 \leq i \leq r$ and $1 \leq j \leq j_i$. As Y_i is an open subset of $\sqcup_{i'=i,\ldots,r} Y_{i'}$, then $\sqcup_{j'=j,\ldots,j_i} Y_{i,j'}$ is an open subset of $Z_{i,j} := (\sqcup_{j'=j,\ldots,j_i} Y_{i,j'}) \sqcup (\sqcup_{j=1,\ldots,j_{i+1}} Y_{i+1,j}) \sqcup \cdots \sqcup (\sqcup_{j=1,\ldots,j_r} Y_{r,j})$. As $Y_{i,j}$ is an open subset of $\sqcup_{j'=j,\ldots,j_i} Y_{i,j'}$, then $Y_{i,j}$ is an open subset of $Z_{i,j}$. □

Proposition 16.3.1.8. *Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$, Y be a subvariety of P and $Y = \sqcup_{i=1,\dots,r} Y_i$ be a stratification of Y. If, for any $i = 1, \dots r$, the complex $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y_i, then the complex $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y and more precisely on the smooth substratification of $Y = \sqcup_{i=1,\dots,r} Y_i$ constructed from the smooth stratifications of Y_i on which $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals.*

Proof. For any $i = 1, \dots, r$, let $Y_i = \sqcup_{j=1,\dots,j_i} Y_{i,j}$ a smooth stratification such that, for any $j = 1, \dots, j_i$, we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y_{ij}}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y_{ij}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. Then we conclude with lemma 16.3.1.7. □

Lemma 16.3.1.9. *Let Y be a subvariety of P. There exists a smooth d-stratification of Y in P.*

Proof. As P is smooth, we reduce to the case where P is integral (indeed, if $(P_n)_n$ are the irreducible components of P, if we have d-stratifications of the form $P_n \cap Y = \sqcup_{i=1,\dots,r_n} Y_{n,i}$, then by setting $Y_i := \sqcup_n Y_{n,i}$ we obtain the d-stratification $Y = \sqcup_{i \geq 1} Y_i$). Let us denote by X the closure of Y. The case where $Y = X$ is obvious. If not, then there exist some divisors $T_1, \dots, T_r$ of P such that $X \setminus Y = \cap_{j=1,\dots,r} T_j$. If y is a generic point of an irreducible component of dimension equal to $\dim Y$, then there exist a j such that $y \notin T_j$. We set then $Y_1 := X \setminus T_j \subset Y$. Proceeding by lexicographic induction on the dimension of Y and on the number of irreducible components of maximal degree, this yields a smooth d-stratification of $Y \setminus Y_1$. Hence we are done. □

Lemma 16.3.1.10. *Let Y be a subvariety of P, Y' be a subvariety of Y. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. If $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y then so is on Y'. In particular, the converse of the proposition 16.3.1.8 is valid.*

Proof. Let $Y = \sqcup_{i=1,\dots,r} Y_i$ be a smooth stratification on which $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals. We get the stratification $Y' = \sqcup_{i=1,\dots,r}(Y_i \cap Y')$. By using 16.3.1.8, we reduce then to the case where Y is smooth and where $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. Following 16.3.1.9, there exists a smooth stratification $Y' = \sqcup_{i=1,\dots,r} Y'_i$. It follows from the lemma 16.3.1.6 that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'_i}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'_i}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. Hence, $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on the stratification $Y' = \sqcup_{i=1,\dots,r} Y'_i$.

□

Proposition 16.3.1.11. *Let Y be a subvariety of P, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. The following conditions are equivalent:*

(a) The complex $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y.

(b) There exists a smooth d-stratification of Y in P of the form $Y = \sqcup_{i=1,\dots,r} Y_i$ such that, for any $i = 1, \dots, r$, we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y_i}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y_i}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$.

Proof. The implication $(b) \Rightarrow (a)$ is tautological. The converse follows from 16.3.1.8, 16.3.1.9 and 16.3.1.10. □

Proposition 16.3.1.12. *Let Y be a subvariety of P.*

(a) Let $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ be devissable in overconvergent isocrystals on Y objects. Then there exists a smooth stratification of Y such that $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ split simultaneously into overconvergent isocrystals on this one.

(b) The full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{1}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ of devissable in overconvergent isocrystals on Y complexes is triangulated.

Proof. a) Let $Y = \sqcup_{i=1,\dots,r} Y_i$ be a smooth stratification of Y on which $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals. Following 16.3.1.10, $\mathcal{F}^{(\bullet)}$ splits into overconvergent isocrystals on each Y_i. Following 16.3.1.8, the complex $\mathcal{F}^{(\bullet)}$ splits into overconvergent isocrystals on a smooth substratification of $Y = \sqcup_{i=1,\dots,r} Y_i$. Following 16.3.1.6, this is also the case of $\mathcal{E}^{(\bullet)}$ on this latter.

b) By devissage in overconvergent isocrystals of $\mathcal{E}^{(\bullet)}, \mathcal{F}^{(\bullet)}$ on the same smooth stratification, we reduce then to the case where Y is smooth and where $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ are objects of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$, which is straightforward. □

Proposition 16.3.1.13. *Let Y, Y' be two subvarieties of P. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. The complex $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on $Y \cup Y'$ if and only if the complex $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y and on Y'.*

Proof. The necessity follows from 16.3.1.10.

Let us suppose that the complex $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y and on Y'. Since $Y \cup Y' = Y \sqcup (Y' \setminus Y)$, since $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on $Y' \subset Y$ (use again 16.3.1.10), then replacing Y' by $Y' \setminus Y$ if necessary we can suppose Y and Y' are disjoint. Let $Y = \sqcup_{i=1,\dots,r} Y_i$ (resp. $Y' = \sqcup_{i'=1,\dots,r'} Y'_{i'}$) be a smooth stratification on which $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals. By symmetry, we can suppose $r' \leq r$. For any $r'+1 \leq i' \leq r$, we set $Y'_{i'} := \emptyset$. For any $j = 1, \dots r$, we set $Y''_j := Y_j \cup Y'_j$. Then we get the smooth stratification $Y \cup Y' = \sqcup_{j=1,\dots,r} Y''_j$. Since $Y_j \cap Y'_j$ is empty, then by using the distinguished triangle of Mayer-Vietoris we get the isomorphism

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Y''_j}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Y_j}(\mathcal{E}^{(\bullet)}) \oplus \mathbb{R}\underline{\Gamma}^{\dagger}_{Y'_j}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y''_j}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}).$$

Hence, we are done. □

16.3.1.14. It follows from 16.3.1.12.(b), that the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ is the smallest full triangulated subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ containing $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$, for any smooth subvariety Y' of P.

Notation 16.3.1.15. Let Y be a subvariety of P. We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y, \mathfrak{P}/\mathcal{V})$ the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ of complexes $\mathcal{E}^{(\bullet)}$ such that there exist an isomorphism of the form $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$.

Lemma 16.3.1.16. *Let Y be a subvariety of P. The category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y, \mathfrak{P}/\mathcal{V})$ is equal to the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ of complexes $\mathcal{E}^{(\bullet)}$ devissable in overconvergent isocrystals on Y and such that there exists an isomorphism of the form $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$.*

Proof. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. If $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y, \mathfrak{P}/\mathcal{V})$, then $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y (see 16.3.1.10). Conversely, suppose $\mathcal{E}^{(\bullet)}$ devissable in overconvergent isocrystals on Y and such that we have an isomorphism of the form $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. Let X and Z be two closed subschemes of P such that $Y = X \setminus Z$. We get the stratification $P = (P \setminus X) \sqcup Y \sqcup (X \cap Z)$. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{P \setminus X}(\mathcal{E}^{(\bullet)}) = 0$, $\mathbb{R}\underline{\Gamma}^{\dagger}_{X \cap Z}(\mathcal{E}^{(\bullet)}) = 0$ (use 13.1.5.6.1 and $\mathbb{R}\underline{\Gamma}^{\dagger}_{\emptyset} = 0$) and $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals on Y, then we conclude via 16.3.1.8. □

Lemma 16.3.1.17. *Let $(Y, X, \mathfrak{P}, Z)$ be a c-frame. For any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{povcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$, there exists a smooth, dense open Y' of Y such that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'}(\mathcal{E}^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(Y', \mathfrak{P}/\mathcal{V})$.*

Proof. Since P is the sum of its irreducible components, we reduce to the case where P is integral. Let $\mathfrak{U}'$ be an affine open formal subscheme of $\mathfrak{P}$ included in $\mathfrak{P} \setminus Z$ such that $Y' := U' \cap X$ is smooth and dense open of Y. Hence, $\mathcal{E}|\mathfrak{U}' \in (F\text{-})\underrightarrow{LM}_{\mathbb{Q},\mathrm{povcoh}}(Y', \mathfrak{U}'/\mathcal{V})$. Via 15.3.1.19 and 15.3.8.26, shrinking $\mathfrak{U}'$ if necessary, we can suppose that $\mathcal{E}|\mathfrak{U}' \in (F\text{-})\mathrm{MIC}^{(\bullet)}(Y', \mathfrak{U}'/\mathcal{V})$. Since $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{povcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$, this yields $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'}(\mathcal{E}^{(\bullet)}) \in (F\text{-})\mathrm{MIC}^{(\bullet)}(Y', \mathfrak{P}/\mathcal{V})$. □

Theorem 16.3.1.18. *Let $(Y, X, \mathfrak{P}, Z)$ be a c-frame. We have the inclusion*

$$(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{povcoh}}(X, \mathfrak{P}, Z/\mathfrak{S}) \subset (F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y, \mathfrak{P}/\mathfrak{S}).$$

Proof. When $\dim Y = 0$, this is obvious. We proceed by induction on the dimension of Y by using the preceding lemma 16.3.1.17. □

16.3.2 Stability by tensor product of the devissability in isocrystals

Let $\mathfrak{P}$ and $\mathfrak{P}'$ be a two separated and smooth $\mathfrak{S}$-formal schemes. Let Y (resp. Y') be a subvariety of $\mathfrak{P}$ (resp. $\mathfrak{P}'$). Set $\mathfrak{P}'' := \mathfrak{P} \times \mathfrak{P}'$ and $Y'' := Y \times Y'$.

Lemma 16.3.2.1. *Suppose Y (resp. Y') is a smooth subvariety of $\mathfrak{P}$ (resp. $\mathfrak{P}'$).*

(a) The bifunctor $-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}-$ induces

$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y''}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}). \tag{16.3.2.1.1}$$

(b) If $\mathfrak{P} = \mathfrak{P}'$ and if $Y \cap Y'$ is smooth, the bifunctor $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}}-$ induces:

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y\cap Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}). \tag{16.3.2.1.2}$$

Proof. As the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y''}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y\cap Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$) is a triangulated subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$), using distinguished triangles of truncation and proceeding by induction on the number of nonzero cohomological space if necessary, we reduce to the case where the complexes of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'})$) are objects of $\mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},T/\mathcal{V})$ (resp. $\mathrm{MIC}^{(\bullet)}(X',\mathfrak{P}',T'/\mathcal{V})$), i.e. to the situation already treated in respectively 16.2.6.1.(a) and 16.2.6.6.1. □

Theorem 16.3.2.2. *We have the factorisations*

$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y,\mathfrak{P}/\mathcal{V}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y',\mathfrak{P}'/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y'',\mathfrak{P}''/\mathcal{V}), \tag{16.3.2.2.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}}-\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y,\mathfrak{P}/\mathcal{V}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y,\mathfrak{P}/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y,\mathfrak{P}/\mathcal{V}). \tag{16.3.2.2.2}$$

Proof. Let us check first 16.3.2.2.1. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y,\mathfrak{P}/\mathcal{V})$ and $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y',\mathfrak{P}'/\mathcal{V})$. Let $Y = \sqcup_{i=1,\dots,r} Y_i$ be a smooth stratification of Y on which $\mathcal{E}^{(\bullet)}$ splits into overconvergent isocrystals. Let $Y' = \sqcup_{j=1,\dots,s} Y'_j$ be a smooth stratification of Y' in P' on which $\mathcal{E}'^{(\bullet)}$ splits into overconvergent isocrystals. As $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y'',\mathfrak{P}''/\mathcal{V})$ is a triangulated subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''})$, we reduce by devissage (see the remark 16.3.1.3) to the case where $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y_i}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ and $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'_j}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'})$. By using 16.3.2.1.1, this yields that $\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}}\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y_i\times Y'_j}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}''}) \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y'',\mathfrak{P}''/\mathcal{V})$ (the inclusion is a consequence of 16.3.1.16). Hence we are done. Let us treat now 16.3.2.2.2. Following 16.3.1.12, there exists $Y = \sqcup_{i=1,\dots,r} Y_i$ a smooth stratification of Y on which $\mathcal{E}^{(\bullet)}$ and $\mathcal{E}'^{(\bullet)}$ both split into overconvergent isocrystals. We proceed then similarly to the check of 16.3.2.2.1 but by using 16.3.2.1.2 instead of 16.3.2.1.1. □

16.3.3 Stability of the overcoherence by pushforward, base change isomorphism

The following proposition improves 13.2.3.4 by removing the hypothesis on realizability of the morphism.

Proposition 16.3.3.1. Let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of separated and smooth $\mathfrak{S}$-formal schemes, $u'\colon X' \hookrightarrow P'$ be a closed immersion with X' integral, Z be a closed subscheme of P. We set $Z' := f^{-1}(Z)$. We suppose $f \circ u'$ proper. For any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X',\mathfrak{P}',Z'/\mathcal{V})$, we have $f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(Z))$.

Proof. By commutation of the local functors with pushforwards (see 13.2.1.4.2), by definition of the overcoherence we can suppose Z is the support of a divisor T of P. We can suppose P' is integral. So either $f^{-1}(T)$ is a divisor of P' or $f^{-1}(T) = P'$. Since the second case is obvious (because we get therefore $({}^{\dagger}f^{-1}(T))(\mathcal{E}'^{(\bullet)}) = 0$, then we can suppose $T' := f^{-1}(T)$ is a divisor of P'. Set $Y' := X' \setminus T'$. Following 16.3.1.18, 16.3.1.11 and by stability of the overcoherence by local functors (see 15.3.7.5.1), there exists a smooth d-stratification $\sqcup_{i=1,\dots,r} Y'_i$ of Y' in P' such that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'_i}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{isoc},Y'_i}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}) \cap$

$\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(X',\mathfrak{P}',T'/\mathcal{V})$, for any for any $i=1,\dots,r$. Since $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ is a triangle subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}})$, we reduce by devissage to check that $g_+\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'_i}(\mathcal{E}'^{(\bullet)})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. We can suppose Y'_i integral. Let X'_i be the closure of Y'_i in P'_i, T'_i be a divisor of P' such that $Y'_i=X'_i\setminus T'_i$. Again by devissage, we reduce to check $g_+\left(H^j(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'_i}(\mathcal{E}'^{(\bullet)}))\right)\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ for any integer $j\in\mathbb{Z}$.

We have $\mathcal{E}'^{(\bullet)}_{i,j}:=H^j(\mathbb{R}\underline{\Gamma}^{\dagger}_{Y'_i}(\mathcal{E}'^{(\bullet)}))\in\mathrm{MIC}^{(\bullet)}(X'_i,\mathfrak{P}',T'_i,/\mathcal{V})\cap\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'/\mathfrak{S}}(T'))$. It follows from de Jong desingularization theorem that there exists a diagram of the form:

$$\begin{array}{ccccccc} X''_i & \xrightarrow{u''} & \mathbb{P}^N_{P'} & \longrightarrow & \widehat{\mathbb{P}}^N_{\mathfrak{P}'} & \xrightarrow{\widehat{\mathbb{P}}^N_g} & \widehat{\mathbb{P}}^N_{\mathfrak{P}} \\ \downarrow{a'} & & \downarrow & & \downarrow{q'} & & \downarrow{q} \\ X'_i & \xrightarrow{u'} & P' & \longrightarrow & \mathfrak{P}' & \xrightarrow{g} & \mathfrak{P}, \end{array} \tag{16.3.3.1.1}$$

where X''_i is k-smooth, q and q' are the canonical projections, u'' is a closed immersion, $a'^{-1}(T'_i\cap X'_i)$ is a strict normal crossing divisor of X''_i, a' is surjective, generically finite and etale. Set $T''_i:=q'^{-1}(T'_i)$, $D':=q'^{-1}(T')$, $D:=q^{-1}(T)$. Put $\mathcal{E}''^{(\bullet)}_{i,j}:=\mathbb{R}\underline{\Gamma}^{\dagger}_{X''_i}q'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}_{i,j})\in\mathrm{MIC}^{(\bullet)}(X''_i,\widehat{\mathbb{P}}^N_{\mathfrak{P}'},T''_i,/\mathcal{V})\cap\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\widehat{\mathbb{P}}^N_{\mathfrak{P}'}/\mathfrak{S}}(D'))$. Following 16.1.11.2, since $\mathcal{E}'^{(\bullet)}_{i,j}\in\mathrm{MIC}^{(\bullet)}(X'_i,\mathfrak{P}',T'_i,/\mathcal{V})$, then $\mathcal{E}'^{(\bullet)}_{i,j}$ is a direct factor of $q'^{(\bullet)}_+(\mathcal{E}''^{(\bullet)}_{i,j})$.

By construction, the morphism $X''_i\to\widehat{\mathbb{P}}^N_{\mathfrak{P}}$ is an immersion (indeed, this is the composition of the graph of $X''_i\to\mathfrak{P}$ with the immersion $X''_i\times\mathfrak{P}\hookrightarrow\widehat{\mathbb{P}}^N_{\mathfrak{P}}$ induced by an immersion of the form $X''_i\hookrightarrow\widehat{\mathbb{P}}^N_{\mathfrak{S}}$). Since X''_i is proper over P then $X''_i\to\widehat{\mathbb{P}}^N_{\mathfrak{P}}$ is more precisely a closed immersion. Since $\mathcal{E}''^{(\bullet)}_{i,j}\in\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\widehat{\mathbb{P}}^N_{\mathfrak{P}'}/\mathfrak{S}}(D'))$, then it follows from 16.1.9.3.(b) that $(\widehat{\mathbb{P}}^N_g)^{(\bullet)}_+(\mathcal{E}''^{(\bullet)}_{i,j})\in\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\widehat{\mathbb{P}}^N_{\mathfrak{P}}/\mathfrak{S}}(D))$. Since q is proper, then $q^{(\bullet)}_+$ preserves the overcoherence and then $g^{(\bullet)}_+q'_+(\mathcal{E}''^{(\bullet)}_{i,j})\xrightarrow{\sim}q^{(\bullet)}_+\circ(\widehat{\mathbb{P}}^N_g)^{(\bullet)}_+(\mathcal{E}''^{(\bullet)}_{i,j})\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$. Since $\mathcal{E}'^{(\bullet)}_{i,j}$ is a direct factor of $q'^{(\bullet)}_+(\mathcal{E}''^{(\bullet)}_{i,j})$, then $g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}_{i,j})$ is a direct factor of $g^{(\bullet)}_+q'^{(\bullet)}_+(\mathcal{E}''^{(\bullet)}_{i,j})$. Hence, $g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}_{i,j})\in\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}/\mathfrak{S}}(T))$ and we have done. □

Proposition 16.3.3.2. Let $f\colon\mathfrak{Y}\to\mathfrak{X}$, and $g\colon\mathfrak{X}'\to\mathfrak{X}$ be two morphisms of separated smooth $\mathfrak{S}$-formal schemes. We suppose f smooth. Let $f'\colon\mathfrak{Y}\times_{\mathfrak{X}}\mathfrak{X}'\to\mathfrak{X}'$, and $g'\colon\mathfrak{Y}\times_{\mathfrak{X}}\mathfrak{X}'\to\mathfrak{Y}$ be the structural projections. Let Z be a closed subscheme of X, $Z':=g^{-1}(Z)$, $Z':=g^{-1}(Z)$, $U':=f'^{-1}(Z')$, $U:=f^{-1}(Z)$. For any $\mathcal{E}'^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}(Z'))$ with proper support over X, we have the base change isomorphism in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}}(U))$ of the form

$$f^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}g'^{(\bullet)}_+f'^{(\bullet)!}(\mathcal{E}'^{(\bullet)}). \tag{16.3.3.2.1}$$

Proof. This is analogue to the proof 13.2.3.7: let $\mathcal{E}'^{(\bullet)}\in\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}(Z'))$ with proper support over X. First, we remark that by using 16.3.3.1, both objects of 16.3.3.2.1 belongs to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}}(U))$. The morphism f is the composition of its graph $\gamma\colon\mathfrak{Y}\hookrightarrow\mathfrak{X}\times\mathfrak{Y}$ with the projection $\pi\colon\mathfrak{X}\times\mathfrak{Y}\to\mathfrak{X}$. Let $g''\colon\mathfrak{X}'\times\mathfrak{Y}\to\mathfrak{X}\times\mathfrak{Y}$, and $\pi'\colon\mathfrak{X}'\times\mathfrak{Y}'\to\mathfrak{X}'$ be the canonical projections. Let $\gamma'\colon\mathfrak{X}'\times_{\mathfrak{X}}\mathfrak{Y}\hookrightarrow\mathfrak{X}'\times\mathfrak{Y}$ be the closed immersion induced by base change via g'' of γ. Following Theorem 9.4.4.3, we have the isomorphism $\pi^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}g''^{(\bullet)}_+\pi'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$. This yields the second isomorphism $\gamma^{(\bullet)}_+f^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}\gamma^{(\bullet)}_+\gamma^{(\bullet)!}\pi^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}\gamma^{(\bullet)}_+\gamma^{(\bullet)!}g''^{(\bullet)}_+\pi'^{!}(\mathcal{E}'^{(\bullet)})$. Using the commutation isomorphism 13.2.1.4.2 and Corollary 13.2.1.5, we get the first isomorphism $\gamma^{(\bullet)}_+\gamma^{(\bullet)!}g'^{(\bullet)}_+\pi'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}g''^{(\bullet)}_+\gamma'^{(\bullet)}_+\gamma'^{(\bullet)!}\pi'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}\gamma^{(\bullet)}_+g'^{(\bullet)}_+f'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$. Hence, by composition, we get the isomorphism $\gamma^{(\bullet)}_+f^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}\gamma^{(\bullet)}_+g'^{(\bullet)}_+f'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$. Following Proposition 16.3.3.1, $f^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})$ and $g'^{(\bullet)}_+f'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$ belongs to $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}}(U))$. Hence we can use Berthelot-Kashiwara theorem of the form 15.3.8.26: by applying $\gamma^{(\bullet)!}$ to the isomorphism $\gamma^{(\bullet)}_+f^{(\bullet)!}g^{(\bullet)}_+(\mathcal{E}'^{(\bullet)})\xrightarrow{\sim}\gamma^{(\bullet)}_+g'^{(\bullet)}_+f'^{(\bullet)!}(\mathcal{E}'^{(\bullet)})$ we get the isomorphism 16.3.3.2.1. □

Chapter 17

Arithmetic $\mathcal{D}$-modules associated with overconvergent F-isocrystals over affine and smooth varieties

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. When we work with F-complex, we suppose there exists an automorphism $\sigma \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining.

17.1 Weak formal schemes

We give some reminders on weak formal schemes defined by Meredith in [Mer72].

17.1.1 p-adic weak completion of commutative algebra, smooth w.c.f.g $\mathcal{V}$-algebra

Let R be a noetherian $\mathcal{V}$-algebra.

Definition 17.1.1.1. Let A be an R-algebra. Let $\widehat{A}$ be the p-adic completion of A and $\iota \colon A \to \widehat{A}$ be the canonical morphism. Let $S \subset A$. We denote by $S^\dagger$ the subset of $\widehat{A}$ consisting of elements x having representations

$$x = \sum_{j=0}^{\infty} P_j(a_1, \ldots, a_n)$$

where $a_1, \ldots, a_n \in \iota(S)$, $P_j \in p^j \cdot R[X_1, \ldots X_n]$ and there exists a constant c such that $\deg P_j \leq c(j+1)$ for all j. Beware that $S^\dagger$ depends on the basis R and we might write $A^{\dagger/R}$ if we would like to clarify.

The "p-adic weak completion as R-algebra of A" is $A^\dagger$. We denote by $w_A \colon A \to A^\dagger$ the canonical $\mathcal{V}$-algebra homomorphism. We say that A is (p-adically) weakly complete (as R-algebra) if the canonical map w_A is a bijection.

We denote by $\mathrm{Alg}(R)$ the category of commutative R-algebras, by $\mathrm{Alg}_{\mathrm{wc}}(R)$ the full subcategory of $\mathrm{Alg}(R)$ consisting of p-adically weakly complete R-algebras.

Example 17.1.1.2. The p-adic weak completion as R-algebra of R is $\widehat{R}$, i.e. $R^{\dagger/R} = \widehat{R}$. Beware in general, $A^{\dagger/R} \neq \widehat{A}$.

Remark 17.1.1.3. We will extend the notion of the weak completion in the non-commutative case later (see 17.7.1.1).

Theorem 17.1.1.4. *Let A be an R-algebra.*

(a) Then $A^\dagger$ is p-adically weakly complete.

(b) $pA^\dagger$ is contained in the Jacobson radical of $A^\dagger$.

(c) *Let $f\colon A \to B$ be a morphism of R-algebra. Then there exists a unique homomorphism $f^\dagger\colon A^\dagger \to B^\dagger$ such that $f^\dagger \circ w_A = w_B \circ f$. If f is surjective, then so is $f^\dagger$.*

Proof. See [MW68, 1] (and the last sentence is obvious). □

Remark 17.1.1.5. Let A be an R-algebra. Since $R^{\dagger/R} = \widehat{R}$, then it follows from 17.1.1.4.(c) that $A^{\dagger/R}$ is an $\widehat{R}$-algebra. Moreover, $A^{\dagger/R} = (A^{\dagger/R})^{\dagger/\widehat{R}}$. However, it is more convenient (e.g. see 17.1.1.13 in the case where A and R are w.c.f.g. $\mathcal{V}$-algebras) not to assume R is p-adically complete.

Definition 17.1.1.6. Let A be an R-algebra. A subset S of A is said to be a set of *weak generators* of the R-algebra A if $A = S^\dagger$. In that case A is said to be a weakly complete R-algebra A which is weakly generated by S.

A *w.c.f.g. R-algebra* is a weakly complete R-algebra A having a finite set of weak generators. The letters "w.c.f.g." stand for weakly complete weakly finitely generated.

A w.c.f.g. R-algebra is naturally equipped with the p-adic topology. A morphism of w.c.f.g. R-algebras is a map between two w.c.f.g. R-algebras which is a (unitary) ring morphism; such a morphism is always continuous.

We denote by $\mathrm{Alg}_{\mathrm{wcfg}}(R)$ the full subcategory of $\mathrm{Alg}(R)$ consisting of w.c.f.g. R-algebras.

Proposition 17.1.1.7. *Let $B(d) := R[t_1, \dots, t_d]^{\dagger/R}$ be the weak completion of the polynomial ring with d variables. Then*

$$B(d) := \left\{ \sum_{\underline{n}\in\mathbb{N}^d} a_{\underline{n}} \underline{t}^{\underline{n}} \mid a_{\underline{n}} \in \widehat{R}, \exists\, \lambda > 0 \text{ such that } v_p(a_{\underline{n}}) \geq \frac{|\underline{n}|}{\lambda} - 1 \right\} \subset \widehat{R}\{t_1, \dots, t_d\}.$$

Proof. See [MW68, 2.3]. □

Example 17.1.1.8. Following Theorem 17.1.1.7, we get the equality $B(d) = \{t_1, \dots, t_d\}^{\dagger/R}$. Hence, $B(d)$ is a w.c.f.g. R-algebra.

Theorem 17.1.1.9. *Let A be a w.c.f.g. R-algebra. Then A is noetherian.*

Proof. See [Ful69]. □

Corollary 17.1.1.10. *A w.c.f.g. R-algebra is a Zariski noetherian ring. In particular, a quotient of a w.c.f.g. R-algebra is a w.c.f.g R-algebra.*

Proof. This follows from theorem 17.1.1.4.(b), 17.1.1.9 and [Mat80, Theorem 56]. □

Corollary 17.1.1.11. *Let A be an R-algebra. The following conditions are equivalent:*

(a) *A is weakly generated by d elements.*

(b) *A is a quotient of $B(d)$.*

Proof. This is more or less contained in the proof of [MW68, 2.2]. □

17.1.1.12. Let A be a finitely generated R-algebra, i.e. is a quotient of $R[t_1, \dots, t_d]$. Then $A^\dagger$ is a quotient of $B(d)$ (see the last statement of 17.1.1.4.(c)). Hence, $A^{\dagger/R}$ is a w.c.f.g. R-algebra (see 17.1.1.11). In other words, the image in $A^\dagger$ of a finite set of generators of A provides a finite set of weak generators of $A^\dagger$.

Conversely, if B is an R-algebra having a finite set S of weak generators, denoting by A the R-subalgebra of B which is generated by S, then $B = A^{\dagger/R}$.

Theorem 17.1.1.13. *Let A be a w.c.f.g. R-algebra. Then the following conditions are equivalent.*

(a) *A is flat over R.*

(b) *$A/\pi A$ is a flat $R/\pi R$-algebra and A is p-torsion free.*

(c) *$A/\pi^{i+1} A$ is flat over $R/\pi^{i+1} R$-algebra, for any $i \in \mathbb{N}$.*

Proof. This follows from [MW68, 2.4] applied in the case where the ideal is πA. □

Theorem 17.1.1.14. *Let A be a w.c.f.g. R-algebra. Then the following conditions are equivalent.*

(a) $A/\pi A$ is a smooth (resp. étale) $R/\pi R$-algebra and A is p-torsion free.

(b) $A/\pi A$ is a smooth (resp. étale) $R/\pi R$-algebra and A is flat over R.

(c) $A/\pi^{i+1}A$ is a smooth (resp. étale) $R/\pi^{i+1}R$-algebra, for any $i \in \mathbb{N}$.

Proof. This follows from [MW68, 2.5] (for the respective case, the proof is the same) and 17.1.1.13. □

Definition 17.1.1.15. Let A be a w.c.f.g. R-algebra. We say that A is a smooth w.c.f.g. R-algebra if the equivalent conditions of 17.1.1.14 hold. Beware that a smooth w.c.f.g. R-algebra is not a smooth R-algebra.

Theorem 17.1.1.16. *Let $A/\pi A$ be a smooth (resp. étale) $R/\pi R$-algebra.*

(a) There exist a smooth R-algebra such that $A/\pi A \xrightarrow{\sim} A_0$. Such an A is called a smooth R-algebra lifting of A_0.

(b) Let A be a smooth R-algebra lifting of A_0. The R-algebra $A^\dagger$ is a smooth w.c.f.g. R-algebra such that $A^\dagger/\pi A^\dagger \xrightarrow{\sim} A_0$. Such an $A^\dagger$ is called a w.c.f.g. smooth R-algebra lifting of A_0.

Proof. This first part is [Ara01, 1.3.1]. The second one is obvious. □

Remark 17.1.1.17. In the practice, we will use the lifting Theorem 17.1.1.16 in the case where $R = \mathcal{V}$. In that case, this lifting theorem was proven by Elkik in [Elk73].

Proposition 17.1.1.18. *Let $A \to B$ and $A \to C$ be two morphisms of w.c.f.g. R-algebras. Then $B \otimes^\dagger_A C$, the weak completion of $B \otimes_A C$, is a w.c.f.g. R-algebra.*

Proof. As B is a quotient of $B(n)$ and C is a quotient of $B(m)$ for some integers n and m (see 17.1.1.11), then $B \otimes_R C$ is a quotient of $B(n) \otimes_R B(m)$. Using the last sentence of 17.1.1.4, this yields that $B \otimes^\dagger_A C$ is a quotient of $B(n) \otimes^\dagger_R B(m)$. Since $B(n) \otimes^\dagger_R B(m) \xrightarrow{\sim} B(n+m)$, then we are done. □

Theorem 17.1.1.19. *Let $f \colon A \to B$ be a homomorphism of w.c.f.g. R-algebras. Let $\overline{f} \colon A/\pi A \to B/\pi B$ be the induced homomorphism.*

(a) If $\overline{f}$ is injective, then so is f.

(b) If $\overline{f}$ is bijective and B is flat over R, then f is bijective.

(c) If $\overline{f}$ is injective and B is flat over R and $\pi^j R/\pi^{j+1}$ is flat over $R/\pi R$ for any $j \in \mathbb{N}$, then f is injective.

Proof. This is a particular case of [MW68, Theorem 3.2]. □

Corollary 17.1.1.20. *Let A_0 be a smooth (resp. étale) $R/\pi R$-algebra. Let A and B be two smooth w.c.f.g. R-algebra lifting of A_0. Then A and B are (non-canonically) isomorphic.*

We recall the notion of very smooth morphisms:

Definition 17.1.1.21. Let $* \in \{\mathrm{wc}, \mathrm{wcfg}\}$. An algebra $A \in \mathrm{Alg}_*(R)$ is said to be "very smooth in $\mathrm{Alg}_*(R)$" when the following two conditions are verified:

(a) The algebra $\overline{A} := A/\pi A$ is smooth on $\overline{R} := R/\pi R$.

(b) For any pair of morphisms of R-algebras $B \overset{p}{\twoheadrightarrow} C \overset{\phi}{\longleftarrow} A$, where $B, C \in \mathrm{Alg}_*(R)$ and p is surjective (of arbitrary kernel), and for each morphism of R-algebras $\overline{h} \colon A/\pi A \to B/\pi B$ verifying $\overline{\phi} = \overline{p} \circ \overline{h}$, there exists a lifting $h \colon A \to B$ of $\overline{h}$, such that $\phi = p \circ h$.

Theorem 17.1.1.22 (Arabia)**.** *Let A_0 be a smooth $R/\pi R$-algebra.*

(a) For any R-smooth lifting A of A_0, the algebra $A^\dagger$ is very smooth in $\mathrm{Alg}_{\mathrm{wc}}(R)$. *For any other R-smooth lifting A' of A_0, the algebras $A^\dagger$ and $A'^\dagger$ are isomorphic.*

(b) Any w.c.f.g. smooth R-algebra lifting of A_0 is very smooth in $\mathrm{Alg}_{\mathrm{wc}}(R)$.

(c) Let A be a smooth w.c.f.g. R-algebra and $B \in \mathrm{Alg}_{\mathrm{wc}}(R)$. Then, any morphism $\overline{h}\colon A/\pi A \to B/\pi B$ admits a lifting $h\colon A \to B$.

Proof. The first assertion is [Ara01, 3.3.2.(b)]. The second one is a consequence of (a) and of 17.1.1.16, 17.1.1.20. The last assertion, is a consequence of (b) and of [Ara01, 3.3.4]. □

Remark 17.1.1.23. For our purpose, we will use 17.1.1.22.(c) in the case where B is w.c.f.g. smooth, which was proved by van der Put (see [vdP86, 2.4.4.(ii)])

17.1.2 Affine weak formal schemes

Let A be a w.c.f.g. $\mathcal{V}$-algebra. We put $A_0 := A/\pi A$. For any $f \in A$, let $\overline{f}$ be the image de f in A_0 and $A_{[f]}$ be the weak completion of A_f.

Definition 17.1.2.1. We denote by $|\mathrm{Spff}(A)|$ the set of open prime ideals of A. We endow $|\mathrm{Spff}(A)|$ with a topology so that the canonical bijection $\iota\colon |\mathrm{Spff}(A)| \to |\operatorname{Spec} A_0|$ given by $\mathfrak{P} \mapsto \mathfrak{P}/\pi\mathfrak{P}$ is bicontinuous. We denote by $D(f) := \{\, \mathfrak{P} \in |\mathrm{Spff}(A)|\,;\, f \notin \mathfrak{P}\}$. Then $\iota(D(f)) = D(\overline{f})$. A principal open subset of $|\mathrm{Spff}(A)|$ is an open of the form $D(f)$ for some $f \in A$.

For any $f, g \in A$, we have $D(\overline{g}) \subset D(\overline{f})$ iff there exists a (unique) A_0-algebra morphism $(A_0)_{\overline{f}} \to (A_0)_{\overline{g}}$ iff $\overline{f} \in (A_0)_{\overline{g}}^\times$ iff $f \in A_{[g]}^\times$ iff there exists a (unique) A-algebra morphism $A_{[f]} \to A_{[g]}$ iff $D(g) \subset D(f)$.

Set $|X| := \mathrm{Spff}(A)$ this topological space. Let $\mathcal{O}_X$ be the presheaf on $|X|$ such that for any $f, g \in A$ such that $D(g) \subset D(f)$, $\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$ is the natural map $A_{[f]} \to A_{[g]}$. Following Meredith, $\mathcal{O}_X$ is in fact a sheaf and the *affine weak formal scheme* associated with A, is the ringed topological space

$$\mathrm{Spff}(A) := (|\mathrm{Spff}(A)|, \mathcal{O}_X).$$

For any $f \in A$, we have $D(f) \xrightarrow{\sim} \mathrm{Spff}(A_{[f]})$.

17.1.2.2. We have type A theorem on $X = \mathrm{Spff}\,(A)$ ([Mer72, 3.3]): the functors $M \mapsto \widetilde{M} := \mathcal{O}_X \otimes_A M$ and $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ are quasi-inverse equivalences between the category of coherent $\mathcal{O}_X$-modules and that of A-modules of finite type. In addition, we have the theorem of type B ([Mer72, 2.14]): for any coherent $\mathcal{O}_X$-module $\mathcal{M}$,for all integer $i > 0$, $H^i(X, \mathcal{M}) = 0$.

Notation 17.1.2.3. Let $\mathfrak{P}$ be an open ideal of A. Put $A_{[\mathfrak{P}]} = \varinjlim_{f \notin \mathfrak{P}} A_{[f]}$ (the inductive system is filtered). This is equipped with the ideal $\mathfrak{P}A_{[\mathfrak{P}]} = \varinjlim_{f \notin \mathfrak{P}} \mathfrak{P}A_{[f]}$.

Proposition 17.1.2.4. Let $\mathfrak{P}$ be an open ideal of A. $A_{[\mathfrak{P}]}$ is a local ring with maximal ideal $\mathfrak{P}A_{[\mathfrak{P}]}$. Its residue field is isomorphic to the field of fractions of $A/\mathfrak{P}$.

Proof. 1) Since $A_{[f]}/\pi A_{[f]} \xrightarrow{\sim} (A_0)_{\overline{f}}$, then for any $f \notin \mathfrak{P}$,

$$A_{[f]}/\mathfrak{P}A_{[f]} \xrightarrow{\sim} A/\mathfrak{P} \otimes_A A_{[f]} \xrightarrow{\sim} A/\mathfrak{P} \otimes_{A_0} A_0 \otimes_A A_{[f]} \xrightarrow{\sim} A/\mathfrak{P} \otimes_{A_0} (A_0)_{\overline{f}} \xrightarrow{\sim} (A_0)_{\overline{f}}/(\mathfrak{P}/\pi A)_{\overline{f}} \neq 0. \tag{17.1.2.4.1}$$

This implies that $\mathfrak{P}A_{[f]} \neq A_{[f]}$ and a fortiori $\mathfrak{P}A_{[\mathfrak{P}]} \neq A_{[\mathfrak{P}]}$ (one verifies that $1 \notin \mathfrak{P}A_{[\mathfrak{P}]}$).

2) Let x be an element of $A_{[\mathfrak{P}]}$ not in $\mathfrak{P}A_{[\mathfrak{P}]}$. To prove the first assertion, we have to check that x is invertible. There exists $f \notin \mathfrak{P}$ such that x comes from an element y of $A_{[f]}$. It follows from 17.1.2.4.1 that the canonical image $\overline{y}$ of y in $A_{[f]}/\pi A_{[f]} \xrightarrow{\sim} (A_0)_{\overline{f}}$ does not belong to $(\mathfrak{P}/\pi A)_{\overline{f}}$. This implies that $\overline{y} = \overline{a}/\overline{f}^r$, where $a \in A \setminus \mathfrak{P}$ and $r \in \mathbb{N}$. Thus, $g := af \notin \mathfrak{P}$ and the canonical image of $\overline{y}$ in $(A_0)_{\overline{g}}$ is invertible. As $\pi A[g]$ is included in the Jacobson ideal of $A[g]$, it follows that the canonical image of y in $A[g]$ is invertible. Hence x is invertible.

Let $S = \widehat{A} \setminus \widehat{\mathfrak{P}}$. By [Gro60, 0.7.6.17], $\widehat{A}\{S\}$ is a local ring with maximal ideal $\widehat{\mathfrak{P}}\widehat{A}\{S\}$. The canonical homomorphism $A_{[\mathfrak{P}]} \to \widehat{A}\{S\}$ is local. It induces a morphism $A_{[\mathfrak{P}]} \to \widehat{A}\{S\}/\widehat{\mathfrak{P}}\widehat{A}\{S\} \xleftarrow{\sim} \mathrm{Frac}(A/\mathfrak{P})$ (since $A/\mathfrak{P} \xrightarrow{\sim} \widehat{A}/\widehat{\mathfrak{P}}\widehat{A}$, then the isomorphism follows from [Gro60, 0.7.6.17]). Since any element of $A/\mathfrak{P}$ comes from an element of $A_{[\mathfrak{P}]}$, then the last homomorphism is surjective. This yields the isomorphism of residue fields: $A_{[\mathfrak{P}]}/\mathfrak{P}A_{[\mathfrak{P}]} \xrightarrow{\sim} \widehat{A}\{S\}/\widehat{\mathfrak{P}}\widehat{A}\{S\}$. □

17.1.2.5. The proposition 17.1.2.4 means that Spff A is a locally ringed space.

Lemma 17.1.2.6. *Let $\phi, \psi\colon A \to B$ be two morphisms of weakly complete $\mathcal{V}$-algebras. Let $\widehat{\phi}, \widehat{\psi}\colon \widehat{A} \to \widehat{B}$ be the induced morphisms.*

(a) We have $\widehat{\phi} = \widehat{\psi}$ if and only if $\phi = \psi$.

(b) Suppose A is a w.c.f.g. $\mathcal{V}$-algebra. The morphism ϕ is an isomorphism if and only if so is $\widehat{\phi}$.

Proof. Since $B \subset \widehat{B}$ then the equality $\widehat{\phi} = \widehat{\psi}$ implies $\phi = \psi$. The converse is obvious. Suppose $\widehat{\phi}$ is an isomorphism. Since $A \subset \widehat{A}$ then ϕ is injective. Let C be a w.c.f.g. $\mathcal{V}$-algebra included in B and containing $\phi(A)$. It follows from [vdP86, 2.4.3] that the morphism $A \to C$ given by $a \mapsto \phi(a)$ admit a left inverse ψ so that the composition $C \to \widehat{C} \to \widehat{B} \xleftarrow{\sim} \widehat{A}$ is the composition of ψ with the inclusion $A \to \widehat{A}$. Hence ψ is injective and therefore bijective. Hence the morphism $A \to C$ given by $a \mapsto \phi(a)$ must be bijective (for any such a choice C) and then $C = B$. □

17.1.2.7. Let $\phi\colon A \to B$ be a morphism of w.c.f.g. $\mathcal{V}$-algebras, $(X, \mathcal{O}_X) :=$ Spff A and $(Y, \mathcal{O}_Y) :=$ Spff B. We get a morphism of topological spaces ${}^a\phi\colon Y \to X$ defined by $\mathfrak{Q} \mapsto \phi^{-1}(\mathfrak{Q})$. For any element f of A, according to 17.1.1.4.(c) we have a unique morphism $A_{[f]} \to B_{[\phi(f)]}$ compatible with the canonical morphism $A_f \to B_{\phi(f)}$. Moreover, for any multiple f' of f, we have the commutative diagram:

$$\begin{array}{ccc} A_{[f]} & \longrightarrow & B_{[\phi(f)]} \\ \downarrow & & \downarrow \\ A_{[f']} & \longrightarrow & B_{[\phi(f')]}. \end{array}$$

As ${}^a\phi^{-1}(D(f)) = D(\phi(f))$, these homomorphisms define therefore a homomorphism of sheaves of rings $\widetilde{\phi}\colon \mathcal{O}_X \to {}^a\phi_*\mathcal{O}_Y$. Thus we have constructed a morphism $({}^a\phi, \widetilde{\phi})\colon$ Spff $B \to$ Spff A of ringed spaces. Moreover, for any open prime ideal $\mathfrak{Q}$ of B, we have a homomorphism of local rings $A_{[\phi^{-1}(\mathfrak{Q})]} \to B_{[\mathfrak{Q}]}$. The homomorphism $({}^a\phi, \widetilde{\phi})$ is thus a homomorphism of locally ringed spacess. Via the proposition above, they are all of this form.

Proposition 17.1.2.8. Let A and B be two w.c.f.g. $\mathcal{V}$-algebras and let $X =$ Spff A, $Y =$ Spff B. A morphism $u = (\psi, \theta)\colon Y \to X$ of ringed spaces is of the form $({}^a\phi, \widetilde{\phi})$, where ϕ is a homomorphism of rings $A \to B$ if and only if u is a morphism of locally ringed spaces.

Proof. By using 17.1.2.7, this is checked similarly to [Gro60, 10.2.2]. □

Remark 17.1.2.9. Let X and Y be two affine $\mathfrak{S}$-weak formal schemes. If $X/\mathfrak{S}$ is smooth and if $f_0\colon Y_0 \to X_0$ is a morphism of k schemes, then there exists a morphism $Y \to X$ of $\mathfrak{S}$-weak formal schemes lifting f_0. Indeed, since X is smooth, then there exists a lifting $\widehat{Y} \to \widehat{X}$ of f_0. We then conclude with [vdP86, 2.4.3].

17.1.3 Weak formal schemes and morphisms of weak formal schemes

Definition 17.1.3.1. The category of $\mathfrak{S}$-weak formal schemes is defined as follows: An $\mathfrak{S}$-*weak formal scheme* is a locally ringed $\mathcal{V}$-algebras space $X = (|X|, \mathcal{O}_X)$ locally isomorphic to an affine $\mathfrak{S}$-weak formal scheme (see [Mer72]). A *morphism* $f\colon Y \to X$ of $\mathfrak{S}$-weak formal schemes is a morphism of locally ringed spaces.

Let $f\colon Y \to X$ a morphism of $\mathfrak{S}$-weak formal schemes. For any integer i, the $\mathcal{V}/\pi^{i+1}\mathcal{V}$-scheme induced by reduction modulo π^{i+1} is denoted by X_i. Moreover, $\widehat{X}$ or $\mathfrak{X}$ denotes the $\mathfrak{S}$-formal scheme obtained by p-adic completion of X, i.e. $\widehat{X} := \varinjlim_i X_i$. Write $f_i\colon Y_i \to X_i$ and $\hat{f}\colon \mathfrak{Y} \to \mathfrak{X}$ for the induced morphisms. By abuse of notations, we sometimes denote them by f. We say that such a morphism is *smooth* (resp. *étale*) if for all i, the f_i are smooth (resp. étale) morphisms. Finally, f is *separated* when f_0 is separated. An $\mathfrak{S}$-weak formal scheme X is separated if so is its structural morphism $X \to$ Spff $\mathcal{V}$. The usual properties of separated morphisms remain true.

An *affine open* $\mathfrak{S}$-weak formal subscheme of X is an open $\mathfrak{S}$-weak formal subscheme Y of X which is also an affine $\mathfrak{S}$-weak formal scheme.

A morphism $f\colon Y \to X$ of $\mathfrak{S}$-weak formal schemes is said to be *affine* if, for all affine open X' of X, $f^{-1}(X')$ is an affine open of Y.

Proposition 17.1.3.2. Let Y be an $\mathfrak{S}$-weak formal scheme, $X = \operatorname{Spff} A$ an affine $\mathfrak{S}$-weak formal scheme. There exists a bijective correspondence between the morphisms of $\mathfrak{S}$-weak formal schemes of the form $Y \to X$ and the ring homomorphisms of the form $A \to \Gamma(Y, \mathcal{O}_Y)$.

Proof. By using proposition 17.1.2.8, we can follows the algebraic or formal proof of [Gro60, 2.2.4 or 10.4.6]. □

Proposition 17.1.3.3. The category of $\mathfrak{S}$-weak formal schemes has fiber products.

Proof. If $A \to B$ and $A \to C$ are two morphisms of w.c.f.g. $\mathcal{V}$-algebras, then it follows from 17.1.3.2 and 17.1.1.4.(c) that the $\mathfrak{S}$-weak formal scheme $\operatorname{Spff}(B) \times_{\operatorname{Spff}(A)} \operatorname{Spff}(C) := \operatorname{Spff}(B \otimes^{\dagger}_A C)$ satisfies the universal property of fiber products. For the general case, one proceeds by glueing (in the same manner as in the case of schemes: see [Gro60]). □

17.1.3.4. Let X be an $\mathfrak{S}$-scheme of finite type. Meredith ([Mer72, 4]) constructs the sheaf $\mathcal{O}^{\dagger}_X$ as follows: if $U \subset V \subset X$ are affine open of X, then $\Gamma(U, \mathcal{O}^{\dagger}_X) := \Gamma(U, \mathcal{O}_X)^{\dagger}$, $\Gamma(V, \mathcal{O}^{\dagger}_X) := \Gamma(V, \mathcal{O}_X)^{\dagger}$ and $\Gamma(V, \mathcal{O}^{\dagger}_X) \to \Gamma(U, \mathcal{O}^{\dagger}_X)$ is the canonical morphism induced by $\Gamma(V, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ (see 17.1.1.4.(c)). Denoting by $|X_0|$ the underlying topological space of $X_0 = X \times_{\operatorname{Spec}\mathcal{V}} \operatorname{Spec}(k)$, one verifies that the ringed space $X^{\dagger} := (|X_0|, \mathcal{O}^{\dagger}_X)$ is an $\mathfrak{S}$-weak formal scheme and that we have a canonical morphism $X^{\dagger} \to X$, which is called the weak completion of X. We can check moreover that the map $X \mapsto X^{\dagger}$ induces canonically (via 17.1.1.4.(c)) a functor from the category of $\mathfrak{S}$-schemes of finite type in that of $\mathfrak{S}$-weak formal schemes.

For instance, $\mathbb{A}^{d\dagger}_{\mathfrak{S}} := (\mathbb{A}^d_{\mathfrak{S}})^{\dagger}$ and $\mathbb{P}^{d\dagger}_{\mathfrak{S}} := (\mathbb{P}^d_{\mathfrak{S}})^{\dagger}$. If Y is an $\mathfrak{S}$-weak formal scheme, $\mathbb{A}^{d\dagger}_Y := \mathbb{A}^{d\dagger}_{\mathfrak{S}} \times Y$ and $\mathbb{P}^{d\dagger}_Y := \mathbb{P}^{d\dagger}_{\mathfrak{S}} \times Y$ (recall 17.1.3.3).

Definition 17.1.3.5. Let U be an $\mathfrak{S}$-weak formal scheme. We say that a finite set $t_1, \ldots, t_d$ of elements of $\Gamma(U, \mathcal{O}_U)$ are "coordinates of $U/\mathfrak{S}$" if the corresponding $\mathfrak{S}$-morphism $U \to \mathbb{A}^{d\dagger}_{\mathfrak{S}}$ is étale.

Proposition 17.1.3.6. *Let U be a p-torsion free $\mathfrak{S}$-weak formal scheme and $\mathfrak{U}$ be the p-adic completion of U. Then U_0/S_0 has coordinates if and only if U/S has coordinates if and only if $\mathfrak{U}/\mathfrak{S}$ has coordinates. Let $t_1, \ldots, t_d \in \Gamma(U, \mathcal{O}_U)$ and $\bar{t}_1, \ldots, \bar{t}_d \in \Gamma(U_0, \mathcal{O}_{U_0})$ be the images. The following conditions are equivalent:*

(a) The map $U_0 \to \mathbb{A}^d_{S_0}$ induced by $\bar{t}_1, \ldots, \bar{t}_d$ is étale ;

(b) The map $\mathfrak{U} \to \widehat{\mathbb{A}}^d_{\mathfrak{S}}$ induced by $t_1, \ldots, t_d$ is étale ;

(c) The map $U \to \mathbb{A}^{d\dagger}_{\mathfrak{S}}$ induced by $t_1, \ldots, t_d$ is étale.

Proof. Let $f_0 \colon U_0 \to \mathbb{A}^d_{S_0}$ be an étale map. We can lift this map to a map $f \colon U \to \mathbb{A}^{d\dagger}_{\mathfrak{S}}$ (use 17.1.3.2 and 17.1.1.4.(c)). Hence, following 17.1.1.14, since U is p-torsion free then f is étale and so is its p-adic completion $\widehat{f} \colon \mathfrak{U} \to \widehat{\mathbb{A}}^d_{\mathfrak{S}}$. Conversely, an étale map $\mathfrak{U} \to \widehat{\mathbb{A}}^d_{\mathfrak{S}}$ induces by base change the étale map $U_0 \to \mathbb{A}^d_{S_0}$. □

Lemma 17.1.3.7. *Let $f, g \colon X \to Y$ be two morphisms of $\mathfrak{S}$-weak formal schemes. We have $\widehat{f} = \widehat{g}$ if and only if $f = g$. Moreover, $\widehat{f}$ is an isomorphism if and only if so is f.*

Proof. Suppose $\widehat{f} = \widehat{g}$. To check that $f = g$, we reduce to the case where Y is affine. Since $\Gamma(X, \mathcal{O}_X) \to \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is injective, then we conclude via the Proposition 17.1.3.2 (and its formal analogue [Gro60, 10.4.6]) that $f = g$.

Suppose $\widehat{f}$ is an isomorphism. To check that f is an isomorphism, we reduce to the case where Y is of the form $Y = \operatorname{Spff} A$. Since $\widehat{f}$ is an isomorphism, then the p-adic completion of $A \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism. Via 17.1.2.6, this implies that $A \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism. For any $a \in A$, let $X_a := f_0^{-1}(D(a))$. Similarly, we check that the canonical morphism $A_{[a]} \to \Gamma(X_a, \mathcal{O}_X)$ is an isomorphism. Since f is an homeomorphism (because $f = f_0$ as morphism of topological spaces), then the family $\{X_a\}_{a \in A}$ is an open neighborhoods basis of $|X|$. Hence the canonical map $f_* \mathcal{O}_X \to \mathcal{O}_Y$ is an isomorphism. This yields the map $X \to Y$ is an isomorphism as morphism of ringed topological spaces. □

Proposition 17.1.3.8. *Let X be a p-torsion free $\mathfrak{S}$-weak formal scheme such that X_0 is affine. Then the canonical morphism $X \to \operatorname{Spff}\Gamma(X, \mathcal{O}_X)$ is an isomorphism. In particular, X is an affine $\mathfrak{S}$-weak formal scheme.*

Proof. Set $Y := \mathrm{Spff}\,\Gamma(X, \mathcal{O}_X)$ and let $f\colon X \to Y$ be the canonical morphism. By hypothesis, f_0 is an isomorphism. Since X_i/S_i and Y_i/S_i are flat, and f_0 is étale then so is f_i. Since $(f_i)_{\mathrm{red}} = (f_0)_{\mathrm{red}}$ is a universal homeomorphism, then so is f_i (see [Gro65, 2.4.3]). Hence f_i is an isomorphism (use [Gro67, 17.9.1]). This yields that $\widehat{f}$ is an isomorphism. We conclude therefore by using 17.1.3.7. □

17.1.4 Immersions

Proposition 17.1.4.1. Let X be a $\mathfrak{S}$-weak formal scheme and $\mathcal{I}$ a coherent ideal of $\mathcal{O}_X$. Let Y be the support of $\mathcal{O}_X/\mathcal{I}$.

(a) Y is a closed subspace of $|X|$.

(b) The ringed topological space $(Y, (\mathcal{O}_X/\mathcal{I})|_Y)$ is a $\mathfrak{S}$-weak formal scheme. More precisely, if X is affine, then $(Y, (\mathcal{O}_X/\mathcal{I})|_Y)$ is canonically isomorphic to $\mathrm{Spff}\,(A/I)$, where $A := \Gamma(X, \mathcal{O}_X)$, $I := \Gamma(X, \mathcal{I})$.

(c) The canonical morphism $(Y, (\mathcal{O}_X/\mathcal{I})|_Y) \to (X, \mathcal{O}_X)$ is affine.

Proof. 1) Suppose X is affine and let $A := \Gamma(X, \mathcal{O}_X)$, $I := \Gamma(X, \mathcal{I})$. Since coherent $\mathcal{O}_X$-modules satisfy theorem of type A, then the canonical morphism $I \otimes_A \mathcal{O}_X \to \mathcal{I}$ is an isomorphism. By flatness of $A \to \mathcal{O}_X$, this yields $A/I \otimes_A \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X/\mathcal{I}$.

i) Let us check the equality: $Y = \{\mathfrak{P} \in \mathrm{Spff}\,A\ ;\ \mathfrak{P} \supset I\}$. Let $x \in |X|$ and $\mathfrak{P}$ be the associated open prime ideal of A. Since $\mathcal{O}_{X,x} = A_{[\mathfrak{P}]}$, then we have $(\mathcal{O}_X/\mathcal{I})_x \xrightarrow{\sim} A/I \otimes_A A_{[\mathfrak{P}]} \xrightarrow{\sim} A_{[\mathfrak{P}]}/IA_{[\mathfrak{P}]}$. Suppose $I \subset \mathfrak{P}$. Since $A_{[\mathfrak{P}]}/IA_{[\mathfrak{P}]} \twoheadrightarrow A_{[\mathfrak{P}]}/\mathfrak{P}A_{[\mathfrak{P}]} \neq 0$, then we get $(\mathcal{O}_X/\mathcal{I})_x \neq 0$. Otherwise, suppose there exists $f \in I \setminus \mathfrak{P}$.Then $A/I \otimes_A A_f \xrightarrow{\sim} 0$. Hence $A/I \otimes_A A_{[\mathfrak{P}]} \xrightarrow{\sim} (A/I \otimes_A A_f) \otimes_{A_f} A_{[\mathfrak{P}]} \xrightarrow{\sim} 0$, and we are done.

ii) Following a), we get the bijection $\varepsilon\colon Y \xrightarrow{\sim} |\mathrm{Spff}\,A/I|$ given by $\mathfrak{P} \to \mathfrak{P}/I$. Let $\iota\colon |\mathrm{Spff}\,(A/I)| \xrightarrow{\sim} |\,\mathrm{Spec}(A_0/I_0)|$ be the canonical bijection given by $\mathfrak{P}/I \to (\mathfrak{P}/\pi A)/I_0$, where $I_0 = I/\pi A$ (see 17.1.2.1). Let $f \in I$ and $\overline{f}$ be its image in A/I. We compute $\varepsilon^{-1}(D(\overline{f})) = D(f) \cap Y$. Hence, ε is an homeomorphism and Y is a closed subspace of $|X|$.

iii) Let us now prove that ε induces the isomorphism of ringed topological spaces of the form $(Y, (A/I \otimes_A \mathcal{O}_X)|_Y) \xrightarrow{\sim} \mathrm{Spff}\,(A/I) =: V(\mathcal{I})$. Since $A/I \otimes_A \mathcal{O}_X$ has its support in the closed subspace Y, then we get for any $f \in A$ the isomorphism:

$$\epsilon\colon \Gamma(D(f) \cap Y, (A/I \otimes_A \mathcal{O}_X)|_Y) = \Gamma(D(f), A/I \otimes_A \mathcal{O}_X) \xrightarrow{\sim} A/I \otimes_A A_{[f]}.$$

On the other hand, $\Gamma(D(\overline{f}), V(\mathcal{I})) \xrightarrow{\sim} (A/I)_{[\overline{f}]}$. Hence, we have to check that for any $f \in A$ and any $g \in A$ which is a multiple of f, we have a canonical isomorphism $A/I \otimes_A A_{[f]} \xrightarrow{\sim} (A/I)_{[\overline{f}]}$ inducing the commutative diagram

$$\begin{array}{ccc} A/I \otimes_A A_{[f]} & \longrightarrow & A/I \otimes_A A_{[g]} \\ \downarrow\wr & & \downarrow\wr \\ (A/I)_{[\overline{f}]} & \longrightarrow & (A/I)_{[\overline{g}]}, \end{array}$$

where $\overline{f}, \overline{g} \in A/I$ are the canonical images of f and g respectively.

The epimorphism $A_f \twoheadrightarrow (A/I)_{\overline{f}}$ induces the following one $(A_f)^\dagger \twoheadrightarrow ((A/I)_{\overline{f}})^\dagger$ (see 17.1.1.4.(c)) and therefore $A/I \otimes_A (A_f)^\dagger \twoheadrightarrow ((A/I)_{\overline{f}})^\dagger$. It is immediate that this is functorial in f. It remains to check injectivity which results from the commutative diagram

$$\begin{array}{ccc} A/I \otimes_A (A_f)^\dagger & \longrightarrow & ((A/I)_{\overline{f}})^\dagger \\ \downarrow & & \downarrow \\ A/I \widehat{\otimes}_A (A_f)^\dagger & \xrightarrow{\sim} & ((A/I)_{\overline{f}})^\wedge, \end{array}$$

where the vertical arrows are injective (because both top objects are w.c.f.g. $\mathcal{V}$-algebras) and the bottom arrow is an isomorphism.

2) It follows from the par 1) that the support Y is a closed subset of X and that $(Y, (\mathcal{O}_X/\mathcal{I})|_Y)$ is a $\mathfrak{S}$-weak formal scheme.

3) Suppose $X = \mathrm{Spff}\,A$ and $I := \Gamma(X, \mathcal{I})$ as in 1). The canonical morphism $A \to A/I$ induces $\mathrm{Spff}\,(A/I) \to \mathrm{Spff}\,(A)$ (see 17.1.2.8). We compute that the composition of this latter morphism with the isomorphism ε is the canonical map $(Y, (\mathcal{O}_X/\mathcal{I})|_Y) \to \mathrm{Spff}\,(A)$. Hence, $(Y, (\mathcal{O}_X/\mathcal{I})|_Y) \to X$ is affine. □

Definition 17.1.4.2. We introduce the following terminology.

(a) A closed $\mathfrak{S}$-weak formal subscheme of an $\mathfrak{S}$-weak formal scheme X is a $\mathfrak{S}$-weak formal scheme of the form $(Y, (\mathcal{O}_X/\mathcal{I})|_Y)$, where $\mathcal{I}$ is a coherent ideal of $\mathcal{O}_X$; we say that this is the closed $\mathfrak{S}$-weak formal subscheme defined by $\mathcal{I}$.

(b) A morphism $f: Y \to X$ of $\mathfrak{S}$-weak formal schemes is a "closed immersion" (resp. "an open immersion") if it factorises into $Y \overset{g}{\to} X' \overset{u}{\to} X$, where X' is a closed $\mathfrak{S}$-weak formal subscheme of X (resp. X' is an open of X) and g an isomorphism.

(c) An "immersion" is the composition of a closed immersion followed by an open immersion.

17.1.4.3. A closed immersion is an affine morphism. Moreover, similarly to [Gro60, 10.14.4], we have the following characterization of a closed immersion. Let $f: Y \to X$ be a morphism of $\mathfrak{S}$-weak formal schemes, and let (X_α) be a covering of $f(Y)$ by affine opens of X, such that the $f^{-1}(X_\alpha)$ are affine opens of Y. The following conditions are therefore equivalent:

(a) f is a closed immersion.

(b) $f(Y)$ is a closed subset of X and, for all α, the homomorphism $\Gamma(X_\alpha, \mathcal{O}_X) \to \Gamma(f^{-1}(X_\alpha), \mathcal{O}_Y)$, induced via 17.1.2.8 by the restriction of f to $f^{-1}(X_\alpha)$, is surjective.

Proposition 17.1.4.4. We have the following properties.

(a) If $f: Z \to Y$ and $g: Y \to X$ are closed immersions (resp. immersions) of $\mathfrak{S}$-weak formal schemes, then $g \circ f$ is a closed immersion (resp. immersion).

(b) Let X, Y, Z be three $\mathfrak{S}$-weak formal schemes, $f: Y \to X$ a closed immersion (resp. immersion) and $Z \to X$ a morphism. The morphism $Y \times_X Z \to Z$ is a closed immersion (resp. immersion).

(c) Let X be an $\mathfrak{S}$-weak formal scheme. Suppose that the X-morphisms $f: Y \to Y'$ and $g: Z \to Z'$ are closed immersions (resp. immersions), then $f \times_X g$ is a closed immersion (resp. immersion).

Proof. 1) First, let us prove the proposition in the non-respective case. Let $A \to C$ be a morphism of w.c.f.g. $\mathcal{V}$-algebras, I an ideal of A and $B := A/I$. The algebra $B \otimes_A C$ is a w.c.f.g. $\mathcal{V}$-algebra. Thus, $C/IC \xrightarrow{\sim} B \otimes_A C \xrightarrow{\sim} B \otimes^\dagger_A C$. With this remark, the proof is analogous to [Gro60, 10.14.5].

2) Let us treat now the respective case. First consider (i). Let $j: Z \to Y$ be an open immersion and $u: Y \to X$ a closed immersion. The image of Z by j is an open subset of Y. Thus there exists an open subset X' of X such that $u^{-1}(X') = j(Z)$. Let $u': j(Z) \to X'$ denote the closed immersion induced by u, $u \circ j$ decomposes into $Z \overset{j}{\to} j(Z) \overset{u'}{\to} X' \subset X$. Thus, $u \circ j$ is the composition of a closed immersion followed by an open immersion.

The assertions (ii) and (iii) follows from (i). □

The following proposition gives examples of closed immersions.

Proposition 17.1.4.5. Let $f: Z \to Y$ and $g: Y \to X$ be morphisms of $\mathfrak{S}$-weak formal schemes. If g is separated then the graph of f, $\Gamma_f = (\mathrm{id}, f)_X : Z \to Z \times_X Y$, is a closed immersion.

Proof. Similar to [Gro60, 10.15.4]. □

Proposition 17.1.4.6. Let $f: X \to Y$ be a morphism of $\mathfrak{S}$-weak formal schemes. The canonical morphism $\delta = (\mathrm{id}, \mathrm{id})_Y : X \to X \times_Y X$ is an immersion. We call it the " diagonal immersion ".

Proof. Let (X_α) and (Y_α) be coverings of respectively X and Y by affine opens such that f factors through $X_\alpha \to Y_\alpha$. By construction of the fiber product $X \times_Y X$, the $\mathfrak{S}$-weak formal scheme $X_\alpha \times_{Y_\alpha} X_\alpha$ is an open subset of $X \times_Y X$ and $\delta^{-1}(X_\alpha \times_{Y_\alpha} X_\alpha) = X_\alpha$. Denote by Y' the union of $X_\alpha \times_{Y_\alpha} X_\alpha$. The morphism δ factors through a morphism $\delta': X \to Y'$. According to the characterisation 17.1.4.3 of closed immersions, in order to prove that δ' is a closed immersion, it suffices to check that the canonical homomorphisms $\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}) \to \Gamma(X_\alpha \times_{Y_\alpha} X_\alpha, \mathcal{O}_{X_\alpha \times_{Y_\alpha} X_\alpha})$ are surjective, which is immediate. □

Corollaire 17.1.4.7. Let $f: X \to S$, $g: Y \to S$ and $\phi: S \to T$ be morphisms of $\mathfrak{S}$-weak formal schemes. The canonical morphism $X \times_S Y \to X \times_T Y$ induced by ϕ is an immersion. In particular, when $S = Y$, the graph of f, $X \to X \times_T Y$, is an immersion.

Proof. This results from $X \times_S Y \xrightarrow{\sim} S \times_{(S \times_T S)} (X \times_T Y)$, 17.1.4.6 and from the fact that the closed immersions are stable by base change (17.1.4.4).

□

17.2 $\mathcal{D}$-modules over smooth $\mathfrak{S}$-weak formal schemes

Let U be a smooth $\mathfrak{S}$-weak formal scheme and $m \in \mathbb{N}$.

17.2.1 Sheaf of differential operators of level m

Lemma 17.2.1.1. *Let U an affine $\mathfrak{S}$-weak formal scheme equipped with coordinates $t_1, \dots, t_d$ and $\tau_1 = 1 \otimes t_1 - t_1 \otimes 1, \dots, \tau_d = 1 \otimes t_d - t_d \otimes 1$. The sequence $\tau_1, \dots, \tau_d$ is a regular sequence of generators of the ideal defining the closed immersion $U \hookrightarrow U \times_{\mathfrak{S}} U$.*

Proof. This results from the "formal case". Indeed, let A be the w.c.f.g. $\mathcal{V}$-algebra of U and I the ideal of $A \otimes^{\dagger}_{\mathcal{V}} A$ corresponding to the closed immersion $U \hookrightarrow U \times U$. The canonical images of $t_1, \dots, t_d$ in $\widehat{A}$ are coordinates of $\widehat{U}$. Hence, it follows from [Gro67, 16.9.3] that the images of $\tau_1, \dots, \tau_d$ in $A \widehat{\otimes}_{\mathcal{V}} A$ is a quasi-regular sequence of $\widehat{I}$ and then by noetherianity (see [Gro67, 16.9.10]) is a regular sequence of $\widehat{I}$. Moreover, by faithful flatness of $A \otimes^{\dagger}_{\mathcal{V}} A \to A \widehat{\otimes}_{\mathcal{V}} A$, as $\widehat{I} \xrightarrow{\sim} I \otimes_{A \otimes^{\dagger}_{\mathcal{V}} A} A \widehat{\otimes}_{\mathcal{V}} A$ is generated by the images of $\tau_1, \dots, \tau_d$, then $\tau_1, \dots, \tau_d$ generates I. In addition, $I/(\tau_1, \dots, \tau_r) \to \widehat{I}/(\tau_1, \dots, \tau_r)$ is injective. As multiplication by τ_{r+1} est injective in $\widehat{I}/(\tau_1, \dots, \tau_r)$, then so is in $I/(\tau_1, \dots, \tau_r)$. □

17.2.1.2. We denote by $\mathcal{I}$ the coherent ideal of the diagonal immersion: $U \hookrightarrow U \times_{\mathfrak{S}} U$. Let $\mathfrak{B}$ be the category of affine opens of U and Alg be that of $\mathcal{V}$-algebra. The contravariant functor $\mathfrak{B} \to$ Alg given by $U' \mapsto \mathcal{P}^n_{(m)}(\Gamma(U' \times_{\mathfrak{S}} U'))$ is a sheaf. We define the *algebra of principal parts of level m and order $\leq n$ of* U which we denote by $\mathcal{P}^n_{U,(m)}$, this sheaf also called the divided power envelop of level m and order $\leq n$ of $\mathcal{I}$. The two canonical projections $U \times_{\mathfrak{S}} U \to U$ induce two structures of O_U-algebras on $\mathcal{P}^n_{U,(m)}$: the left structure and the right structure. It follows from 1.3.3.11 and 17.2.1.1 that, if $U/\mathfrak{S}$ is equipped with coordinates $t_1, \dots, t_d$, for each of these structures of $\mathcal{O}_U$-algebras, the sheaf $\mathcal{P}^n_{U,(m)}$ is a free $\mathcal{O}_U$-module and $\underline{\tau}^{\{\underline{k}\}} = \tau_1^{\{k_1\}} \cdots \tau_d^{\{k_d\}}$, for $\underline{k} \leq n$, form a base.

The *sheaf of differential operators of level m and order $\leq n$ on U*, denote by $\mathcal{D}^{(m)}_{U,n}$, is the $\mathcal{O}_U$-linear dual of $\mathcal{P}^n_{U,(m)}$ for its left structure of $\mathcal{O}_U$-algebras. The *sheaf of differential operators of level m on U* is the union: $\mathcal{D}^{(m)}_U := \cup_{n \in \mathbb{N}} \mathcal{D}^{(m)}_{U,n}$. Similarly to 1.4.2.1, we equip this sheaf with the structure of ring and of two structures of $\mathcal{O}_U$-algebra, one right and the other left.

If $U/\mathfrak{S}$ is equipped with coordinates, with the base of $\mathcal{P}^n_{U,(m)}$ consisting of $\underline{\tau}^{\{\underline{k}\}}$ for $\underline{k} \leq n$, the elements of the corresponding dual base of $\mathcal{D}^{(m)}_{U,n}$ will be denoted by $\underline{\partial}^{\langle \underline{k} \rangle}$. The $\underline{\partial}^{\langle \underline{k} \rangle}$ form thus a base of $\mathcal{D}^{(m)}_U$.

The proposition 1.4.2.7 and the paragraph 1.4.2.8 remain true when we replace "scheme" by "$\mathfrak{S}$-weakly formal scheme". Likewise the definitions and results on the m-PD-stratifications (2.1) are again true when we replace "formal" by "weakly formal".

17.2.2 Coherent $\mathcal{D}^{(m)}_U$-modules

Lemma 17.2.2.1. *Suppose U is affine. The ring $\Gamma(U, \mathcal{D}^{(m)}_U)$ (resp. $\mathcal{D}^{(m)}_{U,x}$ for any $x \in U$) is left and right noetherian.*

Proof. By using 17.1.1.9, we proceed similarly to 1.4.2.11. □

Proposition 17.2.2.2. Let $j \colon U \hookrightarrow P$ be an open immersion of smooth $\mathfrak{S}$-weak formal scheme such that $P_0 \setminus U_0$ is the support of a divisor. The sheaf of rings $\mathcal{D}^{(m)}_P$ and $j_* \mathcal{D}^{(m)}_U$ are left and right coherent.

Proof. The left and right coherence of $\mathcal{D}^{(m)}_P$ is a particular case of that of $j_* \mathcal{D}^{(m)}_U$. Let us prove the left coherence of $j_* \mathcal{D}^{(m)}_U$. There exists a basis $\mathcal{B}$ consisting of open subsets of P such that for any $V \in \mathcal{B}$, the $\mathfrak{S}$-weak formal schemes V and $V \cap U$ are affine. By 17.2.2.1, the ring $\Gamma(V, j_* \mathcal{D}^{(m)}_U) = \Gamma(U \cap V, \mathcal{D}^{(m)}_U)$ is left and right noetherian. Moreover, for any open subsets $V' \subset V$ of $\mathcal{B}$, $\Gamma(U \cap V, \mathcal{O}_U) \to \Gamma(U \cap V', \mathcal{O}_U)$ is flat (since this is a homomorphism of smooth w.c.f.g. $\mathcal{V}$-algebras whose reduction modulo π is flat then this follows from Theorem 17.1.1.13). As $\Gamma(U \cap V', \mathcal{O}_U) \otimes_{\Gamma(U\cap V,\mathcal{O}_U)} \Gamma(V, j_*\mathcal{D}_U^{(m)}) \xrightarrow{\sim} \Gamma(V', j_*\mathcal{D}_U^{(m)})$, $\Gamma(V, j_*\mathcal{D}_U^{(m)}) \to \Gamma(V', j_*\mathcal{D}_U^{(m)})$ is flat. We conclude via 1.4.5.2. □

17.2.2.3. For any integers n and n', we denote by $\mathcal{D}_{U,n}^{(m)} \cdot \mathcal{D}_{U,n'}^{(m)}$ the image of the left and right $\mathcal{O}_U$-linear homomorphism $\mathcal{D}_{U,n}^{(m)} \otimes_{\mathcal{O}_U} \mathcal{D}_{U,n'}^{(m)} \to \mathcal{D}_{U,n+n'}^{(m)}$.

Proposition 17.2.2.4. For $n < 0$, we put $\mathcal{D}_{U,n}^{(m)} := 0$. For any pair $(r, s) \in \mathbb{N}^2$, $\sum_{j=0,\ldots,p^m-1} \mathcal{D}_{U,r-j}^{(m)} \cdot \mathcal{D}_{U,s+j}^{(m)} = \mathcal{D}_{U,r+s}^{(m)}$.

Proof. Same computation as 4.1.3.5. □

Similarly to 4.1.3.7, we introduce the following definition.

Definition 17.2.2.5. Let $\mathcal{M}$ be a $\mathcal{D}_U^{(m)}$-module. A *filtration* of $\mathcal{M}$ is a family $(\mathcal{M}_r)_{r\in\mathbb{N}}$ of sub-$\mathcal{O}_U$-modules of $\mathcal{M}$ such that:

(i) For all $r,\ s \in \mathbb{N}$: $\mathcal{M}_r \subset \mathcal{M}_{r+1}$, $\mathcal{D}_{U,r}^{(m)} \cdot \mathcal{M}_s \subset \mathcal{M}_{r+s}$,

(ii) $\mathcal{M} = \cup_{r\in\mathbb{N}}\mathcal{M}_r$.

Definition 17.2.2.6. Let $\mathcal{M}$ be a left $\mathcal{D}_U^{(m)}$-module equipped with a filtration. The filtration is *good* if and only if:

(i) For any $r \in \mathbb{N}$, $\mathcal{M}_r$ is $\mathcal{O}_U$-coherent;

(ii) There exists an integer $r_1 \in \mathbb{N}$ such that for any integer $r \geq r_1$, we have

$$\mathcal{M}_r = \sum_{j=0}^{p^m-1} \mathcal{D}_{U,r-r_1+j}^{(m)} \cdot \mathcal{M}_{r_1-j}.$$

Example 17.2.2.7. According to proposition 17.2.2.4, the family $(\mathcal{D}_{U,r}^{(m)})_{r\in\mathbb{N}}$ is a filtration of $\mathcal{D}_U^{(m)}$ satisfying condition (ii) of 17.2.2.6 for any integer r_1. Since the condition (i) holds also, then this filtration is therefore good. We call it the *order filtration*.

Proposition 17.2.2.8. A $\mathcal{D}_U^{(m)}$-module which is globally of finite presentation admits a good filtration.

Proof. Similar to 4.1.3.16. □

Theorem 17.2.2.9 (Theorem B)**.** *Suppose U is affine and that $\mathcal{M}$ is a $\mathcal{D}_U^{(m)}$-module which is globally of finite presentation. Then, for all integer $i \neq 0$, $H^i(U, \mathcal{M}) = 0$.*

Proof. This follows from 17.2.2.8, from theorem of type B for coherent $\mathcal{O}_U$-modules and from the fact that, for all integer $i \neq 0$, since U is coherent then the functor $H^i(U, \mathcal{M})$ commutes with filtered inductive limits (see [SGA4.2, VI.5.3]). □

Theorem 17.2.2.10 (Theorem A)**.** *Suppose U is affine.*

(a) The functors $\mathcal{M} \mapsto \Gamma(U, \mathcal{M})$ and $M \mapsto \widetilde{M}$ are exact quasi-inverse equivalences between the category of $\mathcal{D}_U^{(m)}$-modules globally of finite presentation and the category of $\Gamma(U, \mathcal{D}_U^{(m)})$-modules of finite type.

(b) Let $u\colon M \to N$ be a morphism of $\Gamma(U, \mathcal{D}_U^{(m)})$-modules of finite type. The $\mathcal{D}_U^{(m)}$-modules associated to Ker u, Imu, Coker u, *are respectively* Ker $\tilde{u}$, Im$\tilde{u}$, Coker $\tilde{u}$. *In particular, the category of $\mathcal{D}_U^{(m)}$-modules globally of finite presentation is an abelian category.*

Proof. We can copy the proof of 4.1.3.19 to check the first part. The exactness of $M \mapsto \widetilde{M}$ (resp. $\mathcal{M} \mapsto \Gamma(U, \mathcal{M})$) follows from the flatness of $\Gamma(U, \mathcal{D}_U^{(m)}) \to \mathcal{D}_U^{(m)}$ (resp. from 17.2.2.9). We get the second statement by using the exactness of $M \mapsto \widetilde{M}$ and the exact sequences:

$$0 \to \operatorname{Ker} u \to M \to \operatorname{Im} u \to 0, \qquad 0 \to \operatorname{Im} u \to N \to \operatorname{Coker} u \to 0. \tag{17.2.2.10.1}$$

□

Remark 17.2.2.11. As the sheaf of rings $\mathcal{D}_U^{(m)}$ is coherent (17.2.2.2), a $\mathcal{D}_U^{(m)}$-module $\mathcal{M}$ is coherent if and only if it is locally of finite presentation, i.e., according to 17.2.2.10, if and only if it is locally of the form $\widetilde{M}$, where M is a $\Gamma(U, \mathcal{D}_U^{(m)})$-module of finite type.

Theorem 17.2.2.12. *Let $\mathcal{M}$ be a $\mathcal{D}_U^{(m)}$-module. Then $\mathcal{M}$ is coherent if and only if it admits good filtrations locally.*

Proof. Similar to 4.1.3.21. □

Proposition 17.2.2.13. Let $j: U \subset P$ be an open immersion of smooth $\mathfrak{S}$-weak formal scheme such that $P_0 \setminus U_0$ is the support of a divisor and $\mathcal{M}$ a $\mathcal{D}_U^{(m)}$-module which is locally in P of finite presentation (i.e. there exists a covering by open subsets of $P = \cup_\alpha P_\alpha$ such that, for all α, $\mathcal{M}|_{U \cap P_\alpha}$ is globally of finite presentation). Then:

(a) The canonical morphism $j_*\mathcal{M} \to \mathbb{R}j_*\mathcal{M}$ is an isomorphism;

(b) For all smooth morphism $P \to P'$ of $\mathfrak{S}$-weak formal schemes, for all open subset U' of P' such that $U = f^{-1}(U')$, the canonical morphism $j_*(\Omega^\bullet_{U/U'} \otimes_{\mathcal{O}_U} \mathcal{M}) \to \mathbb{R}j_*(\Omega^\bullet_{U/U'} \otimes_{\mathcal{O}_U} \mathcal{M})$ is an isomorphism;

(c) The $j_*\mathcal{D}_U^{(m)}$-module $j_*\mathcal{M}$ is coherent. Moreover, if U is affine and $\mathcal{M}$ is a $\mathcal{D}_U^{(m)}$-module globally of finite presentation, then $j_*\mathcal{M}$ is a $j_*\mathcal{D}_U^{(m)}$-module globally of finite presentation.

Proof. Proof of (a). As $P_0 \setminus U_0$ is a divisor, the assertion being local, we can suppose that P_α and $U \cap P_\alpha$ are affine. For any integer i, $(H^i j_*\mathcal{M})|_{P_\alpha} \xrightarrow{\sim} H^i j_{\alpha *}(\mathcal{M}|_{U \cap P_\alpha})$, where $j_\alpha: U \cap P_\alpha \hookrightarrow P_\alpha$. Since $H^i j_{\alpha *}(\mathcal{M}_{U \cap P_\alpha})$ is the sheaf associated to the presheaf which to any principal open subset P' of P_α associate $H^i(U \cap P', \mathcal{M})$, then by theorem B of the form 17.2.2.9 we conclude the proof of (a).

The assertion (b) follows from (a) and from the fact that $\Omega_{U/U'}$ is a $\mathcal{O}_U$-module which is locally in P free of finite type (one take the open basis consisting of open subsets of P having coordinates above P').

Now we treat the assertion (c). Thanks to 17.2.2.2, it is sufficient to check the last statement. Suppose we have a finite presentation of the form: $(\mathcal{D}_U^{(m)})^r \stackrel{\phi}{\to} (\mathcal{D}_U^{(m)})^s \stackrel{\epsilon}{\to} \mathcal{M} \to 0$. It results from (a) and 17.2.2.10.(b) that the functor j_* is exact on the category of $\mathcal{D}_U^{(m)}$-modules globally of finite presentation. Hence, by applying the functor j_* we get the exact sequence: $j_*(\mathcal{D}_U^{(m)})^r \stackrel{j_*\phi}{\to} j_*(\mathcal{D}_U^{(m)})^s \stackrel{j_*\epsilon}{\to} j_*\mathcal{N} \to 0$. □

17.2.3 Ind-coherent $\mathcal{D}_U^{(m)}$-modules

Suppose U is affine.

Definition 17.2.3.1. Let M be a $\Gamma(U, \mathcal{O}_U)$-module (resp. $\Gamma(U, \mathcal{D}_U^{(m)})$-module). The $\mathcal{O}_U$-module (resp. $\mathcal{D}_U^{(m)}$-module) *associated to* M is $\widetilde{M} := \mathcal{O}_U \otimes_{\Gamma(U,\mathcal{O}_U)} M$ (resp. $\mathcal{D}_U^{(m)} \otimes_{\Gamma(U,\mathcal{D}_U^{(m)})} M$). Such a module is said to be ind-*coherent*.

If $u: M' \to M$ is a homomorphism of $\Gamma(U, \mathcal{O}_U)$-modules (resp. $\Gamma(U, \mathcal{D}_U^{(m)})$-modules), $\widetilde{u}: \widetilde{M'} \to \widetilde{M}$ denotes the canonically induced homomorphism. Since $\Gamma(U, \mathcal{O}_U) \to \mathcal{O}_U$ and $\Gamma(U, \mathcal{D}_U^{(m)}) \to \mathcal{D}_U^{(m)}$ are flat, then both functors $\sim: M \mapsto \widetilde{M}$ are exact.

Remark 17.2.3.2. (a) The sheaf $\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)}$ is not ind-coherent as $\mathcal{D}_U^{(m)}$-module.

(b) A left $\mathcal{D}_U^{(m)}$-module is ind-coherent as $\mathcal{D}_U^{(m)}$-module if and only if it is ind-coherent as $\mathcal{O}_U$-module. Indeed, as $\mathcal{D}_U^{(m)}$ is a filtered inductive limit of coherent $\mathcal{O}_U$-modules and since the functors $\mathcal{O}_U \otimes_{\Gamma(U,\mathcal{O}_U)} -$ and $\Gamma(U, -)$ commute with filtered inductive limit, then we benefit from the canonical morphism $\mathcal{O}_U \otimes_{\Gamma(U,\mathcal{O}_U)} \Gamma(U, \mathcal{D}_U^{(m)}) \to \mathcal{D}_U^{(m)}$ is an isomorphism. This remark can be interpreted via the isomorphism 17.5.1.7.1.

Proposition 17.2.3.3. We have the following properties.

(a) Let $u: M \to N$ be a morphism of $\Gamma(U, \mathcal{D}_U^{(m)})$-modules. The $\mathcal{D}_U^{(m)}$-modules associated to $\operatorname{Ker} u$, $\operatorname{Im} u$, $\operatorname{Coker} u$, are respectively $\operatorname{Ker} \tilde{u}$, $\operatorname{Im} \tilde{u}$, $\operatorname{Coker} \tilde{u}$.

(b) If M is an inductive limit (resp. direct sum) of a family of $\Gamma(U, \mathcal{D}_U^{(m)})$-modules (M_λ), $\widetilde{M}$ is the inductive limit (resp. direct sum) of the family $(\widetilde{M_\lambda})$, up to canonical isomorphism.

(c) If M and N are two $\Gamma(U, \mathcal{D}_U^{(m)})$-modules, then the $\mathcal{D}_U^{(m)}$-modules associated to $M \otimes_{\Gamma(U,\mathcal{O}_U)} N$ and $\mathrm{Hom}_{\Gamma(U,\mathcal{O}_U)}(M, N)$ are respectively $\widetilde{M} \otimes_{\mathcal{O}_U} \widetilde{N}$ and $\mathcal{H}om_{\mathcal{O}_U}(\widetilde{M}, \widetilde{N})$.

We have identical results when $\mathcal{O}_U$ replaces $\mathcal{D}_U^{(m)}$.

Proof. Since the functor $\sim\colon M \mapsto \widetilde{M}$ is exact, then we prove (a) similarly to 17.2.2.10.(b). Assertion (b) results from the fact that inductive limit commutes with tensor product while (c) is easy and is left as an exercise. □

Remark 17.2.3.4. We have the following properties.

(a) A $\Gamma(U, \mathcal{D}_U^{(m)})$-module is the filtered inductive limit of its sub-$\Gamma(U, \mathcal{D}_U^{(m)})$-modules of finite type. Via 17.2.3.3.(b), this yields that an ind-coherent $\mathcal{D}_U^{(m)}$-module is the filtered inductive limit of its sub-$\mathcal{D}_U^{(m)}$-modules globally of finite presentation. Hence, the category of ind-coherent $\mathcal{D}_U^{(m)}$-modules is the smallest category stable by inductive limit and containing the $\mathcal{D}_U^{(m)}$-modules which are globally of finite presentation, which justifies the name.

(b) Let M be a $\Gamma(U, \mathcal{D}_U^{(m)})$-module and U' a principal open subset of U. As tensor product and $\Gamma(U', -)$ commute with filtered inductive limite, it follows from remark (i) and theorem of type A (17.2.2.10) applied to U and U' that $\Gamma(U', \widetilde{M}) \xrightarrow{\sim} \Gamma(U', \mathcal{D}_U^{(m)}) \otimes_{\Gamma(U,\mathcal{D}_U^{(m)})} M$.

Theorem 17.2.3.5 (Theorem B)**.** *For all* ind*-coherent* $\mathcal{D}_U^{(m)}$*-module* $\mathcal{M}$*, for all integer* $i \neq 0$, $H^i(U, \mathcal{M}) = 0$. *In particular, the functor* $\Gamma(U, -)$ *is exact on the category of* ind*-coherent* $\mathcal{D}_U^{(m)}$*-modules.*

Proof. This follows from 17.2.2.9, 17.2.3.4(a) and the fact that for all integer $i \neq 0$, the functor $H^i(U, \mathcal{M})$ commutes with filtered inductive. □

Proposition 17.2.3.6. We have the following properties:

(a) The functors $M \mapsto \widetilde{M}$ and $\mathcal{M} \mapsto \Gamma(U, \mathcal{M})$ induces exact quasi-inverse equivalences between the category of $\Gamma(U, \mathcal{D}_U^{(m)})$-modules and that of ind-coherent $\mathcal{D}_U^{(m)}$-modules.

(b) The category of ind-coherent $\mathcal{D}_U^{(m)}$-modules is a full abelian subcategory of that of $\mathcal{D}_U^{(m)}$-modules.

We get identical results when $\mathcal{O}_U$ replaces $\mathcal{D}_U^{(m)}$.

Proof. a) Let M be a $\Gamma(U, \mathcal{D}_U^{(m)})$-module. According to the remark above 17.2.3.4.(b), $\Gamma(U, \widetilde{M}) \xrightarrow{\sim} M$. Conversely, for any $\mathcal{D}_U^{(m)}$-module $\mathcal{M}$ isomorphic to $\widetilde{M}$, $\Gamma(U, \mathcal{M})^{\sim} \xrightarrow{\sim} \Gamma(U, \widetilde{M})^{\sim} \xrightarrow{\sim} \widetilde{M} \xrightarrow{\sim} \mathcal{M}$. The exactness of $\mathcal{M} \mapsto \Gamma(U, \mathcal{M})$ follows from 17.2.3.5, while the other one is already known.

b) The statement (b) is a consequence of the part (a) and of 17.2.3.3.(a). □

Proposition 17.2.3.7. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of $\mathcal{D}_U^{(m)}$-modules. If two of these modules are ind-coherent, so is the third.

Proof. The case where $\mathcal{E}$ is ind-coherent is deduced from 17.2.3.3.(a). Suppose now $\mathcal{E}'$ and $\mathcal{E}''$ are of the form $\widetilde{E}'$ and $\widetilde{E}''$. Via the theorem B for the ind-coherent $\mathcal{D}_U^{(m)}$-modules, we get that the sequence $0 \to E' \to \Gamma(U, \mathcal{E}) \to E'' \to 0$ is exact. By applying the exact functor $\sim$, using the five lemma we conclude that the canonical homomorphism $\Gamma(U, \mathcal{E})^{\sim} \to \mathcal{E}$ is an isomorphism. □

17.2.4 Extraordinary pullbacks and pushforwards of finite order

Let $g\colon U' \to U$ and $g'\colon U'' \to U'$ be two morphisms of smooth $\mathfrak{S}$-weak formal schemes.

Notation 17.2.4.1. Similarly to 5.1.1.2, we introduce the following transfert sheaves:

(a) The sheaf $g^*\mathcal{D}_U^{(m)}$ is equipped with a canonical structure of $(\mathcal{D}_{U'}^{(m)}, g^{-1}\mathcal{D}_U^{(m)})$-bimodule, which we denote $\mathcal{D}_{U'\to U}^{(m)}$.

(b) We write $\mathcal{D}_{U\leftarrow U'}^{(m)} := \omega_{U'} \otimes_{\mathcal{O}_{U'}} g_{\mathrm{l}}^*(\mathcal{D}_U^{(m)} \otimes_{\mathcal{O}_U} \omega_U^{-1})$, the index l indicating that we choose the left structure of left $\mathcal{D}_U^{(m)}$-module.

17.2.4.2. Similarly to 5.1.1.4, we get the notion of extraordinary pullbacks as follows:

(a) The extraordinary inverse image by g of a complex $\mathcal{E} \in D^-({}^{\mathrm{l}}\mathcal{D}_U^{(m)})$ is defined by putting

$$g^{(m)!}(\mathcal{E}) := \mathcal{D}_{U'\to U}^{(m)} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E}[d_{U'/U}], \tag{17.2.4.2.1}$$

where $d_{U'/U}$ is the relative dimension of U_0'/U_0.

(b) The extraordinary inverse image by g of a complex $\mathcal{M} \in D^-({}^{\mathrm{r}}\mathcal{D}_U^{(m)})$ is defined by putting

$$g^{(m)!}(\mathcal{M}) := g^{-1}\mathcal{M} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_U^{(m)}} \mathcal{D}_{U\leftarrow U'}^{(m)}[d_{U'/U}],$$

(c) Similarly to 5.1.1.5.1, we have for any $\mathcal{M} \in D^-({}^{\mathrm{r}}\mathcal{D}_U^{(m)})$ the isomorphism

$$g^{(m)!}(\mathcal{M}) \otimes_{\mathcal{O}_{U'}} \omega_{U'}^{-1} \xrightarrow{\sim} g^{(m)!}(\mathcal{M} \otimes_{\mathcal{O}_U} \omega_U^{-1}).$$

(d) If there is no confusion on the level, we write $g^!$ instead. We have an isomorphism $g'^! \circ g^! \xrightarrow{\sim} (g \circ g')^!$.

17.2.4.3. Let $\mathcal{E} \in D^-({}^{\mathrm{l}}\mathcal{D}_{U,\mathbb{Q}})$. For any integer $m \leq m'$, since the canonical morphisms $\mathcal{D}_{U,\mathbb{Q}}^{(m)} \to \mathcal{D}_{U,\mathbb{Q}}^{(m')} \to \mathcal{D}_{U,\mathbb{Q}}$ are isomorphisms, then $g^{(m)!}(\mathcal{E}) \to g^{(m')!}(\mathcal{E}) \to g^!(\mathcal{E})$ are isomorphisms, where

$$g^!(\mathcal{E}) := \mathcal{D}_{U'\to U,\mathbb{Q}} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_{U,\mathbb{Q}}} g^{-1}\mathcal{E}[d_{U'/U}] \tag{17.2.4.3.1}$$

and $\mathcal{D}_{U'\to U}$ is the $(\mathcal{D}_{U'}, g^{-1}\mathcal{D}_U)$-bimodule $g^*\mathcal{D}_U$. We have similar result for $\mathcal{M} \in D^-({}^{\mathrm{r}}\mathcal{D}_{U,\mathbb{Q}})$.

Definition 17.2.4.4. A (left or right) $\mathcal{D}_U^{(m)}$-module $\mathcal{E}$ is said to be locally ind-coherent if there exists a open basis $\{U_i\}_{i\in I}$ of U by affine $\mathfrak{S}$-weak formal schemes such that $\mathcal{E}|U_i$ is ind-coherent (see definition 17.2.3.1). Remark that a locally ind-coherent $\mathcal{D}_U^{(m)}$-module is locally a quotient of a free $\mathcal{D}_U^{(m)}$-module. This is therefore a notion in the case of $\mathfrak{S}$-weak formal scheme which is similar to that of quasi-coherence in the case of schemes (e.g. see 17.2.4.6) but beware this is a priori different from the notion of quasi-coherence defined later (see 17.3.1.3).

For any $* \in \{-,+,\mathrm{b}\}$, we denote by $D^*_{\mathrm{lic}}({}^{\mathrm{r}}\mathcal{D}_U^{(m)})$ the strictly full subcategory of $D^*_{\mathrm{lic}}({}^{\mathrm{r}}\mathcal{D}_U^{(m)})$ consisting of objects $\mathcal{E}$ such that for any $j \in \mathbb{N}$ the $\mathcal{D}_U^{(m)}$-module $H^j(\mathcal{E})$ is locally ind-coherent.

Example 17.2.4.5. The sheaf $\mathcal{D}_{U'\to U}^{(m)}$ is a locally ind-coherent left $\mathcal{D}_{U'}^{(m)}$-module. The sheaf $\mathcal{D}_{U\leftarrow U'}^{(m)}$ is a locally ind-coherent right $\mathcal{D}_{U'}^{(m)}$-module. We check that the functors $\omega_U \otimes_{\mathcal{O}_U} -$ and $- \otimes_{\mathcal{O}_U} \omega_U^{-1}$ induce quasi-inverse equivalences between the category of left locally ind-coherent left $\mathcal{D}_U^{(m)}$-modules and that of right locally ind-coherent left $\mathcal{D}_U^{(m)}$-modules.

Proposition 17.2.4.6. *Let $\mathcal{F} \in D({}^{\mathrm{r}}\mathcal{D}_U^{(m)})$ and $\mathcal{G} \in D({}^{\mathrm{l}}g^{-1}\mathcal{D}_U^{(m)})$.*

(i) We have the canonical morphism in $D(\mathbb{Z}_U)$:

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{D}_U^{(m)}} \mathbb{R}g_*(\mathcal{G}) \to \mathbb{R}g_*\left(g^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_U^{(m)}} \mathcal{G}\right). \tag{17.2.4.6.1}$$

Let $\mathcal{D}$ be a sheaf of rings such that $(\mathcal{D}, \mathcal{D}_U^{(m)})$ is right solvable and let $\mathcal{F} \in D_{\mathrm{r-sol}}(\mathcal{D}, \mathcal{D}_U^{(m)})$ (see definition and notation 4.6.3.2). Then the morphism 17.2.4.6.1 can also be viewed as a morphism of $D(\mathcal{D})$.

(ii) Suppose f is quasi-compact and quasi-separated. Suppose one of the following conditions:

(a) either $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{lic}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_U)$, and $\mathcal{G} \in D({}^{\mathrm{l}}g^{-1}\mathcal{D}^{(m)}_U)$,

(b) or $\mathcal{F} \in D^{-}_{\mathrm{lic}}({}^{\mathrm{r}}\mathcal{D}^{(m)}_U)$, and $\mathcal{G} \in D^{-}({}^{\mathrm{l}}g^{-1}\mathcal{D}^{(m)}_U)$.

Then the morphism 17.2.4.6.1 is an isomorphism.

Proof. With the remark of 17.2.4.4, we can copy the proof of 5.1.2.5. □

Corollary 17.2.4.7. *Let $*, ** \in \{\mathrm{l}, \mathrm{r}\}$ such that both are not equal to* r. *Suppose f is quasi-compact and quasi-separated. Suppose moreover one of the following conditions:*

(a) either $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{lic}}({}^{*}\mathcal{D}^{(m)}_U)$, and $\mathcal{G} \in D({}^{**}g^{-1}\mathcal{D}^{(m)}_U)$,

(b) or $\mathcal{F} \in D^{-}_{\mathrm{lic}}({}^{*}\mathcal{D}^{(m)}_U)$, and $\mathcal{G} \in D^{-}({}^{**}g^{-1}\mathcal{D}^{(m)}_U)$.

Then we have the following isomorphism of $D^{-}({}^{**}\mathcal{D}^{(m)}_U)$ (see 5.1.2.7 for the right term):

$$\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{R}g_*(\mathcal{G}) \xrightarrow{\sim} \mathbb{R}g_*\left(g^{-1}\mathcal{F} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} \mathcal{G}\right). \tag{17.2.4.7.1}$$

Proof. Similarly to 5.1.2.8, this is a consequence of 17.2.4.6. □

17.2.4.8. Similarly to 5.1.3.1, we get the notion of pushforward as follows:

(a) As $g_0 \colon U'_0 \to U_0$ is a quasi-separated and quasi-compact morphism between noetherian schemes of finite Krull dimension, the functor $g_{0*} = g_*$ is of finite cohomological dimension. The direct image by g is the functor $g^{(m)}_+ \colon D^{-}({}^{\mathrm{l}}\mathcal{D}^{(m)}_{U'}) \to D^{-}({}^{\mathrm{l}}\mathcal{D}^{(m)}_U)$ which is defined by setting for any $\mathcal{E}' \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(m)}_{U'})$:

$$g^{(m)}_+(\mathcal{E}') := \mathbb{R}g_*(\mathcal{D}^{(m)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}').$$

(b) The direct image by g is the functor $g^{(m)}_+ \colon D^{-}({}^{\mathrm{r}}\mathcal{D}^{(m)}_{U'}) \to D^{-}({}^{\mathrm{r}}\mathcal{D}^{(m)}_U)$ which is defined by setting for any $\mathcal{M}' \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(m)}_{U'})$:

$$g^{(m)}_+(\mathcal{M}') := \mathbb{R}g_*(\mathcal{M}' \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{D}^{(m)}_{U'\to U}).$$

(c) Similarly to 5.1.3.2.1 (indeed, the isomorphism of the form 17.2.4.7.1 holds because it is used in the case where $\mathcal{F} = \omega_U$ is locally free of rank one), we have for any $\mathcal{E}' \in D^{-}({}^{\mathrm{l}}\mathcal{D}^{(m)}_{U'})$ the isomorphism

$$\omega_U \otimes_{\mathcal{O}_U} g^{(m)}_+(\mathcal{E}') \xrightarrow{\sim} g^{(m)}_+(\omega_{U'} \otimes_{\mathcal{O}_{U'}} \mathcal{E}), \tag{17.2.4.8.1}$$

(d) This is also written as g_+. Thanks to 17.2.4.7.1, we can copy the proof in the context of schemes (see 5.1.3.8.1) to construct the canonical isomorphism $g_+ \circ g'_+(\mathcal{E}'') \xrightarrow{\sim} (g \circ g')_+(\mathcal{E}'')$ which is functorial in $\mathcal{E}'' \in D^{-}(\mathcal{D}^{(m)}_{U''})$.

17.2.4.9. For any integer $m \leq m'$, since the canonical morphisms $\mathcal{D}^{(m)}_{U,\mathbb{Q}} \to \mathcal{D}^{(m')}_{U,\mathbb{Q}} \to \mathcal{D}_{U,\mathbb{Q}}$ are isomorphisms, then for any $\mathcal{E}' \in D^{-}({}^{\mathrm{l}}\mathcal{D}_{U',\mathbb{Q}})$, the morphisms $g^{(m)}_+(\mathcal{E}') \to g^{(m')}_+(\mathcal{E}') \to g_+(\mathcal{E}')$ are isomorphisms, where

$$g_+(\mathcal{E}') := \mathbb{R}g_*(\mathcal{D}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}_{U'}} \mathcal{E}')$$

and $\mathcal{D}^{(m)}_{U\leftarrow U'} := \omega_{U'} \otimes_{\mathcal{O}_{U'}} g^*_{\mathrm{l}}(\mathcal{D}_U \otimes_{\mathcal{O}_U} \omega^{-1}_U)$. We have similar result for $\mathcal{M}' \in D^{-}({}^{\mathrm{r}}\mathcal{D}_{U',\mathbb{Q}})$.

17.2.5 Closed immersion: adjunction, preservation of the coherence by push-forward

Let $m \in \mathbb{N}$. Let $v\colon Y \hookrightarrow U$ be a closed immersion of smooth $\mathfrak{S}$-weak formal schemes. Let $\mathcal{I}$ be the ideal of $\mathcal{O}_U$ given by v.

17.2.5.1 (Local description)**.** Suppose $U/\mathfrak{S}$ has coordinates $t_1, \ldots, t_d$ such that $t_{r+1}, \ldots, t_d$ generated $\mathcal{I}$ and the image of $t_1, \ldots, t_r$ in $\Gamma(Y, \mathcal{O}_Y)$ are coordinates of $Y/\mathfrak{S}$. Let $\mathcal{F}^{(m)} \in D(\mathcal{D}^{(m)}_Y)$. Using similar to 5.2.2.4.1 isomorphism, we compute $u^{(m)}_+(\mathcal{F}^{(m)}) \xrightarrow{\sim} \mathcal{V}\{\partial_{r+1}, \ldots, \partial_d\}^{(m)} \otimes_{\mathcal{V}} \mathcal{F}^{(m)}$ (see notation 5.2.2.2). Let $\mathcal{F} \in D(\mathcal{D}_{Y,\mathbb{Q}})$. Using similar to 5.2.2.4.1 isomorphism (of level ∞), we compute $u_+(\mathcal{F}) \xrightarrow{\sim} \mathcal{V}[\partial_{r+1}, \ldots, \partial_d] \otimes_{\mathcal{V}} \mathcal{F}^{(m)}$.

It follows from these local description that the functors u_+ and $u^{(m)}_+$ are exact.

Proposition 17.2.5.2. *Let $* \in \{\mathrm{r}, \mathrm{l}\}$, $\mathcal{M} \in D({}^*\mathcal{D}^{(m)}_U)$, $\mathcal{N} \in D({}^*\mathcal{D}^{(m)}_Y)$. We have the isomorphisms*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_Y}(v^{(m)}_+(\mathcal{N}), \mathcal{M}) \xrightarrow{\sim} v_* \mathbb{R}\mathcal{H}om_{\mathcal{D}^{(m)}_Y}(\mathcal{N}, v^!(\mathcal{M})).$$

Proof. This is checked similarly to 5.2.6.3. □

17.2.5.3. Let $\mathcal{F}^{(m)} \in D(\mathcal{D}^{(m)}_Y)$. It follows from 17.2.5.2 that we have the adjunction morphism

$$\mathcal{F}^{(m)} \to v^{!^{(m)}} \circ v^{(m)}_+(\mathcal{F}^{(m)}). \tag{17.2.5.3.1}$$

Suppose is p-torsion free. Then similarly to 9.3.3.13 we compute that the adjunction morphism 17.2.5.3.1 is an isomorphism. When $\mathcal{F}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_Y)$, the adjunction map 17.2.5.3.1 is compatible with the standard one available on formal schemes of 9.3.2.5, i.e. we get the commutative diagram:

$$\begin{array}{ccc}
\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_Y} \mathcal{F}^{(m)} & \xrightarrow{17.2.5.3.1} & \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_Y} \left(v^{!^{(m)}} \circ v^{(m)}_+(\mathcal{F}^{(m)}))\right) \\
\downarrow{\scriptstyle 9.3.2.5} & & \sim\downarrow{\scriptstyle 17.3.2.6.1} \\
\widehat{v}^{!^{(m)}} \circ \widehat{v}^{(m)}_+ \left(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_Y} \mathcal{F}^{(m)}\right) & \underset{17.3.3.1.1}{\xleftarrow{\sim}} & \widehat{v}^{!^{(m)}} \left(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_U} v^{(m)}_+(\mathcal{F}^{(m)}))\right)
\end{array} \tag{17.2.5.3.2}$$

Let $\mathcal{E}^{(m)} \in D(\mathcal{D}^{(m)}_U)$. It follows from 17.2.5.2 that we have the adjunction morphism

$$v^{(m)}_+ \circ v^{!^{(m)}}(\mathcal{E}^{(m)}) \to \mathcal{E}^{(m)}. \tag{17.2.5.3.3}$$

For any $\mathcal{E}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_U)$ such that $v^{!^{(m)}}(\mathcal{E}^{(m)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_Y)$, we get the commutative diagram:

$$\begin{array}{ccc}
\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_U} v^{(m)}_+ \circ v^{!^{(m)}}(\mathcal{E}^{(m)}) & \xrightarrow{17.2.5.3.3} & \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_U} \mathcal{E}^{(m)} \\
\sim\downarrow{\scriptstyle 17.3.3.1.1} & & \uparrow{\scriptstyle 9.3.2.5} \\
\widehat{v}^{(m)}_+ \left(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_Y} v^{!^{(m)}}(\mathcal{E}^{(m)})\right) & \underset{\sim}{\xrightarrow{17.3.2.6.1}} & \widehat{v}^{(m)}_+ \circ \widehat{v}^{!^{(m)}} \left(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_U} \mathcal{F}^{(m)})\right)
\end{array} \tag{17.2.5.3.4}$$

Proposition 17.2.5.4. Suppose Y and U are affine. For all right globally of finite presentation $\mathcal{D}^{(m)}_Y$-module $\mathcal{M}$, $v_+(\mathcal{M})$ is globally of finite presentation and we have a canonical isomorphism:

$$\Gamma(U, v_+(\mathcal{M})) \xrightarrow{\sim} \Gamma(Y, \mathcal{M}) \otimes_{\Gamma(Y, \mathcal{D}^{(m)}_Y)} \Gamma(Y, \mathcal{D}^{(m)}_{Y \hookrightarrow U}).$$

The same holds when “right module ” is replaced by “ left module”.

Proof. As $\mathcal{M}$ is a right $\mathcal{D}^{(m)}_Y$-module globally of finite presentation, then we get from Theorem of type A of 17.2.2.10 the canonical isomorphism:

$$\mathcal{M} \otimes_{\mathcal{D}^{(m)}_Y} \mathcal{D}^{(m)}_{Y \hookrightarrow U} \xrightarrow{\sim} \Gamma(Y, \mathcal{M}) \otimes_{\Gamma(Y, \mathcal{D}^{(m)}_Y)} \mathcal{D}^{(m)}_{Y \hookrightarrow U}.$$

For any integer n, it follows from the $\mathcal{O}_U$-coherence of $\mathcal{D}^{(m)}_{U,n}$ and theorem of type A for coherent $\mathcal{O}_U$-modules that the canonical morphism $\Gamma(Y, \mathcal{O}_Y) \otimes_{\Gamma(U,\mathcal{O}_U)} \Gamma(U, \mathcal{D}^{(m)}_{U,n}) \to \Gamma(Y, v^*\mathcal{D}^{(m)}_{U,n})$ is an isomorphism. Since tensor products and the functor $\Gamma(U, -)$ commute with filtered inductive limits (see [SGA4.2, VI.5.3]), then the morphism $\Gamma(Y, \mathcal{O}_Y) \otimes_{\Gamma(U,\mathcal{O}_U)} \Gamma(U, \mathcal{D}^{(m)}_U) \to \Gamma(Y, \mathcal{D}^{(m)}_{Y\hookrightarrow U})$ is an isomorphism. Let U' be a principal open subset of U and $Y' := Y \cap U'$. We get $\Gamma(Y', \mathcal{O}_Y) \otimes_{\Gamma(U',\mathcal{O}_U)} \Gamma(U', \mathcal{D}^{(m)}_U) \xrightarrow{\sim} \Gamma(Y', \mathcal{D}^{(m)}_{Y\hookrightarrow U})$. Moreover, $\Gamma(Y', \mathcal{O}_Y) \xrightarrow{\sim} \Gamma(Y, \mathcal{O}_Y) \otimes_{\Gamma(U,\mathcal{O}_U)} \Gamma(U', \mathcal{O}_U)$ (use the part 1) of the proof of 17.1.4.4). Finally, we get:

$$\Gamma(Y, \mathcal{D}^{(m)}_{Y\hookrightarrow U}) \otimes_{\Gamma(U,\mathcal{D}^{(m)}_U)} \Gamma(U', \mathcal{D}^{(m)}_U) \xrightarrow{\sim} \Gamma(Y', \mathcal{D}^{(m)}_{Y\hookrightarrow U}).$$

As the functor "sheafification" commute with v_* (indeed, v is a closed immersion), $v_*(\mathcal{M} \otimes_{\mathcal{D}^{(m)}_Y} \mathcal{D}^{(m)}_{Y\hookrightarrow U})$ is the sheaf associated with the presheaf defined on the principal open subsets U' of U by: $U' \mapsto \Gamma(Y, \mathcal{M}) \otimes_{\Gamma(Y,\mathcal{D}^{(m)}_Y)} \Gamma(Y, \mathcal{D}^{(m)}_{Y\hookrightarrow U}) \otimes_{\Gamma(U,\mathcal{D}^{(m)}_U)} \Gamma(U', \mathcal{D}^{(m)}_U)$. Hence, following Theorem of type A of 17.2.2.10, it remains thus to check that $\Gamma(Y, \mathcal{M}) \otimes_{\Gamma(Y,\mathcal{D}^{(m)}_Y)} \Gamma(Y, \mathcal{D}^{(m)}_{Y\hookrightarrow U})$ is a right $\Gamma(U, \mathcal{D}^{(m)}_U)$-module of finite type. Since tensor product is right exact and $\Gamma(Y, \mathcal{M})$ is a right $\Gamma(Y, \mathcal{D}^{(m)}_Y)$-module of finite type, then it suffices to prove that $\Gamma(Y, \mathcal{D}^{(m)}_{Y\hookrightarrow U})$ is a right $\Gamma(U, \mathcal{D}^{(m)}_U)$-module of finite type. We conclude via $\Gamma(U, \mathcal{D}^{(m)}_U) \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y) \otimes_{\Gamma(U,\mathcal{O}_U)} \Gamma(U, \mathcal{D}^{(m)}_U) \xrightarrow{\sim} \Gamma(Y, \mathcal{D}^{(m)}_{Y\hookrightarrow U})$.

□

Corollaire 17.2.5.5. For all coherent $\mathcal{D}^{(m)}_Y$-module $\mathcal{M}$, $v_+(\mathcal{M})$ is $\mathcal{D}^{(m)}_U$-coherent.

17.3 Comparaison between weak formal and formal cohomological operations without overconvergent singularities

17.3.1 Quasi-coherent complexes over smooth $\mathfrak{S}$-weak formal schemes

Let U be a smooth $\mathfrak{S}$-weak formal scheme and $m \in \mathbb{N}$. We keep notation 17.1.3.1, e.g. $U_i := U \times_{\mathfrak{S}} S_i$ for all $i \in \mathbb{N}$. Let $* \in \{\mathrm{l}, \mathrm{r}\}$.

17.3.1.1. We denote by $U_\bullet$ the topos $\mathrm{Top}(U)_{\mathbb{N}}$ given by the canonical maps $U_{i+1} \to U_i$. The family of maps $\mathcal{D}^{(m)}_{U_{i+1}} \to \mathcal{D}^{(m)}_{U_i}$ induces a sheaf of rings $\mathcal{D}^{(m)}_{U_\bullet}$ on $U_\bullet$ We get the morphism of ringed topoi $\underleftarrow{l}_U \colon (U_\bullet, \mathcal{D}^{(m)}_{U_\bullet}) \to (|U|, \mathcal{D}^{(m)}_U)$. From [Sta22, 07A6] (or see 5.3.5.4), this yields the functors $\mathbb{R}\underleftarrow{l}_{U*} \colon D({}^*\mathcal{D}^{(m)}_{U_\bullet}) \to D({}^*\mathcal{D}^{(m)}_U)$ and $\mathbb{L}\underleftarrow{l}^*_U \colon D({}^*\mathcal{D}^{(m)}_U) \to D({}^*\mathcal{D}^{(m)}_{U_\bullet})$ which are adjoint:

$$\mathrm{Hom}_{D(\mathcal{D}^{(m)}_{U_\bullet})}(\mathbb{L}\underleftarrow{l}^*_U(\mathcal{E}), \mathcal{F}_\bullet) = \mathrm{Hom}_{D(\mathcal{D}^{(m)}_U)}(\mathcal{E}, \mathbb{R}\underleftarrow{l}_{U*}(\mathcal{F}_\bullet)) \tag{17.3.1.1.1}$$

for any $\mathcal{E} \in D({}^*\mathcal{D}^{(m)}_U)$ and any $\mathcal{F}_\bullet \in D({}^*\mathcal{D}^{(m)}_{U_\bullet})$.

We have also the morphism of ringed topoi $\underleftarrow{l}_U \colon (U_\bullet, \mathcal{O}_{U_\bullet}) \to (|U|, \mathcal{O}_U)$. The bijection 17.3.1.1.1 still holds. Both functors $\mathbb{L}\underleftarrow{l}^*_U$ (or $\mathbb{R}\underleftarrow{l}_{U*}$) are canonically isomorphic (modulo the forgetful functor $D({}^*\mathcal{D}^{(m)}_{U_\bullet}) \to D({}^*\mathcal{O}_{U_\bullet})$ and $D({}^*\mathcal{D}^{(m)}_U) \to D({}^*\mathcal{O}_U)$, so this is harmless to write them the same.

Notation 17.3.1.2. Let $\mathcal{E}$ be an object of $D^-({}^{\mathrm{l}}\mathcal{D}^{(m)}_U)$ and $\mathcal{F} \in D^-({}^{\mathrm{r}}\mathcal{D}^{(m)}_U)$. Put $\mathcal{E}_\bullet := \mathbb{L}\underleftarrow{l}^*_U(\mathcal{E}) := \mathcal{D}^{(m)}_{U_\bullet} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} \mathcal{E}$, $\mathcal{F}_\bullet := \mathbb{L}\underleftarrow{l}^*_U(\mathcal{F}) := \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} \mathcal{D}^{(m)}_{U_\bullet}$ and

$$\mathcal{F} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} \mathcal{E} := \mathbb{R}\underleftarrow{l}_{U*}(\mathcal{F}_\bullet \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U_\bullet}} \mathcal{E}_\bullet) = \mathbb{R} \varprojlim_i \mathcal{F}_i \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U_i}} \mathcal{E}_i.$$

The same with $\mathcal{D}^{(m)}_U$ replaced by $\mathcal{O}_U$ or $\widehat{\mathcal{D}}^{(m)}_U$ or more generally a sheaf of rings on U. Choosing a flat resolution of $\mathcal{E}$ and $\mathcal{F}$, we construct a canonical morphism $\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} \mathcal{E} \to \mathcal{F} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} \mathcal{E}$ which is bifunctorial in $\mathcal{E}$ and $\mathcal{F}$ (i.e., a cube is commutative).

Definition 17.3.1.3. We define $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}^{(m)}_U)$, with $* = \mathrm{l}$ or $* = \mathrm{r}$, the full sub-category of $D^{\mathrm{b}}({}^*\mathcal{D}^{(m)}_U)$ consisting of complexes $\mathcal{E}$ such that, for all $i \in \mathbb{N}$, $\mathcal{E}_i \in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}^{(m)}_{U_i})$. Its objects will be called *quasi-coherent complexes.*

Example 17.3.1.4. When U is affine, a ind-coherent $\mathcal{D}_U^{(m)}$-module $\mathcal{E}$ (17.2.3.1) is a quasi-coherent complex. Indeed, this is a consequence of the equivalence of categories 17.2.3.6: taking a flat resolution $P^\bullet$ of $\Gamma(U,\mathcal{E})$ we get $\widetilde{P}^\bullet$ a flat resolution of $\mathcal{E}$ by ind-coherent $\mathcal{D}_U^{(m)}$-module and we are done.

More generally, $D^{\mathrm{b}}_{\mathrm{lic}}({}^*\mathcal{D}_U^{(m)})$ is a full subcategory of $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}_U^{(m)})$ (see notation 17.2.4.4).

17.3.1.5. Let $\mathcal{E}\in D^{\mathrm{b}}({}^*\mathcal{D}_U^{(m)})$. The following properties are equivalent:

(a) $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}_U^{(m)})$

(b) $\mathcal{E}_\bullet:=\mathbb{L}\underleftarrow{l}_U^*\mathcal{E}:=\mathcal{O}_{U_\bullet}\otimes^{\mathbb{L}}_{\mathcal{O}_U}\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_{U_\bullet})$ (see notation 7.3.1.10).

(c) $\mathcal{E}_\bullet:=\mathbb{L}\underleftarrow{l}_U^*\mathcal{E}:=\mathcal{D}_{U_\bullet}^{(m)}\otimes^{\mathbb{L}}_{\mathcal{D}_U^{(m)}}\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U_\bullet}^{(m)})$ (see notation 7.3.1.12).

Lemma 17.3.1.6. *The category $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_U^{(m)})$ is a triangulated subcategory of $D^{\mathrm{b}}(\mathcal{D}_U^{(m)})$. The category $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$ is a triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_U^{(m)})$.*

Proof. This follows from the fact that $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U_i}^{(m)})$ is a triangulated subcategory of $D^{\mathrm{b}}(\mathcal{D}_U^{(m)})$ (resp. $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{U_i}^{(m)})$ is a triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U_i}^{(m)})$ and that $\mathbb{L}\underleftarrow{l}_U^*$ preserves the coherence). □

17.3.1.7. By adjunction (see 17.3.1.1), for any $\mathcal{E}\in D^{\mathrm{b}}({}^*\mathcal{D}_U^{(m)})$, we have the canonical morphism $c_{\mathcal{E}}\colon\mathcal{E}\to\widehat{\mathcal{E}}$ where

$$\widehat{\mathcal{E}}:=\mathbb{R}\underleftarrow{l}_{U*}\mathbb{L}\underleftarrow{l}_U^*(\mathcal{E}).$$

When $*=\mathrm{l}$, we remark that the canonical morphism $\mathcal{O}_U\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_U}\mathcal{E}\to\mathcal{D}_U^{(m)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}_U^{(m)}}\mathcal{E}$ is an isomorphism and they are isomorphic to $\widehat{\mathcal{E}}$; similarly when $*=\mathrm{r}$. We get the functor $D^{\mathrm{b}}({}^*\mathcal{D}_U^{(m)})\to D^{\mathrm{b}}({}^*\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$, given by $\mathcal{E}\mapsto\widehat{\mathcal{E}}$.

Following 7.3.2.10, the functor $\mathbb{R}\underleftarrow{l}_{U*}\colon D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U_\bullet}^{(m)})\to D^{\mathrm{b}}(\mathcal{D}_U^{(m)})$ factorises through the equivalence of categories $\mathbb{R}\underleftarrow{l}_{U*}\colon D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U_\bullet}^{(m)})\cong D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$. Hence, we get the functor $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}_U^{(m)})\to D^{\mathrm{b}}_{\mathrm{qc}}({}^*\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$, given by $\mathcal{E}\mapsto\widehat{\mathcal{E}}$.

Lemma 17.3.1.8. *Let $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}_U^{(m)})$, $\mathcal{F}\in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$ any morphism $f\colon\mathcal{E}\to\mathcal{F}$ a morphism of $D^{\mathrm{b}}({}^*\mathcal{D}_U^{(m)})$. With notation 17.3.1.7, there exists a unique morphism $g\colon\widehat{\mathcal{E}}\to\mathcal{F}$ such that $g\circ c_{\mathcal{E}}=f$.*

Proof. Remark first that for any $\mathcal{G}\in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{D}_U^{(m)})$, we have $\mathcal{G}\in D^{\mathrm{b}}_{\mathrm{qc}}({}^*\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$ if and only if $c_{\mathcal{G}}$ is an isomorphism. Hence, by functoriality of $\mathcal{E}\mapsto c_{\mathcal{E}}$ we can check that $g=\mathbb{R}\underleftarrow{l}_{U*}\mathbb{L}\underleftarrow{l}_U^*(f)$ is the unique factorisation. □

17.3.2 Extraordinary pullbacks

Let $m\in\mathbb{N}$.

Lemma 17.3.2.1. *Let U be a smooth $\mathfrak{S}$-weak formal scheme and $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$. The canonical morphism: $\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)}\otimes_{\mathcal{D}_U^{(m)}}\mathcal{E}\to\widehat{\mathcal{E}}$ is an isomorphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$.*

Proof. With notation 17.3.1.7, since $\widehat{\mathcal{E}}\in D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$ then the morphism $c_{\mathcal{E}}\colon\mathcal{E}\to\widehat{\mathcal{E}}$ induces canonically via the flat extension $\mathcal{D}_U^{(m)}\to\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)}$ the morphism $\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)}\otimes_{\mathcal{D}_U^{(m)}}\mathcal{E}\to\widehat{\mathcal{E}}$. To check that this morphism is an isomorphism is local. Moreover, since the both side functors are wayout left, then we reduce to the case where $\mathcal{E}$ is a free $\mathcal{D}_U^{(m)}$-module of finite type, which is obvious. □

Lemma 17.3.2.2. *Let $A\to A'$ be a morphism of smooth w.c.f.g. $\mathcal{V}$-algebras. Let $g\colon U'\to U$ be the induced morphism of affine smooth $\mathfrak{S}$-weak formal schemes. For any* ind*-coherent left $\mathcal{D}_U^{(m)}$-module $\mathcal{E}$, the left $\mathcal{D}_{U'}^{(m)}$-module $\mathcal{D}_{U'\to U}^{(m)}\otimes_{g^{-1}\mathcal{D}_U^{(m)}}g^{-1}\mathcal{E}$ is* ind*-coherent. Moreover, we have the canonical morphisms:*

$$A'\otimes_A\Gamma(U,\mathcal{E})\to\Gamma(U',\mathcal{D}_{U'\to U}^{(m)})\otimes_{\Gamma(U,\mathcal{D}_U^{(m)})}\Gamma(U,\mathcal{E})\to\Gamma(U',\mathcal{D}_{U'\to U}^{(m)}\otimes_{g^{-1}\mathcal{D}_U^{(m)}}g^{-1}\mathcal{E})\qquad(17.3.2.2.1)$$

are isomorphisms.

Proof. Since $\mathcal{O}_{U'} \otimes_{g^{-1}\mathcal{O}_U} g^{-1}\mathcal{D}_U^{(m)} = \mathcal{D}_{U'\to U}^{(m)}$, then by associativity of the tensor product we get that the morphism $\mathcal{O}_{U'} \otimes_{g^{-1}\mathcal{O}_U} g^{-1}\mathcal{E} \to \mathcal{D}_{U'\to U}^{(m)} \otimes_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E}$ is an isomorphism. Hence, these are ind-coherent. Set $E := \Gamma(U, \mathcal{E})$. Since $\mathcal{E}$ is ind-coherent as $\mathcal{O}_U$-module (see 17.2.3.2.(b)), then the canonical morphism $\mathcal{O}_U \otimes_A E \to \mathcal{E}$ is an isomorphism. This yields, the canonical morphism $\mathcal{O}_{U'} \otimes_A E \to \mathcal{O}_{U'} \otimes_{g^{-1}\mathcal{O}_U} g^{-1}\mathcal{E}$ is an isomorphism. Since $\mathcal{O}_{U'} \otimes_A E \xrightarrow{\sim} \mathcal{O}_{U'} \otimes_{A'} (A' \otimes_A E)$, then it follows from 17.2.3.6 that $\mathcal{E}' := \mathcal{D}_{U'\to U}^{(m)} \otimes_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E}$ is ind-coherent as $\mathcal{O}_{U'}$-module (and therefore as $\mathcal{D}_{U'}^{(m)}$-module) and that the canonical morphism $A' \otimes_A \Gamma(U, \mathcal{E}) \to \Gamma(U', \mathcal{E}')$ is an isomorphism. Taking $\mathcal{E} = \mathcal{D}_U^{(m)}$, this implies that the canonical morphism $A' \otimes_A \Gamma(U, \mathcal{D}_U^{(m)}) \to \Gamma(U', \mathcal{D}_{U'\to U}^{(m)})$ is an isomorphism. Hence, the left morphism of 17.3.2.2.1 is an isomorphism and we are done. □

Lemma 17.3.2.3. *Let $g\colon U' \to U$ be a smooth morphism of affine smooth $\mathfrak{S}$-weak formal schemes. For all $\mathcal{D}_U^{(m)}$-module $\mathcal{E}$ globally of finite presentation, $\mathcal{D}_{U'\to U}^{(m)} \otimes_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E}$ is a $\mathcal{D}_{U'}^{(m)}$-module globally of finite presentation.*

Proof. Via 17.2.2.10, it follows from Lemma 17.3.2.2 that it suffices to prove that $\Gamma(U', \mathcal{D}_{U'\to U}^{(m)}) \otimes_{\Gamma(U, \mathcal{D}_U^{(m)})} \Gamma(U, \mathcal{E})$ is of finite type on $\Gamma(U', \mathcal{D}_{U'}^{(m)})$. Similarly to 5.3.2.1.1, by calculating in local coordinates, we can check that the morphism $\mathcal{D}_{U'}^{(m)} \to \mathcal{D}_{U'\to U}^{(m)}$ is surjective. By 17.2.3.3.(a) and 17.2.3.6, this yields the surjective map $\Gamma(U', \mathcal{D}_{U'}^{(m)}) \twoheadrightarrow \Gamma(U', \mathcal{D}_{U'\to U}^{(m)})$. □

Proposition 17.3.2.4. *Let $g\colon U' \to U$ be a morphism of smooth $\mathfrak{S}$-weak formal schemes and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_U^{(m)})$ (see notation 17.3.1.3).*

(a) Then $g^!(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U'}^{(m)})$ (see notation 17.2.4.2.1) and the canonical morphism

$$\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}_{U'}^{(m)}} g^!(\mathcal{E}) \to \hat{g}^!(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}_U^{(m)}} \mathcal{E}), \tag{17.3.2.4.1}$$

where $\hat{g}^!$ is the extraordinary pullback by $\hat{g}\colon \mathfrak{U}' \to \mathfrak{U}$ of level m (see definition 7.5.5.6.1), is an isomorphism of $D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m)})$. Moreover, these are transitive with respect to the composition of such morphism g.

(b) If $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$ and $g^!(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{U'}^{(m)})$, then the canonical morphism $\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m)} \otimes_{\mathcal{D}_{U'}^{(m)}} g^!(\mathcal{E}) \to \hat{g}^!(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \otimes_{\mathcal{D}_U^{(m)}} \mathcal{E})$ is an isomorphism.

Proof. 1) According to notation 7.5.5.4, we write $\mathcal{D}_{U'_\bullet\to U_\bullet}^{(m)}$ the $(\mathcal{D}_{U'_\bullet}^{(m)}, g^{-1}\mathcal{D}_{U_\bullet}^{(m)})$-bimodule given by the canonical maps $\mathcal{D}_{U'_{i+1}\to U_{i+1}}^{(m)} \to \mathcal{D}_{U'_i\to U_i}^{(m)}$. Since the canonical morphism $\mathcal{O}_{U'_\bullet} \otimes_{\mathcal{O}_{U'}} \mathcal{D}_{U'}^{(m)} \to \mathcal{D}_{U'_\bullet}^{(m)}$ is an isomorphism, then so is the canonical map $\mathcal{D}_{U'_\bullet}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{D}_{U'}^{(m)}} \mathcal{D}_{U'\to U}^{(m)} \to \mathcal{D}_{U'_\bullet\to U_\bullet}^{(m)}$. Hence,

$$\mathcal{D}_{U'_\bullet}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{D}_{U'}^{(m)}} (\mathcal{D}_{U'\to U}^{(m)} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E}) \xrightarrow{\sim} \mathcal{D}_{U'_\bullet\to U_\bullet}^{(m)} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E} \xrightarrow{\sim} \mathcal{D}_{U'_\bullet\to U_\bullet}^{(m)} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_{U_\bullet}^{(m)}} g^{-1}(\mathcal{D}_{U_\bullet}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{D}_U^{(m)}} \mathcal{E}). \tag{17.3.2.4.2}$$

Since $\mathcal{D}_{U_\bullet}^{(m)} \otimes^{\mathbb{L}}_{\mathcal{D}_U^{(m)}} \mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U_\bullet}^{(m)})$, then following 7.5.5.8, the left term of 17.3.2.4.2 belongs to $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U'_\bullet}^{(m)})$. Hence,we obtain from 17.3.2.4.2 that $\mathcal{D}_{U'\to U}^{(m)} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{D}_U^{(m)}} g^{-1}\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U'_\bullet}^{(m)})$ (see 17.3.1.5).

Since the canonical morphism $\widehat{\mathcal{E}} \to \widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}_U^{(m)}} \mathcal{E}$ is an isomorphism, then following 17.3.1.7, we have $\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}_U^{(m)}} \mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)})$. We get the isomorphism of 17.3.2.4.1 by applying $\mathbb{R}\underleftarrow{l}_{U*}$ to 17.3.2.4.2. □

Corollary 17.3.2.5. *Let $g\colon U' \to U$ be a smooth morphism of smooth $\mathfrak{S}$-weak formal schemes. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$.*

(a) $g^{(m)!}(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{U'}^{(m)})$.

(b) The canonical morphism $\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m)} \otimes_{\mathcal{D}_{U'}^{(m)}} g^{(m)!}(\mathcal{E}) \to \hat{g}^{(m)!}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \otimes_{\mathcal{D}_U^{(m)}} \mathcal{E})$ is an isomorphism.

Proof. The first part is a consequence of 17.3.2.3. Via 17.3.2.4, this yields the second assertion. □

Proposition 17.3.2.6. Let m_0 be an integer, $g\colon U' \to U$ a morphism of smooth $\mathfrak{S}$-weak formal schemes. Let $\mathcal{D}_U^{(m_0+\bullet)}$ be the inductive system of rings $\mathcal{D}_U^{(m} \to \mathcal{D}_U^{(m')}$ for any $m' \geq m \geq 0$. Let $\mathcal{E}^{(m_0+\bullet)} \in D^{\mathrm{b}}(\mathcal{D}_U^{(m_0+\bullet)})$. For all integer $m \geq m_0$, put $\widehat{\mathcal{E}}^{(m)} = \widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \otimes_{\mathcal{D}_U^{(m)}} \mathcal{E}^{(m)}$ and $\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m)} = \widehat{\mathcal{D}}_{\mathfrak{U},\mathbb{Q}}^{(m)} \otimes_{\mathcal{D}_U^{(m)}} \mathcal{E}^{(m)}$.

(a) Let $m \geq m_0$ such that $\mathcal{E}_{\mathbb{Q}}^{(m_0)} \to \mathcal{E}_{\mathbb{Q}}^{(m)}$ is an isomorphism. Suppose for $m' \in \{m_0, m\}$ we have $\mathcal{E}^{(m')} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m')})$ and $g^{(m')!}(\mathcal{E}^{(m')}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{U'}^{(m')})$. Then the canonical morphism

$$\widehat{\mathcal{D}}_{\mathfrak{U}',\mathbb{Q}}^{(m)} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{U}',\mathbb{Q}}^{(m_0)}} \hat{g}^{(m_0)!}(\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m_0)}) \to \hat{g}^{(m)!}(\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m)}), \qquad (17.3.2.6.1)$$

where $\hat{g}^{(m')!}$ is defined at 7.5.5.14.1, is an isomorphism.

(b) Suppose for any $m \geq m_0$, the morphisms $\mathcal{E}_{\mathbb{Q}}^{(m_0)} \to \mathcal{E}_{\mathbb{Q}}^{(m)}$ are isomorphisms, $\mathcal{E}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$ and $g^{(m)!}(\mathcal{E}^{(m)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{U'}^{(m)})$. Then the morphism

$$\mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{U}',\mathbb{Q}}^{(m_0)}} \hat{g}^{(m_0)!}(\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m_0)}) \to \hat{g}^{!\dagger}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{U},\mathbb{Q}}^{(m_0)}} \widehat{\mathcal{E}}_{\mathbb{Q}}^{(m_0)}) \qquad (17.3.2.6.2)$$

is an isomorphism. Moreover, these are transitive with respect to the composition of such a morphism g.

Proof. a) If m is an integer such that $\mathcal{E}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$ and $g^{(m)!}(\mathcal{E}^{(m)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{U'}^{(m)})$ then it follows from 17.3.2.4.(b) that the canonical morphism $\widehat{\mathcal{D}}_{\mathfrak{U}',\mathbb{Q}}^{(m)} \otimes_{\mathcal{D}_{\mathfrak{U}',\mathbb{Q}}} g^{!}(\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m)}) \to \hat{g}^{(m)!}(\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m)})$ (see notation 17.2.4.3.1 concerning the functor $g^{!}$) is an isomorphism. This implies that 17.3.2.6.1 is an isomorphism.

b) Using the part a), by tensoring with $\mathbb{Q}$ the following morphism

$$f\colon \widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m_0+\bullet)} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m_0)}} \hat{g}^{(m_0)!}(\widehat{\mathcal{E}}^{(m_0)}) \to \hat{g}^{(m_0+\bullet)!}(\widehat{\mathcal{E}}^{(m_0+\bullet)}), \qquad (17.3.2.6.3)$$

we get an isomorphism. Hence, it follows from 8.4.2.9 that that f is an isomorphism of $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m_0+\bullet)})$.

i) On one hand, we have $\underrightarrow{l}^{*}_{\mathbb{Q}}(\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m_0+\bullet)} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{U}'}^{(m_0)}} \hat{g}^{(m_0)!}(\widehat{\mathcal{E}}^{(m_0)})) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{U}',\mathbb{Q}}^{(m_0)}} \hat{g}^{(m_0)!}(\widehat{\mathcal{E}}_{\mathbb{Q}}^{(m_0)})$.

ii) Since the morphism $\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m_0+\bullet)} \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m_0)}} \widehat{\mathcal{E}}^{(m_0)} \to \widehat{\mathcal{E}}^{(m_0+\bullet)}$ is an isomorphism after tensoring with $\mathbb{Q}$, then this is an isomorphism of $\underrightarrow{D}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m_0+\bullet)})$ (use 8.4.2.9). Hence, $\widehat{\mathcal{E}}^{(m_0+\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m_0+\bullet)})$. Since $\underrightarrow{l}^{*}_{\mathbb{Q}}(\widehat{\mathcal{E}}^{(m_0+\bullet)}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}_{\mathfrak{U},\mathbb{Q}}^{(m_0)}} \widehat{\mathcal{E}}_{\mathbb{Q}}^{(m_0)}$, then then it follows from 9.2.1.24 that by applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ to right term of 17.3.2.6.3, we get the right term of 17.3.2.6.2 up to canonical isomorphism.

iii) Hence, by applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ to 17.3.2.6.3, it follows from i) and ii) that we get the isomorphism 17.3.2.6.2. □

Corollary 17.3.2.7. *Let $g\colon U' \to U$ be a smooth morphism of smooth $\mathfrak{S}$-weak formal schemes and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_U^{(m)})$. The canonical morphism*

$$\mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}_{U'}^{(m)}} g^{!^{(m)}}(\mathcal{E}) \to g^{!\dagger}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}_U^{(m)}} \mathcal{E}) \qquad (17.3.2.7.1)$$

is an isomorphism. Moreover, these are transitive with respect to the composition of such morphism g.

Proof. This follows from17.3.2.5 and 17.3.2.6 (or 9.4.1.7.1 since g is smooth). □

17.3.3 Pushforwards

Let $m \in \mathbb{N}$.

Lemma 17.3.3.1. *Let $g\colon U' \to U$ be a morphism of separated smooth $\mathfrak{S}$-weak formal schemes and $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_{U'}^{(m)})$ (see definition 17.3.1.3).*

(a) We have $g_+^{(m)}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_U)$*.*

(b) The canonical morphism

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} g_+^{(m)}(\mathcal{E}') \to \hat{g}_+^{(m)}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') \tag{17.3.3.1.1}$$

is an isomorphism. Moreover, these are transitive with respect to the composition of such morphism g*. In particular, if* $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{U'})$ *and* $g_+^{(m)}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_U)$ *the canonical morphism* $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}} \otimes_{\mathcal{D}^{(m)}_U} g_+^{(m)}(\mathcal{E}') \to \hat{g}_+^{(m)}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'} \otimes_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')$ *is an isomorphism.*

Proof. a) Via the exact sequence $0 \to \mathcal{O}_U \xrightarrow{\pi^{i+1}} \mathcal{O}_U \to \mathcal{O}_{U_i} \to 0$, we see that the projection map $\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{R}g_*(\mathcal{F}') \to \mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} \mathcal{F}')$ is a functorial in $\mathcal{F}' \in D(g^{-1}\mathcal{D}^{(m)}_U)$ isomorphism. This yields the commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{R}g_*(\mathcal{D}^{(m)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') & \xrightarrow{\sim} & \mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} (\mathcal{D}^{(m)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')) \\
\downarrow & & \downarrow \\
\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{R}g_*(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') & \xrightarrow{\sim} & \mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')) \\
\downarrow & & \downarrow \\
\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_U} \mathbb{R}g_*(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') & \xrightarrow{\sim} & \mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}'))
\end{array} \tag{17.3.3.1.2}$$

Moreover, the canonical morphisms $\mathcal{D}^{(m)}_{U\leftarrow U'} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \to \mathcal{D}^{(m)}_{U_i\leftarrow U'_i}$ induce the isomorphisms: $g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} \mathcal{D}^{(m)}_{U\leftarrow U'} \xrightarrow{\sim} g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \xrightarrow{\sim} \mathcal{D}^{(m)}_{U_i\leftarrow U'_i}$. From this we have a commutative diagram:

$$\begin{array}{ccc}
\mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} (\mathcal{D}^{(m)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')) & \xrightarrow{\sim} & \mathbb{R}g_*(\mathcal{D}^{(m)}_{U_i\leftarrow U'_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'_i}} \mathcal{E}'_i) \\
\downarrow & & \| \\
\mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')) & \xrightarrow{\sim} & \mathbb{R}g_*(\mathcal{D}^{(m)}_{U_i\leftarrow U'_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'_i}} \mathcal{E}'_i) \\
\downarrow & & \| \\
\mathbb{R}g_*(g^{-1}\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{g^{-1}\mathcal{O}_U} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')) & \xrightarrow{\sim} & \mathbb{R}g_*(\mathcal{D}^{(m)}_{U_i\leftarrow U'_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'_i}} \mathcal{E}'_i).
\end{array} \tag{17.3.3.1.3}$$

Hence, by composing the top isomorphisms of 17.3.3.1.2 and 17.3.3.1.3, using 5.1.3.5 we get $g_+^{(m)}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_U)$.

b) By using 17.3.1.8, it follows from the part a) that the arrow 17.3.3.1.1 is therefore the unique morphism of $D^{\mathrm{b}}_{\mathrm{qc}}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}})$ making commutative the diagram below

$$\begin{array}{ccc}
g_+(\mathcal{E}') = \mathbb{R}g_*(\mathcal{D}^{(m)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') & \longrightarrow & \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_U} g_+(\mathcal{E}') \\
\downarrow & & \vdots \\
\mathbb{R}g_*(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') & & \vdots \\
\downarrow & & \downarrow \\
\mathbb{R}g_*(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}\leftarrow \mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') & = & \hat{g}_+(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}')
\end{array} \tag{17.3.3.1.4}$$

Applying $\mathcal{O}_{U_i} \otimes^{\mathbb{L}}_{\mathcal{O}_U} -$ to 17.3.3.1.4, because of 17.3.3.1.2 and 17.3.3.1.3, the composition on the left becomes an isomorphism. It follows by construction that 17.3.3.1.1 is an isomorphism. Finally, the transitivity in g of the construction of 17.3.3.1.4 is easy and is left as an exercice. □

Proposition 17.3.3.2. Let $g\colon U' \to U$ be a morphism of smooth $\mathfrak{S}$-weak formal schemes. For any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{U'})$ such that $g_+^{(m)}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_U)$, the canonical morphism

$$\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_U} g_+^{(m)}(\mathcal{E}') \to g_+^{\dagger}(\mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') \tag{17.3.3.2.1}$$

is an isomorphism. Moreover, these are transitive with respect to the composition of such morphism g.

Proof. Let $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}^{(m)}_{U'})$. It follows from 17.3.3.1,7.5.8.14.1 that the canonical morphism

$$\widehat{\mathcal{D}}^{(m+\bullet)}_{\mathfrak{U}} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U}} g^{(m)}_{+}(\mathcal{E}') \to \hat{g}^{(m+\bullet)}_{+}(\widehat{\mathcal{D}}^{(m+\bullet)}_{\mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') \tag{17.3.3.2.2}$$

is an isomorphism. Suppose $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{U'})$ and $g^{(m)}_{+}(\mathcal{E}') \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{U})$. Therefore, since $\underrightarrow{l}^{*}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(m+\bullet)}_{\mathfrak{U}'} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}') \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{U'}} \mathcal{E}'$ and $\underrightarrow{l}^{*}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(m+\bullet)}_{\mathfrak{U}} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{D}^{(m)}_{U}} g^{(m)}_{+}(\mathcal{E}')) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{U}} g^{(m)}_{+}(\mathcal{E}')$, then by applying the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ to 17.3.3.2.2 we get 17.3.3.2.1. □

17.3.4 Closed immersion: adjunction

Let $m \in \mathbb{N}$. Let $v \colon Y \hookrightarrow U$ be a closed immersion of smooth $\mathfrak{S}$-weak formal schemes.

17.3.4.1. Let $\mathcal{E}^{(m_0+\bullet)} \in D^{\mathrm{b}}(\mathcal{D}^{(m_0+\bullet)}_{U})$ such that for any $m \geq m_0$, the morphisms $\mathcal{E}^{(m_0)}_{\mathbb{Q}} \to \mathcal{E}^{(m)}_{\mathbb{Q}}$ are isomorphisms, $\mathcal{E}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{U})$ and $v^{(m)!}(\mathcal{E}^{(m)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y})$. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{U}} v^{(m)}_{+} v^{!(m_0)}(\mathcal{E}^{(m_0)}) & \xrightarrow[\text{adj}]{17.2.5.3.3} & \mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m_0)}_{U}} \mathcal{E}^{(m_0)} \\
\downarrow \wr & & \| \\
\widehat{v}_{+^{\dagger}} \widehat{v}^{!^{\dagger}}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m_0)}_{U}} \mathcal{E}^{(m_0)}) & \xrightarrow[\text{adj}]{9.3.2.5.1} & \mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(m_0)}_{U}} \mathcal{E}^{(m_0)},
\end{array}$$

where the isomorphism on the right follows from 17.2.5.5, 17.3.2.6 and 17.3.3.2.

17.3.4.2. Let $\mathcal{F}^{(m_0+\bullet)} \in D^{\mathrm{b}}(\mathcal{D}^{(m_0+\bullet)}_{Y})$ such that for any $m \geq m_0$, the morphisms $\mathcal{F}^{(m_0)}_{\mathbb{Q}} \to \mathcal{F}^{(m)}_{\mathbb{Q}}$ are isomorphisms, $\mathcal{F}^{(m)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y})$ and $v^{!(m)} v^{(m)}_{+}(\mathcal{F}^{(m)}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{(m)}_{Y})$. We have the commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{Y}} \mathcal{F}^{(m)} & \xrightarrow{17.2.5.3.1} & \mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{Y}} v^{!(m)} v^{(m)}_{+}(\mathcal{F}^{(m)}) \\
\| & & \downarrow \wr \\
\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{Y}} \mathcal{F}^{(m)} & \xrightarrow[9.3.5.9]{\sim} & v^{!^{\dagger}} v_{+^{\dagger}}(\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{Y}} \mathcal{F}^{(m)}),
\end{array}$$

where the isomorphism on the right follows from 17.3.2.6 and 17.3.3.2. Hence, the top morphism is an isomorphism.

17.4 Comparaison between weak formal and formal cohomological operations with overconvergent singularities

Let $f \colon P' \to P$ be a morphism of separated smooth $\mathfrak{S}$-weak formal schemes, T_0 (resp. T'_0) a divisor of P_0 (resp. P'_0), U (resp. U') the open complement of T_0 (resp. T'_0) in P (resp. P'), $j \colon U \hookrightarrow P$ and $j' \colon U' \hookrightarrow P'$ be the open immersions. Suppose that f factors through $g \colon U' \to U$. Let m be an integer. We have defined the sheaves $\mathcal{O}_{\mathfrak{P}}({}^{\dagger} * T_0)$, $\mathcal{D}^{(m)}_{\mathfrak{P}}({}^{\dagger} * T_0)$ and $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger} * T_0)$ (see 8.7.3.25). Similarly, we set $\mathcal{D}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}} := \mathcal{O}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{D}^{(m)}_{\mathfrak{P}}$, which does not depend on m up to canonical isomorphism. Finally, we write

$$\begin{aligned}
\mathcal{D}^{(m)}_{\mathfrak{P}' \to \mathfrak{P}}({}^{\dagger} * T'_0) &:= \mathcal{O}_{\mathfrak{P}'}({}^{\dagger} * T'_0) \otimes_{f^{-1}\mathcal{O}_{\mathfrak{P}}({}^{\dagger} * T_0)} f^{-1}\mathcal{D}^{(m)}_{\mathfrak{P}}({}^{\dagger} * T_0) \\
&\xleftarrow{\sim} \mathcal{O}_{\mathfrak{P}'}({}^{\dagger} * T'_0) \otimes_{\mathcal{O}_{\mathfrak{P}'}} (\mathcal{O}_{\mathfrak{P}'} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{P}}} f^{-1}\mathcal{D}^{(m)}_{\mathfrak{P}}) = \mathcal{O}_{\mathfrak{P}'}({}^{\dagger} * T'_0) \otimes_{\mathcal{O}_{\mathfrak{P}'}} \mathcal{D}^{(m)}_{\mathfrak{P}' \to \mathfrak{P}}.
\end{aligned}$$

Similarly we put $\mathcal{D}^{(m)}_{\mathfrak{P} \leftarrow \mathfrak{P}'}({}^{\dagger} * T'_0) := \mathcal{O}_{\mathfrak{P}'}({}^{\dagger} * T'_0) \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{D}^{(m)}_{\mathfrak{P} \leftarrow \mathfrak{P}'}$.

17.4.1 Faithful flatness theorem

17.4.1.1. There exists a canonical homomorphism $j_*\mathcal{O}_U \to \mathcal{O}_{\mathfrak{P}}(^\dagger * T_0)$ making commutative the diagram

$$\begin{array}{ccc} j_*\mathcal{O}_U & \longrightarrow & j_*\mathcal{O}_{\mathfrak{U}} \\ & \searrow & \uparrow \\ & & \mathcal{O}_{\mathfrak{P}}(^\dagger * T_0). \end{array} \tag{17.4.1.1.1}$$

Indeed, following 8.7.6.12, the right arrow is faithfully flat and in particular injective. Hence, to check the factorisation we can suppose $P = \operatorname{Spff} A$ affine and there exists $f \in A$ such that $Z := V(\overline{f})$ where $\overline{f}$ is the image of f in $A/\pi A$. We have $\Gamma(P, j_*\mathcal{O}_U) = A[1/f]^{\dagger/V}$, $\Gamma(P, \mathcal{O}_{\mathfrak{P}}(^\dagger * T_0)) = \widehat{A}[1/f]^{\dagger/\widehat{A}}$ (see 8.7.3.26) and $\Gamma(P, j_*\mathcal{O}_{\mathfrak{U}}) = \widehat{A}\{1/f\}$. The inclusion $A[1/f]^{\dagger/V} \subset \widehat{A}\{1/f\}$ factors through $A[1/f]^{\dagger/V} \subset \widehat{A}[1/f]^{\dagger/\widehat{A}}$ and we are done.

Similarly, we computation in local coordinates, there exists a canonical homomorphism $j_*\mathcal{D}_U^{(m)} \to \mathcal{D}_{\mathfrak{P}}^{(m)}(^\dagger * T_0)$ whose composition with $\mathcal{D}_{\mathfrak{P}}^{(m)}(^\dagger * T_0) \to j_*\mathcal{D}_{\mathfrak{U}}^{(m)}$ gives the canonical morphism $j_*\mathcal{D}_U^{(m)} \to j_*\mathcal{D}_{\mathfrak{U}}^{(m)}$.

Proposition 17.4.1.2. *For all integers m, there exists canonical homomorphisms $j_*\mathcal{O}_U \to \mathcal{O}_{\mathfrak{P}}(^\dagger * T_0)$, $j_*\mathcal{D}_U^{(m)} \to \mathcal{D}_{\mathfrak{P}}^{(m)}(^\dagger * T_0)$ and $j_*\mathcal{D}_{U,\mathbb{Q}} \to \mathcal{D}_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}}$ which are left and right faithfully flat. Moreover, we have the canonical left and right flat homomorphisms $j_*\mathcal{D}_U \to \mathcal{D}_{\mathfrak{P}}^\dagger(^\dagger * T_0)$ and $j_*\mathcal{D}_U^{(m)} \to \mathcal{D}_{\mathfrak{P}}^\dagger(^\dagger T_0)_{\mathbb{Q}}$.*

Proof. Consider first the faithfully flatness. As T_0 is a divisor, then there exists a neighborhood basis of P consisting of affine opens of P whose intersection with U are affine. Now, for any affine opens V of U, the extension $\Gamma(V, \mathcal{O}_U) \to \Gamma(V, \mathcal{O}_{\mathfrak{U}})$ is left and right faithfully flat. The extension $j_*\mathcal{O}_U \to j_*\mathcal{O}_{\mathfrak{U}}$ is therefore left and right faithfully flat. Using 8.7.6.12, this yields so is the morphism $j_*\mathcal{O}_U \to \mathcal{O}_{\mathfrak{P}}(^\dagger * T_0)$ making commutative the diagram 17.4.1.1.1.

For any open P' of P, if $P'/\mathfrak{S}$ has coordinates then so is $P' \cap U/\mathfrak{S}$. Hence, we easily compute that the morphism

$$\mathcal{O}_{\mathfrak{P}}(^\dagger * T_0) \otimes_{j_*\mathcal{O}_U} j_*\mathcal{D}_U^{(m)} \to \mathcal{D}_{\mathfrak{P}}^{(m)}(^\dagger * T_0), \tag{17.4.1.2.1}$$

constructed by extension from the map $j_*\mathcal{D}_U^{(m)} \to \mathcal{D}_{\mathfrak{P}}^{(m)}(^\dagger * T_0)$ of 17.4.1.1, is an isomorphism. Hence, the result for the second morphism is a consequence of the first one. We get the third morphism from the second one by applying the functor $\otimes\mathbb{Q}$, which yields the faithfulness.

Next consider the question of flatness. For all m, the extensions $j_*\mathcal{D}_U^{(m)} \to j_*\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)}$ are flat. Passing the limit on the level, this yields that $j_*\mathcal{D}_U \to j_*\mathcal{D}_{\mathfrak{U}}^\dagger$ is flat. Using Proposition 8.7.6.12, this implies the first case. In addition to the previous arguments the flatness of the second morphism results from the flatness of $j_*\widehat{\mathcal{D}}_{\mathfrak{U}}^{(m)} \to j_*\widehat{\mathcal{D}}_{\mathfrak{U},\mathbb{Q}}^{(m)} \to j_*\mathcal{D}_{\mathfrak{U},\mathbb{Q}}^\dagger$ (for the later, following the proof of 8.7.5.3 we have such flatness for global sections on affine formal schemes, then use the fact that T_0 is a divisor as explained above). □

17.4.2 Extraordinary inverse images

Lemma 17.4.2.1. *There exists a canonical morphism $j'_*\mathcal{D}_{U'\to U}^{(m)} \to \mathcal{D}_{\mathfrak{P}'\to\mathfrak{P}}^{(m)}(^\dagger * T'_0)$ of $(j'_*\mathcal{D}_{U'}^{(m)}, j'_*g^{-1}\mathcal{D}_U^{(m)})$-bimodules. Moreover, the induced by extension morphism, $\mathcal{D}_{\mathfrak{P}'}^{(m)}(^\dagger * T'_0) \otimes_{j'_*\mathcal{D}_{U'}^{(m)}} j'_*\mathcal{D}_{U'\to U}^{(m)} \to \mathcal{D}_{\mathfrak{P}'\to\mathfrak{P}}^{(m)}(^\dagger * T'_0)$, is an isomorphism.*

Proof. The canonical injection $\mathcal{D}_{U'\to U}^{(m)} \hookrightarrow \mathcal{D}_{\mathfrak{U}'\to\mathfrak{U}}^{(m)}$ induces the following one $j'_*\mathcal{D}_{U'\to U}^{(m)} \hookrightarrow j'_*\mathcal{D}_{\mathfrak{U}'\to\mathfrak{U}}^{(m)}$. Since the morphism $\mathcal{O}_{\mathfrak{P}'}(^\dagger * T'_0) \to j'_*\mathcal{O}_{\mathfrak{U}'}$ is injective, by a computation in local coordinates, we get the injections (adjunction of j'_* then completion)

$$\mathcal{D}_{\mathfrak{P}'\to\mathfrak{P}}^{(m)}(^\dagger * T_0) \hookrightarrow j'_*\mathcal{D}_{\mathfrak{U}'\to\mathfrak{U}}^{(m)} \hookrightarrow j'_*\widehat{\mathcal{D}}_{\mathfrak{U}'\to\mathfrak{U}}^{(m)}.$$

Since we have the canonical injection $j'_*\mathcal{O}_{U'} \hookrightarrow \mathcal{O}_{\mathfrak{P}'}(^\dagger * T'_0)$, then via a computation in local coordinates we get the factorisation:

$$\begin{array}{ccc} j'_*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'\to\mathfrak{U}} & \nwarrow & \\ \uparrow & & \\ j'_*\mathcal{D}^{(m)}_{U'\to U} & \dashrightarrow & \mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0). \end{array}$$

where the two other maps has been defined above. By extension, we obtain $\mathcal{O}_{\mathfrak{P}'}(^\dagger * T'_0) \otimes_{j'_*\mathcal{O}_{U'}} j'_*\mathcal{D}^{(m)}_{U'\to U} \to \mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0)$. By a computation in local coordinates, we check that this is an isomorphism. We finish using the canonical isomorphism $\mathcal{O}_{\mathfrak{P}'}(^\dagger * T'_0) \otimes_{j'_*\mathcal{O}_{U'}} j'_*\mathcal{D}^{(m)}_{U'\to U} \xrightarrow{\sim} \mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0) \otimes_{j'_*\mathcal{D}^{(m)}_{U'}} j'_*\mathcal{D}^{(m)}_{U'\to U}$ which follows from 17.4.1.2.1. □

Proposition 17.4.2.2. Suppose f is smooth. For any $\mathcal{D}^{(m)}_U$-module $\mathcal{F}$ locally in P of finite presentation, we have a canonical isomorphism of coherent $\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger T'_0)_{\mathbb{Q}}$-modules:

$$\begin{aligned} &\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger T'_0)_{\mathbb{Q}} \otimes_{j'_*\mathcal{D}^{(m)}_{U'}} j'_*(\mathcal{D}^{(m)}_{U'\to U} \otimes_{g^{-1}\mathcal{D}^{(m)}_U} g^{-1}\mathcal{F}) \\ &\xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger T'_0)_{\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}}} f^{-1}(\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{j_*\mathcal{D}^{(m)}_U} j_*\mathcal{F}). \end{aligned} \tag{17.4.2.2.1}$$

In addition, these are transitive with respect to the composition of such morphism f.

Proof. We have the canonical morphisms:

$$\begin{aligned} &\mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0) \otimes_{f^{-1}\mathcal{D}^{(m)}_{\mathfrak{P}}(^\dagger * T_0)} f^{-1}(\mathcal{D}^{(m)}_{\mathfrak{P}}(^\dagger * T_0) \otimes_{j_*\mathcal{D}^{(m)}_U} j_*\mathcal{F}) \\ &\xrightarrow{\sim} \mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0) \otimes_{f^{-1}j_*\mathcal{D}^{(m)}_U} f^{-1}j_*\mathcal{F} \xrightarrow{\text{adj}} \mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0) \otimes_{j'_*g^{-1}\mathcal{D}^{(m)}_U} j'_*g^{-1}\mathcal{F} \\ &\xrightarrow{\sim} \mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0) \otimes_{j'_*\mathcal{D}^{(m)}_{U'}} j'_*\mathcal{D}^{(m)}_{U'\to U} \otimes_{j'_*g^{-1}\mathcal{D}^{(m)}_U} j'_*g^{-1}\mathcal{F} \\ &\xrightarrow{\text{adj}} \mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0) \otimes_{j'_*\mathcal{D}^{(m)}_{U'}} j'_*(\mathcal{D}^{(m)}_{U'\to U} \otimes_{g^{-1}\mathcal{D}^{(m)}_U} g^{-1}\mathcal{F}). \end{aligned} \tag{17.4.2.2.2}$$

where the first adj is induced by canonical morphism $f^{-1}j_* \to j'_*g^{-1}$ (which is constructed by adjunction) and the second adjunction map comes from the adjoint pair (j'^{-1}, j'_*). As $\mathcal{F}$ is locally in P of finite presentation, it follows from 17.3.2.3 that $\mathcal{D}^{(m)}_{U'\to U} \otimes_{g^{-1}\mathcal{D}^{(m)}_U} g^{-1}\mathcal{F}$ is locally in P' of finite presentation. Hence, using 17.2.2.13, the bottom term of 17.4.2.2.2 is locally of finite presentation as $\mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0)$-module, i.e. is a coherent $\mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0)$-module (recall the ring $\mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0)$ is coherent following 8.7.3.27).

It follows from 17.2.2.13 that $\mathcal{D}^{(m)}_{\mathfrak{P}}(^\dagger * T_0) \otimes_{j_*\mathcal{D}^{(m)}_U} j_*\mathcal{F}$ is a coherent $\mathcal{D}^{(m)}_{\mathfrak{P}}(^\dagger * T_0)$-module. Hence, since f is smooth, then similarly to 5.3.2.7 this yields the top term of 17.4.2.2.2 is a coherent $\mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0)$-module. Hence, we get the morphism of coherent.

$$\begin{aligned} &\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger T'_0)_{\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0)} (\mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0) \otimes_{f^{-1}\mathcal{D}^{(m)}_{\mathfrak{P}}(^\dagger * T_0)} f^{-1}(\mathcal{D}^{(m)}_{\mathfrak{P}}(^\dagger * T_0) \otimes_{j_*\mathcal{D}^{(m)}_U} j_*\mathcal{F})) \\ &\to \mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger T'_0)_{\mathbb{Q}} \otimes_{j'_*\mathcal{D}^{(m)}_{U'}} j'_*(\mathcal{D}^{(m)}_{U'\to U} \otimes_{g^{-1}\mathcal{D}^{(m)}_U} g^{-1}\mathcal{F}). \end{aligned} \tag{17.4.2.2.3}$$

As 17.4.2.2.2 is an isomorphism above $\mathfrak{U}'$, then it follows from 8.7.6.14 that 17.4.2.2.3 is an isomorphism.

To conclude the construction of the isomorphism of 17.4.2.2, it suffices therefore to check that the canonical homomorphism $\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger T'_0)_{\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{P}'}(^\dagger * T'_0)} \mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger * T'_0) \to \mathcal{D}^\dagger_{\mathfrak{P}'\to\mathfrak{P}}(^\dagger T'_0)_{\mathbb{Q}}$, induced by extension is an isomorphism. As f is smooth, then this is a morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger T'_0)_{\mathbb{Q}}$-modules. By 8.7.6.11, il suffices to establish that its restriction to $\mathfrak{U}'$ is an isomorphism. Now, as g is smooth, $\mathcal{D}^{(m)}_{\mathfrak{U}'\to\mathfrak{U}}$ is $\mathcal{D}^{(m)}_{\mathfrak{U}'}$-coherent. The morphism $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{U}'}} \mathcal{D}^{(m)}_{\mathfrak{U}'\to\mathfrak{U}} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'\to\mathfrak{U}}$ is therefore an isomorphism. We get the isomorphism: $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}^{(m)}_{\mathfrak{U}'}} \mathcal{D}^{(m)}_{\mathfrak{U}'\to\mathfrak{U}} \xrightarrow{\sim} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'\to\mathfrak{U},\mathbb{Q}}$. It follows from 9.4.1.7.1, that we get the canonical isomorphism $\mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}} \otimes_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}',\mathbb{Q}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{U}'\to\mathfrak{U},\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{U}'\to\mathfrak{U},\mathbb{Q}}$. Hence, we are done. □

Remark 17.4.2.3. We can check that the restriction on $\mathfrak{U}'$ of the morphism 17.4.2.2.1 is canonically isomorphic to 17.3.2.7.1.

17.4.3 Pushforwards

Lemma 17.4.3.1. *For all integer m, there exists a canonical homomorphism $j'_*\mathcal{D}^{(m)}_{U\leftarrow U'} \to \mathcal{D}^{(m)}_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0)$ of $(j'_* g^{-1}\mathcal{D}^{(m)}_U, j'_*\mathcal{D}^{(m)}_{U'})$-bimodules.*

Proof. By adding the sheaf of differential forms of maximum degree, this is verified in a analogous manner as 17.4.2.1. □

Proposition 17.4.3.2. Suppose $T'_0 = f^{-1}(T_0)$. Let $\mathcal{E}'$ be a $\mathcal{D}^{(0)}_{U'}$-module locally in P' of finite presentation such that $g_+(\mathcal{E}')$ is a $\mathcal{D}^{(0)}_U$-module locally in P of finite presentation. We have the canonical morphism:

$$\begin{aligned} &\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger * T_0) \otimes_{j_*\mathcal{D}^{(0)}_U} j_* g_+(\mathcal{E}') \\ &\to \mathbb{R}f_*(\mathcal{D}^\dagger_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger * T'_0)} (\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger * T'_0) \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}')). \end{aligned} \quad (17.4.3.2.1)$$

This is an isomorphism on $\mathfrak{U}$. Moreover, these are transitive with respect to the composition of such morphism f.

Proof. I) Let us construct 17.4.3.2.1. 0) The morphism f is the composition of its graph $\gamma\colon P' \hookrightarrow P' \times P$ followed by the projection $\varpi\colon P' \times_{\mathfrak{S}} P \to P$. Set $P'' := P' \times_{\mathfrak{S}} P$, $U'' := U' \times_{\mathfrak{S}} U$. By 17.2.5.4, since γ is a closed immersion (see 17.1.4.5), then $\gamma_+(\mathcal{E}')$ is a $\mathcal{D}^{(0)}_{U''}$-module locally in P'' of finite presentation. Hence, we reduce to the case where f is either smooth or a closed immersion.

1) Since $g_+(\mathcal{E}') = g_*(\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}')$ is a $\mathcal{D}^{(0)}_U$-module locally in P of finite presentation, then we get the isomorphism:

$$j_*\mathbb{R}g_*(\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}') \overset{17.2.2.13}{\xrightarrow{\sim}} \mathbb{R}j_*\mathbb{R}g_*(\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}') \xrightarrow{\sim} \mathbb{R}f_*\mathbb{R}j'_*(\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}').$$

2) The morphism $j'_*\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}' \to \mathbb{R}j'_*(\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}')$ is an isomorphism.

i) Suppose f is smooth. Similarly to 7.5.10.3.1, we have a canonical quasi-isomorphism $\Omega^\bullet_{U'/U} \otimes_{\mathcal{O}_{U'}} \mathcal{D}^{(0)}_{U'}[d_{U'/U}] \to \mathcal{D}^{(0)}_{U\leftarrow U'}$. In particular, $\mathcal{D}^{(0)}_{U\leftarrow U'}$ admits a bounded resolution by free $\mathcal{D}^{(0)}_{U'}$-modules of finite type and we are done.

ii) If f is a closed immersion, then $\mathcal{D}^{(0)}_{U\leftarrow U'}$ is a free $\mathcal{D}^{(0)}_{U'}$-module and we are done.

3) By composition, we get from 1) and 2) the morphism $j_*g_+(\mathcal{E}') \xrightarrow{\sim} \mathbb{R}f_*(j'_*\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}')$. We have a canonical morphism $\mathcal{D}^{(m)}_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0) \to \mathcal{D}^\dagger_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0)$. By composing with 17.4.3.1, we get $j'_*\mathcal{D}^{(m)}_{U\leftarrow U'} \to \mathcal{D}^\dagger_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0)$. This yields the first morphism:

$$\begin{aligned} \mathbb{R}f_*(j'_*\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}') &\to \mathbb{R}f_*(\mathcal{D}^\dagger_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0) \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}') \\ &\xrightarrow{\sim} \mathbb{R}f_*(\mathcal{D}^\dagger_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger * T'_0)} \mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger * T'_0) \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}') \end{aligned}$$

We obtain by composition: $\theta\colon j_*g_+(\mathcal{E}') \to \mathbb{R}f_*(\mathcal{D}^\dagger_{\mathfrak{P}\leftarrow\mathfrak{P}'}(^\dagger * T_0) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger * T'_0)} \mathcal{D}^\dagger_{\mathfrak{P}'}(^\dagger * T'_0) \otimes^{\mathbb{L}}_{j'_*\mathcal{D}^{(0)}_{U'}} j'_*\mathcal{E}')$. The construction of the morphism of 17.4.3.2.1 follows by extension.

II) Let us prove that the restriction of 17.4.3.2.1 on $\mathfrak{U}$ is an isomorphism. The restriction to $\mathfrak{U}$ of θ corresponds to the morphism: $\mathbb{R}g_*(\mathcal{D}^{(0)}_{U\leftarrow U'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}') \to \mathbb{R}g_*(\mathcal{D}^\dagger_{\mathfrak{U}\leftarrow\mathfrak{U}'} \otimes^{\mathbb{L}}_{\mathcal{D}^{(0)}_{U'}} \mathcal{E}')$, which is an isomorphism according to 17.3.3.2.1. □

17.5 Explicit description of the arithmetic $\mathcal{D}$-module associated to an overconvergent isocrystal on an affine smooth scheme having coordinates

Let $Y^\dagger = \operatorname{Spff} A^\dagger$ be a smooth affine $\mathfrak{S}$-weak formal scheme. By using the lifting property 17.1.1.16 and its unicity 17.1.1.20, there exists a smooth affine $\mathcal{V}$-scheme $Y = \operatorname{Spec} A$ whose weak completion is isomorphic to $Y^\dagger$. We set $D_{Y^\dagger} := \Gamma(Y^\dagger, \mathcal{D}_{Y^\dagger})$ and $D^{(m)}_{Y^\dagger} := \Gamma(Y^\dagger, \mathcal{D}^{(m)}_{Y^\dagger})$ for any $m \in \mathbb{N}$.

17.5.1 Functor $\mathrm{sp}_{Y^\dagger *}$

Notation 17.5.1.1. We introduce the following categories.

(a) We denote by $\mathrm{MIC}(A_K^\dagger/K)$ the category of coherent $A_K^\dagger$-modules endowed with an integral connection, i.e. to the category of left $D_{Y^\dagger,K}$-module which are coherent as $A_K^\dagger$-module.

(b) Let X be the closure of $\mathrm{Spec}\, A$ in some projective space $\mathbb{P}^r_{\mathcal{V}}$ and $j\colon Y_0 \subset X_0$ be the open immersion. Denote by $\mathrm{MIC}(Y_0, X_0, \mathfrak{X}/K)$ the category of coherent $j^\dagger \mathcal{O}_{\mathfrak{X}_K}$ endowed with an integrable connection. Following [Ber96b, 2.5.2], the functor $\Gamma(X_K^{\mathrm{an}}, -)$ induces an equivalence of categories between the category $\mathrm{MIC}(Y_0, X_0, \mathfrak{X}/K)$ and $\mathrm{MIC}(A_K^\dagger/K)$.

(c) We denote by $\mathrm{MIC}^\dagger(A_K^\dagger/K)$ the strictly full subcategory of $\mathrm{MIC}(A_K^\dagger/K)$ so that the functor $\Gamma(X_K^{\mathrm{an}}, -)$ induces an equivalence between the categories $\mathrm{MIC}^\dagger(Y_0, X_0, \mathfrak{X}/K)$ and $\mathrm{MIC}^\dagger(A_K^\dagger/K)$. The objects of $\mathrm{MIC}^\dagger(A_K^\dagger/K)$ are by definition the coherent $A_K^\dagger$-modules endowed with an overconvergent connection.

(d) Denote by $\mathrm{MIC}(Y^\dagger/\mathfrak{S})$ the category of $\mathcal{D}_{Y^\dagger,\mathbb{Q}}$-modules globally of finite presentation which are also $\mathcal{O}_{Y^\dagger,\mathbb{Q}}$-coherent. The functors $\Gamma(Y^\dagger, -)$ and $\mathcal{D}_{Y^\dagger\mathbb{Q}} \otimes_{D_{Y^\dagger\mathbb{Q}}} -$ induce quasi-inverse equivalence between $\mathrm{MIC}(Y^\dagger/\mathfrak{S})$ and $\mathrm{MIC}(A_K^\dagger/K)$. Indeed, for any $E \in \mathrm{MIC}(A_K^\dagger/K)$, the canonical morphism $\mathcal{O}_{Y^\dagger\mathbb{Q}} \otimes_{A_K^\dagger} E \to \mathcal{D}_{Y^\dagger\mathbb{Q}} \otimes_{D_{Y^\dagger\mathbb{Q}}} E$ is an isomorphism. We get therefore by composition the equivalence of categories denoted by : $\mathrm{MIC}(Y_0, X_0, \mathfrak{X}/K) \to \mathrm{MIC}(Y^\dagger/\mathfrak{S})$ denoted by $\mathrm{sp}_{Y^\dagger *}$.

(e) We denote by $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$ the strictly full subcategory of $\mathrm{MIC}(Y^\dagger/\mathfrak{S})$ so that the functor $\mathrm{sp}_{Y^\dagger *}$ induces an equivalence between the categories $\mathrm{MIC}^\dagger(Y_0, X_0, \mathfrak{X}/K)$ and $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$. The functors $\Gamma(Y^\dagger, -)$ and $\mathcal{D}_{Y^\dagger\mathbb{Q}} \otimes_{D_{Y^\dagger\mathbb{Q}}} -$ induce quasi-inverse equivalence between $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$ and $\mathrm{MIC}^\dagger(A_K^\dagger/K)$.

17.5.1.2. When there exists a finite etale morphism of smooth $\mathfrak{S}$-weak formal schemes of the form $Y^\dagger \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$, we will give later a description of $\mathrm{MIC}^\dagger(A_K^\dagger/K)$ in term of $D^\dagger$-module (see 17.7.3.7).

17.5.1.3. Let $b\colon Y'^\dagger \to Y^\dagger$ be a morphism of affine smooth $\mathfrak{S}$-weak formal schemes. Let $A'^\dagger := \Gamma(Y'^\dagger, \mathcal{O}_{Y'^\dagger})$. We have the commutative diagram up to canonical isomorphism:

$$\begin{array}{ccccc}
\mathrm{MIC}^\dagger(Y_0/K) & \xrightarrow[\cong]{17.5.1.1.(b)} & \mathrm{MIC}^\dagger(A_K^\dagger/K) & \xrightarrow[\cong]{\mathcal{D}_{Y^\dagger\mathbb{Q}} \otimes_{D_{Y^\dagger\mathbb{Q}}} -} & \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S}) \\
\downarrow b_0^* & & \downarrow A'^\dagger_K \otimes_{A^\dagger_K} - & & \downarrow \mathcal{D}_{Y'^\dagger \to Y^\dagger,\mathbb{Q}} \otimes_{b^{-1}\mathcal{D}_{Y^\dagger,\mathbb{Q}}} b^{-1}- \\
\mathrm{MIC}^\dagger(Y'_0/K) & \xrightarrow[\cong]{17.5.1.1.(b)} & \mathrm{MIC}^\dagger(A'^\dagger_K/K) & \xrightarrow[\cong]{\mathcal{D}_{Y'^\dagger\mathbb{Q}} \otimes_{D_{Y'^\dagger\mathbb{Q}}} -} & \mathrm{MIC}^{\dagger\dagger}(Y'^\dagger/\mathfrak{S}),
\end{array} \tag{17.5.1.3.1}$$

Indeed, following [Ber96b, 2.5.6], the left square is commutative up to canonical isomorphism. For the right one, this is obvious. For simplicity, set $b^* := \mathcal{D}_{Y'^\dagger \to Y^\dagger,\mathbb{Q}} \otimes_{b^{-1}\mathcal{D}_{Y^\dagger,\mathbb{Q}}} b^{-1}-\colon \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S}) \to \mathrm{MIC}^{\dagger\dagger}(Y'^\dagger/\mathfrak{S})$. When $d_{Y'/Y} = 0$, we prefer the notation $b^!$.

If $b, b'\colon Y'^\dagger \to Y^\dagger$ are two morphisms of affine smooth $\mathfrak{S}$-weak formal schemes such that $b_0 = b'_0$, it follows from the commutative up to canonical isomorphism 17.5.1.3.1 that we have the canonical isomorphism $\tau_{b,b'}\colon b'^* \xrightarrow{\sim} b^*$ of functors $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S}) \to \mathrm{MIC}^{\dagger\dagger}(Y'^\dagger/\mathfrak{S})$. We have $\tau_{b,b} = \mathrm{id}$. Let $b''\colon Y'^\dagger \to Y^\dagger$ be a third lifting of b_0. The cocycle condition $\tau_{b,b'} \circ \tau_{b',b''} = \tau_{b,b''}$ holds. For any morphisms $c\colon Z'^\dagger \to Y'^\dagger$ and $d\colon Y^\dagger \to U^\dagger$ of affine smooth $\mathfrak{S}$-weak formal schemes, we have $c^*(\tau_{b,b'}) = \tau_{b\circ c, b'\circ c}$ and $(\tau_{b,b'}) \circ d^* = \tau_{d\circ b, c\circ b'}$ (modulo the transitivity of pullbacks).

17.5.1.4. Let $b\colon Y'^\dagger \to Y^\dagger$ be a finite and étale morphism of affine smooth $\mathfrak{S}$-weak formal schemes. Let $A'^\dagger := \Gamma(Y'^\dagger, \mathcal{O}_{Y'^\dagger})$. We have the commutative diagram up to canonical isomorphism:

$$\begin{array}{ccccc}
\mathrm{MIC}^\dagger(Y'_0/K) & \xrightarrow[\cong]{17.5.1.1.(b)} & \mathrm{MIC}^\dagger(A'^\dagger_K/K) & \xrightarrow[\cong]{\mathcal{D}_{Y'^\dagger,\mathbb{Q}} \otimes_{D_{Y'^\dagger,\mathbb{Q}}} -} & \mathrm{MIC}^{\dagger\dagger}(Y'^\dagger/\mathfrak{S}) \\
\downarrow b_* & & \downarrow \mathfrak{f} & & \downarrow b_+ = b_*(\mathcal{D}_{Y^\dagger \leftarrow Y'^\dagger,\mathbb{Q}} \otimes_{\mathcal{D}_{Y'^\dagger,\mathbb{Q}}} -) \\
\mathrm{MIC}^\dagger(Y_0/K) & \xrightarrow[\cong]{17.5.1.1.(b)} & \mathrm{MIC}^\dagger(A_K^\dagger/K) & \xrightarrow[\cong]{\mathcal{D}_{Y^\dagger\mathbb{Q}} \otimes_{D_{Y^\dagger\mathbb{Q}}} -} & \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S}),
\end{array} \tag{17.5.1.4.1}$$

where the middle vertical functor $\mathfrak{f}$ is the forgetful one and where the left vertical functor is well defined because b is finite and étale (or, if we would like to avoid the use of pushforwards of overconvergent isocrystals, this functor can be defined so that the left square is commutative).

17.5.1.5. With notation 17.5.1.1.(b) and 11.1.1.7, we get the commutative up to canonical isomorphism diagram:

$$\begin{array}{ccc} \mathrm{MIC}(Y_0, X_0, \mathfrak{X}/K) & \xrightarrow[\cong]{\Gamma(X_K^{\mathrm{an}}, -)} & \mathrm{MIC}(A_K^\dagger/K) \\ \downarrow{\scriptstyle |\mathfrak{Y}_K} & & \downarrow{\scriptstyle \widehat{A}_K \otimes_{A_K^\dagger} -} \\ \mathrm{MIC}(Y_0, Y_0, \mathfrak{Y}/K) & \xrightarrow[\cong]{\Gamma(\mathfrak{Y}_K, -)} & \mathrm{MIC}(\widehat{A}_K/K), \end{array} \tag{17.5.1.5.1}$$

where the top equivalence of categories is 17.5.1.1.(b) and the bottom one is 11.1.1.7. We have such a commutative up to canonical isomorphism diagram 17.6.3.2.2 whose horizontal functors are equivalent by adding some symbol †, i.e replacing MIC by $\mathrm{MIC}^\dagger$.

Proposition 17.5.1.6. *Let* $D_{\mathfrak{Y}/\mathfrak{S}} := \Gamma(\mathfrak{Y}, \mathcal{D}_{\mathfrak{Y}/\mathfrak{S}})$ *and* $D^\dagger_{\mathfrak{Y}/\mathfrak{S}} := \Gamma(\mathfrak{Y}, \mathcal{D}^\dagger_{\mathfrak{Y}/\mathfrak{S}})$. *We assume that* $Y^\dagger/\mathfrak{S}$ *has coordinates. Let* $E \in \mathrm{MIC}^\dagger(A_K^\dagger/K)$.

(a) Then the canonical morphisms

$$\widehat{A}_K \otimes_{A_K^\dagger} E \to D_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E \to D^\dagger_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E \tag{17.5.1.6.1}$$

are isomorphisms.

(b) We have $G := D^\dagger_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E \in \mathrm{MIC}^\dagger(\widehat{A}_K/K)$ *and* $\mathcal{G} := \mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{D_{\mathfrak{Y},K}} G \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathcal{V})$.

Proof. a) Since the canonical morphism $\widehat{A}_K \otimes_{A_K^\dagger} D_{Y^\dagger,K} \to D_{\mathfrak{Y},K}$ is an isomorphism, then we get the first isomorphism of 17.5.1.6.1. Since $E \in \mathrm{MIC}^\dagger(A_K^\dagger/K)$ then $D_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E \in \mathrm{MIC}^\dagger(\widehat{A}_K/K)$ (see 17.5.1.5). Hence, we get the isomorphism

$$D_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E \overset{11.1.1.7.1}{\xrightarrow{\sim}} D^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{D_{\mathfrak{Y},\mathbb{Q}}} (D_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E) \xrightarrow{\sim} D_{\mathfrak{Y},K} \otimes_{D_{Y^\dagger,K}} E.$$

b) It follows from 17.5.1.5 and 11.1.1.7. □

Lemma 17.5.1.7 (Theorem A)**.** *We denote by* $\mathrm{MIC}^{(m)}(A^\dagger/\mathcal{V})$ *the category of coherent* $D^{(m)}_{Y^\dagger}$*-modules which are also coherent as* $A^\dagger$*-module. We denote by* $\mathrm{MIC}^{(m)}(Y^\dagger/\mathfrak{S})$ *the category of globally of finite presentation* $\mathcal{D}^{(m)}_{Y^\dagger}$*-modules which are also coherent as* $\mathcal{O}_{Y^\dagger}$*-module.*

(a) The functors $\mathcal{D}^{(m)}_{Y^\dagger} \otimes_{D^\dagger_{Y^\dagger}} -$ *and* $\Gamma(Y^\dagger, -)$ *are quasi-inverse equivalences between* $\mathrm{MIC}^{(m)}(A^\dagger)$ *and* $\mathrm{MIC}^{(m)}(Y^\dagger/\mathfrak{S})$.

(b) For any $E \in \mathrm{MIC}^{(m)}(A^\dagger)$*, the canonical morphism*

$$\mathcal{O}_{Y^\dagger} \otimes_A E \to \mathcal{D}^{(m)}_{Y^\dagger} \otimes_{D^{(m)}_{Y^\dagger}} E \tag{17.5.1.7.1}$$

are isomorphisms.

Proof. According to 17.1.2.2 and 17.2.2.10, coherent $\mathcal{O}_{Y^\dagger}$-modules and globally of finite presentation $\mathcal{D}^{(m)}_{Y^\dagger}$-modules satisfy theorem of type A. Hence, (a) is a consequence of (b). Since the canonical morphism: $\mathcal{O}_{\mathfrak{Y}^\dagger} \otimes_A D^{(m)}_{Y^\dagger} \to \mathcal{D}^{(m)}_{Y^\dagger}$ is an isomorphism (see 17.2.3.2.(b)), then we are done. □

Proposition 17.5.1.8. Suppose $Y^\dagger/\mathfrak{S}$ has coordinates $t_1, \dots, t_d$. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$ and $m \in \mathbb{N}$. There exists a $\mathcal{E}^{(m)} \in \mathrm{MIC}^{(m)}(Y^\dagger/\mathfrak{S})$ together with a $\mathcal{D}_{Y^\dagger,\mathbb{Q}}$-linear isomorphism $\mathcal{E}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$.

Proof. Write $A^\dagger := \Gamma(Y^\dagger, \mathcal{O}_{Y^\dagger})$ and $E := \Gamma(Y^\dagger, \mathcal{E})$. As E is a projective $A_K^\dagger$-module of finite type, there exists a $A_K^\dagger$-module F, an integer r and a $A_K^\dagger$-linear isomorphism: $E \oplus F \xrightarrow{\sim} (A_K^\dagger)^r$. Write $\widehat{E} := \widehat{A}_K \otimes_{A_K^\dagger} E$ and $\widehat{F} := \widehat{A}_K \otimes_{A_K^\dagger} F$,we obtain the isomorphism $\widehat{E} \oplus \widehat{F} \xrightarrow{\sim} (\widehat{A}_K)^r$ as well as canonical injections $E \hookrightarrow \widehat{E}$ and $F \hookrightarrow \widehat{F}$. By abuse of notations, we consider all the sets included in $(\widehat{A}_K)^r$.

We put $E^{(m)} := \{e \in E | \ \forall \underline{k}, \ \ \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} e \in E \cap (A^\dagger)^r\}$. It follows from the formulas 1.4.2.7.1 and 1.4.2.7.(c) which are still valid in $\mathcal{D}_{Y^\dagger}^{(m)}$, that $E^{(m)}$ is a sub-$D_{Y^\dagger}^{(m)}$-module of $E \cap (A^\dagger)^r$. By noetherianity of $A^\dagger$, $E^{(m)}$ is of finite type as $A^\dagger$-module and therefore as $D_{Y^\dagger}^{(m)}$-module. Since $D_{Y^\dagger}^{(m)}$ is noetherian (see 17.2.2.1), this yields that $E^{(m)}$ is of finite presentation. Hence $E^{(m)} \in \mathrm{MIC}^{(m)}(A^\dagger/\mathcal{V})$. Following 17.5.1.7, the module $\mathcal{E}^{(m)} := \mathcal{D}_{Y^\dagger}^{(m)} \otimes_{\Gamma(Y^\dagger, \mathcal{D}_{Y^\dagger}^{(m)})} E^{(m)}$ is a $\mathcal{D}_{Y^\dagger}^{(m)}$-module globally of finite presentation and is also $\mathcal{O}_{Y^\dagger}$-coherent It remains to prove $\mathcal{E}_{\mathbb{Q}}^{(m)} \xrightarrow{\sim} \mathcal{E}$.

Let v_π be the π-adic norm on $\widehat{A}_K$ given by $\widehat{A}$ (see definition 8.7.1.6). Let $\| * \| := p^{-v_\pi(*)}$, which is a Banach norm on $\widehat{A}_K$. We have $\| a \| \leq 1$ if and only if $a \in \widehat{A}$. $\widehat{A}_K{}^N \twoheadrightarrow \widehat{E}$. We still write $\| - \|$ for the norm on E (which is a Banach norm) by restriction via the inclusion $\widehat{E} \subset \widehat{A}_K$. Hence, for any $e \in \widehat{E}$, if $\| e \| \leq 1$ then $e \in \widehat{E} \cap (\widehat{A})^r$.

With notation 1.2.1.2, according to 8.7.1.7, there exists $\eta < 1$, $c \in \mathbb{R}$ such that $|q_k^{(m)}!| \leq c\eta^k$ for all k. For all $e \in \widehat{E}$ and $\underline{k} \in \mathbb{N}^d$, this yields the inequality: $\| \underline{\partial}^{\langle \underline{k} \rangle_{(m)}} e \| = |q_{\underline{k}}^{(m)}!| \ \| \underline{\partial}^{[\underline{k}]} e \| \leq c^d \eta^{|\underline{k}|} \ \| \underline{\partial}^{[\underline{k}]} e \|$. As $\widehat{E} \in \mathrm{MIC}^\dagger(\widehat{A}_K/K)$ (see 17.5.1.5), then the last term tends to 0 when $|\underline{k}|$ goes to infinity (see 11.1.1.7.2). For all $e \in \widehat{E}$, there exists thus an integer N such that $|\underline{k}| \geq N$ implies $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} e \in \widehat{E} \cap (\widehat{A})^r$.

Since $A^\dagger/\pi A^\dagger \xrightarrow{\sim} \widehat{A}/\pi\widehat{A}$, then $A^\dagger \cap \pi\widehat{A} = \pi A^\dagger$. Since $\widehat{A}$ is p-torsion free, this yields the equality: $A_K^\dagger \cap \widehat{A} = A^\dagger$. Hence, we get the last equality: $E \cap (\widehat{A})^r = E \cap (A_K^\dagger)^r \cap (\widehat{A})^r = E \cap (A^\dagger)^r$. This implies $\forall e \in E$, $\exists N \in \mathbb{N}$, $\forall \underline{k}$ such that $|\underline{k}| \geq N$, $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}} e \in E \cap (A^\dagger)^r$. Hence, $\forall e \in E$, $\exists M \in \mathbb{N}$, $\forall \underline{k}$ we have $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}(p^M e) \in E \cap (A^\dagger)^r$. This yields the canonical inclusion $E_{\mathbb{Q}}^{(m)} \to E$ is in fact an isomorphism. Hence, so is $\mathcal{O}_{Y^\dagger \mathbb{Q}} \otimes_{A^\dagger} E^{(m)} \to \mathcal{O}_{Y^\dagger \mathbb{Q}} \otimes_{A_K^\dagger} E$. Via 17.5.1.1.(d) and 17.5.1.7.1 we are done. □

Proposition 17.5.1.9. *Let $m \in \mathbb{N}$, $\mathcal{E}^{(m)}$ be a left $\mathcal{D}_{Y^\dagger}^{(m)}$-module which is coherent as $\mathcal{O}_{Y^\dagger}$-module.*

(a) We have $\mathcal{E}^{(m)} \in \mathrm{MIC}^{(m)}(Y^\dagger/\mathfrak{S})$.

(b) Let $\widehat{\mathcal{E}}^{(m)}$ be the p-adic completion of $\mathcal{E}^{(m)}$. We have the commutative diagram of isomorphisms:

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{Y^\dagger}} \mathcal{E}^{(m)} & \xrightarrow{\sim} & \widehat{\mathcal{D}}_{\mathfrak{Y}}^{(m)} \otimes_{\mathcal{D}_{Y^\dagger}} \mathcal{E}^{(m)} \\ & \searrow^{\sim} \quad \widehat{\mathcal{E}}^{(m)}. \quad {}^{\sim}\swarrow & \end{array} \tag{17.5.1.9.1}$$

Proof. The first assertion is checked similarly to 7.5.2.1.(a). Since $\mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{Y^\dagger}} \mathcal{E}^{(m)}$ is a coherent $\mathcal{O}_{\mathfrak{Y}}$-module then the left bottom morphism of 17.5.1.9.1 is an isomorphism. Since $\widehat{\mathcal{D}}_{\mathfrak{Y}}^{(m)} \otimes_{\mathcal{D}_{Y^\dagger}} \mathcal{E}^{(m)}$ is a coherent $\widehat{\mathcal{D}}_{\mathfrak{Y}}^{(m)}$-module then the left bottom morphism is an isomorphism. Hence, so is the top morphism of 17.5.1.9.1. □

Corollaire 17.5.1.10. Suppose $Y^\dagger/\mathfrak{S}$ has coordinates $t_1, \ldots, t_d$. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$.

(a) The morphism $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{O}_{Y^\dagger,\mathbb{Q}}} \mathcal{E} \to \mathcal{D}_{\mathfrak{Y},\mathbb{Q}}^\dagger \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \mathcal{E}$ is an isomorphism.

(b) The functor $\mathcal{D}_{\mathfrak{Y},\mathbb{Q}}^\dagger \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} -$ factors through $\mathcal{D}_{\mathfrak{Y},\mathbb{Q}}^\dagger \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} - \colon \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S}) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathfrak{S})$ making commutative up to canonical isomorphism the diagram:

$$\begin{array}{ccc} \mathrm{MIC}^\dagger(Y_0, X_0, \mathfrak{X}/K) & \xrightarrow[\cong]{\mathrm{sp}_{Y^\dagger *}} & \mathrm{MIC}^\dagger(A_K^\dagger/K) \\ \downarrow{\scriptstyle |\mathfrak{Y}_K} & & \downarrow{\scriptstyle \mathcal{D}_{\mathfrak{Y},\mathbb{Q}}^\dagger \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} -} \\ \mathrm{MIC}^\dagger(Y_0, Y_0, \mathfrak{Y}/K) & \xrightarrow[\cong]{\mathrm{sp}_*} & \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathfrak{S}), \end{array} \tag{17.5.1.10.1}$$

where sp_* is the pushforward via the specialisation morphism $\mathrm{sp} \colon \mathfrak{Y}_K \to \mathfrak{Y}$.

(c) The functor $\mathrm{sp}_{Y^\dagger *}$ is exact.

Proof. Let $\mathcal{E}^{(m)} \in \mathrm{MIC}^{(m)}(Y^\dagger/\mathfrak{S})$ together with a $\mathcal{D}_{Y^\dagger,\mathbb{Q}}$-linear isomorphism $\mathcal{E}^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ (see 17.5.1.8). Tensoring with $\mathbb{Q}$ the top isomorphism of 17.5.1.9.1 used for $\mathcal{E}^{(m)}$, we get that the canonical morphism $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{O}_{Y^\dagger,\mathbb{Q}}} \mathcal{E} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \mathcal{E}$ is an isomorphism. We get the first part of the corollary by taking the limit on the level. The second part follows from the first one and from 17.5.1.5. Since the functor $\mathrm{sp}_*\colon \mathrm{MIC}^\dagger(Y_0, Y_0, \mathfrak{Y}/K) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathfrak{S})$ is exact, then we get the last part from the commutativity of 17.5.1.10.1 and the full faithfulness of $\mathcal{O}_{Y^\dagger,\mathbb{Q}} \to \mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$. □

17.5.2 Construction of the functor sp_+

Let $P^\dagger$ be a separated smooth $\mathfrak{S}$-weak formal scheme, T_0 a divisor of P_0, $U^\dagger$ the open complement of T_0 in $P^\dagger$, $j\colon U^\dagger \hookrightarrow P^\dagger$ the open immersion and $v\colon Y^\dagger \hookrightarrow U^\dagger$ a closed immersion of smooth $\mathfrak{S}$-weak formal schemes. Suppose in addition $Y^\dagger$ is affine and $Y^\dagger/\mathfrak{S}$ has coordinates $t_1, \ldots, t_d$. Let X_0 be the schematic closure of Y_0 in P_0 and $(F\text{-})\mathrm{Coh}(X_0, \mathfrak{P}, T_0)$ the category of coherent $(F\text{-})\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$-modules with support in X_0.

Example 17.5.2.1. Let $Y^\dagger$ be an affine $\mathfrak{S}$-weak formal scheme such that $Y^\dagger/\mathfrak{S}$ has coordinates $t_1, \ldots, t_d$. Following 17.1.1.16 and 17.1.1.20, there exists a smooth affine $\mathrm{Spec}(\mathcal{V})$-scheme Y whose weak completion is isomorphic to $Y^\dagger$. Choose a closed immersion $Y \hookrightarrow \mathbb{A}^r_{\mathcal{V}}$. Write $P := \mathbb{P}^r_{\mathcal{V}}$, $U := \mathbb{A}^r_{\mathcal{V}}$ and T the divisor $P \setminus U$. We get a closed immersion $v\colon Y^\dagger \hookrightarrow U^\dagger$ of smooth $\mathfrak{S}$-weak formal schemes and the inclusion $j\colon U^\dagger \subset P^\dagger$.

Remark 17.5.2.2. If X_0 is smooth, then $T_0 \cap X_0$ is a divisor of X_0. Indeed, as X_0 is the direct sum of its irreducible components, we can assume that X_0 is irreducible. Thus since $T_0 \not\supset X_0$, then we are done.

17.5.2.3. Let $E \in \mathrm{MIC}^\dagger(Y_0/K)$ and $\mathcal{E} := \mathrm{sp}_{Y^\dagger *}(E) \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$ be the associated globally of finite presentation $\mathcal{D}_{Y^\dagger,\mathbb{Q}}$-module which is also $\mathcal{O}_{Y^\dagger,\mathbb{Q}}$-coherent (see 17.5.1.1.(e)). We set:

$$\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E) := \widetilde{\mathrm{sp}}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(\mathcal{E}) := \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}} \otimes_{j_* \mathcal{D}_{U^\dagger,K}} j_* v_+(\mathcal{E}). \tag{17.5.2.3.1}$$

where $v_+(\mathcal{E}) := v_*(\mathcal{D}_{U^\dagger \leftarrow Y^\dagger,\mathbb{Q}} \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \mathcal{E})$.

Choose $\mathcal{E}^{(0)}$, a $\mathcal{D}^{(0)}_{Y^\dagger}$-module globally of finite presentation, $\mathcal{O}_{Y^\dagger}$-coherent and satisfying $\mathcal{E}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$ (see 17.5.1.8). Let $\widehat{E} \in \mathrm{MIC}^\dagger(Y_0, Y_0, \mathfrak{Y}/K)$ be the convergent isocrystal on Y_0 induced by E, i.e. $\widehat{E} = E|\mathfrak{Y}_K$. We set $\widehat{\mathcal{E}} := \mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \mathcal{E}$. Following 17.5.1.10.1, we have the isomorphism: $\widehat{\mathcal{E}} \xrightarrow{\sim} \mathrm{sp}_*(\widehat{E})$. By proposition 17.2.5.4, the $\mathcal{D}^{(0)}_{U^\dagger}$-module $v^{(0)}_+(\mathcal{E}^{(0)})$ is globally of finite presentation. According to 17.2.2.13.(c), $j_*\mathcal{D}^{(0)}_{U^\dagger}$-module $j_* v^{(0)}_+(\mathcal{E}^{(0)})$ is globally of finite presentation. Since we have the isomorphism:

$$\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}} \otimes_{j_* \mathcal{D}^{(0)}_{U^\dagger}} j_* v^{(0)}_+(\mathcal{E}^{(0)}), \tag{17.5.2.3.2}$$

then $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E)$ is a $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$-module globally of finite presentation with support in X_0. This yields the functors $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}\colon \mathrm{MIC}^\dagger(Y_0/K) \to \mathrm{Coh}(X_0, \mathfrak{P}, T_0)$ and $\widetilde{\mathrm{sp}}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}\colon \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/K) \to \mathrm{Coh}(X_0, \mathfrak{P}, T_0)$. When T_0 is empty, we omit it in the notation.

Remark 17.5.2.4. The construction of 17.5.2.3.1 is very explicit. However, its defect is (until the proof of the opposite) of not having a generalization for $\mathrm{MIC}^\dagger(Y, X/K)$ instead of $\mathrm{MIC}^\dagger(Y/K)$ when the partial compactification X of Y is not not proper.

Proposition 17.5.2.5. *With notations 17.5.2.3, we have canonical isomorphisms*

$$\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E)|\mathfrak{U} \xrightarrow{\sim} \mathrm{sp}_{Y_0 \hookrightarrow \mathfrak{U}+}(\widehat{E}) \xrightarrow{\sim} v^\dagger_+ \widehat{\mathcal{E}}.$$

Proof. Choose a $\mathcal{D}^{(0)}_{Y^\dagger}$-module $\mathcal{E}^{(0)}$ globally of finite presentation, $\mathcal{O}_{Y^\dagger}$-coherent and satisfying $\mathcal{E}^{(0)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}$. Since $v^{(0)}_+(\mathcal{E}^{(0)})$ is a coherent $\mathcal{D}^{(0)}_{U^\dagger}$-module, then we get the isomorphism: $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E)|_{\mathfrak{U}} = \mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{U^\dagger}} v_+(\mathcal{E}^{(0)}) \overset{17.3.3.2.1}{\xrightarrow{\sim}} v^\dagger_+(\mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{Y^\dagger}} \mathcal{E}^{(0)})$. We conclude via $\widehat{\mathcal{E}} \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{Y^\dagger}} \mathcal{E}^{(0)}$ (17.5.1.10) and 17.5.2.3.2. □

With the remark 17.5.2.2, the following corollary is straightforward.

Corollary 17.5.2.6. *With notations 17.5.2.3, suppose X_0 is smooth. Then we have* $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E) \in \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/\mathcal{V})$ *(see notation 12.2.1.4).*

Remark 17.5.2.7. We will check later that the smoothness hypothesis of corollary 17.5.2.6 is useless (see 17.5.2.6)

Proposition 17.5.2.8. The functor $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}$ is exact and faithful.

Proof. According to 17.2.2.13, the canonical morphism $j_* v_+(\mathcal{E}^{(0)}) \to \mathbb{R}j_* v_+(\mathcal{E}^{(0)})$ is an isomorphism for any $\mathcal{D}^{(0)}_{Y^\dagger}$-module $\mathcal{E}^{(0)}$ globally of finite presentation, $\mathcal{O}_{Y^\dagger}$-coherent. Tensoring by $\mathbb{Q}$, using 17.5.1.8, this yields that the the canonical morphism $j_* v_+(\mathcal{E}) \to \mathbb{R}j_* v_+(\mathcal{E})$ is an isomorphism for any $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$. Following 17.5.1.10.(c) (resp. 17.2.5.1), the functor $\mathrm{sp}_{Y^\dagger *}$ (resp. v_+) is exact. Hence, so is the functor $E \mapsto j_* v_+ \mathrm{sp}_{Y^\dagger *}(E)$. As the extension $\mathcal{D}_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$ is left and right flat, it follows from 17.4.1.2 that $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$ is left and right $j_* \mathcal{D}_{U^\dagger, \mathbb{Q}}$-flat. The functor $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}$ is thus exact.

Next we treat the faithfulness. Via 8.7.6.11, it suffices to establish that the faithfulness of $E \mapsto \mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)|\mathfrak{U} \underset{17.5.2.5}{\overset{\sim}{\longrightarrow}} v^\dagger_+ \widehat{\mathcal{E}} \overset{\sim}{\longrightarrow} v^\dagger_+ \circ \mathrm{sp}_*(\widehat{E})$ (see 17.5.2.3). Since the functor $E \mapsto \widehat{E}$ is faithful (this follows from [Ber96b, 2.1.11]), as $\mathrm{sp}_* \colon \mathrm{MIC}^\dagger(Y_0, Y_0, \mathfrak{Y}/K) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathfrak{S})$ is an equivalence of category and $v^\dagger_+$ is fully faithful on the category of coherent $\mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}}$-modules, then the result follows. □

We close this subsection by the following proposition.

Proposition 17.5.2.9. Consider the commutative diagram of smooth $\mathfrak{S}$-weak formal schemes:

$$\begin{array}{ccccc} Y'^\dagger & \overset{v'}{\hookrightarrow} & U'^\dagger & \overset{j'}{\hookrightarrow} & P'^\dagger \\ \downarrow b & & \downarrow g & & \downarrow f \\ Y^\dagger & \overset{v}{\hookrightarrow} & U^\dagger & \overset{j}{\hookrightarrow} & P^\dagger, \end{array} \tag{17.5.2.9.1}$$

where f and g are smooth morphisms, b is a morphism of affine smooth $\mathfrak{S}$-weak formal schemes, j and j' are open immersions, v and v' are closed immersions. We suppose there exists a divisor T_0 of P_0 (resp. T'_0 of P'_0) such that $U^\dagger = P^\dagger \setminus T_0$ (resp. $U'^\dagger = P'^\dagger \setminus T'_0$). Let X_0 (resp. X'_0) denote the closure of of Y_0 (resp. Y'_0) in P_0 (resp. P'_0).

For any object $E \in \mathrm{MIC}^\dagger(Y_0/K)$ such that $\mathbb{R}\underline{\Gamma}^\dagger_{X'_0} f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T'_0)_{\mathbb{Q}})$, there exists a canonical isomorphism

$$\mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(b^* E)[d_{Y'_0/Y_0}] \overset{\sim}{\longrightarrow} \mathbb{R}\underline{\Gamma}^\dagger_{X'_0} f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)) \tag{17.5.2.9.2}$$

making commutative the diagram

$$\begin{array}{ccc} \mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(b^* E)[d_{Y'_0/Y_0}] & \underset{17.5.2.9.2}{\overset{\sim}{\longrightarrow}} & \mathbb{R}\underline{\Gamma}^\dagger_{X'_0} f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)) \\ \downarrow 17.5.2.5 & & \downarrow 17.5.2.5 \\ \mathrm{sp}_{Y'_0 \hookrightarrow \mathfrak{U}'+}(b^* \widehat{E})[d_{Y'_0/Y_0}] & \overset{12.2.4.1.2}{\longrightarrow} & \mathbb{R}\underline{\Gamma}^\dagger_{Y'_0} g^! \mathrm{sp}_{Y_0 \hookrightarrow \mathfrak{U}+}(\widehat{E}) \end{array} \tag{17.5.2.9.3}$$

In addition, these are transitive with respect to the composition of diagrams of the form 17.5.2.9.1.

Proof. I) We construct the morphism 17.5.2.9.2. Let $\mathcal{E} := \mathrm{sp}_{Y^\dagger *}(E) \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$. Choose a $\mathcal{D}^{(\bullet)}_{Y^\dagger}$-module $\mathcal{E}^{(\bullet)}$ such that for any $m \in \mathbb{N}$ the module $\mathcal{E}^{(m)}$ is p-torsion free, globally of finite presentation as $\mathcal{D}^{(m)}_{Y^\dagger}$-module, coherent as $\mathcal{O}_{Y^\dagger}$-module and satisfying $\mathcal{E}^{(m)}_{\mathbb{Q}} \overset{\sim}{\longrightarrow} \mathcal{E}$ (use 17.5.1.8).

1) Set $\mathcal{E}'^{(\bullet)} := b^{(\bullet)!}(\mathcal{E}^{(\bullet)})[-d_{Y'_0/Y_0}]$. Then for any $m \in \mathbb{N}$ the module $\mathcal{E}'^{(m)}$ is p-torsion free, globally of finite presentation as $\mathcal{D}^{(m)}_{Y'^\dagger}$-module, coherent as $\mathcal{O}_{Y'^\dagger}$-module. Moreover, it follows from 17.5.1.3.1 that we have the isomorphism $\mathcal{E}'^{(m)}_{\mathbb{Q}} \overset{\sim}{\longrightarrow} \mathrm{sp}_{Y'^\dagger *}(b^*(E))$.

2) We have the morphisms:

$$v'^{(0)}_+ b^{(0)!}(\mathcal{E}^{(0)}) \overset{17.2.5.3.1}{\underset{\sim}{\longrightarrow}} v'^{(0)}_+ b^{(0)!} v^{(0)!} v^{(0)}_+(\mathcal{E}^{(0)}) \overset{\sim}{\longrightarrow} v'^{(0)}_+ v'^{(0)!} g^{(0)!} v^{(0)}_+(\mathcal{E}^{(0)}) \overset{17.2.5.3.3}{\longrightarrow} g^{(0)!} v^{(0)}_+(\mathcal{E}^{(0)}) \tag{17.5.2.9.4}$$

3) We get the morphisms:

$$\begin{aligned} \mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(b^*E)[d_{Y'_0/Y_0}] &\overset{\text{step 1)}}{\underset{\sim}{\longrightarrow}} \mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T'_0)_{\mathbb{Q}} \otimes_{j'_* \mathcal{D}^{(0)}_{U'^\dagger}} j'_* v'^{(0)}_+ b^{(0)!}(\mathcal{E}^{(0)}) \\ \underset{17.5.2.9.4}{\longrightarrow} \mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T'_0)_{\mathbb{Q}} \otimes_{j'_* \mathcal{D}^{(0)}_{U'^\dagger}} j'_* g^{(0)!} v^{(0)}_+(\mathcal{E}^{(0)}) &\overset{17.4.2.2}{\underset{\sim}{\longrightarrow}} f^!_{T'_0,T_0}\left(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}} \otimes_{j_* \mathcal{D}^{(0)}_{U^\dagger}} j_* v^{(0)}_+(\mathcal{E}^{(0)})\right) \\ &\overset{\sim}{\longrightarrow} f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)). \end{aligned} \tag{17.5.2.9.5}$$

As $\mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(b^*E) \in \mathrm{Coh}(\mathfrak{P}', T'_0, X'_0)$, then by applying the functor $\mathbb{R}\underline{\Gamma}^\dagger_{X'_0}$ to the composite morphism 17.5.2.9.5 (this is well defined since this composite is a morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T'_0)_{\mathbb{Q}})$), we get the arrow 17.5.2.9.2.

II) It remains to check this is an isomorphism. Set $\widehat{\mathcal{E}} := \mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \mathcal{E}$. Using 8.7.6.11, it remains to check that the restriction on $\mathfrak{U}'$ of the arrow 17.5.2.9.2 is an isomorphism. Since $b^{(\bullet)!}(\mathcal{E}^{(\bullet)}))$ and $v'^{(\bullet)}_+ b^{(\bullet)!}(\mathcal{E}^{(\bullet)}))$ satisfy the required conditions of 17.3.2.6 and 17.3.3.2, we get

$$\mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{U'^\dagger}} v'^{(0)}_+ b^{(0)!}(\mathcal{E}^{(0)}) \overset{\sim}{\longrightarrow} v'_+ b^!(\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{U^\dagger}} \mathcal{E}^{(0)}). \tag{17.5.2.9.6}$$

We have the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(b^*E)|\mathfrak{U}'[d_{Y'_0/Y_0}] & & \xrightarrow{17.5.2.9.5} & & f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E))|\mathfrak{U}' \\
\sim \downarrow \text{step 1)} & & & & \uparrow \sim \\
\mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{U'^\dagger}} v'^{(0)}_+ b^{(0)!}(\mathcal{E}^{(0)}) & \xrightarrow{17.5.2.9.4} & \mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{U'^\dagger}} g^{(0)!} v^{(0)}_+(\mathcal{E}^{(0)}) & \underset{\sim}{\xrightarrow{17.4.2.2}} & g^!(\mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{U^\dagger}} v^{(0)}_+(\mathcal{E}^{(0)})) \\
\sim \downarrow 17.5.2.9.6 & & & & \sim \downarrow 17.3.3.2 \\
v'_+ b^!(\mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{Y^\dagger}} \mathcal{E}^{(0)}) & & \longrightarrow & & g^! v_+(\mathcal{D}^\dagger_{\mathfrak{Y},\mathbb{Q}} \otimes_{\mathcal{D}^{(0)}_{Y^\dagger}} (\mathcal{E}^{(0)})) \\
\downarrow \sim & & & & \downarrow \sim \\
\mathrm{sp}_{Y'_0 \hookrightarrow \mathfrak{U}'+}(b^*\widehat{E})[d_{Y'_0/Y_0}] & & \longrightarrow & & g^! \mathrm{sp}_{Y_0 \hookrightarrow \mathfrak{U}+}(\widehat{E})
\end{array} \tag{17.5.2.9.7}$$

where the top morphism of the bottom square is induced by the morphism $v'_+ b^! \to g^! v_+$ constructed by adjunction similarly to 17.5.2.9.4 (but using the formal version of 9.3.2.5). Indeed, the top square is commutative by construction, that of the middle one follows from 17.2.5.3.2 and 17.2.5.3.4 and the bottom one is tautological. By applying the functor $\mathbb{R}\underline{\Gamma}^\dagger_{Y'_0}$ to 17.5.2.9.7 we get 17.5.2.9.3. Since the bottom morphism of 17.5.2.9.3 is an isomorphism, then so is the top one.

□

Example 17.5.2.10. Suppose the left square of the diagram 17.5.2.9.1 is cartesian. Let $E \in \mathrm{MIC}^\dagger(Y_0/K)$. In that case, the morphism $\mathbb{R}\underline{\Gamma}^\dagger_{X'_0} f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)) \to f^!_{T'_0,T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E))$ is an isomorphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger T'_0)_{\mathbb{Q}})$. Hence, we get the isomorphism:

$$\mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(b^*E)[d_{X'_0/X_0}] \overset{\sim}{\longrightarrow} f^!_{T_0}(\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)).$$

In particular, when $f = \mathrm{id}$, we have the functorial in $E \in \mathrm{MIC}^\dagger(Y_0/K)$ isomorphism: $({}^\dagger T'_0)\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E) \overset{\sim}{\longrightarrow} \mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T'_0,+}(j'^\dagger E)$.

Lemma 17.5.2.11. *Consider the diagram of smooth $\mathfrak{S}$-weak formal schemes*

$$\begin{array}{ccccc} Y'^{\dagger} & \xrightarrow{v'} & U'^{\dagger} & \xrightarrow{j'} & P'^{\dagger} \\ \downarrow b & & \downarrow g & & \downarrow f \\ Y^{\dagger} & \xrightarrow{v} & U^{\dagger} & \xrightarrow{j} & P^{\dagger}, \end{array} \tag{17.5.2.11.1}$$

where f is proper, j and j' are open immersions, the right square is cartesian, v and v' are closed immersions, b is finite and étale and $Y^{\dagger}$ is affine. Suppose in addition that $T_0 := P_0 \backslash U_0$ and $T'_0 := P'_0 \backslash U'_0$ are the support of divisors.

For any $E' \in \mathrm{MIC}^{\dagger}(Y'_0/K)$, with notation 17.5.1.4.1, we have then a canonical isomorphism

$$\mathrm{sp}_{Y^{\dagger} \hookrightarrow U^{\dagger}, T_0, +}(b_* E') \xrightarrow{\sim} f^{\dagger}_{T_0,+}(\mathrm{sp}_{Y'^{\dagger} \hookrightarrow U'^{\dagger}, T'_0, +}(E')). \tag{17.5.2.11.2}$$

Moreover, these are transitive with respect to the composition of diagrams of the form 17.5.2.11.1.

Proof. Set $\mathcal{E}' := \mathrm{sp}_{Y'^{\dagger}*}(E') \in \mathrm{MIC}^{\dagger\dagger}(Y'^{\dagger}/\mathfrak{S})$. Choose a $\mathcal{D}^{(0)}_{Y'^{\dagger}}$-module $\mathcal{E}'^{(0)}$ such that for any $m \in \mathbb{N}$ the module $\mathcal{E}'^{(m)}$ is p-torsion free, globally of finite presentation as $\mathcal{D}^{(m)}_{Y'^{\dagger}}$-module, coherent as $\mathcal{O}_{Y'^{\dagger}}$-module and satisfying $\mathcal{E}'^{(m)}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{E}'$ (use 17.5.1.8). We put $\mathcal{F}'^{(0)} := v'^{(0)}_+(\mathcal{E}'^{(0)})$ and $\mathcal{E}^{(0)} := b^{(0)}_+(\mathcal{E}'^{(0)})$.

As $g^{(0)}_+(\mathcal{F}'^{(0)}) = g^{(0)}_+ v'^{(0)}_+(\mathcal{E}'^{(0)}) \xrightarrow{\sim} v^{(0)}_+ b^{(0)}_+(\mathcal{E}'^{(0)})$, then it follows from 17.2.5.4 that $\mathcal{F}'^{(0)}$ (resp. $g^{(0)}_+(\mathcal{F}'^{(0)})$) is locally in $P'^{\dagger}$ (resp. $P^{\dagger}$) of finite presentation. Hence, tensoring by $\mathbb{Q}$ a morphism of the form 17.4.3.2.1, we have therefore the morphism

$$\begin{aligned} \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}} \otimes_{j_* \mathcal{D}^{(0)}_{U^{\dagger}}} j_* g_+(\mathcal{F}'^{(0)}) &\to \mathbb{R}f_*(\mathcal{D}^{\dagger}_{\mathfrak{P} \leftarrow \mathfrak{P}'}({}^{\dagger}T_0)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T'_0)} \mathcal{D}^{\dagger}_{\mathfrak{P}'}({}^{\dagger}T'_0) \otimes^{\mathbb{L}}_{j'_* \mathcal{D}^{(0)}_{U'^{\dagger}}} j'_* \mathcal{F}'^{(0)}) \\ &\overset{17.5.2.3.2}{\xrightarrow{\sim}} f^{\dagger}_{T_0,+}(\mathrm{sp}_{Y'^{\dagger} \hookrightarrow U'^{\dagger}, T'_0, +}(E')). \end{aligned} \tag{17.5.2.11.3}$$

Since 17.5.2.11.3 is an isomorphism on $\mathfrak{U}$, since this is a morphism of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}})$ (because f is proper), then with 8.7.6.11 we get that 17.5.2.11.3 is an isomorphism.

Since $\mathcal{E}^{(0)}$ is a $\mathcal{D}^{(0)}_{Y^{\dagger}}$-module of globally of finite presentation, is $\mathcal{O}_{Y^{\dagger}}$-coherent and since $\mathcal{E}^{(0)}_{\mathbb{Q}} = b^{(0)}_+(\mathcal{E}'^{(0)})_{\mathbb{Q}} \xrightarrow{\sim} b_+(\mathcal{E}') = b_+ \circ \mathrm{sp}_{Y'^{\dagger}*}(E') \overset{17.5.1.4.1}{\xrightarrow{\sim}} \mathrm{sp}_{Y^{\dagger}*}(b_*(E'))$, then we get the isomorphism:

$$\mathrm{sp}_{Y^{\dagger} \hookrightarrow U^{\dagger}, T_0, +}(b_* E') \overset{17.5.2.3.2}{\xrightarrow{\sim}} \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}} \otimes_{j_* \mathcal{D}^{(0)}_{U^{\dagger}}} j_* v^{(0)}_+(\mathcal{E}^{(0)}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}} \otimes_{j_* \mathcal{D}^{(0)}_{U^{\dagger}}} j_* g_+(\mathcal{F}'^{(0)}). \tag{17.5.2.11.4}$$

By composing 17.5.2.11.4 with 17.5.2.11.3 we obtain the isomorphism 17.5.2.11.2. □

Proposition 17.5.2.12. *Suppose there exists two liftings $v_1, v_2 \colon Y^{\dagger} \hookrightarrow U^{\dagger}$ of $v_0 \colon Y_0 \hookrightarrow U_0$.*

(a) We have the canonical isomorphism

$$\tau_{v_1,v_2} \colon \mathrm{sp}_{Y^{\dagger} \underset{v_2}{\hookrightarrow} U^{\dagger}, T, +} \xrightarrow{\sim} \mathrm{sp}_{Y^{\dagger} \underset{v_1}{\hookrightarrow} U^{\dagger}, T, +} \qquad \tau_{v_1,v_2} \colon \widetilde{\mathrm{sp}}_{Y^{\dagger} \underset{v_2}{\hookrightarrow} U^{\dagger}, T, +} \xrightarrow{\sim} \widetilde{\mathrm{sp}}_{Y^{\dagger} \underset{v_1}{\hookrightarrow} U^{\dagger}, T, +}. \tag{17.5.2.12.1}$$

(b) With notation 17.5.2.5, for any $E \in \mathrm{MIC}^{\dagger}(Y_0/K)$ we have

$$\begin{array}{ccc} \mathrm{sp}_{Y^{\dagger} \underset{v_2}{\hookrightarrow} U^{\dagger}, T, +}(E) & \xrightarrow[\tau_{v_1,v_2}]{17.5.2.12.1} & \mathrm{sp}_{Y^{\dagger} \underset{v_1}{\hookrightarrow} U^{\dagger}, T, +}(E) \\ \downarrow 17.5.2.5 & & \downarrow 17.5.2.5 \\ v^{\dagger}_{2+} \widehat{\mathcal{E}} & \xrightarrow[\tau_{v_1,v_2}]{9.2.2.3.1} & v^{\dagger}_{1+} \widehat{\mathcal{E}} \end{array} \tag{17.5.2.12.2}$$

(c) Let $v_3 \colon Y^{\dagger} \hookrightarrow U^{\dagger}$ be a third lifting of $Y_0 \hookrightarrow U_0$. The cocycle condition $\tau_{v_1,v_2} \circ \tau_{v_2,v_3} = \tau_{v_1,v_3}$ holds. Moreover, $\tau_{v_1,v_1} = \mathrm{id}$.

Proof. a) Denote by $w = (v_1, v_2): Y^\dagger \hookrightarrow U^\dagger \times U^\dagger$, $H_0 := (P_0 \times P_0) \setminus (U_0 \times U_0)$ the divisor of $P_0 \times P_0$, $p_1 : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ and $p_2 : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$ the left and right projections. Following the proposition 17.5.2.9 (to be able to use the second proposition, recall the finiteness property of 17.6.1.7), the functors of the form sp_+ commute with inverse images, i.e., we have the canonical isomorphisms

$$\mathrm{sp}_{Y^\dagger \underset{w}{\hookrightarrow} U^\dagger \times U^\dagger, H_0, +} \overset{17.5.2.9}{\underset{\sim}{\longrightarrow}} \mathbb{R}\underline{\Gamma}^\dagger_{X_0} p^!_{1,H_0,T_0} \circ \mathrm{sp}_{Y^\dagger \underset{v_1}{\hookrightarrow} U^\dagger, T, +}$$
$$\xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{X_0} ({}^\dagger H_0) p^!_{1,T_0} \circ \mathrm{sp}_{Y^\dagger \underset{v_1}{\hookrightarrow} U^\dagger, T, +} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Y_0} p^!_{1,T_0} \circ \mathrm{sp}_{Y^\dagger \underset{v_1}{\hookrightarrow} U^\dagger, T, +}. \qquad (17.5.2.12.3)$$

By using 17.5.2.11.2 (in the case where $b = \mathrm{id}$), we get the isomorphism:

$$\mathrm{sp}_{Y^\dagger \underset{v_2}{\hookrightarrow} U^\dagger, T, +} \overset{17.5.2.11.2}{\underset{\sim}{\longrightarrow}} p_{2,T_0,+} \circ \mathrm{sp}_{Y^\dagger \underset{w}{\hookrightarrow} U^\dagger \times U^\dagger, H_0, +} \overset{17.5.2.12.3}{\underset{\sim}{\longrightarrow}} p_{2,T_0,+} \mathbb{R}\underline{\Gamma}^\dagger_{Y_0} p^!_{1,T_0} \circ \mathrm{sp}_{Y^\dagger \underset{v_1}{\hookrightarrow} U^\dagger, T, +}.$$

Following 16.2.7.6, the functor $p_{2,T_0,+} \mathbb{R}\underline{\Gamma}^\dagger_{Y_0} p^!_{1,T_0}$ is an autoequivalence of the category $\mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K)$. We conclude the construction of the canonical isomorphism 17.5.2.12.1 by using Theorem 17.6.1.7.

b) The commutativity of 17.5.2.12.2 follows from 17.5.2.9.3.

c) The last part is a consequence of the similar properties in the formal context (see 9.2.2.3) and of b). □

17.6 On the equivalence between the category of overcoherent isocrystals to that of overconvergent isocrystals over an affine and smooth variety

17.6.1 Overcoherence of the essential image of $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}$

Definition 17.6.1.1. Let Y_0 be a smooth affine k-scheme, $Y_0 \hookrightarrow X_0$ an open immersion. The open immersion $Y_0 \hookrightarrow X_0$ "ideally desingularizes" or "is ideally desingularisable" if there exists a surjective morphism $a_0 : X'_0 \to X_0$, which decompose into a closed immersion $X'_0 \hookrightarrow \mathbb{P}^r_{X_0}$ followed by the canonical projection $\mathbb{P}^r_{X_0} \to X_0$, such that

(i) X'_0 is smooth ;

(ii) the morphism $Y'_0 := a_0^{-1}(Y_0) \to Y_0$ induced by a_0 is finite and étale ;

(iii) the morphism $Y'_0 \hookrightarrow \mathbb{P}^r_{Y_0}$ induced by $X'_0 \hookrightarrow \mathbb{P}^r_{X_0}$ lifts to a morphism of smooth $\mathfrak{S}$-weak formal scheme of the form $Y'^\dagger \to \mathbb{P}^{r\dagger}_{Y^\dagger}$, where $Y^\dagger$ is a smooth affine $\mathfrak{S}$-weak formal scheme lifting Y_0.

Remark 17.6.1.2. We keep notations of 17.6.1.1.

(a) If X_0 is smooth, then $Y_0 \hookrightarrow X_0$ ideally desingularizes.

(b) The condition (iii) is independent of the choice of the smooth lifting $Y^\dagger$ because these are isomorphic (see 17.1.1.20).

(c) If $Y_0 \hookrightarrow X_0$ ideally desingularizes then for all affine open $\widetilde{Y}_0$ of Y_0 so does the induced open immersion $\widetilde{Y}_0 \hookrightarrow X_0$. Indeed, the morphism $\widetilde{Y}'_0 := a_0^{-1}(\widetilde{Y}_0) \to \widetilde{Y}_0$ induced by a_0 is finite and étale. Moreover, noting $\widetilde{Y}^\dagger$ the open subset of $Y^\dagger$ with underlying space $\widetilde{Y}_0$, the projection $Y'^\dagger \times_{\mathbb{P}^{r\dagger}_{Y^\dagger}} \mathbb{P}^{r\dagger}_{\widetilde{Y}^\dagger} \to \mathbb{P}^{r\dagger}_{\widetilde{Y}^\dagger}$ is a lifting of $\widetilde{Y}'_0 \to \mathbb{P}^r_{\widetilde{Y}_0}$.

17.6.1.3. Let $P^\dagger$ be a separated smooth $\mathfrak{S}$-weak formal scheme, T_0 a divisor of P_0, $U^\dagger$ the open complement of T_0 in $P^\dagger$, $j : U^\dagger \hookrightarrow P^\dagger$ the open immersion and $v : Y^\dagger \hookrightarrow U^\dagger$ a closed immersion of $\mathfrak{S}$-weak formal schemes. Suppose in addition that $Y^\dagger$ is smooth affine. Let X_0 be the schematic closure of Y_0 dans P_0.

Theorem 17.6.1.4. *We keep notations 17.6.1.3 and we suppose that $Y_0 \hookrightarrow X_0$ ideally desingularizes. Then, for any $E \in \mathrm{MIC}^\dagger(Y_0/K)$, the $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$-module $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E) \in \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K)$.*

Proof. By hypotheses, there exists a surjective morphism $a_0 \colon X_0' \to X_0$, which decomposes into a closed immersion $X_0' \hookrightarrow \mathbb{P}^r_{X_0}$ followed by the canonical projection $\mathbb{P}^r_{X_0} \to X_0$, such that X_0' is smooth, the morphism $Y_0' = a_0^{-1}(Y_0) \to Y_0$ is finite and étale and such that there exists a lifting $Y'^\dagger \to \mathbb{P}^{r\dagger}_{Y^\dagger}$ of $Y_0' \hookrightarrow \mathbb{P}^r_{Y_0}$. Composing this lifting with the canonical projection $\mathbb{P}^{r\dagger}_{Y^\dagger} \to Y^\dagger$ (resp. with the closed immersion $\mathbb{P}^{r\dagger}_{Y^\dagger} \hookrightarrow \mathbb{P}^{r\dagger}_{U^\dagger}$), we get a surjective finite and étale morphism $b \colon Y'^\dagger \to Y^\dagger$ (resp. a closed immersion $v' \colon Y'^\dagger \hookrightarrow \mathbb{P}^{r\dagger}_{U^\dagger}$). Write $T_0' := f^{-1}T_0$, $U'^\dagger := \mathbb{P}^{r\dagger}_{U^\dagger}$, $P'^\dagger := \mathbb{P}^{r\dagger}_{P^\dagger}$, $j' \colon U'^\dagger \hookrightarrow P'^\dagger$ the open immersion, $f \colon P'^\dagger \to P^\dagger$ and $g \colon U'^\dagger \to U^\dagger$ the projections, we obtain a commutative diagram of the form 17.5.2.11.1, with b surjective.

Let $E \in \mathrm{MIC}^\dagger(Y_0/K)$. It is sufficient to check that $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(E)$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$-overcoherent. According to 17.5.2.11, there is a canonical isomorphism

$$\mathrm{sp}_+(b_* b^* E) \xrightarrow{\sim} f_{T_0,+^\dagger}(\mathrm{sp}'_+(b^*E)), \tag{17.6.1.4.1}$$

where $\mathrm{sp}_+ = \mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}$ and $\mathrm{sp}'_+ = \mathrm{sp}_{Y'^\dagger \hookrightarrow U'^\dagger, T_0',+}$. As X_0' is smooth, then following 17.5.2.6, $\mathrm{sp}'_+(b^*E) \in \mathrm{MIC}^{\dagger\dagger}(X_0', \mathfrak{P}, T_0'/\mathcal{V})$. Now, by 16.1.1.7, this yields $\mathrm{sp}'_+(b^*E)$ is $\mathcal{D}^\dagger_{\mathfrak{P}'}({}^\dagger f^{-1}T_0)_{\mathbb{Q}}$-overcoherent. With 15.3.6.14 and 17.6.1.4.1, it follows that $\mathrm{sp}_+(b_*b^*E)$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$-overcoherent.

Next, since b is finite, étale and surjective, then E is a direct factor of b_*b^*E. The module $\mathrm{sp}_+(E)$ is thus a direct factor of $\mathrm{sp}_+(b_*b^*E)$ and we are done. □

The following proposition is just a simple extension of [Har70, II.3.1].

Proposition 17.6.1.5. Let k be a field, P a smooth proper k-scheme, U an affine and dense open of P, T the closed subscheme of P complementary to U. Then T is the support of a divisor.

Proof. As P is k-smooth then, according to [E.G.A.IV 17.15.2], P is regular. As a regular local ring is integral, then $\mathcal{O}_{P,x}$ is an integral local ring for all $x \in P$. According to [E.G.A. I.4.5.5], P is then the sum of its irreducible components (i.e., the irreducible components of P do not meet, are open and are identical to the connected components). The fact that U is affine and dense in P is equivalent then to the trace of U on each of the irreducible components of P being affine and dense. Moreover, the assertion T is the support of a divisor is equivalent to the trace of T on each of the components irreducible of P is a divisor. We therefore come back to the case where P is a proper, smooth and integrated scheme. The set T is then the support of a divisor if and only if T has pure codimension equal to 1 in P. Let $\overline{k}$ be an algebraic closure of k. $P \otimes_k \overline{k}$ is a proper and smooth $\overline{k}$-scheme, $U \otimes_k \overline{k}$ is an open affine of $P \otimes_k \overline{k}$ complementary to the closed subscheme $T \otimes_k \overline{k}$. According to [E.G.A.IV.4.1.4], $\dim P = \dim P \otimes_k \overline{k}$. Let P' be an irreducible component of $P \otimes_k \overline{k}$. According to [E.G.A.IV.4.4.1.], the image of P' by the projection morphism $P \otimes_k \overline{k} \to P$ is irreducible and therefore is equal to P. As the projection morphism is quasi-compact, we therefore have $\dim P' \geq \dim P$. As we also have $\dim P' \leq \dim P \otimes_k \overline{k} = \dim P$, then the irreducible components of $P \otimes_k \overline{k}$ all have the same dimension which is equal to that of P. Similarly, if T' is an irreducible component of T then the irreducible components of $T' \otimes_k \overline{k}$ are all of the same dimension which is equal to that of T'.

Let us now show that $T \otimes_k \overline{k}$ is of pure dimension equal to $\dim(P \otimes_k \overline{k}) - 1 = \dim P - 1$. As $P \otimes_k \overline{k}$ is smooth with pure dimension equal to $\dim P$, we can further assume that the scheme is $P \otimes_k \overline{k}$ integral. According to [Har70, II.3.1], $T \otimes_k \overline{k}$ has pure codimension equal to 1. Now, according to [Har77, ex.II.3.20], $T \otimes_k \overline{k}$ is of pure codimension equal to 1 if and only if $T \otimes_k \overline{k}$ is of dimension pure $\dim P \otimes_k \overline{k} - 1$.

We have therefore shown that $T \otimes_k \overline{k}$ has dimension $\dim(P \otimes_k \overline{k}) - 1 = \dim -1$.

Finally, if T' is an irreducible component of T, as the dimension of T' is equal to that of irreducible components of $T' \otimes_k \overline{k}$ (which is a subset of the irreducible components of $T \otimes_k \overline{k}$) we therefore deduce that $\dim T' = \dim P - 1$. According to [Har77, ex.II.3.20], we deduces that T has pure codimension equal to 1. As P is smooth, T is therefore the support of a divisor. □

Proposition 17.6.1.6. With notations 17.6.1.3, suppose X_0 is integral and $P^\dagger$ is either proper or affine.

There exists then a divisor $\widetilde{T}_0$ of P_0 containing T_0 such that $\widetilde{Y}_0 := (P_0 \setminus \widetilde{T}_0) \cap X_0$ is affine dense in Y_0 and the open immersion $\widetilde{Y}_0 \hookrightarrow X_0$ ideally desingularizes.

Proof. By de Jong's desingularisation theorem ([dJ96]) and replacing Y_0 by an affine dense open subset, there exists a projective surjective morphism $a_0 \colon X_0' \to X_0$, which decomposes into a closed immersion $X_0' \hookrightarrow \mathbb{P}^r_{X_0}$ followed by the canonical projection $\mathbb{P}^r_{X_0} \to X_0$, such that

(i) X_0' is integral and smooth ;

(ii) the morphism $b_0 \colon Y_0' := a_0^{-1}(Y_0) \to Y_0$ induced by a_0 is finite and étale.

It remains to check that, shrinking Y_0 if necessary, the property (iii) of 17.6.1.1 holds. Write v_0' for the closed immersion $Y_0' \hookrightarrow \mathbb{P}^r_{Y_0}$. If $x_0, \ldots, x_r$ are the projective coordinates of $\mathbb{P}^r_k$, we write, for all integers $\alpha \in \{0, \ldots, r\}$, D_α for the divisor of $\mathbb{P}^r_k$ defined by the equation $x_\alpha = 0$ and $D_{\alpha, Y_0} := D_\alpha \times_{\mathbb{P}^r_k} \mathbb{P}^r_{Y_0}$. As the intersection of the divisors D_{α, Y_0} is empty, there exists a integer α_0 such that $v_0'(Y_0')$ is not included in D_{α_0, Y_0}. As Y_0' is integral, we obtain $\dim Y_0' \cap D_{\alpha_0, Y_0} < \dim Y_0'$. Via [Gro65, 5.4.2], the finiteness of b_0 implies $\dim b_0(Y_0' \cap D_{\alpha_0, Y_0}) = \dim Y_0' \cap D_{\alpha_0, Y_0}$. As b_0 is surjective, we also have $\dim Y_0' = \dim Y_0$ and thus $\dim b_0(Y_0' \cap D_{\alpha_0, Y_0}) < \dim Y_0$. It follows from the last inequality that there exists an affine open subset $\widetilde{U}_0$ of U_0 such that the open subset $\widetilde{Y}_0 := \widetilde{U}_0 \cap Y_0$ of Y_0 is dense and is included in $Y_0 \setminus b_0(Y_0' \cap D_{\alpha_0, Y_0})$. Set $\widetilde{T}_0 := P_0 \setminus \widetilde{U}_0$. Then using 17.6.1.5 in the case where P is proper or choosing $\widetilde{U}_0$ to be a standard open subset of P_0 in the affine case, we can suppose that $\widetilde{T}_0$ is the support of divisor of P_0.

Put $\widetilde{T}_{X_0} = \widetilde{T}_0 \cap X_0$ and $\widetilde{T}_{X_0'} := a_0^{-1}(\widetilde{T}_{X_0})$. The inclusion $b_0(Y_0' \cap D_{\alpha_0, Y_0}) \subset \widetilde{T}_{X_0}$ (resp. $X_0 \setminus \widetilde{T}_{X_0} \subset Y_0$) implies then $Y_0' \cap D_{\alpha_0, Y_0} \subset \widetilde{T}_{X_0'}$ (resp. $X_0' \setminus \widetilde{T}_{X_0}' \subset Y_0'$). From this follows the factorisation $\widetilde{Y}_0' := X_0' \setminus \widetilde{T}_{X_0'} \hookrightarrow (\mathbb{P}^r_k \setminus D_{\alpha_0}) \times (X_0 \setminus \widetilde{T}_{X_0}) = \mathbb{A}^r_{\widetilde{Y}_0}$.

As $\widetilde{Y}_0 / S_0$ (resp. $\widetilde{Y}_0'/S_0$) is smooth affine, then there exists a very smooth affine $\mathfrak{S}$-weak formal scheme $\widetilde{Y}^\dagger$ (resp. $\widetilde{Y}'^\dagger$) lifting it (see 17.1.1.20 and 17.1.1.22.(c)). We get a lifting $\widetilde{Y}'^\dagger \to \widetilde{Y}^\dagger$ of the map $\widetilde{Y}_0' \to \mathbb{A}^r_{\widetilde{Y}_0}$ (we might invoke 17.1.1.22.(c)) but here this is obvious). Thus the canonical morphism $\widetilde{Y}_0' \to \mathbb{P}^r_{X_0 \setminus \widetilde{T}_{X_0}}$ lifts to a morphism of smooth $\mathfrak{S}$-weak formal scheme of the form $\widetilde{Y}'^\dagger \to \mathbb{P}^{r\dagger}_{\widetilde{Y}^\dagger}$. Hence, the open immersion $\widetilde{Y}_0 \hookrightarrow X_0$ ideally desingularizes. □

Theorem 17.6.1.7. *With the notations of 17.5.2, we have* $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E) \in \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K)$, *for any* $E \in \mathrm{MIC}^\dagger(Y_0/K)$.

Proof. 0 Since this is local in $\mathfrak{P}$, we can suppose that Y_0, U_0 and P_0 are integral and affine. Put $\mathcal{E} := \mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0, +}(E)$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$ be such that $\underrightarrow{l}^*_{\mathbb{Q}}\ \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$.

I) Let Z_0 be a closed (reduced) subscheme of X_0. We prove, by induction on $(\dim Z_0, \mathrm{cmax} Z_0)$ (for the lexicographical order), where $\mathrm{cmax} Z_0$ denotes the number of irreducible components of Z_0 of dimension $\dim Z_0$, the following assertion:

$$\text{“ If } \mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0)), \text{ then } \mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))\text{”} \qquad (17.6.1.7.1)$$

Beware that $\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}})$ is a priori stronger than $\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$. The reason why we need this stronger property is that when we work with coherent complexes, the composition of local cohomological functors (for instance) might be problematic in categories of the form $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}})$ (contrary to overcoherent complexes).

1) Let us treat the case where $\dim Z_0 = 0$. Since $Z_0 \to S_0$ is finite and étale (see 15.3.3.3), then there exists a finite étale morphism $\mathfrak{Z} \to \mathfrak{S}$ which lift $X_0 \to S_0$. We get a lifting $u \colon \mathfrak{Z} \hookrightarrow \mathfrak{P}$ of $Z_0 \to P_0$. If $\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$, then following Berthelot-Kashiwara $u^{(\bullet)!}\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}})$ and then $\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} u_+^{(\bullet)} u^{(\bullet)!}\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ and we are done.

2) i) Suppose now $\dim Z_0 > 0$. Let C_0 be an irreducible component of Z_0 such that $\dim C_0 = \dim Z_0$. Let C_0' be the union of the other irreducible components of Z_0. If $T_0 \supset C_0$ then $\mathbb{R}\underline{\Gamma}^\dagger_{C_0} \mathcal{E}^{(\bullet)} = 0$ which implies $\mathbb{R}\underline{\Gamma}^\dagger_{C_0'} \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)}$ (use 13.1.4.15) and by using the induction hypotheses this yields $\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}})$. We therefore come back to assuming that $T_0 \not\supset C_0$. Hence, $V_0 := C_0 \cap U_0$ and $C_0 \setminus (C_0' \cup T_0)$ are dense opens of C_0. This yields, there exists a principal open subset $\widetilde{\mathfrak{U}}$ of $\mathfrak{U}$ such that $\widetilde{V}_0 := \widetilde{U}_0 \cap C_0 = \widetilde{U}_0 \cap Z_0$ and $\widetilde{V}_0$ is smooth and dense in C_0. By setting $\widetilde{T}_0 := P_0 \setminus \widetilde{U}_0$, remark $\widetilde{T}_0$ is the support of a divisor. Shrinking $\widetilde{U}_0$ if necessary, by 17.6.1.6, we can furthermore suppose the open

immersion $\widetilde{U}_0 \cap C_0 \hookrightarrow C_0$ ideally desingularizes. Denote by $\widetilde{Y}^\dagger$ (resp. $\widetilde{U}^\dagger$) the open subset of $Y^\dagger$ (resp. $U^\dagger$) complementary to $\widetilde{T}_0$ and by $\widetilde{E} := j^\dagger_{\widetilde{T}_0} E \in \mathrm{MIC}^\dagger(\widetilde{Y}_0/K)$. We set $\widetilde{\mathcal{E}}^{(\bullet)} := ({}^\dagger \widetilde{T}_0)\mathcal{E}^{(\bullet)}$ and $\widetilde{\mathcal{E}} := ({}^\dagger \widetilde{T}_0)\mathcal{E}$. We obtain $\underrightarrow{l}^*_{\mathbb{Q}} \widetilde{\mathcal{E}}^{(\bullet)} \xrightarrow{\sim} \widetilde{\mathcal{E}} \xrightarrow{\sim} \mathrm{sp}_{\widetilde{Y}^\dagger \hookrightarrow \widetilde{U}^\dagger, \widetilde{T}_0,+}(\widetilde{E})$ (see 17.5.2.10).

ii) Since by hypothesis $\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$, since $\widetilde{V}_0 := \widetilde{U}_0 \cap C_0 = \widetilde{U}_0 \cap Z_0$, then

$$\mathbb{R}\underline{\Gamma}^\dagger_{C_0}(\widetilde{\mathcal{E}}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{V}_0}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} ({}^\dagger \widetilde{T}_0)(\mathbb{R}\underline{\Gamma}^\dagger_{Z_0} \mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(\widetilde{T}_0)). \qquad (17.6.1.7.2)$$

As $\widetilde{V}_0/S_0$ is smooth affine, there exists a very smooth affine $\mathfrak{S}$-weak formal scheme $\widetilde{V}^\dagger$ lifting it (see 17.1.1.16). Choose $b\colon \widetilde{V}^\dagger \hookrightarrow \widetilde{Y}^\dagger$ a closed immersion lifting $\widetilde{V}_0 \hookrightarrow \widetilde{Y}_0$ (see 17.1.1.22.(c)). We get the commutative diagram:

$$\begin{array}{ccccc}
\widetilde{V}^\dagger & \overset{\widetilde{v}'}{\hookrightarrow} & \widetilde{U}^\dagger & \overset{j'}{\hookrightarrow} & P^\dagger \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\widetilde{Y}^\dagger & \overset{\widetilde{v}}{\hookrightarrow} & \widetilde{U}^\dagger & \overset{j}{\hookrightarrow} & P^\dagger,
\end{array} \qquad (17.6.1.7.3)$$

where $\widetilde{v}\colon \widetilde{Y}^\dagger \hookrightarrow \widetilde{U}^\dagger$ is the closed immersion induced by v and $\widetilde{v}' := \widetilde{v} \circ a$. Via 17.5.2.9, since $\widetilde{V}_0$ is dense in C_0, since following the step i) we have $\widetilde{\mathcal{E}} \xrightarrow{\sim} \mathrm{sp}_{\widetilde{Y}^\dagger \hookrightarrow \widetilde{U}^\dagger, \widetilde{T}_0,+}(\widetilde{E})$, this yields the isomorphism:

$$\mathrm{sp}_{\widetilde{V}^\dagger \hookrightarrow \widetilde{U}^\dagger, \widetilde{T}_0,+}(a^* \widetilde{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{C_0}(\widetilde{\mathcal{E}})[-d_{\widetilde{V}_0/\widetilde{Y}_0}] =: \widetilde{\mathcal{F}}. \qquad (17.6.1.7.4)$$

With 17.6.1.4, this yields $\widetilde{\mathcal{F}} \in \mathrm{MIC}^{\dagger\dagger}(C_0, \mathfrak{P}, \widetilde{T}_0/K)$, in particular $\widetilde{\mathcal{F}}$ is $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}$-overcoherent. Set $\widetilde{\mathcal{F}}^{(\bullet)} := \mathbb{R}\underline{\Gamma}^\dagger_{C_0}({}^\dagger \widetilde{\mathcal{E}}^{(\bullet)})[-d_{\widetilde{V}_0/\widetilde{Y}_0}] \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(\widetilde{T}_0))$, since $\underrightarrow{l}^*_{\mathbb{Q}} \widetilde{\mathcal{F}}^{(\bullet)} \xrightarrow{\sim} \widetilde{\mathcal{F}}$ then following 15.3.6.6 $\widetilde{\mathcal{F}}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(\widetilde{T}_0))$.

iii) Using de Jong's desingularisation theorem, we obtain the commutative diagram (use 16.1.11.1):

$$\begin{array}{ccccccc}
\widetilde{U}_0 & \longrightarrow & U_0 & \longrightarrow & P_0 & \longrightarrow & \mathfrak{P} \\
\uparrow & \square & \uparrow & \square & \uparrow & & \| \\
\widetilde{Y}_0 & \longrightarrow & Y_0 & \longrightarrow & X_0 & \longrightarrow & \mathfrak{P} \\
{\scriptstyle a}\uparrow & \square & \uparrow & \square & \uparrow & & \| \\
\widetilde{V}_0 & \longrightarrow & V_0 & \longrightarrow & C_0 & \longrightarrow & \mathfrak{P} \\
{\scriptstyle b}\uparrow & \square & \uparrow & \square & {\scriptstyle h}\uparrow & & {\scriptstyle q}\uparrow \\
\widetilde{V}'_0 & \longrightarrow & V'_0 & \longrightarrow & C'_0 & \longrightarrow & \widehat{\mathbb{P}}^N_{\mathfrak{P}},
\end{array}$$

in which the squares in the two columns on the left are cartesian, C'_0 is smooth, q is the canonical projection, h is a projective, surjective, generically finite and étale morphism, such that $h^{-1}(C_0 \cap \widetilde{T}_0)$ is a strict normal crossing divisor divisor of C'_0, the horizontal arrows of the right squares are closed immersions. Put $T'_0 := q^{-1}(T_0)$ and $\widetilde{T}'_0 := q^{-1}(\widetilde{T}_0)$. As C'_0/S_0 is smooth and $C'_0 \cap \widetilde{T}'_0$ is a divisor of C'_0, then $C'_0 \cap T_0$ is also a divisor of C'_0. Since $\mathbb{R}\underline{\Gamma}^\dagger_{C'_0} q^!(\widetilde{\mathcal{F}}) \in \mathrm{MIC}^{\dagger\dagger}(C'_0, \widehat{\mathbb{P}}^N_{\mathfrak{P}}, \widetilde{T}'_0/K)$, since C'_0/S_0 is smooth then following 12.2.2.6 there exists a (unique up to isomorphism) $\widetilde{E}' \in \mathrm{MIC}^\dagger(\widetilde{V}'_0, C'_0, \widehat{\mathbb{P}}^N_{\mathfrak{P}}/K)$ such that

$$\mathbb{R}\underline{\Gamma}^\dagger_{C'_0} q^!(\widetilde{\mathcal{F}}) \xrightarrow{\sim} \mathrm{sp}_{C'_0 \hookrightarrow \mathfrak{P}', \widetilde{T}'_0}(\widetilde{E}'). \qquad (17.6.1.7.5)$$

iv) In this step, we check that $\widetilde{E}' \in \mathrm{MIC}^\dagger(\widetilde{V}'_0, C'_0, \widehat{\mathbb{P}}^N_{\mathfrak{P}}/K)$ comes from an object of $\mathrm{MIC}^\dagger(V'_0, C'_0, \widehat{\mathbb{P}}^N_{\mathfrak{P}}/K)$.

Set $\mathfrak{U}' := \widehat{\mathbb{P}}^N_{\mathfrak{P}} \setminus T'_0$. Since V'_0/S_0 is smooth, then using the theorem of contagiosity of Kedlaya (see 16.1.8.1), we reduce to check that the restriction of $\widetilde{E}'$ on $\mathrm{MIC}^\dagger(\widetilde{V}'_0, V'_0, \mathfrak{U}'/K)$ comes from an object of $\mathrm{MIC}^\dagger(V'_0, V'_0, \mathfrak{U}'/K)$. In other words, we reduce to the case where the divisor T_0 is empty, i.e. $U_0 = P_0$, $C_0 = V_0$, $C'_0 = V'_0$ and T'_0 is empty. In that case, denoting by $G \in \mathrm{MIC}^\dagger(Y_0, Y_0/K)$ the convergent isocrystal on Y_0 associated to E, we get from 17.5.2.5 the isomorphism $\mathcal{E} \xrightarrow{\sim} \mathrm{sp}_{Y_0 \hookrightarrow \mathfrak{U}+}(G)$

of $\mathrm{MIC}^{\dagger\dagger}(Y_0,\mathfrak{U}/K)$. With 17.6.1.7.4 and 17.6.1.7.5 we get therefore the isomorphism:

$$\mathrm{sp}_{V'_0\hookrightarrow\mathfrak{U}',\widetilde{T}'_0}(\widetilde{E'}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{V'_0} q^! \mathbb{R}\underline{\Gamma}^{\dagger}_{V_0}({}^{\dagger}\widetilde{T}_0)(\mathrm{sp}_{Y_0\hookrightarrow\mathfrak{U}+}(G))[-d_{\widetilde{V}_0/\widetilde{Y}_0}]$$

$$\xrightarrow{\sim} ({}^{\dagger}\widetilde{T}'_0)\left(\mathbb{R}\underline{\Gamma}^{\dagger}_{V'_0} q^!(\mathrm{sp}_{Y_0\hookrightarrow\mathfrak{U}+}(G))[-d_{\widetilde{V}_0/\widetilde{Y}_0}]\right) \overset{16.2.4.3}{\xrightarrow{\sim}} ({}^{\dagger}\widetilde{T}'_0)(\mathrm{sp}_{V'_0\hookrightarrow\mathfrak{U}'+}(G')) \overset{16.2.4.3}{\xrightarrow{\sim}} \mathrm{sp}_{V'_0\hookrightarrow\mathfrak{U}',\widetilde{T}'_0}(j^{\dagger}_{\widetilde{T}'_0}G')$$

where $G' = h^*(G) \in \mathrm{MIC}^{\dagger}(V'_0, V'_0/K)$. Since the functor $\mathrm{sp}_{V'_0\hookrightarrow\mathfrak{U}',\widetilde{T}'_0}$ is fully faithfull, then we are done.

v) Set $\mathfrak{P}' := \widehat{\mathbb{P}}^N_{\mathfrak{P}}$. Following the step iv), there exists $E' \in \mathrm{MIC}^{\dagger}(V'_0, C'_0, \mathfrak{P}'/K)$ such that the induced object of $\mathrm{MIC}^{\dagger}(\widetilde{V}'_0, C'_0, \mathfrak{P}'/K)$ is isomorphic to $\widetilde{E}'$. Put $\mathcal{E}' := \mathrm{sp}_{C'_0\hookrightarrow\mathfrak{P}',T'_0}(E')$. Let $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'_0))$ be such that $\underrightarrow{l}^*_{\mathbb{Q}}\ \mathcal{E}'^{(\bullet)} \xrightarrow{\sim} \mathcal{E}'$. It follows from 17.6.1.7.6 the isomorphism:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{C'_0} q^!(\widetilde{\mathcal{F}}) \xrightarrow{\sim} ({}^{\dagger}\widetilde{T}'_0)(\mathcal{E}'). \tag{17.6.1.7.6}$$

Since $\widetilde{\mathcal{F}}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(\widetilde{T}_0))$, then $\mathbb{R}\underline{\Gamma}^{\dagger}_{C'_0} q^{(\bullet)!}(\widetilde{\mathcal{F}}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(\widetilde{T}'_0))$. Since $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'_0))$, then $({}^{\dagger}\widetilde{T}'_0)(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(\widetilde{T}'_0))$. Since $\underrightarrow{l}^*_{\mathbb{Q}}$ is fully faithful on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(\widetilde{T}'_0))$, then we get from 17.6.1.7.6 the isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(\widetilde{T}'_0))$:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{C'_0} q^{(\bullet)!}(\widetilde{\mathcal{F}}^{(\bullet)}) \xrightarrow{\sim} ({}^{\dagger}\widetilde{T}'_0)(\mathcal{E}'^{(\bullet)}). \tag{17.6.1.7.7}$$

Since $({}^{\dagger}\widetilde{T}'_0)(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'}(T'_0))$, then so is $\mathbb{R}\underline{\Gamma}^{\dagger}_{C'_0} q^{(\bullet)!}(\widetilde{\mathcal{F}}^{(\bullet)})$. By 15.3.6.14, this yields $q^{(\bullet)}_{+}\mathbb{R}\underline{\Gamma}^{\dagger}_{C'_0} q^{(\bullet)!}(\widetilde{\mathcal{F}}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$. By 16.1.11.2, $\widetilde{\mathcal{F}}^{(\bullet)}$ is a direct summand of $q^{(\bullet)}_{+}\mathbb{R}\underline{\Gamma}^{\dagger}_{C'_0} q^{(\bullet)!}(\widetilde{\mathcal{F}}^{(\bullet)})$. This yields $\widetilde{\mathcal{F}}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$.

vi) Write the localisation triangle in $\widetilde{T}_0$ of $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0}\mathcal{E}$ as:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0\cap\widetilde{T}_0}\mathcal{E}^{(\bullet)} \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0}\mathcal{E}^{(\bullet)} \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0}({}^{\dagger}\widetilde{T}_0)\mathcal{E}^{(\bullet)} \to +1. \tag{17.6.1.7.8}$$

The middle term of 17.6.1.7.8 is coherent by hypothesis. Moreover, via the isomorphism 17.6.1.7.2, following the step v) we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0}({}^{\dagger}\widetilde{T}_0)\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$. Hence, from 17.6.1.7.8 we get $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0\cap\widetilde{T}_0}\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$. As $\widetilde{V}_0$ is dense in C_0, then $\dim C_0\cap\widetilde{T}_0 < \dim C_0 = \dim Z_0$. Hence, either $\dim Z_0\cap\widetilde{T}_0 < \dim Z_0$ or $\dim Z_0\cap\widetilde{T}_0 = \dim Z_0$ and $\mathrm{cmax}(Z_0\cap\widetilde{T}_0) < \mathrm{cmax}(Z_0)$. By the induction hypotheses, it follows that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z_0\cap\widetilde{T}_0}\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}}(T_0))$. We conclude the induction via 17.6.1.7.8.

II) Applying 17.6.1.7.1 to the case $Z_0 = X_0$, we obtain the $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}}$-overcoherence of $\mathcal{E}$. Hence, using 17.5.2.5 we get $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K)$. □

17.6.2 The functor $\mathrm{sp}_{Y\hookrightarrow U^{\dagger},T,+}$

Let $P^{\dagger}$ be a separated and smooth $\mathfrak{S}$-weak formal scheme with special fiber P_0, let T_0 be a divisor of P_0, $U^{\dagger}$ the open set of $P^{\dagger}$ complementary to T_0, U_0 the special fiber of $U^{\dagger}$, $j\colon U^{\dagger}\hookrightarrow P^{\dagger}$ the open immersion, $v_0\colon Y_0\hookrightarrow U_0$ be a closed immersion of smooth k-schemes. We denote by X_0 the closure of Y_0 in P_0. In this subsection, using 17.1.1.22.(c) we fix the following lifting choices:

17.6.2.1. Let $(P^{\dagger}_{\alpha})_{\alpha\in\Lambda}$ be an open covering of $P^{\dagger}$. We set $P^{\dagger}_{\alpha\beta} := P^{\dagger}_{\alpha}\cap P^{\dagger}_{\beta}$, $P^{\dagger}_{\alpha\beta\gamma} := P^{\dagger}_{\alpha}\cap P^{\dagger}_{\beta}\cap P^{\dagger}_{\gamma}$, $U^{\dagger}_{\alpha\beta} := U^{\dagger}\cap P^{\dagger}_{\alpha\beta}$, $U^{\dagger}_{\alpha\beta\gamma} := U^{\dagger}\cap P^{\dagger}_{\alpha\beta\gamma}$. We denote by $Y_{\alpha 0} := Y_0\cap U_{\alpha 0}$, $Y_{\alpha\beta 0} := Y_0\cap U_{\alpha\beta 0}$, $Y_{\alpha\beta\gamma 0} := Y_0\cap U_{\alpha\beta\gamma 0}$. We set $T_{\alpha 0} := T_0\cap P_{\alpha 0}$, $T_{\alpha\beta 0} := T_0\cap P_{\alpha\beta 0}$, $T_{\alpha\beta\gamma 0} := T_0\cap P_{\alpha\beta\gamma 0}$.

We suppose that for every $\alpha\in\Lambda$, $P^{\dagger}_{\alpha}$, $U^{\dagger}_{\alpha}$ are affine and $U^{\dagger}_{\alpha}/\mathfrak{S}$ has coordinates.

For any 3uple $(\alpha,\beta,\gamma)\in\Lambda^3$, fix $Y^{\dagger}_{\alpha}$ (resp. $Y^{\dagger}_{\alpha\beta}$, $Y^{\dagger}_{\alpha\beta\gamma}$) some smooth $\mathfrak{S}$-weak formal scheme lifting $Y_{\alpha 0}$ (resp. $Y_{\alpha\beta 0}$, $Y_{\alpha\beta\gamma 0}$), $p^{\alpha\beta}_1\colon Y^{\dagger}_{\alpha\beta}\to Y^{\dagger}_{\alpha}$ (resp. $p^{\alpha\beta}_2\colon Y^{\dagger}_{\alpha\beta}\to Y^{\dagger}_{\beta}$) some lifting of the inclusions $Y_{\alpha\beta 0}\to Y_{\alpha 0}$ (resp. $Y_{\alpha\beta 0}\to Y_{\beta 0}$).

Similarly, for any $(\alpha,\beta,\gamma)\in\Lambda^3$, fix some lifting $p^{\alpha\beta\gamma}_{12}\colon Y^{\dagger}_{\alpha\beta\gamma}\to Y^{\dagger}_{\alpha\beta}$, $p^{\alpha\beta\gamma}_{23}\colon Y^{\dagger}_{\alpha\beta\gamma}\to Y^{\dagger}_{\beta\gamma}$, $p^{\alpha\beta\gamma}_{13}\colon Y^{\dagger}_{\alpha\beta\gamma}\to Y^{\dagger}_{\alpha\gamma}$, $p^{\alpha\beta\gamma}_{1}\colon Y^{\dagger}_{\alpha\beta\gamma}\to Y^{\dagger}_{\alpha}$, $p^{\alpha\beta\gamma}_{2}\colon Y^{\dagger}_{\alpha\beta\gamma}\to Y^{\dagger}_{\beta}$, $p^{\alpha\beta\gamma}_{3}\colon Y^{\dagger}_{\alpha\beta\gamma}\to Y^{\dagger}_{\gamma}$, $v_{\alpha}\colon Y^{\dagger}_{\alpha}\hookrightarrow U^{\dagger}_{\alpha}$, $v_{\alpha\beta}\colon Y^{\dagger}_{\alpha\beta}\hookrightarrow U^{\dagger}_{\alpha\beta}$ and $v_{\alpha\beta\gamma}\colon Y^{\dagger}_{\alpha\beta\gamma}\hookrightarrow U_{\alpha\beta\gamma}$.

Definition 17.6.2.2. For any $\alpha \in \Lambda$, let $\mathcal{E}_\alpha \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger_\alpha/\mathcal{V})$. A *glueing data* on $(\mathcal{E}_\alpha)_{\alpha\in\Lambda}$ is the data for any α, $\beta \in \Lambda$ of an isomorphism of $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger_{\alpha\beta}/\mathcal{V})$ of the form:

$$\theta_{\alpha\beta} \colon p_2^{\alpha\beta !}(\mathcal{E}_\beta) \xrightarrow{\sim} p_1^{\alpha\beta !}(\mathcal{E}_\alpha),$$

satisfying the cocycle condition: $\theta_{13}^{\alpha\beta\gamma} = \theta_{12}^{\alpha\beta\gamma} \circ \theta_{23}^{\alpha\beta\gamma}$, where $\theta_{12}^{\alpha\beta\gamma}$, $\theta_{23}^{\alpha\beta\gamma}$ and $\theta_{13}^{\alpha\beta\gamma}$ are the isomorphisms making commutative the following diagram

$$\begin{array}{ccc}
p_{12}^{\alpha\beta\gamma !} p_2^{\alpha\beta !}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\tau} & p_2^{\alpha\beta\gamma !}(\mathcal{E}_\beta) \\
\sim\downarrow p_{12}^{\alpha\beta\gamma !}(\theta_{\alpha\beta}) & & \downarrow \theta_{12}^{\alpha\beta\gamma} \\
p_{12}^{\alpha\beta\gamma !} p_1^{\alpha\beta !}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\tau} & p_1^{\alpha\beta\gamma !}(\mathcal{E}_\alpha),
\end{array}
\quad
\begin{array}{ccc}
p_{23}^{\alpha\beta\gamma !} p_2^{\beta\gamma !}(\mathcal{E}_\gamma) & \xrightarrow[\sim]{\tau} & p_3^{\alpha\beta\gamma !}(\mathcal{E}_\gamma) \\
\sim\downarrow p_{23}^{\alpha\beta\gamma !}(\theta_{\beta\gamma}) & & \downarrow \theta_{23}^{\alpha\beta\gamma} \\
p_{23}^{\alpha\beta\gamma !} p_1^{\beta\gamma !}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\tau} & p_2^{\alpha\beta\gamma !}(\mathcal{E}_\beta),
\end{array}
\quad
\begin{array}{ccc}
p_{13}^{\alpha\beta\gamma !} p_2^{\alpha\gamma !}(\mathcal{E}_\gamma) & \xrightarrow[\sim]{\tau} & p_3^{\alpha\beta\gamma !}(\mathcal{E}_\gamma) \\
\sim\downarrow p_{13}^{\alpha\beta\gamma !}(\theta_{\alpha\gamma}) & & \downarrow \theta_{13}^{\alpha\beta\gamma} \\
p_{13}^{\alpha\beta\gamma !} p_1^{\alpha\gamma !}(\mathcal{E}_\alpha) & \xrightarrow[\sim]{\tau} & p_1^{\alpha\beta\gamma !}(\mathcal{E}_\alpha),
\end{array}
\tag{17.6.2.2.1}$$

where τ are the glueing isomorphisms defined at 17.5.1.3.

Definition 17.6.2.3. We define the category $\mathrm{MIC}^{\dagger\dagger}((Y^\dagger_\alpha)_{\alpha\in\Lambda}/\mathcal{V})$ as follows:

(a) an object is a family $(\mathcal{E}_\alpha)_{\alpha\in\Lambda}$ of objects $\mathcal{E}_\alpha \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger_\alpha/\mathcal{V})$ for each $\alpha \in \Lambda$ together with a glueing data $(\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}$,

(b) a morphism $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \to ((\mathcal{E}'_\alpha)_{\alpha\in\Lambda}, (\theta'_{\alpha\beta})_{\alpha,\beta\in\Lambda})$ is a family of morphisms $f_\alpha \colon \mathcal{E}_\alpha \to \mathcal{E}'_\alpha$ of $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger_\alpha/\mathcal{V})$ commuting with glueing data, i.e., such that the following diagrams are commutative:

$$\begin{array}{ccc}
p_2^{\alpha\beta !}(\mathcal{E}_\beta) & \xrightarrow[\sim]{\theta_{\alpha\beta}} & p_1^{\alpha\beta !}(\mathcal{E}_\alpha) \\
p_2^{\alpha\beta !}(f_\beta)\downarrow & & \downarrow p_1^{\alpha\beta !}(f_\alpha) \\
p_2^{\alpha\beta !}(\mathcal{E}'_\beta) & \xrightarrow[\sim]{\theta'_{\alpha\beta}} & p_1^{\alpha\beta !}(\mathcal{E}'_\alpha).
\end{array}
\tag{17.6.2.3.1}$$

17.6.2.4. It follows from 17.5.1.1 that we get an equivalence of categories $\mathrm{sp}_{Y_0 *} \colon \mathrm{MIC}^\dagger(Y/K) \to \mathrm{MIC}^{\dagger\dagger}((Y^\dagger_\alpha)_{\alpha\in\Lambda}/\mathcal{V})$ by setting $\mathrm{sp}_{Y_0 *}(E) := ((\mathrm{sp}_{Y^\dagger_\alpha *}(E|Y_{\alpha 0}))_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}))$, where

$$\theta_{\alpha\beta} \colon p_2^{\alpha\beta !}(\mathcal{E}_\beta) = p_2^{\alpha\beta !}(\mathrm{sp}_{Y^\dagger_\beta *}(E|Y_{\beta 0})) \overset{17.5.1.3.1}{\xrightarrow{\sim}} \mathrm{sp}_{Y^\dagger_{\alpha\beta} *}(E|Y_{\alpha\beta 0})) \overset{17.5.1.3.1}{\xrightarrow{\sim}} p_1^{\alpha\beta !}(\mathrm{sp}_{Y^\dagger_\alpha *}(E|Y_{\alpha 0})) = p_1^{\alpha\beta !}(\mathcal{E}_\alpha),$$

17.6.2.5. We define the functor $\widetilde{\mathrm{sp}}_{Y_0 \hookrightarrow U^\dagger, T, +} \colon \mathrm{MIC}^{\dagger\dagger}((Y^\dagger_\alpha)_{\alpha\in\Lambda}/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K)$ as follows: Let $((\mathcal{E}_\alpha)_{\alpha\in\Lambda}, (\theta_{\alpha\beta})_{\alpha,\beta\in\Lambda}) \in \mathrm{MIC}^{\dagger\dagger}((Y^\dagger_\alpha)_{\alpha\in\Lambda}/\mathcal{V})$. Let $w_1^{\alpha\beta} \colon U^\dagger_{\alpha\beta} \cap Y^\dagger_\alpha \hookrightarrow U^\dagger_{\alpha\beta}$ be the the restriction of v_α. The morphism $w_1^{\alpha\beta}$ is a closed immersion of affine and smooth $\mathfrak{S}$-weak formal scheme which is a lifting of $Y_{\alpha\beta 0} \hookrightarrow U_{\alpha\beta 0}$. Since $p_1^{\alpha\beta} \colon Y^\dagger_{\alpha\beta} \to Y^\dagger_\alpha$ is a lifting of the open inclusion $Y_{\alpha\beta 0} \subset Y_{\alpha 0}$ then $p_1^{\alpha\beta}$ factor through $q_1^{\alpha\beta} \colon Y^\dagger_{\alpha\beta} \to Y^\dagger_\alpha \cap U^\dagger_{\alpha\beta}$. Setting $v_1^{\alpha\beta} := w_1^{\alpha\beta} \circ q_1^{\alpha\beta}$ and $j_1^{\alpha\beta} \colon U^\dagger_{\alpha\beta} \subset U^\dagger_\alpha$, we get the equality $j_1^{\alpha\beta} \circ v_1^{\alpha\beta} = u_\alpha \circ v_1^{\alpha\beta}$. Hence, it follows from 17.5.1.3 and 17.5.2.9 the first isomorphism:

$$\widetilde{\mathrm{sp}}_{Y^\dagger_\alpha \underset{v_\alpha}{\hookrightarrow} U^\dagger_\alpha, T_\alpha, +}(\mathcal{E}_\alpha)|\mathfrak{P}_{\alpha\beta} \xrightarrow{\sim} \widetilde{\mathrm{sp}}_{Y^\dagger_{\alpha\beta} \underset{v_1^{\alpha\beta}}{\hookrightarrow} U^\dagger_{\alpha\beta}, T_{\alpha\beta}, +}(p_1^{\alpha\beta !}(\mathcal{E}_\alpha)) \overset{17.5.2.12}{\xrightarrow{\sim}} \widetilde{\mathrm{sp}}_{Y^\dagger_{\alpha\beta} \underset{v_{\alpha\beta}}{\hookrightarrow} U^\dagger_{\alpha\beta}, T_{\alpha\beta}, +}(p_1^{\alpha\beta !}(\mathcal{E}_\alpha)).
\tag{17.6.2.5.1}$$

Similarly to 17.6.2.5.1, we get the first isomorphism:

$$\tau_{\alpha\beta} \colon \widetilde{\mathrm{sp}}_{Y^\dagger_\beta \underset{v_\beta}{\hookrightarrow} U^\dagger_\beta, T_\beta, +}(\mathcal{E}_\beta)|\mathfrak{P}_{\alpha\beta} \xrightarrow{\sim} \widetilde{\mathrm{sp}}_{Y^\dagger_{\alpha\beta} \underset{v_{\alpha\beta}}{\hookrightarrow} U^\dagger_{\alpha\beta}, T_{\alpha\beta}, +}(p_2^{\alpha\beta !}(\mathcal{E}_\beta))$$

$$\overset{\theta_{\alpha\beta}}{\xrightarrow{\sim}} \widetilde{\mathrm{sp}}_{Y^\dagger_{\alpha\beta} \underset{v_{\alpha\beta}}{\hookrightarrow} U^\dagger_{\alpha\beta}, T_{\alpha\beta}, +}(p_1^{\alpha\beta !}(\mathcal{E}_\alpha)) \overset{17.6.2.5.1}{\xrightarrow{\sim}} \widetilde{\mathrm{sp}}_{Y^\dagger_\alpha \underset{v_\alpha}{\hookrightarrow} U^\dagger_\alpha, T_\alpha, +}(\mathcal{E}_\alpha)|\mathfrak{P}_{\alpha\beta}.
\tag{17.6.2.5.2}$$

It follows from the commutativity of the diagram 17.5.2.12.2 that so is the following one:

$$\begin{array}{ccccccc}
\widetilde{\mathrm{sp}}_{Y^\dagger_\beta \underset{v_\beta}{\hookrightarrow} U^\dagger_\beta, T_\beta, +}(\mathcal{E}_\beta)|\mathfrak{U}_{\alpha\beta} & & \xrightarrow{\qquad\qquad\tau_{\alpha\beta}\qquad\qquad} & & & & \widetilde{\mathrm{sp}}_{Y^\dagger_\alpha \underset{v_\alpha}{\hookrightarrow} U^\dagger_\alpha, T_\alpha, +}(\mathcal{E}_\alpha)|\mathfrak{U}_{\alpha\beta} \\
\downarrow 17.5.2.5 & & & & & & \downarrow 17.5.2.5 \\
\widehat{v}_{\beta +}(\widehat{\mathcal{E}}_\beta) & \xrightarrow{\sim} & \widehat{v}_{\alpha\beta +}(\widehat{p}_2^{\alpha\beta !}(\widehat{\mathcal{E}}_\beta)) & \xrightarrow[\theta_{\alpha\beta}]{\sim} & \widehat{v}_{\alpha\beta +}(\widehat{p}_1^{\alpha\beta !}(\widehat{\mathcal{E}}_\alpha)) & \xrightarrow{\sim} & \widehat{v}_{\alpha +}(\widehat{\mathcal{E}}_\alpha),
\end{array}
\tag{17.6.2.5.3}$$

where the composition of the bottom morphisms is the morphism $\tau_{\alpha\beta}$ constructed at 9.3.7.6.2. Hence, since the family $(\widehat{v}_{\alpha +}(\widehat{\mathcal{E}}_\alpha))_{\alpha\in\Lambda}$ glues (see 9.3.7.6) then so is $(\widetilde{\mathrm{sp}}_{Y^\dagger_\alpha \underset{v_\alpha}{\hookrightarrow} U^\dagger_\alpha, T_\alpha,+}(\mathcal{E}_\alpha))_{\alpha\in\Lambda}$.

17.6.2.6. By setting $\mathrm{sp}_{Y_0\hookrightarrow U^\dagger,T_0,+} := \widetilde{\mathrm{sp}}_{Y_0\hookrightarrow U^\dagger,T_0,+}\circ \mathrm{sp}_{Y_0*}$, we get the functor

$$\mathrm{sp}_{Y_0\hookrightarrow U^\dagger,T_0,+}\colon \mathrm{MIC}^\dagger(Y_0/K)\to \mathrm{MIC}^{\dagger\dagger}(X_0,\mathfrak{P},T_0/K). \qquad (17.6.2.6.1)$$

Remark 17.6.2.7. The functor $\mathrm{sp}_{Y^\dagger\hookrightarrow U^\dagger,T,+}$ is the one we will need to check main Theorem 17.7.4.6 of the whole chapter, i.e. it is be used when Y_0 has a smooth lifting. However, the proof of the comparison between both functors sp_+ (see 17.6.3.2) requires a priori to slightly extend the functor $\mathrm{sp}_{Y^\dagger\hookrightarrow U^\dagger,T,+}$ by glueing in the context of 17.6.2.6.1.

Proposition 17.6.2.8. *Consider the commutative diagram :*

$$\begin{array}{ccccc} Y'_0 & \overset{v'}{\hookrightarrow} & U'^\dagger & \overset{j'}{\hookrightarrow} & P'^\dagger \\ \downarrow b & & \downarrow g & & \downarrow f \\ Y_0 & \overset{v}{\hookrightarrow} & U^\dagger & \overset{j}{\hookrightarrow} & P^\dagger, \end{array} \qquad (17.6.2.8.1)$$

where f and g are smooth morphisms of separated and smooth $\mathfrak{S}$-weak formal schemes, b is a morphism of smooth k-varieties, j and j' are open immersions, v and v' are closed immersions. We suppose there exists a divisor T_0 of P_0 (resp. T'_0 of P'_0) such that $U^\dagger = P^\dagger\setminus T_0$ (resp. $U'^\dagger = P'^\dagger\setminus T'$). Denote by X'_0 the closure of Y'_0 in P'_0. Then we have, for any $E\in\mathrm{MIC}^\dagger(Y/K)$, the canonical isomorphism

$$\mathrm{sp}_{Y'_0\hookrightarrow U'^\dagger,T',+}\circ b^*(E)\xrightarrow{\sim}\mathbb{R}\underline{\Gamma}^\dagger_{Y'_0}\circ f^!\circ \mathrm{sp}_{Y_0\hookrightarrow U^\dagger,T,+}(E)[-d_{Y'_0/Y_0}]. \qquad (17.6.2.8.2)$$

Proof. This is a consequence by glueing of 17.5.2.10 and 17.5.2.9. □

17.6.3 Comparison between $\mathrm{sp}_{Y\hookrightarrow U^\dagger,T,+}$ and $\mathrm{sp}_{X\hookrightarrow\mathfrak{P},T,+}$

Lemma 17.6.3.1. *With the notations of 17.5.2, suppose $U^\dagger$ is affine. Let $\mathcal{M}\in\mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$. Set $M := \Gamma(Y^\dagger,\mathcal{M})\in\mathrm{MIC}^\dagger(A^\dagger_K/K)$ (see 17.5.1.1), $D_{Y^\dagger} := \Gamma(Y^\dagger,\mathcal{D}_{Y^\dagger})$ $D_{U^\dagger\leftarrow Y^\dagger} = \Gamma(Y^\dagger,\mathcal{D}_{U^\dagger\leftarrow Y^\dagger})$, $v_+(M) := D_{U^\dagger\leftarrow Y^\dagger,K}\otimes_{D_{Y^\dagger,K}} M$. We have the canonical isomorphism*

$$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}\otimes_{D_{U^\dagger,K}} v_+(M)\xrightarrow{\sim}\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}\otimes_{j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}} j_*v_+(\mathcal{M}). \qquad (17.6.3.1.1)$$

Proof. We have, for any left $D_{U^\dagger}$-module M, the functorial in M morphism of the form

$$j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} M\to j_*(\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} M). \qquad (17.6.3.1.2)$$

When M is a free $D_{U^\dagger,K}$-module, this is clearly an isomorphism (when M is of finite type this is obvious, otherwise because of the coherence of $U^\dagger$ both functors commute with filtered inductive limits: use[SGA4.2, VI.5.3]). Following 17.2.2.13, the functors $j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} -$ and $j_*(\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} -)$ are right exact on the abelian category of left $D_{U^\dagger,K}$-modules. Hence, by the using the five lemma, this yields the morphism 17.6.3.1.2 is an isomorphism. In particular we get the canonical isomorphism $j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} v_+(M)\xrightarrow{\sim} j_*(\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} v_+(M))$.

Moreover, it follows from 17.2.5.4 (switching from the right to the left module), we have the isomorphism: $\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} v_+(M)\xrightarrow{\sim} v_+(\mathcal{M})$. Hence, $j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}\otimes_{D_{U^\dagger,K}} v_+(M)\xrightarrow{\sim} j_*v_+(\mathcal{M})$. By applying the functor $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}}\otimes_{j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}} -$ to this latter isomorphism, we get 17.6.3.1.1. □

The following proposition 17.6.3.2 will be improved later in 17.6.3.3.

Proposition 17.6.3.2. *With notation 17.6.2, the diagram*

$$\begin{array}{ccc} \mathrm{MIC}^\dagger(Y_0/K) & \xrightarrow{|(Y_0,X_0)} & \mathrm{MIC}^\dagger(Y_0,X_0/K) \\ & {\scriptstyle \mathrm{sp}_{Y_0\hookrightarrow U^\dagger,T,+}}\searrow \qquad \swarrow{\scriptstyle \mathrm{sp}_{X_0\hookrightarrow\mathfrak{P},T_0,+}}\ \cong & \\ & \mathrm{MIC}^{\dagger\dagger}(Y_0,X_0/K) & \end{array} \qquad (17.6.3.2.1)$$

where $|(Y_0,X_0)$ is the canonical restriction, is commutative up to canonical isomorphism.

Proof. I) Firstly, suppose $P^\dagger$ affine and T_0 defined by a equation local and $Z_0 := X_0 \cap T_0$ is a divisor of X_0. Let $U^\dagger$ be the open of $P^\dagger$ which is topologically U_0 and let $j\colon U^\dagger \subset P^\dagger$ be the inclusion. Moreover, let $u\colon X^\dagger \hookrightarrow P^\dagger$ be the closed immersion of affine smooth $\mathfrak{S}$-weak formal schemes which is a lifting of $X_0 \hookrightarrow P_0$. Let $v\colon Y^\dagger \hookrightarrow U^\dagger$ be the induced by u closed immersion. Let $\alpha\colon Y^\dagger \hookrightarrow X^\dagger$ be the open inclusion. Following 17.4.1.1, we have the injections $j_*\mathcal{D}_{Y^\dagger,\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}}$ and $j_*\mathcal{O}_{Y^\dagger,\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}}$. Let $E \in \mathrm{MIC}^\dagger(Y_0/K)$ and with 17.5.1.1.(e) let $\mathcal{M} := \mathrm{sp}_{Y^\dagger *}(E) \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$ (we have changed the letter in order to avoid confusion between $\Gamma(Y^\dagger, \mathcal{M})$ and E).

1) We get the commutative diagram up to canonical isomorphism:

$$\begin{array}{ccc}
\mathrm{MIC}^\dagger(Y_0/K) & \xrightarrow[\cong]{\mathrm{sp}_{Y^\dagger *}} & \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S}) \\
\downarrow{\scriptstyle |(Y_0,X_0)} & & \downarrow{\scriptstyle \mathcal{O}_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{O}_{Y^\dagger,\mathbb{Q}}} \alpha_* -} \\
\mathrm{MIC}^\dagger(Y_0, X_0, \mathfrak{X}/K) & \xrightarrow[\cong]{\mathrm{sp}_{\mathfrak{X}*}} & \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{X}, Z_0/K).
\end{array} \tag{17.6.3.2.2}$$

By construction, we have: $\mathrm{sp}_{X_0 \hookrightarrow \mathfrak{P}, T_0,+}(E|(Y_0, X_0)) = u^\dagger_+ \mathrm{sp}_{\mathfrak{X}*}(E|(Y_0, X_0))$. Since $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}}$-module whose restriction over $\mathfrak{Y}$ belongs to $\mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathfrak{S})$ (see the corollary 17.5.1.10), then $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M}) \in \mathrm{MIC}^{\dagger\dagger}(X, \mathfrak{X}, Z/K)$ (see 12.2.1.5). In particular, this latter is $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}}$-coherent. Since the canonical arrow

$$\mathcal{O}_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{O}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M}) \to \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M})$$

is a morphism of coherent $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}}$-modules which is an isomorphism outside Z_0, then this is an isomorphism. This yields the canonical isomorphism:

$$\mathrm{sp}_{X_0 \hookrightarrow \mathfrak{P}, T_0,+}(E|(Y_0, X_0)) \xrightarrow{\sim} u^\dagger_+ \left(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z_0)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M}) \right). \tag{17.6.3.2.3}$$

2) Using 17.6.3.1 and with its notations, we get the isomorphism: $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, Z_0,+}(E) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \otimes_{D_{U^\dagger,\mathbb{Q}}} v_+(M)$. Using the canonical map $D_{U^\dagger \leftarrow Y^\dagger,\mathbb{Q}} \to \Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{P} \leftarrow \mathfrak{X}}({}^\dagger T_0)_{\mathbb{Q}})$, we get the arrow

$$v_+(M) \to u^\dagger_+(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M}))$$

This yields by extension the canonical morphism

$$\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T,+}(E) \to u^\dagger_+(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z)_{\mathbb{Q}} \otimes_{\alpha_*\mathcal{D}_{Y^\dagger,\mathbb{Q}}} \alpha_*(\mathcal{M})). \tag{17.6.3.2.4}$$

This arrow 17.6.3.2.4 is a morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules whose restriction outside T is an isomorphism (this is a consequence of 17.5.1.10 and 17.5.2.5). Following 8.7.6.11, this implies that the morphism 17.6.3.2.4 is in fact an isomorphism. Hence we are done for the first case.

II) Let us now prove the general case. Let $(P^\dagger_\alpha)_{\alpha \in \Lambda}$ be an affine open covering of $P^\dagger$ such that $T \cap P_\alpha$ is defined by a equation. We denote by $U^\dagger_\alpha := P^\dagger_\alpha \cap U^\dagger$, $Y_{\alpha 0} := P_{\alpha 0} \cap Y_0$, $T_{\alpha 0} := P_{\alpha 0} \cap T_0$, $X_{\alpha 0} := P_\alpha \cap X_0$. Since X_{α_0} is affine and smooth, using the lifting theorem 17.1.1.20, there exists an $\mathfrak{S}$-formal scheme weak affine and smooth $X^\dagger_\alpha$ which is a lifting of $X_{\alpha 0}$. We denote by $j_\alpha\colon P^\dagger_\alpha \subset P^\dagger$ the canonical open immersion. Following 17.1.1.22.(c), there exist $u_\alpha\colon X^\dagger_\alpha \hookrightarrow P^\dagger_\alpha$ a closed immersion which is a lifting of $X_{\alpha 0} \hookrightarrow P_{\alpha 0}$. Since $U^\dagger_\alpha$ is a principal open set of $P^\dagger_\alpha$, then the open set $X^\dagger_\alpha \cap U^\dagger_\alpha$ of $X^\dagger_\alpha$ is also principal. In particular, it is affine. Setting $Y^\dagger_\alpha := X^\dagger_\alpha \cap U^\dagger_\alpha$, we get the open immersion $\alpha_\alpha\colon Y^\dagger_\alpha \hookrightarrow X^\dagger_\alpha$ and the closed immersion $v_\alpha\colon Y^\dagger_\alpha \hookrightarrow U^\dagger_\alpha$ such that $u_\alpha \circ \lambda_\alpha = j_\alpha \circ v_\alpha$. Similarly, we choose lifting of $X_{\alpha\beta 0}$ and $X_{\alpha\beta\gamma 0}$ and we reduce by glueing the general case from the first treated case of I). ☐

Theorem 17.6.3.3. *With notation 17.6.2, the diagram*

$$\begin{array}{ccc}
\mathrm{MIC}^\dagger(Y_0/K) & \longrightarrow & \mathrm{MIC}^\dagger(X_0, \mathfrak{P}, T_0/K) \\
& {\scriptstyle \mathrm{sp}_{Y_0 \hookrightarrow U^\dagger, T_0,+}} \searrow \quad \swarrow {\scriptstyle \mathrm{sp}_{X_0 \hookrightarrow \mathcal{P}, T_0,+}}\ \cong & \\
& \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K) &
\end{array} \tag{17.6.3.3.1}$$

is commutative up to canonical isomorphism.

Proof. Let $E \in \mathrm{MIC}^\dagger(Y_0/K)$ and $E|(Y_0, X_0)$ the induced object of $\mathrm{MIC}^\dagger(Y_0, X_0/K) = \mathrm{MIC}^\dagger(X_0, \mathfrak{P}, T_0/K)$. With the lemma 16.1.2.9 (and the remark 16.1.2.10) we reduce to the case where X_0 is integral and Y_0 is dense in X_0. Via the theorem of desingularisation of de Jong (and with the help of the lemma 16.1.6.10), there exist a divisor $\widetilde{T}_0$ containing T_0 and a diagram of the form

$$\begin{array}{ccccc} Y_0^{(0)} & \xrightarrow{j_0^{(0)}} & X_0^{(0)} & \xrightarrow{u_0^{(0)}} & P_0^{(0)\dagger} \\ \downarrow b_0 & \square & \downarrow a_0 & & \downarrow f \\ Y_0 & \xrightarrow{j_0} & X_0 & \xrightarrow{u_0} & P^\dagger, \end{array} \tag{17.6.3.3.2}$$

where the left square is cartesian, f is a proper and smooth morphism of separated and smooth $\mathfrak{S}$-weak formal schemes, a_0 is a proper, surjective and generically finite étale morphism of k-varieties with $X_0^{(0)}$ smooth, j_0 and $j_0^{(0)}$ are open immersions, u_0 and $u_0^{(0)}$ are closed immersions. We have the canonical isomorphisms:

$$\mathrm{sp}_{Y^{(0)} \hookrightarrow U^{(0)\dagger}, T^{(0)},+} \circ b^*(E) \overset{17.6.2.8}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^\dagger_{Y^{(0)}} \circ f^! \circ \mathrm{sp}_{Y \hookrightarrow U^\dagger, T,+}(E);$$

$$\mathrm{sp}_{X^{(0)} \hookrightarrow \mathcal{P}^{(0)}, T^{(0)},+}(a^*(E|(Y,X))) \overset{16.1.8.6.1}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^\dagger_{Y^{(0)}} \circ f^! \mathrm{sp}_{X \hookrightarrow \mathcal{P}, T,+}(E|(Y,X)). \tag{17.6.3.3.3}$$

As $X^{(0)}$ is smooth, then following 17.6.3.2, the left terms of 17.6.3.3.3 are canonically isomorphic. Then so are similarly right terms. We get moreover the isomorphisms:

$$\mathrm{sp}_{Y_0 \hookrightarrow U^\dagger, T_0,+}(E)|\mathfrak{U} \xrightarrow{\sim} \mathrm{sp}_{Y_0 \hookrightarrow U^\dagger, \emptyset,+}(E) \overset{17.6.3.2}{\xrightarrow{\sim}} \mathrm{sp}_{X_0 \hookrightarrow \mathfrak{U}, \emptyset,+}(E|(Y_0, Y_0)) \xrightarrow{\sim} \mathrm{sp}_{X_0 \hookrightarrow \mathcal{P}, T_0,+}(E|(Y_0, X_0))|\mathfrak{U}.$$

Thanks to the fully faithfulness theorem 16.1.5.2, this yields the result. □

Corollary 17.6.3.4. *With notation 17.6.2, we suppose P_0 proper. The functor $\mathrm{sp}_{Y_0 \hookrightarrow U^\dagger, T_0,+}$ induced then the equivalence of categories:*

$$\mathrm{sp}_{Y_0 \hookrightarrow U^\dagger, T_0,+} \colon \mathrm{MIC}^\dagger(Y/K) \cong \mathrm{MIC}^{\dagger\dagger}(Y_0/K). \tag{17.6.3.4.1}$$

Proof. This is a consequence of 16.1.8.4 and 17.6.3.3. □

17.6.4 Commutation of the tensor product with sp_+ in the weakly smooth case

17.6.4.1. Let $P^\dagger$, $P'^\dagger$ be two smooth weak formal $\mathcal{V}$-schemes and separated, T_0 (resp. T'_0) a divisor of P_0 (resp. P'_0), $U^\dagger$ (resp. $U'^\dagger$) the open of $P^\dagger$ (resp. $P'^\dagger$) complementary to T_0 (resp. T'_0), $j \colon U^\dagger \hookrightarrow P^\dagger$ (resp. $j' \colon U'^\dagger \hookrightarrow P'^\dagger$) the open immersion and $v \colon Y_0 \hookrightarrow U_0$ (resp. $v' \colon Y'_0 \hookrightarrow U'_0$) a closed immersion of k-schemes smooth. We denote by $P''^\dagger := P^\dagger \times P'^\dagger$, $U''^\dagger := U^\dagger \times U'^\dagger$, T''_0 the divisor reduced of P''_0 of space topological $P''_0 \setminus U''_0$, $Y''_0 := Y_0 \times Y'_0$, $b \colon Y''_0 \to Y_0$ and $b' \colon Y''_0 \to Y'_0$ the canonical projections.

Let $E \in \mathrm{MIC}^\dagger(Y_0/K)$ and $E' \in \mathrm{MIC}^\dagger(Y'_0/K)$. We have the canonical functors $b^* \colon \mathrm{MIC}^\dagger(Y_0/K) \to \mathrm{MIC}^\dagger(Y''_0/K)$ and $b'^* \colon \mathrm{MIC}^\dagger(Y'_0/K) \to \mathrm{MIC}^\dagger(Y''_0/K)$ (see [Ber96b, 2.3.6] and [Car07, 1.4.1])). The exterior tensor product of E and E' is defined by setting $E \boxtimes E' := b^*(E) \otimes b'^*(E')$, which gives the bifunctor

$$- \boxtimes - \colon \mathrm{MIC}^\dagger(Y_0/K) \times \mathrm{MIC}^\dagger(Y'_0/K) \to \mathrm{MIC}^\dagger(Y''_0/K).$$

Proposition 17.6.4.2. *With the notations 17.6.4.1, we have a canonical isomorphism*

$$\mathrm{sp}_{Y''_0 \hookrightarrow U''^\dagger, T''_0+}(E \boxtimes E') \xrightarrow{\sim} \mathrm{sp}_{Y_0 \hookrightarrow U^\dagger,+}(E) \overset{\mathbb{L}}{\boxtimes}{}^\dagger_{\mathcal{O}_{\mathfrak{S}}, T_0, T'_0} \mathrm{sp}_{Y'_0 \hookrightarrow U'^\dagger,'+}(E'). \tag{17.6.4.2.1}$$

Proof. Let us denote by X_0 (resp. X'_0, resp. X''_0) the closure of Y_0 in P_0 (resp. of Y'_0 in P'_0, resp. of Y''_0 in P''_0). By using 16.2.6.5, the right square of the canonical diagram

$$\begin{array}{ccccc} \mathrm{MIC}^\dagger(Y_0/K) \times \mathrm{MIC}^\dagger(Y'_0/K) & \longrightarrow & \mathrm{MIC}^\dagger(X_0, \mathfrak{P}, T_0/K) \times \mathrm{MIC}^\dagger(\mathcal{P}', T'_0, X'_0/K) & \xrightarrow{\mathrm{sp}_+ \times \mathrm{sp}_+} & \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K) \times \mathrm{MIC}^\dagger(\mathcal{P}', T'_0, X'_0/K \\ \downarrow -\boxtimes- & & \downarrow -\boxtimes- & & \downarrow -\boxtimes- \\ \mathrm{MIC}^\dagger(Y''_0/K) & \longrightarrow & \mathrm{MIC}^\dagger(\mathcal{P}'', T''_0, X''_0/K) & \xrightarrow{\mathrm{sp}_{X''_0 \hookrightarrow \mathcal{P}'', T''_0,+}} & \mathrm{MIC}^{\dagger\dagger}(\mathcal{P}'', T''_0, X''_0/K). \end{array} \tag{17.6.4.2.2}$$

is commutative, up to canonical isomorphism. As the left square is also commutative, then the theorem 17.6.3.3 allow us to conclude. □

Proposition 17.6.4.3. *With the notations of 17.6.4.1, we suppose moreover on $P^\dagger = P'^\dagger$ and $Y_0 \cap Y'_0$ smooth (e.g., $Y_0 = Y'_0$). By denoting $i^*\colon \mathrm{MIC}^\dagger(Y_0/K) \to \mathrm{MIC}^\dagger(Y_0 \cap Y'_0/K)$ and $i'^*\colon \mathrm{MIC}^\dagger(Y'_0/K) \to \mathrm{MIC}^\dagger(Y_0 \cap Y'_0/K)$ the canonical functors, then we have the canonical isomorphism:*

$$\mathrm{sp}_{Y_0\cap Y'_0\hookrightarrow U^\dagger\cap U'^\dagger, T_0\cup T'_0,+}(i^*(E)\otimes i'^*(E')) \tag{17.6.4.3.1}$$

$$\xrightarrow{\sim} ({}^\dagger T_0 \cup T'_0)\circ \mathrm{sp}_{Y_0\hookrightarrow U^\dagger,+}(E)\overset{\mathbb{L}}{\otimes}{}^\dagger_{\mathcal{O}_{\mathcal{P}_0}({}^\dagger T_0\cup T'_0)_{\mathbb{Q}}}({}^\dagger T_0\cup T'_0)\circ \mathrm{sp}_{Y'_0\hookrightarrow U'^\dagger,'+}(E')[d_{Y_0}+d_{Y'_0}-d_{Y_0\cap Y'_0}-d_{P_0}]. \tag{17.6.4.3.2}$$

Proof. This is a consequence of 16.2.6.6 and of 17.6.3.3. □

17.7 Application: characterization of overconvergent isocrystals on certain subschemes of the affine space

The purpose of this section is to prove Theorem 17.7.4.6 which gives a sufficient condition for that an arithmetical $\mathcal{D}$-module to be an isocrystal overconvergent. This is the main ingredient in the proof of the stability of the holonomicity for projective and smooth $\mathcal{S}$-formal schemes (see the proof of Theorem 18.3.3.1). Roughly speaking, on certain subschemes of projective space, it the arithmetical $\mathcal{D}$-module comes from a convergent isocrystal outside the singularities then it comes from an overconvergent isocrystal. Set $D^{(m)} := \Gamma(\mathbb{A}^n_k, \mathcal{D}^{(m)}_{\mathbb{A}^{n\dagger}_{\mathcal{V}}})$ and $D := \Gamma(\mathbb{A}^n_k, \mathcal{D}_{\mathbb{A}^{n\dagger}_{\mathcal{V}}})$.

17.7.1 p-adic weak completion of non-commutative $\mathcal{V}$-algebra

Let us recall the definition of the p-adic weak completion given by Noot-Huyghe in [Huy03, 1.3] in the case of a not necessarily commutative $\mathcal{V}$-algebra:

Definition 17.7.1.1. For any integer N, we denote by B_N the non-commutative $\mathcal{V}$-algebra of polynomials with N variables with coefficients in $\mathcal{V}$ (i.e. the tensor $\mathcal{V}$-algebra of $\mathcal{V}^N$). Let A be a not necessarily commutative $\mathcal{V}$-algebra. We denote by $\widehat{A}$ the p-adic completion of A and $A^\dagger$ the subset of $\widehat{A}$ of elements z such that there exist a constant $c \in \mathbb{R}$, some elements $x_1, \ldots, x_n \in \mathrm{Im}(A \to \widehat{A})$ and, for any $j \in \mathbb{N}$, some polynomials $P_j \in \pi^j B_n$ such that $\deg P_j \le c(j+1)$ and

$$z = \sum_{j\in\mathbb{N}} P_j(x_1, \ldots, x_n). \tag{17.7.1.1.1}$$

The set $A^\dagger$ is a $\mathcal{V}$-subalgebra of $\widehat{A}$ and is called "the p-adic weak completion of A as $\mathcal{V}$-algebra " or "the p-adic weak completion of A" if there is no ambiguity with basis $\mathcal{V}$. We also say that "z is weakly completely generated on $\mathcal{V}$ by the elements $x_1, \ldots, x_n$ of A".

We denote by $w_A\colon A \to A^\dagger$ the canonical $\mathcal{V}$-algebra homomorphism. We say that A is p-adically weakly complete as $\mathcal{V}$-algebra if the canonical map w_A is a bijection. We write $A_K := A \otimes_{\mathcal{V}} K$ and $A^\dagger_K := A^\dagger \otimes_{\mathcal{V}} K$.

Remark 17.7.1.2. When A is commutative, we retrieve the definition 17.1.1.1.

With the notations of 17.7.1.1, the polynomial P_j appearing in the sum 17.7.1.1.1 can be chosen so that each monomials have a π-adic valuation equal to j. Indeed, let us denote by $R_0 = 0$ and, for any $j \in \mathbb{N}$, define by induction on $j \ge 0$ the polynomials Q_j and R_{j+1} by setting: $R_j + P_j = Q_j + R_{j+1}$ where Q_j is the sum of monomials of $R_j + P_j$ having π-adic valuation equal to j whereas R_{j+1} is the sum of other terms. We have the equality $z = \sum_{j\in\mathbb{N}} Q_j(x_1, \ldots, x_n)$ with $\deg Q_j \le c(j+1)$ and all monomials of Q_j have a π-adic valuation equal to j (also, $\deg R_{j+1} \le c(j+1)$ and $R_{j+1} \in \pi^{j+1} B_n$).

The next proposition gives an important example that we will need by using its corollary 17.7.4.2.

Proposition 17.7.1.3 (Huyghe)**.** *Let $\mathfrak{P}$ be a smooth $\mathfrak{S}$-formal scheme and T_0 be a divisor of P_0. With notation 8.7.3.25, $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger * T_0)$ is a sheaf of weakly complete $\mathcal{V}$-algebras.*

Proof. See a proof in [Huy07, 2.2]. □

Remark 17.7.1.4. In fact, Huyghe proved that The sheaf $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger} * T_0)$ is a sheaf of weakly complete $\mathcal{O}_{\mathfrak{P}}$-algebras, but for our purpose we prefer to focus on the notion of weakly complete $\mathcal{V}$-algebras.

Proposition 17.7.1.5. *Let A be a $\mathcal{V}$-algebra, $c \in \mathbb{R}$, $y_1, \ldots, y_n \in A^{\dagger}$, and, for any $j \in \mathbb{N}$, $P_j \in \pi^j B_n$ be some polynomials such that $\deg P_j \leq c(j+1)$. Hence the element $z := \sum_{j\in\mathbb{N}} P_j(y_1, \ldots, y_n)$ belongs to $A^{\dagger}$. More precisely, if $y_1, \ldots, y_n$ are all weakly completely generated on $\mathcal{V}$ by the elements $x_1, \ldots, x_m$ of A, then z is weakly completely generated on $\mathcal{V}$ by $x_1, \ldots, x_m$. In particular, we have the equality $A^{\dagger} = (A^{\dagger})^{\dagger}$, i.e., the p-adic weak completion of a $\mathcal{V}$-algebra is p-adically weakly complete.*

Proof. 0) Increasing c if necessary, by definition of $A^{\dagger}$, there exist $x_1, \ldots, x_m \in \mathrm{Im}(A \to \widehat{A})$ such that for any $i' = 1, \ldots, n$, for any $k \in \mathbb{N}$, there exist some polynomials $P_{i',k} \in \pi^k B_m$ such that $\deg P_{i',k} \leq c(k+1)$ and $y_{i'} = \sum_{k\in\mathbb{N}} P_{i',k}(x_1, \ldots, x_m)$.

1) For any $j \in \mathbb{N}$, P_j is a finite sum of monomials of the form $P(Y_1, \ldots, Y_n) = \lambda \cdot Y_{\phi(1)} Y_{\phi(2)} \cdots Y_{\phi(r)}$ with $r \in \mathbb{N}$, $\phi(1), \ldots, \phi(r) \in \{1, \ldots, n\}$ and $\lambda \in \mathcal{V}$ are such that $v_\pi(\lambda) \geq j$ and $r \leq c(j+1)$. For such monomial P, we get

$$P(y_1, \ldots, y_n) = \sum_{k_1\in\mathbb{N},\ldots,k_r\in\mathbb{N}} \lambda \cdot P_{\phi(1),k_1}(x_1, \ldots, x_m) \cdots P_{\phi(r),k_r}(x_1, \ldots, x_m).$$

Let us denote by $P_{(k_1,\ldots,k_r)}(X_1, \ldots, X_m) := \lambda \cdot P_{\phi(1),k_1}(X_1, \ldots, X_m) \cdots P_{\phi(r),k_r}(X_1, \ldots, X_m)$ the polynomials of B_m appearing in this sum. Then we have $\deg P_{(k_1,\ldots,k_r)} = \sum_{i=1}^r \deg P_{\phi(i),k_i} \leq c \sum_{i=1}^r (k_i + 1)$. Let us denote by $v_\pi(P_{(k_1,\ldots,k_r)})$ the minimal element of the set of π-adic valuations of the coefficients of $P_{(k_1,\ldots,k_r)}$. By construction, $v_\pi(P_{(k_1,\ldots,k_r)}) \geq j + \sum_{i=1}^r k_i$.

2) Fix $J \in \mathbb{N}$. Moreover, since for any j the polynomial P_j is a finite sum of monomials P such as in the step 1), as $v_\pi(P_{(k_1,\ldots,k_r)}) \geq j + \sum_{i=1}^r k_i$ (see the step 1), then there exist a finite number of such polynomials $P_{(k_1,\ldots,k_r)}$ and such that $v_\pi(P_{(k_1,\ldots,k_r)}) = J$. Let $Q_J(X_1, \ldots, X_m)$ be the sum of all polynomials $P_{(k_1,\ldots,k_r)}(X_1, \ldots, X_m)$ such that $v_\pi(P_{(k_1,\ldots,k_r)}) = J$. Hence $v_\pi(Q_J) \geq J$ and $z = \sum_{J\in\mathbb{N}} Q_J(x_1, \ldots, x_m)$.

3) It remains to find an upper bound of the degree of Q_J: Let $P_{(k_1,\ldots,k_r)}$ be one of these polynomials such that $v_\pi(P_{(k_1,\ldots,k_r)}) = J$. As $r \leq c(j+1)$, as $J \geq j + \sum_{i=1}^r k_i$, this yields then $\deg P_{(k_1,\ldots,k_r)} \leq c(J - j + c(j+1)) \leq c(J + 1 + c(J+1)) = c(1+c)(J+1)$. Let us denote by $C = c(1+c)$. Then we have established $\deg Q_J \leq C(J+1)$. Hence we are done. □

The lemma below is left as an exercise:

Lemma 17.7.1.6. *Let $N_1, N_2 \in \mathbb{N}$, $c \in \mathbb{R}$ and, for any $j \in \mathbb{N}$, let $P_j, Q_j \in \pi^j B_n$ be such that $\deg P_j \leq c(j + N_1)$, $\deg Q_j \leq c(j + N_2)$. We define the following elements of $\mathcal{V}[t_1, \ldots, t_n]^{\dagger}$ by setting: $z = \sum_{j\in\mathbb{N}} P_j(t_1, \ldots, t_n)$, $u = \sum_{j\in\mathbb{N}} Q_j(t_1, \ldots, t_n)$.*

(a) For any $\underline{k} \in \mathbb{N}^n$, there exists a family $\{\widetilde{P}_j\}_{j\in\mathbb{N}}$ of polynomials such that $\widetilde{P}_j \in \pi^j B_n$, $\deg \widetilde{P}_j \leq c(j+N_1)$ and $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}(z) = \sum_{j\in\mathbb{N}} \widetilde{P}_j(t_1, \ldots, t_n)$.

(b) There exists a family $\{R_j\}_{j\in\mathbb{N}}$ of polynomials such that $R_j \in \pi^j B_n$, $\deg R_j \leq c(j + \max\{N_1, N_2\})$ and $z + u = \sum_{j\in\mathbb{N}} R_j(t_1, \ldots, t_n)$.

(c) There exists a family $\{\widetilde{R}_j\}_{j\in\mathbb{N}}$ of polynomials such that $\widetilde{R}_j \in \pi^j B_n$, $\deg \widetilde{R}_j \leq c(j + N_1 + N_2)$ and $zu = \sum_{j\in\mathbb{N}} \widetilde{R}_j(t_1, \ldots, t_n)$.

Proposition 17.7.1.7. *Let A be a $\mathcal{V}$-algebra.*

(a) If B is p-adically separated quotient of $A^{\dagger}$ then B is p-adically weakly complete.

(b) For any $i \in \mathbb{N}$, the canonical map $A/\pi^{i+1}A \to A^{\dagger}/\pi^{i+1}A^{\dagger}$ is an isomorphism.

(c) Let $f \colon A \to B$ be a morphism of $\mathcal{V}$-algebra. Then there exists a unique homomorphism $f^{\dagger} \colon A^{\dagger} \to B^{\dagger}$ such that $f^{\dagger} \circ w_A = w_B \circ f$. If f is surjective, then so is $f^{\dagger}$.

(d) $pA^{\dagger}$ is contained in the Jacobson radical of $A^{\dagger}$.

Proof. This is checked similarly to the commutative case of [MW68, 1]. □

Remark 17.7.1.8. Let A be a $\mathcal{V}$-algebra. If A is left Noetherian then so is its Rees ring $\widetilde{A} := \oplus_{n\in\mathbb{N}} p^n A$. Indeed, if A is left Noetherian then so is the polynomial ring $A[T]$, where T is commuting with A. Since the map $A[T] \to \widetilde{A}$ given by $\sum a_k T^k \mapsto \sum p^k a_k$ is a ring epimorphism then we are done.

Beware that if A is left Noetherian then $A^\dagger$ may not be Noetherian. For instance, take $A = \Gamma(\mathbb{A}^n_k, \mathcal{D}_{\mathbb{A}^{n\dagger}_{\mathcal{V}}})$.

If A is a left Noetherian weakly complete $\mathcal{V}$-algebra then it follows from 17.7.1.7.(d) and from the first remark that A is a Zariskian ring (see [LvO96, II.2.1 Definition]). Beware that $D^\dagger$ is not a Zariskian ring so we are not able to use properties on Zariskian rings in the context of the weak completion of a differential operators ring.

17.7.1.9. Let $\mathfrak{A}$ be the category of $\mathcal{V}$-algebras and $\mathfrak{A}^\dagger$ be the category of p-adic weakly complete $\mathcal{V}$-algebras. It follows from 17.7.1.7.(c) that the functor weak completion $\dagger\colon \mathfrak{A} \to \mathfrak{A}^\dagger$ is a left adjoint of the forgetful functor $\mathfrak{f}\colon \mathfrak{A}^\dagger \to \mathfrak{A}$. Hence, the functor $\dagger\colon \mathfrak{A} \to \mathfrak{A}^\dagger$ commutes with inductive limits. Since the functor $\dagger$ is essentially surjective and since $\mathfrak{A}$ admits injective limits, this yields that $\mathfrak{A}^\dagger$ admits injective limits and if $F\colon I \to \mathfrak{A}^\dagger$ is a functor then $\varinjlim F = (\varinjlim(\mathfrak{f}\circ F))^\dagger$.

When I is filtered then since the set of the elements $x_1, \ldots, x_n$ satisfying 17.7.1.1.1 is finite, then $\varinjlim(\mathfrak{f}\circ F) \in \mathfrak{A}^\dagger$ (this makes a difference with respect to p-adically complete $\mathcal{V}$-algebras). Hence, in that case, both inductive limits computed in $\mathfrak{A}^\dagger$ or in $\mathfrak{A}$ are equal. For example, since $D \xrightarrow{\sim} \varinjlim D^{(m)}$, then we get $D^\dagger \xrightarrow{\sim} \varinjlim_{m\in\mathbb{N}} D^{(m)\dagger}$, where $\varinjlim_{m\in\mathbb{N}}$ is the inductive limit computed in $\mathfrak{A}$.

17.7.2 Weak completion of the global section of the sheaf of differential operators on a finite and etale scheme over an affine space

Let $X^\dagger$ be a smooth $\mathfrak{S}$-weak formal scheme. It follows from Kedlaya's work on etale covers of affines spaces (see [Ked02] or [Ked05]) and from the lifting theorem 17.1.1.20, that there exist an affine dense open $U^\dagger$ of $X^\dagger$ and a finite etale morphism of the form $g\colon U^\dagger \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$. We prove in this section that the canonical homomorphisms $D^{(m)\dagger} \to \Gamma(U^\dagger, \mathcal{D}^{(m)\dagger}_{U^\dagger})$ are $D^\dagger \to \Gamma(U^\dagger, \mathcal{D}^\dagger_{U^\dagger})$ are finite, right and left fully faithful (see 17.7.2.4).

Lemma 17.7.2.1. *Let $f\colon \mathfrak{Y}' \to \mathfrak{Y}$ be an etale morphism of affine smooth $\mathfrak{S}$-formal schemes such that $\mathfrak{Y}/\mathfrak{S}$ is endowed with coordinates. Set $A := \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$, $A' := \Gamma(\mathfrak{Y}', \mathcal{O}_{\mathfrak{Y}'})$, $\widehat{D}^{(m)}_{\mathfrak{Y}} := \Gamma(\mathfrak{Y}, \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}})$ $\widehat{D}^{(m)}_{\mathfrak{Y}'} := \Gamma(\mathfrak{Y}', \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'})$, $D^\dagger_{\mathfrak{Y}} := \Gamma(\mathfrak{Y}, \mathcal{D}^\dagger_{\mathfrak{Y}})$ and $D^\dagger_{\mathfrak{Y}'} := \Gamma(\mathfrak{Y}', \mathcal{D}^\dagger_{\mathfrak{Y}'})$. Since f is etale then following 9.2.1.22 we get the ring homomorphism: $f^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}} \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Y}'}$ and $f^{-1}\mathcal{D}^\dagger_{\mathfrak{Y}} \to \mathcal{D}^\dagger_{\mathfrak{Y}'}$.*

(a) The induced morphisms $A'\widehat{\otimes}_A \widehat{D}^{(m)}_{\mathfrak{Y}} \to \widehat{D}^{(m)}_{\mathfrak{Y}'}$, $\widehat{D}^{(m)}_{\mathfrak{Y}} \widehat{\otimes}_A A' \to \widehat{D}^{(m)}_{\mathfrak{Y}'}$ are isomorphisms.

(b) If f is moreover finite, then the induced morphisms $A' \otimes_A D^\dagger_{\mathfrak{Y}} \to D^\dagger_{\mathfrak{Y}'}$, $D^\dagger_{\mathfrak{Y}} \otimes_A A' \to D^\dagger_{\mathfrak{Y}'}$ are isomorphisms.

Proof. Let us denote by $f_i\colon Y'_i \to Y_i$ (resp. A_i, A'_i) the reduction modulo π^{i+1} of f (resp. A, A'). Since f_i is étale, then following 5.1.3.6 the canonical morphisms $A'_i \otimes_{A_i} D^{(m)}_{Y_i} \to D^{(m)}_{Y'_i}$, $D^{(m)}_{Y_i} \otimes_{A_i} A'_i \to D^{(m)}_{Y'_i}$ are isomorphisms. Hence we are done by taking projective and inductive limits. □

Lemma 17.7.2.2. *Let $g\colon U'^\dagger \to U^\dagger$ an etale morphism of affine and smooth $\mathfrak{S}$-weak formal schemes.*

(a) The canonical morphism of left $\mathcal{D}^{(m)}_{U'^\dagger}$-modules $\mathcal{D}^{(m)}_{U'^\dagger} \to g^*\mathcal{D}^{(m)}_{U^\dagger}$ is an isomorphism.

(b) The composite morphism $g^{-1}\mathcal{D}^{(m)}_{U^\dagger} \to g^*\mathcal{D}^{(m)}_{U^\dagger} \xleftarrow{\sim} \mathcal{D}^{(m)}_{U'^\dagger}$ is a morphism of $\mathcal{V}$-algebras.

We have similar results replacing $\mathcal{D}^{(m)}$ by $\mathcal{D}$.

Proof. Since this is local, we can suppose $U'^\dagger$ are affine and $U^\dagger/\mathfrak{S}$ has coordinates. Similarly to 5.1.3.6, this is an easy computation (more precisely, see 17.7.2.3). □

17.7.2.3. With notation 17.7.2.2, suppose that $U^\dagger$ and $U'^\dagger$ are affine and $U^\dagger/\mathfrak{S}$ has coordinates. Set $A^\dagger := \Gamma(U^\dagger, \mathcal{O}_{U^\dagger})$, $A'^\dagger := \Gamma(U^\dagger, \mathcal{O}_{U'^\dagger})$. Let E be either $(\Gamma(U, \mathcal{D}^{(m)}_{U^\dagger})$ or $(\Gamma(U, \mathcal{D}_{U^\dagger})$. Let E' be either $(\Gamma(U', \mathcal{D}^{(m)}_{U'^\dagger})$ or $(\Gamma(U', \mathcal{D}_{U'^\dagger})$. By taking the global sections, it follows from 17.7.2.2 that we get a morphism of $\mathcal{V}$-algebras $D^{(m)}_{U^\dagger} \to D^{(m)}_{U'^\dagger}$. Moreover, we compute that the induced maps

$$A'^\dagger \otimes_{A^\dagger} E \to E', \qquad E \otimes_{A^\dagger} A'^\dagger \to E' \tag{17.7.2.3.1}$$

are isomorphisms (which proves locally 17.7.2.2).

By functoriality of the p-adic weak completion of $\mathcal{V}$-algebras, we get the morphism $D^{(m)\dagger}_{U^\dagger} \to D^{(m)\dagger}_{U'^\dagger}$. Taking the inductive limit on the level, this gives the canonical morphism of $\mathcal{V}$-algebras: $D^\dagger_{U^\dagger} \to D^\dagger_{U'^\dagger}$.

Proposition 17.7.2.4. *Let $g\colon U^\dagger \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$ be a finite etale morphism of smooth $\mathcal{S}$-weak formal schemes and $A^\dagger := \Gamma(U^\dagger, \mathcal{O}_{U^\dagger})$. Set $D^{(m)}_{U^\dagger} := \Gamma(U^\dagger, \mathcal{D}^{(m)}_{U^\dagger})$, $D_{U^\dagger} := \Gamma(, U^\dagger, \mathcal{D}_{U^\dagger})$. The induced (from 17.7.2.3) canonical $A^\dagger$-linear morphisms*

$$A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} D^{(m)\dagger} \to D^{(m)\dagger}_{U^\dagger}, \qquad D^{(m)\dagger} \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger \to D^{(m)\dagger}_{U^\dagger}, \tag{17.7.2.4.1}$$

$$A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} D^\dagger \to D^\dagger_{U^\dagger}, \qquad D^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger \to D^\dagger_{U^\dagger} \tag{17.7.2.4.2}$$

are isomorphisms.

Proof. By symmetry and by passage to the inductive limit on the level (use 17.7.1.9), we reduce to check that the canonical morphism $\theta\colon A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} D^{(m)\dagger} \to D^{(m)\dagger}_{U^\dagger}$ is an isomorphism.

As $A^\dagger$ is a finite $\mathcal{V}[\underline{t}]^\dagger$-algebra, then $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger \xrightarrow{\sim} \widehat{A}$. This yields the first isomorphism:

$$A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} \widehat{D}^{(m)} \xrightarrow{\sim} \widehat{A} \otimes_{\mathcal{V}\{\underline{t}\}} \widehat{D}^{(m)} \xrightarrow{\sim} \widehat{A} \widehat{\otimes}_{\mathcal{V}\{\underline{t}\}} \widehat{D}^{(m)} \overset{17.7.2.1}{\xrightarrow{\sim}} \widehat{D}^{(m)}_{\mathfrak{U}}.$$

This composition isomorphism fits in the commutative diagram:

$$\begin{array}{ccc} A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} \widehat{D}^{(m)} & \xrightarrow{\sim} & \widehat{D}^{(m)}_{\mathfrak{U}} \\ \uparrow & & \uparrow \\ A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} D^{(m)\dagger} & \xrightarrow{\theta} & D^{(m)\dagger}_{U^\dagger}. \end{array} \tag{17.7.2.4.3}$$

As $A^\dagger$ is a flat extension of $\mathcal{V}[\underline{t}]^\dagger$, since $D^{(m)\dagger}$ and $D^{(m)\dagger}_{U^\dagger}$ are separated (for the p-adic topology), this yields that the vertical arrows of 17.7.2.4.3 are injective. So we get the injectivity of θ.

Let us now check the surjectivity of θ via the following steps :

0) *Let us fix some notations.* Let $x_1, \ldots, x_s \in A^\dagger$ generating $A^\dagger$ as $\mathcal{V}[t_1, \ldots, t_n]^\dagger$-module. Let us denote by X the column vector of coordinates $x_1, \ldots, x_s$.

i) For any $\underline{a} = (a_1, \ldots, a_n) \leq p^m$ (i.e. $a_1, \ldots, a_n \leq p^m$), let $A^{(\underline{a})} = (a^{(\underline{a})}_{ij})_{1 \leq i,j \leq s} \in M_s(\mathcal{V}[t_1, \ldots, t_n]^\dagger)$ be such that

$$\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}(X) = A^{(\underline{a})} X, \tag{17.7.2.4.4}$$

where $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}(X)$ is the column vector of coordinates $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}(x_1), \ldots, \underline{\partial}^{\langle \underline{a} \rangle_{(m)}}(x_s)$. There exists a real constant $c \geq 1$ and, for any $k \in \mathbb{N}$, some polynomials $P^{(\underline{a})}_{ijk} \in \pi^k B_n$ such that $\deg P^{(\underline{a})}_{ijk} \leq c(k+1)$ and $a^{(\underline{a})}_{ij} = \sum_{k \in \mathbb{N}} P^{(\underline{a})}_{ijk}(t_1, \ldots, t_n)$.

ii) For any $1 \leq b \leq s$, let $B^{(b)} = (b^{(b)}_{ij})_{1 \leq i,j \leq s} \in M_s(\mathcal{V}[t_1, \ldots, t_n]^\dagger)$ such that

$$x_b \cdot X = B^{(b)} X, \tag{17.7.2.4.5}$$

where $x_b \cdot X$ is the column vector of coordinates $x_b x_1, \ldots, x_b x_s$. Increasing c if necessary, there exist for any $k \in \mathbb{N}$, some polynomials $P^{(b)}_{ijk} \in \pi^k B_n$ such that $\deg P^{(b)}_{ijk} \leq c(k+1)$ and satisfying $b^{(b)}_{ij} = \sum_{k \in \mathbb{N}} P^{(b)}_{ijk}(t_1, \ldots, t_n)$.

1) Let $z \in D^{(m)\dagger}_{U^\dagger}$. Then there exists some elements $y_1, \ldots, y_e$ of $D^{(m)}_{U^\dagger}$ such that z is weakly completely generated by $y_1, \ldots, y_e$. Since $D^{(m)}_{U^\dagger}$ is a $\mathcal{V}[t_1, \ldots, t_n]^\dagger$-algebra generated by $x_1, \ldots, x_s$ and by

$\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}$ for $\underline{a} \leq p^m$, then every elements $y_1, \dots, y_e$ are weakly completely generated by $x_1, \dots, x_s$, by $t_1, \dots, t_n$ and by $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}$ for $\underline{a} \leq p^m$. Using 17.7.1.5 this yields that z is weakly completely generated by $x_1, \dots, x_s$, $t_1, \dots, t_n$ and $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}$ for $\underline{a} \leq p^m$. Let us denote by $N := n + s + n(p^m + 1)$. Hence, there exists for any $J \in \mathbb{N}$, some polynomials $Z_J \in \pi^J B_N$ such that $\deg Z_J \leq c(J+1)$ and $z = \sum_{J \in \mathbb{N}} Z_J(t_1, \dots, t_n, x_1, \dots, x_s, \underline{\partial}^{\langle \underline{a} \rangle_{(m)}}, \underline{a} \leq p^m)$.

2) Following the formulas 1.4.2.7.1 and 1.4.2.7.(d) the passage from the right to the left of a polynomial in $t_1, \dots, t_n$ with respect to an operator of the form $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}$ do not increase the degree in $t_1, \dots, t_n$. We can then suppose that Z_J is a *finite* sum of monomials of the form :

$$M_J = \pi^J P(t_1, \dots, t_n) Q_1(\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}, \underline{a} \leq p^m) P_1(x_1, \dots, x_s) \cdots Q_r(\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}, \underline{a} \leq p^m) P_r(x_1, \dots, x_s),$$

where P is a monomial of B_n and, for $i = 1, \dots, r$, P_i is a unitary monomial of B_s and Q_i is a unitary monomial $B_{n(p^m+1)}$. We have $\deg M_J = \deg P + \sum_{u=1}^r (\deg P_u + \deg Q_u) \leq c(J+1)$.

3) *Linearisation of* $P_1(x_1, \dots, x_s), \dots, P_r(x_1, \dots, x_s)$. Let $1 \leq u \leq r$. When $\deg(P_u) \leq 1$, then we keep it. If $\deg(P_u) \geq 2$, then there exist a line vector $L^{(u)} = (l_1^{(u)}, \dots, l_s^{(u)})$ with coefficients in $\mathcal{V}[t_1, \dots, t_n]^\dagger$ and, for any $1 \leq i \leq s$ and $j \in \mathbb{N}$, some polynomials $L_{ij}^{(u)} \in \pi^j B_n$ such that $\deg(L_{ij}^{(u)}) \leq c(j + \deg(P_u) - 1)$, $l_i^{(u)} = \sum_{j \in \mathbb{N}} L_{ij}^{(u)}(t_1, \dots, t_n)$ and $P_u(x_1, \dots, x_s) = L^{(u)} X$. Indeed, we proceed by induction on $\deg(P_u)$. When $\deg(P_u) = 2$, this is a consequence of 17.7.2.4.5. The heredity of the induction follows from 17.7.2.4.5 and 17.7.1.6.(b-c).

4) *Passage from the right to the left handside with respect to* $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}$ *of linear combinations of* $x_1, \dots, x_s$ *with coefficients in* $\mathcal{V}[t_1, \dots, t_n]^\dagger$. For any $\underline{a} \leq p^m$, we have the equalities:

$$\begin{aligned} \underline{\partial}^{\langle \underline{a} \rangle_{(m)}} L^{(u)} X &\overset{1.4.2.7.1}{=} \sum_{\underline{h} \leq \underline{a}} \left\{ \begin{smallmatrix} \underline{a} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{a} - \underline{h} \rangle_{(m)}} (L^{(u)} X) \underline{\partial}^{\langle \underline{h} \rangle_{(m)}} \\ &\overset{4.1.1.2.1}{=} \sum_{\underline{h} \leq \underline{a}} \left\{ \begin{smallmatrix} \underline{a} \\ \underline{h} \end{smallmatrix} \right\} \sum_{\underline{h}' \leq \underline{a} - \underline{h}} \left\{ \begin{smallmatrix} \underline{a} - \underline{h} \\ \underline{h}' \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{a} - \underline{h} - \underline{h}' \rangle_{(m)}} (L^{(u)}) A^{(\underline{h}')} X \underline{\partial}^{\langle \underline{h} \rangle_{(m)}} \\ &= \sum_{\underline{h} \leq \underline{a}} L_{\underline{h}}^{(u, \underline{a})} X \underline{\partial}^{\langle \underline{h} \rangle_{(m)}}, \end{aligned} \tag{17.7.2.4.6}$$

where $L_{\underline{h}}^{(u,\underline{a})} := (l_{\underline{h},1}^{(u,\underline{a})}, \dots, l_{\underline{h},s}^{(u,\underline{a})}) := \left\{ \begin{smallmatrix} \underline{a} \\ \underline{h} \end{smallmatrix} \right\} \sum_{\underline{h}' \leq \underline{a} - \underline{h}} \left\{ \begin{smallmatrix} \underline{a} - \underline{h} \\ \underline{h}' \end{smallmatrix} \right\} \underline{\partial}^{\langle \underline{a} - \underline{h} - \underline{h}' \rangle_{(m)}} (L^{(u)}) A^{(\underline{h}')}$ is a line vector with coefficients in $\mathcal{V}[t_1, \dots, t_n]^\dagger$. Moreover, following 17.7.1.6, there exist, for any $i = 1, \dots, s$ and every $j \in \mathbb{N}$, some polynomials $L_{\underline{h},ij}^{(u,\underline{a})} \in \pi^j B_n$ such that $\deg(L_{\underline{h},ij}^{(u,\underline{a})}) \leq c(j + \deg(P_u))$ and $l_{\underline{h},i}^{(u,\underline{a})} = \sum_{j \in \mathbb{N}} L_{\underline{h},ij}^{(u,\underline{a})}(t_1, \dots, t_n)$.

To sum-up: the passage from the right to the left with respect to $\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}$ of linear combinations of $x_1, \dots, x_s$ needs the add of "1" in the inequality of the form $\deg(L_{\underline{h},ij}^{(u,\underline{a})}) \leq c(j + \deg(P_u))$. This number 1 corresponds also to the degree of the monomials $\underline{\partial}^{\langle \underline{h} \rangle_{(m)}}$ (because $\underline{h} \leq p^m$).

5) By reiterating the process of the step 4) (and also using again 17.7.2.4.5 and 17.7.1.6.(b-c) when we multiply two linear combinations of $x_1, \dots, x_s$), the monomial M_J is therefore equal to a finite sum of terms of the form:

$$R_J = \pi^J L X Q(\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}, \underline{a} \leq p^m)$$

where $Q \in B_{n(p^m+1)}$ with $\deg(Q) \leq \deg(Q_1) + \cdots + \deg(Q_r) \leq \deg(M_J)$, $L = (l_1, \dots, l_s)$ is a line vector with coefficients in $\mathcal{V}[t_1, \dots, t_n]^\dagger$ such that for any $i = 1, \dots, s$ and every $j \in \mathbb{N}$, there exist some polynomials $L_{ij} \in B_n$ such that $\deg(L_{ij}) \leq c(j + \deg(M_J))$ and $l_i = \sum_{j \in \mathbb{N}} \pi^j L_{ij}(t_1, \dots, t_n)$.

6) *Conclusion.* Let us denote by $S_{J,j} := L_{ij}(t_1, \dots, t_n) Q(\underline{\partial}^{\langle \underline{a} \rangle_{(m)}}, \underline{a} \leq p^m) \in B_{n+n(p^m+1)}$. So,

$$R_J = \sum_{i=1}^s x_i \sum_{j \in \mathbb{N}} \pi^{J+j} S_{J,j}.$$

As $\deg(M_J) \leq \deg(P_J) \leq c(J+1)$, then $\deg(Q) \leq c(J+1)$ and $\deg(L_{ij}) \leq c(j + c(J+1))$. Hence : $\deg(S_{J,j}) \leq c(j + c(J+1)) + c(J+1) \leq c(1+c)(J+j+1)$.

When j and J are fixed, the set of polynomials of the form $S_{J,j}$ defined as above has a finite cardinal. Let us denote by $\widetilde{S}_{J,j}$ the finite sum of elements of this set. We get then the sum

$$z = \sum_{i=1}^{s} x_i \sum_{J,j\in\mathbb{N}} \pi^{J+j} \widetilde{S}_{J,j}.$$

As $\deg(\widetilde{S}_{J,j}) \leq c(1+c)(J+j+1)$, this yields that $\sum_{J,j\in\mathbb{N}} \pi^{J+j}\widetilde{S}_{J,j} \in D^{(m)\dagger}$. □

Corollary 17.7.2.5. *Let $g\colon U^\dagger \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$ a finite etale morphism of smooth $\mathcal{S}$-weak formal schemes. The canonical homomorphisms $D^{(m)\dagger} \to D^{(m)\dagger}_{U^\dagger}$ are $D^\dagger \to D^\dagger_{U^\dagger}$ are right and left fully faithful (see 17.7.2.3).*

Proof. It follows from 17.7.2.4 that the functors $D^{(m)\dagger}_{U^\dagger} \otimes_{D^{(m)\dagger}} -$, $-\otimes_{D^{(m)\dagger}} D^{(m)\dagger}_{U^\dagger}$ $D^{(m)\dagger}_{U^\dagger} \otimes_{D^{(m)\dagger}} -$ and $-\otimes_{D^{(m)\dagger}} D^{(m)\dagger}_{U^\dagger}$ are canonically isomorphic to $A^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} -$. Since $\mathcal{V}[\underline{t}]^\dagger \to A^\dagger$ is faithfully flat (because so is $\mathcal{V}\{\underline{t}\} \to \widehat{A}$), then we are done. □

Proposition 17.7.2.6. *We keep the notations and hypotheses of 17.7.2.4. The canonical morphisms*

$$D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^\dagger} D^\dagger_{U^\dagger} \to D^\dagger_{\mathfrak{U}}, \qquad D^\dagger_{U^\dagger} \otimes_{D^\dagger} D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \to D^\dagger_{\mathfrak{U}}$$

are isomorphisms.

Proof. By applying the functor $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^\dagger} -$ to 17.7.2.4 we get:

$$D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^\dagger} D^\dagger_{U^\dagger} \overset{17.7.2.4}{\underset{\sim}{\longleftarrow}} D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^\dagger} D^\dagger \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger \overset{\sim}{\longleftarrow} D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger.$$

Moreover, as $A^\dagger$ is a finite $\mathcal{V}[\underline{t}]^\dagger$-algebra, then $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger \xrightarrow{\sim} \widehat{A}$. Hence, $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{\mathcal{V}\{\underline{t}\}} \widehat{A} \xrightarrow{\sim} D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger$. The isomorphism $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{\mathcal{V}\{\underline{t}\}} \widehat{A} \xrightarrow{\sim} D^\dagger_{\mathfrak{U}}$ of 17.7.2.1 allows us to conclude by composing these isomorphisms. □

17.7.3 Explicit description of overconvergent isocrystals on finite and etale schemes over affine spaces

We give a description of overconvergent isocrystals on the affine space (see 17.7.3.3 and 17.7.3.5). Next we deduce, thanks to the preceding section, a description of overconvergent isocrystals on the finite and etale schemes over the affine space (see 17.7.3.7).

In this subsection, we will keep the following notations : let $\mathfrak{P} := \widehat{\mathbb{P}}^n_{\mathcal{V}}$ be the formal projective space on $\mathcal{V}$ of dimension n, $u_0, \dots, u_n$ the projective coordinates of $\mathfrak{P}$, H_0 the hyperplan defined by $u_0 = 0$, i.e., $H_0 := \mathbb{P}^n_k \setminus \mathbb{A}^n_k$. We denote by $\mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$ (resp. $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$) the sheaf of fonctions (resp. differential operators of finite level) on $\mathfrak{P}$ with overconvergent singularities along of H_0 (see [Ber96c, 4.2]). We set moreover on $\mathcal{D}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} := \mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{P},\mathbb{Q}}} \mathcal{D}_{\mathfrak{P},\mathbb{Q}}$, where $\mathcal{D}_{\mathfrak{P}}$ is the sheaf usuel of differential operators on $\mathfrak{P}$.

Following the theorem of comparison of Noot-Huyghe (see [Huy97] or [Huy98]), we have in this geometrical situation the isomorphism: $D^\dagger_K := D^\dagger_{\mathbb{A}^{n\dagger}_{\mathcal{V}},K} \xrightarrow{\sim} \Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}})$. She established moreover on the formula:

$$\Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}) = \left\{ \sum_{\underline{k},\underline{l}\in\mathbb{N}^n} a_{\underline{k},\underline{l}} \underline{t}^{\underline{k}} \underline{\partial}^{[\underline{l}]} \mid a_{\underline{k},\underline{l}} \in K, \exists \eta < 1, \exists c \geq 1 \text{ such that } |a_{\underline{k},\underline{l}}| < c\eta^{|\underline{k}|+|\underline{l}|} \right\},$$

where $t_1 = \frac{u_1}{u_0}, \dots, t_n = \frac{u_n}{u_0}$ are the coordinates on the affine space. Moreover, we set: $\mathcal{V}[\underline{t}]^\dagger_K := \mathcal{V}[t_1, \dots, t_n]^{\dagger/\mathcal{V}} \otimes_{\mathcal{V}} K \xrightarrow{\sim} \Gamma(\mathbb{A}^{n\dagger}_{\mathcal{V}}, \mathcal{O}_{\mathbb{A}^{n\dagger}_{\mathcal{V}},\mathbb{Q}}) \xrightarrow{\sim} \Gamma(,\mathfrak{P}, \mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}})$. We have also: $\Gamma(,\mathfrak{P}, \mathcal{D}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}) \xrightarrow{\sim} D_K$.

17.7.3.1 (Huyghe's theorems of type A)**.** Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-module.

(a) Following the theorem of type A for coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules (see [Huy97]), $E := \Gamma(\mathfrak{P}, \mathcal{E})$ is a coherent $D^\dagger_K$-module and the canonical morphism

$$\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} E \to \mathcal{E} \tag{17.7.3.1.1}$$

is an isomorphism. Moreover, the functors $\Gamma(\mathfrak{P}, -)$ and $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} -$ induce quasi-inverse equivalences between the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules and that of coherent $D^\dagger_K$-modules.

(b) We have Similarly, the functors $\Gamma(\mathfrak{P}, -)$ and $\mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{\mathcal{V}[\underline{t}]^\dagger_K} -$ (resp. $\mathcal{D}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{\mathcal{V}[\underline{t}]^\dagger_K} -$) induce quasi-inverse equivalences between the category of coherent $\mathcal{D}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules (resp. coherent $\mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules) and that of coherent D_K-modules (resp. coherent $\mathcal{V}[\underline{t}]^\dagger_K$-modules).

(c) For any affine open $\mathfrak{U}' \subset \widehat{\mathbb{A}}^n_{\mathcal{V}}$, following the theorem of type A for coherent $\mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}}$-modules (see 8.7.5.5), the canonical morphism

$$\mathcal{D}^\dagger_{\mathfrak{U}',\mathbb{Q}} \otimes_{D^\dagger_{\mathfrak{U}',K}} \Gamma(\mathfrak{U}', \mathcal{E}) \to \mathcal{E}|\mathfrak{U}' \tag{17.7.3.1.2}$$

is an isomorphism.

(d) By combining 17.7.3.1.2 and 17.7.3.1.1, this yields that the canonical morphism

$$D^\dagger_{\mathfrak{U}',K} \otimes_{D^\dagger_K} E \to \Gamma(\mathfrak{U}', \mathcal{E}) \tag{17.7.3.1.3}$$

is an isomorphism.

17.7.3.2. Using Theorems of 17.7.3.1, since the homomorphism $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \to j_* \mathcal{D}^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$ is faithfully flat, then so is the global version analogue $D^\dagger_K \to D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$ where $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}} := \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, \mathcal{D}^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}})$ (use also 8.7.6.2).

17.7.3.3. Denote by $\mathrm{MIC}(\mathfrak{P}, H_0/\mathcal{V})$ the category of $\mathcal{D}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules, which are also coherent as $\mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules It follows from the theorems of type A of 17.7.3.1 that the functors $\Gamma(\mathfrak{P}, -)$ and $\mathcal{D}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} -$ induce quasi-inverse equivalences between the category $\mathrm{MIC}(\mathfrak{P}, H_0/\mathcal{V})$ and $\mathrm{MIC}(\mathcal{V}[\underline{t}]^\dagger_K/K)$, the category of D_K-modules which are $\mathcal{V}[\underline{t}]^\dagger_K$-coherent (see notation 17.5.1.1).

By using 11.2.1.14, the category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$ is equal to that of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-modules, which are also coherent as $\mathcal{O}_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$-module. Using theorems of type A of 17.7.3.1, this yields the functors $\Gamma(\mathfrak{P}, -)$ and $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} -$ induce quasi-inverse equivalences between the category $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$ and the category of coherent $D^\dagger_K$-modules which are $\mathcal{V}[\underline{t}]^\dagger_K$-coherent.

17.7.3.4. By definition(see 17.5.1.1.(c) used in the case where $Y^\dagger = \mathbb{A}^{n\dagger}_{\mathcal{V}}$, $X = \mathbb{P}^n_{\mathcal{V}}$ and then $X^{\mathrm{an}}_K = \mathfrak{P}_K$), the functor $\Gamma(\mathfrak{P}_K, -)$ induces an equivalence between the category $\mathrm{MIC}^\dagger(\mathbb{A}^n_k, \mathbb{P}^n_k, \mathfrak{P}/K)$ and $\mathrm{MIC}^\dagger(\mathcal{V}[\underline{t}]^\dagger_K/K)$. On The functor sp_*, where $\mathrm{sp}\colon \mathfrak{P}_K \to \mathfrak{P}$ is the specialisation morphism, gives an equivalence between $\mathrm{MIC}^\dagger(\mathbb{A}^n_k, \mathbb{P}^n_k, \mathfrak{P}/K)$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$. Since $\Gamma(\mathfrak{P}, -) \circ \mathrm{sp}_* = \Gamma(\mathfrak{P}_K, -)$, then we deduce from 17.7.3.3 that the functor $\Gamma(\mathfrak{P}, -)$ induces an equivalence between $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$ and $\mathrm{MIC}^\dagger(\mathcal{V}[\underline{t}]^\dagger_K/K)$. This yields that $\mathrm{MIC}^\dagger(\mathcal{V}[\underline{t}]^\dagger_K/K)$ is equal to the category of coherent $D^\dagger_K$-modules which are $\mathcal{V}[\underline{t}]^\dagger_K$-coherent

Lemma 17.7.3.5. *Let $\mathcal{E} \in \mathrm{MIC}(\mathfrak{P}, H_0/\mathcal{V})$ and $E := \Gamma(\mathfrak{P}, \mathcal{E}) \in \mathrm{MIC}(\mathcal{V}[\underline{t}]^\dagger_K/K)$. The following assertions are equivalent:*

(a) $E \in \mathrm{MIC}^\dagger(\mathcal{V}[\underline{t}]^\dagger_K/K)$;

(b) $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$;

(c) The canonical morphism $E \to D^\dagger_K \otimes_{D_K} E$ is an isomorphism ;

(d) E is endowed with a structure of coherent $D^\dagger_K$-module extending its structure of D_K-module.

Proof. The equivalence $(b) \Leftrightarrow (a)$ follows from 17.7.3.4. Suppose $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$ and let us check that E satisfies (c). Following 11.2.1.9.2, the canonical morphism $\mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}} \mathcal{E}$ is an isomorphism. We conclude using theorems of type A for respectively coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-modules and coherent $\mathcal{D}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-modules (see 17.7.3.1). By noetherianity of D_K, E is therefore D_K-coherent. Hence, $(c) \Rightarrow (d)$. Using 17.7.3.3 and Theorem of type A (see 17.7.3.1), we get the implication: $(d) \Rightarrow (b)$. ☐

Proposition 17.7.3.6. *Let $g\colon U^{\dagger} \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$ be a finite etale morphism of smooth $\mathfrak{S}$-weak formal schemes, $A^{\dagger} := \Gamma(U^{\dagger}, \mathcal{O}_{U^{\dagger}})$ and $D_{U^{\dagger}} := \Gamma(U^{\dagger}, \mathcal{D}_{U^{\dagger}})$. Let $E \in \mathrm{MIC}(A^{\dagger}_K/K)$ (see 17.5.1.1). Let us denote by $g_*(E)$ the module E viewed as an element of $\mathrm{MIC}(\mathcal{V}[\underline{t}]^{\dagger}_K/K)$ via the ring homomorphism $D_K \to D_{U^{\dagger},K}$. The following assertions are equivalent:*

(a) The connection of E is overconvergente, i.e. $E \in \mathrm{MIC}^{\dagger}(A^{\dagger}_K/K)$;

(b) The connection of $g_*(E)$ is overconvergente, i.e. $g_*(E) \in \mathrm{MIC}^{\dagger}(\mathcal{V}[\underline{t}]^{\dagger}_K/K)$;

(c) E is endowed with a structure of coherent $D^{\dagger}_{U^{\dagger},K}$-module extending its structure of $D_{U^{\dagger},K}$-module ;

(d) The canonical $D_{U^{\dagger},K}$-linear morphism $E \to D^{\dagger}_{U^{\dagger},K} \otimes_{D_{U^{\dagger},K}} (E)$ is an isomorphism.

Proof. By using [LS07, 7.2.15], we get the equivalence between the two first assertions. Following 17.7.2.3.1, we have that the canonical isomorphism: $D_K \otimes_{\mathcal{V}[\underline{t}]^{\dagger}} A^{\dagger} \xrightarrow{\sim} D_{U^{\dagger},K}$. By using 17.7.2.4.2, this yields that the canonical morphism : $D^{\dagger}_K \otimes_{D_K} D_{U^{\dagger},K} \to D^{\dagger}_{U^{\dagger},K}$ is an isomorphism. Via 17.7.3.5, this yields the equivalence between (b) and (d). The implication $(d) \Rightarrow (c)$ is obvious. Moreover, if E is endowed with a structure of coherent $D^{\dagger}_{U^{\dagger},K}$-module extending its structure of $D_{U^{\dagger},K}$-module then g_*E is endowed with a structure of coherent $D^{\dagger}_K$-module extending its structure of D_K-module. Via 17.7.3.5, this proves the implication $(b) \Rightarrow (c)$. ☐

Corollary 17.7.3.7. *Let $g\colon U^{\dagger} \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$ be a finite etale morphism of smooth $\mathfrak{S}$-weak formal schemes, $A^{\dagger} := \Gamma(U^{\dagger}, \mathcal{O}_{U^{\dagger}})$ and $D_{U^{\dagger}} := \Gamma(U^{\dagger}, \mathcal{D}_{U^{\dagger}})$.*

(a) Let $\phi\colon E \to F$ be a $D_{U^{\dagger},K}$-linear map between two objects of $\mathrm{MIC}^{\dagger}(A^{\dagger}_K/K)$. Then ϕ is $D^{\dagger}_{U^{\dagger},K}$-linear.

(b) The category $\mathrm{MIC}^{\dagger}(A^{\dagger}_K/K)$ is equal to the strictly full subcategory of that of $D^{\dagger}_{U^{\dagger},K}$-modules consisting of coherent $D^{\dagger}_{U^{\dagger},K}$-modules which are also coherent as $A^{\dagger}_K$-module.

Lemma 17.7.3.8. *Let $\mathcal{F} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$. Set $F := \Gamma(\mathfrak{P}, \mathcal{F}) \in \mathrm{MIC}^{\dagger}(\mathcal{V}[\underline{t}]^{\dagger}_K/K)$. The canonical morphisms:*

$$\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^{\dagger}} F \to D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^{\dagger}} F \to \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, \mathcal{F})$$

are isomorphisms.

Proof. Using respectively the theorems of type A for coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-modules and coherent $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-modules (see 17.7.3.1), we get the homomorphisms: $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^{\dagger}} F \to \Gamma(\widehat{\mathbb{A}}^n_{\mathfrak{S}}, \mathcal{F})$ and $D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^{\dagger}} F \to \Gamma(\widehat{\mathbb{A}}^n_{\mathfrak{S}}, \mathcal{F})$ are isomorphisms. Hence, we are done.

☐

Proposition 17.7.3.9. *Let $g\colon U^{\dagger} \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$ be a finite etale morphism of smooth $\mathfrak{S}$-weak formal schemes, $A^{\dagger} := \Gamma(U^{\dagger}, \mathcal{O}_{U^{\dagger}})$ and $D_{U^{\dagger}} := \Gamma(U^{\dagger}, \mathcal{D}_{U^{\dagger}})$. Set $D_{\mathfrak{U}} := \Gamma(\mathfrak{U}, \mathcal{D}_{\mathfrak{U}})$ and $D^{\dagger}_{\mathfrak{U}} := \Gamma(\mathfrak{U}, \mathcal{D}^{\dagger}_{\mathfrak{U}})$. Let $E \in \mathrm{MIC}^{\dagger}(A^{\dagger}_K/K)$.*

(a) Then the canonical morphism

$$\widehat{A}_K \otimes_{A^{\dagger}_K} E \to D^{\dagger}_{\mathfrak{U},K} \otimes_{D^{\dagger}_{U^{\dagger},K}} E$$

is an isomorphism.

(b) We have $G := D^\dagger_{\mathfrak{U},K} \otimes_{D^\dagger_{U^\dagger,K}} E \in \mathrm{MIC}^\dagger(\widehat{A}_K/K)$ *(see notation 11.1.1.7) and* $\mathcal{G} := \mathcal{D}^\dagger_{\mathfrak{U},\mathbb{Q}} \otimes_{D^\dagger_{\mathfrak{U},K}} G \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{U}/\mathcal{V})$.

Proof. a) Let us check the first assertion. i) Following 17.7.3.6, $g_*E \in \mathrm{MIC}^\dagger(\mathcal{V}[\underline{t}]^\dagger_K/K)$. Let $\mathcal{F} := \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} g_*E$ (and then $g_*E = \Gamma(\mathfrak{P}, \mathcal{F})$). Following 17.7.3.5, we have $\mathcal{F} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}, H_0/\mathcal{V})$. Using 17.7.3.8, this yields the canonical morphism $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} g_*E \to D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^\dagger} g_*E$ is an isomorphism.

ii) As $A^\dagger$ is a finite $\mathcal{V}[\underline{t}]^\dagger$-algebra, then $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} A^\dagger \xrightarrow{\sim} \widehat{A}$. Hence, the canonical morphism $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} g_*E \to \widehat{A}_K \otimes_{A^\dagger_K} E$ is an isomorphism. It follows from 17.7.2.6 that the canonical morphism $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}}} \otimes_{D^\dagger} g_*E \to D^\dagger_{\mathfrak{U},K} \otimes_{D^\dagger_{U^\dagger,K}} E$ is an isomorphism. Hence, using the last isomorphism of the part i) of the proof we are done.

b) This follows from a) and from 17.7.3.9. □

Lemma 17.7.3.10. *We keep notation 17.7.3.9. Let* $E \in \mathrm{MIC}^\dagger(A^\dagger_K/K)$*,* $\widehat{F} \in \mathrm{MIC}^\dagger(\widehat{A}_K/K)$.

We suppose that there exist a $D_{U^\dagger,K}$*-linear morphism* $E \to \widehat{F}$ *and such that the induced morphism* $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} E \to \widehat{F}$ *is bijective. The induced morphism* $\widehat{A}_K \otimes_{A^\dagger_K} E \to \widehat{F}$ *is therefore an isomorphism of* $\mathrm{MIC}^\dagger(\widehat{A}_K/K)$.

Proof. The canonical morphism $\mathcal{V}\{\underline{t}\} \otimes_{\mathcal{V}[\underline{t}]^\dagger} E \to \widehat{A}_K \otimes_{A^\dagger_K} E \to E$ is an isomorphism. Hence, this follows from 17.7.3.9. □

17.7.4 The main result

We keep notations of 17.7.3. Moreover, let $U^\dagger$ be an affine open subspace of $\mathbb{A}^{n\dagger}_{\mathcal{V}}$, $j\colon U^\dagger \subset \mathbb{P}^{n\dagger}_{\mathcal{V}}$ the open immersion, $\mathfrak{U}$ its p-adic completion, $T_0 := \mathbb{P}^n_k \setminus U_0$ be the reduced divisor of $\mathbb{P}^n_k$ whose support is complementary to U_0 (see 17.6.1.5). We set $D_{U^\dagger} := \Gamma(U^\dagger, \mathcal{D}_{U^\dagger})$ and $D^{(m)}_{U^\dagger} := \Gamma(U^\dagger, \mathcal{D}^{(m)}_{U^\dagger})$ (and similarly replacing $U^\dagger$ by another affine smooth $\mathfrak{S}$-weak formal scheme). Let $v\colon Y^\dagger \hookrightarrow U^\dagger$ be a closed immersion of affine and smooth $\mathfrak{S}$-weak formal schemes, with Y_0 integral and $\dim Y_0 = n - r$ for some integer r. Let X_0 be the closure of Y_0 in P. We suppose moreover there exist a finite etale morphism $g_0\colon U_0 \to \mathbb{A}^n_k$ such that $g_0(Y_0) \subset \mathbb{A}^{n-r}_k$, where $\mathbb{A}^{n-r}_k$ is the closed subscheme defined by the equations $t_1 = 0, \ldots, t_r = 0$. We denote by $g\colon U^\dagger \to \mathbb{A}^{n\dagger}_{\mathcal{V}}$ a lifting of g_0. The p-adic completions of v or g will still be denoted by respectively v or g. The main result of this section is the characterization of 17.7.4.6 of overconvergent isocrystals on Y_0.

17.7.4.1. As $U^\dagger$ is an open subset of $\mathbb{A}^{n\dagger}_{\mathcal{V}}$, we obtain the canonical morphism of restriction (for any level m) $D^{(m)}_{\mathbb{A}^{n\dagger}_{\mathcal{V}}} \to D^{(m)}_{U^\dagger}$. By functoriality of the p-adic weak completion this yields $D^{(m)\dagger}_{\mathbb{A}^{n\dagger}_{\mathcal{V}}} \to D^{(m)\dagger}_{U^\dagger}$. Since $D_{U^\dagger} \xrightarrow{\sim} \varinjlim_m D^{(m)}_{U^\dagger}$, then following 17.7.1.9 we have $D^\dagger_{U^\dagger} \xrightarrow{\sim} \varinjlim_m D^{(m)\dagger}_{U^\dagger}$ (and similarly replacing $U^\dagger$ by $\mathbb{A}^{n\dagger}_{\mathcal{V}}$). By passage to the limit on the level, we get therefore the canonical morphism :

$$\Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}) \xrightarrow{\sim} D^\dagger_{\mathbb{A}^{n\dagger}_{\mathcal{V}}} \to D^\dagger_{U^\dagger}. \tag{17.7.4.1.1}$$

Lemma 17.7.4.2. *We have the canonical maps* $D^\dagger_U \to \Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger * T_0))$ *and* $D^\dagger_{U,K} \to \Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}})$.

Proof. Following 17.4.1.2, we have the homomorphism $j_*\mathcal{D}_U \to \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger * T_0)$. This yields the map $D_U \to \Gamma(\mathcal{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger * T_0))$. Since $\Gamma(\mathcal{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger * T_0))$ is a weakly complete $\mathcal{V}$-algebra (see 17.7.1.3), then using the universal property of a weak completion, we get the first map. Since $\Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger * T_0)_{\mathbb{Q}}) = \Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T_0)_{\mathbb{Q}})$, then we get the second one. □

Remark 17.7.4.3. The $\mathcal{V}$-algebra $\Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}(T_0))$ is weakly complete (use 17.7.1.9 and the fact that a complete $\mathcal{V}$-algebra is a weak $\mathcal{V}$-algebra). However, we do not have a map of the form $D^\dagger_U \to \Gamma(\mathfrak{P}, \mathcal{D}^\dagger_{\mathfrak{P}}(T_0))$.

Lemma 17.7.4.4. *Let* $\alpha\colon \widehat{\mathbb{P}}^{n-r}_{\mathcal{V}} \hookrightarrow \widehat{\mathbb{P}}^n_{\mathcal{V}}$ *be the closed immersion defined by* $u_1 = 0, \ldots, u_r = 0$*. Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}}$*-module with support in* $\mathbb{P}^{n-r}_k$ *and* $E := \Gamma(\widehat{\mathbb{P}}^n_{\mathcal{V}}, \mathcal{E})$*. Then* $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{E})) \xrightarrow{\sim} \cap_{i=1}^r \ker(t_i\colon E \to E)$.

Proof. Let us denote by $\mathfrak{P}' := \widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}$, $\mathcal{I}$ the ideal of $\mathcal{O}_{\mathfrak{P}}$ given by the closed immersion α. By taking the p-adic completion of $\mathcal{D}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}} \xrightarrow{\sim} \alpha^{-1}(\mathcal{D}^{(m)}_{\mathfrak{P}}/\mathcal{I}\mathcal{D}^{(m)}_{\mathfrak{P}})$, we get $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}'\to\mathfrak{P}} \xrightarrow{\sim} \alpha^{-1}(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}}/\mathcal{I}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{P}})$. By adding overconvergent singularities along of H_0, by taking the limit on the level and by tensorizing by $\mathbb{Q}$, we get then the isomorphism $\mathcal{D}^{\dagger}_{\mathfrak{P}'\to\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}} \xrightarrow{\sim} \alpha^{-1}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}/\mathcal{I}\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}})$. This implies

$$\alpha^!(\mathcal{E}) = \mathcal{D}^{\dagger}_{\mathfrak{P}'\to\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\alpha^{-1}\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}} \alpha^{-1}\mathcal{E}[-r] \xrightarrow{\sim} \alpha^{-1}(\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}/\mathcal{I}\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}} \mathcal{E})[-r].$$

The ideal $\mathcal{I}\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$ of $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$ is generated by the global sections $t_1, \dots, t_r$, i.e. via the theorem A of coherent $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-modules (see 17.7.3.1), $\Gamma(\mathfrak{P}, \mathcal{I}\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}})$ is the ideal of $\mathcal{V}[\underline{t}]^{\dagger}_K$ generated by $t_1, \dots, t_r$. Via the Koszul resolution induced by the regular sequence of elements $t_1, \dots, t_r$ which generate the ideal $\mathcal{I}\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$ of $\mathcal{O}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$, we compute $H^0(\alpha_*\alpha^!(\mathcal{E})) \xrightarrow{\sim} \cap_{i=1}^r \ker(t_i \colon \mathcal{E} \to \mathcal{E})$. Moreover, since $\mathcal{E}$ has its support in $\mathbb{P}^{n-r}_k$, using the theorem of Berthelot-Kashiwara of 9.3.5.9), we get $H^0(\alpha^!(\mathcal{E})) \xrightarrow{\sim} \alpha^!(\mathcal{E})$. Hence $\alpha_*\alpha^!(\mathcal{E}) \xrightarrow{\sim} \cap_{i=1}^r \ker(t_i \colon \mathcal{E} \to \mathcal{E})$. We conclude by applying to this latter the global sections functor. □

Lemma 17.7.4.5. *Let $\alpha\colon \widehat{\mathbb{P}}^{n-r}_{\mathcal{V}} \hookrightarrow \widehat{\mathbb{P}}^n_{\mathcal{V}}$ be the closed immersion defined by $u_1 = 0, \dots, u_r = 0$. Let $\beta\colon \widehat{\mathbb{A}}^{n-r}_{\mathcal{V}} \hookrightarrow \widehat{\mathbb{A}}^n_{\mathcal{V}}$ be the morphism induced by α. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-module with support in $\mathbb{P}^{n-r}_k$. The canonical diagram*

$$\begin{array}{ccc} \Gamma(\mathbb{P}^{n-r}_k, \alpha^!(\mathcal{E})) & \xrightarrow[17.7.4.4]{\sim} & \cap_{i=1}^r \ker(t_i \colon \Gamma(\mathbb{P}^n_k, \mathcal{E}) \to \Gamma(\mathbb{P}^n_k, \mathcal{E})) \\ \downarrow & & \downarrow \\ \Gamma(\mathbb{A}^{n-r}_k, \alpha^!(\mathcal{E})) = \Gamma(\mathbb{A}^{n-r}_k, \beta^!(\mathcal{E}|\mathbb{A}^n_k)) & \xrightarrow[9.3.1.20.2]{\sim} & \cap_{i=1}^r \ker(t_i \colon \Gamma(\mathbb{A}^n_k, \mathcal{E}) \to \Gamma(\mathbb{A}^n_k, \mathcal{E}), \end{array} \qquad (17.7.4.5.1)$$

where the horizontal isomorphisms come from 17.7.4.4 and 9.3.1.20.2, is commutative.

Proof. This is a consequence of the construction of horizontal isomorphisms. □

Theorem 17.7.4.6. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-module such that $\mathcal{E}|\mathfrak{U} \in \mathrm{MIC}^{\dagger\dagger}(Y_0, \mathfrak{U}/K)$. Then there exists $G \in \mathrm{MIC}^{\dagger}(Y_0/K)$ and a $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T_0)_{\mathbb{Q}}$-linear isomorphism of the form:*

$$\mathrm{sp}_{Y^{\dagger}\hookrightarrow U^{\dagger}, T_0, +}(G) \xrightarrow{\sim} \mathcal{E}({}^{\dagger}T_0).$$

In other words, $\mathcal{E}({}^{\dagger}T_0) \in \mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, T_0/K)$.

Proof. Put $A^{\dagger} := \Gamma(U^{\dagger}, \mathcal{O}_{U^{\dagger}})$, $E := \Gamma(\mathfrak{P}, \mathcal{E})$ and $E' := D^{\dagger}_{U^{\dagger}} \otimes_{D^{\dagger}} E$ where the extension $D^{\dagger} \to D^{\dagger}_{U^{\dagger}}$ defining the tensor product is the one induced by the open immersion $U^{\dagger} \subset A^{n\dagger}_{\mathcal{V}}$, i.e., that of 17.7.4.1.1. Let us denote by $\widetilde{Y}^{\dagger} := g^{-1}(\mathbb{A}^{n-r\dagger}_{\mathcal{V}})$, $a\colon \widetilde{Y}^{\dagger} \to \mathbb{A}^{n-r\dagger}_{\mathcal{V}}$ the finite etale morphism induced by g. Let $w\colon Y^{\dagger} \hookrightarrow \widetilde{Y}^{\dagger}$ (resp. $\widetilde{v}\colon \widetilde{Y}^{\dagger} \hookrightarrow U^{\dagger}$) a lifting of the closed immersion $Y_0 \hookrightarrow \widetilde{Y}_0$ (resp. $\widetilde{Y}_0 \hookrightarrow U_0$). Let us denote by $\beta\colon \mathbb{A}^{n-r\dagger}_{\mathcal{V}} \hookrightarrow \mathbb{A}^{n\dagger}_{\mathcal{V}}$ and $\alpha\colon \mathbb{P}^{n-r\dagger}_{\mathcal{V}} \hookrightarrow \mathbb{P}^{n\dagger}_{\mathcal{V}}$ the canonical closed immersions given by respectively $t_1 = 0, \dots, t_r = 0$ and $u_1 = 0, \dots, u_r = 0$. The p-adic completions of morphisms of smooth $\mathcal{S}$-weak formal schemes are still abusively designated by the same letter, e.g., $\beta\colon \widehat{\mathbb{A}}^{n-r}_{\mathcal{V}} \hookrightarrow \widehat{\mathbb{A}}^n_{\mathcal{V}}$ or $\alpha\colon \widehat{\mathbb{P}}^{n-r}_{\mathcal{V}} \hookrightarrow \widehat{\mathbb{P}}^n_{\mathcal{V}}$. Finally, let us denote by $x_1, \dots, x_n$ the local coordinates of $U^{\dagger}$ corresponding to $t_1, \dots, t_n$ via g^*. Let us denote by $G := \cap_{i=1}^r \ker(x_i \colon E' \to E')$.

I) The module $G \in \mathrm{MIC}^{\dagger}(A^{\dagger}_K/K)$ (see notation 17.5.1.1).

One main problem is to establish that the module G is a coherent $\Gamma(Y^{\dagger}, \mathcal{O}_{Y^{\dagger},\mathbb{Q}})$-module. The idea is to reduce via the morphism g to the case where the compactification of Y_0 in P_0 is smooth.

0) i) The morphism g induces a ring homomorphism $\rho_g\colon D^{\dagger}_K \to D^{\dagger}_{\mathfrak{U},K}$ (see the end of 17.7.2.3). We denote by $g_*(E')$ the module E' viewed as a $D^{\dagger}_K$-module via ρ_g. We construct a canonical isomorphism of coherent $D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K}$-modules of the form $D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^{\dagger}_K} g_*(E') \xrightarrow{\sim} \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, g_+(\mathcal{E}|\mathfrak{U}))$ as follows: We have the $D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K}$-linear isomorphism:

$$D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^{\dagger}_K} g_*(E') \overset{17.7.2.6}{\xrightarrow{\sim}} D^{\dagger}_{\mathfrak{U},K} \otimes_{D^{\dagger}_{U^{\dagger},K}} E' \xrightarrow{\sim} D^{\dagger}_{\mathfrak{U},K} \otimes_{D^{\dagger}_K} E \overset{17.7.3.1.3}{\xrightarrow{\sim}} \Gamma(\mathfrak{U}, \mathcal{E}). \qquad (17.7.4.6.1)$$

Moreover, as g is finite etale, then following 9.2.4.15.2 we have the isomorphism $g_+(\mathcal{E}|\mathfrak{U}) \xrightarrow{\sim} g_*(\mathcal{E}|\mathfrak{U})$ of $\mathcal{D}^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$-modules. Since g is proper and $\mathcal{E}|\mathfrak{U}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}}$-module, then $g_+(\mathcal{E}|\mathfrak{U})$ is more precisely a coherent $\mathcal{D}^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$-module. Hence, following the theorem of type A (e.g. see 17.7.3.1.2), we get the isomorphisms of coherent $D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$-modules:

$$\Gamma(\mathfrak{U},\mathcal{E}) \xrightarrow{\sim} \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, g_*(\mathcal{E}|\mathfrak{U})) \xrightarrow{\sim} \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, g_+(\mathcal{E}|\mathfrak{U})). \tag{17.7.4.6.2}$$

Composing 17.7.4.6.1 and 17.7.4.6.2, we are done.

ii) Since the extension $D^{\dagger}_K \to D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K}$ is faithfully flat (see 17.7.3.2), since $D^{\dagger}_K$ is a coherent ring, then we deduce from i) that $g_*(E')$ is a coherent $D^{\dagger}_K$-module. Hence, we construct a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-module by setting

$$\mathcal{F} := \mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}} \otimes_{D^{\dagger}_K} g_*(E').$$

1) *Let us check the isomorphism* $\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}} \xrightarrow{\sim} g_+(\mathcal{E}|\mathfrak{U})$.

Since $\mathcal{F}|\mathfrak{U} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}} \otimes_{D^{\dagger}_K} g_*(E')$ is a coherent $\mathcal{D}^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$-modules, then it follows from theorem of type A the first isomorphism:

$$\Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathcal{F}) \xrightarrow{\sim} D^{\dagger}_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^{\dagger}_K} g_*(E') \overset{0.i)}{\xrightarrow{\sim}} \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, g_+(\mathcal{E}|\mathfrak{U})).$$

Since $\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}}$ and $g_+(\mathcal{E}|\mathfrak{U})$ are both coherent, then we conclude via theorem of type A.

2) Put $\mathcal{H} := v^!(\mathcal{E}|\mathfrak{U})$. Since $\mathcal{H} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{Y}/\mathcal{V})$ (see notation 11.1.1.3), then $\mathcal{H}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-module which is also coherent as $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-module (see 11.1.1.2). Moreover, as $\dim \widetilde{Y}_0 = \dim Y_0$ and as $\widetilde{Y}_0$ is smooth, Y_0 is then a connected component of $\widetilde{Y}_0$. This yields that $\widetilde{\mathcal{H}} := w_+(\mathcal{H})$ is a coherent $\mathcal{D}^{\dagger}_{\widetilde{\mathfrak{Y}},\mathbb{Q}}$-module which is coherent as $\mathcal{O}_{\widetilde{\mathfrak{Y}},\mathbb{Q}}$-module, i.e. $\widetilde{\mathcal{H}} \in \mathrm{MIC}^{\dagger\dagger}(\widetilde{\mathfrak{Y}}/\mathcal{V})$.

3) We have the isomorphism $\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}} \xrightarrow{\sim} \beta_+ a_+(\widetilde{\mathcal{H}})$. Indeed, since $\mathcal{E}|\mathfrak{U}$ has its support in Y then following the theorem of Berthelot-Kashiwara (see 9.3.5.9), we have the canonical isomorphism $\mathcal{E}|\mathfrak{U} \xrightarrow{\sim} v_+(\mathcal{H})$. Moreover, following the step 1), $\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}} \xrightarrow{\sim} g_+(\mathcal{E}|\mathfrak{U})$. This yields: $\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}} \xrightarrow{\sim} g_+ v_+(\mathcal{H}) \xrightarrow{\sim} g_+\widetilde{v}_+(\widetilde{\mathcal{H}}) \xrightarrow{\sim} \beta_+ a_+(\widetilde{\mathcal{H}})$.

4) *The sheaf* $\mathcal{F}$ *has its support in* $\mathbb{P}^{n-r}_k$.

Let us denote by $H_1,\dots,H_r$ the hyperplans of $\mathbb{P}^n_k$ corresponding to $u_1 = 0,\dots,u_r = 0$. It follows from the isomorphism of the step 3) that $\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}}$ has its support in $\mathbb{A}^{n-r}_k$. So, for any $s = 1,\dots,r$, $\mathcal{F}({}^{\dagger}H_s)$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_s \cup H_0)_{\mathbb{Q}}$-module which is null outside $H_s \cup H_0$. By 8.7.6.11, this implies $\mathcal{F}({}^{\dagger}H_s) = 0$. By using the triangle of localisation with respect to H_s, this yields the isomorphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_s}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. Since $\mathbb{P}^{n-r}_k = H_1 \cap \cdots \cap H_r$, using 13.1.5.6.1 we get therefore the isomorphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{\mathbb{P}^{n-r}_k}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. This is equivalent to saying that $\mathcal{F}$ has its support in $\mathbb{P}^{n-r}_k$ and we are done.

5) *Let us check that* $\mathcal{F} \in \mathrm{MIC}^{\dagger\dagger}(P', \mathfrak{P}, H_0/\mathcal{V})$ *and* $\alpha^!(\mathcal{F}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}', H_0/\mathcal{V})$*, i.e. they are associated to an isocrystal of* $\mathrm{MIC}^{\dagger}(\mathbb{A}^{n-r}_k/K)$.

Following the step 4), it is sufficient to establish that $\alpha^!(\mathcal{F}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}', H_0/\mathcal{V})$. Following the theorem of Berthelot-Kashiwara, it follows from the step 4) that $\alpha^!(\mathcal{F})$ is a coherent $\mathcal{D}^{\dagger}_{\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}}({}^{\dagger}H_0 \cap \mathbb{P}^{n-r}_k)_{\mathbb{Q}}$-module and we have the isomorphism: $\alpha_+ \circ \alpha^!(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. Following the characterization 11.2.1.14.(e) of an overconvergent isocrystal, to check that $\alpha^!(\mathcal{F}) \in \mathrm{MIC}^{\dagger\dagger}(P', \mathfrak{P}', H_0/\mathcal{V})$, it is therefore sufficient to establish that $\alpha^!(\mathcal{F})|\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}} \in \mathrm{MIC}^{\dagger\dagger}(\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}}/\mathcal{V})$. We have the isomorphisms: $\alpha^!(\mathcal{F})|\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}} \xrightarrow{\sim} \beta^!(\mathcal{F}|\widehat{\mathbb{A}}^n_{\mathcal{V}}) \underset{3)}{\xrightarrow{\sim}} \beta^!\beta_+ a_+(\widetilde{\mathcal{H}}) \xrightarrow{\sim} a_+(\widetilde{\mathcal{H}})$. Moreover, following the step 2), $\widetilde{\mathcal{H}} \in \mathrm{MIC}^{\dagger\dagger}(\widetilde{\mathfrak{Y}}/\mathcal{V})$. As a is finite etale, then the functor a_+ factors through $a_+ \colon \mathrm{MIC}^{\dagger\dagger}(\widetilde{\mathfrak{Y}}/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}}/\mathcal{V})$. Hence, we are done.

6) *Set* $\widetilde{A}^{\dagger} := \Gamma(\widetilde{Y}^{\dagger}, \mathcal{O}_{\widetilde{Y}^{\dagger}})$. *The module* G *belongs to* $\mathrm{MIC}^{\dagger}(\widetilde{A}^{\dagger}_K/K)$ *(see notation 17.5.1.1) and* $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F})) = a_*(G)$.

a) As $\alpha^!(\mathcal{F}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}', H_0/\mathcal{V})$, following 17.7.3.3 (with n replaced by $n-r$), $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F}))$ is a coherent $D^\dagger_{\mathbb{A}^{n-r\dagger}_{\mathcal{V}},K}$-module, $\mathcal{V}[t_{r+1},\dots,t_n]^\dagger\otimes_{\mathcal{V}}K$-coherent, i.e. $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}},\alpha^!(\mathcal{F}))\in \mathrm{MIC}^{\dagger\dagger}(\mathbb{A}^{n-r\dagger}_{\mathcal{V}}/\mathcal{V})$.

b) In this step, we check that $G\in \mathrm{MIC}(\widetilde{A}^\dagger_K/K)$.

i) First, let us construct a structure of left $D_{\widetilde{Y}^\dagger,K}$-module on G extending its structure of $\Gamma(\widetilde{Y}^\dagger,\mathcal{O}_{\widetilde{Y}^\dagger})$-module A priori this structure depends on the choice of the coordinates $x_1,\dots,x_r$ (we will not need it but this is left to the reader to check its canonicity).

Let $\widetilde{I}$ be the ideal of $A^\dagger$ generated by $x_1,\dots,x_r$. Then $\widetilde{I}$ is the ideal defining the closed immersion $\widetilde{v}$, i.e. $\Gamma(\widetilde{Y}^\dagger,\mathcal{O}_{\widetilde{Y}^\dagger})=A^\dagger/\widetilde{I}=:\widetilde{B}^\dagger$.

We denote by $D_{\widetilde{v},\underline{x},K}$ the free $A^\dagger_K$-module with the basis $\{\underline{\partial}^{[(\underline{0},\underline{i})]}\mid \underline{i}\in\mathbb{N}^{n-r}\}$, where $\underline{0}:=(0,\dots,0)\in\mathbb{N}^r$. The sheaf $D_{\widetilde{v},\underline{x},K}$ is equal to the sub-$\mathcal{V}$-algebra of $D_{U^\sharp,K}$ which is generated by $A^\dagger_K$, by ∂_i for any $i=n-r+1,\dots,n$.

Since $D_{U^\dagger,K}$ is a free left $D_{\widetilde{v},\underline{x},K}$-module with the basis $\{\underline{\partial}^{[(\underline{h},\underline{0})]}\mid \underline{h}\in\mathbb{N}^r\}$, where $\underline{0}:=(0,\dots,0)\in\mathbb{N}^{n-r}$. This means that the canonical homomorphism of $\mathcal{V}$-algebras

$$\mu\colon \mathcal{V}[\partial_1,\dots,\partial_r]\otimes_{\mathcal{V}}D_{\widetilde{v},\underline{x},K}\xrightarrow{\sim}D_{U^\dagger,K}\tag{17.7.4.6.3}$$

given by $P\otimes Q\mapsto PQ$ is also an isomorphism of left $D_{\widetilde{v},\underline{x},K}$-modules.

Since $x_1,\dots,x_r$ generate $\widetilde{I}$ and are in the center of $D_{\widetilde{v},\underline{x},K}$, then we compute $\widetilde{I}D_{\widetilde{v},\underline{x},K}=D_{\widetilde{v},\underline{x},K}\widetilde{I}$. Hence, we get a canonical $\mathcal{V}$-algebra structure on $D_{\widetilde{v},\underline{x},K}/\widetilde{I}D_{\widetilde{v},\underline{x},K}$ induced by that of $D_{\widetilde{v},\underline{x},K}$. Since $\widetilde{I}D_{\widetilde{v},\underline{x},K}=D_{\widetilde{v},\underline{x},K}\cap\widetilde{I}D_{U^\dagger,K}$, then we get the inclusion

$$D_{\widetilde{v},\underline{x},K}/\widetilde{I}D_{\widetilde{v},\underline{x},K}\hookrightarrow D_{U^\dagger,K}/\widetilde{I}D_{U^\dagger,K}.$$

The morphism 17.7.4.6.3 induces the isomorphism of left $D_{\widetilde{v},\underline{x},K}/\widetilde{I}D_{\widetilde{v},\underline{x},K}$-modules:

$$\mathcal{V}[\partial_1,\dots,\partial_r]\otimes_{\mathcal{V}}(D_{\widetilde{v},\underline{x},K}/\widetilde{I}D_{\widetilde{v},\underline{x},K})\xrightarrow{\sim}D_{U^\dagger,K}/\widetilde{I}D_{U^\dagger,K}.\tag{17.7.4.6.4}$$

Taking the structure of $A^\dagger_K$-module induced by the structure of left $D_{U^\dagger,K}$-module on $D_{\widetilde{v},\underline{x},K}$ (resp. induced by the structure of left $D_{\widetilde{Y}^\dagger,K}$-module on $D_{\widetilde{Y}^\dagger,K}$), we denote by

$$\sigma_{\widetilde{u},\widetilde{x}}\colon D_{\widetilde{v},\underline{x},K}\to D_{\widetilde{Y}^\dagger,K},\tag{17.7.4.6.5}$$

the $A^\dagger_K$-linear morphism given by $\sigma_{\widetilde{u},\widetilde{x}}(\underline{\partial}^{[(\underline{0},\underline{i})]})=\underline{\partial}^{[\underline{i}]}$, for any $\underline{i}\in\mathbb{N}^{n-r}$.

The kernel of $\sigma_{\widetilde{u},\widetilde{x}}$ is $\widetilde{I}D_{\widetilde{v},\underline{x},K}$ and we get therefore the $\widetilde{B}^\dagger_K$-linear isomorphism (for the left structures):

$$\overline{\sigma}_{\widetilde{u},\widetilde{x}}\colon D_{\widetilde{v},\underline{x},K}/\widetilde{I}D_{\widetilde{v},\underline{x},K}=D_{\widetilde{v},\underline{x},K}/D_{\widetilde{v},\underline{x},K}\widetilde{I}\xrightarrow{\sim}D_{\widetilde{Y}^\dagger,K}.\tag{17.7.4.6.6}$$

which satisfies the formula $\overline{\sigma}_{\widetilde{u},\widetilde{x}}(\underline{\xi}^{[(\underline{0},\underline{i})]})=\underline{\partial}^{[\underline{i}]}$, for any $\underline{i}\in\mathbb{N}^{n-r}$, where $\underline{\xi}^{[(\underline{0},\underline{i})]}$ is the image of $\underline{\partial}^{[(\underline{0},\underline{i})]}$. We denote by $\vartheta:=(\overline{\sigma}_{\widetilde{u},\widetilde{x}})^{-1}$ by $[-]_{\widetilde{Y}^\dagger}\colon D_{\widetilde{v},\underline{x},K}\to D_{\widetilde{v},\underline{x},K}/\widetilde{I}D_{\widetilde{v},\underline{x},K}$ the projection.

We define the structure of $D_{\widetilde{Y}^\dagger,K}$-module on G as follows: Let $Q\in D_{\widetilde{Y}^\dagger,K}$. Choose any $Q_{U^\dagger}\in D_{\widetilde{v},\underline{x},K}$ such that $\vartheta(Q)=[Q_{U^\dagger}]_{\widetilde{Y}^\dagger}$. For any $x\in G$, we set

$$Q\cdot x:=Q_{U^\dagger}\cdot x.\tag{17.7.4.6.7}$$

Since G is annihilated by $\widetilde{I}$, then we can check this is well defined.

ii) Since a is finite then $\widetilde{A}^\dagger_K$ is a finite $\mathcal{V}[t_{r+1},\dots,t_n]^\dagger$-algebra. This yields that G is a coherent $\widetilde{A}^\dagger_K$-module, i.e. $G\in\mathrm{MIC}(\widetilde{A}^\dagger_K/K)$.

c) Following 17.7.4.4, as $g_*(E')=\Gamma(\widehat{\mathbb{P}}^n_{\mathcal{V}},\mathcal{F})$, then we obtain the equalities:

$$\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}},\alpha^!(\mathcal{F}))=\cap_{i=1}^r\ker(t_i\colon g_*(E')\to g_*(E'))=\cap_{i=1}^r\ker(x_i\colon E'\to E')=G.\tag{17.7.4.6.8}$$

We denote by $a_*(G)$ the induced coherent $D_{\mathbb{A}^{n-r\dagger}_{\mathcal{V}},K}$-module via the canonical morphism $D_{\mathbb{A}^{n-r\dagger}_{\mathcal{V}},K}\to D_{\widetilde{Y}^\dagger,K}$ induced by a. Comparing the formulas 17.7.4.6.7 and 9.3.1.20.3, we compute that the equality

17.7.4.6.8 is an equality of $D_{\mathbb{A}^{n-r\dagger}_{\mathcal{V}},K}$-modules, i.e. $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F})) = a_*(G)$. Using 17.7.3.6 and the step a) and b), this yields that $G \in \mathrm{MIC}^\dagger(\widetilde{A}^\dagger_K/K)$.

7) We denote by $O_{\widetilde{\mathfrak{Y}}} := \Gamma(\widetilde{\mathfrak{Y}}, \mathcal{O}_{\widetilde{\mathfrak{Y}}})$ the p-adic completion of $\widetilde{A}^\dagger$. We set $\widehat{G} := D^\dagger_{\widetilde{\mathfrak{Y}},K} \otimes_{D^\dagger_{\widetilde{Y}^\dagger,K}} G \in \mathrm{MIC}^\dagger(O_{\widetilde{\mathfrak{Y}},K}/K)$ (see notation 17.7.3.9). In this step, we check the isomorphism $\widehat{G} \xrightarrow{\sim} \Gamma(\widetilde{\mathfrak{Y}}, \widetilde{\mathcal{H}})$ of $\mathrm{MIC}^\dagger(O_{\widetilde{\mathfrak{Y}},K}/K)$.

a) Following 17.7.2.6, the canonical morphism of $(D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K}, D^\dagger_{U^\dagger,K})$-bimodules $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^\dagger_K} D^\dagger_{U^\dagger,K} \to D^\dagger_{\mathfrak{U},K}$ is an isomorphism. This yields the $D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K}$-linear isomorphism : $D^\dagger_{\mathfrak{U},K} \otimes_{D^\dagger_{U^\dagger,K}} E' \xleftarrow{\sim} D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^\dagger_K} E'$. Since g is proper, then it follows from 13.2.3.7.1 (see also example 13.2.3.2) that we have the base change isomorphism: $\alpha^! \circ g_+ \xrightarrow{\sim} a_+ \circ \widetilde{v}^!$. Since g and a are finite and etale then $g_+ = g_*$ and $a_+ = a_*$. With 9.3.1.20.2 and via the theorem A for coherent $\mathcal{D}^\dagger_{\widehat{\mathbb{A}}^{n-r},\mathbb{Q}}$-modules (resp. coherent $\mathcal{D}^\dagger_{\widehat{\mathbb{A}}^n,\mathbb{Q}}$-modules), for any $D^\dagger_{\mathfrak{U},K}$-module M the equality $a_*(\cap_{i=1}^r \ker(x_i \colon M \to M)) = \cap_{i=1}^r \ker(t_i \colon g_*M \to g_*M)$ is an equality as $D^\dagger_{\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}},K}$-modules. Hence, we obtain the $D^\dagger_{\widehat{\mathbb{A}}^{n-r},K}$-linear bottom morphism of the following commutative diagram

$$\begin{array}{ccc} G = \cap_{i=1}^r \ker(x_i \colon E' \to E') & = & \cap_{i=1}^r \ker(t_i \colon E' \to E') \\ \downarrow & & \downarrow \\ \cap_{i=1}^r \ker(x_i \colon D^\dagger_{\mathfrak{U},K} \otimes_{D^\dagger_{U^\dagger,K}} E' \to D^\dagger_{\mathfrak{U},K} \otimes_{D^\dagger_{U^\dagger,K}} E') & \xleftarrow{\sim} & \cap_{i=1}^r \ker(t_i \colon D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^\dagger_K} E' \to D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^\dagger_K} E'). \end{array} \tag{17.7.4.6.9}$$

b) It follows from the step 2) and 17.7.3.9 that $\Gamma(\widetilde{\mathfrak{Y}}, \widetilde{\mathcal{H}}) \in \mathrm{MIC}^\dagger(O_{\widetilde{\mathfrak{Y}},K}/K)$. Moreover, the left bottom term of 17.7.4.6.9 is canonically isomorphic to $\Gamma(\widetilde{\mathfrak{Y}}, \widetilde{\mathcal{H}})$. Indeed, as $\widetilde{\mathcal{H}} \xrightarrow{\sim} \widetilde{v}^!(\mathcal{E}|\mathfrak{U})$, then using 9.3.1.20.2 we get the isomorphism: $\Gamma(\widetilde{\mathfrak{Y}}, \widetilde{\mathcal{H}}) \xrightarrow{\sim} \cap_{i=1}^r \ker(x_i \colon \Gamma(\mathfrak{U}, \mathcal{E}) \to \Gamma(\mathfrak{U}, \mathcal{E}))$. Moreover, we have checked in 17.7.4.6.1 the isomorphism $D^\dagger_{\mathfrak{U},K} \otimes_{D^\dagger_{U^\dagger,K}} E' \xrightarrow{\sim} \Gamma(\mathfrak{U}, \mathcal{E})$. Hence we are done.

c) We compute moreover that the left (resp. right) arrow of 17.7.4.6.9 is $D_{\widetilde{Y}^\dagger,K}$-linear (resp. $D_{\mathbb{A}^{n-r\dagger},K}$-linear).

d) According to the theorem of type A of 17.7.3.1.1, we get $\Gamma(\widehat{\mathbb{P}}^n_{\mathcal{V}}, \mathcal{F}) \xrightarrow{\sim} g_*E'$. Via 17.7.3.8, this yields that the injection $\Gamma(\widehat{\mathbb{P}}^n_{\mathcal{V}}, \mathcal{F}) \subset \Gamma(\widehat{\mathbb{A}}^n_{\mathcal{V}}, \mathcal{F})$ is canonically isomorphic the morphism $E' \to D^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},K} \otimes_{D^\dagger_K} E'$. Following 17.7.4.5.1, this yields that the right arrow of 17.7.4.6.9 is isomorphic to the injection $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F})) \subset \Gamma(\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F}))$. Since $\alpha^!(\mathcal{F}) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{P}', H_0/\mathcal{V})$ (see step 5), then $\Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F})) \subset \Gamma(\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F}))$ induced by extension the isomorphism $\mathcal{V}\{t_{r+1}, \ldots, t_n\} \otimes_{\mathcal{V}[t_{r+1},\ldots,t_n]^\dagger} \Gamma(\widehat{\mathbb{P}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F})) \xrightarrow{\sim} \Gamma(\widehat{\mathbb{A}}^{n-r}_{\mathcal{V}}, \alpha^!(\mathcal{F}))$ (see 17.7.3.8). This yields that the injection $G \hookrightarrow \Gamma(\widetilde{\mathfrak{Y}}, \widetilde{\mathcal{H}})$ (left arrow of 17.7.4.6.9 modulo the isomorphism of 7.b)) induced by extension the isomorphism

$$\mathcal{V}\{t_{r+1}, \ldots, t_n\} \otimes_{\mathcal{V}[t_{r+1},\ldots,t_n]^\dagger} G \xrightarrow{\sim} \Gamma(\widetilde{\mathfrak{Y}}, \widetilde{\mathcal{H}}).$$

e) Via a), b), c), d), we conclude thanks to the lemma 17.7.3.10.

8) *The module* $G \in \mathrm{MIC}^\dagger(A^\dagger_K/K)$.

Using the equivalence of categories of 17.5.1.1.(e), we get from the step 6) $\widetilde{\mathcal{G}} := \mathcal{D}_{\widetilde{Y}^\dagger,\mathbb{Q}} \otimes_{D_{\widetilde{Y}^\dagger,K}} G \in \mathrm{MIC}^{\dagger\dagger}(\widetilde{Y}^\dagger/\mathcal{V})$ and from 17.5.1.6, $\mathcal{D}^\dagger_{\widetilde{\mathfrak{Y}},\mathbb{Q}} \otimes_{\mathcal{D}_{\widetilde{Y}^\dagger,K}} \widetilde{\mathcal{G}} \in \mathrm{MIC}^{\dagger\dagger}(\widetilde{\mathfrak{Y}}/\mathcal{V})$. Via the equivalence of categories of 17.5.1.1.(e), the step 7) means that we have the isomorphism: $\mathcal{D}^\dagger_{\widetilde{\mathfrak{Y}},\mathbb{Q}} \otimes_{\mathcal{D}_{\widetilde{Y}^\dagger,K}} \widetilde{\mathcal{G}} \xrightarrow{\sim} \widetilde{\mathcal{H}}$ of $\mathrm{MIC}^{\dagger\dagger}(\widetilde{Y}^\dagger/\mathcal{V})$. As $\widetilde{\mathcal{H}} = w_+(\mathcal{H}) = w_*(\mathcal{H})$, then the restriction of $\widetilde{\mathcal{H}}$ on $\mathfrak{Y}^\flat := \widetilde{\mathfrak{Y}} \setminus \mathfrak{Y}$ is null. Since the functor $\mathcal{D}^\dagger_{\widetilde{\mathfrak{Y}},\mathbb{Q}} \otimes_{\mathcal{D}_{\widetilde{Y}^\dagger,K}} - \colon \mathrm{MIC}^{\dagger\dagger}(\widetilde{Y}^\dagger/\mathcal{V}) \to \mathrm{MIC}^{\dagger\dagger}(\widetilde{\mathfrak{Y}}/\mathcal{V})$ is faithful (see 17.5.1.6 and use the full flatness of $\widetilde{A}^\dagger_K \to O_{\widetilde{\mathfrak{Y}},K}$), this yields $\widetilde{\mathcal{G}}|\mathfrak{Y}^\flat = 0$. As $w^!(\widetilde{\mathcal{G}}) \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathcal{V})$, then we get the second equality: $G = \Gamma(\widetilde{Y}^\dagger, \widetilde{\mathcal{G}}) = \Gamma(Y^\dagger, w^!(\widetilde{\mathcal{G}})) \in \mathrm{MIC}^{\dagger\dagger}(A^\dagger_K/K)$.

II) Construction of the isomorphism $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(G_{Y_0}) \xrightarrow{\sim} \mathcal{E}(^\dagger T_0)$*, where* G_{Y_0} *means the object of* $\mathrm{MIC}^\dagger(Y_0/K)$ *associated with* $G \in \mathrm{MIC}^\dagger(A^\dagger_K/K)$ *via the equivalence of categories 17.5.1.1.(c))*

1) Let $\mathcal{G} := \mathcal{D}_{Y^\dagger,\mathbb{Q}} \otimes_{D_{Y^\dagger,K}} G \in \mathrm{MIC}^{\dagger\dagger}(Y^\dagger/\mathcal{V})$. Since $\Gamma(Y,\mathcal{G}) \xrightarrow{\sim} G$, then with notation 17.6.3.1.1 we get the isomorphism:

$$\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{D_{U^\dagger,K}} v_+(G) \overset{17.6.3.1.1}{\xrightarrow{\sim}} \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{j_*\mathcal{D}_{U^\dagger,\mathbb{Q}}} j_* v_+(\mathcal{G}) = \mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(G_{Y_0}). \qquad (17.7.4.6.10)$$

2) Let us denote by $\widetilde{\mathcal{G}} := w_+(\mathcal{G})$ the direct image of $\mathcal{G}$ by w (beware as $\mathcal{D}$-module and not as $\mathcal{D}^\dagger$-module), $\widetilde{v}_+(G) := \Gamma(U^\dagger, \mathcal{D}_{U^\dagger \leftarrow \widetilde{Y}^\dagger,\mathbb{Q}}) \otimes_{D_{\widetilde{Y}^\dagger,K}} G$. Let $\partial_1, \ldots, \partial_n$ be the derivations corresponding to the coordinates $x_1, \ldots, x_n$. Similarly to 5.2.6.1, as $G = \cap_{i=1}^r \ker(x_i \colon E' \to E')$ we get the canonical $D_{U^\dagger,K}$-linear morphism $\widetilde{v}_+(G) \to E'$. Modulo the isomorphism $\widetilde{v}_+(G) \xrightarrow{\sim} K[\partial_1, \ldots, \partial_r] \otimes_K G$ (same computation as in 5.2.6.1), the map is $\widetilde{v}_+(G) \to E'$ is given by $\partial_i \otimes x \mapsto \partial_i \cdot x$.

By transitivity of the direct image, we obtain $v_+(\mathcal{G}) \xrightarrow{\sim} \widetilde{v}_+(\widetilde{\mathcal{G}})$. As $\widetilde{\mathcal{G}}$ is a coherent $\mathcal{D}_{\widetilde{Y}^\dagger,\mathbb{Q}}$-module such that $\Gamma(\widetilde{Y}^\dagger, \widetilde{\mathcal{G}}) = \Gamma(Y^\dagger, \mathcal{G}) = G$, this yields (thanks to 17.2.5.4): $v_+(G) \xrightarrow{\sim} \widetilde{v}_+(G)$. Hence the $D_{U^\dagger,K}$-linear morphism : $v_+(G) \to E'$. This yields the $D^\dagger_{U^\dagger,K}$-linear morphism :

$$D^\dagger_{U^\dagger,K} \otimes_{D_{U^\dagger,K}} v_+(G) \to E' = D^\dagger_{U^\dagger,K} \otimes_{D^\dagger_K} E. \qquad (17.7.4.6.11)$$

Moreover, we have following 17.7.4.2 the map $D^\dagger_{U^\dagger,\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}}$. By applying the functor $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{D^\dagger_{U^\dagger,K}} -$ to the map 17.7.4.6.11, we obtain: $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{D_{U^\dagger,K}} v_+(G) \to \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} E$. Moreover,

$$\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} E \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger H_0)_{\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger H_0)_{\mathbb{Q}} \otimes_{D^\dagger_K} E \overset{17.7.3.1.1}{\xrightarrow{\sim}} \mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger H_0)_{\mathbb{Q}}} \mathcal{E} = \mathcal{E}(^\dagger T_0),$$

Hence, we get by composition the map:

$$\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}} \otimes_{D_{U^\dagger,K}} v_+(G) \to \mathcal{E}(^\dagger T_0). \qquad (17.7.4.6.12)$$

3) By composing the inverse of 17.7.4.6.10 with 17.7.4.6.12, we obtain therefore the canonical morphism ϕ: $\mathrm{sp}_{Y^\dagger \hookrightarrow U^\dagger, T_0,+}(G_{Y_0}) \to \mathcal{E}(^\dagger T_0)$. Via 17.5.2.5, we check that ϕ is an isomorphism apart from T_0. Following 8.7.6.11, since ϕ is a morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}}$-modules, this yields that ϕ is an isomorphism. □

Remark 17.7.4.7. In the proof of 17.7.4.6, we have used the theorem of type A of the form 17.7.3.1.1 (see the step II.2). Hence, unless some new idea arises, the hypothesis that the coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger T_0)_{\mathbb{Q}}$-module $\mathcal{E}(^\dagger T_0)$ comes by extension from a coherent $\mathcal{D}^\dagger_{\mathfrak{P}}(^\dagger H_0)_{\mathbb{Q}}$-module is essential (what is important in fact is that H_0 is ample).

Chapter 18

Coefficients stable under Grothendieck's six operations with Frobenius structure

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. When we work with F-complex, we suppose there exists an automorphism $\sigma \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining.

18.1 Overholonomicity

18.1.1 Overholonomicity (after any base change) in a smooth $\mathcal{V}$-formal scheme

Definition 18.1.1.1. Let $\mathcal{E}$ be an object of $(F\text{-})D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. We define by induction on the integer $r \geq 0$, the notion on of *r-overholonomicity in* $\mathfrak{P}$, as follows:

(a) The complex $\mathcal{E}$ is "0-overholonomic in $\mathfrak{P}$" if $\mathcal{E}$ is overcoherent in $\mathfrak{P}$;

(b) for any integer $r \geq 1$, $\mathcal{E}$ is " r-overholonomic in $\mathfrak{P}$" if $\mathcal{E}$ is $r-1$-overholonomic in $\mathfrak{P}$ and for any divisor T of X, the complex $\mathbb{D} \circ ({}^\dagger T)(\mathcal{E})$ is $r-1$-overholonomic in $\mathfrak{P}$.

We say that $\mathcal{E}$ is "∞-overholonomic in $\mathfrak{P}$" or "overholonomic in $\mathfrak{P}$" if $\mathcal{E}$ is r-overholonomic in $\mathfrak{P}$ for any integer $r \geq 0$. Finally, for any $r \in \mathbb{N} \cup \{+\infty\}$, a $(F\text{-})\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module is r-overholonomic in $\mathfrak{P}$ if so is as an object of $(F\text{-})D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

Definition 18.1.1.2. Let $r \in \mathbb{N} \cup \{+\infty\}$ and $\mathcal{E}$ be an object of $(F\text{-})D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. We say that $\mathcal{E}$ is "r-overholonomic in $\mathfrak{P}$ after any base change" if, for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$, denoting by $\mathfrak{S}' := \operatorname{Spf} \mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f \colon \mathfrak{P}' \to \mathfrak{P}$ the projection, the complex $\mathcal{D}^\dagger_{\mathfrak{P}'/\mathfrak{S}',\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}} f^{-1}\mathcal{E}$ is r-overholonomic in $\mathfrak{P}'$, i.e. the base change $f^!(\mathcal{E})$ of $\mathcal{E}$ via $\mathcal{V} \to \mathcal{V}'$ is r-overholonomic in $\mathfrak{P}'$ (see 9.2.7.1). An $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module is r-overholonomic in $\mathfrak{P}$ after any base change if so is as an object of $(F\text{-})D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. We denote by $D^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ the full subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ consisting of overholonomic after any base change in $\mathfrak{P}$ complexes.

Proposition 18.1.1.3. *Let $\mathcal{E} \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ and $r \in \mathbb{N} \cup \{+\infty\}$. Then, the complex $\mathcal{E}$ is r-overholonomic in $\mathfrak{P}$ after any base change if and only if for any integer $j \in \mathbb{Z}$ the modules $H^j(\mathcal{E})$ are r-overholonomic in $\mathfrak{P}$ after any base change.*

Proof. It comes from 15.3.5.4 that r-overholonomic after any base change complexes in $\mathfrak{P}$ are holonomic. Then we check the corollary by induction on r by using the fact that the dual functor $\mathbb{D}$ (resp. the localisation functor $({}^\dagger T)$ outside a divisor T of P) is exact on the category of holonomic (resp. coherent and then holonomic) $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules. □

18.1.2 Overholonomicity (after any base change) and stability

We denote by $\mathfrak{P}$ an smooth $\mathfrak{S}$-formal scheme.

Definition 18.1.2.1. Let $\mathcal{E}$ be an object of $(F\text{-})D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. We define by induction on the integer $r \geq -1$, the notion of r-*overholonomicity*, as follows:

(a) $\mathcal{E}$ is 0-*overholonomic* if and only if $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ (see notation 15.3.6.2);

(b) For any integer $r \geq 1$, $\mathcal{E}$ is r-overholonomic if and only if $\mathcal{E}$ is $r-1$-overholonomic and, for any smooth morphism $f \colon \mathfrak{P}' \to \mathfrak{P}$ of $\mathfrak{S}$-formal schemes, for any divisor T' of X', the complex $\mathbb{D}(^\dagger T')f^!(\mathcal{E})$ is $r-1$-overholonomic.

(c) We say that $\mathcal{E}$ is ∞-*overholonomic* or simply *overholonomic* if $\mathcal{E}$ is r-overholonomic for any integer r. We denote by $(F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ the full subcategory of $(F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ consisting of overholonomic complexes.

(d) A $(F\text{-})\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module is r-overholonomic (resp. overholonomic) if so is as an object of $(F\text{-})D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

Definition 18.1.2.2. Let $r \in \mathbb{N} \cup \{+\infty\}$ and $\mathcal{E}$ be an object of $(F\text{-})D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. We say that $\mathcal{E}$ is "r-overholonomic after any base change" if, for any morphism $\mathcal{V} \to \mathcal{V}'$ of $\mathrm{DVR}(\mathcal{V})$, denoting by $\mathfrak{S}' := \mathrm{Spf}\,\mathcal{V}'$, $\mathfrak{P}' := \mathfrak{P} \times_{\mathfrak{S}} \mathfrak{S}'$, $f \colon \mathfrak{P}' \to \mathfrak{P}$ the projection, the complex $\mathcal{D}^\dagger_{\mathfrak{P}'/\mathfrak{S}',\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{P}/\mathfrak{S},\mathbb{Q}}} f^{-1}\mathcal{E}$ is r-overholonomic, i.e. the base change $f^!(\mathcal{E})$ of $\mathcal{E}$ via $\mathcal{V} \to \mathcal{V}'$ is r-overholonomic (see 9.2.7.1). An $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module is r-overholonomic after any base change if so is as an object of $(F\text{-})D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. We denote by $D^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ the full subcategory of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ consisting of overholonomic after any base change complexes.

Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. We say that $\mathcal{E}^{(\bullet)}$ is r-overholonomic (after any base change) if $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ and if $\underrightarrow{l}^*_{\mathbb{Q}}(\mathcal{E}^{(\bullet)})$ is r-overholonomic (after any base change). We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$) the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ consisting of overholonomic (resp. overholonomic after any base) complexes.

18.1.2.3. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$. It follows from 15.3.6.6 that $\mathcal{E}^{(\bullet)}$ is 0-overholonomic if and only if $\mathcal{E}^{(\bullet)}$ is overcoherent in the sense of definition 15.3.6.1. Using the dual functor 9.2.4.21, we could have defined the overholonomicity by copying the definition 18.1.2.1 in the context of categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ (similarly to 15.3.6.1). The stability properties of the overholonomicity of the subsection is still valid replacing categories of the form $D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ by that of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$.

Remark 18.1.2.4. (a) Let $r' \geq r$ be two nonnegative integers and let $\mathcal{E}$ be an r'-overholonomic complex of $(F\text{-})D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. Then $\mathcal{E}$ is r-overholonomic. In particular $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

(b) This is an open question wether the category $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ is stable by dual functors (for any smooth $\mathfrak{S}$-formal scheme $\mathfrak{P}$). If this was true then, for any integer r, the notion of r-overholonomicity would be equivalent to that of overcoherence (i.e. 0-overholonomicity) and then to that of overholonomicity. By 15.3.2.8, then this would also imply that overcoherent complexes are holonomic.

Proposition 18.1.2.5. *Let $\mathcal{E} \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ and $r \in \mathbb{N} \cup \{+\infty\}$. Then $\mathcal{E}$ is r-overholonomic if and only if, for any smooth morphism $f \colon \mathfrak{P}' \to \mathfrak{P}$ of $\mathfrak{S}$-formal schemes, $f^!(\mathcal{E})$ is r-overholonomic in $\mathfrak{P}'$.*

Proof. The case concerning the overcoherence results from the commutation of the extraordinary inverse image with the functor of localisation (see 13.2.1.4.1). The case of the r-overholonomic can be checked by induction on r. We proceed in a similar way by using moreover the isomorphism 11.3.5.1.1 checked by Abe. □

Corollary 18.1.2.6. *Let $\mathcal{E} \in D^{\mathrm{b}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ and $r \in \mathbb{N} \cup \{+\infty\}$. Then, the complex $\mathcal{E}$ is r-overholonomic after any base change if and only if for any integer $j \in \mathbb{Z}$ the modules $H^j(\mathcal{E})$ are r-overholonomic after any base change.*

Proof. This follows from 18.1.1.3, from the characterization of 18.1.2.5 and from the fact that the functor f^* is acyclic for any smooth morphism $f \colon \mathfrak{P}' \to \mathfrak{P}$ on the category of coherent $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules. □

Remark 18.1.2.7. The isomorphism 11.3.5.1.1 of Abe is necessary in order to check the characterization 18.1.2.5 of the overholonomicity. However, it is possible to check directly the corollary 18.1.2.6 from 15.3.5.4 without using this isomorphism by using the arguments of acyclicity of the dual functor, localisation and extraordinary inverse image by a smooth morphism.

Proposition 18.1.2.8. *The $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$ is overholonomic.*

Proof. By induction on $n \geq 0$, let us prove the following property (P_n): the $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{O}_{\mathfrak{P},\mathbb{Q}}$ is n-overholonomic. The assertion P_0 is exactly 15.3.6.3 . For $n \geq 0$, assuming P_n true, we prove P_{n+1}. As the constant coefficient is stable under extraordinary inverse images and base change, then it is a question of proving that $\mathbb{D} \circ (^{\dagger}T)(\mathcal{O}_{\mathfrak{P},\mathbb{Q}})$ is n-overholonomic. Since this is local in $\mathfrak{P}$, we can suppose $\mathfrak{P}$ is affine.

1) i) First suppose that $T = \emptyset$. Following 11.2.6.3.4, $\mathbb{D}(\mathcal{O}_{\mathfrak{P},\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}^{\vee}_{\mathfrak{P},\mathbb{Q}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{P},\mathbb{Q}}$. We then conclude by induction hypothesis.

ii) It follows from i) that $\mathbb{D} \circ (^{\dagger}T)(\mathcal{O}_{\mathfrak{P},\mathbb{Q}})$ is n-overholonomic if and only if $\mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$ is n-overholonomic.

2) Let us now treat the case where T is a smooth divisor. Since T is affine and smooth, then there exists an affine and smooth $\mathfrak{S}$-formal schemes $i\colon \mathfrak{T} \hookrightarrow \mathfrak{P}$ which is a lifting of the closed immersion $T \hookrightarrow P$. By applying the dual functor $\mathbb{D}^{(\bullet)}$ to the localization triangle of $\mathcal{O}^{(\bullet)}_{\mathfrak{P}}$ with respect to T, we obtain the distinguished triangle

$$\mathbb{D}^{(\bullet)}(^{\dagger}T)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \to \mathbb{D}^{(\bullet)}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \to \mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \to \mathbb{D}^{(\bullet)}(^{\dagger}T)(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})[1]. \tag{18.1.2.8.1}$$

We have the isomorphism:

$$\mathbb{D}^{(\bullet)}\circ\mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \underset{13.2.1.5.1}{\xrightarrow{\sim}} \mathbb{D}^{(\bullet)}\circ i^{(\bullet)}_{+}\circ i^{(\bullet)!}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \xrightarrow{\sim} \mathbb{D}^{(\bullet)}\circ i^{(\bullet)}_{+}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}}[-1]) \underset{9.4.5.2}{\xrightarrow{\sim}} i^{(\bullet)}_{+}\circ\mathbb{D}^{(\bullet)}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}})[1] \underset{11.2.6.3.4}{\xrightarrow{\sim}} i^{(\bullet)}_{+}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}})[1].$$

Since $\mathcal{O}^{(\bullet)}_{\mathfrak{T}}$ is n-overholonomic by induction hypothesis, then it follows from the theorem 18.1.2.15 that $i^{(\bullet)}_{+}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}})[1]$ is n-overholonomic and therefore that $\mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$ is n-overholonomic. Hence, thanks to 1)ii), we are done.

3) Now suppose that T is a strict normal crossing divisor (see 4.5.2.5) Denote by $T_1, \dots, T_r$ the irreducible components of T. Let us then proceed with a (second) recurrence on $r \geq 1$. The case where $r = 1$ was treated in ii). Consider the distinguished triangle from Mayer-Vietoris (13.1.4.15.2):

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{T_1}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \oplus \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=1}^{r}T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})[1]. \tag{18.1.2.8.2}$$

Let us choose a lifting $i_1\colon \mathfrak{T}_1 \hookrightarrow \mathfrak{P}$ of the closed immersion $T_1 \hookrightarrow P$. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{T_1} \xrightarrow{\sim} i^{(\bullet)}_{1+}i^{(\bullet)!}_{1}$ (see 13.2.1.5.1), by ??, we obtain the isomorphism

$$\begin{aligned}\mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) &\underset{9.3.5.13}{\xrightarrow{\sim}} i^{(\bullet)}_{1+}i^{(\bullet)!}_{1}\mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \\ &\underset{13.2.1.4.1}{\xrightarrow{\sim}} i^{(\bullet)}_{1+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(i^{(\bullet)!}_{1}\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \xrightarrow{\sim} i^{(\bullet)}_{1+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}_1}[-1]).\end{aligned} \tag{18.1.2.8.3}$$

As $\cup_{i=2}^{r}T_1 \cap T_i$ is a strict normal crossing divisor of T_1, then by using the induction hypothesis on r and 1.ii), we obtain that $\mathbb{D}^{(\bullet)}\circ\mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}_1}[-1])$ is n-overholonomic. It follows from the preservation of n-overholonomy by direct image by a proper morphism (see 18.1.2.15), that

$$\mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \underset{18.1.2.8.3}{\xrightarrow{\sim}} \mathbb{D}^{(\bullet)} \circ i^{(\bullet)}_{1+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}_1}[-1]) \underset{9.4.5.2}{\xrightarrow{\sim}} i_{1+} \circ \mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_1\cap T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{T}_1}[-1]) \tag{18.1.2.8.4}$$

is n-overholonomic. We further obtain by induction hypothesis on r (and again using 1.ii) that $\mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{T_1}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}}) \oplus \mathbb{D}^{(\bullet)}\circ\mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{i=2}^{r}T_i}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$ is n-overholonomic. By applying the functor $\mathbb{D}^{(\bullet)}$ to the distinguished triangle 18.1.2.8.2, this implies that the complex $\mathbb{D}^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{T}(\mathcal{O}^{(\bullet)}_{\mathfrak{P}})$ is n-overholonomic. Then, using 1.ii), we are done.

4) Finally, let's move on to the case where T is any divisor but different from the empty set. Using de Jong's desingularization theorem ([dJ96]), there exist a projective smooth morphism $f\colon \mathcal{Q}' \to \mathfrak{P}$, a smooth scheme X' over k, a closed immersion $\iota_0'\colon P' \hookrightarrow \mathcal{Q}'$, a projective, surjective, generically finite and étale morphism $a_0\colon P' \to P$ such that $a_0 = f_0 \circ \iota_0'$ and $T' := a_0^{-1}(T)$ is a strict normal crossing divisor of P'. Since $({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{P}, T/\mathcal{V})$, then following 16.1.11.2, $({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)})$ is a direct summand of $f_+^{(\bullet)} \mathbb{R}\underline{\Gamma}^\dagger_{P'} f^{(\bullet)!}(({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)}))$. Hence, $\mathbb{D}^{(\bullet)} \circ ({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)})$ is a direct summand of

$$\mathbb{D}^{(\bullet)} \circ f_+^{(\bullet)} \mathbb{R}\underline{\Gamma}^\dagger_{P'} f^{(\bullet)!}(({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)})) \underset{9.4.5.2}{\xrightarrow{\sim}} f_+^{(\bullet)} \circ \mathbb{D}^{(\bullet)} \mathbb{R}\underline{\Gamma}^\dagger_{P'} f^{(\bullet)!}(({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)})).$$

Since the morphism f is proper, then using 18.1.2.15 we reduce therefore to check the n-overholonomicity of $\mathbb{D}^{(\bullet)} \mathbb{R}\underline{\Gamma}^\dagger_{P'} f^{(\bullet)!}(({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)}))$. Since this is local on $\mathcal{Q}'$, then we can suppose that there exists a lifting $\iota'\colon \mathfrak{P}' \hookrightarrow \mathcal{Q}'$ of ι_0'.

$$\begin{aligned} &\mathbb{D}^{(\bullet)} \mathbb{R}\underline{\Gamma}^\dagger_{P'} f^{(\bullet)!}(({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)})) \underset{13.2.1.5.1}{\xrightarrow{\sim}} \mathbb{D}^{(\bullet)} \circ \iota_+'^{(\bullet)} \circ \iota'^{(\bullet)!} f^{(\bullet)!}(({}^\dagger T)(\mathcal{O}_{\mathfrak{P}}^{(\bullet)})) \\ &\underset{13.2.1.4.1}{\xrightarrow{\sim}} \mathbb{D}^{(\bullet)} \circ \iota_+^{(\bullet)} \circ ({}^\dagger T) \circ \iota'^{(\bullet)!} f^{(\bullet)!}(\mathcal{O}_{\mathfrak{P}}^{(\bullet)}) \xrightarrow{\sim} \mathbb{D}^{(\bullet)} \circ \iota_+'^{(\bullet)} \circ ({}^\dagger T')(\mathcal{O}_{\mathfrak{P}'}^{(\bullet)}) \underset{9.4.5.2}{\xrightarrow{\sim}} \iota_+'^{(\bullet)} \circ \mathbb{D}^{(\bullet)} \circ ({}^\dagger T')(\mathcal{O}_{\mathfrak{P}'}^{(\bullet)}). \end{aligned} \tag{18.1.2.8.5}$$

Following the step 3), $\mathbb{D}^{(\bullet)} \circ ({}^\dagger T')(\mathcal{O}_{\mathfrak{P}'}^{(\bullet)})$ is n-overholonomic. Hence, thanks to 18.1.2.8.5 and 18.1.2.15. □

Proposition 18.1.2.9. *Let $r \in \mathbb{N} \cup \{\infty\}$.*

(a) Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ be an $r+1$-overholonomic complex. Then $\mathbb{D}(\mathcal{E})$ is r-overholonomic

(b) For any $r \in \mathbb{N} \cup \{\infty\}$, a direct summand of an r-overholonomic complex is r-overholonomic.

(c) Let $r \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}'[1]$ be a distinguished triangle of $D(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. If two of the three complexes are r-overholonomic then so is the third. In particular, $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ is a triangulated subcategory of $D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

(d) Let $r \geq 1$ be an integer or $r = \infty$ and $\mathcal{E}$ be an r-overholonomic $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module. Then $\mathcal{E}$ is holonomic.

(e) Let $r \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ be an r-overholonomic complex. Then, for any subscheme Y of P, the complex $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E})$ is r-overholonomic.

(f) For any smooth morphism $f\colon \mathfrak{P}' \to \mathfrak{P}$ of $\mathfrak{S}$-formal schemes, for any r-overholonomic complex $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$, the complex $f^!(\mathcal{E})$ is r-overholonomic.

Proof. The property a) is tautological (with the notations of 18.1.2.1, it is sufficient to take T' empty and f equal to the identity). The properties b) and c) are already known when $r = 0$ (see 15.3.6 when the divisor T is empty). The general case easily follows by induction on $r \in \mathbb{N}$. Now let us deal with d). Following the remark 18.1.2.4.(a), it is sufficient to check it when $r = 1$. Thanks to a), if $\mathcal{E}$ is a 1-overholonomic $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module then $\mathbb{D}(\mathcal{E})$ is overcoherent and therefore has finite extraordinary fibers (see 15.3.6.12). By 15.3.2.8, this implies that $\mathcal{E}$ is holonomic. Hence d). Now, let's establish e). Since this is local, we can suppose P is integral. The case $r = 0$ has already been treated (see 15.3.6.9). Let $r \geq 1$ be an integer and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ be an r-overholonomic complex.

Let T be a divisor of P. We obtain from the induction hypothesis that $\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E})$ and $({}^\dagger T)(\mathcal{E})$ are $r-1$-overholonomic. Moreover, for any smooth morphism $f\colon \mathfrak{P}' \to \mathfrak{P}$ of $\mathfrak{S}$-formal schemes, for any divisor T' of X', we have the isomorphisms:

$$\mathbb{D}({}^\dagger T') f^! ({}^\dagger T)(\mathcal{E}) \overset{13.2.1.4.1}{\xrightarrow{\sim}} \mathbb{D}({}^\dagger T')({}^\dagger f^{-1}(T)) f^!(\mathcal{E}) \overset{13.1.5.6.1}{\xrightarrow{\sim}} \mathbb{D}({}^\dagger T' \cup f^{-1}(T)) f^!(\mathcal{E}).$$

Since $T' \cup f^{-1}(T)$ is a divisor of P', as $\mathcal{E}$ is r-overholonomic, then by definition $\mathbb{D}({}^\dagger T' \cup f^{-1}(T)) f^!(\mathcal{E})$ is $r-1$-overholonomic (see 18.1.2.1). Hence so is the complex $\mathbb{D}({}^\dagger T') f^! ({}^\dagger T)(\mathcal{E})$. This means that $({}^\dagger T)(\mathcal{E})$ is r-overholonomic. By using the triangle of localisation in T (see 13.1.5.6.3) and via c), this yields that $\mathbb{R}\underline{\Gamma}^\dagger_T(\mathcal{E})$ is r-overholonomic.

Let Z be a closed subscheme of X. Following 13.1.3.6, there exists some divisors $T_1, \ldots, T_n$ of X such that $Z = \cap_{i=1,\ldots,n} T_i$. By setting $Z' := \cap_{i=2,\ldots,n} T_i$, we have the distinguished triangle of Mayer-Vietoris (see 13.1.4.15.1):

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}(\mathcal{E}) \oplus \mathbb{R}\underline{\Gamma}^{\dagger}_{T_1}(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z' \cup T_1}(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E})[1].$$

Since Z' and $Z' \cup T_1$ are the intersections of $n-1$ divisors, we end the proof thanks to c) and by proceeding by induction on n.

Finally, the statement (f) is obvious by definition. □

Corollary 18.1.2.10. *Let T be a divisor of X and $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Then $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$ if and only if $\mathbb{D}_T(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$.*

Proof. a) Suppose $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$. The canonical morphism $\mathcal{E} \to ({}^{\dagger}T)(\mathcal{E})$ is a morphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}T)_{\mathbb{Q}}$-modules which is an isomorphism apart from T. By 8.7.6.11, this implies that $\mathcal{E} \to ({}^{\dagger}T)(\mathcal{E})$ is an isomorphism. However, following 9.2.4.22.3 dual functors commute with localisation, i.e., $\mathbb{D}_T \circ ({}^{\dagger}T)(\mathcal{E}) \xrightarrow{\sim} ({}^{\dagger}T) \circ \mathbb{D}(\mathcal{E})$. We obtain then $\mathbb{D}_T(\mathcal{E}) \xrightarrow{\sim} ({}^{\dagger}T) \circ \mathbb{D}(\mathcal{E})$. Hence, using 18.1.2.9.(a) and 18.1.2.9.(e) this yields $\mathbb{D}_T(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$.

b) The converse comes from the biduality isomorphism $\mathbb{D}_T \circ \mathbb{D}_T(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ (see 8.7.7.3) and from the part a). □

Like coherence, we verify that the notion of overholonomicity is local on $\mathfrak{P}$. Moreover, we extend the standard properties of coherent modules to overholonomic modules. For example, we have the following two propositions:

Proposition 18.1.2.11. *Let $\Phi \colon \mathcal{E} \to \mathcal{F}$ be a morphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules and $r \in \mathbb{N} \cup \{\infty\}$. If $\mathcal{E}$ and $\mathcal{F}$ are r-overholonomic, then so are* $\operatorname{Ker}\Phi$, $\operatorname{Coker}\Phi$ *and* $\operatorname{Im}\Phi$.

Proof. We proceed by induction on $r \geq 0$. The case $r = 0$ is 15.3.6.4.(a). Suppose now the theorem holds for $r-1 \geq 0$ and let us check it for r. Let $f \colon \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism of $\mathfrak{S}$-formal schemes, T' be a divisor of P'. Since the functor $({}^{\dagger}T') \circ f^*$ is exact and preserves the r-overholonomic (see 18.1.2.9), then by induction hypothesis we reduce to check that $\mathbb{D}(\ker\Phi)$, $\mathbb{D}(\operatorname{Coker}\Phi)$ and $\mathbb{D}(\operatorname{Im}\Phi)$ are $r-1$-overholonomic. By 18.1.2.9.(d), since $r \geq 1$, then Φ is a morphism of holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules. Hence, it follows from 15.2.4.14 that $\ker\Phi$, $\operatorname{Coker}\Phi$ and $\operatorname{Im}\Phi$ are holonomic. Let us recall that following the homological criterion of holonomicity (see 15.2.4.8), if $\mathcal{G}$ is a holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module, then by setting $\mathcal{G}^* := H^0\mathbb{D}(\mathcal{G})$ we get $\mathcal{G}^* \xrightarrow{\sim} \mathbb{D}(\mathcal{G})$. We get the morphism $\Phi^* := H^0\mathbb{D}(\Phi) \colon \mathcal{F}^* \to \mathcal{E}^*$. Since the functor $H^0\mathbb{D}$ is exact on the category of holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules, then $(\operatorname{Im}(\phi))^* \xrightarrow{\sim} \operatorname{Im}(\phi^*)$, $(\ker\Phi)^* \xrightarrow{\sim} \operatorname{Coker}(\Phi^*)$, $(\operatorname{Coker}\Phi)^* \xrightarrow{\sim} \ker(\Phi^*)$. By definition of the r-overholonomic, Φ^* is a morphism of $r-1$-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules. By induction hypothesis, this yields that $(\operatorname{Im}(\phi))^*$, $(\ker\Phi)^*$, $(\operatorname{Coker}\Phi)^*$ are $r-1$-overholonomic, q.e.d. □

Proposition 18.1.2.12. *Let $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \mathcal{E}_4 \to \mathcal{E}_5$ an exact sequence of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules. For any $r \in \mathbb{N} \cup \{\infty\}$, if $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$ are r-overholonomic, then $\mathcal{E}_3$ is r-overholonomic.*

Proof. The case $r = 0$ is already known (see 15.3.6.4.(b)). Suppose now the proposition holds for $r-1 \geq 0$ and let us prove it for r. Let $f \colon \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism of $\mathfrak{S}$-formal schemes, T' be a divisor of P'. Since the functor $({}^{\dagger}T') \circ f^*$ is exact and preserves the r-overholonomic (see 18.1.2.9), then by induction hypothesis we reduce to check that $\mathbb{D}(\mathcal{E}_3)$ is $r-1$-overholonomic.

By 18.1.2.9.(d), since $r \geq 1$, then $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$ are holonomic. Denote by $\mathcal{F}_2$ (resp. $\mathcal{F}_3$) the image of $\mathcal{E}_2 \to \mathcal{E}_3$ (resp. $\mathcal{E}_3 \to \mathcal{E}_4$). Using 15.2.4.14, since $\mathcal{E}_2$ (resp. $\mathcal{E}_4$) is holonomic, then so is $\mathcal{F}_2$ (resp. $\mathcal{F}_3$). Since we have the exact sequence $0 \to \mathcal{F}_2 \to \mathcal{E}_3 \to \mathcal{F}_3 \to 0$ with $\mathcal{F}_2$ and $\mathcal{F}_3$ holonomic then so is $\mathcal{E}_3$ (thanks to 15.2.4.14). Hence, $\mathbb{D}(\mathcal{E}_3) \xrightarrow{\sim} \mathcal{E}_3^*$. Since the functor $H^0\mathbb{D}$ is exact on the category of holonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules (see 15.2.4.15), then we get the exact sequence $\mathcal{E}_5^* \to \mathcal{E}_4^* \to \mathcal{E}_3^* \to \mathcal{E}_2^* \to \mathcal{E}_1^*$. By induction hypothesis, $\mathcal{E}_3^*$ is therefore $r-1$-overholonomic. □

Corollary 18.1.2.13. *Let $N \in \mathbb{N}$. For $N \geq p, q \geq 0$, $r_0 \geq 1$, let $\mathcal{E}_{r_0}^{p,q} \Rightarrow \mathcal{E}^n$ be a spectral sequence of coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules. If, for any p, q, the $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules $\mathcal{E}_{r_0}^{p,q}$ are r-overholonomic, then so are the $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules $\mathcal{E}^n$ for any $n \in \mathbb{N}$.*

Proof. It follows from 18.1.2.11 that the $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules $\mathcal{E}^{p,q}_{\infty}$ are r-overholonomic. With 18.1.2.12, this implies that so are $\mathcal{E}^n$. $\square$

Theorem 18.1.2.14. *Let $f\colon \mathfrak{Y} \to \mathfrak{P}$ be a morphism of smooth $\mathfrak{S}$-formal schemes, $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$, $\in \mathbb{N} \cup \{\infty\}$ and $r \in \mathbb{N}$. If $\mathcal{E}$ is r-overholonomic then so is $f^!(\mathcal{E})$.*

Proof. We proceed by induction on $r \geq 0$. For $r = 0$, this is 15.3.6.12. Now let us suppose the theorem holds for $r-1 \geq 0$ and let us prove it for r.

Since f factors as the composition of an immersion (its graph $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \times_{\mathfrak{S}} \mathfrak{P}$) followed by the smooth projection $\mathfrak{Y} \times_{\mathfrak{S}} \mathfrak{P} \to \mathfrak{P}$, then it is sufficient to treat the case where f is smooth and where f is a closed immersion. Since the case where f is smooth is obvious (see 18.1.2.9.(f)) then we reduce to treat the case where f is a closed immersion.

Let $g\colon \mathfrak{Y}' \to \mathfrak{Y}$ be a smooth morphism and Z' be a closed subscheme of Y'. By induction hypothesis on r, it is sufficient to check that $\mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} \circ g^!(f^!(\mathcal{E}))$ is $r-1$-overholonomic. Indeed, taking Z' equal to Y', this implies that $\mathbb{D} \circ g^!(f^!(\mathcal{E}))$ is $r-1$-overholonomic. By applying the dual functor $\mathbb{D}$ to the triangle of localisation of $g^!(f^!(\mathcal{E}))$ in Z' (see 13.1.5.6.3), then it follows from 18.1.2.9.c that $\mathbb{D} \circ ({}^{\dagger}Z') \circ g^!(f^!(\mathcal{E}))$ is $r-1$-overholonomic.

Let us prove now that $\mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} \circ g^!(f^!(\mathcal{E}))$ is $r-1$-overholonomic. Since this is local on $\mathfrak{Y}'$, we can then suppose that g factors in a closed immersion $\mathfrak{Y}' \hookrightarrow \widehat{\mathbb{A}}^n_{\mathfrak{Y}}$ followed by the projection $\widehat{\mathbb{A}}^n_{\mathfrak{Y}} \to \mathfrak{Y}$. Let $p\colon \widehat{\mathbb{A}}^n_{\mathfrak{P}} \to \mathfrak{P}$ be the projection and i the closed immersion $\mathfrak{Y}' \hookrightarrow \widehat{\mathbb{A}}^n_{\mathfrak{P}}$. Since, $f \circ g = p \circ i$, then we get the first isomorphism:

$$\mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} \circ g^!(f^!(\mathcal{E})) \xrightarrow{\sim} \mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} \circ i^! \circ p^!(\mathcal{E}) \underset{13.2.1.4.1}{\xrightarrow{\sim}} \mathbb{D} \circ i^! \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} \circ p^!(\mathcal{E}) \underset{9.3.5.11.1}{\xrightarrow{\sim}} i^! \circ \mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} \circ p^!(\mathcal{E}), \tag{18.1.2.14.1}$$

where the last isomorphism follows from the fact that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} p^!(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Q}',\mathbb{Q}})$ is coherent with support in Y' (because $\mathcal{E}$ is overcoherent). Following 18.1.2.9, since $\mathcal{E}$ is r-overholonomic then the complex $\mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} p^!(\mathcal{E})$ is $r-1$-overholonomic. By induction hypothesis, this yields $i^! \mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} p^!(\mathcal{E})$ is also $r-1$-overholonomic. Hence, we are done. $\square$

Theorem 18.1.2.15. *Let $f\colon \mathfrak{Y} \to \mathfrak{P}$ be a realizable (in the sense of 13.2.3.1) morphism of smooth $\mathcal{V}$-formal schemes. For any $\mathcal{F} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}})$ with proper support over P, for any integer r, if $\mathcal{F}$ is r-overholonomic then $f_+(\mathcal{F})$ is r-overholonomic.*

Proof. We proceed by induction on the integer $r \geq 0$. The case $r = 0$ has been checked in 16.3.3.1 (in fact, the version of 15.3.6.14 is sufficient). Now, let us suppose the theorem holds for $r-1 \geq 0$ and let us check it for r. Let $\mathcal{F}$ be an r-overholonomic complex of $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}})$, $g\colon \mathfrak{P}' \to \mathfrak{P}$ be a smooth morphism of $\mathfrak{S}$-formal schemes and T' be a divisor of P'. Denote by $\mathfrak{Y}' := \mathfrak{Y} \times_{\mathfrak{P}} \mathfrak{P}'$, $f'\colon \mathfrak{Y}' \to \mathfrak{P}'$ and $g'\colon \mathfrak{Y}' \to \mathfrak{Y}$ the respective projections. We have to check that $\mathbb{D}({}^{\dagger}T')g^! f_+(\mathcal{F})$ is $r-1$ overholonomic. However, by using the isomorphism of base change of the direct image of a realisable morphism by a smooth morphism $g^! f_+(\mathcal{F}) \xrightarrow{\sim} f'_+ g'^!(\mathcal{F})$ (see 13.2.3.7), the commutation of the direct image with the local cohomological functor (see 13.2.1.4.2) and to the relative duality theorem (see 13.2.4.1), we get the isomorphisms:

$$\mathbb{D}({}^{\dagger}T')g^! f_+(\mathcal{F}) \xrightarrow{\sim} \mathbb{D}({}^{\dagger}T')f'_+ g'^!(\mathcal{F}) \xrightarrow{\sim} \mathbb{D} f'_+({}^{\dagger}f'^{-1}T')g'^!(\mathcal{F}) \xrightarrow{\sim} f'_+ \mathbb{D}({}^{\dagger}f'^{-1}T')g'^!(\mathcal{F}).$$

Following the proposition 18.1.2.9 and the theorem 18.1.2.14, the complex $\mathbb{D}({}^{\dagger}f'^{-1}T')g'^!(\mathcal{F})$ is $r-1$-overholonomic. By induction hypothesis, this yields that the complex $f'_+\mathbb{D}({}^{\dagger}f'^{-1}T')g'^!(\mathcal{F})$ is $r-1$-overholonomic. $\square$

Theorem 18.1.2.16 (Overholonomic version of Berthelot-Kashiwara theorem)**.** *Let $f\colon \mathfrak{Y} \to \mathfrak{P}$ be a closed immersion of smooth $\mathcal{V}$-formal schemes, $r \in \mathbb{N} \cup \{\infty\}$.*

(a) For any r-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module $\mathcal{E}$ with support in Y, any r-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-module $\mathcal{F}$ and any integer $k \neq 0$, $H^k f_+(\mathcal{F}) = 0$ and $H^k f^!(\mathcal{E}) = 0$.

(b) The functors f_+ and $f^!$ induce canonically quasi-inverse equivalences between the category of r-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-modules with support in Y and that of r-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-modules.

Proof. The a) part results from the coherent version of the theorem of Kashiwara due to Berthelot (see 9.3.5.9)) and of the fact that a r-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-module (resp. a r-overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module with support in Y) is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-module (resp. a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module with support in Y). The assertion b) comes from of 9.3.5.9) and from the stability of the r-overholonomicity by direct image and extraordinary inverse image by a closed immersion (see 18.1.2.14 and of 18.1.2.15). □

18.2 Acyclicity of u_+ and $u_!$ when the underlying formal scheme morphism is the identity

Let $m \in \mathbb{N} \cup \{\infty\}$, $\mathfrak{S} := \operatorname{Spf} \mathcal{V}$. Let $\mathfrak{X} \to \mathfrak{S}$ be a smooth morphism of formal schemes. Let $\mathfrak{Z} \subset \mathfrak{X}$ be a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor (see definition 4.5.2.7). Let $\mathfrak{Y}$ be the open of $\mathfrak{X}$ complementary to $\mathfrak{Z}$ and $j \colon \mathfrak{Y} \subset \mathfrak{X}$ be the open immersion. Set $\mathfrak{X}^{\sharp} := (\mathfrak{X}, M(\mathfrak{Z}))$. Let $\mathfrak{Z}^{\flat}, \mathfrak{D} \subset \mathfrak{Z}$ be subdivisors of $\mathfrak{Z}$ such that $\mathfrak{Z} = \mathfrak{Z}^{\flat} \cup \mathfrak{D}$ and the irreducible components of $\mathfrak{Z}^{\flat}$ and $\mathfrak{D}$ are disjoint. Set $\mathfrak{X}^{\flat} := (\mathfrak{X}, M(\mathfrak{Z}^{\flat}))$. Let $u \colon \mathfrak{X}^{\sharp} \to \mathfrak{X}^{\flat}$ be the canonical morphism. The underlying morphism of smooth $\mathfrak{S}$-formal schemes is the identity of $\mathfrak{X}$.

The goal of this section is to establish the theorem 18.2.3.13. The first part of the theorem is a consequence of 18.2.3.1 (which follows from 18.2.1.6). The second part of the theorem 18.2.3.13 (or also 18.2.3.11) corresponds in some way (modulo the twist "$(\mathfrak{Z})$") to an isomorphism of (logarithmic) relative duality to the canonical morphism $\mathfrak{X}^{\sharp} \to \mathfrak{X}$. It can be proved similarly to the complex case of [CMNM05, 3.1.2] by using the isomorphism of associativity of 4.5.3.3 (or more precisely the induced isomorphism 4.5.3.8.1).

18.2.1 Finite order case

Let $\mathcal{B}_{\mathfrak{X}}$ be an $\mathcal{O}_X$-algebra endowed with a compatible structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\flat}}$-module. We get a structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}}$-module on $\mathcal{B}_{\mathfrak{X}} = u^*\mathcal{B}_{\mathfrak{X}}$ whose stratification $(\varepsilon_n^{u^*\mathcal{B}_{\mathfrak{X}}})_n$ given by the pullbacks via the algebra homomorphisms $\mathcal{P}^n_{\mathfrak{X}^{\flat}/\mathfrak{S}(m)} \to \mathcal{P}^n_{\mathfrak{X}^{\sharp}/\mathfrak{S}(m)}$ (see 4.4.2). Hence, $\varepsilon_n^{u^*\mathcal{B}_{\mathfrak{X}}}$ are $\mathcal{P}^n_{\mathfrak{X}^{\sharp}/\mathfrak{S}(m)}$-algebra isomorphisms, i.e. the induced structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}}$-module of $\mathcal{B}_{\mathfrak{X}}$ is compatible. Hence, for any $0 \le m' \le m$, we get the $\mathcal{O}_{\mathfrak{S}}$-algebras by setting $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^{\sharp}} := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^{\sharp}}$ and $\widetilde{\mathcal{D}}^{(m')}_{\mathfrak{X}^{\flat}} := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m')}_{\mathfrak{X}^{\flat}}$. Recall, the canonical morphism $\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/S} \to \widetilde{\mathcal{D}}^{(m)}_{X^{\flat}/S}$ is a ring morphism (use a similar to 4.5.2.18.3 diagram).

18.2.1.1 (Local description)**.** Suppose there exist nice coordinates $t_1, \dots, t_d$ of $\mathfrak{X}^{\sharp}/\mathfrak{S}$ so that $\mathfrak{Z}$ is empty or $\mathfrak{Z}$ is cut out by $\prod_{1 \le j \le r} t_j$ for some $r \ge 1$, and $\mathfrak{Z}^{\flat}$ is either empty of cut out by $\prod_{1 \le j \le s} t_j$ for some $r \ge s \ge 1$ (which is always locally possible following 4.5.2.14). Then the coordinates $t_1, \dots, t_d$ of $\mathfrak{X}^{\sharp}/\mathfrak{S}$ are semi-nice coordinates of $\mathfrak{X}^{\flat}/\mathfrak{S}$ (because when $r > s$, t_r is not invertible). We get the description 4.5.2.18.(b). We get the bases $\{\underline{\partial}^{\underline{k}}_{(r)} : \underline{k} \in \mathbb{N}^d\}$ of $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}$ and $\{\underline{\partial}^{\underline{k}}_{(s)} : \underline{k} \in \mathbb{N}^d\}$ of $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\flat}/\mathfrak{S}}$. According to 4.5.1.9, in the computations of the subsection, when we refer to some logarithmic formulas, we will mean its semi-logarithmic avatar.

We have the commutative canonical diagram

$$\begin{array}{ccc} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\flat}} & \xrightarrow{\sigma} & \operatorname{gr} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\flat}} \\ \uparrow & & \uparrow \\ \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}} & \xrightarrow{\sigma} & \operatorname{gr} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\sharp}} \end{array} \qquad (18.2.1.1.1)$$

where the vertical arrows are injective. For any $i = 1, \dots, d$, we set $\xi_{i,(s)} := \sigma(\partial_{i,(s)})$ and $\xi_{i,(r)} := \sigma(\partial_{i,(r)})$ (this latter can be viewed as an element of $\operatorname{gr} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\flat}}$).

Lemma 18.2.1.2. *We keep hypotheses and notation of 18.2.1.1 and we suppose $t_1, \dots, t_d$ is a regular sequence of $\mathcal{B}_{\mathfrak{X}}$ (e.g. $\mathcal{O}_{\mathfrak{X}}$).*

(a) The sequence $\xi_{1,(r)}, \dots, \xi_{d,(r)}$ of $\operatorname{gr} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^{\flat}}$ is regular.

(b) We have the equality

$$\sigma(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}(\partial_{1,(r)},\ldots,\partial_{d,(r)})) = (\operatorname{gr}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat})(\xi_{1,(r)},\ldots,\xi_{d,(r)}). \tag{18.2.1.2.1}$$

Proof. a) Since $t_1,\ldots,t_d$ nice coordinates of $\mathfrak{X}^\sharp/\mathfrak{S}$, then they are semi-logarithmic coordinates of $\mathfrak{X}^\flat/\mathfrak{S}$ and we get: $\operatorname{gr}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} = \mathcal{B}_{\mathfrak{X}}[\xi_{1,(s)},\ldots,\xi_{d,(s)}]$. For any integer i such that $1\le i\le s$ and $r+1\le i\le d$, we have $\xi_{i,(r)}=\xi_{i,(s)}$. For any integer i such that $s+1\le i\le r$, we have $\xi_{i,(r)} = t_i\xi_{i,(s)}$. Hence, for any $1\le i\le s$, $\operatorname{gr}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}/(\xi_{1,(r)},\ldots,\xi_{i,(r)}) = \mathcal{B}_{\mathfrak{X}}[\xi_{i+1,(s)},\ldots,\xi_{d,(s)}]$; for any $r+1\le i\le s$, $\operatorname{gr}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}/(\xi_{1,(r)},\ldots,\xi_{i,(r)}) = \big(\mathcal{O}_{\mathfrak{X}}[\xi_{r+1,(s)},\ldots,\xi_{i,(s)}]/(t_{r+1}\xi_{r+1,(s)},\ldots,t_i\xi_{i,(s)})\big)[\xi_{i+1,(s)},\ldots,\xi_{d,(s)}]$. Since the sequence $t_1,\ldots,t_d$ of $\mathcal{B}_{\mathfrak{X}}$ is regular, then $t_{i+1}\xi_{i+1,(s)}$ is a nonzerodivisor of $\operatorname{gr}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}/(\xi_{1,(r)},\ldots,\xi_{i,(r)})$. Hence, we are done.

b) Let us prove now the equality 18.2.1.2.1. To simplify notation, for any integers $1\le i\le d$ and $n\ge 0$, we set $\delta_i := \partial_{i,(r)}$, $\xi_i := \xi_{i,(r)}$, $\mathcal{D} := \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}$, $\mathcal{I} := \mathcal{D}(\delta_1,\ldots,\delta_d)$, $\mathcal{D}_n := \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat,n}$. Let $\mathcal{D}^d\to\mathcal{I}$ be the epimorphism given by $(P_1,\ldots,P_d)\mapsto\sum_i P_i\delta_i$ and let $\mathcal{I}_n$ be the image of $\mathcal{D}^d_{n-1}$ by this map. We get $\sigma(\mathcal{D}(\delta_1,\ldots,\delta_d) = \oplus_{n\ge1}(\mathcal{I}\cap\mathcal{D}_n+\mathcal{D}_{n-1})/\mathcal{D}_{n-1}$ and $\operatorname{gr}\mathcal{D}(\xi_1,\ldots,\xi_d) = \oplus_{n\ge1}\mathcal{I}_n+\mathcal{D}_{n-1}/\mathcal{D}_{n-1}$. Hence, we have to check $\mathcal{I}\cap\mathcal{D}_n+\mathcal{D}_{n-1} = \mathcal{I}_n+\mathcal{D}_{n-1}$ for any $n\ge 1$. Since $\mathcal{I}_n\subset\mathcal{I}\cap\mathcal{D}_n$ then $\mathcal{I}_n+\mathcal{D}_{n-1}\subset\mathcal{I}\cap\mathcal{D}_n+\mathcal{D}_{n-1}$ is obvious. Conversely, let $P\in\mathcal{I}\cap\mathcal{D}_n$. Let $P_1,\ldots,P_d\in\mathcal{D}$ such that $P=\sum_i P_i\delta_i$. We prove $P\in\mathcal{I}_n+\mathcal{D}_{n-1}$ by induction on the maximal order N of the operators P_i $(i=1,\ldots,d)$, i.e. on $N:=\min\{n\ge1$; such that $P_i\in\mathcal{D}_n$ for any $i=1,\ldots,d\}$. If $N\le n-1$, then $P\in\mathcal{I}_n$ and we are done. Suppose now $N\le n$. Let K be the the set of subindices j such that P_j has order N. If $\sum_{i\in K}\sigma(P_i)\xi_i\ne 0$, then $\sigma(P)=\sum_{i\in K}\sigma(P_i)\xi_i\in\mathcal{D}_{N+1}$, which is absurd. Hence, $\sum_{i\in K}\sigma(P_i)\xi_i = 0$. We set $G_i:=\sigma(P_i)$ for any $i\in K$, and $G_i=0$ for any $i\notin K$. This yields $\sum_{i=1}^d G_i\xi_i = 0$. As the sequence $\xi_1,\ldots,\xi_d$ of $\operatorname{gr}\mathcal{D} = \mathcal{B}_{\mathfrak{X}}[\xi_{1,(s)},\ldots,\xi_{d,(s)}]$ is regular (from the part a), then there exists homogeneous polynomials $(G_{ij})_{1\le i<j\le d}$ of $\operatorname{gr}\mathcal{D}$ of order $N-1$ such that

$$(G_1,\ldots,G_d) = \sum_{1\le i<j\le d} G_{ij}(\xi_j\vec{e}_i - \xi_i\vec{e}_j),$$

where $\vec{e}_i = (0,\ldots,0,1,0,\ldots,0)$ is the element of $\mathbb{N}^d$ with 1 at the ith position. We choose for any $1\le i<j\le d$ operators $Q_{ij}\in\mathcal{D}_{N-1}$ such that $\sigma(Q_{ij}) = G_{ij}$. We set $(Q_1,\ldots,Q_n) := (P_1,\ldots,P_n) - \sum_{1\le i<j\le d} Q_{ij}(\delta_j\vec{e}_i-\delta_i\vec{e}_j)$. Let $\sigma_N\colon\mathcal{D}_N\to\operatorname{gr}_N\mathcal{D}$ be the canonical map. For $i=1,\ldots,d$, we compute $\sigma_N(P_i) = G_i$ and $\sigma_N(Q_{ij}) = \sigma(Q_{ij}) = G_{ij}$. Since σ_N is $\mathcal{B}_{\mathfrak{X}}$-linear and multiplicative (with respect to the product of two elements staying in $\mathcal{D}_N$), this yields $(\sigma_N(Q_1),\ldots,\sigma_N(Q_n)) = (Q_1,\ldots,Q_n) - \sum_{1\le i<j\le d} G_{ij}(\xi_j\vec{e}_i-\xi_i\vec{e}_j) = 0$. Hence, $Q_1,\ldots,Q_d\in\mathcal{D}_{N-1}$. Moreover, since the family $(\xi_i)_i$ commutes two by two, we get:

$$\sum_{i=1}^d Q_i\delta_i = \sum_{i=1}^d P_i\delta_i - \sum_{1\le i<j\le d} Q_{ij}(\xi_i\xi_j-\xi_j\xi_i) = 0.$$

By using the induction hypothesis, we get $(Q_1,\ldots,Q_d)$. □

Theorem 18.2.1.3. *We suppose $\mathcal{B}_{\mathfrak{X}}$ is $\mathcal{O}_{\mathfrak{X}}$-flat. Let $\mathcal{E}$ be a left coherent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}$-module which is coherent and flat on $\mathcal{B}_{\mathfrak{X}}$. Then, the complex $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}}\operatorname{Sp}^\bullet_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}}(\mathcal{E})$, where $\operatorname{Sp}^\bullet_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}}(\mathcal{E})$ is the Spencer complex of 4.7.3.7.1 is acyclic.*

Proof. 0) We have to check the exactness of the sequence:

$$0\to\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{B}_{\mathfrak{X}}}\wedge^d\widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E}\xrightarrow{\epsilon}\cdots\xrightarrow{\epsilon}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{B}_{\mathfrak{X}}}\wedge^1\widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E}\xrightarrow{\epsilon}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E}\xrightarrow{\varpi}\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}}\mathcal{E}\to 0, \tag{18.2.1.3.1}$$

where ϖ is the canonical epimorphism and where ϵ is induced by extension from the Spencer complex (see the formula of 4.7.3.4), i.e. is the morphism of left $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}$-modules

$$\epsilon\colon\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{B}_{\mathfrak{X}}}\wedge^i\widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E}\to\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{B}_{\mathfrak{X}}}\wedge^{i-1}\widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{E} \tag{18.2.1.3.2}$$

given by

$$
\begin{aligned}
\epsilon\left(P \otimes (v_1 \wedge \cdots \wedge v_i) \otimes u\right) = & \sum_{a=1}^{i} (-1)^{a-1} P v_a \otimes (v_1 \wedge \cdots \wedge \widehat{v_a} \wedge \cdots \wedge v_i) \otimes u \\
& - \sum_{a=1}^{i} (-1)^{a-1} P \otimes (v_1 \wedge \cdots \wedge \widehat{v_a} \wedge \cdots \wedge v_i) \otimes v_a u \\
& + \sum_{1 \le a < b \le i} (-1)^{a+b} P \otimes ([v_a, v_b] \wedge v_1 \wedge \cdots \wedge \widehat{v_a} \wedge \cdots \wedge \widehat{v_b} \wedge \cdots \wedge v_i) \otimes u
\end{aligned}
$$

1) As $\mathcal{E}$ is flat on $\mathcal{B}_{\mathfrak{X}}$, we get the following filtration of 18.2.1.3.1 for $n \in \mathbb{N}$:

$$
\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n-d} \otimes_{\mathcal{B}_{\mathfrak{X}}} \wedge^d \widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n-1} \otimes_{\mathcal{B}_{\mathfrak{X}}} \wedge^1 \widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\epsilon} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\varpi} \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}). \tag{18.2.1.3.3}
$$

The induced morphism by ϵ morphism

$$
\overline{\epsilon}_{n,i} \colon \operatorname{gr}_{n-i}(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \wedge^i \widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E} \to \operatorname{gr}_{n-i+1}(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \wedge^{i-1} \widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E} \tag{18.2.1.3.4}
$$

is given by

$$
\overline{\epsilon}_{n,i}\left(\overline{P} \otimes (v_1 \wedge \cdots \wedge v_i) \otimes u\right) = \sum_{a=1}^{i} (-1)^{a-1} \overline{P v_a} \otimes (v_1 \wedge \cdots \wedge \widehat{v_a} \wedge \cdots \wedge v_i) \otimes u.
$$

Moreover, since $\mathcal{E}$ is flat as $\mathcal{B}_{\mathfrak{X}}$-module, then we get

$$
\varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}) / \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n-1} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}) \xrightarrow{\sim} \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n}) / \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n-1}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}),
$$

where $\varpi \colon \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}} \mathcal{B}_{\mathfrak{X}}$ is the canonical morphism (i.e. this is ϖ in the case where $\mathcal{E} = \mathcal{B}_{\mathfrak{X}}$). Hence, the graded complex induced by the filtration 18.2.1.3.3 is canonically isomorphic to the graded complex induce by the filtration 18.2.1.3.3 where $\mathcal{E}$ is equal to $\mathcal{B}_{\mathfrak{X}}$ tensored by $\mathcal{E}$ above $\mathcal{B}_{\mathfrak{X}}$. As $\mathcal{E}$ is flat on $\mathcal{B}_{\mathfrak{X}}$, we reduce therefore to the case where $\mathcal{E} = \mathcal{B}_{\mathfrak{X}}$.

2) Let us check the graded complex induced by the filtration 18.2.1.3.3 is acyclic when $\mathcal{E} = \mathcal{B}_{\mathfrak{X}}$. Since this is local, we can suppose the local situation of 18.2.1.1 holds and we use the notation of the proof of 18.2.1.2.1. Since $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp} / \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}(\delta_1, \ldots, \delta_d) \xrightarrow{\sim} \mathcal{B}_{\mathfrak{X}}$, then $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}} \mathcal{B}_{\mathfrak{X}} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} / \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}(\partial_{1,(r)}, \ldots, \partial_{d,(r)}) = \mathcal{D}/\mathcal{I}$. Hence, $\varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n}) = \mathcal{D}_n + \mathcal{I}/\mathcal{I}$ and therefore $\varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n}) / \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n-1}) = \mathcal{D}_n + \mathcal{I}/\mathcal{D}_{n-1} + \mathcal{I} = \mathcal{D}_n/(\mathcal{I} \cap \mathcal{D}_n + \mathcal{D}_{n-1})$. Since $\mathcal{B}_{\mathfrak{X}}$ is a flat $\mathcal{O}_{\mathfrak{X}}$-algebra, then we get from 18.2.1.2.1 the equality $\operatorname{gr} \mathcal{D} / \operatorname{gr} \mathcal{D}(\xi_1, \ldots, \xi_d) = \operatorname{gr} \mathcal{D} / \sigma(\mathcal{I})$. As $\sigma(\mathcal{I}) = \oplus_{n \ge 0} (\mathcal{I} \cap \mathcal{D}_n + \mathcal{D}_{n-1}) / \mathcal{D}_{n-1}$, this yields:

$$
\operatorname{gr} \mathcal{D} / \operatorname{gr} \mathcal{D}(\xi_1, \ldots, \xi_d) = \operatorname{gr} \mathcal{D} / \sigma(\mathcal{I}) = \oplus_{n \ge 0} \mathcal{D}_n / (\mathcal{I} \cap \mathcal{D}_n + \mathcal{D}_{n-1}) = \oplus_{n \ge 0} \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n}) / \varpi(\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat, n-1}). \tag{18.2.1.3.5}
$$

Since $\widetilde{\mathcal{T}}_{\mathfrak{X}^\sharp}$ is a free $\mathcal{B}_{\mathfrak{X}}$-module whose basis is given by $\delta_1, \ldots, \delta_d$, then the map $\oplus_{n \in \mathbb{N}} \overline{\epsilon}_{n,i}$ (where $\overline{\epsilon}_{n,i}$ is defined at 18.2.1.3.4 with $\mathcal{E} = \mathcal{B}_{\mathfrak{X}}$) is the i-th map given by the Koszul complex of the ring $\operatorname{gr} \mathcal{D}$ with respect to the sequence $\xi_1, \ldots, \xi_d$. Since the sequence $\xi_1, \ldots, \xi_d$ of $\operatorname{gr} \mathcal{D}$ is regular (see 18.2.1.2), then the Koszul complex is acyclic. Using 18.2.1.3.5, we are done.

□

Corollary 18.2.1.4. *We suppose $\mathcal{B}_{\mathfrak{X}}$ is $\mathcal{O}_{\mathfrak{X}}$-flat. Let $\mathcal{E}$ be a left coherent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}$-module which is coherent and flat on $\mathcal{B}_{\mathfrak{X}}$. The canonical homomorphism $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}} \mathcal{E} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat} \otimes_{\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}} \mathcal{E}$ is then an isomorphism.*

Proof. This is a consequence of 4.7.3.6 and 18.2.1.3. □

Remark 18.2.1.5. Following a counter-example of Noot-Huyghe, the extension $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp} \to \widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\flat}$ is not flat. The corollary 18.2.1.4 is then not valid for any left coherent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}$-module.

Example 18.2.1.6. Let T be a divisor of X. Then $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ is $\mathcal{O}_{\mathfrak{X}}$-flat. We have $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} := \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^\sharp} = \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(0)}_{\mathfrak{X}^\sharp}$. Let $\mathcal{E}$ be a left coherent $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is locally projective of finite type as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module. The canonical homomorphism $\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \to \mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}$ is therefore an isomorphism.

Notation 18.2.1.7. According to 3.4.5.1, the sheaf $\omega_{\mathfrak{X}^\flat}$ (resp. $\omega_{\mathfrak{X}^\flat}$) is endowed with a canonical structure of right $\mathcal{D}_{\mathfrak{X}^\sharp}$-module (resp. right $\mathcal{D}_{\mathfrak{X}^\flat}$-module). Via the homomorphism $\mathcal{D}_{\mathfrak{X}^\sharp} \subset \mathcal{D}_{\mathfrak{X}^\flat}$, $\omega_{\mathfrak{X}^\flat}$ can also be viewed as a right $\mathcal{D}_{\mathfrak{X}^\sharp}$-module. Hence, we get from 4.2.3.5.(c), two left $\mathcal{D}_{\mathfrak{X}^\sharp}$-modules by setting:

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^\flat}, \omega_{\mathfrak{X}^\sharp}), \mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^\sharp}, \omega_{\mathfrak{X}^\flat}). \tag{18.2.1.7.1}$$

Both sheaves $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$ and $\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D})$ are locally free $\mathcal{O}_{\mathfrak{X}}$-modules of rank 1 (buy beware neither $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$ nor $\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D})$ are sheaves of rings). Let $\mathfrak{U}$ be an open of $\mathfrak{X}$ such that there exist nice coordinates $t_1, \dots, t_d$ of $\mathfrak{U}^\sharp/\mathfrak{S}$ so that $\mathfrak{Z} \cap \mathfrak{U}$ is empty or $\mathfrak{Z} \cap \mathfrak{U}$ is cut out by $\prod_{1 \le j \le r} t_j$ for some $r \ge 1$, and $\mathfrak{Z}^\flat \cap \mathfrak{U}$ is either empty of cut out by $\prod_{1 \le j \le s} t_j$ for some $r \ge s \ge 1$. By applying $j^* = |\mathfrak{Y}$ to the canonical morphisms $\omega_{\mathfrak{X}} \to \omega_{\mathfrak{X}^\sharp}$ and $\omega_{\mathfrak{X}} \to \omega_{\mathfrak{X}^\sharp}$ we get isomorphisms. Indeed, since this is local in $\mathfrak{X}$, we reduce to check it on $\mathfrak{U} \cap \mathfrak{Y}$. The canonical morphisms $\omega_{\mathfrak{U}} \to \omega_{\mathfrak{U}^\sharp}$ is given by $dt_1 \wedge \cdots \wedge dt_d \to t_1 d\log t_1 \wedge \cdots \wedge t_r d\log t_r \wedge dt_{r+1} \wedge \cdots \wedge t_d$ and the canonical morphisms $\omega_{\mathfrak{U}} \to \omega_{\mathfrak{U}^\flat}$ is given by $dt_1 \wedge \cdots \wedge dt_d \to t_1 d\log t_1 \wedge \cdots \wedge t_s d\log t_s \wedge dt_{s+1} \wedge \cdots \wedge t_d$, which conclude the check. Hence, we get the $\mathcal{D}_{\mathfrak{Y}}$-linear isomorphism: $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})|\mathfrak{Y} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{Y}}, \omega_{\mathfrak{Y}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{Y}}$. Hence we get by adjunction the $\mathcal{D}_{\mathfrak{X}^\sharp}$-linear map $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \to j_* \mathcal{O}_{\mathfrak{Y}}$. If $\mathfrak{X} = \mathfrak{U}$, then $V(t_{s+1} \cdots t_r) = \mathfrak{D}$ and we compute that $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \to j_* \mathcal{O}_{\mathfrak{Y}}$ is given by $1 \mapsto \frac{1}{t_{s+1} \cdots t_r}$, which justifies the notation.

Using 4.2.4.11.1, we get the $\mathcal{D}_{\mathfrak{X}^\sharp}$-linearity of 18.2.1.7.2 .

$$\mathrm{ev}\colon \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp},\ \mathrm{ev}\colon \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\flat}. \tag{18.2.1.7.2}$$

By evaluating twice, we obtain from 18.2.1.7.2 the $\mathcal{D}_{\mathfrak{X}^\sharp}$-linear isomorphism : $\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp}$. Hence, we get the $\mathcal{D}_{\mathfrak{X}^\sharp}$-linear isomorphism :

$$\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}. \tag{18.2.1.7.3}$$

Notation 18.2.1.8. For any integer $n \in \mathbb{N}$, this yields of left $\mathcal{D}_{\mathfrak{X}^\sharp}$-modules by setting: $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(\mathfrak{D})^{\otimes n}$ and $\mathcal{O}_{\mathfrak{X}}(-n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(-\mathfrak{D})^{\otimes n}$, where $\otimes n$ means that we tensorise n-times as $\mathcal{O}_{\mathfrak{X}}$-module. For any $n, n' \in \mathbb{Z}$, using 18.2.1.7.3, the canonical isomorphisms $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n'\mathfrak{D}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}((n+n')\mathfrak{D})$ are $\mathcal{D}_{\mathfrak{X}^\sharp}$-linear.

According to 4.3.4.4, we get the left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module $\mathcal{B}_{\mathfrak{X}}(n\mathfrak{D}) := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D})$. Let $\mathcal{E}$ (resp. $\mathcal{M}$) is a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module and $n \in \mathbb{Z}$. Following 4.3.4.12, we define a left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module by setting $\mathcal{E}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\sim} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}$ (resp. $\mathcal{M}(n\mathfrak{D}) := \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})$).

Following 4.2.5.1.1, we have the so called transposition isomorphism of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-bimodule $\gamma_{\mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})}\colon \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D}) \xrightarrow{\sim} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$. Following 4.2.5.1.2, then we check the formula

$$\gamma_{\mathcal{B}_{\mathfrak{X}}(nZ)}(\underline{\partial}_\sharp^{\langle \underline{k} \rangle} \otimes e) = \sum_{\underline{h} \le \underline{k}} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{\partial}_\sharp^{\langle \underline{k}-\underline{h} \rangle} e \otimes \underline{\partial}_\sharp^{\langle \underline{h} \rangle}.$$

We get a $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-bimodule by setting $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$. Using 4.2.4.3, we have the canonical isomorphisms: $\mathcal{E}(n\mathfrak{D}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(n\mathfrak{D}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{E}$ and $\mathcal{M}(n\mathfrak{D}) \xrightarrow{\sim} \mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})) \overset{\gamma_{\mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})}}{\xrightarrow{\sim}} \mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(n\mathfrak{D})$. This yields the isomorphisms:

$$\mathcal{M}(n\mathfrak{D}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(n\mathfrak{D}) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{E}(n\mathfrak{D}). \tag{18.2.1.8.1}$$

18.2.1.9. The functor $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} -$ is an equivalence of category between the category from the left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules to that of right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules, and the functor $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat} = \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^\flat}, -)$ is a quasi-inverse. Indeed, for any left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module $\mathcal{E}$, right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module $\mathcal{M}$, using the right part of 4.2.4.11.2 (resp. 4.2.4.11.1), we get the left (resp. right) canonical morphism

$$\mathcal{E} \to \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^\flat}, \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}), \qquad \mathrm{ev}\colon \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\omega_{\mathfrak{X}^\flat}, \mathcal{M}) \to \mathcal{M} \tag{18.2.1.9.1}$$

is $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-linear. Since $\omega_{\mathfrak{X}^\flat}$ is locally free of rank one, then these morphisms are isomorphisms and we are done.

Lemma 18.2.1.10. *Let $\mathcal{E}$ be a left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module and $\mathcal{M}$ be a right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module. We have the following canonical isomorphisms of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules :*

$$\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}(\mathfrak{D})\xrightarrow{\sim}\omega_{\mathfrak{X}^\sharp}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E},\qquad \omega_{\mathfrak{X}^\sharp}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}(-\mathfrak{D})\xrightarrow{\sim}\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}, \tag{18.2.1.10.1}$$

$$\mathcal{E}(\mathfrak{D})\xrightarrow{\sim}(\omega_{\mathfrak{X}^\sharp}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E})\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat},\qquad \mathcal{E}(-\mathfrak{D})\xrightarrow{\sim}(\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E})\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\sharp}, \tag{18.2.1.10.2}$$

$$\mathcal{M}(\mathfrak{D})\xrightarrow{\sim}\omega_{\mathfrak{X}^\sharp}\otimes_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat}),\qquad \mathcal{M}(-\mathfrak{D})\xrightarrow{\sim}\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\sharp}). \tag{18.2.1.10.3}$$

Proof. By applying the functor $-_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}$ to the isomorphism 18.2.1.7.2, via the associativity isomorphism of the tensor product of 4.2.4.1, we get the isomorphisms 18.2.1.10.1. Via 4.3.5.4 and 18.2.1.9, this yields the other isomorphisms of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules. □

Notation 18.2.1.11. According to 5.1.1.2, we set $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat\leftarrow\mathfrak{X}^\sharp}:=\omega_{\mathfrak{X}^\sharp}\overset{\mathrm{r}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat})$ viewed as $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat},\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$-bimodule, where the symbol "r" means that to compute the tensor product we choose the right structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-module of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat}$. Following 18.2.1.10.3, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat\leftarrow\mathfrak{X}^\sharp}$ is canonically isomorphic to $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$, which justifies the notation. We can also denote by $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\rightarrow\mathfrak{X}^\flat}:=\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$ considered as a $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp},\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})$-bimodule.

Definition 18.2.1.12. We denote by $\widetilde{\mathfrak{X}}^\sharp$ (resp. $\widetilde{\mathfrak{X}}^\flat$) the ringed $\mathcal{V}$-logarithmic formal scheme $(\mathfrak{X}^\sharp,\mathcal{B}_{\mathfrak{X}})$ (resp. $(\mathfrak{X}^\sharp,\mathcal{B}_{\mathfrak{X}})$), and by $\tilde{u}\colon\widetilde{\mathfrak{X}}^\sharp\to\widetilde{\mathfrak{X}}^\flat$ the induced morphism of ringed $\mathcal{V}$-logarithmic formal schemes by the diagram. Similarly to 5.1.3.1 (we had not considered the "algebraic version" of pushforwards on formal schemes), for any $\mathcal{E}\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M}\in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we define respectively the *direct image* by $\widetilde{u}$ of $\mathcal{E}$ and $\mathcal{M}$ by setting:

$$\widetilde{u}^{(m)}_{+}(\mathcal{E}):=\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat\leftarrow\mathfrak{X}^\sharp}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}\mathcal{E}\xrightarrow{\sim}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}))\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}}\mathcal{E},$$
$$\widetilde{u}^{(m)}_{+}(\mathcal{M}):=\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp\rightarrow\mathfrak{X}^\flat}=\mathcal{M}\otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}. \tag{18.2.1.12.1}$$

18.2.1.13. By reversing the roles of $\mathfrak{D}$ and $\mathfrak{Z}^\flat$, we define similarly to 18.2.1.7.1 and 18.2.1.8 the left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-modules $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)$ for any $n\in\mathbb{Z}$. In fact, by an easy computation, we can check that $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)$ is a left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-submodule of $j_*\mathcal{O}_{\mathfrak{Y}}$. Let $\mathcal{E}^\sharp$ be a left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module.

By applying $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}-$ to the canonical map of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules $\mathcal{E}^\sharp\to\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\mathcal{E}^\sharp$, we get the morphism of of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp\to\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\mathcal{E}^\sharp)$. This yields by extension the morphism of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-modules

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp)\to\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\mathcal{E}^\sharp). \tag{18.2.1.13.1}$$

This morphism is an isomorphism. Indeed, we construct an inverse map as follows. Using 18.2.1.13.1 for $-n$ and with $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp$ instead of $\mathcal{E}^\sharp$ we get the morphism:

$$\begin{aligned}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\mathcal{E}^\sharp&\xrightarrow{\sim}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{O}_{\mathfrak{X}}(-n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp)\\ &\underset{18.2.1.13.1}{\longrightarrow}\mathcal{O}_{\mathfrak{X}}(-n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\left(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp)\right).\end{aligned} \tag{18.2.1.13.2}$$

By applying $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}-$ to 18.2.1.13.2 we get the morphism

$$\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}\mathcal{E}^\sharp)\to\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}\otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp). \tag{18.2.1.13.3}$$

The morphism 18.2.1.13.3 is an inverse of 18.2.1.13.1 and we are done.

SImilarly, for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we construct the canonical isomorphisms:

$$\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} (\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\sharp) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{E}^\sharp), \tag{18.2.1.13.4}$$

$$(\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M}^\sharp) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}). \tag{18.2.1.13.5}$$

Proposition 18.2.1.14. *For any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have the canonical isomorphisms:*

$$\widetilde{u}^{(m)}_+(\mathcal{M}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat} \xrightarrow{\sim} \widetilde{u}^{(m)}_+(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}), \qquad \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{u}^{(m)}_+(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^{(m)}_+(\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{18.2.1.14.1}$$

Moreover, if $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$ (resp. $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$) then $\widetilde{u}^{(m)}_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})$ (resp. $\widetilde{u}^{(m)}_+(\mathcal{M}) \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})$).

Proof. We construct the left isomorphism of 18.2.1.14.1 as follows:

$$\widetilde{u}^{(m)}_+(\mathcal{M}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat} \underset{4.2.4.3}{\xrightarrow{\sim}} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}) \underset{4.3.5.6.1}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^\sharp} \overset{\mathrm{l}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat})) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} (\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp})$$
$$\underset{4.2.5.6.3}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^\sharp} \overset{\mathrm{r}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat})) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} (\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}) = \widetilde{u}^{(m)}_+(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}),$$

The symbol "l" and "r" meaning respectively that to compute the tensor product we choose the left and right structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-module of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}$. This yields by passage from right to left (i.e., via the equivalences of categories of 4.3.5.5) the right isomorphism of 18.2.1.14.1. Concerning the the preservation of the perfectness, The case of right modules is straightforward. As equivalence of categories between left and right modules of the form 4.3.5.7 are exact and preserve the local projectivity of finite type, then they also preserve perfect complexes. Hence, the left case is then a consequence of 18.2.1.14.1. □

Definition 18.2.1.15. For any $* \in \{\mathrm{l}, \mathrm{r}\}$, we define the dual functors $\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp} \colon D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}) \to D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$ as follows. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$. Similarly to 5.1.4.1 (we are working with formal schemes), we define *the $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-linear duals* of $\mathcal{E}$ and of $\mathcal{M}$ by setting

$$\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{E}) = \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{E}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}[d_X], \qquad \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{M}) = \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{M}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})[d_X].$$

We also consider the $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-linear duals: $\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\flat} \colon D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}) \to D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})$.

18.2.1.16. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$. Similarly to 5.1.4.4, we have the biduality isomorphism: $\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp} \circ \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ (similarly for $\mathcal{M}$). Similarly to 5.1.4.3, we get the isomorphisms: $\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{E})$ and $\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}) \xrightarrow{\sim} \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{M}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}$.

18.2.1.17. It follows from 18.2.1.9 that for any $\mathcal{E} \in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}, \mathcal{M}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{E}, \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}), \tag{18.2.1.17.1}$$

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}, \mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{M}, \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{18.2.1.17.2}$$

Definition 18.2.1.18. For any $* \in \{\mathrm{l}, \mathrm{r}\}$, we define the *extraordinary direct image* by $\widetilde{u}$ the functor $\widetilde{u}^{(m)}_! \colon \widetilde{u}^{(m)}_! \colon D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}) \to D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})$ by setting

$$\widetilde{u}^{(m)}_! := \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\flat} \circ \widetilde{u}^{(m)}_+ \circ \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}.$$

Proposition 18.2.1.19. *For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have the canonical isomorphisms*

$$\widetilde{u}^{(m)}_!(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}) \xrightarrow{\sim} \widetilde{u}^{(m)}_!(\mathcal{M}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}, \qquad \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{u}^{(m)}_!(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^{(m)}_!(\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}). \tag{18.2.1.19.1}$$

Proof. This comes by composition from 18.2.1.14 and 18.2.1.16. □

Proposition 18.2.1.20. *For any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have the canonical isomorphisms:*

$$\widetilde{u}^{(m)}_{!}(\mathcal{E}) \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{E},\ \widetilde{u}^{(m)}_{!}(\mathcal{M}) \xrightarrow{\sim} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} (\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}). \tag{18.2.1.20.1}$$

Proof. Set $\mathcal{F} := \mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\sharp}(\mathcal{E})$. By definition, $\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\flat} \circ \widetilde{u}^{(m)}_{+}(\mathcal{F}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}}((\omega_{\mathfrak{X}^\sharp} \overset{\mathrm{r}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat})) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat})[d]$, the symbol "r" meaning that to compute the internal tensor product we choose the right structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-module of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}$. Via the transposition isomorphism $\beta_{\widetilde{\mathfrak{X}}^\flat} : \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}$ (see 4.2.5.6.3), we get the isomorphisms of $(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$-bimodules:

$$\omega_{\mathfrak{X}^\sharp} \overset{\mathrm{r}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}) \underset{\beta_{\widetilde{\mathfrak{X}}^\flat}}{\xrightarrow{\sim}} \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}, \tag{18.2.1.20.2}$$

where the structure of left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-module (resp. left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module) of the right term is the twisted structure induced by the right $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$-module (resp. the left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-module) structure of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$. Hence, we get:

$$\begin{aligned}
&\mathbb{D}^{(m)}_{\widetilde{\mathfrak{X}}^\flat} \circ \widetilde{u}^{(m)}_{+}(\mathcal{F}) \overset{\beta_{\widetilde{\mathfrak{X}}^\flat}}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}}((\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat})[d].\\
&\underset{18.2.1.17.2}{\xrightarrow{\sim}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}}((\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}) \overset{\mathrm{l}}{\otimes}{}^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{F}, \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\flat})[d] \underset{4.2.5.6.2}{\xrightarrow{\sim}}\\
&\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}}((\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}) \overset{\mathrm{l}}{\otimes}{}^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})[d] \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}((\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}) \overset{\mathrm{l}}{\otimes}{}^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})[d] \underset{4.2.5.6.1}{\xrightarrow{\sim}}\\
&\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat})[d] \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}, \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})[d]\\
&\underset{18.2.3.6.1}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{F}, (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp})_{\mathrm{r}})[d]\\
&\underset{4.2.5.6.1}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{F}, (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp})_{\mathrm{l}})[d] \underset{4.6.6.4}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathbb{D}_{\mathfrak{X}^\sharp}(\mathcal{F}).
\end{aligned} \tag{18.2.1.20.3}$$

where the second isomorphism is 18.2.1.17.2 used in the case where $\sharp = \flat$, where the symbol "r" (resp. "l") means that to compute $\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}$ we choose the right (resp. left) structure of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}$. Since by biduality $\mathbb{D}_{\mathfrak{X}^\sharp}(\mathcal{F}) \xrightarrow{\sim} \mathcal{E}$ (see 18.2.1.16), then we get the first isomorphism of 18.2.1.20.1. In order to establish the second one, we can suppose $\mathcal{M} = \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$. We get:

$$\begin{aligned}
&u_{!}(\mathcal{M}) \underset{18.2.1.19.1}{\xrightarrow{\sim}} \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} u_{!}(\mathcal{E}) \underset{18.2.1.20.1}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{E} \underset{\delta}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}) \overset{\mathrm{l}}{\otimes}{}^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{E}\\
&\underset{4.3.5.6.1}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} ((\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}) \overset{\mathrm{l}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^\sharp}) \underset{18.2.1.10.2}{\xrightarrow{\sim}} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} (\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}).
\end{aligned}$$

□

The following proposition means that the relative duality isomorphism of the canonical morphism $\mathfrak{X}^\sharp \to \mathfrak{X}^\flat$ necessity of use a twist (see 18.2.1.21).

Proposition 18.2.1.21. *For any $* \in \{\mathrm{l}, \mathrm{r}\}$, for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{*}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have the canonical isomorphism:*

$$\widetilde{u}^{(m)}_{+}(\mathcal{E}) \xrightarrow{\sim} \widetilde{u}^{(m)}_{!}(\mathcal{E}(\mathfrak{D})). \tag{18.2.1.21.1}$$

Proof. With 18.2.1.14.1 and 18.2.1.19.1, it is sufficient to treat the left case (i.e. $* = l$). Let $\mathcal{P}^\flat$ be a resolution of $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$ by flat $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-bimodule and $\mathcal{P}^\sharp$ be a resolution of $\mathcal{E}$ by flat left $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}$-modules. Hence we have isomorphisms:

$$\begin{aligned}
&(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \xleftarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^\sharp) \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{P}^\flat \overset{4.5.3.8.1}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P}^\flat) \overset{\mathrm{l}}{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{P}^\sharp\\
&\xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}) \overset{\mathrm{l}}{\otimes}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{P}^\sharp \underset{4.2.5.6.1}{\xrightarrow{\sim}} \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{P}^\sharp) \xrightarrow{\sim} \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} (\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}} \mathcal{E}).
\end{aligned}$$

With 18.2.1.12.1, 18.2.1.20, we get therefore $\widetilde{u}_{+}^{(m)}(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{u}_{!}^{(m)}(\mathcal{E})$. By applying to it the functor $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}^\flat}^{-1}$, via 18.2.1.10.2, 18.2.1.14.1 this yields the isomorphism: $\widetilde{u}_{+}^{(m)}(\mathcal{E}(-\mathfrak{D})) \xrightarrow{\sim} \widetilde{u}_{!}^{(m)}(\mathcal{E})$. □

Corollary 18.2.1.22. *For any $* \in \{\mathrm{l}, \mathrm{r}\}$, for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have the canonical isomorphism:*

$$\widetilde{u}_{+}^{(m)}(\mathcal{E}(\mathfrak{Z}^\flat)) \xrightarrow{\sim} (\widetilde{u}_{+}^{(m)}(\mathcal{E}))(\mathfrak{Z}^\flat), \qquad \widetilde{u}_{!}^{(m)}(\mathcal{E}(\mathfrak{Z}^\flat)) \xrightarrow{\sim} (\widetilde{u}_{!}^{(m)}(\mathcal{E}))(\mathfrak{Z}^\flat). \tag{18.2.1.22.1}$$

Proof. This is a consequence of 18.2.1.13.4, 18.2.1.13.5, 18.2.1.20 and 18.2.1.21. □

18.2.1.23. Using 18.2.1.12.1, 18.2.1.20 and 18.2.1.21, for any $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we get the isomorphism

$$(\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathfrak{Z})) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \mathcal{E} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} (\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}),$$
$$(\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(-\mathfrak{Z})) \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \xrightarrow{\sim} \mathcal{M} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} (\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}).$$

Proposition 18.2.1.24. *We suppose $\mathcal{B}_{\mathfrak{X}}$ is $\mathcal{O}_{\mathfrak{X}}$-flat. Let $\mathcal{E}$ be a left coherent $\widetilde{\mathcal{D}}^{(0)}_{\mathfrak{X}^\sharp}$-module which is coherent and flat on $\mathcal{B}_{\mathfrak{X}}$. For any $l \neq 0$,*

$$H^l(\widetilde{u}_{+}^{(m)}(\mathcal{E})) = 0, \qquad H^l(\widetilde{u}_{!}^{(m)}(\mathcal{E})) = 0. \tag{18.2.1.24.1}$$

Proof. For any $l \neq 0$, the equality $H^l(\widetilde{u}_{!}^{(m)}(\mathcal{E})) = 0$ follows from 18.2.1.20 and 18.2.1.4. Moreover, $\widetilde{u}_{+}^{(m)}(\mathcal{E}) \underset{18.2.1.21.1}{\xrightarrow{\sim}} \widetilde{u}_{!}^{(m)}(\mathcal{E}(\mathfrak{D})) \underset{18.2.1.20}{\xrightarrow{\sim}} \widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat} \otimes^{\mathbb{L}}_{\widetilde{\mathcal{D}}_{\mathfrak{X}^\sharp}} (\mathcal{E}(\mathfrak{D}))$. Since $\mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})$ is a flat $\mathcal{B}_{\mathfrak{X}}$-module, since $\mathcal{E}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\sim} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{E}$, then $\mathcal{E}(n\mathfrak{D})$ is a flat $\mathcal{B}_{\mathfrak{X}}$-module. Hence, we conclude using 18.2.1.4. □

18.2.2 Preliminaries on complete level m context

We keep notation 18.2.1 and we suppose $\mathcal{B}_{\mathfrak{X}}$ be an $\mathcal{O}_{\mathfrak{X}}$-algebra commutative satisfying the hypotheses of 7.5.3. We set $\mathcal{D}_\sharp := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp}$, $\widehat{\mathcal{D}}_\sharp := \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp}$, $\mathcal{D}_\flat := \mathcal{B}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\flat}$, $\widehat{\mathcal{D}}_\flat := \mathcal{B}_{\mathfrak{X}} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\flat}$, which are coherent sheaves of rings (see 7.5.1.3). By p-adic completion, we get from the ring homomorphism $\mathcal{D}_\sharp \to \mathcal{D}_\flat$, the ring homomorphism $\widehat{\mathcal{D}}_\sharp \to \widehat{\mathcal{D}}_\flat$.

The purpose of this subsection is mainly to prove 18.2.2.4.2 which will be used to check later 18.2.3.5.2, a key ingredient in the proof of 18.2.3.11.

18.2.2.1. Since $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D})$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module then it is p-adically complete and therefore endowed with a structure of $\widehat{\mathcal{D}}_\sharp$-module extending its structure of $\mathcal{D}$-module. Moreover, the canonical morphism $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D} \to \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}_\sharp$ is an isomorphism. Hence, $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}_\sharp$ is endowed with a structure of $\widehat{\mathcal{D}}_\sharp$-bimodule extending canonically its structure of $\mathcal{D}$-bimodule. Similarly, $\widehat{\mathcal{D}}_\sharp \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D})$ is endowed with a canonical structure of $\widehat{\mathcal{D}}_\sharp$-bimodule. By p-adic completion of the transposition isomorphism of 4.2.5.1.1, we get $\widehat{\gamma}_{\mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})} \colon \widehat{\mathcal{D}}_\sharp \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}_\sharp$. This yields a $\widehat{\mathcal{D}}_\sharp$-bimodule by setting $\widehat{\mathcal{D}}_\sharp(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}_\sharp$.

Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\widehat{\mathcal{D}}_\sharp$-module and $n \in \mathbb{Z}$. Using 4.2.4.3, we compute that the canonical isomorphisms $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{D}}_\sharp(n\mathfrak{D}) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}$ and $\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \xrightarrow{\sim} \mathcal{M} \otimes_{\widehat{\mathcal{D}}_\sharp} (\widehat{\mathcal{D}}_\sharp \otimes_{\mathcal{B}_{\mathfrak{X}}} \mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})) \overset{\widehat{\gamma}_{\mathcal{B}_{\mathfrak{X}}(n\mathfrak{D})}}{\xrightarrow{\sim}} \mathcal{M} \otimes_{\widehat{\mathcal{D}}_\sharp} \widehat{\mathcal{D}}_\sharp(n\mathfrak{D})$ are $\mathcal{D}$-linear. Hence, $\mathcal{E}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ (resp. $\mathcal{M}(n\mathfrak{D}) := \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D})$) is canonically endowed with a structure of left (resp. right) $\widehat{\mathcal{D}}_\sharp$-module extending its structure of left (resp. right) $\mathcal{D}_\sharp$-module. Similarly to 18.2.1.8.1, we get the isomorphisms:

$$\mathcal{M}(n\mathfrak{D}) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\widehat{\mathcal{D}}_\sharp} \widehat{\mathcal{D}}_\sharp(n\mathfrak{D}) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}(n\mathfrak{D}). \tag{18.2.2.1.1}$$

18.2.2.2. We keep notation 18.2.2.1.

(a) Since $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp$ is p-adically complete, then they are endowed with a structure of right $\widehat{\mathcal{D}}_\sharp$-bimodule extending there structure of right $\mathcal{D}$-modules.

(b) It follows from 4.2.4.3 that if $\mathcal{E}$ is a left $\widehat{\mathcal{D}}_\sharp$-module, then $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}$ is endowed with a structure of right $\widehat{\mathcal{D}}_\sharp$-module extending canonically its structure of right $\mathcal{D}$-module making $\widehat{\mathcal{D}}_\sharp$-linear the isomorphism:

$$\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}. \tag{18.2.2.2.1}$$

(c) Via 4.2.4.4, if $\mathcal{M}$ is a right $\widehat{\mathcal{D}}_\sharp$-module, then $\mathcal{M} \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\flat}^{-1} = \mathcal{H}om_{\mathcal{O}_\mathfrak{X}}(\omega_{\mathfrak{X}^\flat}, \mathcal{M})$ is endowed with a structure of right $\widehat{\mathcal{D}}_\sharp$-module extending canonically its structure of left $\mathcal{D}$-module making $\widehat{\mathcal{D}}_\sharp$-linear the isomorphism:

$$\mathcal{H}om_{\mathcal{O}_\mathfrak{X}}(\omega_{\mathfrak{X}^\flat}, \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\widehat{\mathcal{D}}_\sharp}(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp, \mathcal{M}). \tag{18.2.2.2.2}$$

(d) Since $\omega_{\mathfrak{X}^\flat}$ is locally free (of rank one), the canonical $\mathcal{O}_\mathfrak{X}$-linear morphism, then we have the $\widehat{\mathcal{D}}_\sharp$-linear isomorphism

$$\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{H}om_{\mathcal{O}_\mathfrak{X}}(\omega_{\mathfrak{X}^\flat}, \mathcal{M}) \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{H}om_{\widehat{\mathcal{D}}_\sharp}(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp, \mathcal{M}) \xrightarrow{\sim} \mathcal{M}, \tag{18.2.2.2.3}$$

where the last isomorphism is the evaluation one. We have the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-linear isomorphism:

$$\mathcal{E} \xrightarrow{\sim} \mathcal{H}om_{\widehat{\mathcal{D}}_\sharp}(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp, \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_\mathfrak{X}}(\omega_{\mathfrak{X}^\flat}, \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}). \tag{18.2.2.2.4}$$

(e) Similarly to 4.3.5.6, this yields that for any left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^\sharp/S^\sharp}$-module $\mathcal{E}$ (resp. $\mathcal{M}$), we have the following isomorphism of $\mathcal{O}_S$-modules:

$$\mathcal{M} \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}) \otimes_{\widehat{\mathcal{D}}_\sharp} (\mathcal{M} \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\flat}^{-1}). \tag{18.2.2.2.5}$$

(f) As for 4.3.5.7, using the above results, we can check that the functors $- \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\flat}^{-1} = \mathcal{H}om_{\mathcal{O}_\mathfrak{X}}(\omega_{\mathfrak{X}^\flat}, -)$ and $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} -$ induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\widehat{\mathcal{D}}_\sharp$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\widehat{\mathcal{D}}_\sharp$-modules. These equivalences extends to complexes.

Lemma 18.2.2.3. *Let $\mathcal{E}$ be a left $\widehat{\mathcal{D}}_\sharp$-module and $\mathcal{M}$ be a right $\widehat{\mathcal{D}}_\sharp$-module. Via the structure of $\widehat{\mathcal{D}}_\sharp$-modules defined at 18.2.2.2 and 7.5.1.13, we have the following canonical isomorphisms of $\widehat{\mathcal{D}}_\sharp$-modules:*

$$\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}, \qquad \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}(-\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}, \tag{18.2.2.3.1}$$

$$\mathcal{E}(\mathfrak{D}) \xrightarrow{\sim} (\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}) \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\flat}^{-1}, \qquad \mathcal{E}(-\mathfrak{D}) \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}) \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\sharp}^{-1}, \tag{18.2.2.3.2}$$

$$\mathcal{M}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} (\mathcal{M} \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\flat}^{-1}), \qquad \mathcal{M}(-\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} (\mathcal{M} \otimes_{\mathcal{O}_\mathfrak{X}} \omega_{\mathfrak{X}^\sharp}^{-1}). \tag{18.2.2.3.3}$$

Proof. It follows by functoriality from 18.2.1.10.1, that we have the canonical isomorphism of right $\mathcal{D}$-bimodules $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp$. Since they are p-adically complete, then this is in fact an isomorphism of right $\widehat{\mathcal{D}}_\sharp$-bimodules. Hence, we get the third isomorphism:

$$\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}(\mathfrak{D}) \underset{\sim}{\overset{18.2.2.2.1}{\longrightarrow}} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp) \otimes_{\widehat{\mathcal{D}}_\sharp} (\widehat{\mathcal{D}}_\sharp(\mathfrak{D}) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E})$$
$$\xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp(\mathfrak{D})) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} \widehat{\mathcal{D}}_\sharp) \otimes_{\widehat{\mathcal{D}}_\sharp} \mathcal{E} \underset{\sim}{\overset{7.5.1.13}{\longrightarrow}} \omega_{\mathfrak{X}^\sharp} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{E}.$$

We get the other isomorphisms by using 18.2.2.2 and 7.5.1.13. □

18.2.2.4. Let $\mathcal{E}^\sharp \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}_\sharp)$ and $\mathcal{F}^\flat \in D({}^{\mathrm{l}}\widehat{\mathcal{D}}_\flat)$. Using 18.2.2.2.(f), by applying $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{B}_\mathfrak{X}} -$ to the morphism $\mathcal{E}^\sharp \to \widehat{\mathcal{D}}_\flat \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}^\sharp$ of $D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\widehat{\mathcal{D}}_\sharp)$, we get the morphism of $D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{r}}\widehat{\mathcal{D}}_\sharp)$: $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{B}_\mathfrak{X}} \mathcal{E}^\sharp \to \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{B}_\mathfrak{X}} (\widehat{\mathcal{D}}_\flat \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}^\sharp)$. This yields by extension the homomorphism of coherent right $\widehat{\mathcal{D}}_\flat$-modules :

$$(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{B}_\mathfrak{X}} \mathcal{E}^\sharp) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_\sharp} \widehat{\mathcal{D}}_\flat \to \omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{B}_\mathfrak{X}} (\widehat{\mathcal{D}}_\flat \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}^\sharp). \tag{18.2.2.4.1}$$

Following 4.5.3.8.2, this map is isomorphism modulo $\mathfrak{m}^{i+1}$. Since the complexes are coherent and therefore quasi-coherent, then 18.2.2.4.1 is an isomorphism. Using 7.5.1.13.7 instead of 4.3.5.10.1, similarly to 4.5.3.8.1 we get from 18.2.2.4.1 the isomorphism:

$$(\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\sharp) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_\sharp} \mathcal{F}^\flat \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}^\flat) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}_\sharp} \mathcal{E}^\sharp. \tag{18.2.2.4.2}$$

18.2.3 Holonomicity and acyclicity of the image by u_+ and $u_!$ of overconvergent log isocrystals

Let T be a divisor of X. Applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ to the ring homomorphism $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T)$ (see 18.2.2), we get the ring homomorphism $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}$.

Theorem 18.2.3.1. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module of finite type. The canonical morphism*

$$\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \to \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \tag{18.2.3.1.1}$$

is an isomorphism.

Proof. Since $\mathcal{E}$ be a left coherent $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module, then according to 18.2.1.6 the canonical morphism

$$\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}_{\mathfrak{X}^\sharp,\mathbb{Q}}} \mathcal{E} \to \mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp,\mathbb{Q}}} \mathcal{E} \tag{18.2.3.1.2}$$

is an isomorphism. Since the extensions $\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}$, $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$ are flat, this yields that the canonical morphism

$$\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} (\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}) \to \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} (\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}) \tag{18.2.3.1.3}$$

is an isomorphism. We conclude via 11.2.1.9.2. □

18.2.3.2. Tensoring with $\mathbb{Q}$ and passing to the inductive limit on the level, we get from 18.2.2.1 the following properties: $\mathcal{O}_{\mathfrak{X}}(n\mathfrak{D})$ is endowed with a structure of $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module extending its structure of $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module. The transposition isomorphism of 18.2.2.1 induces the isomorphism of $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-bimodule of the form: $\gamma\colon \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$. We get a $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-bimodule by setting $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$.

Let $\mathcal{E}$ (resp. $\mathcal{M}$) be a left (resp. right) $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module and $n \in \mathbb{Z}$. The sheaves $\mathcal{E}(n\mathfrak{D}) := \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ (resp. $\mathcal{M}(n\mathfrak{D}) := \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{D})$) is canonically endowed with a structure of left (resp. right) $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module extending its structure of left (resp. right) $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module. Similarly to 18.2.1.8.1, we get the isomorphisms:

$$\mathcal{M}(n\mathfrak{D}) \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}(n\mathfrak{D}) \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}(n\mathfrak{D}). \tag{18.2.3.2.1}$$

Let $\mathcal{M} \in D^-({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, $\mathcal{F} \in D^-({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$. By taking flat resolutions of $\mathcal{M}$ and $\mathcal{F}$, we construct from 18.2.3.2.1 the canonical isomorphism:

$$\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} (\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{Z}) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{F}) \xrightarrow{\sim} (\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{Z})) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{F}. \tag{18.2.3.2.2}$$

18.2.3.3. We keep notation 18.2.3.2.

(a) Tensoring with $\mathbb{Q}$ and passing to the inductive limit on the level, we get from 18.2.2.2 that $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$ is endowed with a structure of right $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-bimodule extending there structure of right $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-modules.

(b) It follows from 4.2.4.3 that if $\mathcal{E}$ is a left $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module, then $\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$ is endowed with a structure of right $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module extending canonically its structure of right $\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module making $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-linear the isomorphism:

$$\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}) \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}. \tag{18.2.3.3.1}$$

(c) Via 4.2.4.4, if $\mathcal{M}$ is a right $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-module, then $\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\flat}} = \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^{\flat}},\mathcal{M})$ is endowed with a structure of right $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-module extending canonically its structure of left $\mathcal{D}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-module making $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-linear the isomorphism:

$$\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^{\flat}},\mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}(\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}},\mathcal{M}). \tag{18.2.3.3.2}$$

(d) Since $\omega_{\mathfrak{X}^{\flat}}$ is locally free (of rank one), the canonical $\mathcal{O}_{\mathfrak{X}}$-linear morphism, then we have the $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-linear isomorphism

$$\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^{\flat}},\mathcal{M}) \xrightarrow{\sim} (\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}(\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}},\mathcal{M}) \xrightarrow{\sim} \mathcal{M}, \tag{18.2.3.3.3}$$

where the last isomorphism is the evaluation one. We have the canonical $\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/S^{\sharp}}$-linear isomorphism:

$$\mathcal{E} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}(\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}},\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{E}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^{\flat}},\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}). \tag{18.2.3.3.4}$$

(e) Similarly to 4.3.5.6, this yields that for any left (resp. right) $\widetilde{\mathcal{D}}^{(m)}_{X^{\sharp}/S^{\sharp}}$-module $\mathcal{E}$ (resp. $\mathcal{M}$), we have the following isomorphism of $\mathcal{O}_S$-modules:

$$\mathcal{M}\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E})\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\flat}}). \tag{18.2.3.3.5}$$

(f) As for 4.3.5.7, using the above results, we can check that the functors $-\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\flat}} = \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}^{\flat}},-)$ and $\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_X}-$ induce quasi-inverse equivalences between the category of (resp. coherent, resp. flat, resp. locally projective of finite type) left $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules and that of (resp. coherent, resp. flat, resp. locally projective of finite type) right $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules. These equivalences extends to complexes.

Lemma 18.2.3.4. *Let $\mathcal{E}$ be a left $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-module and $\mathcal{M}$ be a right $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-module. Via the structure of $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules defined at 18.2.3.3 and 8.7.2.2, we have the following canonical isomorphisms of $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-modules:*

$$\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}, \qquad \omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}(-\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}, \tag{18.2.3.4.1}$$

$$\mathcal{E}(\mathfrak{D}) \xrightarrow{\sim} (\omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E})\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\flat}}, \qquad \mathcal{E}(-\mathfrak{D}) \xrightarrow{\sim} (\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E})\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\sharp}}, \tag{18.2.3.4.2}$$

$$\mathcal{M}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\flat}}), \qquad \mathcal{M}(-\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^{\sharp}}). \tag{18.2.3.4.3}$$

Proof. It follows by functoriality from 18.2.1.10.1, that we have the canonical isomorphism of right $\mathcal{D}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-bimodules $\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}(\mathfrak{D}) \xrightarrow{\sim} \omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$. Since they are p-adically complete, then this is in fact an isomorphism of right $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}$-bimodules. Hence, we get the third isomorphism:

$$\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}(\mathfrak{D}) \overset{18.2.2.2.1}{\xrightarrow{\sim}} (\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}(\mathfrak{D})\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{E})$$

$$\xrightarrow{\sim} (\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}(\mathfrak{D}))\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{E} \xrightarrow{\sim} (\omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})\otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}}}\mathcal{E} \overset{7.5.1.13}{\xrightarrow{\sim}} \omega_{\mathfrak{X}^{\sharp}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}.$$

We get the other isomorphisms by using 18.2.2.2 and 7.5.1.13. □

18.2.3.5. Let $\mathcal{E}^{\sharp} \in D^{\mathrm{b}}_{\mathrm{coh}}({}^{\mathrm{l}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\sharp}}({}^{\dagger}T)_{\mathbb{Q}})$. Let $\lambda_0\colon \mathbb{N}\to\mathbb{N}$ an increasing map such that $p^{\lambda_0(0)} \geq e/(p-1)$ (see 1.2.4.2) and $\lambda_0(m) \geq m$, for any $m\in\mathbb{N}$. Let $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}}(T)$, $\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}}(T) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T)\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\sharp}}$. It follows from the equivalence 8.7.5.4.1, that there exists $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}(T))$ such that $\underrightarrow{l}^{*}_{\mathbb{Q}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}$. By definition (see 8.4.1.1), we can suppose there exists $\lambda \in L(\mathbb{N})$ large enough such that $\mathcal{E}^{(\bullet)} \in D^{\mathrm{b}}_{\mathrm{coh}}(\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}(T))$. Let $\mathcal{F}^{(\bullet)} \in D^{\mathrm{b}}({}^{\mathrm{l}}\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\flat}/\mathfrak{S}}(T),{}^{\mathrm{r}}\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\flat}/\mathfrak{S}}(T))$ and $\mathcal{F}^{\flat} := \underrightarrow{l}^{*}_{\mathbb{Q}}({}^{\mathrm{l}}\mathcal{F}^{(\bullet)}) \in D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\flat}}({}^{\dagger}T)_{\mathbb{Q}},{}^{\mathrm{r}}\mathcal{D}^{\dagger}_{\mathfrak{X}^{\flat}}({}^{\dagger}T)_{\mathbb{Q}})$. It follows from 18.2.2.4.2 that we have the canonical isomorphism of $D^{\mathrm{b}}({}^{\mathrm{r}}\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\flat}/\mathfrak{S}}(T))$:

$$(\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)})\otimes^{\mathbb{L}}_{\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}(T)}\mathcal{F}^{(\bullet)} \xrightarrow{\sim} (\omega_{\mathfrak{X}^{\flat}}\otimes_{\mathcal{B}_{\mathfrak{X}}}\mathcal{F}^{(\bullet)})\otimes^{\mathbb{L}}_{\lambda^{*}\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}/\mathfrak{S}}(T)}\mathcal{E}^{(\bullet)}). \tag{18.2.3.5.1}$$

By applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ this yields the isomorphism of $D^{\mathrm{b}}({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}})$:

$$(\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^\sharp)\otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{F}^\flat \xrightarrow{\sim} (\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{F}^\flat)\otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{E}^\sharp). \tag{18.2.3.5.2}$$

18.2.3.6. It follows from 18.2.3.3 that for any $\mathcal{E}\in D({}^{\mathrm{l}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, $\mathcal{M}\in D({}^{\mathrm{r}}\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E},\mathcal{M}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{E},\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat}), \tag{18.2.3.6.1}$$

$$\mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat},\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}}(\mathcal{M},\omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}). \tag{18.2.3.6.2}$$

Definition 18.2.3.7. Let $*\in\{\mathrm{l},\mathrm{r}\}$. The pushforward by u with coefficients in $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ of finite level (as in 18.2.1.12) is denoted by $u^{\mathrm{alg}}_{T,+}\colon D^-({}^*\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})\to D^-({}^*\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}})$. This later functor do not have to be confound with the usual pushforward $u_{T,+}\colon D^-({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})\to D^-({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}})$ (see 9.2.4.14), which is defined by setting for any $\mathcal{F}\in D^-({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, $\mathcal{M}\in D^-({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$,

$$u_{T,+}(\mathcal{F}) := \mathcal{D}^\dagger_{\mathfrak{X}^\flat\leftarrow\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}\otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{F}, \qquad u_{T,+}(\mathcal{M}) := \mathcal{M}\otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp\to\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}},$$

where $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp\to\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} := \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}$ as $(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}},\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}})$-bimodule and where $\mathcal{D}^\dagger_{\mathfrak{X}^\flat\leftarrow\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$ is the $(\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}},\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$-bimodule

$$\mathcal{D}^\dagger_{\mathfrak{X}^\flat\leftarrow\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}} := \omega_{\mathfrak{X}^\sharp}\overset{\mathrm{r}}{\otimes}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat}) \underset{18.2.3.4.3}{\xrightarrow{\sim}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \underset{18.2.3.2}{\xrightarrow{\sim}} \mathcal{O}_{\mathfrak{X}}(\mathfrak{D})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}.$$

Concerning the dual functors, we keep notation 9.2.4.22.1 and 9.2.4.22.2.

This yields the extraordinary pushforward by u with coefficients in $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$ of finite level or not: $u^{\mathrm{alg}}_{T,!}\colon D^-({}^*\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})\to D^-({}^*\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}})$ and $u_{T,!}\colon D^-({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})\to D^-({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}})$ by setting $u^{\mathrm{alg}}_{T,!} = \mathbb{D}^{\mathrm{alg}}_{\mathfrak{X}^\flat,T}\circ u^{\mathrm{alg}}_{T,+}\circ\mathbb{D}^{\mathrm{alg}}_{\mathfrak{X}^\sharp,T}$ and $u_{T,!}=\mathbb{D}_{\mathfrak{X}^\flat,T}\circ u_{T,+}\circ\mathbb{D}_{\mathfrak{X}^\sharp,T}$.

These functors preserves perfectness.

Proposition 18.2.3.8. *For any $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, $\mathcal{M}\in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, we have the canonical isomorphisms*

$$u_!(\mathcal{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\sharp}) \xrightarrow{\sim} u_!(\mathcal{M})\otimes_{\mathcal{O}_{\mathfrak{X}}}\omega^{-1}_{\mathfrak{X}^\flat}, \qquad \omega_{\mathfrak{X}^\flat}\otimes_{\mathcal{O}_{\mathfrak{X}}}u_!(\mathcal{E}) \xrightarrow{\sim} u_!(\omega_{\mathfrak{X}^\sharp}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}). \tag{18.2.3.8.1}$$

Proof. Following 9.2.4.8.2 and 9.2.4.17 (resp. 9.2.4.20.3 and 9.2.4.22.(f)), we have such commutation isomorphisms for pushforward (duality). Hence, the proposition follows by composition. □

Proposition 18.2.3.9. *For any $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, $\mathcal{M}\in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, we have the canonical isomorphisms:*

$$u_!(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}\otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{E},\ u_!(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}\otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}(\mathcal{O}_{\mathfrak{X}}(-\mathfrak{D})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}). \tag{18.2.3.9.1}$$

Proof. Using 8.7.2.4 (resp. 18.2.3.6.2, resp. 18.2.3.4) instead of 4.2.5.6 (resp. 18.2.1.17.2, resp. 18.2.1.10), we can copy the proof of 18.2.1.20. □

Lemma 18.2.3.10. *Let $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$ and $\mathcal{M}\in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$. We have the canonical isomorphisms:*

$$u^{\mathrm{alg}}_{T,+}(\mathcal{M})\otimes_{\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}}\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \xrightarrow{\sim} u_{T,+}(\mathcal{M}\otimes_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}), \tag{18.2.3.10.1}$$

$$\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}\otimes_{\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}}u^{\mathrm{alg}}_{T,+}(\mathcal{E}) \xrightarrow{\sim} u_{T,+}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}\otimes_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}}\mathcal{E}). \tag{18.2.3.10.2}$$

In the same way by replacing the direct image by the extraordinary direct image.

Proof. As the dual functor commutes with scalar extensions (see 4.6.4.4.1), then it is sufficient to treat the case of the direct image. The isomorphism 18.2.3.10.1 is straightforward. Setting $\mathcal{D}^{(m)}_{\mathfrak{X}^\flat}(T) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\flat}$, $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{D}^{(m)}_{\mathfrak{X}^\flat}$, we have the canonical bijection

$$\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T)\otimes_{\mathcal{D}^{(m)}_{\mathfrak{X}^\flat}(T)}((\mathcal{D}^{(m)}_{\mathfrak{X}^\flat}(T)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})) \xrightarrow{\sim} (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T)\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})) \tag{18.2.3.10.3}$$

given by associativity of the tensor product. Since the source and the target are $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T)$-coherent, then they are p-adically complete. Hence, it follows from 4.2.4.3.1 by p-adic completion that 18.2.3.10.3 is an isomorphism of $(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T), \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(T))$-bimodules. Tensoring by $\mathbb{Q}$ and taking inductive limit on the level, this yields the isomorphism of $(\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q}, \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q})$-bimodules:

$$\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q}} ((\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{X}(\mathfrak{D})) \xrightarrow{\sim} (\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{X}(\mathfrak{D})) \tag{18.2.3.10.4}$$

The isomorphism 18.2.3.10.2 is built as follows:

$$\begin{aligned}\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q}} u^{\mathrm{alg}}_{T,+}(\mathcal{E}) &\xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q}} ((\mathcal{D}_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{X}(\mathfrak{D})) \otimes^\mathbb{L}_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}) \\ &\xrightarrow[18.2.3.10.4]{\sim} (\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{X}(\mathfrak{D})) \otimes^\mathbb{L}_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E} \xrightarrow{\sim} \\ &\xrightarrow{\sim} (\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{X}(\mathfrak{D})) \otimes^\mathbb{L}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q} \otimes^\mathbb{L}_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E} \xrightarrow{\sim} u_{T,+}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{D}_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}).\end{aligned}$$

□

The following proposition means that the isomorphism of relative duality to the canonical morphism $\mathfrak{X}^\sharp \to \mathfrak{X}^\flat$ necessity of use a twist (see 18.2.3.11).

Proposition 18.2.3.11. *We have, for any $\mathcal{E} \in D^\mathrm{b}_\mathrm{perf}({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q})$, of the canonical isomorphism:*

$$u_+(\mathcal{E}) \xrightarrow{\sim} u_!(\mathcal{E}(\mathfrak{D})). \tag{18.2.3.11.1}$$

Proof. Using 9.2.4.8.2 and 9.2.4.17, 18.2.3.8, it is sufficient to treat the left case (i.e. $* = l$). Replacing 4.5.3.8.1 (resp. 4.2.5.6, 18.2.1.12.1, 18.2.1.20, 18.2.1.10.2) by 18.2.3.5.2 (resp. 8.7.2.4, 18.2.3.7, 18.2.3.9.1, 18.2.3.4), we can copy the proof of 18.2.1.21.1.

□

Corollary 18.2.3.12. *Let $\mathcal{E} \in D^\mathrm{b}_\mathrm{perf}(\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q})$. We have the canonical quasi-isomorphism:*

$$\Omega^\bullet_{\mathfrak{X}^\sharp/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \Omega^\bullet_{\mathfrak{X}/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} u_{T+}(\mathcal{E}). \tag{18.2.3.12.1}$$

Proof. It follows from 18.2.3.9 and 18.2.3.11 that we have the isomorphism:

$$u_{T,+}(\mathcal{E}) \xrightarrow{\sim} u_{T,!}(\mathcal{E}(\mathfrak{D})) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes^\mathbb{L}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}(\mathfrak{D}). \tag{18.2.3.12.2}$$

We have isomorphisms:

$$\begin{aligned}\Omega^\bullet_{\mathfrak{X}^\sharp,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E} &\xrightarrow{\sim} (\Omega^\bullet_{\mathfrak{X}^\sharp,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}) \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E} \\ \xrightarrow[8.7.7.5.a)]{\sim} \omega_{\mathfrak{X}^\sharp,\mathbb{Q}}({}^\dagger T) \otimes^\mathbb{L}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}[-d] &\xrightarrow[18.2.1.10.1]{\sim} (\omega_{\mathfrak{X},\mathbb{Q}}({}^\dagger T) \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{Z})) \otimes^\mathbb{L}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}[-d] \\ \xrightarrow[18.2.3.2.2]{\sim} \omega_{\mathfrak{X},\mathbb{Q}}({}^\dagger T) \otimes^\mathbb{L}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}(\mathfrak{Z})[-d] &\xrightarrow[18.2.3.12.2]{\sim} \omega_{\mathfrak{X},\mathbb{Q}}({}^\dagger T) \otimes^\mathbb{L}_{\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger T)_\mathbb{Q}} u_{T,+}(\mathcal{E})[-d] \\ &\xrightarrow[8.7.7.5.a)]{\sim} \Omega^\bullet_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} u_{T,+}(\mathcal{E}).\end{aligned} \tag{18.2.3.12.3}$$

□

Theorem 18.2.3.13. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}$-module which is a locally projective $\mathcal{O}_\mathfrak{X}({}^\dagger T)_\mathbb{Q}$-module of finite type.*

(a) For any $l \neq 0$,

$$H^l(u_{T,+}(\mathcal{E})) = 0, \qquad H^l(u_{T,!}(\mathcal{E})) = 0. \tag{18.2.3.13.1}$$

(b) We have isomorphisms $u_{T,+}(\mathcal{E}) \xrightarrow{\sim} u_{T,!}(\mathcal{E}(\mathfrak{D})) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_\mathbb{Q} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_\mathbb{Q}} \mathcal{E}(\mathfrak{D})$.

(c) We have the canonical isomorphism:

$$\mathbb{D}_{\mathfrak{X},T} \circ u_{T,+}(\mathcal{E}) \xrightarrow{\sim} u_{T,+}(\mathcal{E}^\vee(-\mathfrak{D})). \tag{18.2.3.13.2}$$

(d) *The sheaves $u_{T,+}(\mathcal{E})$ and $u_{T,!}(\mathcal{E})$ are $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-holonomic (see 15.2.4.5).*

Proof. The (a), (b) assertions are a consequence of 18.2.3.1 and 18.2.3.12.2. Let us check (c). By biduality (see 8.7.7.3), it follows from $u_{T,+}(\mathcal{E}) \xrightarrow{\sim} u_{T,!}(\mathcal{E}(\mathfrak{D}))$ the isomorphism $\mathbb{D}_{\mathfrak{X},T} \circ u_{T,+}(\mathcal{E}) \xrightarrow{\sim} u_{T,+} \circ \mathbb{D}_{\mathfrak{X}^\sharp,T}(\mathcal{E}(\mathfrak{D}))$ (see the notation 9.2.4.22). By 11.2.6.3.4, $\mathbb{D}_{\mathfrak{X}^\sharp,T}(\mathcal{E}(\mathfrak{D})) \xrightarrow{\sim} (\mathcal{E}(\mathfrak{D}))^\vee \xrightarrow{\sim} \mathcal{E}^\vee(-\mathfrak{D})$. Hence, we are done. Finally, (d) is a easy consequence of (a) and (c). □

18.2.3.14. Let $n \in \mathbb{Z}$. Similarly to 18.2.1.13.4, 18.2.1.13.5, for any $\mathcal{E}^\sharp \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{l}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, $\mathcal{M}^\sharp \in D^{\mathrm{b}}_{\mathrm{perf}}({}^{\mathrm{r}}\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}})$, we construct the canonical isomorphisms:

$$\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} (\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}^\sharp) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}^\sharp), \tag{18.2.3.14.1}$$

$$(\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M}^\sharp) \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^\flat) \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}). \tag{18.2.3.14.2}$$

Hence, it follows from 18.2.3.9 and 18.2.3.11 that for any $* \in \{\mathrm{l}, \mathrm{r}\}$, for any $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{perf}}({}^*\widetilde{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp})$, we have the canonical isomorphism:

$$u_{T,+}(\mathcal{E}(n\mathfrak{Z}^\flat)) \xrightarrow{\sim} (u_{T,+}(\mathcal{E}))(n\mathfrak{Z}^\flat), \qquad u_{T,!}(n\mathcal{E}(\mathfrak{Z}^\flat)) \xrightarrow{\sim} (u_{T,!}(\mathcal{E}))(n\mathfrak{Z}^\flat). \tag{18.2.3.14.3}$$

For $\natural \in \{\mathrm{qc}, \mathrm{coh}\}$, we have the functor $(\mathfrak{Z}^\flat)\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T))$ by setting for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T))$:

$$(\mathfrak{Z}^\flat)(\mathcal{E}^{(\bullet)}) = \mathcal{O}^{(\bullet)}_{\mathfrak{X}}(\mathfrak{Z}^\flat) \otimes_{\mathcal{O}^{(\bullet)}_{\mathfrak{X}}} \mathcal{E}^{(\bullet)}, \tag{18.2.3.14.4}$$

where $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}(\mathfrak{Z}^\flat)$ is the constant sheaf given by $\mathcal{O}_{\mathfrak{X}}(\mathfrak{Z}^\flat)$. For any divisor T' of X, we have the canonical isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\natural}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T))$:

$$({}^\dagger T') \circ (\mathfrak{Z}^\flat) \xrightarrow{\sim} (\mathfrak{Z}^\flat) \circ ({}^\dagger T'), \qquad \mathbb{R}\underline{\Gamma}^\dagger_{T'} \circ (\mathfrak{Z}^\flat) \xrightarrow{\sim} (\mathfrak{Z}^\flat) \circ \mathbb{R}\underline{\Gamma}^\dagger_{T'}. \tag{18.2.3.14.5}$$

Indeed, the left isomorphism of 18.2.3.14.5 is a consequence of the isomorphism 9.1.1.5.2. By construction of the local cohomological functor with strict support over a divisor (see 13.1.1.4), using [BBD82, 1.1.9] this yields the right canonical isomorphism of 18.2.3.14.5.

When T is empty, $D^{\mathrm{b}}_{\mathrm{perf}}({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp,\mathbb{Q}} = D^{\mathrm{b}}_{\mathrm{coh}}({}^*\mathcal{D}^\dagger_{\mathfrak{X}^\sharp,\mathbb{Q}})$ (see 8.7.7.7 and 1.4.3.29). Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}({}^*\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$. It follows from 18.2.3.14.3 that we get the isomorphism:

$$u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)}(n\mathfrak{Z}^\flat)) \xrightarrow{\sim} (u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)}))(n\mathfrak{Z}^\flat), \qquad u^{(\bullet)}_{!}(\mathcal{E}^{(\bullet)}(n\mathfrak{Z}^\flat)) \xrightarrow{\sim} (u^{(\bullet)}_{!}(\mathcal{E}^{(\bullet)}))(\mathfrak{Z}^\flat). \tag{18.2.3.14.6}$$

Proposition 18.2.3.15. *The canonical ring homomorphisms $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(T \cup D) \to \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T \cup D)$ and $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T \cup D) \to \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\flat}(T \cup D)$ are injective. Via theses injections, we get the inclusion: $p\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T \cup D) \subset \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp}(T \cup D)$. In particular, the ring homomorphism $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T \cup D) \to \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T \cup D)$ is an isomorphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T \cup D))$. The canonical morphism $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T \cup D)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T \cup D)_{\mathbb{Q}}$ is a ring isomorphism.*

Proof. The assertion is local on $\mathfrak{X}$. We can suppose that there exists local coordinates $t_1, \ldots, t_d$ of $\mathfrak{X}$ such that $\mathfrak{Z} = V(t_1 \ldots t_r)$, $\mathfrak{Z}^\flat = V(t_1 \ldots t_s)$ and $\mathfrak{D} = V(t_{s+1} \ldots t_r)$ for some $0 \le s \le r$. Then $t_1, \ldots, t_d$ of $\mathfrak{X}^\sharp/\mathfrak{S}$ are semi-nice coordinates of both $\mathfrak{X}^\sharp/\mathfrak{S}$ and $\mathfrak{X}^\flat/\mathfrak{S}$. We get the description 7.5.1.6. For instance, we get the bases $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{(r)} : \underline{k} \in \mathbb{N}^d\}$ of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T \cup D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\sharp/\mathfrak{S}}$ and $\{\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{(s)} : \underline{k} \in \mathbb{N}^d\}$ of $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T \cup D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}_{\mathfrak{X}^\flat/\mathfrak{S}}$. For any integer $m \ge 0$, using the description 7.5.1.6, since $\underline{\partial}^{\langle \underline{k} \rangle_{(m)}}_{(r)} \in \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T \cup D)$ for any $\underline{k} \in \mathbb{N}^d$ (recall 4.5.1.1.1), then we get $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\sharp}(T \cup D) \subset \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T \cup D)$.

For any integer $k \ge 0$, we denote by $q^{(m)}_k$, $q^{(m+1)}_k$, $r^{(m)}_k$, $r^{(m+1)}_k$, $\tilde{r}^{(m)}_k$ the integers satisfying the following conditions: $k = p^m q^{(m)}_k + r^{(m)}_k$, $0 \le r^{(m)}_k < p^m$, $k = p^{m+1} q^{(m+1)}_k + r^{(m+1)}_k$, $0 \le r^{(m+1)}_k < p^{m+1}$, $q^{(m)}_k = p q^{(m+1)}_k + \tilde{r}^{(m)}_k$, $0 \le \tilde{r}^{(m)}_k < p$. We recall that the p-adic valuation of $k!$ is $v_p(k!) = (k - \sigma(k))/(p-1)$, where $\sigma(k) = \sum_i a_i$ if $k = \sum_i a_i p^i$ with $0 \le a_i < p$. We compute: $v_p(q^{(m)}_k!) - v_p(q^{(m+1)}_k!) = (q^{(m)}_k - q^{(m+1)}_k - \tilde{r}^{(m)}_k)/(p-1) = q^{(m+1)}_k$. Let $u_k \in \mathbb{Z}_p^\times$ be a unit such that $q^{(m)}_k!/q^{(m+1)}_k! = u_k p^{q^{(m+1)}_k}$. Let $s+1 \le$

$i \leq r$ be an integer. By 1.4.2.5.2 (and via $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}} \subset \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X},\mathbb{Q}}$), we have: $\partial_i^{\langle k \rangle_{(m)}} = q_k^{(m)}!/q_k^{(m+1)}!\partial_i^{\langle k \rangle_{(m+1)}}$. Hence, we compute:

$$p\partial_i^{\langle k \rangle_{(m)}} = u_k p^{q_k^{(m+1)}+1} \partial_i^{\langle k \rangle_{(m+1)}} = u_k t_i^{p^{m+1}-r_k^{(m+1)}} \left(\frac{p}{t_i^{p^{m+1}}} \right)^{q_k^{(m+1)}+1} t_i^k \partial_i^{\langle k \rangle_{(m+1)}} \in \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp}(T \cup D).$$

This yields the inclusion $p^d \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T \cup D) \subset \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X}^\sharp}(T \cup D)$.

Let $\lambda \in L(\mathbb{N})$ given by $m \mapsto m+1$ and $\chi \in M(\mathbb{N})$ given by $m \mapsto m+1$. Then we get the commutative square:

$$\begin{array}{ccc} \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T \cup D) & \longrightarrow & \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T \cup D) \\ \downarrow & \swarrow & \downarrow \\ \lambda^* \chi^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T \cup D) & \longrightarrow & \lambda^* \chi^* \widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T \cup D) \end{array} \qquad (18.2.3.15.1)$$

Then, by applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ to 18.2.3.15.1, we get the canonical morphism $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T \cup D)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T \cup D)_{\mathbb{Q}}$ is an isomorphism. □

Corollary 18.2.3.16. *Let $\mathfrak{X}$ (resp. $\mathfrak{X}'$) be a smooth $\mathfrak{S}$-formal scheme and $\mathfrak{D} \subset \mathfrak{X}$ (resp. $\mathfrak{D}' \subset \mathfrak{X}'$) be a relative to $\mathfrak{X}/\mathfrak{S}$ (resp. $\mathfrak{X}'/\mathfrak{S}$) strict normal crossing divisor. Set $\mathfrak{X}^\sharp := (\mathfrak{X}, M(\mathfrak{D}))$ and $\mathfrak{X}'^\sharp := (\mathfrak{X}', M(\mathfrak{D}'))$. Let $f\colon \mathfrak{X}^\sharp \to \mathfrak{X}'^\sharp$ be a morphism of log-smooth $\mathfrak{S}$-log formal schemes. Let T, T' be two divisors such that $f_0^{-1}(T) \subset T'$, $Z \subset T$ and $Z' \subset T'$. We have the equalities*

(a) The categories $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp)$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathfrak{S}^\sharp)$ (resp. $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}'^\sharp, T'/\mathfrak{S}^\sharp)$ and $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}', T'/\mathfrak{S}^\sharp)$) are canonically equivalent. Moreover, modulo theses identifications, the functors $f^{\sharp *}_{T',T}\colon \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S}^\sharp) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}'^\sharp, T'/\mathfrak{S})$ and $f^*_{T',T}\colon \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, T/\mathfrak{S}^\sharp) \to \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}', T'/\mathfrak{S})$ of 11.2.3.5 and the functors $\mathbb{L}f^{\sharp *(\bullet)}_{T',T}\colon \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp, T/\mathfrak{S}) \to \mathrm{MIC}^{(\bullet)}(\mathfrak{X}'^\sharp, T'/\mathfrak{S})$ and $\mathbb{L}f^{*(\bullet)}_{T',T}\colon \mathrm{MIC}^{(\bullet)}(\mathfrak{X}, T/\mathfrak{S}) \to \mathrm{MIC}^{(\bullet)}(\mathfrak{X}', T'/\mathfrak{S})$ of 11.2.3.5.1 are equals.

(b) Let $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}^\sharp/\mathfrak{S})$ and $\mathcal{E} := \underrightarrow{l}^*_{\mathbb{Q}}\, \mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp/\mathfrak{S})$. We have therefore the isomorphism of coherent $\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}$-modules, $\mathcal{O}_{\mathfrak{X}'}({}^\dagger T')_{\mathbb{Q}}$-coherent:

$$({}^\dagger T')(f^\sharp(\mathcal{E})) \xrightarrow{\sim} f^*_{T',T}(\mathcal{E}({}^\dagger T)), \qquad ({}^\dagger T')(f^{\sharp(\bullet)}(\mathcal{E}^{(\bullet)})) \xrightarrow{\sim} f^{(\bullet)*}_{T',T}(\mathcal{E}^{(\bullet)}({}^\dagger T)). \qquad (18.2.3.16.1)$$

Proof. The first part is a consequence of 18.2.3.15. Using 13.2.1.4.1, this yields the second part. □

18.2.3.17. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module which is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module of finite type. We get the map:

$$\mathcal{E}(\mathfrak{D}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \to \mathcal{O}_{\mathfrak{X}}({}^\dagger T \cup D)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T \cup D)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} = \mathcal{E}({}^\dagger T \cup D). \qquad (18.2.3.17.1)$$

Since $\mathcal{E}(\mathfrak{D})$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}$-module, this yields by extension the morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T \cup D)_{\mathbb{Q}}$-module:

$$\mathcal{E}(\mathfrak{D})({}^\dagger T \cup D) = \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T \cup D)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} (\mathcal{E}(\mathfrak{D})) \to \mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T \cup D)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E} = \mathcal{E}({}^\dagger T \cup D). \qquad (18.2.3.17.2)$$

Since the map 18.2.3.17.2 is an isomorphism outside $T \cup D$, then it is an isomorphism. It follows from 18.2.3.15 that $\mathcal{E}({}^\dagger T \cup D)$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T \cup D)_{\mathbb{Q}}$. Hence, we get by extension from 18.2.3.17.1 the last $\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}$-linear morphism:

$$\rho_{\mathcal{E}}\colon u_{T,+}(\mathcal{E}) \underset{18.2.3.13.(b)}{\xrightarrow{\sim}} \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}({}^\dagger T)_{\mathbb{Q}}} \mathcal{E}(\mathfrak{D}) \to \mathcal{E}({}^\dagger T \cup D), \qquad (18.2.3.17.3)$$

$\rho_{\mathcal{E}}$ being by definition the composite morphism. This yields by extension from 18.2.3.17.3 the homomorphism of coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T \cup D)_{\mathbb{Q}}$-modules:

$$({}^\dagger T \cup D)(u_{T,+}(\mathcal{E})) = \mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T \cup D)_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\flat}({}^\dagger T)_{\mathbb{Q}}} u_{T,+}(\mathcal{E}) \to \mathcal{E}({}^\dagger T \cup D), \qquad (18.2.3.17.4)$$

Since the map 18.2.3.17.4 is an isomorphism outside $T \cup D$, then it is an isomorphism.

Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T))$ such that $\underrightarrow{l}^*_{\mathbb{Q}}\ \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$. It follows from 18.2.3.15 that the functor $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T \cup D) \otimes^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T\cup D)} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp}(T \cup D)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T \cup D))$ and we can identify them. Then similarly to 18.2.3.17.4 we construct the morphism of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\flat}(T \cup D))$ of the form:

$$({}^\dagger D) \circ u^{(\bullet)}_{T,+}(\mathcal{E}^{(\bullet)}) \to ({}^\dagger D)(\mathcal{E}^{(\bullet)}), \qquad (18.2.3.17.5)$$

which is an isomorphism thanks to 18.2.3.17.4.

18.3 Stability of the (over)holonomicity with a Frobenius structure

18.3.1 A comparison theorem between log-de Rham complexes and de Rham complexes

Let $\mathfrak{X}$ be a separated, quasi-compact and smooth $\mathcal{V}$-formal scheme and $g\colon \mathfrak{X} \to \operatorname{Spf}\mathcal{V}$ be the structural morphism. Let $\mathfrak{Z}$ be a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor, $\mathfrak{Y} := \mathfrak{X} \setminus \mathfrak{Z}$, and let $j\colon \mathfrak{Y} \subset \mathfrak{X}$ be the open immersion. Let $\mathfrak{Z}_1, \dots, \mathfrak{Z}_r$ be the irreducible components of $\mathfrak{Z}$ and $\mathfrak{Z}^\flat := \cup_{i\geq 2} \mathfrak{Z}_i$. We set $\mathfrak{X}^\sharp := (\mathfrak{X}, M(\mathfrak{Z}))$, $\mathfrak{X}^\flat := (\mathfrak{X}, M(\mathfrak{Z}^\flat))$. We get a relative strict normal crossing divisor on $\mathfrak{Z}_1/\mathfrak{S}$ defined by $\mathfrak{D}_1 := i_1^{-1}(\mathfrak{Z}^\flat) := \mathfrak{Z}_1 \cap \mathfrak{Z}^\flat := \bigcup_{i=2}^r (\mathfrak{Z}_1 \cap \mathfrak{Z}_i)$ and we set $\mathfrak{Z}_1^\flat := (\mathfrak{Z}_1, M(\mathfrak{D}_1))$. Let $i_1\colon \mathfrak{Z}_1 \hookrightarrow \mathfrak{X}$ be the corresponding closed immersion and $i_1^\flat\colon \mathfrak{Z}_1^\flat \hookrightarrow \mathfrak{X}^\flat$ be the induced exact closed immersion. We put $u\colon \mathfrak{X}^\sharp \to \mathfrak{X}$, $u_1\colon \mathfrak{X}^\sharp \to \mathfrak{X}^\flat$, $v_1\colon \mathfrak{X}^\flat \to \mathfrak{X}$, $w_1\colon \mathfrak{Z}_1^\flat \to \mathfrak{Z}_1$ be the canonical morphisms.

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. When we work with F-complex, we suppose there exists an automorphism $\sigma\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining.

Let T be a divisor of X such that $U := T \cap Z_1$ is a divisor of Z_1. Let $\lambda_0\colon \mathbb{N} \to \mathbb{N}$ an increasing map such that $p^{\lambda_0(0)} \geq e/(p-1)$ (see why at 1.2.4.2) and $\lambda_0(m) \geq m$, for any $m \in \mathbb{N}$. We put then $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{X}}(T)$, $\widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}_1}(U) := \mathcal{B}^{(\lambda_0(m))}_{\mathfrak{Z}_1}(U)$, $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}(T) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{X}}(T) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\flat}$, $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}_1^\flat}(U) := \widetilde{\mathcal{B}}^{(m)}_{\mathfrak{Z}_1}(U) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Z}_1}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{Z}_1^\flat}$.

Definition 18.3.1.1. We denote by $\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{X}^\sharp, T/\mathfrak{S})$ the full subcategory of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ consisting of objects which are locally projective of finite type as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module and having nilpotent residues along each irreducible components of D not included in T.

We denote by $\mathrm{MIC}^{\dagger\dagger}_{\mathrm{NL}}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ the full subcategory of $\mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ consisting of objects which are locally projective of finite type as $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-module and such that the following conditions hold:

(a) none of differences of exponents along each irreducible components of D not included in T of $\mathcal{E}$ is a p-adic Liouville number,

(b) none of exponents along each irreducible components of D not included in T of $\mathcal{E}$ is a p-adic Liouville number.

When the divisor is empty, we simply write $\mathrm{MIC}^{\dagger\dagger}_{\natural}(\mathfrak{X}^\sharp/\mathfrak{S})$, with $\natural \in \{\mathrm{NL}, 0\}$.

Proposition 18.3.1.2. *The category* $\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{X}^\sharp, T/\mathfrak{S})$ *is abelian.*

Proof. When T is empty, this is [Ked07, 3.2.14] (see also the definition [Ked07, 2.3.7]). Let $\mathfrak{U}$ be the open of $\mathfrak{X}$ complementary to T. Let $\mathcal{E} \to \mathcal{F}$ be a morphism $\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{X}^\sharp, T/\mathfrak{S})$. We have to check that its kernel and cokernel belong to $\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{X}^\sharp, T/\mathfrak{S})$. Since this is Zariski local on $\mathfrak{X}$, we can suppose $\mathfrak{X}$ and T defined by an equation. This yields $\mathfrak{U}$ is affine. Using theorem of type A concerning coherent $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules, we reduce to check that $\Gamma(\mathfrak{X}, \mathcal{E})$ is a projective $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$-module of finite type.

Following the case where T is empty, we get that $\mathcal{E}$ is a locally projective $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-module of finite type. Since $\mathfrak{U}$ is affine, then via theorem of type A for coherent $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-modules we get $\Gamma(\mathfrak{U}, \mathcal{E})$ is a projective $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U},\mathbb{Q}})$-module of finite type. Theorem of type A concerning coherent $\mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}$-modules implies that the canonical morphism $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})} \Gamma(\mathfrak{X}, \mathcal{E}) \to \Gamma(\mathfrak{U}, \mathcal{E})$ is an isomorphism. Since $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) = \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U},\mathbb{Q}})$, since $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}}) \to \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U},\mathbb{Q}})$ is faithfully flat (this is checked in the proof of 8.7.6.8), then this implies that $\Gamma(\mathfrak{X}, \mathcal{E})$ is a projective $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}({}^\dagger T)_{\mathbb{Q}})$-module of finite type. □

Remark 18.3.1.3. The category $\mathrm{MIC}^{\dagger\dagger}_{\mathrm{NL}}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ is not abelian. For instance, $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ and $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{Z}_1)$ are objects of $\mathrm{MIC}^{\dagger\dagger}_{\mathrm{NL}}(\mathfrak{X}^\sharp/\mathfrak{S})$ but not the cokernel of the canonical inclusion $\mathcal{O}_{\mathfrak{X},\mathbb{Q}} \to \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(\mathfrak{Z}_1)$.

18.3.1.4 (Inverse image)**.** Let $\mathcal{V} \to \mathcal{V}'$ be a morphism of mixed characteristic complete discrete valuation rings, $k \to k'$ be the induced morphism of perfect residue fields, $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme, $\mathfrak{X}'$ be a smooth formal $\mathcal{V}'$-scheme and $\mathfrak{Z}$ (resp. $\mathfrak{Z}'$) be a relative to $\mathfrak{X}/\mathfrak{S}$ (resp. relative to $\mathfrak{X}'/\operatorname{Spf}\mathcal{V}'$) strict normal crossing divisor. Let $f_0\colon (X', M(\mathfrak{Z}')) \to (X, M(\mathfrak{Z}))$ be a morphism of log-schemes over $\operatorname{Spec} k$ (see notation 4.5.2.14). We have a canonical inverse image functor under f_0 denoted by f_0^*: $I_{\mathrm{conv,et}}((X, M(\mathfrak{Z}))/\operatorname{Spf}\mathcal{V}) \to I_{\mathrm{conv,et}}((X', M(\mathfrak{Z}'))/\operatorname{Spf}\mathcal{V}')$ (this is obvious from the definition [Shi02, 2.1.5, 2.1.6]). We get from 11.2.1.5.(b) an inverse image functor under f_0, also denoted by f_0^*, from the category of coherent $\mathcal{D}^\dagger_{(\mathfrak{X},M(\mathfrak{Z})),\mathbb{Q}}$-modules, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ to the category of coherent $\mathcal{D}^\dagger_{(\mathfrak{X}',M(\mathfrak{Z}')),\mathbb{Q}}$-modules, locally projective of finite type over $\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$. When there exists a lifting $f\colon (\mathfrak{X}', M(\mathfrak{Z}')) \to (\mathfrak{X}, M(\mathfrak{Z}))$ of $(X', M(\mathfrak{Z}')) \to (X, M(\mathfrak{Z}))$ then f_0^* is canonically isomorphic to the usual functor f^*.

18.3.1.5 (Frobenius structure)**.** Suppose now that $\mathcal{V} \to \mathcal{V}'$ is σ (which is a fixed lifting of the ath Frobenius power of k) and f_0 is $F_{(X,Z)}$ (or simply F) the ath power of the absolute Frobenius of (X, Z). A "coherent F-$\mathcal{D}^\dagger_{(\mathfrak{X},M(\mathfrak{Z})),\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$" or "coherent $\mathcal{D}^\dagger_{(\mathfrak{X},M(\mathfrak{Z})),\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ and endowed with a Frobenius structure" is a coherent $\mathcal{D}^\dagger_{(\mathfrak{X},M(\mathfrak{Z})),\mathbb{Q}}$-module $\mathcal{E}$, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ and endowed with a $\mathcal{D}^\dagger_{(\mathfrak{X},M(\mathfrak{Z})),\mathbb{Q}}$-linear isomorphism $\mathcal{E} \xrightarrow{\sim} F^*(\mathcal{E})$. This notion is compatible (via the equivalence of categories 11.2.1.5.(b)) with Shiho's notion of convergent F-log-isocrystal on (X, Z) (see [Shi02, 2.4.2]). By [Shi02, 2.4.3], an F-log-isocrystal on (X, Z) is strikingly locally free.

We translate 10.3.4.23 (beware we have changed the notation) in the arithmetic $\mathcal{D}$-modules side as follows:

Theorem 18.3.1.6. *We assume that $g : \mathfrak{X} \to \mathfrak{S}$ factors through a smooth morphism $g_1\colon \mathfrak{X} \to \mathfrak{Z}_1$ over $\mathfrak{S}$ such that $g_1 \circ i_1 = \mathrm{id}$ and $\mathfrak{Z}^\flat = g_1^{-1} \circ i_1^{-1}(\mathfrak{Z}^\flat)$. Let $g_1^\sharp\colon \mathfrak{X}^\sharp \to \mathfrak{Z}_1^\flat$ be the morphism induced by g_1. Let $\mathcal{E} \in \mathrm{MIC}^{\dagger\dagger}_{\mathrm{NL}}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ (see notation 11.2.1.4).*

(i) We suppose that none of exponents along Z_1 of $\mathcal{E}$ is a nonnegative integer. Then the canonical morphism $g^\sharp_{1,T+}(\mathcal{E}) \to g^\sharp_{1,T\cup Z_1+}(\mathcal{E}({}^\dagger Z_1))$ is an isomorphism.

(ii) Suppose there exist $n \in \mathbb{Z}$ and $\mathcal{G} \in \mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{X}^\sharp, T/\mathfrak{S})$ such that $\mathcal{E} := \mathcal{G}(n\mathfrak{Z}_1)$. Then

$$\operatorname{Cone}\left(g^\sharp_{1,T+}(\mathcal{E}) \to g^\sharp_{1,T\cup Z_1+}(\mathcal{E}({}^\dagger Z_1))\right) \tag{18.3.1.6.1}$$

is isomorphic to a complex of $D^{\mathrm{b}}(\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{Z}_1^\flat, T/\mathfrak{S}))$.

Proof. 1) We set $E := \mathrm{sp}^*(\mathcal{E})$, $Y_1 := X \setminus Z_1$, $j_1\colon Y_1 \subset X$ be the open immersion. Since $\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} E \cong j^\dagger\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{j^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} E$, since the functor $\underline{\Gamma}^\dagger_{]Z_1[_\mathfrak{X}}$ is exact, since mapping cones commute with the functor $\mathbb{R}g_{1K*}(\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} -)$ then we obtain the isomorphisms:

$$\begin{aligned}\mathbb{R}g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_\mathfrak{X}}\left(\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} E\right) &\cong \mathbb{R}g_{1K*}\left(\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} \underline{\Gamma}^\dagger_{]Z_1[_\mathfrak{X}} E\right)\\ &\cong \mathbb{R}g_{1K*}\left(\operatorname{Cone}\left(\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} E \to \Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} j_1^\dagger E\right)[-1]\right)\\ \cong \operatorname{Cone}\left(\mathbb{R}g_{1K*}(\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} E) \right.&\left.\to \mathbb{R}g_{1K*}(\Omega^\bullet_{\mathfrak{X}^\sharp_K/\mathfrak{Z}^\flat_{1K}} \otimes_{\mathcal{O}_{]X[_\mathfrak{X}}} j_1^\dagger E)\right)[-1]\end{aligned} \tag{18.3.1.6.2}$$

2) We have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{R}\mathrm{sp}_*\mathbb{R}g_{1K*}(\Omega^{\bullet}_{\mathfrak{X}^{\sharp}_K/\mathfrak{Z}^{\flat}_{1K}} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} E) & \longrightarrow & \mathbb{R}\mathrm{sp}_*\mathbb{R}g_{1K*}(\Omega^{\bullet}_{\mathfrak{X}^{\sharp}_K/\mathfrak{Z}^{\flat}_{1K}} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} j^{\dagger}_{Y_1} E) \\
\downarrow \sim & & \downarrow \sim \\
\mathbb{R}g_{1*}(\Omega^{\bullet}_{\mathfrak{X}^{\sharp}/\mathfrak{Z}^{\flat}_1,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathrm{sp}_*(E)) & \longrightarrow & \mathbb{R}g_{1*}(\Omega^{\bullet}_{\mathfrak{X}^{\sharp}/\mathfrak{Z}^{\flat}_1,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathrm{sp}_*(j^{\dagger}_{Y_1} E)), \\
\downarrow \sim & & \sim \downarrow 11.2.7.2.1 \\
\mathbb{R}g_{1*}(\Omega^{\bullet}_{\mathfrak{X}^{\sharp}/\mathfrak{Z}^{\flat}_1,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}) & \longrightarrow & \mathbb{R}g_{1*}(\Omega^{\bullet}_{\mathfrak{X}^{\sharp}/\mathfrak{Z}^{\flat}_1,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}(^{\dagger}Z_1)), \\
\sim \downarrow 9.4.1.6.1 & & \sim \downarrow 9.4.1.6.1 \\
g^{\sharp}_{1,T+}(\mathcal{E})[-1] & \longrightarrow & g^{\sharp}_{1,T\cup Z_1+}(\mathcal{E}(^{\dagger}Z_1))[-1],
\end{array} \qquad (18.3.1.6.3)$$

where both top vertical arrows can be constructed by using the canonical isomorphism $\mathbb{R}\mathrm{sp}_* \circ \mathbb{R}g_{1K*} \xrightarrow{\sim} \mathbb{R}g_{1*} \circ \mathbb{R}\mathrm{sp}_*$ and by using 10.1.2.3.2.

3) By applying the functor $\mathbb{R}\mathrm{sp}_*$ to the left top term of 18.3.1.6.2, following 10.3.4.23 (and via the equivalence of categories 11.2.1.5.(a)) we get either the null complex in the case (i) of the theorem or a complex isomorphic to a complex of coherent $\mathcal{D}^{\dagger}_{\mathfrak{Z}^{\flat}_1}(^{\dagger}U)_{\mathbb{Q}}$-modules belonging to $\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{Z}^{\flat}_1, U/\mathfrak{S})$ in the second case (see the description of the right part of 10.3.4.23.1, we have two terms with nilpotent residues on $\mathfrak{Z}^{\flat}_1$). Hence, we conclude thanks to the steps 1) and 2). □

18.3.1.7. Since $i^{\flat *}_1(\mathcal{O}_{\mathfrak{X}}(n\mathfrak{Z}^{\flat})) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{Z}^{\flat}_1}(n\mathfrak{D}_1)$, then for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$ we obtain the isomorphism:

$$i^{\flat(\bullet)!}_1(\mathcal{E}^{(\bullet)}(n\mathfrak{Z}^{\flat})) \xrightarrow{\sim} i^{\flat(\bullet)!}_1(\mathcal{E}^{(\bullet)})(n\mathfrak{D}_1)). \qquad (18.3.1.7.1)$$

By using Berthelot-Kashiwara theorem of the form 9.3.5.13, for any $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^{\sharp}_1}(U))$ we get from 18.3.1.7.2 the isomorphism:

$$i^{\flat(\bullet)}_{1+}(\mathcal{F}^{(\bullet)})(n\mathfrak{Z}^{\flat}) \xrightarrow{\sim} i^{\flat(\bullet)}_{1+}(\mathcal{F}^{(\bullet)}(n\mathfrak{D}_1)). \qquad (18.3.1.7.2)$$

The following paragraph means that the context of Theorem 18.3.1.6 is Zariski local up to an étale morphism.

18.3.1.8 (Local situation)**.** Suppose $\mathfrak{X} := \mathrm{Spf}\, A$ affine and there exists $\phi\colon \mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathcal{V}}$ be an étale morphism of $\mathcal{V}$-formal schemes such that, denoting by $t_1, \ldots, t_d \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ the coordinates given by ϕ, for any $i = 1, \ldots, r$, we have $\mathfrak{Z}_i = V(t_i)$.

Let $\bar{t}_2, \ldots, \bar{t}_d$ be the global sections of $\mathcal{O}_{\mathfrak{Z}_1}$ induced by $t_2, \ldots, t_d$ and let $\phi_1\colon \mathfrak{Z}_1 \to \widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}$ be the étale morphism induced by the coordinates $\bar{t}_2, \ldots, \bar{t}_d$. We get the morphism of smooth $\mathcal{V}$-formal schemes $\mathrm{id} \times \phi_1\colon \widehat{\mathbb{A}}^1_{\mathcal{V}} \times \mathfrak{Z}_1 \to \widehat{\mathbb{A}}^1_{\mathcal{V}} \times \widehat{\mathbb{A}}^{d-1}_{\mathcal{V}} = \widehat{\mathbb{A}}^d_{\mathcal{V}}$. Let $\iota_0\colon \mathrm{Spf}\, \mathcal{V} \hookrightarrow \widehat{\mathbb{A}}^1_{\mathcal{V}}$ be the closed immersion given by 0 and $\iota_0 := \iota_0 \times \mathrm{id}\colon \mathfrak{Z}_1 = \mathrm{Spf}\, \mathcal{V} \times \mathfrak{Z}_1 \hookrightarrow \widehat{\mathbb{A}}^1_{\mathcal{V}} \times \mathfrak{Z}_1$. By putting $\mathfrak{X}'' := (\widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1) \times_{\widehat{\mathbb{A}}^d} \mathfrak{X}$, we get the closed immersion $i'_1\colon \mathfrak{Z}_1 \hookrightarrow \mathfrak{X}'$ making commutative the left diagram:

$$\begin{array}{ccccc}
\mathfrak{Z}_1 & \overset{i''_1}{\dashrightarrow} & \mathfrak{X}'' & \overset{\phi''}{\longrightarrow} & \widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1 \\
& \underset{i_1}{\searrow} & \downarrow f' & \square & \downarrow \mathrm{id} \times \phi_1 \\
& & \mathfrak{X} & \underset{\phi}{\longrightarrow} & \widehat{\mathbb{A}}^d
\end{array}\ (\mathfrak{Z}_1 \xrightarrow{\iota_0} \widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1), \qquad
\begin{array}{ccccc}
\mathfrak{Z}_1 & \overset{i'_1}{\dashrightarrow} & \mathfrak{X}' & \overset{\phi'}{\longrightarrow} & \widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1 \\
& \underset{i_1}{\searrow} & \downarrow f & & \downarrow \mathrm{id} \times \phi_1 \\
& & \mathfrak{X} & \underset{\phi}{\longrightarrow} & \widehat{\mathbb{A}}^d
\end{array}\ (\mathfrak{Z}_1 \xrightarrow{\iota_0} \widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1) \qquad (18.3.1.8.1)$$

whose square is cartesian and has étale morphisms and where f' and ϕ'' are the canonical projections. Putting $\mathfrak{Z}''_1 := f'^{-1}(\mathfrak{Z}_1) = \mathfrak{Z}_1 \times_{\mathfrak{X}} \mathfrak{X}''$, we get a section $(\mathrm{id}, i''_1)\colon \mathfrak{Z}_1 \hookrightarrow \mathfrak{Z}''_1$ of the étale projection $\mathfrak{Z}''_1 \to \mathfrak{Z}_1$. Hence, by using SGA1 Exp. I Corollary 5.3., there exists a dense open $\mathfrak{X}'$ of $\mathfrak{X}''$ such that $\mathfrak{Z}_1 \xrightarrow{\sim} \mathfrak{Z}_1 \times_{\mathfrak{X}} \mathfrak{X}'$. We get therefore the right diagram of 18.3.1.8.1. Let $g_1\colon \mathfrak{X}' \to \mathfrak{Z}_1$ be the canonical morphism, i.e. the composition of ϕ' and the projection $\widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1 \to \mathfrak{Z}_1$. By construction, i'_1 is a section of g_1.

We put $T' := f^{-1}(T)$, $\mathfrak{Z}' := f^{-1}(\mathfrak{Z})$, $\mathfrak{Z}'^{\flat} := f^{-1}(\mathfrak{Z}^{\flat})$, $\mathfrak{X}'^{\sharp} := (\mathfrak{X}', M(\mathfrak{Z}'))$, $\mathfrak{X}'^{\flat} := (\mathfrak{X}', M(\mathfrak{Z}'^{\flat}))$. We denote by $f^{\flat}\colon \mathfrak{X}'^{\flat} \to \mathfrak{X}^{\flat}$, $f^{\sharp}\colon \mathfrak{X}'^{\sharp} \to \mathfrak{X}^{\sharp}$, $u'_1\colon \mathfrak{X}'^{\sharp} \to \mathfrak{X}'^{\flat}$, $g^{\sharp}_1\colon \mathfrak{X}'^{\sharp} \to \mathfrak{Z}^{\flat}_1$, $g^{\flat}_1\colon \mathfrak{X}'^{\flat} \to \mathfrak{Z}^{\flat}_1$, $i'^{\flat}_1\colon \mathfrak{Z}^{\flat}_1 \to \mathfrak{X}'^{\flat}$,

$v_1' \colon \mathfrak{X}'^{\flat} \to \mathfrak{X}'$ the associated morphisms. We summarize the notation and the above properties in the following commutative diagrams:

$$\begin{array}{ccccccc} \mathfrak{Z}_1 & \overset{i_1'}{\underset{g_1}{\rightleftarrows}} & \mathfrak{X}' & \overset{v_1'}{\longleftarrow} & \mathfrak{X}'^{\flat} & \overset{g_1^{\flat}}{\underset{i_1'^{\flat}}{\rightleftarrows}} & \mathfrak{Z}_1^{\flat} \\ \| & \square & \downarrow f & \square & \downarrow f^{\flat} & \square & \| \\ \mathfrak{Z}_1 & \overset{i_1}{\hookrightarrow} & \mathfrak{X} & \underset{v_1}{\longleftarrow} & \mathfrak{X}^{\flat} & \overset{i_1^{\flat}}{\longleftarrow} & \mathfrak{Z}_1^{\flat}, \end{array} \quad (w_1 \colon \mathfrak{Z}_1^{\flat} \to \mathfrak{Z}_1) \qquad \begin{array}{ccc} \mathfrak{X}'^{\flat} & \overset{u_1'}{\longleftarrow} & \mathfrak{X}'^{\sharp} \\ f^{\flat} \downarrow & \square & \downarrow f^{\sharp} \\ \mathfrak{X}^{\flat} & \underset{u_1}{\longleftarrow} & \mathfrak{X}^{\sharp} \end{array} \tag{18.3.1.8.2}$$

where squares (including the below "square" containing w_1) are cartesian, where the commutativity of the top part means that i_1' is a section of g_1 and $i_1'^{\flat}$ is a section of $g_1^{\flat}$.

We remark that $\mathfrak{Z}'^{\flat} = g_1^{-1}(\mathfrak{Z}_1^{\flat})$. Indeed, since $\mathfrak{Z}^{\flat} = \phi^{-1}(V(x_2 \dots x_r))$, then using the commutativity of the square of the right diagram of 18.3.1.8.1 we get $\mathfrak{Z}'^{\flat} = f^{-1}\phi^{-1}(V(x_2 \dots x_d)) = \phi'^{-1}(id \times \phi_1)^{-1}(V(x_2 \dots x_r)) = \phi'^{-1}(\widehat{\mathbb{A}}^1 \times \mathfrak{Z}_1^{\flat}) = g_1^{-1}(\mathfrak{Z}_1^{\flat})$. Moreover, since $i_1'^{-1}(\mathfrak{Z}'^{\flat}) = i_1^{-1}(\mathfrak{Z}^{\flat}) = \mathfrak{Z}_1^{\flat}$, this implies $\mathfrak{Z}'^{\flat} = g_1^{-1} \circ i_1'^{-1}(\mathfrak{Z}'^{\flat})$.

Remark 18.3.1.9. With the notation 18.3.1.6, let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}^{\mathrm{b}}_{\mathbb{Q},\mathrm{perf}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$ an object such that $\underrightarrow{l}^*_{\mathbb{Q}}\, \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$ (see 8.4.5.6). We have the isomorphism (see 13.1.5.6.3):

$$\underrightarrow{l}^*_{\mathbb{Q}} g_{1+}^{\sharp(\bullet)} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z_1}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Cone}\left(g^{\sharp}_{1T+}(\mathcal{E}) \to g^{\sharp}_{1T\cup Z_1+}(\mathcal{E}({}^{\dagger}T \cup Z_1))\right)[-1]. \tag{18.3.1.9.1}$$

Corollary 18.3.1.10. *Let $\mathcal{G} \in \mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{X}^{\sharp}, T/\mathfrak{S})$ (see notation 18.3.1.1). Let $n \in \mathbb{Z}$ and $\mathcal{E} := \mathcal{G}(n\mathfrak{Z}_1)$. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\sharp}}(T))$ such that $\underrightarrow{l}^*_{\mathbb{Q}}\, \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}$. The following properties are then satisfied:*

(a) We have $i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}_1^{\flat}}(U))$.

(b) If $n \geq 0$, then $i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$ and the canonical morphism $u_{1+}(\mathcal{E}) \to \mathcal{E}({}^{\dagger}Z_1)$ is an isomorphism.

(c) Suppose we are in the situation 18.3.1.8. Then, the complex $i_1^{\flat!} \circ u_{1+}(\mathcal{E})$ is isomorphic to a complex (with two terms) of $D^{\mathrm{b}}(\mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{Z}_1^{\flat}, U/\mathfrak{S}))$.

Proof. 0) Since the proposition is Zariski local on $\mathfrak{X}$, then we can suppose we are in the situation 18.3.1.8 and we take its notation. By using the right cartesian square of 18.3.1.8.2 and thanks to the base change isomorphism of a proper pushforward by a smooth pullback (more precisely, see 13.2.3.7) which gives the isomorphism $f^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}(\bullet)) \xrightarrow{\sim} u_{1+}'^{(\bullet)} \circ f^{\sharp(\bullet)!}(\mathcal{E}(\bullet))$, since $\underrightarrow{l}^*_{\mathbb{Q}} \circ f^{\sharp(\bullet)!}(\mathcal{E}(\bullet)) \xrightarrow{\sim} f^{\sharp!}(\mathcal{E}) \in \mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{X}'^{\sharp}, T'/\mathfrak{S})$, then we reduce to the case where there exists a smooth morphism $g_1 \colon \mathfrak{X} \to \mathfrak{Z}_1$ such that $g_1 \circ i_1 = \mathrm{id}$ and $\mathfrak{Z}^{\flat} = g_1^{-1} \circ i_1^{-1}(\mathfrak{Z}^{\flat})$.

1) Let us check (a) and (c). Since $i_1^{\flat}$ is an exact immersion, then following 13.2.1.5.1 we have $i_{1+}^{\flat(\bullet)} \circ i_1^{\flat(\bullet)!} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z_1}$. As $({}^{\dagger}Z_1) \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}({}^{\dagger}Z_1)$ (see 18.2.3.17.5), then we get from 13.1.1.5 the distinguished triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\flat}}(T))$:

$$i_{1+}^{\flat(\bullet)} \circ i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \to u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}({}^{\dagger}Z_1) \to i_{1+}^{\flat(\bullet)} \circ i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)})[1]. \tag{18.3.1.10.1}$$

By applying the functor $g_{1+}^{\flat(\bullet)}$ to the triangle 18.3.1.10.1, as $g_{1+}^{\flat(\bullet)} \circ i_{1+}^{\flat(\bullet)} = \mathrm{id}$, since $g_{1+}^{\flat(\bullet)} \circ u_{1+}^{(\bullet)} = g_{1+}^{\sharp(\bullet)}$, we get then the distinguished triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}_1^{\flat}}(U))$:

$$i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \to g_{1+}^{\sharp(\bullet)}(\mathcal{E}^{(\bullet)}) \to g_{1+}^{\sharp(\bullet)}(\mathcal{E}^{(\bullet)}({}^{\dagger}Z_1)) \to i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)})[1]. \tag{18.3.1.10.2}$$

As $u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LM}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\flat}}(T))$, then $\underrightarrow{l}^*_{\mathbb{Q}} \circ i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} i_1^{\flat!} \circ u_{1,T+}(\mathcal{E})$. Hence, by applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$ to 18.3.1.10.2, we obtain:

$$i_1^{\flat!} \circ u_{1,T+}(\mathcal{E}) \to g^{\sharp}_{1,T+}(\mathcal{E}) \to g^{\sharp}_{1,T\cup Z_1+}(\mathcal{E}({}^{\dagger}Z_1)) \to i_1^{\flat!} \circ u_{1,T+}(\mathcal{E})[1]. \tag{18.3.1.10.3}$$

Using Theorem 18.3.1.6.(ii), we deduce from 18.3.1.10.3 that $i_1^{\flat !} \circ u_{1+}(\mathcal{E})$ is isomorphic to a complex of $D^{\mathrm{b}}(\mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{Z}_1^\flat, T/\mathfrak{S}))$. Using 14.3.3.9, 14.3.3.1 and 18.3.1.2, this yields $i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}_1^\flat}(U))$.

2) It remains to check (b). When $n \geq 0$, then $\mathcal{E}$ satisfies the conditions (a), (b) and (p) of 18.3.1.6.(i), and then the canonical morphism $g^\sharp_{1,T+}(\mathcal{E}) \to g^\sharp_{1,T\cup Z_1+}(\mathcal{E}(^\dagger Z_1))$ is an isomorphism. With 18.3.1.10.3, this implies that $i_1^{\flat !} \circ u_{1,T+}(\mathcal{E}) \xrightarrow{\sim} 0$. Since $i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}_1^\flat}(U))$, since $\underrightarrow{l}^*_{\mathbb{Q}} \circ i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} i_1^{\flat !} \circ u_{1,T+}(\mathcal{E})$, then $i_1^{\flat(\bullet)!} \circ u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} 0$. With 18.3.1.10.1, this yields $u_{1+}^{(\bullet)}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}(^\dagger Z_1)$ is an isomorphism. By applying $\underrightarrow{l}^*_{\mathbb{Q}}$ we are done. □

Corollary 18.3.1.11. *Let $\mathcal{G} \in \mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ (see notation 18.3.1.1). The canonical morphism $u_{T,+}(\mathcal{G}) \to \mathcal{G}(^\dagger Z)$ of 18.2.3.17.3 is an isomorphism.*

Proof. We proceed by induction on r as follows. We have $u_+(\mathcal{G}) \xrightarrow{\sim} v_{1,+}(u_{1,+}(\mathcal{G})) \underset{18.3.1.10.(b)}{\xrightarrow{\sim}} v_{1,+}(\mathcal{G}(^\dagger Z_1)$. By induction hypothesis, since $\mathcal{G}(^\dagger Z_1) \in \mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{X}^\flat, T \cup Z_1/\mathfrak{S})$ (see 18.2.3.17.3), then we get $v_{1,+}(\mathcal{G}(^\dagger Z_1)) \xrightarrow{\sim} \mathcal{G}(^\dagger Z_1)(^\dagger Z^\flat) \xrightarrow{\sim} \mathcal{G}(^\dagger Z)$.

□

Remark 18.3.1.12. Let $\mathcal{G}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\sharp}(^\dagger T)_{\mathbb{Q}}$-module which is a locally projective $\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}$-module of finite type. Moreover, if we do not make some assumptions on the exponents, the homomorphism the canonical morphism $u_+(\mathcal{G}) \to \mathcal{G}(^\dagger Z)$ of 18.2.3.17.3 is not always an isomorphism, i.e. the hypothesis $\mathcal{G} \in \mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{X}^\sharp, T/\mathfrak{S})$ in 18.3.1.11 is crucial. Here are two counter-examples:

(a) When $\mathfrak{Z}$ is non-empty, we check that $\rho_{\mathcal{O}_{\mathfrak{X}}(^\dagger T)_{\mathbb{Q}}(-\mathfrak{Z})}$ is not an isomorphism. Indeed, by 8.7.6.11, we reduce to the case where T is empty. Let us suppose $\mathfrak{X}^\sharp/\mathfrak{S}$ that there exist nice local coordinates $t_1, \dots, t_d$ such that $\mathfrak{Z} = V(t_1 \dots t_s)$ (see definition 4.5.2.15). We compute $\mathcal{D}_{\mathfrak{X}^\sharp,\mathbb{Q}}/\mathcal{D}_{\mathfrak{X}^\sharp,\mathbb{Q}}(\partial_{\sharp,1}, \dots, \partial_{\sharp,d}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. Following 11.1.1.6.(c), this yields :

$$\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}/\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(t_1\partial_1, \dots, t_d\partial_d) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\sharp,\mathbb{Q}}} \mathcal{O}_{\mathfrak{X},\mathbb{Q}} \underset{18.2.3.13.(b)}{\xrightarrow{\sim}} u_{T,+}(\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(-\mathfrak{Z})).$$

Moreover, we deduce from 12.1.2.3.1 the first isomorphism:

$$\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}/\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(\partial_1 t_1, \dots, \partial_s t_s, \partial_{s+1}, \dots, \partial_d) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}} \xrightarrow{\sim} (\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(-\mathfrak{Z}))(^\dagger Z).$$

When $s \geq 1$, we conclude then noticing $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(t_1\partial_1, \dots, t_d\partial_d) \neq \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(\partial_1 t_1, \dots, \partial_s t_s, \partial_{s+1}, \dots, \partial_d)$.

(b) When $\mathfrak{X}$ is proper and T is empty, the fact that $\rho_{\mathcal{G}}$ is an isomorphism implies that the rigid cohomology of the overconvergent isocrystal $\mathcal{G}(^\dagger Z)$ would be of finite dimension. Hence we can consider the overconvergent isocrystal (which comes from a log-isocrystal convergent) described by Berthelot in the last remark of [Ber96c] to notice that this is not always the case.

Corollary 18.3.1.13. *Let $\mathcal{E}$ be an isocrystal on $\mathfrak{X}^\sharp$ overconvergent along T with nilpotent residues. Suppose that there exists a smooth morphism $\mathfrak{X} \to \mathfrak{T}$ of smooth formal schemes over $\mathfrak{S}$ such that $\mathfrak{Z}$ is a relative strict normal crossing divisor of $\mathfrak{X}$ over $\mathfrak{T}$. Then the canonical morphism $\Omega^\bullet_{\mathfrak{X}^\sharp/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E} \to \Omega^\bullet_{\mathfrak{X}/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}(^\dagger Z)$ is a quasi-isomorphism.*

Proof. This follows from 18.3.1.11 and 18.2.3.12. □

18.3.2 Overholonomicity of overconvergent F-isocrystals and stability

Theorem 18.3.2.1. *Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme, $\mathfrak{Z}$ a relative to $\mathfrak{X}/\mathfrak{S}$ strict normal crossing divisor, Let $\mathfrak{X}^\sharp := (\mathfrak{X}, M(\mathfrak{Z}))$ be the induced log smooth $\mathcal{V}$-log formal scheme and $u\colon \mathfrak{X}^\sharp \to \mathfrak{X}$ the canonical morphism. Let $\mathfrak{Z}_1, \dots, \mathfrak{Z}_l$ be the irreducible components of $\mathfrak{Z}$. Let $\mathcal{G} \in \mathrm{MIC}_0^{\dagger\dagger}(\mathfrak{X}^\sharp/\mathfrak{S})$ (see notation 18.3.1.1). Let $n \in \mathbb{Z}$ and $\mathcal{E} := \mathcal{G}(n\mathfrak{Z})$. Then $u_+(\mathcal{E})$ is overholonomic.*

Proof. For convenience, an object $\mathcal{F}$ of $D(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$ is said to be -1-*overholonomic* if $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$. Let $r \geq -1$, $n \geq 0$ be two integers and let us consider the next properties:

$(P_{n,r})$ If $\dim X \leq n$ then the module $u_+(\mathcal{E})$ is r-overholonomic (see 18.1.2.1).

$(Q_{n,r})$ If $\dim X \leq n$ then the complex $\mathbb{R}\underline{\Gamma}^\dagger_Z u_+(\mathcal{E})$ is r-overholonomic.

$(R_{n,r})$ If $\dim X \leq n$ then the module $\mathcal{E}({}^\dagger Z)$ is r-overholonomic.

(I) *$(P_{n,-1})$, $(Q_{n,-1})$ and $(R_{n,-1})$ are satisfied.*

Since the functor u_+ preserves the coherence, then $(P_{n,-1})$ is satisfied. Following 18.3.1.11, $u_+(\mathcal{G}) \xrightarrow{\sim} \mathcal{G}({}^\dagger Z)$. Since $\mathcal{G}({}^\dagger Z) \xrightarrow{\sim} \mathcal{E}({}^\dagger Z)$ (see 18.2.3.17.2), then $(R_{n,-1})$ is satisfied. By using the exact triangle of localisation with respect to Z, this yields that $(Q_{n,-1})$ is satisfied.

(II) *For any $n \geq 1$, $r \geq 0$, we check the implication $(P_{n-1,r}) \Rightarrow (Q_{n,r})$.*

Let $\mathcal{G}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$ such that $\underrightarrow{l}^*_{\mathbb{Q}}\ \mathcal{G}^{(\bullet)} \xrightarrow{\sim} \mathcal{G}$. We set $\mathcal{E}^{(\bullet)} := \mathcal{G}^{(\bullet)}(n\mathfrak{Z})$. Since this is local, we can suppose we are in the local situation 18.3.1.8 and we use its notation (within notation 18.3.1). and $u_{1+} \circ (l\mathfrak{Z}^\flat) \xrightarrow{\sim} (l\mathfrak{Z}^\flat) \circ u_{1+}$, and since $\mathbb{R}\underline{\Gamma}^\dagger_{Z_1}$ commutes with pushforwards, then we get the isomorphism Since $\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\sharp})$, then we get the second isomorphism:

$$\begin{aligned}\mathbb{R}\underline{\Gamma}^\dagger_{Z_1} \circ u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)}) &\xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Z_1} \circ v^{(\bullet)}_{1+} \circ u^{(\bullet)}_{1+} \circ (l\mathfrak{Z}^\flat) \circ (\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1))\\ &\underset{18.2.3.14.6}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^\dagger_{Z_1} \circ v^{(\bullet)}_{1+} \circ (l\mathfrak{Z}^\flat) \circ u^{(\bullet)}_{1+} \circ (\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1))\\ &\underset{13.2.1.4.2}{\xrightarrow{\sim}} v^{(\bullet)}_{1+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{Z_1} \circ (l\mathfrak{Z}^\flat) \circ u^{(\bullet)}_{1+} \circ (\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1))\\ &\underset{18.2.3.14.5}{\xrightarrow{\sim}} v^{(\bullet)}_{1+} \circ (l\mathfrak{Z}^\flat) \circ \mathbb{R}\underline{\Gamma}^\dagger_{Z_1} \circ u^{(\bullet)}_{1+}(\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1)).\end{aligned} \tag{18.3.2.1.1}$$

Set $\mathcal{F}^{(\bullet)}_1 := i^{\flat(\bullet)!}_1 \circ u^{(\bullet)}_{1+}(\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1))$ and $\mathcal{F}_1 := \underrightarrow{l}^*_{\mathbb{Q}}\ \mathcal{F}^{(\bullet)}_1$. Following 18.3.1.10.(a), $\mathcal{F}^{(\bullet)}_1 \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{Z}^\flat_1})$ and according to 18.3.1.10.c, $\mathcal{F}_1$ is isomorphic to a complex (with two terms) of $D^{\mathrm{b}}(\mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{Z}^\flat_1/\mathfrak{S}))$. Since $i^\flat_1$ is an exact immersion, then following 13.2.1.5.1 we have $i^{\flat(\bullet)}_{1+} \circ i^{\flat(\bullet)!}_1 \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Z_1}$. Hence, we get the first isomorphism:

$$\begin{aligned} v^{(\bullet)}_{1+} \circ (l\mathfrak{Z}^\flat) \circ \mathbb{R}\underline{\Gamma}^\dagger_{Z_1} \circ u^{(\bullet)}_{1+}(\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1)) &\underset{13.2.1.5.1}{\xrightarrow{\sim}} v^{(\bullet)}_{1+} \circ (l\mathfrak{Z}^\flat) \circ i^{\flat(\bullet)}_{1+} \circ i^{\flat(\bullet)!}_1 \circ u^{(\bullet)}_{1+}(\mathcal{G}^{(\bullet)}(l\mathfrak{Z}_1))\\ = v^{(\bullet)}_{1+} \circ (l\mathfrak{Z}^\flat) \circ i^{\flat(\bullet)}_{1+}(\mathcal{F}^{(\bullet)}_1) &\underset{18.3.1.7.2}{\xrightarrow{\sim}} v^{(\bullet)}_{1+} \circ i^{\flat(\bullet)}_{1+} \circ (l\mathfrak{D}_1)(\mathcal{F}^{(\bullet)}_1) \xrightarrow{\sim} i^{(\bullet)}_{1+} \circ w^{(\bullet)}_{1+} \circ (l\mathfrak{D}_1)(\mathcal{F}^{(\bullet)}_1).\end{aligned} \tag{18.3.2.1.2}$$

Composing 18.3.2.1.1 and 18.3.2.1.2 and applying the functor $\underrightarrow{l}^*_{\mathbb{Q}}$, we get the isomorphism: $\mathbb{R}\underline{\Gamma}^\dagger_{Z_1} u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} i^{(\bullet)}_{1+} \circ w^{(\bullet)}_{1+} \circ (l\mathfrak{D}_1)(\mathcal{F}^{(\bullet)}_1)$. Hence, it follows from 9.4.2.4 that $\mathbb{R}\underline{\Gamma}^\dagger_{Z_1} u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. The hypothesis $(P_{n-1,r})$ implies that $w_{1+} \circ (l\mathfrak{D}_1)(\mathcal{F}_1)$ is r-overholonomic. Hence, so is $\mathbb{R}\underline{\Gamma}^\dagger_{Z_1} u_+(\mathcal{E}) \xrightarrow{\sim} i_{1+} \circ w_{1+} \circ (l\mathfrak{D}_1)(\mathcal{F}_1)$ and therefore $\mathbb{R}\underline{\Gamma}^\dagger_{Z_1} u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)})$. Symmetrically, we obtain for any $i = 1, \ldots, r$ that $\mathbb{R}\underline{\Gamma}^\dagger_{Z_i} u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)})$ is r-overholonomic. Using Mayer-Vietoris exact triangle and the stability of the r-overholonomicity under local cohomological functors, this implies that $\mathbb{R}\underline{\Gamma}^\dagger_{Z} u^{(\bullet)}_{+}(\mathcal{E}^{(\bullet)})$ is r-overholonomic.

(III). *We prove the implication $(P_{n,r-1}) \Rightarrow (R_{n,r})$ for any $n \geq 0$, $r \geq 0$.*

We suppose $\dim X \leq n$. We suppose $r = 0$ (resp. $r \geq 1$). Since $\mathcal{G}({}^\dagger Z) \xrightarrow{\sim} \mathcal{E}({}^\dagger Z)$, then we reduce to the case where $\mathcal{G} = \mathcal{E}$, i.e $l = 0$. Moreover, in the respective case, this implies that $\mathcal{G}({}^\dagger Z)$ is $r-1$-overholonomic (and in particular overcoherent). Let $\alpha\colon \mathfrak{P} \to \mathfrak{X}$ be a smooth morphism. Then $\alpha^{-1}(\mathfrak{Z})$ is a relative to $\mathfrak{P}/\mathfrak{S}$ strict normal crossing divisor. Let $\mathfrak{P}^\sharp := (\mathfrak{P}, M(\alpha^{-1}(\mathfrak{Z})))$ be the induced log smooth $\mathcal{V}$-log formal scheme and $\alpha^\sharp\colon \mathfrak{P}^\sharp \to \mathfrak{X}^\sharp$ be the induced morphism. Then following 18.2.3.16.1, $\alpha^{\sharp!}(\mathcal{G})[-d_{P/X}] \in \mathrm{MIC}^{\dagger\dagger}_0(\mathfrak{P}^\sharp/\mathfrak{S})$ and $({}^\dagger\alpha^{-1}(Z)) \circ \alpha^{\sharp!}(\mathcal{G}) \xrightarrow{\sim} \alpha^! \circ ({}^\dagger(Z))(\mathcal{G})$. Hence, it is sufficient to prove that for any divisor T of X, $\mathcal{G}({}^\dagger Z \cup T)$ is $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathfrak{X}}(\mathcal{G}({}^\dagger Z \cup T))$ is $(r-1)$-overholonomic). Using de Jong's desingularization theorem ([dJ96]), there exist a projective smooth morphism $f\colon \mathfrak{P}' \to \mathfrak{X}$, a smooth scheme X' over k, a closed immersion $\iota'_0\colon X' \hookrightarrow \mathfrak{P}'$, a projective, surjective, generically finite and étale morphism $a_0\colon X' \to X$ such that $a_0 = f_0 \circ \iota'_0$ and $Z'' := a_0^{-1}(Z \cup T)$ is a strict normal crossing divisor of X'. Since $\mathcal{G}({}^\dagger Z \cup T) \in \mathrm{MIC}^{\dagger\dagger}(\mathfrak{X}, Z \cup T/\mathcal{V})$, then following 16.1.11.2 $\mathcal{G}({}^\dagger Z \cup T)$ is a direct summand of $f_+\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^!(\mathcal{G}({}^\dagger Z \cup T))$. By using the stability of the coherence under pushforwards (resp. the stability of the $(r-1)$-overholonomicity under pushforwards and the fact that f_+ commutes with $\mathbb{D}_{\mathfrak{X}}$ because f is proper following 13.2.4.1), it remains to prove that $\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^!(\mathcal{G}({}^\dagger Z \cup T))$ is $\mathcal{D}^\dagger_{\mathfrak{P}',\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathfrak{P}'} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} \circ f^{!}(\mathcal{G}(^{\dagger}Z \cup T))$ is $(r-1)$-overholonomic. Since this is local on $\mathfrak{P}'$, then we can suppose that there exists a lifting $\iota' \colon \mathfrak{X}' \hookrightarrow \mathfrak{P}'$ of ι'_0 and that Z'' lifts to a relative to $\mathfrak{X}'/\mathfrak{S}$ strict normal crossing divisor $\mathfrak{Z}''$ (see 4.5.2.17). We set $\mathfrak{X}'^{\sharp} := (\mathfrak{X}', M(\mathfrak{Z}''))$. Since this is Zariski local on X', we can suppose the canonical morphism $a_0^{\sharp} \colon X'^{\sharp} \to X^{\sharp}$ (because $Z'' = a_0^{-1}(Z \cup T)$) lifts to a morphism $a^{\sharp} \colon \mathfrak{X}'^{\sharp} \to \mathfrak{X}^{\sharp}$ (see 4.5.2.17). We denote by $a \colon \mathfrak{X}' \to \mathfrak{X}$ the underlying morphism of formal schemes.

We have the isomorphism:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{(\bullet)!}(\mathcal{G}^{(\bullet)}(^{\dagger}Z \cup T)) \underset{13.2.1.5.1}{\xrightarrow{\sim}} \iota'^{(\bullet)}_{+} \iota'^{(\bullet)!} f^{(\bullet)!}(\mathcal{G}^{(\bullet)}(^{\dagger}Z \cup T)) \xrightarrow{\sim} \iota'^{(\bullet)}_{+} a^{(\bullet)!}(\mathcal{G}^{(\bullet)}(^{\dagger}Z \cup T)).$$

Hence, by using the stability of the coherence (resp. $(r-1)$-overholonomicity) under pushforwards, we come down to prove that $a^{(\bullet)!}(\mathcal{G}^{(\bullet)}(^{\dagger}Z \cup T)) = a^{(\bullet)*}(\mathcal{G}^{(\bullet)}(^{\dagger}Z \cup T)) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'})$ (resp. $\mathbb{D}_{\mathfrak{X}'} a^{!}_{Z \cup T}(\mathcal{G}(^{\dagger}Z \cup T))$ is $r-1$-overholonomic). Following 11.2.3.5.1, $a^{(\bullet)!}(\mathcal{G}^{(\bullet)}(^{\dagger}Z \cup T)) \in \mathrm{MIC}^{(\bullet)}(\mathfrak{X}', Z \cup T/\mathfrak{S})$. Using 9.1.6.3, we reduce therefore in the non respective case to check that $a^{!}_{Z \cup T}(\mathcal{G}(^{\dagger}Z \cup T)) = a^{*}_{Z \cup T}(\mathcal{G}(^{\dagger}Z \cup T))$ is $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$-coherent. We get from 18.2.3.16.1 the following isomorphism: $a^{*}_{Z \cup T}(\mathcal{G}(^{\dagger}Z \cup T)) \xrightarrow{\sim} (^{\dagger}Z'')(a^{\sharp *}(\mathcal{G}))$. Thus, it remains to prove that $(^{\dagger}Z'')(a^{\sharp *}(\mathcal{G}))$ is $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathfrak{X}'} \circ (^{\dagger}Z'')(a^{\sharp *}(\mathcal{G}))$ is $(r-1)$-overholonomic). We check this separately:

Non-respective case. Since $\mathcal{G} \in \mathrm{MIC}^{\dagger\dagger}_{0}(\mathfrak{X}^{\sharp}/\mathfrak{S})$, then $a^{\sharp *}(\mathcal{G}) \in \mathrm{MIC}^{\dagger\dagger}_{0}(\mathfrak{X}'^{\sharp}/\mathfrak{S})$ (for the nilpotence of the residues, see 10.3.4.4.a). Hence, using 18.3.1.11 this yields $u'_{+}(a^{\sharp *}(\mathcal{G})) \xrightarrow{\sim} (^{\dagger}Z'')(a^{\sharp *}(\mathcal{G}))$. This yields that $(^{\dagger}Z'')(a^{\sharp *}(\mathcal{G}))$ is $\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}}$-coherent.

Respective case. The log-relative duality isomorphism of 18.2.3.13.2 gives: $\mathbb{D}_{\mathfrak{X}'} \circ u'_{+}(a^{\sharp *}(\mathcal{G})) \xrightarrow{\sim} u'_{+}((a^{\sharp *}(\mathcal{G}))^{\vee}(-\mathfrak{Z}''))$. Since $(a^{\sharp *}(\mathcal{G}))^{\vee} \in \mathrm{MIC}^{\dagger\dagger}_{0}(\mathfrak{X}'^{\sharp}/\mathfrak{S})$, then by using $(P_{n,r-1})$ we obtain that $u'_{+}((a^{\sharp *}(\mathcal{G}))^{\vee}(-\mathfrak{Z}''))$ is $(r-1)$-overholonomic. Since $\mathbb{D}_{\mathfrak{X}'} \circ (^{\dagger}Z'')(a^{\sharp *}(\mathcal{G})) \xrightarrow{\sim} \mathbb{D}_{\mathfrak{X}'} \circ u'_{+}(a^{\sharp *}(\mathcal{G}))$, then this yields that $\mathbb{D}_{\mathfrak{X}'} \circ (^{\dagger}Z'')(a^{\sharp *}(\mathcal{G}))$ is $(r-1)$-overholonomic.

(III). *Conclusion.*

For any $n \geq 0$, we know that $(P_{n,-1})$ is true. Also, for any $r \geq -1$, $(P_{0,r})$ is already known (see 18.1.2.8).

We get from the two previous steps that, for any $r \geq 0$ and $n \geq 1$, $(P_{n-1,r}) + (P_{n,r-1}) \Rightarrow (Q_{n,r}) + (R_{n,r})$. Using the exact triangle of localization of $u_{+}(\mathcal{E})$ with respect to Z we get $(Q_{n,r}) + (R_{n,r}) \Rightarrow (P_{n,r})$. Thus, $(P_{n,r-1}) + (P_{n-1,r}) \Rightarrow (P_{n,r})$. This implies that $(P_{n,r})$ is true for any $r \geq -1$ and $n \geq 0$. □

The following is the main result of [CT12]:

Theorem 18.3.2.2 (Caro-Tsuzuki)**.** *Let $(Y, X, \mathfrak{P}, Z)/\mathfrak{S}$ be a smooth c-frame over $\mathfrak{S}$ (see definition 16.2.1.8). Let $E \in F\text{-}\mathrm{MIC}^{\dagger}(X, \mathfrak{P}, Z/K)$. With notation 16.2.1.10.1, $\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z,+}(E)$ is overholonomic after any base change.*

Proof. Since $\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z,+}$ commutes with base change, then we reduce to prove the overholonomicity of $\mathrm{sp}^{(\bullet)}_{X \hookrightarrow \mathfrak{P}, Z,+}(E)$. Since this is local in $\mathfrak{P}$, we can suppose P is integral. Hence, writing Z as a finite intersection of divisor of P, using 16.2.7.4.1, we reduce to the case where Z is a divisor. Since E admits a semistable reduction (see 10.3.3.3), then there exists a commutative diagram of the form:

$$\begin{array}{ccccc} Y' & \longrightarrow & X' & \xrightarrow{\iota'_0} & \mathfrak{P}' \\ \downarrow b_0 & & \downarrow a_0 & & \downarrow f \\ Y & \longrightarrow & X & \xrightarrow{\iota_0} & \mathfrak{P}, \end{array} \qquad (18.3.2.2.1)$$

such that f is a proper smooth morphism of smooth $\mathcal{V}$-formal schemes, the left square is cartesian, X' is a smooth scheme over k, ι'_0 is a closed immersion, a_0 is a projective, surjective, generically finite and étale morphism, $a_0^{-1}(Z)$ is a strict normal crossing divisor of X' and the F-isocrystal $a_0^{*}(E)$ on Y' overconvergent along $a_0^{-1}(Z)$ is log-extendable on X'. We set $\mathcal{E} := \mathrm{sp}_{X \hookrightarrow \mathfrak{P}, T,+}(E)$. We have $\mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{!}_{T}(\mathcal{E}) \xrightarrow{\sim} \mathrm{sp}_{X' \hookrightarrow \mathfrak{P}', f^{-1}(T),+}(a_0^{*}(E))$ (see 16.2.4.3). Then by 16.1.11.2 we check that $\mathcal{E}$ is a direct factor of $f_{T,+}\mathrm{sp}_{X' \hookrightarrow \mathfrak{P}', f^{-1}(T),+}(a_0^{*}(E))$. Since the overholonomicity is stable under direct image by a proper morphism (see 18.1.2.15), it is therefore sufficient to prove that $\mathrm{sp}_{X' \hookrightarrow \mathfrak{P}', f^{-1}(T),+}(a_0^{*}(E))$ is overholonomic. This last statement is local on $\mathfrak{P}'$. Then, we can suppose that there exists a lifting $\iota' \colon \mathfrak{X}' \hookrightarrow \mathfrak{P}'$ of ι'_0 and that $a_0^{-1}(Z)$ lifts to a relative strict normal crossing divisor $\mathfrak{Z}'$ of $\mathfrak{X}'$ over $\mathfrak{S}$.

Then, $\mathrm{sp}_{X' \hookrightarrow \mathfrak{P}', f^{-1}(T),+}(a_0^*(E)) \xrightarrow{\sim} \iota'_+ \mathrm{sp}_*(a_0^*(E))$, where $\mathrm{sp}\colon \mathfrak{X}'_K \to \mathfrak{X}'$ is the specialization morphism of $\mathfrak{X}'$. It remains to check that $\mathrm{sp}_*(a_0^*(E))$ is overholonomic. But since $a_0^*(E)$ is an F-isocrystal on Y' overconvergent along $a_0^{-1}(Z)$ which is log-extendable on X', it follows from 18.3.2.1 that $\mathrm{sp}_*(a_0^*(E))$ is overholonomic. □

Theorem 18.3.2.3. *Let $(Y, X, \mathfrak{P}, Z)$ be a c-frame. Let $\mathcal{E}^{(\bullet)} \in F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ such that $\mathbb{R}\underline{\Gamma}^\dagger_Y \mathcal{E}^{(\bullet)} \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. Then the following assertions are equivalent:*

(a) The F-complex $\mathcal{E}^{(\bullet)}$ belongs to $F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{povcoh}}(X, \mathfrak{P}, Z/\mathfrak{S})$;

(b) The F-complex $\mathcal{E}^{(\bullet)}$ belongs to $F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$

(c) The F-complex $\mathcal{E}^{(\bullet)}$ belongs to $F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(X, \mathfrak{P}, Z/\mathfrak{S})$

(d) The F-complex $\mathcal{E}^{(\bullet)}$ belongs to $F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y, \mathfrak{P}/\mathfrak{S})$.

Proof. Following 16.3.1.18, we have $(a) \Rightarrow (d)$. By 18.3.2.2, we get by devissage $(d) \Rightarrow (c)$. Finally, the implications $(c) \Rightarrow (b) \Rightarrow (a)$ are tautological. □

Remark 18.3.2.4. The theorem 18.3.2.3 is wrong without Frobenius structure even when $Z = T$ is a divisor. Indeed, by using the counter-example of Berthelot given at the end of [Ber96c], there exists some $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules $\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-coherent which are not $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-coherent. Moreover, following 16.1.1.7, coherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-modules $\mathcal{O}_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-coherent are $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-overcoherent. This yields that an overcoherent $\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger T)_{\mathbb{Q}}$-module is not necessarily $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-coherent nor a fortiori $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-overcoherent.

Notation 18.3.2.5. Let $\mathfrak{P}$ be a separated and smooth $\mathfrak{S}$-formal scheme. Let Y be a subvariety of P.

We denote by $(F\text{-})D^{\mathrm{b}}_{\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V})$ (resp. $(F\text{-})D^{\mathrm{b}}_{\mathrm{h}}(Y, X, \mathfrak{P}, Z/\mathcal{V})$) the full subcategory of $(F\text{-})D^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ of $(F\text{-})$complexes $\mathcal{E}$ such that there exists an isomorphism of the form $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$.

We denote by $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V})$ the strictly full subcategory of $(F\text{-})\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}({}^{\mathrm{l}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$ of complexes $\mathcal{E}^{(\bullet)}$ such that there exist an isomorphism of the form $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. With notation 16.3.1.15, we get the equality $F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V}) = F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V})$.

Corollary 18.3.2.6. *Let $\mathfrak{P}$ and $\mathfrak{P}'$ be a two separated and smooth $\mathfrak{S}$-formal schemes. Let Y (resp. Y') be a subvariety of $\mathfrak{P}$ (resp. $\mathfrak{P}'$). Set $\mathfrak{P}'' := \mathfrak{P} \times \mathfrak{P}'$ and $Y'' := Y \times Y'$. We have the factorisations*

$$-\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{S}}} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y, \mathfrak{P}/\mathcal{V}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y', \mathfrak{P}'/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}(Y'', \mathfrak{P}''/\mathcal{V}), \tag{18.3.2.6.1}$$

$$-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}^{(\bullet)}_{\mathfrak{P}}} -\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V}). \tag{18.3.2.6.2}$$

Proof. By using 16.3.1.18 and 18.3.2.3, this is a straightforward consequence of 16.3.2.2.2. □

18.3.3 Berthelot's conjectures on the holonomicity stability on projective and smooth $\mathcal{V}$-formal schemes

The theorem 18.3.3.1 below means that the conjecture [Ber02, 5.3.6.D)] of Berthelot is satisfied when the divisor is ample.

Theorem 18.3.3.1. *Let $\mathfrak{P}$ be a proper and smooth $\mathcal{V}$-formal scheme, H_0 be an ample divisor of P_0, $\mathfrak{A}$ the open of $\mathfrak{P}$ complementary to H_0. Let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P}}({}^\dagger H_0)_{\mathbb{Q}})$ such that $\mathcal{E}|\mathfrak{A} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{A},\mathbb{Q}})$. Hence $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.*

Proof. There exists a closed immersion $\alpha_0\colon P_0 \hookrightarrow \mathbb{P}^n_k$ such that $(\mathbb{P}^n_k \setminus \mathbb{A}^n_k) \cap P_0 = H_0$. Following the theorem of Berthelot-Kashiwara $\alpha_0^! \circ \alpha_{0+}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ (see 9.3.5.9). Moreover $\mathcal{E}|\mathfrak{A}$ is holonomic if and only if so is $\alpha_{0+}(\mathcal{E})|\widehat{\mathbb{A}}^n_{\mathcal{V}}$ (for the holonomic version of the theorem of Berthelot-Kashiwara, see 15.2.4.19). As $\alpha_{0+}(\mathcal{E})$ is a coherent $F\text{-}\mathcal{D}^\dagger_{\widehat{\mathbb{P}}^n_{\mathcal{V}}}({}^\dagger\mathbb{P}^n_k \setminus \mathbb{A}^n_k)_{\mathbb{Q}}$-module such that $\alpha_{0+}(\mathcal{E})|\widehat{\mathbb{A}}^n_{\mathcal{V}}$ is a holonomic $F\text{-}\mathcal{D}^\dagger_{\widehat{\mathbb{A}}^n_{\mathcal{V}},\mathbb{Q}}$-module, since the overholonomicity is closed by extraordinary inverse image (see 18.1.2.14), we reduce then to the case where $\mathfrak{P} = \widehat{\mathbb{P}}^n_{\mathcal{V}}$ and $H_0 = \mathbb{P}^n_k \setminus \mathbb{A}^n_k$.

We proceed now by induction on the lexicographical order $(\dim \mathrm{Supp}(\mathcal{E}), N_{\mathrm{cmax}})$, where $\dim \mathrm{Supp}(\mathcal{E})$ is the dimension of the support of $\mathcal{E}$ and N_{cmax} means the number of irreducible components of the support of $\mathcal{E}$ whose dimension is $\dim \mathrm{Supp}(\mathcal{E})$, i.e., its maximal dimension. The case where $\dim \mathrm{Supp}(\mathcal{E}) \leq 1$ is a consequence of 15.3.3.4, 15.3.3.9 and 18.3.2.3. Then let us suppose $\dim \mathrm{Supp}(\mathcal{E}) \geq 1$.

For any integer j, $H^j(\mathcal{E})$ is a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-module such that $\mathcal{E}|\mathfrak{A}$ is a holonomic $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{A},\mathbb{Q}}$-module. Moreover, to establish that $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$, it is sufficient to check that for any integer j, $H^j(\mathcal{E})$ is an overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-module. We reduce therefore to the case where the complex $\mathcal{E}$ is in fact a module.

Let us denote by X_0 the support of $\mathcal{E}$. Let $\mathfrak{U}$ be an affine open of $\mathfrak{P}$ included in $\mathfrak{A}$ such that $Y_0 := X_0 \cap U_0$ is integral, smooth and dense in an irreducible component of X of dimension $\dim X$. There exist a lifting $v \colon \mathfrak{Y} \hookrightarrow \mathfrak{U}$ of the closed immersion $Y_0 \hookrightarrow U_0$. Following the theorem of Berthelot-Kashiwara (see 15.2.4.19), as $\mathcal{E}|\mathfrak{U}$ is holonomic and with support in Y_0, then $v^!(\mathcal{E}|\mathfrak{U})$ is a holonomic $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-module. By 15.3.1.20, shrinking $\mathfrak{U}$ and $\mathfrak{Y}$ if necessary, we can suppose that $v^!(\mathcal{E}|\mathfrak{U})$ is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Let us denote by T_0 the reduced divisor of P_0 complementary to U_0 (see 17.6.1.5). Shrinking $\mathfrak{U}$ if necessary, we can suppose $\mathfrak{U}$ endowed with local coordinates $x_1, \ldots, x_n$ such that $\mathfrak{Y}$ is defined by the ideal generated by $x_1, \ldots, x_r$. Moreover, via [Ked05, Theorem 2] (applied to the point 0 and with the irreducible divisors defined by $x_1 = 0, \ldots, x_r = 0$), shrinking $\mathfrak{U}$ if necessary, we can suppose that there exist a finite etale morphism $g_0 \colon U_0 \to \mathbb{A}^n_k$ such that $g_0(Y_0) \subset \mathbb{A}^{n-r}_k$. Thanks to theorems 17.7.4.6, 17.6.3.3 and 18.3.2.2, this yields that $\mathcal{E}({}^{\dagger}T_0)$ is overholonomic. We conclude the induction by using the triangle of localisation of $\mathcal{E}$ in T_0. □

Remark 18.3.3.2. The theorem 18.3.3.1 is false if the complex $\mathcal{E}$ is not endowed with a Frobenius structure. Indeed, it is about using the example gives every to the end of [Ber96c] by Berthelot where $\mathfrak{P} = \widehat{\mathbb{P}}^1_{\mathcal{V}}$ and $A = \mathbb{G}_{m,k}$. Following this example, there exist a coherent $\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-module $\mathcal{E}$ such that $\mathcal{E}|\mathfrak{A}$ is $\mathcal{O}_{\mathfrak{A},\mathbb{Q}}$-coherent and then $\mathcal{D}^{\dagger}_{\mathfrak{A},\mathbb{Q}}$-holonomic and $\mathcal{D}^{\dagger}_{\mathfrak{A},\mathbb{Q}}$-overholonomic. But, this module $\mathcal{E}$ is same not $\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}}$-coherent.

With this same counter-example, the theorem 18.3.3.4 is false without Frobenius structure.

Corollary 18.3.3.3. *Let $\mathfrak{P}$ be a proper and smooth $\mathcal{V}$-formal scheme, H_0 be an ample divisor of P_0, $\mathfrak{A}$ the open of $\mathfrak{P}$ complementary to H_0, X_0 be a closed smooth subscheme of P_0 and $Y_0 := X_0 \setminus H_0$. Then* $\mathrm{MIC}^{\dagger\dagger}(X_0, \mathfrak{P}, H_0/\mathcal{V})$ *(see notation 12.2.1.4) is equal to the full subcategory of* $\mathrm{Coh}(X_0, \mathfrak{P}, H_0/\mathcal{V})$ *(see notation 9.3.7.4) whose objects $\mathcal{E}$ satisfy the following condition: $\mathcal{E}|\mathfrak{A} \in \mathrm{MIC}^{\dagger\dagger}(Y_0, \mathfrak{A}/\mathcal{V})$.*

Proof. It follows from 15.1.5.4 and from the holonomic version of the Berthelot-Kashiwara theorem of 15.2.4.19 that $\mathcal{E}|\mathfrak{A} \in \mathrm{MIC}^{\dagger\dagger}(Y_0, \mathfrak{A}/\mathcal{V})$ Following 18.3.3.1, this implies that a coherent $F\text{-}\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}}$-module $\mathcal{E}$ tel that $\mathcal{E}|\mathfrak{A}$ is in the essential image of $\mathrm{sp}_{Y_0 \hookrightarrow \mathfrak{A},+}$ is a F-isocrystal overconvergent on Y_0. The converse is straightforward.

□

The theorem 18.3.3.1 remains valid by replacing "holonomic " by "having finite extraordinary fibers" (see the definition 15.3.2.1):

Theorem 18.3.3.4. *Let $\mathfrak{P}$ be a proper and smooth $\mathcal{V}$-formal scheme, H_0 be an ample divisor of P_0, $\mathfrak{A}$ the open of $\mathfrak{P}$ complementary to H_0 and $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{P}}({}^{\dagger}H_0)_{\mathbb{Q}})$. If $\mathcal{E}|\mathfrak{A}$ has finite extraordinary fibers (see 15.3.2.1) then $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^{\dagger}_{\mathfrak{P},\mathbb{Q}})$.*

Proof. The proof is similar to that of 18.3.3.1 but with slight changes: we reduce similarly to the case where $P_0 = \mathbb{P}^n_k$ and $H_0 = \mathbb{P}^n_k \setminus \mathbb{A}^n_k$. We still proceed by induction on the lexicographical order of $(\dim \mathrm{Supp}(\mathcal{E}), N_{\mathrm{cmax}})$. The case $\dim \mathrm{Supp}(\mathcal{E}) \leq 1$ is still a consequence of 15.3.3.4, 15.3.3.9 and 18.3.2.3. The difference here is that we can not reduce directly to the case where $\mathcal{E}$ is a module because the property of having finite extraordinary fibers does not "a priori" hold for the cohomological spaces $H^l(\mathcal{E})$, $l \in \mathbb{Z}$. Let us denote by X_0 the support of $\mathcal{E}$. By replacing 15.3.1.20 by 15.3.2.5, we check similarly (to the proof of 18.3.3.1) that there exist an affine open formal subscheme $\mathfrak{U}$ of $\mathfrak{P}$ included in $\mathfrak{A}$ such that $Y_0 := X_0 \cap U_0$ is integral, smooth and dense in a irreducible component of X of dimension $\dim X$ and such that $v^!(\mathcal{E}|\mathfrak{U}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{Y},\mathbb{Q}})$. Moreover, since $\mathcal{E}|\mathfrak{U} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}})$ and has its support in Y_0, then following the theorem of Berthelot-Kashiwara (see 9.3.5.9), for any integer r, $v^!(H^r(\mathcal{E}|\mathfrak{U})) \xrightarrow{\sim} H^r(v^!(\mathcal{E}|\mathfrak{U}))$, Hence, $v^!(H^r(\mathcal{E})|\mathfrak{U})$ is $\mathcal{O}_{\mathfrak{Y},\mathbb{Q}}$-coherent. Let us denote by T_0 the reduced divisor of P_0 complementary to U_0. By using [Ked05, Theorem 2], shrinking U_0 if necessary, we can suppose that there exist a finite etale morphism $g_0 \colon U_0 \to \mathbb{A}^n_k$ such that $g_0(Y_0) \subset \mathbb{A}^{n-r}_k$. Via the theorems 17.7.4.6, 17.6.3.3 and 18.3.2.2, this yields that $(H^r\mathcal{E})(^\dagger T_0)$ is overholonomic. As $(H^r\mathcal{E})(^\dagger T_0) \xrightarrow{\sim} H^r(\mathcal{E}(^\dagger T_0))$, the complex $\mathcal{E}(^\dagger T_0)$ is then overholonomic. We conclude the induction by using the triangle of localisation of $\mathcal{E}$ with respect to T_0. □

Theorem 18.3.3.5. *Let $\mathfrak{P}$ be a projective and smooth $\mathcal{V}$-formal scheme, $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. The following assertions are equivalent :*

(a) The F-complex $\mathcal{E}$ belongs to $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

(b) The F-complex $\mathcal{E}$ has finite extraordinary fibers.

(c) The F-complex $\mathcal{E}$ belongs to $F\text{-}D^{\mathrm{b}}_{\mathrm{ovcoh}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

(d) The F-complex $\mathcal{E}$ belongs to $F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$.

Proof. Following 18.3.2.3, we already know $(d) \Leftrightarrow (c)$. The implication $(c) \Rightarrow (b)$ is clear. Using 18.1.2.9.(d) and 18.1.2.6, we get the implication $(d) \Rightarrow (a)$. Let us check now $(a) \Rightarrow (d)$. Let us suppose then $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. Since $\mathfrak{P}$ is projective, using the theorem of Berthelot-Kashiwara, we reduce to the case where $\mathfrak{P} = \widehat{\mathbb{P}}^n_{\mathcal{V}}$. Let $\mathcal{H}$ be the hyperplan of $\widehat{\mathbb{P}}^n_{\mathcal{V}}$ defined by $u_0 = 0$, i.e. $H_0 := \mathbb{P}^n_k \setminus \mathbb{A}^n_k$. Hence, following 18.3.3.1, $\mathcal{E}(^\dagger H_0)$ is overholonomic. Via the triangle of localisation of $\mathcal{E}$ in H_0, this yields that $\mathbb{R}\underline{\Gamma}^\dagger_{H_0}(\mathcal{E})$ is also holonomic. Denoting by $\alpha \colon \mathcal{H} \hookrightarrow \widehat{\mathbb{P}}^n_{\mathcal{V}}$ the canonical closed immersion we get $\alpha^!(\mathcal{E}) \xrightarrow{\sim} \alpha^!(\mathbb{R}\underline{\Gamma}^\dagger_{H_0}(\mathcal{E}))$. Following the holonomic version of the theorem of Berthelot-Kashiwara (see 15.2.4.19) this yields that $\alpha^!(\mathcal{E})$ is holonomic. By proceeding by induction on n, we get then the overholonomicity of $\alpha^!(\mathcal{E})$. Since $\mathbb{R}\underline{\Gamma}^\dagger_{H_0}(\mathcal{E}) \xrightarrow{\sim} \alpha_+\alpha^!(\mathcal{E})$ (see 14.3.3.1), then using 15.2.4.19 we get the overholonomicity of $\mathbb{R}\underline{\Gamma}^\dagger_{H_0}(\mathcal{E})$. Via the localisation triangle of $\mathcal{E}$ with respect to H_0, this yields that $\mathcal{E}$ is also overholonomic. Hence, we have checked $(a) \Rightarrow (d)$. Finally, to establish the implication $(b) \Rightarrow (d)$ we proceed similarly to the proof of $(a) \Rightarrow (d)$ by replacing the use of theorem 18.3.3.1 by 18.3.3.4. □

Via the stability 18.1.2.15 and 18.1.2.14 of the overholonomicity, we obtain the following corollary which answers positively in the projective case to the conjectures [Ber02, 5.3.6.A),B)] of Berthelot.

Corollary 18.3.3.6. *Let $f \colon \mathfrak{P}' \to \mathfrak{P}$ be a morphism of $\mathcal{V}$-formal schemes projective smooth, $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$, $\mathcal{E}' \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P}',\mathbb{Q}})$. Hence $f_+(\mathcal{E}') \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ and $f^!(\mathcal{E}) \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P}',\mathbb{Q}})$.*

This will give the stability under Grothendieck six operations in the context of quasi-projective varieties (see 19.2.4).

Chapter 19

Coefficients stable under Grothendieck's six operations

Suppose the residue field k of $\mathcal{V}$ is a perfect field of characteristic $p > 0$. When we work with F-complex, we suppose there exists an automorphism $\sigma \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ which is a lifting of the sth Frobenius power of k. The data s and σ are fixed in the remaining.

19.1 Data of coefficients

19.1.1 Definitions

Definition 19.1.1.1. A *data of coefficients* $\mathfrak{C}$ *over* $\mathcal{V}$ will be the data for any object $\mathcal{W}$ of DVR$(\mathcal{V})$ (see notation 9.2.6.12), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$ of a strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$, which will be denoted by $\mathfrak{C}(\mathfrak{X}/\mathcal{W})$, or simply $\mathfrak{C}(\mathfrak{X})$ if there is no ambiguity with the base $\mathcal{W}$. If there is no ambiguity with $\mathcal{V}$, we simply say a *data of coefficients*.

Examples 19.1.1.2. We have the following data of coefficients.

(a) We define the data of coefficients $\mathfrak{B}_{\emptyset}$ as follows: for any object $\mathcal{W}$ of DVR$(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{B}_{\emptyset}(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ whose unique object is $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ (where $\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ is the constant object $\mathcal{O}^{(m)}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$ for any $m \in \mathbb{N}$ with the identity as transition maps).

(b) We will need the larger data of coefficients $\mathfrak{B}_{\mathrm{div}}$ defined as follows: for any object $\mathcal{W}$ of DVR$(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{B}_{\mathrm{div}}(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ whose objects are of the form $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T)$, where T is any divisor of the special fiber of $\mathfrak{X}$. From Corollary 12.2.7.2, we have $\mathcal{B}^{(\bullet)}_{\mathfrak{X}}(T) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$.

(c) We define $\mathfrak{B}_{\mathrm{cst}}$ as follows: for any object $\mathcal{W}$ of DVR$(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{B}_{\mathrm{cst}}(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ whose objects are of the form $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\mathcal{O}^{(\bullet)}_{\mathfrak{X}}$, where Y is a subvariety of the special fiber of $\mathfrak{X}$ and the functor $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}$ is defined in 13.1.5.1. Recall following 13.1.5.5, theses objects are coherent.

(d) We define $\mathfrak{M}_{\emptyset}$ (resp. $\mathfrak{M}_{\mathrm{sncd}}$, resp. $\mathfrak{M}_{\mathrm{div}}$) as follows: for any object $\mathcal{W}$ of DVR$(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{M}_{\emptyset}(\mathfrak{X})$ (resp. $\mathfrak{M}_{\mathrm{sncd}}(\mathfrak{X})$, resp. $\mathfrak{M}_{\mathrm{div}}(\mathfrak{X})$) is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ consisting of objects of the form $({}^{\dagger}T)(\mathcal{E}^{(\bullet)})$, where $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(Z, \mathfrak{X}/K)$ (see notation 12.2.1.6), with Z is a smooth subvariety of the special fiber of $\mathfrak{X}$, and where T is an empty divisor (resp. a strict normal crossing divisor, resp. a divisor) of Z. Recall that following 13.2.2.1, these objects are indeed coherent.

Definition 19.1.1.3. In order to be precise, let us fix some terminology. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients over $\mathcal{V}$.

(a) We say that the data of coefficients $\mathfrak{C}$ is stable under pushforwards if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any *realizable* (see 13.2.3.1) morphism $g\colon \mathfrak{X}' \to \mathfrak{X}$ of smooth formal schemes over $\mathcal{W}$, for any object $\mathcal{E}'^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X}')$ with proper support over X via g, the complex $g_+(\mathcal{E}'^{(\bullet)})$ is an object of $\mathfrak{C}(\mathfrak{X})$.

(b) We say that the data of coefficients $\mathfrak{C}$ is stable under extraordinary pullbacks (resp. under *smooth* extraordinary pullbacks) if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any morphism (resp. *smooth* morphism) $f\colon \mathfrak{Y} \to \mathfrak{X}$ of smooth formal schemes over $\mathcal{W}$, for any object $\mathcal{E}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X})$, we have $f^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$.

(c) We still say that the data of coefficients $\mathfrak{C}$ satisfies the first property (resp. the second property) of Berthelot-Kashiwara theorem or satisfies $BK^!$ (resp. BK_+) for short if the following property is satisfied: for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any closed immersion $u\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ of smooth formal schemes over $\mathcal{W}$, for any object $\mathcal{E}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X})$ with support in $\mathfrak{Z}$, we have $u^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Z})$ (resp. for any object $\mathcal{G}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{Z})$, we have $u_+(\mathcal{G}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{X})$). Remark that $BK^!$ and BK_+ hold if and only if the data of coefficients $\mathfrak{C}$ satisfies (an analogue of) Berthelot-Kashiwara theorem, which justifies the terminology.

(d) We say that the data of coefficients $\mathfrak{C}$ is stable under base change if for any morphism $\mathcal{W} \to \mathcal{W}'$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any object $\mathcal{E}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X})$, we have $\mathcal{W}'\widehat{\otimes}^{\mathbb{L}}_{\mathcal{W}}\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X} \times_{\operatorname{Spf}\mathcal{W}} \operatorname{Spf}\mathcal{W}')$.

(e) We say that the data of coefficients $\mathfrak{C}$ is stable under tensor products (resp. duals) if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any objects $\mathcal{E}^{(\bullet)}$ and $\mathcal{F}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X})$ we have $\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$ (resp. $\mathbb{D}_{\mathfrak{X}}(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{X})$).

(f) We say that the data of coefficients $\mathfrak{C}$ is stable under local cohomological functors (resp. under localizations outside a divisor), if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any object $\mathcal{E}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X})$, for any subvariety Y (resp. for any divisor T) of the special fiber of $\mathfrak{X}$, we have $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$ (resp. $({}^{\dagger}T)(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{X})$).

(g) We say that the data of coefficients $\mathfrak{C}$ is stable under cohomology if, for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any object $\mathcal{E}^{(\bullet)}$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$, the property $\mathcal{E}^{(\bullet)}$ is an object of $\mathfrak{C}(\mathfrak{X})$ is equivalent to the fact that, for any integer n, $H^n(\mathcal{E}^{(\bullet)})$ is an object of $\mathfrak{C}(\mathfrak{X})$.

(h) We say that the data of coefficients $\mathfrak{C}$ is stable under shifts if, for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any object $\mathcal{E}^{(\bullet)}$ of $\mathfrak{C}(\mathfrak{X})$, for any integer n, $\mathcal{E}^{(\bullet)}[n]$ is an object of $\mathfrak{C}(\mathfrak{X})$.

(i) We say that the data of coefficients $\mathfrak{C}$ is stable by devissages if $\mathfrak{C}$ is stable by shifts and if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any exact triangle $\mathcal{E}_1^{(\bullet)} \to \mathcal{E}_2^{(\bullet)} \to \mathcal{E}_3^{(\bullet)} \to \mathcal{E}_1^{(\bullet)}[1]$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$, if two objects are in $\mathfrak{C}(\mathfrak{X})$, then so is the third one.

(j) We say that the data of coefficients $\mathfrak{C}$ is stable under direct summands if, for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$ we have the following property: any direct summand in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ of an object of $\mathfrak{C}(\mathfrak{X})$ is an object of $\mathfrak{C}(\mathfrak{X})$.

(k) We say that $\mathfrak{C}$ contains $\mathfrak{D}$ (or $\mathfrak{D}$ is contained in $\mathfrak{C}$) if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$ the category $\mathfrak{D}(\mathfrak{X})$ is a full subcategory of $\mathfrak{C}(\mathfrak{X})$.

(l) We say that the data of coefficients $\mathfrak{C}$ is local if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any open covering $(\mathfrak{X}_i)_{i\in I}$ of $\mathfrak{X}$, for any object $\mathcal{E}^{(\bullet)}$ of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$, we have $\mathcal{E}^{(\bullet)} \in \operatorname{Ob}\mathfrak{C}(\mathfrak{X})$ if and only if $\mathcal{E}^{(\bullet)}|\mathfrak{X}_i \in \operatorname{Ob}\mathfrak{C}(\mathfrak{X}_i)$ for any $i \in I$. For instance, it follows from 8.4.5.8.b that the data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$ is local.

(m) We say that the data of coefficients $\mathfrak{C}$ is quasi-local if for any object $\mathcal{W}$ of DVR($\mathcal{V}$), for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, for any open immersion $j\colon \mathfrak{Y} \hookrightarrow \mathfrak{X}$ for any object $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$, we have $j^{(\bullet)!}\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{Y})$.

We finish the subsection with some notation.

19.1.1.4 (Duality)**.** Let $\mathfrak{C}$ be a data of coefficients. We define its dual data of coefficients $\mathfrak{C}^\vee$ as follows: for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{C}^\vee(\mathfrak{X})$ is the subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ of objects $\mathcal{E}^{(\bullet)}$ such that $\mathbb{D}_{\mathfrak{X}}(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{X})$.

Notation 19.1.1.5. Let $\mathfrak{C}$ be a data of coefficients. We denote by $\mathfrak{C}^+$ the smallest data of coefficients containing $\mathfrak{C}$ and stable under shifts. We define by induction on $n \in \mathbb{N}$ the data of coefficients $\Delta_n(\mathfrak{C})$ as follows: for $n = 0$, we put $\Delta_0(\mathfrak{C}) = \mathfrak{C}^+$. Suppose $\Delta_n(\mathfrak{C})$ is constructed for $n \in \mathbb{N}$. For any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\Delta_{n+1}(\mathfrak{C})(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ of objects $\mathcal{E}^{(\bullet)}$ such that there exists an exact triangle of the form $\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to \mathcal{E}^{(\bullet)}[1]$ such that $\mathcal{F}^{(\bullet)}$ and $\mathcal{G}^{(\bullet)}$ are objects of $\Delta_n(\mathfrak{C})(\mathfrak{X})$. Finally, we put $\Delta(\mathfrak{C}) := \cup_{n\in\mathbb{N}}\Delta_n(\mathfrak{C})$. The data of coefficients $\Delta(\mathfrak{C})$ is the smallest data of coefficients containing $\mathfrak{C}$ and stable under devissage.

Example 19.1.1.6. Using the isomorphisms 13.1.5.6.2, and Theorem 13.2.1.4, we check that $\mathfrak{B}^+_{\mathrm{cst}}$ satisfies BK_+, and is stable under local cohomological functors, extraordinary pullbacks and tensor products.

The following lemma is obvious.

Lemma 19.1.1.7. *Let $\mathfrak{D}$ be a data of coefficients over $\mathcal{V}$. If $\mathfrak{D}$ is stable under pushforwards (resp. extraordinary pullbacks, resp. smooth extraordinary pullbacks, resp. tensor products, resp. base change, resp. local cohomological functors, resp. localisation outside a divisor) then so is $\Delta(\mathfrak{D})$. If $\mathfrak{D}$ satisfies BK_+ (resp. is quasi-local) then so is $\Delta(\mathfrak{D})$. If $\mathfrak{D}$ satisfies $BK^!$ and is stable under local cohomological functors then so is $\Delta(\mathfrak{D})$.*

19.1.1.8. Beware also that if $\mathfrak{D}$ is local (resp. stable under cohomology, resp. satisfies $BK^!$), then it is not clear that so is $\Delta(\mathfrak{D})$.

Since the converse of 19.1.1.7 is not true, let us introduce the following definition.

Definition 19.1.1.9. Let $\mathfrak{D}$ be a data of coefficients over $\mathcal{V}$. Let P be one of the stability property of 19.1.1.3. We say that $\mathfrak{D}$ is Δ-stable under P if there exists a data of coefficients $\mathfrak{D}'$ over $\mathcal{V}$ such that $\Delta(\mathfrak{D}') = \Delta(\mathfrak{D})$ and $\mathfrak{D}'$ is stable under P.

Lemma 19.1.1.10. *The data of coefficients $\mathfrak{D}$ is Δ-stable under pushforwards (resp. extraordinary pullbacks, resp. smooth extraordinary pullbacks, resp. tensor products, resp. base change, resp. local cohomological functors, resp. localisation outside a divisor) if and only if $\Delta(\mathfrak{D})$ is stable under pushforwards (resp. extraordinary pullbacks, resp. smooth extraordinary pullbacks, resp. tensor products, resp. base change, resp. local cohomological functors, resp. localisation outside a divisor). The data of coefficients $\mathfrak{D}$ satisfies Δ-BK_+ (resp. is Δ-quasi local) if and only if $\Delta(\mathfrak{D})$ satisfies BK_+ (is quasi-local).*

Proof. This is a translation of Lemma 19.1.1.7. □

Beware, it is not clear that if $\mathfrak{D}$ satisfies Δ-$BK^!$ and is Δ-stable under local cohomological functors then $\Delta(\mathfrak{D})$ satisfies $BK^!$.

19.1.2 Overcoherence, (over)holonomicity (after any base change) revisited and complements

Definition 19.1.2.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients.

(a) We denote by $S_0(\mathfrak{D},\mathfrak{C})$ the data of coefficients defined as follows: for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $S_0(\mathfrak{D},\mathfrak{C})(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ of objects $\mathcal{E}^{(\bullet)}$ satisfying the following properties:

   ($\star$) for any smooth morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ of smooth $\mathcal{W}$-formal schemes, for any object $\mathcal{F}^{(\bullet)} \in \mathfrak{D}(\mathfrak{Y})$, we have $\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$.

(b) We denote by $S(\mathfrak{D},\mathfrak{C})$ the data of coefficients defined as follows: for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $S(\mathfrak{D},\mathfrak{C})(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ of objects $\mathcal{E}^{(\bullet)}$ satisfying the following ($\star\star$) property

(⋆⋆) for any morphism $\mathcal{W} \to \mathcal{W}'$ of $\mathrm{DVR}(\mathcal{V})$, we have $\mathcal{W}'\widehat{\otimes}^{\mathbb{L}}_{\mathcal{W}}\mathcal{E}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X} \times_{\mathrm{Spf}\,\mathcal{W}} \mathrm{Spf}\,\mathcal{W}')$.

(c) Let $\sharp$ be a symbol so that either $S_\sharp = S_0$ or $S_\sharp = S$.

Examples 19.1.2.2. We retrieve previous notions as follows:

(a) We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}} = S_0(\mathfrak{B}_{\mathrm{div}}, \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}})$ (see the second example of 19.1.1.2). We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}} = S(\mathfrak{B}_{\mathrm{div}}, \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}})$. These notions correspond to that of the overcoherence (after any base change) as defined in 15.3.6.1. More precisely, for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, we get the equality $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\mathfrak{X}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$.

(b) We put $\mathfrak{H}_0 := S(\mathfrak{B}_{\mathrm{div}}, \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}})$ and by induction on $i \in \mathbb{N}$, we put $\mathfrak{H}_{i+1} := \mathfrak{H}_i \cap S(\mathfrak{B}_{\mathrm{div}}, \mathfrak{H}_i^{\vee})$ (see Notation 19.1.1.4). The coefficients of $\mathfrak{H}_i$ are called *i-overholonomic after any base change.* We get the data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}} := \mathfrak{H}_\infty := \cap_{i\in\mathbb{N}}\mathfrak{H}_i$ whose objects are called *overholonomic after any base change.* This correspond to that of the overholonomicity as defined in 18.1.2.2, i.e. for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, we get the equality $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}(\mathfrak{X}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$.

(c) Replacing S by S_0 in the definition of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}$, we get a data of coefficients that we will denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}$. This correspond to that of the overholonomicity after any base change as defined in 18.1.2.2, i.e. for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, we get the equality $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(\mathfrak{X}) = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$.

(d) Finally, we set $\underrightarrow{LM}_{\mathbb{Q},\star} := \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star} \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}$, for $\star \in \{\mathrm{ovcoh}, \mathrm{oc}, \mathrm{h}, \mathrm{ovhol}\}$.

Remark 19.1.2.3. (a) Let $\mathfrak{C}$ be a data of coefficients. The data of coefficients $\mathfrak{C}$ is stable under smooth extraordinary inverse image and localizations outside a divisor (resp. under smooth extraordinary inverse image, localizations outside a divisor, and base change) if and only if $S_0(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C}) = \mathfrak{C}$ (resp. $S(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C}) = \mathfrak{C}$).

(b) By construction, we remark that $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}$ is the biggest data of coefficients which contains $\mathfrak{B}_{\mathrm{div}}$, is stable by devissage, dual functors and the operation $S_0(\mathfrak{B}_{\mathrm{div}}, -)$. Moreover, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}$ is the biggest data of coefficients which contains $\mathfrak{B}_{\mathrm{div}}$, is stable by devissage, dual functors and the operation $S(\mathfrak{B}_{\mathrm{div}}, -)$.

We will need later the following Lemmas.

Lemma 19.1.2.4. *Let $\mathfrak{C}$ be a data of coefficients stable under devissage.*

(a) We have the equality $\Delta(\mathfrak{B}_{\mathrm{div}}) = \Delta(\mathfrak{B}_{\mathrm{cst}})$ (see Notation 19.1.1.2).

(b) The data of coefficients $\mathfrak{C}$ is stable under local cohomological functors if and only if it is stable under localizations outside a divisor (see Definitions 19.1.1.3).

Proof. Both statements are checked by using exact triangles of localisation 13.1.4.3.3 and Mayer-Vietoris exact triangles 13.1.4.15.2. □

Lemma 19.1.2.5. *Let $\mathfrak{C}$ be a data of coefficients stable under local cohomological functors. Then the data of coefficients $\mathfrak{C}$ is stable under smooth extraordinary pullbacks and satisfies $BK^!$ if and only if $\mathfrak{C}$ is stable under extraordinary pullbacks.*

Proof. Since the converse is obvious, let us check that if $\mathfrak{C}$ is stable under smooth extraordinary pullbacks and satisfies $BK^!$ then $\mathfrak{C}$ is stable under extraordinary pullbacks. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of smooth formal schemes over $\mathcal{W}$, and $\mathcal{E}^{(\bullet)}$ be an object of $\mathfrak{C}(\mathfrak{X})$. Since f is the composition of its graph $\mathfrak{Y} \hookrightarrow \mathfrak{Y}\times\mathfrak{X}$ followed by the projection $\mathfrak{Y}\times\mathfrak{X} \to \mathfrak{X}$ which is smooth, then using the stability under smooth extraordinary pullbacks, we reduce to the case where f is a closed immersion. From the stability under local cohomological functors, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$. Since $\mathfrak{C}$ satisfies $BK^!$, then $f^!\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{Y})$. We conclude using the isomorphism $f^!\mathbb{R}\underline{\Gamma}^{\dagger}_{Y}\mathcal{E}^{(\bullet)} \xrightarrow{\sim} f^!(\mathcal{E}^{(\bullet)})$ (use 13.2.1.4). □

Lemma 19.1.2.6. *Let $\mathfrak{D}$ be a data of coefficients over $\mathcal{V}$. If $\mathfrak{D}$ contains $\mathfrak{B}_{\mathrm{div}}$, and if $\mathfrak{D}$ is stable under tensor products, then $\mathfrak{D}$ is stable under localizations outside a divisor.*

Proof. This is a consequence of the isomorphism 13.1.5.6.2 (we use the case where $\mathcal{E}^{(\bullet)} = \mathcal{O}_{\mathfrak{X}}^{(\bullet)}$). □

Lemma 19.1.2.7. *Let $\mathfrak{C}$ be a data of coefficients. If the data of coefficients $\mathfrak{C}$ is local (resp. is stable under devissages, resp. is stable under direct summands, resp. is stable under pushforwards, resp. is stable under base change, resp. satisfies $BK^!$), then so is $\mathfrak{C}^\vee$ (see Notation 19.1.1.4).*

Proof. The stability under pushforwards is a consequence of the relative duality isomorphism (see 13.2.4.1). It follows from 9.3.5.11.1, that if $\mathfrak{C}$ satisfies $BK^!$, then so is $\mathfrak{C}^\vee$. The other stability properties are straighforward. □

Lemma 19.1.2.8. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients.*

(a) If $\mathfrak{D} \subset \mathfrak{C}$ then $\mathfrak{D}^\vee \subset \mathfrak{C}^\vee$.

(b) We have the equality $\Delta(\mathfrak{C})^\vee = \Delta(\mathfrak{C}^\vee)$.

Proof. The first statement is obvious. Moreover, since $\mathfrak{C} \subset \Delta(\mathfrak{C})$ then $\mathfrak{C}^\vee \subset \Delta(\mathfrak{C})^\vee$. From 19.1.2.7, $\Delta(\mathfrak{C})^\vee$ is stable under devissage. Hence $\Delta(\mathfrak{C}^\vee) \subset \Delta(\mathfrak{C})^\vee$. By replacing in the inclusion $\mathfrak{C}$ by $\mathfrak{C}^\vee$, since $(\mathfrak{C}^\vee)^\vee = \mathfrak{C}$, we get $\Delta(\mathfrak{C}) \subset \Delta(\mathfrak{C}^\vee)^\vee$. Hence, $\Delta(\mathfrak{C})^\vee \subset (\Delta(\mathfrak{C}^\vee)^\vee)^\vee = \Delta(\mathfrak{C}^\vee)$.

□

Lemma 19.1.2.9. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients. With the notation of 19.1.2.1, we have the following properties.*

1. *If $\mathfrak{D}$ contains $\mathfrak{B}_\emptyset$ (see Notation 19.1.1.2) then $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is contained in $\mathfrak{C}$. If $\mathfrak{D}$ contains $\mathfrak{B}_{\mathrm{div}}$, then $S_0(\mathfrak{D}, \mathfrak{C})$ (resp $S(\mathfrak{D}, \mathfrak{C})$) is included in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$).*
2. *If $\mathfrak{C} \subset \mathfrak{C}'$ and $\mathfrak{D}' \subset \mathfrak{D}$, then $S_\sharp(\mathfrak{D}, \mathfrak{C}) \subset S_\sharp(\mathfrak{D}', \mathfrak{C}')$.*
3. *If either $\mathfrak{C}$ or $\mathfrak{D}$ is stable under devissages (resp. shifts), then so is $S_\sharp(\mathfrak{D}, \mathfrak{C})$ and we have the equality $S_\sharp(\Delta(\mathfrak{D}), \mathfrak{C}) = S_\sharp(\mathfrak{D}, \mathfrak{C})$ (resp. $S_\sharp(\mathfrak{D}^+, \mathfrak{C}) = S_\sharp(\mathfrak{D}, \mathfrak{C})$).*
4. *Suppose that $\mathfrak{D}$ is stable under smooth extraordinary pullbacks, tensor products (resp. and base change), and that $\mathfrak{C}$ contains $\mathfrak{D}$.*
   (a) The data of coefficients $S_0(\mathfrak{D}, \mathfrak{C})$ contains $\mathfrak{D}$ (resp. $S(\mathfrak{D}, \mathfrak{C})$ contains $\mathfrak{D}$).
   (b) If $\mathfrak{D}$ contains $\mathfrak{B}_\emptyset$, if either $\mathfrak{C}$ or $\mathfrak{D}$ is stable under shifts, then $S_0(\mathfrak{D}, \mathfrak{C}) = S_0(\mathfrak{D}, S_0(\mathfrak{D}, \mathfrak{C}))$ (resp. $S(\mathfrak{D}, \mathfrak{C}) = S(\mathfrak{D}, S(\mathfrak{D}, \mathfrak{C}))$).
   (c) If either $\mathfrak{C}$ or $\mathfrak{D}$ is stable under shifts then $S_0(S_0(\mathfrak{D}, \mathfrak{C}), S_0(\mathfrak{D}, \mathfrak{C}))$ (resp. $S(S(\mathfrak{D}, \mathfrak{C}), S(\mathfrak{D}, \mathfrak{C}))$) contains $\mathfrak{D}$.

Proof. The assertions 1), 2), 3), 4.a) and 4.b) are obvious. Let us prove 4)c). Let us suppose moreover $\mathfrak{C}$ stable under shifts. Since tensor products and extraordinary inverse images commute with base change, to check the second part, we reduce to establish the non-respective case. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$, and $\mathcal{E}^{(\bullet)} \in \mathfrak{D}(\mathfrak{X})$. Let $f \colon \mathfrak{Y} \to \mathfrak{X}$ be a smooth morphism of smooth $\mathcal{W}$-formal schemes. Let $\mathcal{F}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{Y})$. We have to check that $\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^!(\mathcal{E}^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{Y})$. Let $g \colon \mathfrak{Z} \to \mathfrak{Y}$ be a smooth morphism of smooth $\mathcal{W}$-formal schemes, and $\mathcal{G}^{(\bullet)} \in \mathfrak{D}(\mathfrak{Z})$. We have the isomorphisms

$$\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} g^! \left(\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^!(\mathcal{E}^{(\bullet)}) \right) \underset{9.2.1.27.1}{\xrightarrow{\sim}} \left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} g^! \mathcal{F}^{(\bullet)} \right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} (f \circ g)^!(\mathcal{E}^{(\bullet)})[-d_{Z/Y}]$$
$$\xrightarrow{\sim} \left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} (f \circ g)^!(\mathcal{E}^{(\bullet)}) \right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} g^! \mathcal{F}^{(\bullet)}[-d_{Z/Y}]. \qquad (\star)$$

Since $\mathfrak{D}$ is stable under smooth extraordinary pullbacks and tensor products, then $\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} (f \circ g)^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{D}(\mathfrak{Z})$. Since $\mathcal{F}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{Y})$, then $\left(\mathcal{G}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} (f \circ g)^!(\mathcal{E}^{(\bullet)}) \right) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Z}}} g^! \mathcal{F}^{(\bullet)}[-d_{Z/Y}] \in \mathfrak{C}(\mathfrak{Z})$. Hence, using $(\star)$ we conclude. The respective case is treated similarly. □

Remark 19.1.2.10. Let $\mathfrak{C}$, $\mathfrak{D}$ be two data of coefficients.

1. If $\mathfrak{C}$ is stable under devissages, then using 19.1.2.9.3 and 19.1.2.4 we get $S_\sharp(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C}) = S_\sharp(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})$.

2. Let $\mathfrak{D}'$ be a data of coefficients such that $\Delta(\mathfrak{D}') = \Delta(\mathfrak{D})$. If $\mathfrak{C}$ is stable under devissages, then $S_\sharp(\mathfrak{D}', \mathfrak{C}) = S_\sharp(\mathfrak{D}, \mathfrak{C})$. Hence, in the case of stable properties appearing in Lemma 19.1.1.10 and when $\mathfrak{C}$ is stable under devissages, to study $S_\sharp(\mathfrak{D}, \mathfrak{C})$ it is enough to consider Δ-stable properties instead of stable properties satisfied by $\mathfrak{D}$ (e.g. see the beginning of the proof of 19.1.2.12).

3. If $\mathfrak{D}$ is stable under smooth extraordinary pullbacks, tensor products, and that $\mathfrak{D}$ contains $\mathfrak{B}_{\mathrm{div}}$ and is contained in $\mathfrak{C}$, if moreover either $\mathfrak{C}$ or $\mathfrak{D}$ is stable under shifts, then using 19.1.2.9 (1, 2 and 4.b), we get
$$S_0(\mathfrak{D}, \mathfrak{C}) = S_0\left(\mathfrak{D}, S_0(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C})\right) = S_0\left(\mathfrak{D}, S_0(\mathfrak{D}, \mathfrak{C})\right). \qquad (19.1.2.10.1)$$

   If moreover $\mathfrak{D}$ is stable under base change, then
$$S(\mathfrak{D}, \mathfrak{C}) = S\left(\mathfrak{D}, S(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C})\right) = S\left(\mathfrak{D}, S(\mathfrak{D}, \mathfrak{C})\right). \qquad (19.1.2.10.2)$$

Lemma 19.1.2.11. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients. We have the following properties.*

1. *If $\mathfrak{C}$ is local and if $\mathfrak{D}$ is quasi-local then $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is local. If $\mathfrak{C}$ is stable under direct summands, then so is $S_\sharp(\mathfrak{D}, \mathfrak{C})$.*

2. *The data of coefficients $S_0(\mathfrak{D}, \mathfrak{C})$ (resp. $S(\mathfrak{D}, \mathfrak{C})$) is stable under smooth extraordinary pullbacks (resp. and under base change).*

3. *If $\mathfrak{D}$ is stable under local cohomological functors (resp. localizations outside a divisor), then so is $S_\sharp(\mathfrak{D}, \mathfrak{C})$.*

4. *Suppose that $\mathfrak{C}$ is stable under pushforwards and shifts. Suppose that $\mathfrak{D}$ is stable under extraordinary pullbacks. Then the data of coefficients $S_\sharp(\mathfrak{D}, \mathfrak{C})$ are stable under pushforwards.*

5. *Suppose that $\mathfrak{C}$ is stable under shifts, and satisfies $BK^!$. Moreover, suppose that $\mathfrak{D}$ satisfies BK_+. Then the data of coefficients $S_\sharp(\mathfrak{D}, \mathfrak{C})$ satisfies $BK^!$.*

Proof. a) Using 8.4.5.8 (beware that tensor products do not preserve coherence), we check that if $\mathfrak{C}$ is local and if $\mathfrak{D}$ is quasi-local then $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is local. The rest of the assertions 1) and 2) are obvious.

b) Let us check 3). From the commutation of the base change with local cohomological functors, we reduce to check that $S_0(\mathfrak{D}, \mathfrak{C})$ is stable under local cohomological functors (resp. localisations outside a divisor). Using 13.1.5.6.2 and the commutation of local cohomological functors with extraordinary inverse images (see 13.2.1.4), we check the desired properties.

c) Let us check 4). From the commutation of base changes with pushforwards (see 9.2.6), we reduce to check the stability of $S_0(\mathfrak{D}, \mathfrak{C})$ under pushforwards. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$. Let $g\colon \mathfrak{X}' \to \mathfrak{X}$ be a realizable morphism of smooth $\mathcal{W}$-formal schemes. Let $\mathcal{E}'^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X}')$ with proper support over X. We have to check that $g_+(\mathcal{E}'^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$. Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a smooth morphism of smooth $\mathcal{W}$-formal schemes. Let $f'\colon \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}' \to \mathfrak{X}'$, and $g'\colon \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}' \to \mathfrak{Y}$ be the structural projections. Let $\mathcal{F}^{(\bullet)} \in \mathfrak{D}(\mathfrak{Y})$. We have to check $\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^! g_+(\mathcal{E}'^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$. Using the hypotheses on $\mathfrak{C}$ and $\mathfrak{D}$, via the isomorphisms
$$\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^! g_+(\mathcal{E}'^{(\bullet)}) \underset{13.2.3.7.1}{\xrightarrow{\sim}} \mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} g'_+ f'^!(\mathcal{E}'^{(\bullet)}) \underset{9.4.3.1.1}{\xrightarrow{\sim}} g'_+\left(g'^!(\mathcal{F}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f'^!(\mathcal{E}'^{(\bullet)})\right)[-d_{X'/X}],$$
we check that $\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^! g_+(\mathcal{E}'^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$.

d) Let us check 5) (we might remark the similarity with the proof of 15.3.6.12). Since extraordinary pullbacks commute with base change, we reduce to check that $S_0(\mathfrak{D}, \mathfrak{C})$ satisfies $BK^!$. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, and $u\colon \mathfrak{X} \hookrightarrow \mathfrak{P}$ be a closed immersion of smooth formal schemes over $\mathcal{W}$. Let $\mathcal{E}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{P})$ with support in $\mathfrak{X}$. We have to check that $u^!(\mathcal{E}^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$. We already know that $u^!(\mathcal{E}^{(\bullet)}) \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ (thanks to Berthelot-Kashiwara theorem 9.3.5.13). Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a smooth morphism of smooth $\mathcal{W}$-formal schemes, and $\mathcal{F}^{(\bullet)} \in \mathfrak{D}(\mathfrak{Y})$. We have to check $\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^!(u^! \mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$.

The morphism f is the composite of its graph $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \times \mathfrak{X}$ with the projection $\mathfrak{Y} \times \mathfrak{X} \to \mathfrak{X}$. We denote by v the composite of $\mathfrak{Y} \hookrightarrow \mathfrak{Y} \times \mathfrak{X}$ with $id \times u \colon \mathfrak{Y} \times \mathfrak{X} \hookrightarrow \mathfrak{Y} \times \mathfrak{P}$. Let $g \colon \mathfrak{Y} \times \mathfrak{P} \to \mathfrak{P}$ be the projection. Set $\mathfrak{U} := \mathfrak{Y} \times \mathfrak{P}$. Since $\mathfrak{D}$ satisfies BK_+, then $v_+(\mathcal{F}^{(\bullet)}) \in \mathfrak{D}(\mathfrak{U})$. Since $\mathcal{E}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{P})$ and g is smooth, this yields $v_+(\mathcal{F}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{U}}} g^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{U})$. Since $\mathfrak{C}$ satisfies $BK^!$, this implies $v^! \left(v_+(\mathcal{F}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{U}}} g^!(\mathcal{E}^{(\bullet)}) \right) \in \mathfrak{C}(\mathfrak{Y})$. Since $v^! \left(v_+(\mathcal{F}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{U}}} g^!(\mathcal{E}^{(\bullet)}) \right) \xrightarrow{\sim} v^! v_+(\mathcal{F}^{(\bullet)}) \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} v^! g^!(\mathcal{E}^{(\bullet)})[r]$ with r an integer (see 9.2.1.27.1), since $v^! v_+(\mathcal{F}^{(\bullet)}) \xrightarrow{\sim} \mathcal{F}^{(\bullet)}$ (see Berthelot-Kashiwara theorem 9.3.5.13), since $\mathfrak{C}$ is stable under shifts, since by transitivity $v^! g^! \xrightarrow{\sim} f^! u^!$, then we get $\mathcal{F}^{(\bullet)} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^! u^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$. □

Proposition 19.1.2.12. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients. We suppose $\mathfrak{D}$ contains $\mathfrak{B}_{\mathrm{div}}$, satisfies Δ-BK_+, and is Δ-stable under extraordinary pullbacks and tensor products (resp. $\mathfrak{D}$ contains $\mathfrak{B}_{\emptyset}$, satisfies Δ-BK_+, and is Δ-stable under extraordinary pullbacks and local cohomological functors). We suppose $\mathfrak{C}$ is local, satisfies $BK^!$, is stable under devissages, pushforwards, and direct summands.*

In that case, the data of coefficients $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is local, stable under devissages, direct summands, local cohomological functors, extraordinary pullbacks, pushforwards ($S(\mathfrak{D}, \mathfrak{C})$ is moreover stable under base change).

Proof. Let us check the non-respective case. Using 19.1.1.10, $\Delta(\mathfrak{D})$ satisfies the same properties than $\mathfrak{D}$ without the symbol Δ. Following 19.1.2.9.3, since $\mathfrak{C}$ is stable under devissage, then $S_\sharp(\mathfrak{D}, \mathfrak{C}) = S_\sharp(\Delta(\mathfrak{D}), \mathfrak{C})$. Hence, we reduce to the case $\Delta(\mathfrak{D}) = \mathfrak{D}$. Using 19.1.2.11 (except 3), we get $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is local, is stable under devissages, direct summands, smooth extraordinary pullbacks, pushforwards, satisfies $BK^!$ and $S(\mathfrak{D}, \mathfrak{C})$ is moreover stable under base change. Using 19.1.2.6, we check $\mathfrak{D}$ is stable under localizations outside a divisor. This yields by 19.1.2.4.(b) that $\mathfrak{D}$ is stable under under local cohomological functors. Hence, using 19.1.2.11.3 so is $S_\sharp(\mathfrak{D}, \mathfrak{C})$. By applying 19.1.2.5 this yields that $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is stable under extraordinary pullbacks. Similarly, the respective case is a straightforward consequence of 19.1.2.5 and 19.1.2.11. □

Corollary 19.1.2.13. *Let $i \in \mathbb{N} \cup \{\infty\}$. The data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$, resp. $\mathfrak{H}_i$) contains $\mathfrak{B}_{\mathrm{cst}}$, is local, stable under devissages, direct summands, local cohomological functors, extraordinary pullbacks, pushforwards (resp. and base change). Moreover, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}$ is stable under duality.*

Proof. For any $i \in \mathbb{N}$, since $\mathfrak{H}_{i+1} \subset \mathfrak{H}_i \cap \mathfrak{H}_i^\vee$, we get the stability under duality of $\mathfrak{H}_\infty$. Using 19.1.2.10.1, we get $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}} = S_0(\mathfrak{B}^+_{\mathrm{cst}}, \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}} = S(\mathfrak{B}^+_{\mathrm{cst}}, \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}})$, resp. $S(\mathfrak{B}_{\mathrm{div}}, \mathfrak{H}_i^\vee) = S(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{H}_i^\vee)$). Since $\mathfrak{B}^+_{\mathrm{cst}}$ satisfies BK_+, and is stable under local cohomological functors, extraordinary pullbacks and tensor products (see 19.1.1.6), since $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$ is local, satisfies $BK^!$, is stable under devissages, pushforwards, direct summands then we conclude by applying 19.1.2.7 and 19.1.2.12.

□

Notation 19.1.2.14. Let $\mathfrak{C}$ be a data of coefficients. We denote by $\mathfrak{C}^0$ the data of coefficients defined as follows. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$. Then $\mathfrak{C}^0(\mathfrak{X}) := \mathfrak{C}(\mathfrak{X}) \cap \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$.

Lemma 19.1.2.15. *Let $\mathfrak{C}$ be a data of coefficients.*

(a) If $\mathfrak{C}$ is stable under cohomology, then $\Delta(\mathfrak{C}) = \Delta(\mathfrak{C}^0)$.

(b) If $\mathfrak{C}$ is stable under devissages and cohomology, then for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{C}^0(\mathfrak{X})$ is an abelian strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ which is stable under extensions (i.e. is a weak Serre subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ in the sense of [Sta22, 02MP]).

(c) If for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\mathfrak{C}^0(\mathfrak{X})$ contains the zero objects, is an abelian strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ which is stable under extensions, then $\mathfrak{D} := \Delta(\mathfrak{C}^0)$ is stable under devissages and cohomology and $\mathfrak{D}^0 = \mathfrak{C}^0$.

Proof. a) Since $\Delta(\mathfrak{C}^0) \subset \Delta(\mathfrak{C})$, it remains to check $\mathfrak{C} \subset \Delta(\mathfrak{C}^0)$. Let $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$. By using some exact triangles of truncations (for the canonical t-structure on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ as explained in 13.1.4.18), we check $\mathcal{E}^{(\bullet)} \in \Delta_n(\mathfrak{C}^0)(\mathfrak{X})$ where n is the cardinal of $\{j \in \mathbb{Z}\ ;\ H^j(\mathcal{E}^{(\bullet)}) \neq 0\}$.

b) Let $f\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\mathfrak{C}^0(\mathfrak{X})$. By considering the mapping cone of f and by using the stability properties of $\mathfrak{C}$, we check that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are objects of $\mathfrak{C}^0(\mathfrak{X})$. The stability under extensions is obvious. Hence, we are done.

c) Denote by $\mathfrak{D}(\mathfrak{X})$ the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ consisting of objects $\mathcal{E}^{(\bullet)}$ such that, for any integer n, $H^n(\mathcal{E}^{(\bullet)})$ is an object of $\mathfrak{C}^0(\mathfrak{X})$. By devissage, we get $\Delta(\mathfrak{C}^0) \supset \mathfrak{D}(\mathfrak{X})$. Conversely, since $\mathfrak{C}^0(\mathfrak{X})$ is a weak Serre category (see [Sta22, 0754]), then for any exact triangle of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ whose two objects belong to $\mathfrak{D}(\mathfrak{X})$, using the induced long exact sequence, then we prove that the third object is an object of $\mathfrak{D}(\mathfrak{X})$, i.e. $\mathfrak{D}(\mathfrak{X})$ is stable under devissage. Since $\mathfrak{C}^0 \subset \mathfrak{D}(\mathfrak{X})$, this yields that $\Delta(\mathfrak{C}^0) \subset \mathfrak{D}(\mathfrak{X})$. Hence, we have proved $\mathfrak{D}(\mathfrak{X}) = \Delta(\mathfrak{C}^0)$. This yields that $\Delta(\mathfrak{C}^0)$ is stable under devissage and cohomology. □

Proposition 19.1.2.16. *Let $\mathfrak{X}$ be a smooth $\mathcal{V}$-formal scheme. Let $\mathfrak{A}$ be an abelian strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ containing the zero object. We denote by $\overline{\mathfrak{A}}$ the (strictly) full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ consisting of object $\mathcal{E}^{(\bullet)}$ such that there exists a filtration $0 = \mathcal{E}^{(\bullet)}_0 \subset \mathcal{E}^{(\bullet)}_1 \subset \dots \mathcal{E}^{(\bullet)}_r = \mathcal{E}^{(\bullet)}$ of length $r \geq 1$ of $\mathcal{E}^{(\bullet)}$ by objects $\mathcal{E}^{(\bullet)}_i$ of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ such that $\mathcal{E}^{(\bullet)}_{i+1}/\mathcal{E}^{(\bullet)}_i$ belong to $\mathfrak{A}$ for any $0 \leq i \leq r-1$. Then $\overline{\mathfrak{A}}$ is the smallest weak Serre subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ containing $\mathfrak{A}$ (in the sense of [Sta22, 02MP]).*

Proof. 0) $\overline{\mathfrak{A}}$ contains $\mathfrak{A}$.

1) We easily check that if $0 \to \mathcal{F}^{(\bullet)} \to \mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to 0$ is an exact sequence of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$, if $\mathcal{F}^{(\bullet)} \in \overline{\mathfrak{A}}$ admits a filtration of length r and $\mathcal{G}^{(\bullet)} \in \overline{\mathfrak{A}}$ admits a filtration of length s then $\mathcal{E}^{(\bullet)} \in \overline{\mathfrak{A}}$ admits a filtration of length $r+s$. In particular, $\overline{\mathfrak{A}}$ is closed under extension.

2) Let $\phi\colon \mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}$ be a morphism of $\overline{\mathfrak{A}}$. Let us prove $\ker \phi \in \overline{\mathfrak{A}}$. Set $\mathcal{K}^{(\bullet)} := \ker \phi$.

a) Suppose $\mathcal{F}^{(\bullet)} \in \mathfrak{A}$. Let $0 = \mathcal{E}^{(\bullet)}_0 \subset \mathcal{E}^{(\bullet)}_1 \subset \dots \mathcal{E}^{(\bullet)}_r = \mathcal{E}^{(\bullet)}$ be a filtration of $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}_{i+1}/\mathcal{E}^{(\bullet)}_i \in \mathfrak{A}$ for any $0 \leq i \leq r-1$. We proceed by induction on r. The case $r=1$, i.e. $\mathcal{E}^{(\bullet)} \in \mathfrak{A}$ follows from the abelianity of $\mathfrak{A}$. Suppose $r \geq 2$ and the property checked for $r-1$. Let $\mathcal{K}^{(\bullet)}_1$ be the kernel of the map $\mathcal{E}^{(\bullet)}_1 \to \mathcal{F}^{(\bullet)}$ induced by ϕ. Then $\mathcal{K}^{(\bullet)}_1 = \mathcal{K}^{(\bullet)} \cap \mathcal{E}^{(\bullet)}_1 \in \mathfrak{A}$. By induction hypothesis, $\mathcal{K}^{(\bullet)}/\mathcal{K}^{(\bullet)}_1$, the kernel of the map $\mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)}_1 \to \mathcal{F}^{(\bullet)}/\phi(\mathcal{E}^{(\bullet)}_1)$ induced by ϕ, belongs to $\overline{\mathfrak{A}}$. Hence, using the step 1), this yields $\mathcal{K}^{(\bullet)} \in \overline{\mathfrak{A}}$.

b) Suppose $\mathcal{E}^{(\bullet)} \in \mathfrak{A}$. Let $0 = \mathcal{F}^{(\bullet)}_0 \subset \mathcal{F}^{(\bullet)}_1 \subset \dots \mathcal{F}^{(\bullet)}_s = \mathcal{F}^{(\bullet)}$ be a filtration of $\mathcal{F}^{(\bullet)}$ such that $\mathcal{F}^{(\bullet)}_{j+1}/\mathcal{F}^{(\bullet)}_j \in \mathfrak{A}$ for any $0 \leq j \leq s-1$. We proceed by induction in s. When $s=1$, this follows from the abelianity of $\mathfrak{A}$. Suppose $s \geq 2$. By induction hypothesis, $\mathcal{E}^{(\bullet)}_1 := \ker \left(\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}/\mathcal{F}^{(\bullet)}_1\right) \in \overline{\mathfrak{A}}$. Then $\mathcal{K}^{(\bullet)}$ is equal to the kernel of the map $\mathcal{E}^{(\bullet)}_1 \to \mathcal{F}^{(\bullet)}_1$. Since $\mathcal{F}^{(\bullet)}_1 \in \mathfrak{A}$ and $\mathcal{E}^{(\bullet)}_1 \in \overline{\mathfrak{A}}$, then following a), we get $\mathcal{K}^{(\bullet)} \in \overline{\mathfrak{A}}$.

c) Let us treat now the general case. Let $0 = \mathcal{E}^{(\bullet)}_0 \subset \mathcal{E}^{(\bullet)}_1 \subset \dots \mathcal{E}^{(\bullet)}_r = \mathcal{E}^{(\bullet)}$ be a filtration of $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}_{i+1}/\mathcal{E}^{(\bullet)}_i \in \mathfrak{A}$ for any $0 \leq i \leq r-1$. We proceed by induction on r. The case $r=1$, i.e. $\mathcal{E}^{(\bullet)} \in \mathfrak{A}$, is the step b). Suppose $r \geq 2$ and the property holds for $r-1$. Let $\mathcal{K}^{(\bullet)}_1 = \mathcal{K}^{(\bullet)} \cap \mathcal{E}^{(\bullet)}_1$ be the kernel of the map $\mathcal{E}^{(\bullet)}_1 \to \mathcal{F}^{(\bullet)}$ induced by ϕ. Then following b) we have $\mathcal{K}^{(\bullet)}_1 \in \overline{\mathfrak{A}}$. By induction hypothesis, $\mathcal{K}^{(\bullet)}/\mathcal{K}^{(\bullet)}_1$, the kernel of the map $\mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)}_1 \to \mathcal{F}^{(\bullet)}/\phi(\mathcal{E}^{(\bullet)}_1)$ induced by ϕ, belongs to $\overline{\mathfrak{A}}$. Hence, using the step 1), this yields $\mathcal{K}^{(\bullet)} \in \overline{\mathfrak{A}}$.

3) Let $\mathcal{E}^{(\bullet)} \subset \mathcal{F}^{(\bullet)}$ be a monomorphism of $\overline{\mathfrak{A}}$. Let us check that $\mathcal{F}^{(\bullet)}/\mathcal{E}^{(\bullet)} \in \overline{\mathfrak{A}}$.

a) Suppose $\mathcal{E}^{(\bullet)} \in \mathfrak{A}$. Let $0 = \mathcal{F}^{(\bullet)}_0 \subset \mathcal{F}^{(\bullet)}_1 \subset \dots \mathcal{F}^{(\bullet)}_s = \mathcal{F}^{(\bullet)}$ be a filtration of $\mathcal{F}^{(\bullet)}$ such that $\mathcal{F}^{(\bullet)}_{j+1}/\mathcal{F}^{(\bullet)}_j \in \mathfrak{A}$ for any $0 \leq j \leq s-1$. We proceed by induction in s. Since $\mathcal{E}^{(\bullet)} \cap \mathcal{F}^{(\bullet)}_{s-1}$ is the kernel of the morphism $\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}/\mathcal{F}^{(\bullet)}_{s-1}$ of $\mathfrak{A}$, then $\mathcal{E}^{(\bullet)} \cap \mathcal{F}^{(\bullet)}_{s-1}$ belongs to $\mathfrak{A}$. Hence, $(\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)})/\mathcal{F}^{(\bullet)}_{s-1} \xleftarrow{\sim} \mathcal{E}^{(\bullet)}/\mathcal{E}^{(\bullet)} \cap \mathcal{F}^{(\bullet)}_{s-1} \in \mathfrak{A}$. Using the exact sequence $0 \to (\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)})/\mathcal{F}^{(\bullet)}_{s-1} \to \mathcal{F}^{(\bullet)}/\mathcal{F}^{(\bullet)}_{s-1} \to \mathcal{F}^{(\bullet)}/(\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)}) \to 0$, this yields that $\mathcal{F}^{(\bullet)}/(\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)}) \in \mathfrak{A}$. Via the exact sequence $0 \to (\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)})/\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}/\mathcal{E}^{(\bullet)} \to \mathcal{F}^{(\bullet)}/(\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)}) \to 0$, since $(\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)})/\mathcal{E}^{(\bullet)} \xleftarrow{\sim} \mathcal{F}^{(\bullet)}_{s-1}/(\mathcal{E}^{(\bullet)} \cap \mathcal{F}^{(\bullet)}_{s-1}) \in \overline{\mathfrak{A}}$ (induction hypothesis), since $\mathcal{F}^{(\bullet)}/(\mathcal{F}^{(\bullet)}_{s-1} + \mathcal{E}^{(\bullet)}) \in \mathfrak{A}$, then we get from the step 1) that $\mathcal{F}^{(\bullet)}/\mathcal{E}^{(\bullet)} \in \overline{\mathfrak{A}}$.

b) Let us go back to the general case. Let $0 = \mathcal{E}^{(\bullet)}_0 \subset \mathcal{E}^{(\bullet)}_1 \subset \dots \mathcal{E}^{(\bullet)}_r = \mathcal{E}^{(\bullet)}$ be a filtration of $\mathcal{E}^{(\bullet)}$ such that $\mathcal{E}^{(\bullet)}_{i+1}/\mathcal{E}^{(\bullet)}_i \in \mathfrak{A}$ for any $0 \leq i \leq r-1$. Using the exact sequence $0 \to \mathcal{F}^{(\bullet)}/\mathcal{E}^{(\bullet)}_1 \to \mathcal{F}^{(\bullet)}/\mathcal{E}^{(\bullet)} \to$

$\mathcal{E}^{(\bullet)}/\mathcal{E}_1^{(\bullet)} \to 0$, since $\mathcal{E}^{(\bullet)}/\mathcal{E}_1^{(\bullet)} \in \overline{\mathfrak{A}}$, since following the step a) we have $\mathcal{F}^{(\bullet)}/\mathcal{E}_1^{(\bullet)} \in \overline{\mathfrak{A}}$, then using the step 1) we get $\mathcal{F}^{(\bullet)}/\mathcal{E}^{(\bullet)} \in \overline{\mathfrak{A}}$.

4) It follows from the steps 2) and 3) that for any morphism ϕ of $\overline{\mathfrak{A}}$, $\mathrm{Coker}\,\phi \in \overline{\mathfrak{A}}$. □

Proposition 19.1.2.17. *Let $\mathfrak{A}$ be a data of coefficients over $\mathcal{V}$ such that for any object $\mathcal{W}$ of* $\mathrm{DVR}(\mathcal{V})$ *(see notation 9.2.6.12), for any smooth $\mathcal{W}$-formal scheme $\mathfrak{X}$, $\mathfrak{A}(\mathfrak{X})$ is an abelian strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$. Then, $\Delta(\mathfrak{A})$ is the smallest data of coefficients containing $\mathfrak{A}$ and stable under devissage and cohomology.*

Proof. Let $\overline{\mathfrak{A}}$ be the data of coefficients over $\mathcal{V}$ such that for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$ (see notation 9.2.6.12), for any smooth $\mathcal{W}$-formal scheme $\mathfrak{X}$, $\overline{\mathfrak{A}}(\mathfrak{X})$ is the smallest weak Serre subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$ containing $\mathfrak{A}(\mathfrak{X})$ (see 19.1.2.16). Since $\mathfrak{A} \subset \overline{\mathfrak{A}}$, then $\Delta(\overline{\mathfrak{A}}) \supset \Delta(\mathfrak{A})$. Moreover, since $\overline{\mathfrak{A}} \subset \Delta(\mathfrak{A})$, then $\Delta(\overline{\mathfrak{A}}) \subset \Delta(\mathfrak{A})$. Hence, $\Delta(\overline{\mathfrak{A}}) = \Delta(\mathfrak{A})$. The proposition is therefore an obvious consequence of 19.1.2.16 and 19.1.2.15. □

Proposition 19.1.2.18. *Let $\mathfrak{A}$ be a data of coefficients over $\mathcal{V}$ such that for any object $\mathcal{W}$ of* $\mathrm{DVR}(\mathcal{V})$ *(see notation 9.2.6.12), for any smooth $\mathcal{W}$-formal scheme $\mathfrak{X}$, $\mathfrak{A}(\mathfrak{X})$ is an abelian strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$. We denote by $\overline{\mathfrak{A}}_{\mathrm{loc}}$ the data of coefficients over $\mathcal{V}$ such that for any object $\mathcal{W}$ of* $\mathrm{DVR}(\mathcal{V})$*, for any smooth $\mathcal{W}$-formal scheme $\mathfrak{X}$, $\overline{\mathfrak{A}}_{\mathrm{loc}}(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$ consisting of objects which are locally in $\mathfrak{X}$ in the smallest weak Serre subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$ containing $\mathfrak{A}(\mathfrak{X})$ (see 19.1.2.16). Then, $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})$ is the smallest data of coefficients containing $\mathfrak{A}$ which is local and stable under devissage and cohomology.*

Proof. By definition, $\overline{\mathfrak{A}}_{\mathrm{loc}}$ is local. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$ and $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$. Let $0 \to \mathcal{F}^{(\bullet)} \to \mathcal{E}^{(\bullet)} \to \mathcal{G}^{(\bullet)} \to 0$ be an exact sequence of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$ such that $\mathcal{F}^{(\bullet)}, \mathcal{G}^{(\bullet)} \in \overline{\mathfrak{A}}_{\mathrm{loc}}(\mathfrak{X})$. Since the property $\mathcal{E}^{(\bullet)} \in \overline{\mathfrak{A}}_{\mathrm{loc}}(\mathfrak{X})$ is local, we can suppose that $\mathcal{F}^{(\bullet)}, \mathcal{G}^{(\bullet)} \in \overline{\mathfrak{A}}(\mathfrak{X})$ (see notation of the proof of 19.1.2.17). Hence, following 19.1.2.16, $\mathcal{E}^{(\bullet)} \in \overline{\mathfrak{A}}_{\mathrm{loc}}(\mathfrak{X})$. Similarly, we get the remaining properties to check that $\overline{\mathfrak{A}}_{\mathrm{loc}}(\mathfrak{X})$ is a weak Serre subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$. Hence, following 19.1.2.15.(c), $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})$ is stable and under devissage and cohomology and $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})^0 = \overline{\mathfrak{A}}_{\mathrm{loc}}$. Hence, $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})(\mathfrak{X})$ is equal to the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$ consisting of objects $\mathcal{E}^{(\bullet)}$ such that, for any integer n, $H^n(\mathcal{E}^{(\bullet)})$ is an object of $\overline{\mathfrak{A}}_{\mathrm{loc}}(\mathfrak{X})$. Since $\overline{\mathfrak{A}}_{\mathrm{loc}}$ is local, this yields that so is $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})$. Hence, $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})$ is a data of coefficients containing $\mathfrak{A}$ which is local, stable and under devissage and cohomology. Let $\mathfrak{D}$ is a data of coefficients containing $\mathfrak{A}$ which is local, stable and under devissage and cohomology. Then $\mathfrak{D}$ must contains $\overline{\mathfrak{A}}_{\mathrm{loc}}$ and therefore $\Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})$. Hence, we are done. □

Notation 19.1.2.19. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$ and $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$. Let $F\text{-}\mathrm{H}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$ be the category of overholonomic $\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules endowed with a Frobenius structure. We get a data of coefficients $\mathfrak{A}$ over $\mathcal{V}$ by setting $\mathfrak{A}(\mathfrak{X}) := F\text{-}\mathrm{H}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$. Since this is an abelian category then following 19.1.2.18 and according to its notation we get $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{h}} := \Delta(\overline{\mathfrak{A}}_{\mathrm{loc}})$ the smallest data of coefficients over $\mathcal{V}$ which is local and stable under devissage and cohomology and containing $\mathfrak{A}$. Then it follows from 18.3.2 that $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{h}}$ is local, stable by devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change, tensor products, duals, cohomology. Such objects are called of Frobenius type. [1]

Lemma 19.1.2.20. *Let $\mathfrak{C}$ be a data of coefficients stable under devissages and cohomology and which is local. Let $\mathcal{W}$ be an object of* $\mathrm{DVR}(\mathcal{V})$*, $\mathfrak{T} := \mathrm{Spf}\ \mathcal{W}$, $\mathfrak{X}$ and $\mathfrak{Y}$ be two smooth formal schemes over $\mathcal{W}$, $\mathcal{E}^{0(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{X}}^{(\bullet)})$, and $\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}_{\mathfrak{Y}}^{(\bullet)})$. Recall exterior tensor products is defined in 9.2.5.8.1.*

(a) The following properties are equivalent

(i) for any $n \in \mathbb{Z}$, $\mathcal{E}^{0(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}} H^n(\mathcal{F}^{(\bullet)}) \in \mathfrak{C}^0(\mathfrak{X})$;

(ii) $\mathcal{E}^{0(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}} \mathcal{F}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$.

(b) If for any $n \in \mathbb{Z}$ we have $\mathcal{E}^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}} H^n(\mathcal{F}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{X})$, then $\mathcal{E}^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}} \mathcal{F}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X})$.

[1] This has previously been called 'F-able' in the literature in the context of overconvergent isocrystal.

Proof. Since this is local, we can suppose $\mathfrak{X}$ affine. The first statement is an obvious consequence of 9.2.5.12.(b). Following 8.4.5.11, there exists $\mathcal{G}^{(\bullet)} \in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{T}}))$ such that $\mathcal{G}^{(\bullet)}$ is isomorphic to $\mathfrak{e}(\mathcal{E}^{(\bullet)})$ (resp. $\mathcal{H}^{(\bullet)} \in D^{\mathrm{b}}(\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{Y}/\mathfrak{T}}))$ such that $\mathcal{H}^{(\bullet)}$ is isomorphic to $\mathfrak{e}(\mathcal{F}^{(\bullet)})$ where $\mathfrak{e}$ is defined at 8.1.5.14.1). Following 9.2.5.12.(a), we get the spectral sequence in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}\times\mathfrak{Y}/\mathfrak{T}})$ of the form $H^r(\mathcal{G}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}} H^s(\mathcal{H}^{(\bullet)}) =: E_2^{r,s} \Rightarrow E^n := H^n\left(\mathcal{G}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}}\mathcal{H}^{(\bullet)}\right)$. Since $\mathfrak{C}^0(\mathfrak{X}\times\mathfrak{Y})$ is an abelian strictly full subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widetilde{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}\times\mathfrak{Y}/\mathfrak{S}})$ closed under extensions (see 19.1.2.15.b), since $H^r(\mathcal{G}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}} H^s(\mathcal{H}^{(\bullet)}) \in \mathfrak{C}^0(\mathfrak{X}\times\mathfrak{Y})$ (use 8.4.5.7.1 and the part (a) of the Lemma), then $H^n\left(\mathcal{G}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}}\mathcal{H}^{(\bullet)}\right) \in \mathfrak{C}^0(\mathfrak{X}\times\mathfrak{Y})$. Using 9.2.5.11.1 and 8.4.5.7.1, this yields $H^n\left(\mathcal{E}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{T}}}\mathcal{F}^{(\bullet)}\right) \in \mathfrak{C}^0(\mathfrak{X}\times\mathfrak{Y})$.

□

Proposition 19.1.2.21. *Let $\mathfrak{C}$ be a data of coefficients. Suppose that $\mathfrak{C}$ is stable under cohomology, and devissage. Then $S_\sharp(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})$ is stable under devissages and cohomology.*

Proof. Following 19.1.2.10, $S_\sharp(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C}) = S_\sharp(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C})$. Since localizations outside a divisor and the functor $f^{(\bullet)*}$ when f is any smooth morphism are t-exact (for the canonical t-structure of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$), then this is straightforward. □

Corollary 19.1.2.22. *The data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}$, and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$ are stable under cohomology.*

Proposition 19.1.2.23. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients. Suppose that $\mathfrak{D}$ is stable under cohomology, smooth extraordinary pullbacks, and that $\mathfrak{C}$ is local and stable under cohomology, devissage, extraordinary pullbacks. Then $S_\sharp(\mathfrak{D}, \mathfrak{C})$ is stable under devissages and cohomology.*

Proof. 1) We prove the case where $\sharp = 0$. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a smooth morphism of smooth $\mathcal{W}$-formal schemes, $\mathcal{F}^{(\bullet)} \in \mathfrak{D}(\mathfrak{Y})$.

a) Suppose that $\mathcal{E}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$. Since $\mathfrak{D}$ is stable under cohomology and smooth extraordinary pullbacks, then $H^r(\mathcal{F}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{W}} f^!(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y}\times_{\mathrm{Spf}\,\mathcal{W}}\mathfrak{Y})$, for any $r \in \mathbb{Z}$. Since $\mathfrak{C}$ is local and is stable under cohomology and devissage, then using 19.1.2.20.a, we get $H^r(\mathcal{F}^{(\bullet)})\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{W}} H^s(f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) \in \mathfrak{C}^0(\mathfrak{Y}\times_{\mathrm{Spf}\,\mathcal{W}}\mathfrak{Y})$, for any $r, s \in \mathbb{Z}$. Hence, since $H^s(f^{(\bullet)!}(\mathcal{E}^{(\bullet)})) \xrightarrow{\sim} f^{*(\bullet)}H^{s-d_f}(\mathcal{E}^{(\bullet)})$, using 19.1.2.20.a, we get $\mathcal{F}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{W}} f^{*(\bullet)}H^s(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y}\times_{\mathrm{Spf}\,\mathcal{W}}\mathfrak{Y})$, for any $s \in \mathbb{Z}$. Since $\mathfrak{C}$ is stable under extraordinary pullbacks and shifts, this yields $\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^{*(\bullet)}H^s(\mathcal{E}^{(\bullet)})) \xrightarrow{\sim} \mathbb{L}\delta^{*(\bullet)}(\mathcal{F}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{W}} f^{*(\bullet)}H^s(\mathcal{E}^{(\bullet)})) \in \mathfrak{C}(\mathfrak{Y})$, where $\delta\colon \mathfrak{Y} \hookrightarrow \mathfrak{Y}\times_{\mathrm{Spf}\,\mathcal{W}}\mathfrak{Y}$ is the diagonal immersion. Hence, $H^s(\mathcal{E}^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$, for any $s \in \mathbb{Z}$.

b) Conversely, suppose $H^s(\mathcal{E}^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$, for any $s \in \mathbb{Z}$. Then $\mathcal{F}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{W}} f^{*(\bullet)}H^s(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y}\times_{\mathrm{Spf}\,\mathcal{W}}\mathfrak{Y})$. Using 19.1.2.20.b, this yields $\mathcal{F}^{(\bullet)}\widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{W}} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y}\times_{\mathrm{Spf}\,\mathcal{W}}\mathfrak{Y})$. Hence, $\mathcal{F}^{(\bullet)}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{Y}}} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{Y})$.

2) For any morphism $\mathcal{W} \to \widetilde{\mathcal{W}}$ of $\mathrm{DVR}(\mathcal{V})$, set $\widetilde{\mathfrak{X}} := \mathfrak{X}\times_{\mathrm{Spf}\,\mathcal{W}} \mathrm{Spf}\,\widetilde{\mathcal{W}}$, and $\widetilde{\mathcal{E}}^{(\bullet)} := \widetilde{\mathcal{W}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{W}}\mathcal{E}^{(\bullet)}$. The property $\mathcal{E}^{(\bullet)} \in S(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$ is equivalent to the property $\widetilde{\mathcal{E}}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\widetilde{\mathfrak{X}})$ (for any such morphism $\mathcal{W} \to \widetilde{\mathcal{W}}$). The property $\widetilde{\mathcal{E}}^{(\bullet)} \in S_0(\mathfrak{D}, \mathfrak{C})(\widetilde{\mathfrak{X}})$ is equivalent from the first part to $H^n(\widetilde{\mathcal{E}}^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\widetilde{\mathfrak{X}})$ for any $n \in \mathbb{Z}$. Since the functors H^n commute with base change, this latter property is equivalent to $\widetilde{\mathcal{W}}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{W}} H^n(\mathcal{E}^{(\bullet)}) \in S_0(\mathfrak{D}, \mathfrak{C})(\widetilde{\mathfrak{X}})$ for any $n \in \mathbb{Z}$, i.e. $H^n(\mathcal{E}) \in S(\mathfrak{D}, \mathfrak{C})(\widetilde{\mathfrak{X}})$ for any $n \in \mathbb{Z}$.

□

19.1.2.24. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. Following 9.2.4.20.1, we have the dual functor $\mathbb{D}^{(\bullet)}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$. Similarly to 15.2.4.8, we say that $\mathcal{E}^{(\bullet)}$ is holonomic if for any $i \neq 0$, $H^i(\mathbb{D}^{(\bullet)}(\mathcal{E}^{(\bullet)})) = 0$. We denote by $\underrightarrow{LM}_{\mathbb{Q},\mathrm{hol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ the strictly subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ of holonomic $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}$-modules. By copying 15.2.4.14, we check $\underrightarrow{LM}_{\mathbb{Q},\mathrm{hol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ is in fact a Serre subcategory of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$.

We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{hol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ the strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ consisting of complexes $\mathcal{E}^{(\bullet)}$ such that $H^n\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{hol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$ for any $n \in \mathbb{Z}$. This yields the t-exact equivalence of categories $\mathbb{D}^{(\bullet)}\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{hol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}}) \cong \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{hol}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}/\mathfrak{S}})$. It follows from 15.3.5.4 that $\underrightarrow{LM}_{\mathbb{Q},\mathrm{oc}} \subset \underrightarrow{LM}_{\mathbb{Q},\mathrm{hol}}$. This yields

$$\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}} \subset \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{hol}}. \qquad (19.1.2.24.1)$$

Lemma 19.1.2.25. *Let $\mathfrak{C}$ be a data of coefficients stable under cohomology and included in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$. Then $\mathfrak{C}^\vee$ is stable under cohomology.*

Proof. This is a straightforward consequence of 19.1.2.24.1. □

Corollary 19.1.2.26. *Let $i \in \mathbb{N} \cup \{\infty\}$. The data of coefficients $\mathfrak{H}_i$ is stable under cohomology.*

19.1.2.27. Let $\mathfrak{C}$ be a data of coefficients stable under devissages and cohomology. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$. We get a canonical t-structure on $\mathfrak{C}(\mathfrak{X}/\mathcal{W})$ whose heart is $\mathfrak{C}^0(\mathfrak{X}/\mathcal{W})$ and so that the t-structure of $\mathfrak{C}(\mathfrak{X}/\mathcal{W})$ is induced by that of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ (which is defined at 13.1.4.18), i.e. the truncation functors are the same and $\mathfrak{C}^{\geq n}(\mathfrak{X}/\mathcal{W}) := \underrightarrow{LD}^{\geq n}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \cap \mathfrak{C}(\mathfrak{X}/\mathcal{W})$, $\mathfrak{C}^{\leq n}(\mathfrak{X}/\mathcal{W}) := \underrightarrow{LD}^{\leq n}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \cap \mathfrak{C}(\mathfrak{X}/\mathcal{W})$.

For instance, using 19.1.2.22 and 19.1.2.26, we get for $\star \in \{\mathrm{ovcoh}, \mathrm{oc}, \mathrm{h}, \mathrm{hol}\}$ a canonical t-structure on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}$. The heart of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\star}$ is $\underrightarrow{LM}^{\mathrm{b}}_{\mathbb{Q},\star}$.

Corollary 19.1.2.28. *The data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}$, and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$ are stable under special descent of the base.*

Definition 19.1.2.29. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients.

(a) We denote by $\boxtimes_0(\mathfrak{D}, \mathfrak{C})$ the data of coefficients defined as follows: for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\boxtimes_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ consisting of objects $\mathcal{E}^{(\bullet)}$ satisfying the following property:

   ($\star$) for any smooth $\mathcal{W}$-formal scheme $\mathfrak{Y}$, for any object $\mathcal{F}^{(\bullet)} \in \mathfrak{D}(\mathfrak{Y})$, we have $\mathcal{E}^{(\bullet)} \widehat{\boxtimes}^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Spf}\,\mathcal{W}}} \mathcal{F}^{(\bullet)} \in \mathfrak{C}(\mathfrak{X} \times_{\mathrm{Spf}\,\mathcal{W}} \mathfrak{Y})$.

(b) We denote by $\boxtimes(\mathfrak{D}, \mathfrak{C})$ the data of coefficients defined as follows: for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$, for any smooth formal scheme $\mathfrak{X}$ over $\mathcal{W}$, the category $\boxtimes(\mathfrak{D}, \mathfrak{C})(\mathfrak{X})$ is the full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ consisting of objects $\mathcal{E}^{(\bullet)}$ satisfying the following property:

   ($\star\star$) for any morphism $\mathcal{W} \to \mathcal{W}'$ of $\mathrm{DVR}(\mathcal{V})$, $\mathcal{W}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{W}} \mathcal{E}^{(\bullet)} \in \boxtimes_0(\mathfrak{D}, \mathfrak{C})(\mathfrak{X} \times_{\mathrm{Spf}\,\mathcal{W}} \mathrm{Spf}\,\mathcal{W}')$.

(c) Let $\sharp$ be a symbol so that either $\boxtimes_\sharp = \boxtimes_0$ or $\boxtimes_\sharp = \boxtimes$.

Lemma 19.1.2.30. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two data of coefficients.*

(a) If $\mathfrak{D}$ contains $\mathfrak{B}_\emptyset$, then $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C})$ is contained in $\mathfrak{C}$.

(b) If $\mathfrak{C} \subset \mathfrak{C}'$ and $\mathfrak{D}' \subset \mathfrak{D}$, then $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C}) \subset \boxtimes_\sharp(\mathfrak{D}', \mathfrak{C}')$.

(c) If $\mathfrak{C}$ is stable under devissage then so is $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C})$. Moreover, $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C}) = \boxtimes_\sharp(\Delta(\mathfrak{D}), \mathfrak{C})$.

(d) The data $\boxtimes(\mathfrak{D}, \mathfrak{C})$ is stable under base change.

(e) If $\mathfrak{C}$ is stable under pushforwards (resp. satisfies $BK^!$, resp. is local, resp. is stable under direct summands), then so is $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C})$.

(f) If $\mathfrak{C}$ is local and is stable under devissages and cohomology, and if $\mathfrak{D}$ is stable under cohomology, then $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C})$ is stable under cohomology.

Proof. The first forth statements are obvious. The non-respective case and the first respective case of the fifth one is a consequence of 9.4.4.1. The other cases are obvious. It remains to check the sixth one. Following 19.1.2.15.(a), since $\mathfrak{D}$ is stable under cohomology, then $\Delta(\mathfrak{D}) = \Delta(\mathfrak{D}^0)$. Hence, from the third part of our lemma, we get $\boxtimes_\sharp(\mathfrak{D}, \mathfrak{C}) = \boxtimes_\sharp(\mathfrak{D}^0, \mathfrak{C})$. Then, we conclude by using 19.1.2.20.(a). □

19.1.3 Constructions of stable data of coefficients

Following 19.1.2.19, we have already a data of coefficients stable under Grothendieck operations for objects of "Frobenius type". However, we would like such stability for more general coefficients, e.g. containing convergent isocrystals (without Frobenius structure).

Definition 19.1.3.1. Let $\mathfrak{D}$ be a data of coefficients over $\mathcal{V}$. We say that $\mathfrak{D}$ is almost stable under dual functors if the following property holds: for any data of coefficients $\mathfrak{C}$ over $\mathcal{V}$ which is local, stable under devissages, direct summands and pushforwards, if $\mathfrak{D} \subset \mathfrak{C}$ then $\mathfrak{D}^\vee \subset \mathfrak{C}$. Remark from the biduality isomorphism that the inclusion $\mathfrak{D}^\vee \subset \mathfrak{C}$ is equivalent to the following one $\mathfrak{D} \subset \mathfrak{C}^\vee$.

Lemma 19.1.3.2. *Let $\mathfrak{D}$ be a data of coefficients over $\mathcal{V}$. The data $\mathfrak{D}$ is almost stable under dual functors if and only if $\Delta(\mathfrak{D})$ is almost stable under dual functors.*

Proof. This is a consequence of 19.1.2.8. □

Lemma 19.1.3.3. *With notation 19.1.1.2, we have the equalities $\mathfrak{M}_\emptyset^\vee = \mathfrak{M}_\emptyset$, $(\Delta(\mathfrak{M}_\emptyset))^\vee = \Delta(\mathfrak{M}_\emptyset)$ and $\Delta(\mathfrak{M}_{\mathrm{sncd}}) = \Delta(\mathfrak{M}_\emptyset)$.*

Proof. The first equality is a consequence of 12.2.5.6. The second one follows from 19.1.2.8. It remains to check the inclusion $\mathfrak{M}_{\mathrm{sncd}} \subset \Delta(\mathfrak{M}_\emptyset)$. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$, Z be a smooth subvariety of the special fiber of $\mathfrak{X}$, T be a strict normal crossing divisor of Z, and $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(Z, \mathfrak{X}/K)$. We have to prove that $({}^\dagger T)(\mathcal{E}^{(\bullet)}) \in \Delta(\mathfrak{M}_\emptyset)(\mathfrak{X})$. We proceed by induction on the dimension of T and next the number of irreducible components of T. Let T_1 be one irreducible component of T and T' be the union of the other irreducible components. We have the localisation triangle

$$({}^\dagger T' \cap T_1)\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) \to ({}^\dagger T')(\mathcal{E}^{(\bullet)}) \to ({}^\dagger T)(\mathcal{E}^{(\bullet)}) \to +1 \qquad (19.1.3.3.1)$$

Following 12.2.1.9, we have $\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)})[1] \in \mathrm{MIC}^{(\bullet)}(T_1, \mathfrak{X}/K)$. Hence, since $T' \cap T_1$ is a strict normal crossing divisor of T_1, by induction hypothesis we get $({}^\dagger T' \cap T_1)\mathbb{R}\underline{\Gamma}^\dagger_{T_1}(\mathcal{E}^{(\bullet)}) \in \Delta(\mathfrak{M}_\emptyset)(\mathfrak{X})$. By induction hypothesis, we have also $({}^\dagger T')(\mathcal{E}^{(\bullet)}) \in \Delta(\mathfrak{M}_\emptyset)(\mathfrak{X})$. Hence, by devissage, we get $({}^\dagger T)(\mathcal{E}^{(\bullet)}) \in \Delta(\mathfrak{M}_\emptyset)(\mathfrak{X})$. □

Proposition 19.1.3.4. *The data of coefficients $\mathfrak{B}_{\mathrm{div}}$, $\mathfrak{M}_{\mathrm{div}}$, and $\mathfrak{B}_{\mathrm{cst}}$ are almost stable under duality.*

Proof. I) Since $\Delta(\mathfrak{B}_{\mathrm{cst}}) = \Delta(\mathfrak{B}_{\mathrm{div}})$ (see 19.1.2.4.a) and using 19.1.3.2, since the case $\mathfrak{B}_{\mathrm{div}}$ is checked similarly, we reduce to prove the almost dual stability of $\mathfrak{M}_{\mathrm{div}}$.

II) Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{M}_{\mathrm{div}}$, and which is local, stable under devissages, direct summands and pushforwards. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$, $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$, Z be a smooth subvariety of the special fiber of $\mathfrak{X}$, T be a divisor of Z, and $\mathcal{E}^{(\bullet)} \in \mathrm{MIC}^{(\bullet)}(Z, \mathfrak{X}/K)$ such that $({}^\dagger T)(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}^\vee(\mathfrak{X})$. We have to check that $({}^\dagger T)(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}^\vee(\mathfrak{X})$.

0) Let l be the residue field of $\mathcal{W}$. Since this is local in $\mathfrak{X}$, we can suppose Z integral. Following the de Jong 's desingularisation theorem (of the form 10.4.1.2), there exist a smooth integral l-variety Z', a projective generically finite and etale morphism of l-varieties $\phi\colon Z' \to Z$ such that Z' is quasi-projective and $T' := \phi^{-1}(T)$ is a strict normal crossing divisor of Z'. Hence, there exists a closed immersion of the form $u\colon Z' \hookrightarrow \mathbb{P}^n_Z$ whose composition with the projection $\mathbb{P}^n_Z \to Z$ is ϕ. Let $\mathfrak{X}' := \widehat{\mathbb{P}}^n_{\mathfrak{X}}$, $f\colon \mathfrak{X}' \to \mathfrak{X}$ be the projection. Following 16.1.11.2, $\mathcal{E}^{(\bullet)}$ is a direct summand of $f_+^{(\bullet)}\mathbb{R}\underline{\Gamma}^\dagger_{Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$. Hence, $({}^\dagger T)(\mathcal{E}^{(\bullet)})$ is a direct summand of $({}^\dagger T) f_+^{(\bullet)}\mathbb{R}\underline{\Gamma}^\dagger_{Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$ Using the commutation of localisation functor with pushforwards, this yields $({}^\dagger T)(\mathcal{E}^{(\bullet)})$ is a direct summand of $f_+^{(\bullet)}({}^\dagger T')\mathbb{R}\underline{\Gamma}^\dagger_{Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)})$.

1) Since $\mathcal{E}'^{(\bullet)} := \mathbb{R}\underline{\Gamma}^\dagger_{Z'} f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \in \mathrm{MIC}^{(\bullet)}(Z', \mathfrak{X}'/K)$ (use 12.2.1.9), then $({}^\dagger T')(\mathcal{E}'^{(\bullet)}) \in \mathfrak{M}_{\mathrm{sncd}}(\mathfrak{X}')$. Since $\mathfrak{C}$ contains $\mathfrak{M}_{\mathrm{sncd}}$ and is stable under devissages, then using 19.1.3.3 we get $\mathfrak{M}_{\mathrm{sncd}} \subset \mathfrak{C}^\vee$. Hence, $({}^\dagger T')(\mathcal{E}'^{(\bullet)}) \in \mathfrak{C}^\vee(\mathfrak{X}')$. Since $\mathfrak{C}$ is stable under direct summands and pushforwards, we are done. □

Notation 19.1.3.5. Let $\mathfrak{C}, \mathfrak{D}$ be two data of coefficients. We put $T_0(\mathfrak{D}, \mathfrak{C}) := S(\mathfrak{D}, \mathfrak{C})$. By induction on $i \in \mathbb{N}$, we set $U_i(\mathfrak{D}, \mathfrak{C}) := T_i(\mathfrak{D}, \mathfrak{C}) \cap T_i(\mathfrak{D}, \mathfrak{C})^\vee$, $\widetilde{T}_i(\mathfrak{D}, \mathfrak{C}) := S(\mathfrak{D}, U_i(\mathfrak{D}, \mathfrak{C}))$ and $T_{i+1}(\mathfrak{D}, \mathfrak{C}) := S(\widetilde{T}_i(\mathfrak{D}, \mathfrak{C}), \widetilde{T}_i(\mathfrak{D}, \mathfrak{C}))$. We put $T(\mathfrak{D}, \mathfrak{C}) := \cap_{i \in \mathbb{N}} T_i(\mathfrak{D}, \mathfrak{C})$.

Theorem 19.1.3.6. *Let $\mathfrak{B}_{\mathrm{div}} \subset \mathfrak{D} \subset \mathfrak{C}$ be two data of coefficients. We suppose*

1. *The data $\mathfrak{D}$ is stable under cohomology ;*

2. *The data $\Delta(\mathfrak{D})$ satisfies BK_+, is stable under extraordinary pullbacks, base change, tensor products and is almost stable under dual functors ;*

3. *The data $\mathfrak{C}$ satisfies $BK^!$, is local and stable under devissages, direct summands, pushforwards, cohomology.*

Then, the data of coefficients $T(\mathfrak{D}, \mathfrak{C})$ (see Definition 19.1.3.5) is included in $\mathfrak{C}$, contains $\mathfrak{D}$, is local, stable by devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change, tensor products, duals, cohomology.

Proof. I) First, we check by induction on $i \in \mathbb{N}$ that the data of coefficients $T_i(\mathfrak{D}, \mathfrak{C})$ contains $\mathfrak{D}$, is contained in $\mathfrak{C}$, is local, stable under devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change and cohomology (which implies such stability properties for $T(\mathfrak{D}, \mathfrak{C})$).

a) Let us verify that $T_0(\mathfrak{D}, \mathfrak{C})$ satisfies these properties. Using 19.1.2.9.3, we get from 19.1.2.9.1 (resp. 19.1.2.9.4) that $T_0(\mathfrak{D}, \mathfrak{C})$ is included in $\mathfrak{C}$ (resp. contains $\mathfrak{D}$). Using 19.1.1.6 and (the non-respective case of) 19.1.2.12, we check that $S(\mathfrak{D}, \mathfrak{C})$ and $S(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})$ are both local, stable under devissages, direct summands, local cohomological functors, extraordinary pullbacks, pushforwards and base change. Following 19.1.2.21 $S(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})$ is also stable under cohomology. Moreover, it follows from 19.1.2.23 that $S\left(\mathfrak{D}, S(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})\right)$ is also stable under cohomology. Using 19.1.2.10 (specially the left equality of 19.1.2.10.2), we get the equalities $S(\mathfrak{D}, \mathfrak{C}) = S(\Delta(\mathfrak{D}), \mathfrak{C}) = S\left(\Delta(\mathfrak{D}), S(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})\right) = S\left(\mathfrak{D}, S(\mathfrak{B}^+_{\mathrm{cst}}, \mathfrak{C})\right)$. Hence, we are done.

b) Suppose that the properties hold for $T_i(\mathfrak{D}, \mathfrak{C})$ for some $i \in \mathbb{N}$.

i) Since $\mathfrak{D}$ is almost stable under duals, then $U_i(\mathfrak{D}, \mathfrak{C})$ contains $\mathfrak{D}$. Since $\Delta(\mathfrak{D})$ is stable by tensor products, extraordinary pullbacks, and base change then, using 19.1.2.9.4 (where $\mathfrak{C}$ is replaced by $U_i(\mathfrak{D}, \mathfrak{C})$ which is stable under devissage), this implies that $\mathfrak{D}$ is contained in $\widetilde{T}_i(\mathfrak{D}, \mathfrak{C})$ and $T_{i+1}(\mathfrak{D}, \mathfrak{C})$. Using 19.1.2.9.1, we get that $\widetilde{T}_i(\mathfrak{D}, \mathfrak{C})$ and $T_{i+1}(\mathfrak{D}, \mathfrak{C})$ are included in $\mathfrak{C}$.

ii) From Lemma 19.1.2.7, $U_i(\mathfrak{D}, \mathfrak{C})$ satisfies $BK^!$, is local, stable under pushforwards, under devissages, direct summands, base change/ It follows from 19.1.2.9.1 that $T_i(\mathfrak{D}, \mathfrak{C})$ is included in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$. Using 19.1.2.25, this yields that $U_i(\mathfrak{D}, \mathfrak{C})$ is stable under cohomology. Hence, by applying the step I)a) in the case where $\mathfrak{C}$ is replaced by $U_i(\mathfrak{D}, \mathfrak{C})$, we get that $\widetilde{T}_i(\mathfrak{D}, \mathfrak{C})$ is local, stable under devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change and cohomology.

Using (the respective case of) 19.1.2.12, this yields that $T_{i+1}(\mathfrak{D}, \mathfrak{C})$ is local, stable under devissages, direct summands, local cohomological functors, extraordinary pullbacks, pushforwards and base change. By applying 19.1.2.23 we check moreover that $T_{i+1}(\mathfrak{D}, \mathfrak{C})$ is stable under cohomology.

II) From 19.1.2.9.1, $T_{i+1}(\mathfrak{D}, \mathfrak{C})$ is contained in $\widetilde{T}_i(\mathfrak{D}, \mathfrak{C})$ and $\widetilde{T}_i(\mathfrak{D}, \mathfrak{C})$ is contained in $T_i(\mathfrak{D}, \mathfrak{C}) \cap T_i(\mathfrak{D}, \mathfrak{C})^\vee$. Hence, by construction, the tensor product of two objects of $T_{i+1}(\mathfrak{D}, \mathfrak{C})$ is an object of $T_i(\mathfrak{D}, \mathfrak{C})$ and the dual of an object of $T_{i+1}(\mathfrak{D}, \mathfrak{C})$ is an object of $T_i(\mathfrak{D}, \mathfrak{C})$. □

Example 19.1.3.7. We can choose $\mathfrak{D} = \mathfrak{B}^+_{\mathrm{div}}$ and $\mathfrak{C} = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$.

Lemma 19.1.3.8. *Let $\mathfrak{C}$ be a data of coefficients which contains $\mathfrak{M}_{\mathrm{div}}$ and is stable under shifts. We have the inclusions:*

(a) The data $\mathfrak{M}^+_{\mathrm{div}}$ is stable under base change, smooth extraordinary pullbacks and tensor products.

(b) $\mathfrak{M}_{\mathrm{div}} \subset S(\mathfrak{B}_{\mathrm{div}}, \mathfrak{C})$.

(c) $\mathfrak{M}_{\mathrm{div}} \subset \boxtimes\left(\boxtimes(\mathfrak{M}_{\mathrm{div}}, \mathfrak{C}), \boxtimes(\mathfrak{M}_{\mathrm{div}}, \mathfrak{C})\right)$.

Proof. The first statement is a consequence of 12.2.1.9 and 12.2.1.13. The other ones are easy consequences of the first statement. □

Notation 19.1.3.9. We put $T_0 := S(\mathfrak{B}_{\mathrm{div}}, \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}})$. By induction on $i \in \mathbb{N}$, we set $U_i := T_i \cap T_i^\vee$, $\widetilde{U}_i := \boxtimes(\boxtimes(\mathfrak{M}_{\mathrm{div}}, U_i), \boxtimes(\mathfrak{M}_{\mathrm{div}}, U_i))$, and $T_{i+1} := S(\mathfrak{B}_{\mathrm{div}}, \widetilde{U}_i)$. We put $T := \cap_{i \in \mathbb{N}} T_i$.

Theorem 19.1.3.10. *The data of coefficients T contains $\mathfrak{M}_{\mathrm{div}}$, is local, stable by devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change, tensor products, duals and cohomology*

Proof. I) We prove by induction on i that T_i contains $\mathfrak{M}_{\mathrm{div}}$, is local, stable under devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change and cohomology.

a) For $T_0 = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{oc}}$, this comes from the step I)a) of the proof of 19.1.3.6, and of 19.1.3.8.b.

b) Suppose that this is true for T_i for some $i \in \mathbb{N}$.

i) Since $\mathfrak{M}_{\mathrm{div}}$ is almost stable under duality (see 19.1.3.4), then U_i contains $\mathfrak{M}_{\mathrm{div}}$. Hence, using 19.1.3.8, $\widetilde{U}_i$ contains $\mathfrak{M}_{\mathrm{div}}$.

ii) Similarly to the first part of the step I)b)ii) of the proof of 19.1.3.6, we check that U_i satisfies $BK^!$, is local, stable under pushforwards, under devissages, direct summands, base change and cohomology. Hence, following 19.1.2.30, we check that so is $\boxtimes(\mathfrak{M}_{\mathrm{div}}, U_i)$ and then so is $\widetilde{U}_i$. Hence, using the step I)a) of the proof of 19.1.3.6, we get that T_{i+1} satisfied the desired properties.

II) Using first 19.1.2.9.1, and next 19.1.2.30.a we get the inclusions $T_{i+1} \subset \widetilde{U}_i \subset \boxtimes(\mathfrak{M}_{\mathrm{div}}, U_i) \subset U_i \subset T_i$. Hence, by construction, the exterior tensor product of two objects of T_{i+1} is an object of T_i and the dual of an object of T_{i+1} is an object of T_i.

□

19.2 Grothendieck six operations over realizable pairs

We shall define here Grothendieck six operations for arithmetic $\mathcal{D}$-modules over realizable pairs.

Let $\mathcal{W}$ be an object of DVR($\mathcal{V}$), and l be its residue field.

19.2.1 Data of coefficients over frames

Definition 19.2.1.1. We define the category of frames over $\mathcal{W}$ as follows.

(a) A *frame* $(Y, X, \mathfrak{P})$ over $\mathcal{W}$ means that $\mathfrak{P}$ is a realizable smooth formal scheme over $\mathcal{W}$ (see definition 13.2.3.1), X is a closed subscheme of the special fiber P of $\mathfrak{P}$ and Y is an open subscheme of X. Let $(Y', X', \mathfrak{P}')$ and $(Y, X, \mathfrak{P})$ be two frames over $\mathcal{W}$. A morphism $\theta = (b, a, f): (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$ of frames over $\mathcal{W}$ is the data of a morphism $f: \mathfrak{P}' \to \mathfrak{P}$ of realizable smooth formal schemes over $\mathcal{W}$, a morphism $a: X' \to X$ of l-schemes, and a morphism $b: Y' \to Y$ of l-schemes inducing the commutative diagram

$$\begin{array}{ccccc} Y' & \hookrightarrow & X' & \hookrightarrow & \mathfrak{P}' \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} \\ Y & \hookrightarrow & X & \hookrightarrow & \mathfrak{P}. \end{array}$$

If there is no ambiguity with $\mathcal{W}$, we simply say frame or morphism of frames.

(b) A morphism $\theta = (b, a, f): (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$ of frames over $\mathcal{W}$ is said to be *complete* (resp. *strictly complete*) if a is proper (resp. f and a are proper).

Remark 19.2.1.2. To avoid confusion with the notion of frames in the context of isocrystals (see 10.1.1.4), one might call a frame $(Y, X, \mathfrak{P})$ as defined in 19.2.1.1 a *realizable* frame. However, hoping that this does not cause any confusion, in order not to overload the notations we prefer not to add the term realizable.

Definition 19.2.1.3. (a) We define the category of *realizable pairs* over $\mathcal{W}$ as follow. A realizable pair (Y, X) over $\mathcal{W}$ means the two first data of a frame over $\mathcal{W}$ of the form $(Y, X, \mathfrak{P})$. A frame of the form $(Y, X, \mathfrak{P})$ is said to be enclosing (Y, X). A morphism of realizable pairs $u = (b, a): (Y', X') \to (Y, X)$ over $\mathcal{W}$ is the data of a morphism of l-schemes of the form $a: X' \to X$ such that $a(Y') \subset Y$ and $b: Y' \to Y$ is the induced morphism.

(b) A morphism of realizable pairs $u = (b, a): (Y', X') \to (Y, X)$ over $\mathcal{W}$ is said to be *complete* if a is proper.

Remark 19.2.1.4. (a) Let $u = (b, a)\colon (Y', X') \to (Y, X)$ be a complete morphism of realizable pairs over $\mathcal{W}$. Then there exists a strictly complete morphism of frames over $\mathcal{W}$ of the form $\theta = (b, a, f)\colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$. Indeed, by definition, there exist some frames over $\mathcal{W}$ of the form $(Y', X', \mathfrak{P}'')$ and $(Y, X, \mathfrak{P})$. There exists an immersion $\mathfrak{P}'' \hookrightarrow \mathfrak{Q}''$ with $\mathfrak{Q}''$ a proper and smooth $\mathcal{W}$-formal scheme. Hence, put $\mathfrak{P}' := \mathfrak{Q}'' \times \mathfrak{P}$ and let $f\colon \mathfrak{P}' \to \mathfrak{P}$ be the projection. Since a is proper, $X \hookrightarrow \mathfrak{P}$ is proper, and f is proper, then the immersion $X' \hookrightarrow \mathfrak{P}'$ is also proper.

(b) Let $u = (b, a)\colon (Y', X') \to (Y, X)$ be a morphism of realizable pairs over $\mathcal{W}$. Similarly, we check that there exists a morphism of frames over $\mathcal{W}$ of the form $\theta = (b, a, f)\colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$.

Notation 19.2.1.5. Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$. Let $(Y, X, \mathfrak{P})$ be a frame over $\mathcal{W}$. We denote by $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ the full subcategory of $\mathfrak{C}(\mathfrak{P})$ of objects $\mathcal{E}$ such that there exists an isomorphism of the form $\mathcal{E} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Y}(\mathcal{E})$. We remark that $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ only depend on the immersion $Y \hookrightarrow \mathfrak{P}$ which explains the notation. We might choose X equal to the closure of Y in P.

Notation 19.2.1.6. Let $\mathfrak{C}$ be a data of coefficients stable under devissages and cohomology. Let $(Y, X, \mathfrak{P})$ be a frame over $\mathcal{W}$. Choose $\mathfrak{U}$ an open set of $\mathfrak{P}$ such that Y is closed in $\mathfrak{U}$.

(a) We define a canonical t-structure on $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ as follows. We denote by $\mathfrak{C}^{\leq n}(Y, \mathfrak{P}/\mathcal{W})$ (resp. $\mathfrak{C}^{\geq n}(Y, \mathfrak{P}/\mathcal{W})$) the full subcategory of $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ of complexes $\mathcal{E}$ such that $\mathcal{E}|\mathfrak{U} \in \mathfrak{C}^{\leq n}(Y, \mathfrak{U}/\mathcal{W}) := \mathfrak{C}(Y, \mathfrak{U}/\mathcal{W}) \cap \mathfrak{C}^{\leq n}(\mathfrak{U}/\mathcal{W})$ (resp. $\mathcal{E}|\mathfrak{U} \in \mathfrak{C}^{\geq n}(Y, \mathfrak{U}/\mathcal{W}) := \mathfrak{C}(Y, \mathfrak{U}/\mathcal{W}) \cap \mathfrak{C}^{\geq n}(\mathfrak{U}/\mathcal{W})$), where the t-structure on $\mathfrak{C}(\mathfrak{U}/\mathcal{W})$ is the canonical one (see 19.1.2.27). Since the restriction on an open formal subscheme is r-acyclic, then we get the independance on the choice of the open $\mathfrak{U}$ such that Y is closed in $\mathfrak{U}$. The heart of this t-structurewill be denoted by $\mathfrak{C}^{0}(Y, \mathfrak{P}/\mathcal{W})$. Finally, we denote by $\mathcal{H}^{i}_{\mathrm{t}}$ the ith space of cohomology with respect to this canonical t-structure.

(b) Suppose Y is smooth. Then, we denote by $\mathfrak{C}_{\mathrm{isoc}}(Y, \mathfrak{P}/\mathcal{W})$ (resp. $\mathfrak{C}^{\geq n}_{\mathrm{isoc}}(Y, \mathfrak{P}/\mathcal{W})$, resp. $\mathfrak{C}^{\leq n}_{\mathrm{isoc}}(Y, \mathfrak{P}/\mathcal{W})$, resp. $\mathfrak{C}^{0}_{\mathrm{isoc}}(Y, \mathfrak{P}/\mathcal{W})$) the full subcategory of (resp. $\mathfrak{C}^{\geq n}(Y, \mathfrak{P}/\mathcal{W})$, resp. $\mathfrak{C}^{\leq n}(Y, \mathfrak{P}/\mathcal{W})$, resp. $\mathfrak{C}^{0}(Y, \mathfrak{P}/\mathcal{W})$) consisting of complexes $\mathcal{E}^{(\bullet)}$ such that $H^{i}(\mathcal{E}^{(\bullet)}|\mathfrak{U}) \in \mathrm{MIC}^{(\bullet)}(Y, \mathfrak{U}/K)$. We refer "isoc" as isocrystals. Remark that when $\mathfrak{C}$ is included in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}$ then $\mathfrak{C}^{0}_{\mathrm{isoc}}(Y, \mathfrak{P}/\mathcal{W}) = \mathfrak{C}^{0}(Y, \mathfrak{P}/\mathcal{W}) \cap \mathrm{MIC}^{(\bullet)}(Y, X, \mathfrak{P}, Z/K)$ is X is the closure of Y in P and $Z = X \setminus Y$ (see 16.2.1.1).

Remark 19.2.1.7. Let $\mathfrak{C}$ be a data of coefficients stable under devissages and cohomology. Let $\mathfrak{P}$ be a smooth $\mathcal{W}$-formal scheme, Y be a subscheme of P, Z be a closed subscheme of Y, and $Y' := Y \setminus Z$.

(a) We get the t-exact functor $({}^{\dagger}Z)\colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y', \mathfrak{P}/\mathcal{W})$. Beware the functor $({}^{\dagger}Z)\colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ is not always t-exact.

(b) We say that Z locally comes from a divisor of P if locally in P, there exists a divisor T of P such that $Z = Y \cap T$ (this is equivalent to saying that locally in P, the ideal defining $Z \hookrightarrow Y$ is generated by one element). In that case, we get the t-exact functor $({}^{\dagger}Z)\colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$. Indeed, by construction of our t-structures, we can suppose Y is closed in $\mathfrak{P}$ (and then we reduce to the case where the t-structure on $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ is induced by the standard t-structure of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$). Since the property is local, we can suppose there exists a divisor T such that $Z = T \cap Y$. Then both functors $({}^{\dagger}Z)$ and $({}^{\dagger}T)$ of $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ are isomorphic. Since $({}^{\dagger}T)$ is t-exact, we are done.

19.2.2 Grothendieck six operations over realizable pairs

Theorem 19.2.2.1 (Independence)**.** *Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, and under local cohomological functors.*

Let $\theta = (b, a, f)\colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$ be a morphism of frames over $\mathcal{W}$ such that a and b are proper.

(a) For any $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$, for any $\mathcal{E}'^{(\bullet)} \in \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$ (recall notation 19.2.1.5, we have

$$\mathrm{Hom}_{\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})}(f^{(\bullet)}_{+}(\mathcal{E}'^{(\bullet)}), \mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})}(\mathcal{E}'^{(\bullet)}, \mathbb{R}\underline{\Gamma}^{\dagger}_{Y'} f^{!(\bullet)}(\mathcal{E}^{(\bullet)})).$$

(b) Suppose that $Y' = Y$ and b is the identity, and that $\mathfrak{C}$ is stable under cohomology. Then, for any $\mathcal{E}^{(\bullet)} \in \mathfrak{C}^0(Y, \mathfrak{P}/\mathcal{W})$, for any $\mathcal{E}'^{(\bullet)} \in \mathfrak{C}^0(Y, \mathfrak{P}'/\mathcal{W})$, for any $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\mathcal{H}^n_t \mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) = 0, \qquad \mathcal{H}^n_t f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) = 0.$$

(c) Suppose that $Y' = Y$ and b is the identity. For any $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$, for any $\mathcal{E}'^{(\bullet)} \in \mathfrak{C}(Y, \mathfrak{P}'/\mathcal{W})$, the adjunction morphisms $\mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)!} f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) \to \mathcal{E}'^{(\bullet)}$ and $f^{(\bullet)}_+ \mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ are isomorphisms. In particular, the functors $\mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)!}$ and $f^{(\bullet)}_+$ induce quasi-inverse equivalences of categories between $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ and $\mathfrak{C}(Y, \mathfrak{P}'/\mathcal{W})$.

Proof. I) Let us check the first statement. Replacing X and X' by the closure of Y in P and Y' in P', we can suppose Y is dense in X and Y' is dense in X'. Let $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$, and $\mathcal{E}'^{(\bullet)} \in \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$. Since a is proper, using 13.2.4.2.2, the stability of $\mathfrak{C}$ under extraodinary pullbacks, and the equivalence of categories 8.4.5.6, we get the bijection

$$\mathrm{Hom}_{\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})}(f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}), \mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathrm{Hom}_{\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'})}(\mathcal{E}'^{(\bullet)}, f^{!(\bullet)}(\mathcal{E}^{(\bullet)})).$$

Since a and b are proper, then the open immersion $Y' \subset a^{-1}(Y)$ is proper. Since Y' is dense in X', then $Y' = a^{-1}(Y)$. Hence, the functors $\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^{!(\bullet)}$ and $\mathbb{R}\underline{\Gamma}^\dagger_{Y'} f^{!(\bullet)}$ (resp. $f^{(\bullet)}_+$ and $\mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)}_+$) are isomorphic over $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ (resp. $\mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$). Hence, the functor $\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^{!(\bullet)}$ (resp. $f^{(\bullet)}_+$) induces $\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^{!(\bullet)} \colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$ (resp. $f^{(\bullet)}_+ \colon \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$). Since $\mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$ is a strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'})$, we conclude using the equality

$$\mathrm{Hom}_{\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}'})}(\mathcal{E}'^{(\bullet)}, f^{!(\bullet)}(\mathcal{E}^{(\bullet)})) = \mathrm{Hom}_{\mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})}(\mathcal{E}'^{(\bullet)}, \mathbb{R}\underline{\Gamma}^\dagger_{X'} f^{!(\bullet)}(\mathcal{E}^{(\bullet)})).$$

II) Now let us check at the same time the last two statements. Using the stability properties that $\mathfrak{C}$ satisfies, we check that the functors $f^{(\bullet)}_+ \colon \mathfrak{C}(Y, \mathfrak{P}'/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ and $\mathbb{R}\underline{\Gamma}^\dagger_Y f^{!(\bullet)} \colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}'/\mathcal{W})$ are well defined. Since $\mathfrak{C}$ is included in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}$, we reduce to check the case where $\mathfrak{C} = \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}$. Choose $\mathfrak{U}$ (resp. $\mathfrak{U}'$) an open set of $\mathfrak{P}$ (resp. $\mathfrak{P}'$) such that Y is closed in $\mathfrak{U}$ (resp. Y is closed in $\mathfrak{U}'$), and such that $f(\mathfrak{U}') \subset \mathfrak{U}$. The functor $|\mathfrak{U} \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathcal{W}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{U}/\mathcal{W})$ is t-exact, and the same with some primes. Moreover, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathcal{W})$ (or $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathcal{W})$), the property $\mathcal{E}^{(\bullet)} = 0$ is equivalent to $\mathcal{E}^{(\bullet)}|\mathfrak{U} = 0$. Hence, we can suppose $\mathfrak{U} = \mathfrak{P}$ and $\mathfrak{U}' = \mathfrak{P}'$, i.e. $Y \hookrightarrow P$ and $Y \hookrightarrow P'$ are closed immersions.

1) In this case we treat the case where Y is smooth. Since the theorem is local, we can suppose there exists a smooth formal scheme $\mathfrak{Y}$ which is a lifting of Y. Hence, this is an obvious consequence of Berthelot-Kashiwara theorem 9.3.5.13.

2) Let us go back to the general case. We proceed by induction on the dimension of Y. When $\dim Y = 0$, then using 15.3.3.3, this is a consequence of the step 1).

Now, suppose $\dim Y \geq 1$ and the theorem valid when Y is replaced by a variety of inferior dimension. Since l is perfect, since the theorem is local on $\mathfrak{P}$, we can suppose $\mathfrak{P}$ integral and affine, and there exists a (reduced) divisor T of P such that, putting $Z := Y \cap T$, we have $Y \setminus Z$ is a smooth l-variety, and $\dim Z < \dim Y$.

3) We check in this step that for any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}'/\mathcal{W})$, for any integer $r \neq 0$, $\mathcal{H}^r f^{(\bullet)}_+(\mathcal{E}'^{(\bullet)}) = 0$.

Since Z locally comes from a divisor of P', then the functor $({}^\dagger Z) \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}'/\mathcal{W}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}'/\mathcal{W})$ is t-exact (see 19.2.1.7.b). Hence, the localisation triangle in Z of $\mathcal{E}'^{(\bullet)}$ induces the exact sequence in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}'/\mathcal{W})$:

$$0 \to \mathcal{H}^{\dagger,0}_Z(\mathcal{E}'^{(\bullet)}) \to \mathcal{E}'^{(\bullet)} \to ({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \to \mathcal{H}^{\dagger,1}_Z(\mathcal{E}'^{(\bullet)}) \to 0. \tag{19.2.2.1.1}$$

Let $\mathcal{F}'^{(\bullet)}$ be the kernel of the epimorphism $({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \to \mathcal{H}^{\dagger,1}_Z(\mathcal{E}'^{(\bullet)})$. We get the exact sequence in $\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}'/\mathcal{W})$

$$0 \to \mathcal{F}'^{(\bullet)} \to ({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \to \mathcal{H}^{\dagger,1}_Z(\mathcal{E}'^{(\bullet)}) \to 0.$$

By applying the functor $f_+^{(\bullet)}$ to this latter exact sequence, we get a long exact sequence. We have $({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y \setminus Z, \mathfrak{P}'/\mathcal{W})$ and $\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Z, \mathfrak{P}'/\mathcal{W})$. Hence, following the step 1), using the induction hypothesis, using the long exact sequence, we check that for any integer $r \notin \{0,1\}$, we have $\mathcal{H}^r(f_+^{(\bullet)})(\mathcal{F}'^{(\bullet)}) = 0$. Moreover, $\mathcal{H}^1(f_+^{(\bullet)})(\mathcal{F}'^{(\bullet)}) = 0$ if and only if the morphism $s\colon \mathcal{H}^0(f_+^{(\bullet)})(({}^\dagger Z)(\mathcal{E}'^{(\bullet)})) \to \mathcal{H}^0(f_+^{(\bullet)})(\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}))$ is an epimorphism. We split the check of this latter property in the following two steps a) and b).

3.a) In this step, we check that the morphism $s' := \mathcal{H}^0(\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!})(s)$ is an epimorphism. Since $({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y \setminus Z, \mathfrak{P}'/\mathcal{W})$, since the functors $\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!}$ and $\mathbb{R}\underline{\Gamma}_{Y\setminus Z}^\dagger \circ f^{(\bullet)!}$ are canonically isomorphic over $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y \setminus Z, \mathfrak{P}/\mathcal{W})$ then following the step 1), the canonical morphism

$$({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \to \mathcal{H}^0(\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!}) \circ \mathcal{H}^0(f_+^{(\bullet)})(({}^\dagger Z)(\mathcal{E}'^{(\bullet)}))$$

is an isomorphism. Since $\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}) \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Z, \mathfrak{P}'/\mathcal{W})$, since the functors $\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!}$ and $\mathbb{R}\underline{\Gamma}_Z^\dagger \circ f^{(\bullet)!}$ are canonically isomorphic over $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Z, \mathfrak{P}/\mathcal{W})$ then by induction hypothesis the canonical morphism

$$\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}) \to \mathcal{H}^0(\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!}) \circ \mathcal{H}^0(f_+^{(\bullet)})(\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}))$$

is an isomorphism. Since $({}^\dagger Z)(\mathcal{E}'^{(\bullet)}) \to \mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)})$ is an epimorphism, this yields that so is s'.

3.b) Let us check that s is an epimorphism. Let $\mathcal{F}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathcal{W})$ be the image of s, and i be the canonical monomorphism $\mathcal{F}^{(\bullet)} \hookrightarrow \mathcal{H}^0(f_+^{(\bullet)})(\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}))$. Since $\mathcal{H}^0(f_+^{(\bullet)})(\mathcal{H}_Z^{\dagger,1}(\mathcal{E}'^{(\bullet)}))$ has his support in Z, then i is in fact a monomorphism of $\underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Z, \mathfrak{P}/\mathcal{W})$. Using the induction hypothesis, since the functors $\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!}$ and $\mathbb{R}\underline{\Gamma}_Z^\dagger \circ f^{(\bullet)!}$ are canonically isomorphic over $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Z, \mathfrak{P}/\mathcal{W})$ this yields that $i' := \mathcal{H}^0(\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!})(i)$ is a monomorphism. Since s' is an epimorphism, then so is i'. Hence, the morphism i' is an isomorphism. Using the induction hypothesis, this implies that i is an isomorphism. This yields that s is an epimorphism.

3.c) Hence, we have checked that for any integer $r \neq 0$, we have $\mathcal{H}^r(f_+^{(\bullet)})(\mathcal{F}'^{(\bullet)}) = 0$. From 19.2.2.1.1, we get the exact sequence $0 \to \mathcal{H}_Z^{\dagger,0}(\mathcal{E}'^{(\bullet)}) \to \mathcal{E}'^{(\bullet)} \to \mathcal{F}'^{(\bullet)} \to 0$. By applying the functor $f_+^{(\bullet)}$ to this latter sequence, we get a long exact sequence. Looking at this later one, we remark that the property "for any $r \neq 0$, $\mathcal{H}^r(f_+^{(\bullet)})(\mathcal{F}'^{(\bullet)}) = 0$ and $\mathcal{H}^r(f_+^{(\bullet)})(\mathcal{H}_Z^{\dagger,0}(\mathcal{E}'^{(\bullet)})) = 0$", implies that "for any $r \neq 0$, $\mathcal{H}^r(f_+^{(\bullet)})(\mathcal{E}'^{(\bullet)}) = 0$".

4) Similarly to the step 3), we check that for any $r \neq 0$, for any $\mathcal{E}^{(\bullet)} \in \underrightarrow{LM}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathcal{W})$, we have $\mathcal{H}^r(\mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!})(\mathcal{E}^{(\bullet)}) = 0$.

5) It remains to check the last statement of the theorem. Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}/\mathcal{W})$. Using the localisation triangle with respect to Z, to check that the morphism $f_+^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)}$ is an isomorphism, we reduce to check we get an isomorphism after applying $\mathbb{R}\underline{\Gamma}_Z^\dagger$ and $({}^\dagger Z)$. Using 13.2.1.4 and 13.1.5.6.1, after applying $\mathbb{R}\underline{\Gamma}_Z^\dagger$, we get a morphism canonically isomorphic to the canonical morphism $f_+^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}_Z^\dagger \circ f^{(\bullet)!}(\mathbb{R}\underline{\Gamma}_Z^\dagger \mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}_Z^\dagger \mathcal{E}^{(\bullet)}$. By induction hypothesis, this latter is an isomorphism. Moreover, after applying $({}^\dagger Z)$, we get the morphism $f_+^{(\bullet)} \circ \mathbb{R}\underline{\Gamma}_{Y\setminus Z}^\dagger \circ f^{(\bullet)!}(\mathbb{R}\underline{\Gamma}_{Y\setminus Z}^\dagger \mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}_{Y\setminus Z}^\dagger \mathcal{E}^{(\bullet)}$, which is an isomorphism following the step 1).

We proceed similarly to check that the canonical morphism $\mathcal{E}'^{(\bullet)} \to \mathbb{R}\underline{\Gamma}_Y^\dagger \circ f^{(\bullet)!} \circ f_+^{(\bullet)}(\mathcal{E}'^{(\bullet)})$ is an isomorphism for any $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovcoh}}(Y, \mathfrak{P}'/\mathcal{W})$.

□

Corollary 19.2.2.2. *Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, and local cohomological functors. Let $\mathbb{Y} := (Y, X)$ be a realizable pair over $\mathcal{W}$.*

(a) Choose a frame of the form $(Y, X, \mathfrak{P})$. The category $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ does not depend, up to a canonical equivalence of categories, on the choice of the frame $(Y, X, \mathfrak{P})$ over $\mathcal{W}$ enclosing (Y, X). Hence, we can simply write $\mathfrak{C}(\mathbb{Y}/\mathcal{W})$ instead of $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ without ambiguity (up to canonical equivalence of categories).

(b) If moreover $\mathfrak{C}$ is stable under cohomology, then we get a canonical t-structure on $\mathfrak{C}(\mathbb{Y}/\mathcal{W})$.

Proof. Let $(Y, X, \mathfrak{P})$ and $(Y, X, \mathfrak{P}')$ be two frames over $\mathcal{W}$ enclosing (Y, X). The closed immersions $X \hookrightarrow \mathfrak{P}$ and $X \hookrightarrow \mathfrak{P}'$ induce $X \hookrightarrow \mathfrak{P} \times \mathfrak{P}'$. Denoting by $\pi_1 \colon \mathfrak{P} \times \mathfrak{P}' \to \mathfrak{P}$ and $\pi_2 \colon \mathfrak{P} \times \mathfrak{P}' \to \mathfrak{P}'$ the structural projections, we get two morphisms of frames over $\mathcal{W}$ of the form $(id, id, \pi_1) \colon (Y, X, \mathfrak{P} \times \mathfrak{P}') \to (Y, X, \mathfrak{P})$ and $(id, id, \pi_2) \colon (Y, X, \mathfrak{P} \times \mathfrak{P}') \to (Y, X, \mathfrak{P}')$. From 19.2.2.1, the functors $\pi^{(\bullet)}_{2+} \mathbb{R}\underline{\Gamma}^\dagger_Y \pi_1^{!(\bullet)}$ and $\pi^{(\bullet)}_{1+} \mathbb{R}\underline{\Gamma}^\dagger_Y \pi_2^{(\bullet)!}$ are canonically quasi-inverse equivalences of categories between $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ and $\mathfrak{C}(Y, \mathfrak{P}'/\mathcal{W})$. When $\mathfrak{C}$ is stable under cohomology then these equivalences are t-exact. Hence we are done. □

Lemma 19.2.2.3. *Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, local cohomological functors, and duals. Let $\mathbb{Y} := (Y, X)$ be a realizable pair over $\mathcal{W}$. Choose a frame of the form $(Y, X, \mathfrak{P})$. The functor $\mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}} \colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ does not depend, up to canonical isomorphism, of 19.2.2.2 (more precisely, we have the commutative diagram 19.2.2.3.1 up to canonical isomorphism), on the choice of the frame enclosing (Y, X). Hence, we will denote by $\mathbb{D}_{\mathbb{Y}} \colon \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}/\mathcal{W})$ the functor $\mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}}$.*

Proof. As in the beginning of the proof, 19.2.2.2, let $(Y, X, \mathfrak{P}_1)$ and $(Y, X, \mathfrak{P}_2)$ be two frames over $\mathcal{W}$ enclosing (Y, X). Let $\pi_1 \colon \mathfrak{P}_1 \times \mathfrak{P}_2 \to \mathfrak{P}_1$ and $\pi_2 \colon \mathfrak{P}_1 \times \mathfrak{P}_2 \to \mathfrak{P}_2$ be the structural projections. We have to check that the diagram

$$\begin{array}{ccccc}
\mathfrak{C}(Y, \mathfrak{P}_1/\mathcal{W}) & \xrightarrow[\mathbb{R}\underline{\Gamma}^\dagger_Y \pi_1^{!(\bullet)}]{\cong} & \mathfrak{C}(Y, \mathfrak{P}_1 \times \mathfrak{P}_2/\mathcal{W}) & \xrightarrow[\pi^{(\bullet)}_{2+}]{\cong} & \mathfrak{C}(Y, \mathfrak{P}_2/\mathcal{W}) \\
\downarrow {\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}_1}} & & \downarrow {\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}} & & \downarrow {\scriptstyle \mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}_2}} \\
\mathfrak{C}(Y, \mathfrak{P}_1/\mathcal{W}) & \xrightarrow[\cong]{\mathbb{R}\underline{\Gamma}^\dagger_Y \pi_1^{!(\bullet)}} & \mathfrak{C}(Y, \mathfrak{P}_1 \times \mathfrak{P}_2/\mathcal{W}) & \xrightarrow[\cong]{\pi^{(\bullet)}_{2+}} & \mathfrak{C}(Y, \mathfrak{P}_2/\mathcal{W})
\end{array} \qquad (19.2.2.3.1)$$

is commutative, up to canonical isomorphisms. Let $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(Y, \mathfrak{P}_1 \times \mathfrak{P}_2/\mathcal{W})$. From 13.2.4.1, we have the isomorphism $\mathbb{D}_{\mathfrak{P}_2} \pi^{(\bullet)}_{2+}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \pi^{(\bullet)}_{2+} \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)})$. Hence, by applying the functor $\mathbb{R}\underline{\Gamma}^\dagger_Y$ to this isomorphism, we get the first one $\mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}_2} \pi^{(\bullet)}_{2+}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Y \pi^{(\bullet)}_{2+} \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)}) \underset{13.2.1.4}{\xrightarrow{\sim}} \pi^{(\bullet)}_{2+} \mathbb{R}\underline{\Gamma}^\dagger_{\pi_2^{-1}(Y)} \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)})$. The immersion $Y \hookrightarrow \pi_2^{-1}(Y)$ is in fact a closed immersion. This yields $Y = \overline{Y} \cap \pi_2^{-1}(Y)$, where $\overline{Y}$ is the closure of Y in $P_1 \times P_2$. Since $\mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)})$ has in support in $\overline{Y}$, then $\mathbb{R}\underline{\Gamma}^\dagger_{\pi_2^{-1}(Y)} \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{\pi_2^{-1}(Y)} \mathbb{R}\underline{\Gamma}^\dagger_{\overline{Y}} \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Y \mathbb{D}_{\mathfrak{P}_1 \times \mathfrak{P}_2}(\mathcal{E}^{(\bullet)})$. Hence, we have checked the commutativity, up to commutative isomorphism, of the right square of 19.2.2.3.1. From 19.2.2.1, $\pi^{(\bullet)}_{1+}$ is canonically a quasi-inverse of the equivalence of categories $\mathbb{R}\underline{\Gamma}^\dagger_Y \pi_1^{!(\bullet)} \colon \mathfrak{C}(Y, \mathfrak{P}_1 \times \mathfrak{P}_2/\mathcal{W}) \cong \mathfrak{C}(Y, \mathfrak{P}_1/\mathcal{W})$ (we means that we have canonical isomorphisms $\pi^{(\bullet)}_{1+} \mathbb{R}\underline{\Gamma}^\dagger_Y \pi_1^{!(\bullet)} \xrightarrow{\sim} \mathrm{id}$ and $id \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Y \pi_1^{!(\bullet)} \pi^{(\bullet)}_{1+}$). Hence, we get the commutativity, up to canonical isomorphism, of the left square of 19.2.2.3.1. □

Lemma 19.2.2.4. *Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, and local cohomological functors. Let $u = (b, a) \colon (Y', X') \to (Y, X)$ be a morphism of realizable pairs over $\mathcal{W}$. Put $\mathbb{Y} := (Y, X)$ and $\mathbb{Y}' := (Y', X')$. Let us choose a morphism of frames $\theta = (b, a, f) \colon (Y', X', \mathfrak{P}') \to (Y, X, \mathfrak{P})$ over $\mathcal{W}$ enclosing u.*

(a) The functor $\theta^{!(\bullet)} := \mathbb{R}\underline{\Gamma}^\dagger_{Y'} \circ f^{(\bullet)!} \colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$ does not depend on the choice of such θ enclosing u (up to canonical equivalences of categories). Hence, it will be denoted by $u^! \colon \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}'/\mathcal{W})$.

(b) Suppose that u is complete, i.e. that $a \colon X' \to X$ is proper. The functor $\theta_+ := f^{(\bullet)}_+ \colon \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ does not depend on the choice of such θ enclosing u (up to canonical equivalences of categories). Hence, it will be denoted by $u_+ \colon \mathfrak{C}(\mathbb{Y}'/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}/\mathcal{W})$.

Proof. To check the first assertion, we proceed as in the proof of 19.2.2.3 (use also the commutation of local cohomological functors with extraordinary inverse images given in 13.2.1.4). Let us check that the functor $f^{(\bullet)}_+ \colon \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ is well defined. Let $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$. Since a is proper, then $f^{(\bullet)}_+(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(\mathfrak{P})$. We compute $\mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)}_+(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} f^{(\bullet)}_+ \mathbb{R}\underline{\Gamma}^\dagger_{f^{-1}Y}(\mathcal{E}^{(\bullet)})$. Since Y' is included in $f^{-1}(Y)$ and $\mathcal{E}^{(\bullet)} \in \mathfrak{C}(Y', \mathfrak{P}'/\mathcal{W})$, then $\mathbb{R}\underline{\Gamma}^\dagger_{f^{-1}Y}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}^{(\bullet)}$. Hence, $\mathbb{R}\underline{\Gamma}^\dagger_Y f^{(\bullet)}_+(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} f^{(\bullet)}_+(\mathcal{E}^{(\bullet)})$, which implies that $f^{(\bullet)}_+(\mathcal{E}^{(\bullet)}) \in \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$. To check that the functor does not depend on the choice of θ enclosing u, we proceed as in the proof of 19.2.2.3. □

Lemma 19.2.2.5. *Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, and tensor products. Let $\mathbb{Y} := (Y, X)$ be a realizable pair over $\mathcal{W}$. Choose a frame of the form $(Y, X, \mathfrak{P})$. The bifunctor $-\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{P}}} - [-\dim P] \colon \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \times \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W}) \to \mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ does not depend, up to the canonical equivalences of categories of 19.2.2.2, on the choice of the frame enclosing (Y, X). It will be denoted by $\widetilde{\otimes}_{\mathbb{Y}} \colon \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \times \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}/\mathcal{W})$.*

Proof. From Lemmas 19.1.2.4.b and 19.1.2.6, the data of coefficients $\mathfrak{C}$ is also stable under local cohomological functors. From 9.2.1.27.1 (resp. 13.1.5.6.2), extraordinary inverse images (resp. local cohomological functors) commute with tensor products (up to a shift). Proceeding as in the proof of 19.2.2.3 with its notation, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} \pi_1^{!(\bullet)}$ and $\mathbb{R}\underline{\Gamma}^{\dagger}_{Y} \pi_2^{!(\bullet)}$ commute with tensor products and then so are $\pi_{1+}^{(\bullet)}$ and $\pi_{2+}^{(\bullet)}$.

□

19.2.2.6 (Grothendieck six operations)**.** Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, duals, and tensor products. To sum-up the above Lemmas we can define Grothendieck six operations on realizable pairs as follows. Let $u = (b, a) \colon (Y', X') \to (Y, X)$ be a morphism of realizable pairs over $\mathcal{W}$. Put $\mathbb{Y} := (Y, X)$ and $\mathbb{Y}' := (Y', X')$.

(a) We have the dual functor $\mathbb{D}_{\mathbb{Y}} \colon \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}/\mathcal{W})$ (see 19.2.2.3).

(b) We have the extraordinary pullback $u^! \colon \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}'/\mathcal{W})$ (see 19.2.2.4). We get the pullbacks $u^+ := \mathbb{D}_{\mathbb{Y}'} \circ u^! \circ \mathbb{D}_{\mathbb{Y}}$.

(c) Suppose that u is complete. Then, we have the functor $u_+ \colon \mathfrak{C}(\mathbb{Y}'/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}/\mathcal{W})$ (see 19.2.2.4). We denote by $u_! := \mathbb{D}_{\mathbb{Y}} \circ u_+ \circ \mathbb{D}_{\mathbb{Y}'}$, the extraordinary pushforward by u.

(d) We have the tensor product $-\widetilde{\otimes}_{\mathbb{Y}} - \colon \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \times \mathfrak{C}(\mathbb{Y}/\mathcal{W}) \to \mathfrak{C}(\mathbb{Y}/\mathcal{W})$ (see 19.2.2.5)

Examples 19.2.2.7. (a) We recall the data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}$ and $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}$ are defined respectively in 19.1.2.2.b and 19.1.2.2.c. Using Lemmas 19.1.2.5 and 19.1.2.11 (and 19.1.2.10), they are stable under local cohomological functors, pushforwards, extraordinary pullbacks, and duals. Hence, with the notation 19.2.2.2, using Lemmas 19.2.2.4, 19.2.2.5, and 19.2.2.3, for any frame $(Y, X, \mathfrak{P})$ over $\mathcal{W}$, we get the categories of the form $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(Y, \mathfrak{P}/\mathcal{W})$, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{h}}(\mathbb{Y}/\mathcal{W})$, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}(Y, \mathfrak{P}/\mathcal{W})$ or $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{ovhol}}(\mathbb{Y}/\mathcal{W})$ endowed with five of Grothendieck cohomological operations (the tensor product is a priori missing).

(b) With notation 19.1.2.19, the data of coefficients $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{h}}$ is local, stable by devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change, tensor products, duals, cohomology. Hence, $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{h}}$ is endowed with Grothendieck six cohomological operations.

(c) Following theorem 19.1.3.10, there exist a data of coefficients T which contains $\mathfrak{M}_{\mathrm{div}}$, is local, stable by devissages, direct summands, local cohomological functors, pushforwards, extraordinary pullbacks, base change, tensor products, duals, cohomology and special descent of the base. Hence, for any frame $(Y, X, \mathfrak{P})$ over $\mathcal{W}$, we get the triangulated category $T(Y, \mathfrak{P}/\mathcal{W})$ or $T(\mathbb{Y}/\mathcal{W})$, endowed with a t-structure and Grothendieck six operations.

19.2.3 Grothendieck six operations over realizable varieties

Definition 19.2.3.1 (Proper compactification)**.** (a) A frame $(Y, X, \mathfrak{P})$ over $\mathcal{W}$ is said to be *proper* if $\mathfrak{P}$ is proper. The category of proper frames over $\mathcal{W}$ is the full subcategory of the category of frames over $\mathcal{W}$ whose objects are proper frames over $\mathcal{W}$.

(b) The category of *proper realizable pairs* over $\mathcal{W}$ is the full subcategory of the category of realizable pairs over $\mathcal{W}$ whose objects (Y, X) are such that X is proper. We remark that if (Y, X) is a proper realizable pair over $\mathcal{W}$ then there exists a proper frame over $\mathcal{W}$ of the form $(Y, X, \mathfrak{P})$.

(c) A *realizable variety* over $\mathcal{W}$ is a l-scheme Y such that there exists a proper frame over $\mathcal{W}$ of the form $(Y, X, \mathfrak{P})$. For such frame $(Y, X, \mathfrak{P})$, we say that the proper frame $(Y, X, \mathfrak{P})$ encloses Y or that the proper realizable pair (Y, X) encloses Y.

19.2.3.2 (Grothendieck six operations)**.** Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, duals, and tensor products. Similarly to Lemma 19.2.2.2, we check using Theorem 19.2.2.1 that the category $\mathfrak{C}(Y, \mathfrak{P}/\mathcal{W})$ (resp. $\mathfrak{C}(Y, X/\mathcal{W})$) does not depend, up to a canonical equivalence of categories, on the choice of the proper frame $(Y, X, \mathfrak{P})$ (resp. the proper realizable pair (Y, X)) over $\mathcal{W}$ enclosing Y. Hence, we simply denote it by $\mathfrak{C}(Y/\mathcal{W})$. As for 19.2.2.6, we can define Grothendieck six operations on realizable varieties as follows. Let $u\colon Y' \to Y$ be a morphism of realizable varieties over $\mathcal{W}$.

(a) We have the dual functor $\mathbb{D}_Y\colon \mathfrak{C}(Y/\mathcal{W}) \to \mathfrak{C}(Y/\mathcal{W})$ (see 19.2.2.3).

(b) We have the extraordinary pullback $u^!\colon \mathfrak{C}(Y/\mathcal{W}) \to \mathfrak{C}(Y'/\mathcal{W})$ (see 19.2.2.4). We get the pullbacks $u^+ := \mathbb{D}_{Y'} \circ u^! \circ \mathbb{D}_Y$.

(c) We have the functor $u_+\colon \mathfrak{C}(Y'/\mathcal{W}) \to \mathfrak{C}(Y/\mathcal{W})$ (see 19.2.2.4). We denote by $u_! := \mathbb{D}_Y \circ u_+ \circ \mathbb{D}_{Y'}$, the extraordinary pushforward by u.

(d) We have the tensor product $-\widetilde{\otimes}_Y -\colon \mathfrak{C}(Y/\mathcal{W}) \times \mathfrak{C}(Y/\mathcal{W}) \to \mathfrak{C}(Y/\mathcal{W})$ (see 19.2.2.5)

19.2.4 Stability of the holonomicity by Grothendieck's six operations on quasi-projective k-varieties, algebraic stacks, crystallin companion

Let us introduce a projective version of the notion of data of coefficients of 19.1.1.1.

Definition 19.2.4.1. A *projective data of coefficients* $\mathfrak{C}$ *over* $\mathcal{V}$ will be the data for any object $\mathcal{W}$ of $\mathrm{DVR}(\mathcal{V})$ (see notation 9.2.6.12), for any projective smooth formal scheme $\mathfrak{P}$ over $\mathcal{W}$ of a strictly full subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{P}})$, which will be denoted by $\mathfrak{C}(\mathfrak{P}/\mathcal{W})$, or simply $\mathfrak{C}(\mathfrak{P})$ if there is no ambiguity with the base $\mathcal{W}$. If there is no ambiguity with $\mathcal{V}$, we simply say a *projective data of coefficients*.

Replacing data of coefficients by projective data of coefficients in 19.1.1.3, we get similar stability notions.

Notation 19.2.4.2. Let $\mathcal{W}$ be an object of $\mathrm{DVR}(\mathcal{V})$ and $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{W}$. Let $F\text{-}\mathrm{Hol}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$ be the category of holonomic $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-modules endowed with a Frobenius structure. We get a data of coefficients $\mathfrak{A}$ over $\mathcal{V}$ by setting $\mathfrak{A}(\mathfrak{X}) := F\text{-}\mathrm{Hol}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$. Since this is an abelian category then following 19.1.2.17 we get $\Delta(\mathfrak{A})$ the smallest data of coefficients over $\mathcal{V}$ which is stable under devissage and cohomology and containing $\mathfrak{A}$. We denote by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}$ the underlying projective data of coefficients of $\Delta(\mathfrak{A})$.

19.2.4.3 (Complexes of arithmetic $\mathcal{D}$-modules with bounded and F-holonomic cohomology on quasi-projective k-variety)**.** Let Y be a variety quasi-projective on k. Then there exists a projective and smooth $\mathcal{V}$-formal scheme $\mathfrak{P}$, such that Y is a subscheme of the special fiber P of $\mathfrak{P}$. We denote by $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y, \mathfrak{P}/\mathcal{V})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y, \mathfrak{P}/\mathcal{V})$) the full subcategory of $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(\mathfrak{P}/\mathcal{V})$) of complexes $\mathcal{E}$ such that there exists an isomorphism of the form $\mathbb{R}\underline{\Gamma}^\dagger_Y(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. Similarly to 19.2.2.1, we can check that the category $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y, \mathfrak{P}/\mathcal{V})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y, \mathfrak{P}/\mathcal{V})$) is independent, up to canonical isomorphism, of the choice of the immersion $Y \hookrightarrow \mathfrak{P}$ into a projective and smooth $\mathcal{V}$-formal scheme. We write it $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V})$ (resp. $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V})$).

19.2.4.4 (Operations cohomological on quasi-projective k-varieties)**.** Let $b\colon Y' \to Y$ be a morphism of quasi-projective varieties on k. Similarly to 19.2.3.2 (use 18.3.3.5 and the stability of the overholonomicity with Frobenius structure), we get the functors $b_+, b_!\colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y'/\mathcal{V}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V})$ called respectively direct image and extraordinary direct image by b ; we have the functors $b^+, b^!\colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y'/\mathcal{V})$ called respectively inverse image and extraordinary inverse image by b, the dual functor $\mathbb{D}_Y\colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V})$ and the tensor product $-\widetilde{\otimes}_{\mathcal{O}_Y} -\colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V}) \times F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V})$.

Similarly to 19.2.3.2, we get the functors $b^+, b^! \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y'/\mathcal{V})$ called respectively inverse image and extraordinary inverse image by b, the dual functor $\mathbb{D}_Y \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V})$ and the tensor product $-\widetilde{\otimes}_{\mathcal{O}_Y} - \colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V}) \times \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{hol}}(Y/\mathcal{V})$.

19.2.4.5 (Arithmetic $\mathcal{D}$-modules for algebraic stacks)**.** In [Abe18, 2], T. Abe construct construct a p-adic cohomology theory for algebraic stacks from the quasi-projective case. He first needs to construct a theory for algebraic spaces, and making use of the theory, he may construct that for algebraic stacks.

19.2.4.6 (Crystalline companion)**.** The Weil conjectures were finally proven by P. Deligne in the 1970s, culminating in the theory of weights for l-adic cohomology in his celebrated paper [Del80]. A p-adic analogue of such a theory of weights were proved by K. Kedlaya for overconvergent isocrystals in [Ked06b], and by T. Abe and the author for arithmetic $\mathcal{D}$-modules in [AC18]. Moreover, in his famous paper, Deligne made the following conjecture on the existence of compatible systems :

Conjecture (Deligne [Del80, 1.2.10])**.** Soient X un schéma normal connexe de type fini sur $\mathbb{F}_p$, et $\mathcal{F}$ un faisceau lisse irréducible dont le déterminant est défini par un caract'ere d'ordre fini du groupe fondamental.

(ii) Il existe un corps de nombres $E \subset \overline{\mathbb{Q}}_l$ tel que le polynôme $\det(1 - F_x t, \mathcal{F})$ pour $x \in |X|$, soit à coefficients dans E.

(v) Pour E convenable (peut-être plus grand qu'en (ii)), et chaque place non archimédienne λ première à p, il existe un E_λ-faisceau compatible à $\mathcal{F}$ (mêmes valeurs propres des Frobenius).

(vi) Pour λ divisant p, on espère des petits camarades cristallins.

The part (vi) is written vaguely because a good theory of p-adic cohomology was not available at the time Deligne conjectured it. R. Crew made this conjecture more precise in [Cre92, 4.13] after P. Berthelot's foundational works in p-adic cohomology theory. This conjecture has been one of the driving forces in developing a p-adic cohomology theory over fields of positive characteristic parallel to the l-adic cohomology theory (e.g., see the introduction of [Chi98]). When X is a curve, all parts of the the conjecture except for (vi) are consequences of the Langlands correspondence, which was proven by V. Drinfeld in the rank 2 case and by L. Lafforgue in the higher rank case. Moreover, Deligne and Drinfeld proved all parts of the conjecture except for (vi) for any smooth scheme X as a consequence of the Langlands correspondence. When X is a curve, T. Abe proves a correspondence between irreducible overconvergent F-isocrystals with finite determinant on an open dense subscheme of X and cuspidal automorphic representations of the function field of X with finite central character (see [Abe18, Theorem 4.2.2]). This implies in particular the part (vi) of Deligne's conjecture in the curve case. T. Abe also proves in [Abe18, Theorem 4.4.5] the converse of Deligne's conjecture when X is smooth using the techniques of Deligne and Drinfeld in [EK12] and [Dri12] assuming the Bertini-type conjecture (more precisely, see [Abe18, Theorem 4.4.2]): for any overconvergent F-isocrystal over a smooth scheme, there exists an l-adic companion for any $l \neq p$. His strategy of proof uses similar ingredients to the l-adic case: for instance, he needs the product formula for epsilon factors, which was proven in the p-adic setting by Abe and A. Marmora in [AM].

19.2.5 Constructible t-structure

For completeness, we introduce the notion of constructibility. Let $\mathfrak{C}$ be a data of coefficients over $\mathcal{V}$ which contains $\mathfrak{B}_{\mathrm{div}}$, which is stable under devissages, pushforwards, extraordinary pullbacks, duals, tensor products, and cohomology.

19.2.5.1 (Constructible t-structure)**.** Let $\mathbb{Y} := (Y, X)$ be a realizable pair. Choose a frame $(Y, X, \mathfrak{P})$. If $Y' \to Y$ is an immersion, then we denote by $i_{Y'} \colon (Y', X', \mathfrak{P}) \to (Y, X, \mathfrak{P})$ the induced morphism where X' is the closure of Y' in X. We define on $\mathfrak{C}(\mathbb{Y}/\mathcal{W})$ the constructible t-structure as follows. An object $\mathcal{E} \in \mathfrak{C}(\mathbb{Y}/\mathcal{W})$ belongs to $\mathfrak{C}^{c,\geq 0}(\mathbb{Y}/\mathcal{W})$ (resp. $\mathfrak{C}^{c,\leq 0}(\mathbb{Y}/\mathcal{W})$) if there exists a special morphism $\mathcal{W} \to (\mathcal{W}', \mathcal{W}^\flat)$ such that, denoting by $\mathfrak{P}' := \mathfrak{P} \times_{\mathrm{Spf}\,\mathcal{W}} \mathrm{Spf}\,\mathcal{W}'$, $Y' := (Y \times_{\mathrm{Spf}\,\mathcal{W}} \mathrm{Spf}\,\mathcal{W}')_{\mathrm{red}}$ and $\mathcal{E}'^{(\bullet)} := \mathcal{W}' \widehat{\otimes}^{\mathbb{L}}_{\mathcal{W}} \mathcal{E}^{(\bullet)}$, there exists a smooth stratification (see 16.3.1.2) $(Y'_i)_{i=1,\ldots,r}$ of Y' such that for any i, the complex $i^+_{Y'_i}(\mathcal{E}'^{(\bullet)})[d_{Y'_i}]$ (see notation 19.2.2.6) belongs to $\mathfrak{C}^{\geq 0}_{\mathrm{isoc}}(Y'_i, \mathfrak{P}'/\mathcal{W})$ (resp. $\mathfrak{C}^{\leq 0}_{\mathrm{isoc}}(Y'_i, \mathfrak{P}'/\mathcal{W})$).

Proposition 19.2.5.2. *Let $\mathbb{Y} := (Y, X)$ be a realizable pair.*

(a) Let $\mathcal{E}'^{(\bullet)} \to \mathcal{E} \to \mathcal{E}''^{(\bullet)} \to \mathcal{E}'^{(\bullet)}[1]$ be an exact triangle in $\mathfrak{C}(\mathbb{Y}/\mathcal{W})$. If $\mathcal{E}'^{(\bullet)}$ and $\mathcal{E}''^{(\bullet)}$ are in $\mathfrak{C}^{c,\geq 0}(\mathbb{Y}/\mathcal{W})$ (resp. $\mathfrak{C}^{c,\leq 0}(\mathbb{Y}/\mathcal{W})$) then so is $\mathcal{E}$.

(b) Suppose that Y is smooth. Let $\mathcal{E} \in \mathfrak{C}_{isoc}(\mathbb{Y}/\mathcal{W})$. Then $\mathcal{E} \in \mathfrak{C}^{c,\geq 0}(\mathbb{Y}/\mathcal{W})$ (resp. $\mathcal{E} \in \mathfrak{C}^{c,\leq 0}(\mathbb{Y}/\mathcal{W})$) if and only if $\mathcal{E} \in \mathfrak{C}^{\geq d_X}_{isoc}(\mathbb{Y}/\mathcal{W})$ (resp. $\mathcal{E} \in \mathfrak{C}^{\leq d_X}_{isoc}(\mathbb{Y}/\mathcal{W})$).

Proof. This is left to the reader. □

19.3 Comparison between p-adic de Rham type cohomologies

19.3.1 A comparison between rigid and $\mathcal{D}$-module cohomology for constructible isocrystals of Frobenius type

19.3.1.1. Let $\mathfrak{P}$ be a smooth separated formal scheme over $\mathfrak{S}$. Let Z and X be two closed subschemes of P such that $X \setminus Z$ is smooth over $\operatorname{Spec} k$. Following 16.2.7.11.1, we have constructed the canonical equivalence of categories

$$\mathrm{sp}^{(\bullet)}_{Y,+} = \mathrm{sp}^{(\bullet)}_{X\hookrightarrow\mathfrak{P},Z,+} \colon (F\text{-})\mathrm{MIC}^{\dagger}(X,\mathfrak{P},Z/K) \cong (F\text{-})\mathrm{MIC}^{(\bullet)}(X,\mathfrak{P},Z/\mathcal{V}).$$

Let Y be a realisable smooth k-variety. By embedding Y inside a smooth and proper formal $\mathcal{V}$-scheme $\mathfrak{P}$, it follows from 18.3.2.2 that the above functor sp_+ has the factorisation:

$$\mathrm{sp}_{Y,+} \colon F\text{-}\mathrm{Isoc}^{\dagger}(Y,\mathfrak{P}/\mathcal{V}) \to F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V}),$$
$$\mathrm{sp}_{Y,+} \colon \mathrm{Isoc}^{\dagger}_{F}(Y,\mathfrak{P}/\mathcal{V}) \to D^{\mathrm{b}}_{F\text{-}\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V}),$$

where $\mathrm{Isoc}^{\dagger}_{F}(Y,\mathfrak{P}/\mathcal{V})$ is the category of overconvergent isocrystals on Y of Frobenius type, where $F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V})$ is the category of complexes of $D^{\mathrm{b}}_{\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V})$ having a Frobenius structure, and $D^{\mathrm{b}}_{F\text{-}\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V})$ is the category of overholonomic complexes after any base change on $Y/\mathcal{V}$ of Frobenius type (i.e. according to notation 19.1.2.19, the functor $\underrightarrow{l}^{*}_{\mathbb{Q}}$ induces the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},F\text{-}\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V}) \cong D^{\mathrm{b}}_{F\text{-}\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V})$). It follows from 11.3.5.3 (similarly to the remark of [AL22, 7.1.2]), that for any $E \in F\text{-}\mathrm{MIC}^{\dagger}(Y,\mathfrak{P}/K)$, we have the isomorphism of $F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(Y,\mathfrak{P}/\mathcal{V})$:

$$\mathrm{sp}_{Y+}(E^{\vee}) \xrightarrow{\sim} \mathbb{D}_{Y} \circ \mathrm{sp}_{Y+}(E)(d_Y). \tag{19.3.1.1.1}$$

In fact, thanks to a descent construction of T. Abe [Abe19], the fully faithful functor $\mathrm{sp}_{Y,+}$ can be extended to any (non necessarily smooth) realisable k-variety. In the Rieman-Hilbert's correspondence point of vue, using notation 8.8.3.3, we write

$$\mathrm{sp}^{\mathrm{RH}}_{Y,+} := \mathrm{sp}_{Y,+}[-d_Y](-d_Y)^{2}. \tag{19.3.1.1.2}$$

19.3.1.2 (Constructible isocrystals)**.** In [LS14, LS16], B. Le Stum's introduced a theory of constructible isocrystals. Le Stum's category of constructible isocrystals on $\mathfrak{P}$ corresponds to a rigid cohomological analogue of the category of constructible ℓ-adic sheaves, and consists of convergent ∇-modules on $\mathfrak{P}_K$, which become locally free on the tubes of each stratum in some stratification of $\mathfrak{P}$.

Theorem 19.3.1.3 (Abe-Lazda)**.** *Let $\mathfrak{P}$ be a realisable smooth formal $\mathcal{V}$-scheme, and $\mathcal{F}$ is a constructible isocrystal of Frobenius type. Then $\mathrm{sp}^{\mathrm{RH}}_{!}\mathcal{F}$ is a dual constructible complex of overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{P}\mathbb{Q}}$-modules. If $\mathcal{F}$ is supported on some locally closed subscheme $Y \hookrightarrow P$, then so is $\mathrm{sp}_{!}\mathcal{F}$, where $\mathrm{sp}^{\mathrm{RH}}_{!} := \mathbb{R}\mathrm{sp}_{*}[\dim \mathfrak{P}]$.*

Proof. The strategy is to use de Jong's theory of alterations to reduce to the case of overconvergent isocrystals on completely smooth d-frames. To get the overholonomicity, the fundamental result of 18.3.2.2 is needed. This descent requires also a rather delicate verification of the compatibility of the functor $\mathrm{sp}_{!}$ with the rigid analytic trace map defined in [AL20]. For more details, see [AL22, 6.2.1]. □

[2] Beware in [AL22], $\mathrm{sp}^{\mathrm{RH}}_{Y,+}$ (resp. $\mathrm{sp}_{Y,+}$) is denoted by sp_{+} (resp. $\widetilde{\mathrm{sp}}_{+}$)

Abe and Lazda also prove the compatibility of $\mathrm{sp}_!$ with arbitrary pullback, as well as pushforward along a locally closed immersion. Both functors sp_+ and $\mathrm{sp}_!$ are compatible as follows, which justifies the notation:

Theorem 19.3.1.4 (Abe-Lazda)**.** *Let Y be a realisable smooth k-variety, and E an overconvergent isocrystal of Frobenius type on Y. Then there is a natural isomorphism*

$$\mathrm{sp}_+^{\mathrm{RH}} E \cong \mathbb{D}_Y \mathrm{sp}_!^{\mathrm{RH}} E^\vee$$

in $D^{\mathrm{b}}_{F\text{-}\mathrm{h}}(Y)$.

Proof. Given this descent result, and compatibility with pullbacks, the proof of Theorem 19.3.1.4 is actually relatively straightforward. The point is that the objects involved lie in an abelian category which satisfies h-descent. Thus to construct such an isomorphism locally for the h-topology on Y. Via some de Jong's alterationd, they reduce to the case when Y has a smooth compactification, which then locally lifts to a smooth formal $\mathcal{V}$-scheme. It is then possible to compare $\mathrm{sp}_!^{\mathrm{RH}}$ with $\mathrm{sp}_+^{\mathrm{RH}}$ more or less directly. For more details, see [AL22, 7.1.1]. □

Theorem 19.3.1.5 (Abe-Lazda)**.** *Let Y/k be a realisable k-variety, $f\colon Y \to \operatorname{Spec} k$ its structure morphism. Let E be an overconvergent isocrystal on Y of Frobenius type. We denote by $H^*_{\mathrm{rig}}(Y, E)$ (resp. $H^*_{c,\mathrm{rig}}(Y, E)$) the rigid cohomology (the rigid cohomology with compact support) of E.*

(a) There is a canonical isomorphism

$$H^*_{c,\mathrm{rig}}(Y, E) \cong H^*(f_! \circ \mathrm{sp}^{\mathrm{RH}}_{Y,+}(E))$$

of finite dimensional, graded K-vector spaces. When $E \in F\text{-}\mathrm{Isoc}^\dagger(Y, \mathfrak{P}/\mathcal{V})$, this is compatible with Frobenius.

(b) Suppose Y is either *smooth* or *proper. There is a canonical isomorphism*

$$H^*_{\mathrm{rig}}(Y, E) \cong H^*(Y, f_+ \circ \mathrm{sp}^{\mathrm{RH}}_{Y,+}(E))$$

of finite dimensional, graded K-vector spaces. When $E \in F\text{-}\mathrm{Isoc}^\dagger(Y, \mathfrak{P}/\mathcal{V})$, this is compatible with Frobenius.

Proof. Let us only give few ingredients of Abe-Lazda's proof. Given an embedding of $i_Y : Y \hookrightarrow \mathfrak{P}$ into a smooth and proper formal $\mathcal{V}$-scheme, the proof of the part (a) boils down to computing the de Rham cohomology of the constructible ∇-module $i_{Y!}E$ on $\mathfrak{P}_K$. Using 19.3.1.4, we reduce to check a similar isomorphism with $\mathrm{sp}^{\mathrm{RH}}_{Y,!}$ instead of $\mathrm{sp}^{\mathrm{RH}}_{Y,+}$. The functor $\mathrm{sp}_!^{\mathrm{RH}}$ has a genuinely global definition which make easier to prove this comparison theorem for compactly supported cohomology. A key ingredient in achieving this is the trace morphism in rigid analytic geometry defined in [AL20], which enables to interpret the de Rham cohomology of $i_{Y!}E$ as the rigid Borel–Moore homology of Y with coefficients in E. The part (b) is a consequence of the part (a) and Poincaré duality. See [AL22, 7.2.1 and 7.2.2] for more details. □

19.3.1.6. Kedlaya proved that the rigid cohomology with or without compact support $H^*_{c,\mathrm{rig}}(Y, E)$ and $H^*_{c,\mathrm{rig}}(Y, E)$ of an overconvergent F-isocrystals E, have finite cohomological dimension (see [Ked06a]), which is the culmination of a long series of finiteness results of these rigid cohomologies: We first got the first finiteness result by Crew in the case where Y is an affine and smooth curve (see [Cre98]). When E is equal to the constant coefficient, i.e. $E = j^\dagger \mathcal{O}_{\mathfrak{P}_K}$, the finiteness of $H^*_{\mathrm{rig}}(Y/K)$, the rigid cohomology with or without compact support was first proved Berthelot proved in [Ber97b] in the case where Y/S is smooth. Via a proper cohomological descent and using de Jong's desingularisation theorem, Tsuzuki proved in [Tsu03], the finiteness of $H^*_{c,\mathrm{rig}}(Y/K)$ and $H^*_{c,\mathrm{rig}}(Y/K)$ for any k-scheme of finite type Y. Grosse Könne gave a second proof of this later result in [GK02] via his theory of dagger spaces"dagger spaces".

19.3.2 L-functions associated with arithmetic $\mathcal{D}$-modules

Let $q = p^s$ be a power of a prime number. We assume the residual field k is equal to a field with q elements $\mathcal{F}_q$. Let Y be a variety quasi-projective on k. Choose a projective and smooth $\mathcal{V}$-formal scheme $\mathfrak{P}$, such that Y is a subscheme of the special fiber P of $\mathfrak{P}$. Let X be the closure of Y in P and $j: Y \subset X$ be the open immersion.

19.3.2.1. Let Y^0 be the set of closed points of Y. Let $y \in Y^0$. Then we denote by $i_y\colon \operatorname{Spec} k(y) \hookrightarrow Y$ the induced closed immersion. Choose a morphism (by abuse of notation still denoted by i_y) of smooth $\mathcal{V}$-formal schemes of the form $i_y\colon \operatorname{Sp} \mathcal{V}(y) \hookrightarrow \mathfrak{P}$ which is a lifting of the closed immersion $\operatorname{Spec} k(y) \hookrightarrow Y$ induced by i_y. Let $i_{y,K}\colon \operatorname{Sp} K(y) \hookrightarrow \mathfrak{P}_K$ be the induced morphism of rigid analytic K-spaces.

Definition 19.3.2.2. With notation 19.2.4.3, let $(\mathcal{E},\ \Phi) \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V})$. We define the L-function associated with $(\mathcal{E},\ \Phi)$ [3] by setting:

$$L(Y, \mathcal{E}, t) = \prod_{y\in Y^0}\prod_{r\in\mathbb{Z}} \det{}_K \left(1 - t^{\deg y} F^{\deg y} | H^r(i_y^+(\mathcal{E}))\right)^{(-1)^{r+1}/\deg y}.$$

19.3.2.3 (Link with the L-functions associated with overconvergent F-isocrystals)**.** Let $(E, \Phi) \in F\text{-}\operatorname{Isoc}^\dagger(Y, \mathfrak{P}/K)$. For any closed point y of Y, let $E_y := H^0_{\mathrm{rig}}(\operatorname{Spec} k(y), i^*_{y,K}E)$ the fiber of E in y and $F|E_y$ its Frobenius automorphism.

In [ÉLS93, 2.3], the L-function associated with (E, Φ) is given by

$$L(Y, E, t) := \prod_{y\in Y^0} \det{}_K(1 - t^{\deg y} F^{\deg y} | E_y)^{-1/\deg y}. \tag{19.3.2.3.1}$$

They proved in [ÉLS93] the cohomological formula:

$$L(Y, E, t) := \prod_{i} \det{}_K(1 - t^{\deg y} F | H^i_{c,\mathrm{rig}}(Y, E))^{(-1)^{i+1}}. \tag{19.3.2.3.2}$$

Viewing $H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_p)$ as the kernel of the action of $1-F$ acting on $H^i_{c,\mathrm{rig}}(Y/K)$ they checked in [ÉLS97] Katz's conjecture is valid for the constant coefficient, i.e. the function

$$Z(X, t)/\det(1 - tF | H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_p))^{(-1)^{i+1}}$$

has neither zero nor pole on the unit disk ($\overline{k}$ denotes an algebraic closure of $\mathbb{F}_q$).

Building on work of Crew, Kedlaya proved in [Ked06b] a rigid cohomological analogue of the main result of Deligne's of "Weil II" (i.e. [Del80]), which gives a purely p-adic proof of the Weil conjectures. Ingredients include a p-adic analogue of Laumon's application of the geometric Fourier transform in the l-adic setting or that of Huyghe's one as exposed in 9.4.6. This gives a rigid avatar of Deligne's theory of weights. Such a theory has an arithmetic $\mathcal{D}$-modules analogue (see [AC18]).

The computation of Weil's zeta functions via p-adic methods leads to new algorithms (e.g. see [Ked04c], [Har07]).

Proposition 19.3.2.4. *Suppose Y is smooth. Let $(E, \Phi) \in F\text{-}\operatorname{Isoc}^\dagger(Y, \mathfrak{P}/K)$. Since the functor sp_{Y+} commutes with pullbacks (see 16.2.4.3.1), then we get a Frobenius structure on $\mathrm{sp}_{Y+}E$. We have the equality:*

$$L(Y, E, t) = L(\mathfrak{Y}, \mathrm{sp}^{\mathrm{RH}}_{Y+}(E), t).$$

Proof. Using 19.3.1.1.1 and 19.3.1.1.2, we reduce to check

$$L(Y, E^\vee, t) = L(\mathfrak{Y}, \mathbb{D}_{\mathfrak{X},T}(\mathrm{sp}_{Y+}E)[d_Y], t).$$

Let y be a closed point of Y. By transitivity of the isomorphism 16.2.4.3.1, the canonical functorial in E isomorphism: $\mathrm{sp}_{y+} \circ i^*_{y,K}(E) \xrightarrow{\sim} i^!_{y\,T} \circ \mathrm{sp}_{Y+}(E)[d_{\mathfrak{X}}]$ is therefore compatible with Frobenius, where sp_{y+} is equal to the pushforward sp_{y*} by the specialisation morphism $\mathrm{sp}_y\colon \operatorname{Sp} K(y) \to \operatorname{Spf} \mathcal{V}(y)$. Furthermore, as E is a $j^\dagger\mathcal{O}_{\mathfrak{P}_K}$-locally free module of finite type, we have a $K(y)$-linear isomorphism compatible with Frobenius: $\mathrm{sp}_{y+} \circ i^*_{y,K}(E^\vee) \xrightarrow{\sim} \mathbb{D}_y \circ \mathrm{sp}_{y+} \circ i^*_{y,K}(E)$. Hence: $\mathrm{sp}_{y+} \circ i^*_{y,K}(E^\vee) \xrightarrow{\sim} \mathbb{D}_y \circ i^!_{y,\,T}(\mathrm{sp}_{Y+}E)[d_X]$. Using the isomorphism compatible with Frobenius $i^+_y(\mathbb{D}_Y(\mathrm{sp}_{Y+}E)) \xrightarrow{\sim} \mathbb{D}_y \circ i^!_{y,\,T}(\mathrm{sp}_{Y+}E)$, we then conclude the proof. ☐

[3]Compared to [Car06b, 3.1.1 and 3.1.3], we have multiplied by $(-1)^{d_Y}$ both L-function and cohomological function

Example 19.3.2.5. Let $j^\dagger\mathcal{O}_{]X[_{\mathfrak{P}}}$ be the constant isocrystal on Y/K and let $K_Y := \mathrm{sp}_{Y+}(j^\dagger\mathcal{O}_{]X[_{\mathfrak{P}}})$ be the associated object of $F\text{-}D^{\mathrm{b}}_{\mathrm{h}}(Y, \mathfrak{P}/\mathcal{V})$. Then

$$L(Y, K_Y, t) = L(Y, j^\dagger\mathcal{O}_{]X[_{\mathfrak{P}}}, t) = Z(Y, t),$$

where $Z(Y, t)$ it the Weil zeta function.

Theorem 19.3.2.6. *For any complex $(\mathcal{E},\ \Phi) \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/\mathcal{V})$, we have the equality:*

$$L(Y, \mathcal{E}, t) = \prod_{r\in\mathbb{Z}} \det_K (1 - tF|H^r(f_{T,!}\mathcal{E}))^{(-1)^{r+1}}.$$

Proof. By devissage, we reduce to the case where Y is smooth and $(\mathcal{E},\ \Phi)$ comes from an overconvergent F-isocrystal. In that case, we have the cohomological interpretation of Étesse et Le Stum (see 19.3.2.3.2). Using the comparison between both L-functions (19.3.2.4) and cohomology (see 19.3.1.5.(b)), then we are done. □

19.3.3 Comparison with some other cohomologies

19.3.3.1. E. Grosse-Klönne in [GK00] defines a category of "rigid spaces with overconvergent structures sheaf", which he calls "dagger spaces". He develops a theory of coherent analytic sheaves for his spaces, and shows that it satisfies an analogue of Serre duality. He also defines de Rham cohomology, and de Rham cohomology with compact support, and shows that these satisfy Poincaré duality and the Künneth formula. Finally, he compares this de Rham cohomology theory to Berthelot's rigid cohomology and proved its finiteness (see [GK02]).

19.3.3.2 (Comparison with overconvergent de Rham-Witt cohomology)**.** Let $W = W(k)$ be the ring of Witt vectors over k, and let $K = W[1/p]$. Let X be a smooth variety over k. The rigid cohomology groups $H^i_{\mathrm{rig}}(X/K)$ are finite-dimensional K-vector spaces. One might wonder whether there is an integral cohomology theory with similar finiteness properties. If X is proper over k, then the crystalline cohomology $H^i_{\mathrm{crys}}(X/W)$ is a finitely generated W-module, and its image in $H^i_{\mathrm{rig}}(X/K)$ is a lattice in $H^i_{\mathrm{rig}}(X/K)$. However, without the properness assumption, $H^i_{\mathrm{crys}}(X/W)$ is not always finitely generated. L. Illusie introduced in [Ill79] a canonical sheaf of differential graded algebras $W\Omega^\bullet_{X/k}$ on the Zariski site of X, called the de Rham-Witt complex, whose degree zero component is the ring of Witt vectors $W\mathcal{O}_X$ of the structure sheaf $\mathcal{O}_X$. He proved that its cohomology $H^i(X, W\Omega^\bullet_{X/k})$ is isomorphic to $H^i_{\mathrm{crys}}(X/W)$. Later, A. Langer, and T. Zink generalize Illusie's definition of the de Rham-Witt complex to a relative situation, i.e. more precisely for a proper and smooth morphism (see [LZ04]).

In [DLZ11, DLZ12], C. Davis, A. Langer, and T. Zink introduced a natural differential graded subalgebra $W^\dagger\Omega^\bullet_{X/k} \subset W\Omega^\bullet_{X/k}$, called the overconvergent de Rham-Witt complex, and shows that its cohomology tensored with $\mathbb{Q}$ computes the rigid cohomology of X in case X is quasi-projective, i.e. $H^i(X, W^\dagger\Omega^\bullet_{X/k}) \otimes_W K$ is isomorphic to $H^i_{\mathrm{rig}}(X/W)$. The degree zero term $W^\dagger\mathcal{O}_X$ is called the sheaf of overconvergent Witt vectors, and its main properties were already developed in a previous work by the same authors; it is defined in terms of a growth condition for the components of the Witt vectors that is too technical to reproduce here. These growth conditions are then extended to the whole de Rham-Witt complex, to define the overconvergent subcomplex.

The torsion subgroup of $H^i(X, W^\dagger\Omega^\bullet_{X/k})$ can be infinitely generated (e.g. in the case $X = \mathbb{A}^1_k$ and $i = 1$). However, one might still wonder if the image of $H^i(X, W^\dagger\Omega^\bullet_{X/k})$ in $H^i_{\mathrm{rig}}(X/W)$, the ith space of the integral overconvergent de Rham-Witt cohomology modulo torsion, is a lattice. In [ES20], V. Ertl and A. Shiho show that if X is a smooth affine curve over k whose smooth compactification has genus zero, then the image of $H^i(X, W^\dagger\Omega^\bullet_{X/k})$ in $H^i_{\mathrm{rig}}(X/W)$ is a finitely generated W-module. However, they construct a counter-example if the genus is instead positive. They conjecture that if the genus is positive, then the image is never finitely generated.

Generalizing a definition of Bloch, V. Ertl introduced overconvergent de Rham-Witt connections in [Ert16]. This provides a tool to extend the comparison morphisms of Davis, Langer and Zink between overconvergent de Rham-Witt cohomology and Monsky-Washnitzer respectively rigid cohomology to coefficients. Another step toward an interpretation of F-isocrystals as overconvergent de Rham-Witt connections has been made by R. Muñoz--Bertrand in [MB22] and [MB24].

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